APPLICATION OF THE TWO-VARIABLE EXPANSION PROCEDURE TO PROBLEMS IN CELESTIAL MECHANICS

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ABSTRACT

This work illustrates the application of the two-variable expansion procedure of reference 1 to the solution of two representative problems in celestial mechanics.

The expansion procedure is applied first to the problem of aerodynamic perturbations of a satellite orbit. The case of planar motion is considered with both lift and drag perturbations acting on the satellite. A simplified model of the earth is used, but the motion is expected to exhibit a similar qualitative behavior in the more general case. It is found that the effect of drag causes the satellite to spiral toward the center of attraction while the orbit is tending to become circular. The effect of lift, to the order computed, is felt only by a slow advance of the apse.

The second application of the expansion procedure is to the problem of third-body perturbations of a satellite orbit. A special case of the restricted three-body problem is used in which the plane of the satellite's orbit is coincident with the orbital plane of the two larger bodies. The two-variable expansion is applied to approximate equations which are valid for satellite orbits close to the smaller of the two large bodies. The results are in exact agreement with those of reference 2 and DePontécoulant's lunar theory. The solution of this problem is to serve as a preliminary step in establishing the choice of variables for the more general case in which the satellite's orbit has a high inclination to the orbital plane of the two larger bodies.

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I. INTRODUCTION

This work illustrates the application of the two-variable expansion procedure of reference 1 to the solution of the following two representative problems in celestial mechanics.

1) Aerodynamic perturbations of a satellite orbit.
2) Third-body perturbation of a satellite orbit.

In both of the above problems, certain simplifying assumptions have been made in formulating the physical problem in order to easily establish the basic ideas of the technique and provide the necessary preliminary step in the solution of the more general cases.

Although in both of these problems the motion is Keplerian in the absence of the perturbing forces, the nature of the perturbations and the behavior of the solutions are quite different in each case.

In problem (1), the perturbation due to aerodynamic drag is dissipative; and this causes the semi-major axis of the osculating ellipse to decrease slowly with time. The two-variable expansion procedure was specifically developed for problems with dissipative perturbations where other methods, such as the method of variation of elements, fail to provide uniformly valid approximations for large times.

In problem (2), the presence of a distant third body introduces conservative perturbations in the sense that the equations of motion possess a conservation law (the Jacobi integral). As a result, the elements of the osculating ellipse undergo bounded variations in some precessing coordinate system.

This problem was solved by essentially Poincaré's procedure in reference 2, and it will be shown that the present approach yields the same results in a more systematic manner. In addition, it is anticipated that the difficulties inherent in the Poincaré method for orbits having arbitrary inclinations will be eliminated by using the two-variable expansion procedure. The present work has, in this sense, established the appropriate approach to the solution of this more general case.

II. AERODYNAMIC PERTURBATIONS OF A SATELLITE ORBIT

2.1 Formulation of the Problem

The problem of drag perturbations of a satellite orbit has been solved in reference 1 by the two-variable expansion procedure. In this section of the present work, the perturbations of a satellite orbit due to aerodynamic lift and drag will be determined by using the two-variable expansion procedure.

The physical model to be used will be the same as that in reference 1. A satellite is assumed to move in a planar orbit about a homogeneous, spherical, and non-rotating earth surrounded by a constant density atmosphere. It is also assumed that no other celestial bodies influence the motion of the satellite. The drag coefficient and lift-drag ratio of the satellite are taken to be constants. This physical model admittedly departs considerably from realism, and is presented only to illustrate the qualitative behavior of the motion. The more realistic model with a variable density, and aerodynamic coefficients depending on Mach number and angle of attack can be treated by this method at the cost of considerable algebraic complication.

2.2 Equation of Motion and Initial Conditions

The dimensional equations of motion may be written as:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 = -\frac{GM}{r^2} - \frac{C_D\rho S}{2m}\left[\frac{dr}{dt} - nr\frac{d\varphi}{dt}\right]\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right]^{1/2} \qquad (2.1a)$$

$$\frac{rd^2\varphi}{dt^2} + 2\,\frac{dr}{dt}\,\frac{d\varphi}{dt} = -\,\frac{C_D\rho S}{2m}\left[r\,\frac{d\varphi}{dt} + n\,\frac{dr}{dt}\right]\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right]^{1/2} \qquad (2.1b)$$

where

C_D = drag coefficient of the satellite

G = universal gravitation constant

M = mass of the earth

m = mass of the satellite

n = lift-drag ratio of the satellite

r = radial distance from center of earth to satellite

S = reference area of the satellite

t = time

ρ = atmospheric density

φ = true anomaly

If the following dimensionless quantities are chosen

$$r^* = \frac{r}{R} \qquad t^* = \frac{t}{T} \qquad \epsilon = \frac{C_D\rho SR}{2m}$$

where

R = initial radial distance of the satellite

T = period of the Keplerian orbit corresponding to the initial conditions = $\sqrt{R^3/GM}$

ϵ = aerodynamic parameter (ratio of drag to centrifugal force initially)

equations (2.1) may be rewritten in the non-dimensional form:

$$\frac{d^2 r^*}{dt^{*2}} - r^*\left(\frac{d\varphi}{dt^*}\right)^2 = -\frac{1}{r^{*2}} - \epsilon\left[\frac{dr^*}{dt^*} - nr^*\frac{d\varphi^*}{dt^*}\right]\left[\left(\frac{dr^*}{dt^*}\right)^2 + r^{*2}\left(\frac{d\varphi}{dt^*}\right)^2\right]^{1/2} \tag{2.2a}$$

$$\frac{r^* d^2\varphi}{dt^{*2}} + \frac{2dr^*}{dt^*}\frac{d\varphi}{dt^*} = -\epsilon\left[r^*\frac{d\varphi}{dt^*} + \frac{ndr^*}{dt^*}\right]\left[\left(\frac{dr^*}{dt^*}\right)^2 + r^{*2}\left(\frac{d\varphi}{dt^*}\right)^2\right]^{1/2} \tag{2.2b}$$

These equations are next transformed so that u $(=1/r^*)$ and t^* become the dependent variables and φ the independent variable. Hence,

$$\frac{d^2u}{d\varphi^2} + u = u^4\left(\frac{dt^*}{d\varphi}\right)^2 - \frac{\epsilon n}{u^3}\left[\left(\frac{du}{d\varphi}\right)^2 + u^2\right]^{3/2} \tag{2.3a}$$

$$u^2\frac{d^2t^*}{d\varphi^2} + 2u\frac{du}{d\varphi}\frac{dt^*}{d\varphi} = \epsilon\frac{dt^*}{d\varphi}\left[1 - \frac{n}{u}\frac{du}{d\varphi}\right]\left[\left(\frac{du}{d\varphi}\right)^2 + u^2\right]^{1/2} \tag{2.3b}$$

The following initial conditions are adopted:

$$u(0) = 1 \qquad \frac{du(0)}{d\varphi} = 0 \qquad t^*(0) = 0 \qquad \frac{dt^*}{d\varphi}(0) = 1 \tag{2.4}$$

2.3 The Two-Variable Expansion

As indicated in reference 1, the decay of the orbit due to drag will be exhibited in the solution by the behavior of u with respect to the slow variable $\tilde{\varphi} = \epsilon\varphi$; and it will be required that u be a bounded function of φ for fixed $\tilde{\varphi}$. Equations (2.3) will be solved by the two-variable expansion procedure where u and t^* will be expanded in terms of the independent variables φ and $\tilde{\varphi}$, and the indeterminacy introduced by $\tilde{\varphi}$ will be eliminated by appropriate boundness conditions. Let

$$u = U(\varphi, \tilde{\varphi}, \epsilon) = U^{(0)}(\varphi, \tilde{\varphi}) + \epsilon U^{(1)}(\varphi, \tilde{\varphi}) + O(\epsilon^2) \tag{2.5a}$$

$$t^* = t^*(\varphi, \tilde{\varphi}, \epsilon) = t^{(0)}(\varphi, \tilde{\varphi}) + \epsilon t^{(1)}(\varphi, \tilde{\varphi}) + O(\epsilon^2) \tag{2.5b}$$

Then

$$\frac{du}{d\varphi} = U_1^{(0)} + \epsilon \left(U_1^{(1)} + U_2^{(0)} \right) + O(\epsilon^2) \tag{2.6a}$$

$$\frac{d^2u}{d\varphi^2} = U_{11}^{(0)} + \epsilon \left(U_{11}^{(1)} + 2U_{12}^{(0)} \right) + O(\epsilon^2) \tag{2.6b}$$

with similar expressions for $dt^*/d\varphi$ and $d^2t^*/d\varphi^2$. The subscripts one and two indicate partial derivatives with respect to φ and $\tilde{\varphi}$, respectively. Expressions (2.5) and (2.6) are inserted in equations (2.3), and the limit process of section 2.3 of reference 1 is applied to obtain:

$$U_{11}^{(0)} + U^{(0)} = \left(U^{(0)} \right)^4 \left(t_1^{(0)} \right)^2 \tag{2.7a}$$

$$\left(U^{(0)} \right)^2 t_{11}^{(0)} + 2U^{(0)} U_1^{(0)} t_1^{(0)} = 0 \tag{2.7b}$$

$$U_{11}^{(1)} + 2U_{12}^{(0)} + U^{(1)} = 2 \left(U^{(0)} \right)^4 t_1^{(0)} \left(t_1^{(1)} + t_2^{(0)} \right) + 4 \left(U^{(0)} \right)^3 U^{(1)} \left(t_1^{(0)} \right)^2$$
$$- \frac{n}{\left(U^{(0)} \right)^3} \left[\left(U_1^{(0)} \right)^2 + \left(U^{(0)} \right)^2 \right]^{3/2} \tag{2.8a}$$

$$\left(U^{(0)} \right)^2 \left(t_{11}^{(1)} + 2t_{12}^{(0)} \right) + 2U^{(0)} U^{(1)} t_{11}^{(0)} + 2U^{(0)} \left[U_1^{(0)} \left(t_1^{(1)} + t_2^{(0)} \right) + t_1^{(0)} \left(U_1^{(1)} + U_2^{(0)} \right) \right]$$
$$+ 2U^{(1)} U_1^{(0)} t_1^{(0)} = t_1^{(0)} \left[1 - \frac{nU_1^{(0)}}{U^{(0)}} \right] \left[\left(U_1^{(0)} \right)^2 + \left(U^{(0)} \right)^2 \right]^{1/2} \tag{2.8b}$$

Equations (2.7) give the solution for $\epsilon = 0$, where the motion of the satellite is described by a Keplerian elliptical orbit. Equations (2.8) give the solutions for the first-order perturbations due to aerodynamic forces.

2.4 Solution for the Keplerian Orbit

Integration of equation (2.7b) gives

$$\left(U^{(0)}\right)^2 t_1^{(0)} = f(\tilde{\varphi}) \tag{2.9a}$$

With this result, the solution of equation (2.7a) is

$$U^{(0)} = f^2[1-e\cos(\varphi-\beta)] \tag{2.9b}$$

where

$$f^2 = \frac{1}{a(1-e^2)} \quad \text{(by definition)}$$

$a = a(\tilde{\varphi})$ = dimensionless semi-major axis

$e = e(\tilde{\varphi})$ = eccentricity

$\beta = \beta(\tilde{\varphi})$ = longitude of the apse

Using equation (2.9b), equation (2.9a) may be integrated to give

$$t^{(0)} = a^{3/2}\left[\frac{e(1-e^2)^{\frac{1}{2}}\sin(\varphi-\beta)}{1-e\cos(\varphi-\beta)} + \cos^{-1}\frac{\cos(\varphi-\beta)-e}{1-e\cos(\varphi-\beta)}\right] + \tau \tag{2.10}$$

The quantities a, e, β and τ which correspond to the four integration constants for Keplerian motion, are now functions of $\tilde{\varphi}$ and will be evaluated subsequently.

2.5 Solution for the First-Order Perturbations

By making use of equation (2.9a), equation (2.8b) can be written in the form:

$$\frac{\partial}{\partial\varphi}\left[\left(U^{(0)}\right)^2\left(t_1^{(1)}+t_2^{(0)}\right)+2f\frac{U^{(1)}}{U^{(0)}}\right]+\frac{df}{d\tilde{\varphi}}=\frac{f}{\left(U^{(0)}\right)^2}\left[1-n\frac{U_1^{(0)}}{U^{(0)}}\right]\left[\left(U_1^{(0)}\right)^2+\left(U^{(0)}\right)^2\right]^{\frac{1}{2}} \tag{2.11}$$

Now let

$$\frac{\partial P}{\partial\varphi}=-\frac{f}{\left(U^{(0)}\right)^2}\left[1-n\frac{U_1^{(0)}}{U^{(0)}}\right]\left[\left(U_1^{(0)}\right)^2+\left(U^{(0)}\right)^2\right]^{\frac{1}{2}} \tag{2.12}$$

so that integration of equation (2.11) gives

$$\left(U^{(0)}\right)^2\left(t_1^{(1)}+t_2^{(0)}\right)+\frac{2fU^{(1)}}{U^{(0)}}+P+\frac{df}{d\tilde{\varphi}}\varphi=g(\tilde{\varphi}) \tag{2.13}$$

The quantity P may be obtained by direct integration of equation (2.12). The interest of this work is in orbits of small eccentricity, so that only terms which are linear in e will be retained. Thus

$$\frac{\partial P}{\partial\varphi}=-\frac{1}{f}\left[1+e\cos(\varphi-\beta)-ne\sin(\varphi-\beta)\right]+O(e^2) \tag{2.14}$$

$$P=-\frac{1}{f}\left[\varphi+e\sin(\varphi-\beta)+ne\cos(\varphi-\beta)\right]+O(e^2) \tag{2.15}$$

Then equation (2.8a) becomes

$$U_{11}^{(1)}+U^{(1)}=-2U_{12}^{(0)}+2f\left[-\frac{2fU^{(1)}}{U^{(0)}}+\frac{1}{f}\left[\varphi+e\sin(\varphi-\beta)+ne\cos(\varphi-\beta)\right]\right.$$

$$\left.-\frac{df}{d\tilde{\varphi}}\varphi+g\right]+\frac{4f^2U^{(1)}}{U^{(0)}}-\frac{n}{\left(U^{(0)}\right)^3}\left[\left(U_1^{(0)}\right)^2+\left(U^{(0)}\right)^2\right]^{\frac{3}{2}} \tag{2.16}$$

but

$$U_{12}^{(0)}=\left(\frac{f^2de}{d\tilde{\varphi}}+2f\frac{df}{d\tilde{\varphi}}e\right)\sin(\varphi-\beta)-f^2e\frac{d\beta}{d\tilde{\varphi}}\cos(\varphi-\beta)$$

so that

$$U_{11}^{(1)} + U^{(1)} = \left(-2f^2 \frac{de}{d\tilde{\varphi}} - 4f \frac{df}{d\tilde{\varphi}} e + 2e\right)\sin(\varphi-\beta) + \left(2f^2 e \frac{d\beta}{d\tilde{\varphi}} + 2ne\right)\cos(\varphi-\beta)$$

$$+ 2\left(1-f\frac{df}{d\tilde{\varphi}}\right)\varphi + 2fg-n \tag{2.17}$$

In order that $U^{(1)}$ be a bounded function of φ, the terms proportional to φ, $\sin(\varphi-\beta)$, and $\cos(\varphi-\beta)$ in equation (2.17) must vanish. Hence:

$$1-f\frac{df}{d\tilde{\varphi}} = 0 \tag{2.18a}$$

$$e-f^2\frac{de}{d\tilde{\varphi}} - 2f\frac{df}{d\tilde{\varphi}} e = 0 \tag{2.18b}$$

$$f^2\frac{d\beta}{d\tilde{\varphi}} + n = 0 \tag{2.18c}$$

The solutions of equations (2.18) correct to order (e) and satisfying the initial conditions (2.4) are:

$$f^2 = 2\tilde{\varphi} + 1 \tag{2.19a}$$

$$e = \frac{e_0}{(2\tilde{\varphi}+1)^{1/2}} \tag{2.19b}$$

$$\beta = \ln\frac{1}{(2\tilde{\varphi}+1)^{n/2}} \tag{2.19c}$$

The quantity e_0 is the initial eccentricity and is assumed to be small compared to unity.

Equation (2.17) then becomes

$$U_{11}^{(1)} + U^{(1)} = 2fg-n \tag{2.20}$$

which has the solution

$$U^{(1)} = A(\tilde{\varphi}) \sin(\varphi-\beta) + B(\tilde{\varphi}) \cos(\varphi-\beta) + 2fg-n \tag{2.21}$$

where A and B must be determined by the boundedness requirement on $U^{(2)}$. Insertion of equation (2.21) into equation (2.13) yields

$$t_1^{(1)} = -t_2^{(0)} - \frac{2f[A\sin(\varphi-\beta) + B\,\mathrm{Cos}(\varphi-\beta) + 2fg-n]}{\left(U^{(0)}\right)^3} + \frac{e\sin(\varphi-\beta)}{f\left(U^{(0)}\right)^2}$$
$$+ \frac{ne\cos(\varphi-\beta)}{f\left(U^{(0)}\right)^2} + \frac{g}{\left(U^{(0)}\right)^2} \tag{2.22}$$

or

$$t_1^{(1)} = -\left[\frac{d\tau}{d\tilde{\varphi}} + 3a^2 g + 3a^{\frac{5}{2}} eB + 3a^{\frac{5}{2}}\beta - a^{\frac{5}{2}} n\right] + 3a^{\frac{5}{2}}\varphi + \left(9a^{\frac{5}{2}} e - 2a^{\frac{5}{2}} A\right)\sin(\varphi-\beta)$$
$$-\left(2a^{\frac{5}{2}} B - 5a^{\frac{5}{2}} en + 10a^2 eg\right)\cos(\varphi-\beta) - 3a^{\frac{5}{2}} eA\sin 2(\varphi-\beta) - 3a^{\frac{5}{2}} eB\cos 2(\varphi-\beta) \tag{2.23}$$

and integration gives

$$t^{(1)} = -\left[\frac{d\tau}{d\tilde{\varphi}} + 3a^2 g + 3a^{\frac{5}{2}} eB + 3a^{\frac{5}{2}}\beta - a^{\frac{5}{2}} n\right]\varphi + \frac{3a^{\frac{5}{2}}\varphi^2}{2}$$
$$-\left(9a^{\frac{5}{2}} e - 2a^{\frac{5}{2}} A\right)\cos(\varphi-\beta) - \left(2a^{\frac{5}{2}} B - 5a^{\frac{5}{2}} en + 10a^2 eg\right)\sin(\varphi-\beta)$$
$$+ \frac{3a^{\frac{5}{2}} eA}{2}\cos 2(\varphi-\beta) - \frac{3a^{\frac{5}{2}} eB}{2}\sin 2(\varphi-\beta) + h(\tilde{\varphi}) \tag{2.24}$$

In this equation, the term linear in φ may be eliminated by requiring that

$$\frac{d\tau}{d\tilde{\varphi}} = -3a^2 g - 3a^{\frac{5}{2}} eB - 3a^{\frac{5}{2}}\beta + a^{\frac{5}{2}} n \tag{2.25}$$

This expression determines τ after the functions g and B are obtained from the boundedness requirement on $U^{(2)}$. In equation (2.24), the term quadratic in φ cannot be eliminated. It is therefore characteristic of this problem that the time is an unbounded function of φ.

2.6 Summary and Discussion of Results

The result of the preceding analysis may be summarized as follows:

$$u = \frac{1}{a}\left[1-e\cos(\varphi-\beta)\right] + \epsilon\left[A\sin(\varphi-\beta)+B\cos(\varphi-\beta)+2fg-n\right]+O(\epsilon^2) \tag{2.26a}$$

$$\begin{aligned} t = a^{\frac{3}{2}}\left[\varphi-\beta+2e\sin(\varphi-\beta)\right]+\tau+\epsilon&\left[\frac{3a^{\frac{5}{2}}\varphi^2}{2} - \left(9a^{\frac{5}{2}}e-2a^{\frac{5}{2}}A\right)\cos(\varphi-\beta)\right. \\ &-\left(2a^{\frac{5}{2}}B-5a^{\frac{5}{2}}en+10a^2eg\right)\sin(\varphi-\beta)+\frac{3a^{\frac{5}{2}}eA}{2}\cos 2(\varphi-\beta) \\ &\left.-\frac{3a^{\frac{5}{2}}eB}{2}\sin 2(\varphi-\beta)+h\right]+O(\epsilon^2) \end{aligned} \tag{2.26b}$$

where

$$a = \frac{1}{2\tilde{\varphi}+1} \tag{2.27a}$$

$$e = \frac{e_0}{(2\tilde{\varphi}+1)^{1/2}} \tag{2.27b}$$

$$\beta = \ln\frac{1}{(2\tilde{\varphi}+1)^{n/2}} \tag{2.27c}$$

$$\tau = \tau(\tilde{\varphi}) \quad \text{(given by equation 2.25)} \tag{2.27d}$$

The quantities A, B, g, and τ are the functions obtained from the integration of the first-order perturbation equations (2.8) and can be explicitly determined by requirements on $U^{(2)}$ and $t^{(2)}$. The dominant behavior of the motion is already exhibited by the form of the functions a, e and β.

The effect of drag is to cause the semi-major axis and eccentricity to tend to zero monotonically as $\tilde{\varphi}$ increases. Thus the orbit gradually spirals toward the center of attraction and tends to become circular. This behavior justifies the omission of quadratic terms in e in the solution if the initial eccentricity is sufficiently small. The effect of lift is, to this order, only felt by a slow advance of the apse as shown by equation (2.27c). In order to evaluate the period of this spiraling motion, it is necessary to carry the computations one step further and calculate τ which depends on B and g.

It is expected that for the more realistic problem in which the density and aerodynamic coefficients vary, the motion will exhibit a similar qualitative behavior.

III. THIRD-BODY PERTURBATION OF A SATELLITE ORBIT

3.1 Formulation of the Problem

This problem has been solved in reference 2 by a method similar to that of Poincaré for periodic systems. In the method of reference 2, various coordinate transformations are made in order to account for the perturbations in the satellite orbit due to the sun's gravitational field. In the present work, it will be shown that the two-variable expansion procedure provides a more systematic approach requiring no prior knowledge of the solution.

The restricted three-body formulation will be used in which the sun and planet are assumed to move in circular orbits about their common center of mass. The mass of the planet is assumed to be much smaller than the mass of the sun. The analysis of the present work will further be restricted to the case where the plane of the satellite's orbit coincides with the orbital plane of the sun-planet system.

3.2 Equations of Motion

The dimensional equations of motion of the satellite with respect to a coordinate system centered at the mass-center of the sun-planet system and revolving with the planet are: (see figure 1)

$$\frac{d^2\xi}{d\bar{t}^2} - 2\omega\frac{d\eta}{d\bar{t}} - \omega^2\xi = \frac{Gm_p}{r_p^3}(\xi_p - \xi) - \frac{Gm_S}{r_S^3}(\xi - \xi_S) \tag{3.1a}$$

$$\frac{d^2\eta}{d\bar{t}^2} + 2\omega\frac{d\xi}{d\bar{t}} - \omega^2\eta = -\frac{Gm_p\eta}{r_p^3} - \frac{Gm_S\eta}{r_S^3} \tag{3.1b}$$

where

G = universal gravitation constant

m_p = mass of the planet

m_S = mass of the sun

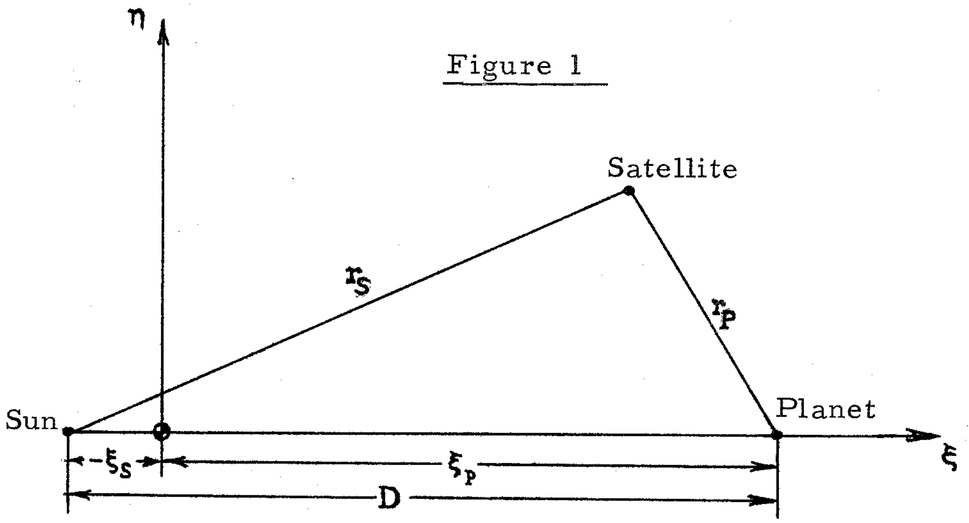
r_p = distance from satellite to planet = $\sqrt{(\xi_p - \xi)^2 + \eta^2}$

r_S = distance from satellite to sun = $\sqrt{(\xi_S - \xi)^2 + \eta^2}$

$\bar{t}$ = time

ω = angular velocity of planet with respect to mass-center

Figure 1



It is shown in reference 2 that for a satellite in close proximity to the planet, the following approximate equations can be derived from equations (3.1):

$$\frac{d^2x}{dt^2} = \frac{-x}{r^3} + 2\epsilon\frac{dy}{dt} + 3\epsilon^2 x + O(\epsilon^4) \tag{3.2a}$$

$$\frac{d^2y}{dt^2} = -\frac{y}{r^3} - 2\epsilon\frac{dx}{dt} + O(\epsilon^4) \tag{3.2b}$$

where

$$+\frac{3}{2}\,\epsilon^2\left(\frac{dt}{d\varphi^*}\right)^2\left[\frac{dS}{d\varphi^*}\sin 2\varphi^* - S\cos 2\varphi^*\right] + O(\epsilon^4) \tag{3.3a}$$

$$\frac{Sd^2t}{d\varphi^{*2}} + 2\,\frac{dS}{d\varphi^*}\,\frac{dt}{d\varphi^*} = -2\epsilon\,\frac{dS}{d\varphi^*}\left(\frac{dt}{d\varphi^*}\right)^2 + \frac{3}{2}\,\epsilon^2 S\left(\frac{dt}{d\varphi^*}\right)^3\sin 2\varphi^* + O(\epsilon^4) \tag{3.3b}$$

3.3 The Two-Variable Expansion

Since φ^* appears explicitly in equations (3.3), it will be used as the fast variable. In order to allow for slight variations in the arguments of the trigonometric terms appearing in the solution, the slow variable will be chosen in the form $\tilde{\varphi} = \epsilon[1+\alpha\epsilon+O(\epsilon^2)]\varphi^*$, where α is an unknown constant to be determined by requiring the solution to be bounded with respect to $\tilde{\varphi}$. The following expansions are then assumed:

$$S = S(\varphi^*, \tilde{\varphi}, \epsilon) = S^{(0)}(\varphi^*, \tilde{\varphi}) + \epsilon S^{(1)}(\varphi^*, \tilde{\varphi}) + \epsilon^2 S^{(2)}(\varphi^*, \tilde{\varphi}) + \ldots \tag{3.4a}$$

$$t = t(\varphi^*, \tilde{\varphi}, \epsilon) = t^{(0)}(\varphi^*, \tilde{\varphi}) + \epsilon t^{(1)}(\varphi^*, \tilde{\varphi}) + \epsilon^2 t^{(2)}(\varphi^*, \tilde{\varphi}) + \ldots \tag{3.4b}$$

Substituting these expansions into equations (3.3) and carrying out the limit process of section 2.3 of reference 1 give

$$S_{11}^{(0)} + S^{(0)} = \left(S^{(0)}\right)^4\left(t_1^{(0)}\right)^2 \tag{3.5a}$$

$$S^{(0)}\,t_{11}^{(0)} + 2S_1^{(0)}\,t_1^{(0)} = 0 \tag{3.5b}$$

$$S_{11}^{(1)} + S^{(1)} + 2S_{12}^{(0)} = 2\left(S^{(0)}\right)^4 t_1^{(0)}\left(t_1^{(1)} + t_2^{(0)}\right) + 4\left(S^{(0)}\right)^3\left(t_1^{(0)}\right)^2 S^{(1)}$$

$$-2S^{(0)}t_1^{(0)} - \frac{2\left(S_1^{(0)}\right)^2 t_1^{(0)}}{S^{(0)}} \tag{3.6a}$$

$$S^{(0)}\left(t_{11}^{(1)}+2t_{12}^{(0)}\right)+S^{(1)}t_{11}^{(0)}+2t_{1}^{(0)}\left(S_{1}^{(1)}+S_{2}^{(0)}\right)+2S_{1}^{(0)}\left(t_{1}^{(1)}+t_{2}^{(0)}\right)=-2S_{1}^{(0)}\left(t_{1}^{(0)}\right)^{2} \tag{3.6b}$$

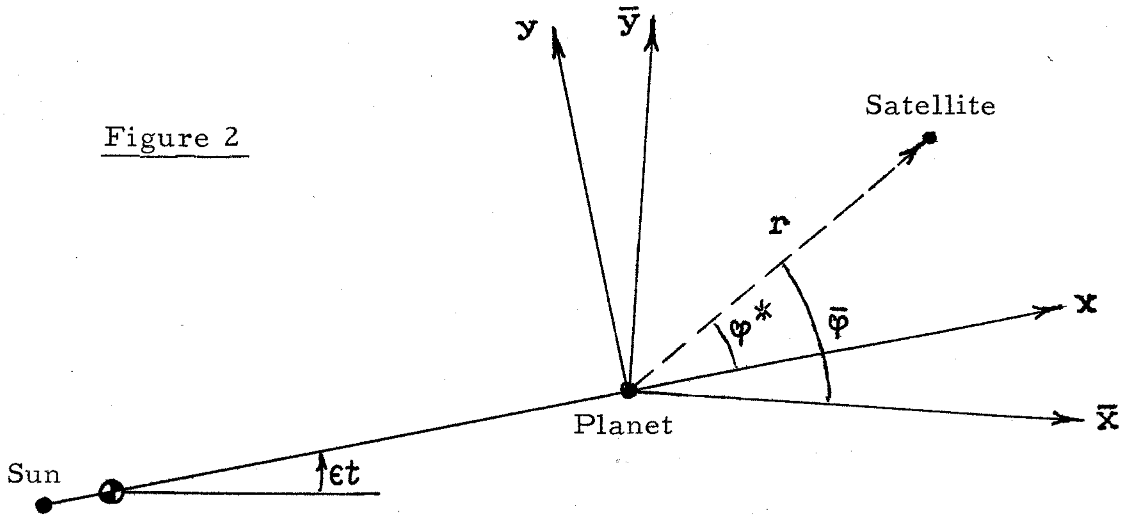
$$\begin{aligned}
S_{11}^{(2)}+S^{(2)}+2S_{12}^{(1)}+S_{22}^{(0)}+2aS_{12}^{(0)} &= 2\left(S^{(0)}\right)^{4}t_{1}^{(0)}\left(t_{1}^{(2)}+t_{2}^{(1)}+at_{2}^{(0)}\right) \\
&+\left(S^{(0)}\right)^{4}\left(t_{1}^{(1)}+t_{2}^{(0)}\right)^{2}+8\left(S^{(0)}\right)^{3}S^{(1)}t_{1}^{(0)}\left(t_{1}^{(1)}+t_{2}^{(0)}\right)+4\left(S^{(0)}\right)^{3}S^{(2)}\left(t_{1}^{(0)}\right)^{2} \\
&+6\left(S^{(0)}\right)^{2}\left(t_{1}^{(0)}\right)^{2}\left(S^{(1)}\right)^{2}-2S^{(0)}\left(t_{1}^{(1)}+t_{2}^{(0)}\right)-2S^{(1)}t_{1}^{(0)}-\frac{4S_{1}^{(0)}t_{1}^{(0)}}{S^{(0)}}\left(S_{1}^{(1)}+S_{2}^{(0)}\right) \\
&-\frac{2\left(S_{1}^{(0)}\right)^{2}\left(t_{1}^{(1)}+t_{2}^{(0)}\right)}{S^{(0)}}+\frac{2\left(S_{1}^{(0)}\right)^{2}t_{1}^{(0)}S^{(1)}}{\left(S^{(0)}\right)^{2}}-\frac{3S^{(0)}}{2}\left(t_{1}^{(0)}\right)^{2} \\
&+\frac{3}{2}\left(t_{1}^{(0)}\right)^{2}\left[S_{1}^{(0)}\sin 2\varphi^{*}-S^{(0)}\cos 2\varphi^{*}\right]
\end{aligned} \tag{3.7a}$$

$$\begin{aligned}
&S^{(0)}\left(t_{11}^{(2)}+2t_{12}^{(1)}+t_{22}^{(0)}+2at_{12}^{(0)}\right)+S^{(1)}\left(t_{11}^{(1)}+2t_{12}^{(0)}\right)+S^{(2)}t_{11}^{(0)} \\
&+2t_{1}^{(0)}\left(S_{1}^{(2)}+S_{2}^{(1)}+aS_{2}^{(0)}\right)+2S_{1}^{(0)}\left(t_{1}^{(2)}+t_{2}^{(1)}+at_{2}^{(0)}\right)+2\left(S_{1}^{(1)}+S_{2}^{(0)}\right)\left(t_{1}^{(1)}+t_{2}^{(0)}\right) \\
&=-4S_{1}^{(0)}t_{1}^{(0)}\left(t_{1}^{(1)}+t_{2}^{(0)}\right)-2\left(t_{1}^{(0)}\right)^{2}\left(S_{1}^{(1)}+S_{2}^{(0)}\right)+\frac{3}{2}S^{(0)}\left(t_{1}^{(0)}\right)^{3}\sin 2\varphi^{*}
\end{aligned} \tag{3.7b}$$

3.4 Initial Conditions

The initial conditions for equations (3.5), (3.6) and (3.7) will be chosen so that the results of this work will be comparable to those of reference 2. In order to do this, it is necessary to develop a relationship between the angles $\bar{\varphi}$ and φ^{*} as shown in figure 2.

Figure 2



In this figure, the plane of the satellite orbit and the plane of the sun-planet orbit coincide with the plane of the paper. The notation is the same as that of reference 2, with the exception of φ^*. The (x, y) frame is the one used in the present work. The initial conditions of reference 2 are given for the $(\bar{x}, \bar{y})$ frame.

From reference 2, $\bar{\varphi}$ is defined by:

$$\bar{\varphi} = \varphi + \epsilon^2 \nu_1 t \tag{3.8a}$$

and

$$\varphi = \varphi^* + \epsilon t - \epsilon^2 \lambda_1 t \tag{3.8b}$$

thus

$$\bar{\varphi} = \varphi^* + \epsilon t + \epsilon^2 (\nu_1 - \lambda_1) t \tag{3.8c}$$

so that

$$\frac{d\bar{\varphi}}{d\varphi^*} = 1 + \epsilon \frac{dt}{d\bar{\varphi}} + O(\epsilon^2) \tag{3.9}$$

Hence

$$\frac{dS}{d\varphi^*} = \frac{dS}{d\bar{\varphi}} \frac{d\bar{\varphi}}{d\varphi^*} = \frac{dS}{d\bar{\varphi}} \left[1 + \epsilon \frac{dt}{d\bar{\varphi}} + O(\epsilon^2) \right] \tag{3.10a}$$

$$\frac{dt}{d\varphi^*} = \frac{dt}{d\bar{\varphi}} \frac{d\bar{\varphi}}{d\varphi^*} = \frac{dt}{d\bar{\varphi}} \left[1 + \epsilon \frac{dt}{d\bar{\varphi}} + O(\epsilon^2) \right] \tag{3.10b}$$

The initial conditions at $\overline{\varphi} = 0$ as given in reference 2 are:

$$S(0) = \frac{1}{a_0(1+e_0)} \tag{3.11a}$$

$$\frac{dS(0)}{d\overline{\varphi}} = 0 \tag{3.11b}$$

$$t(0) = 0 \tag{3.11c}$$

$$\frac{dt}{d\overline{\varphi}}(0) = \frac{a_0^{3/2}(1+e_0)^2}{(1-e_0^2)^{1/2}} \tag{3.11d}$$

Hence, the initial conditions for the present formulation are:

$$S(0) = \frac{1}{a_0(1+e_0)} \tag{3.12a}$$

$$\frac{dS(0)}{d\varphi^*} = 0 \tag{3.12b}$$

$$t(0) = 0 \tag{3.12c}$$

$$\frac{dt(0)}{d\varphi^*} = \frac{a_0^{3/2}(1+e_0)^2}{(1-e_0^2)^{1/2}} + \frac{\epsilon a_0^3(1+e_0)^4}{1-e_0^2} + O(\epsilon^2) \tag{3.12d}$$

where

a_0 = dimensionless initial semi-major axis

e_0 = initial eccentricity

Finally, the expansions for S and t given by (3.4) and for $dS/d\varphi^*$ and $dt/d\varphi^*$ given below

$$\frac{dS}{d\varphi^*} = S_1^{(0)} + \epsilon\left(S_1^{(1)}+S_2^{(0)}\right) + O(\epsilon^2) \tag{3.13a}$$

$$\frac{dt}{d\varphi^*} = t_1^{(0)} + \epsilon\left(t_1^{(1)}+t_2^{(0)}\right) + O(\epsilon^2) \tag{3.13b}$$

impose the following eight initial conditions on equations (3.5) and (3.6):

$$S^{(0)}(0,0) = \frac{1}{a_0(1+e_0)} \qquad S^{(1)}(0,0) = 0 \tag{3.14a}$$

$$S_1^{(0)}(0,0) = 0 \qquad S_1^{(1)}(0,0) = -S_2^{(0)}(0,0) \tag{3.14b}$$

$$t^{(0)}(0,0) = 0 \qquad t^{(1)}(0,0) = 0 \tag{3.14c}$$

$$t_1^{(0)}(0,0) = \frac{a_0^{\frac{3}{2}}(1+e_0)^2}{(1-e_0^2)^{1/2}} \qquad t_1^{(1)}(0,0) = -t_2^{(0)}(0,0) + \frac{a_0^3(1+e_0)^4}{1-e_0^2} \tag{3.14d}$$

3.5 Solution for the Keplerian Orbit

The solution of equations (3.5) is again

$$S^{(0)} = f^2[1-e\cos(\varphi^*-\beta)] \tag{3.15a}$$

$$t^{(0)} = a^{3/2}\left[\frac{e(1-e^2)^{1/2}\sin(\varphi^*-\beta)}{1-e\cos(\varphi^*-\beta)} + \cos^{-1}\frac{\cos(\varphi^*-\beta)-e}{1-e\cos(\varphi^*-\beta)}\right] + \tau \tag{3.15b}$$

where the conservation relation

$$t_1^{(0)} = \frac{f}{\left|S^{(0)}\right|^2} \tag{3.16}$$

has been used. The function f is defined as

$$f(\tilde{\varphi}) = \frac{1}{[a(1-e^2)]^{1/2}} \tag{3.17}$$

The quantities e, a, β and τ are functions of $\tilde{\varphi}$ to be determined by requirements on $S^{(1)}$ and $t^{(1)}$.

3.6 Solution for the First-Order Perturbations

Equation (3.6b) may now be written as:

$$\frac{\partial}{\partial\varphi^*}\left[\left(S^{(0)}\right)^2\left(t_1^{(1)}+t_2^{(0)}\right)+\frac{2fS^{(1)}}{S^{(0)}}-\frac{f^2}{\left(S^{(0)}\right)^2}\right]+\frac{df}{d\tilde{\varphi}}=0 \tag{3.18}$$

and can be directly integrated to the form:

$$\left(S^{(0)}\right)^2\left(t_1^{(1)}+t_2^{(0)}\right)=-\frac{2fS^{(1)}}{S^{(0)}}+\frac{f^2}{\left(S^{(0)}\right)^2}-\frac{df\varphi^*}{d\tilde{\varphi}}+g(\tilde{\varphi}) \tag{3.19}$$

Insertion of (3.19) into (3.6a) yields:

$$S_{11}^{(1)}+S^{(1)}=-2S_{12}^{(0)}+\frac{2f^3}{\left(S^{(0)}\right)^2}-\frac{2f}{S^{(0)}}-2f\frac{df}{d\tilde{\varphi}}\varphi^*+2fg-\frac{2f\left(S_1^{(0)}\right)^2}{\left(S^{(0)}\right)^3} \tag{3.20}$$

Expanding equation (3.20) in terms of e and retaining terms to order (e), gives:

$$S_{11}^{(1)}+S^{(1)}=-2\left(f^2\frac{de}{d\tilde{\varphi}}+2f\frac{df}{d\tilde{\varphi}}e\right)\sin(\varphi^*-\beta)+\left(2f^2e\frac{d\beta}{d\tilde{\varphi}}+\frac{2e}{f}\right)\cos(\varphi^*-\beta)$$

$$-2f\frac{df}{d\tilde{\varphi}}\varphi^*+2fg \tag{3.21}$$

In order that $S^{(1)}$ be a bounded function of φ^*, the coefficients of the terms φ^*, $\sin(\varphi^*-\beta)$ and $\cos(\varphi^*-\beta)$ must vanish. Thus,

$$f\frac{df}{d\tilde{\varphi}}=0 \tag{3.22a}$$

$$f^2\frac{de}{d\tilde{\varphi}}+2f\frac{df}{d\tilde{\varphi}}e=0 \tag{3.22b}$$

$$f^2\frac{d\beta}{d\tilde{\varphi}}+\frac{1}{f}=0 \tag{3.22c}$$

The solutions of these equations are

$$f = \text{constant} = f_0 = \frac{1}{[a_0(1-e_0^{\ 2})]^{1/2}} \tag{3.23a}$$

$$e = \text{constant} = e_0 \tag{3.23b}$$

$$\beta = -a_0^{\ 3/2}\,\tilde{\varphi} + O(e) \tag{3.23c}$$

The solution for $S^{(1)}$ is then

$$S^{(1)} = A\sin(\varphi^{*}-\beta) + B\cos(\varphi^{*}-\beta) + 2f_0 g \tag{3.24}$$

where A and B are functions of $\tilde{\varphi}$ to be determined.

Equation (3.19) now becomes

$$t_1^{(1)} = 2\left(e_0 a_0^{\ 3} - 5e_0 g a_0^{\ 2} - a_0^{\frac{5}{2}} B\right)\cos(\varphi^{*}-\beta) - 2a_0^{\frac{5}{2}} A\sin(\varphi^{*}-\beta)$$
$$-\left(3ga_0^{\ 2} + 3e_0 a_0^{\frac{5}{2}} B + \frac{d\tau}{d\tilde{\varphi}}\right) - 3e_0 a_0^{\frac{5}{2}}\left[A\sin 2(\varphi^{*}-\beta) + B\cos 2(\varphi^{*}-\beta)\right] \tag{3.25}$$

and can be integrated to:

$$t^{(1)} = 2\left(e_0 a_0^{\ 3} - 5e_0 a_0^{\ 2} g - a_0^{\frac{5}{2}} B\right)\sin(\varphi^{*}-\beta) + 2a_0^{\frac{5}{2}} A\cos(\varphi^{*}-\beta)$$
$$-\left(3ga_0^{\ 2} + 3e_0 a_0^{\frac{5}{2}} B + \frac{d\tau}{d\tilde{\varphi}}\right)\varphi^{*} + \frac{3}{2} e_0 a_0^{\frac{5}{2}}\left[A\cos 2(\varphi^{*}-\beta) - B\sin 2(\varphi^{*}-\beta)\right] + h(\tilde{\varphi}) \tag{3.26}$$

In this expression, the term linear in φ^{*} is eliminated by requiring that

$$\frac{d\tau}{d\tilde{\varphi}} = -3\left(ga_0^{\ 2} + e_0 a_0^{\frac{5}{2}} B\right) \tag{3.27}$$

which can be integrated to give τ after g and B have been determined.

3.7 Solution for the Second-Order Perturbations

With the use of equations (3.16) and (3.19), equation (3.7b) may be written in the form:

$$\frac{\partial}{\partial\varphi^*}\left[\left(S^{(0)}\right)^2\left(t_1^{(2)}+t_2^{(1)}+at_2^{(0)}\right)+\frac{2f_0S^{(2)}}{S^{(0)}}-\frac{3f_0\left(S^{(1)}\right)^2}{\left(S^{(0)}\right)^2}+\frac{4f_0^{\,2}S^{(1)}}{\left(S^{(0)}\right)^3}+\frac{2gS^{(1)}}{S^{(0)}}\right.$$

$$\left.-\frac{f_0^{\,3}}{\left(S^{(0)}\right)^4}-\frac{2f_0g}{\left(S^{(0)}\right)^2}\right]+\frac{\partial}{\partial\tilde{\varphi}}\left[\left(S^{(0)}\right)^2\left(t_1^{(1)}+t_2^{(0)}+at_1^{(0)}\right)+\frac{2f_0S^{(1)}}{S^{(0)}}-\frac{f_0^{\,2}}{\left(S^{(0)}\right)^2}\right]$$

$$=\frac{3}{2}\frac{f_0^{\,3}}{\left(S^{(0)}\right)^4}\sin 2\varphi^* \tag{3.28}$$

Now let

$$\frac{\partial Q}{\partial\varphi^*}=\frac{3}{2}\frac{f_0^{\,3}}{\left(S^{(0)}\right)^4}\sin 2\varphi^* \tag{3.29}$$

Then integration of equation (3.28) gives

$$\left(S^{(0)}\right)^2\left(t_1^{(2)}+t_2^{(1)}+at_2^{(0)}\right)=\frac{-2f_0S^{(2)}}{S^{(0)}}+\frac{3f_0\left(S^{(1)}\right)^2}{\left(S^{(0)}\right)^2}-\frac{4f_0^{\,2}S^{(1)}}{\left(S^{(0)}\right)^3}-\frac{2gS^{(1)}}{S^{(0)}}$$

$$+\frac{f_0^{\,3}}{\left(S^{(0)}\right)^4}+\frac{2f_0g}{\left(S^{(0)}\right)^2}-\frac{dg}{d\tilde{\varphi}}\varphi^*+Q+k(\tilde{\varphi}) \tag{3.30}$$

where Q is given by

$$Q=\frac{-3e_0}{f_0^{\,5}}\cos(\varphi^*+\beta)-\frac{3}{4f_0^{\,5}}\cos 2\varphi^*-\frac{e_0}{f_0^{\,5}}\cos(3\varphi^*-\beta) \tag{3.31}$$

Equations (3.30) and (3.31) are inserted in (3.7a) to give:

$$S^{(2)}_{11}+S^{(2)}=-2S^{(1)}_{12}-S^{(0)}_{22}-2aS^{(0)}_{12}+\frac{2f_0^{\,4}}{\left(S^{(0)}\right)^4}+\frac{4f_0^{\,2}g}{\left(S^{(0)}\right)^2}-\frac{2f_0 dg\varphi^*}{d\tilde{\varphi}}$$

$$-\frac{6e_0}{f_0^{\,4}}\cos(\varphi^*+\beta)-\frac{3}{2f_0^{\,4}}\cos 2\varphi^*-\frac{2e_0}{f_0^{\,4}}\cos(3\varphi^*-\beta)+2f_0k+\frac{f_0^{\,4}}{\left(S^{(0)}\right)^4}+g^2$$

$$-\frac{4f_0^{\,3}S^{(1)}}{\left(S^{(0)}\right)^3}+\frac{2f_0^{\,2}g}{\left(S^{(0)}\right)^2}+\frac{4f_0S^{(1)}}{\left(S^{(0)}\right)^2}-\frac{2f_0^{\,2}}{\left(S^{(0)}\right)^3}-\frac{2g}{S^{(0)}}-\frac{2f_0S^{(1)}}{\left(S^{(0)}\right)^2}-\frac{3}{2}\frac{f_0^{\,2}}{\left(S^{(0)}\right)^3}$$

$$-\frac{4f_0S^{(0)}_1}{\left(S^{(0)}\right)^3}\left(S^{(1)}_1+S^{(0)}_2\right)-\frac{2\left(S^{(0)}_1\right)^2}{S^{(0)}}\left[\left(t^{(1)}_1+t^{(0)}_2\right)-\frac{f_0S^{(1)}}{\left(S^{(0)}\right)^3}\right]$$

$$+\frac{3}{2}f_0^{\,2}\frac{S^{(0)}_1}{\left(S^{(0)}\right)^4}\sin 2\varphi^*-\frac{3}{2}\frac{f_0^{\,2}}{\left(S^{(0)}\right)^3}\cos 2\varphi^* \tag{3.32}$$

Evaluating all terms in the above equation and retaining terms to order (e), yields

$$S^{(2)}_{11}+S^{(2)}=-2f_0\frac{dg}{d\tilde{\varphi}}\varphi^*+\frac{2dB}{d\tilde{\varphi}}\sin(\varphi^*-\beta)+\left(-\frac{2dA}{d\tilde{\varphi}}+\frac{e_0}{2f_0^{\,4}}-\frac{6e_0g}{f_0^{\,2}}-\frac{2ae_0}{f_0}\right)\cdot$$

$$\cdot\cos(\varphi^*-\beta)-\frac{15}{2}\frac{e_0}{f_0^{\,4}}\cos(\varphi^*+\beta)-\frac{5e_0}{f_0^{\,4}}\cos(3\varphi^*-\beta)-\frac{6Ae_0}{f_0^{\,3}}\sin 2(\varphi^*-\beta)$$

$$-\frac{6Be_0}{f_0^{\,3}}\cos 2(\varphi^*-\beta)-\frac{3}{f_0^{\,4}}\cos 2\varphi^*+2f_0k+g^2-\frac{1}{2f_0^{\,4}}-\frac{2Be_0}{f_0^{\,3}} \tag{3.33}$$

Here it should be noted that φ^* was chosen as the fast variable so that the trigonometric forcing functions of equations (3.7) would not introduce terms of order $(\frac{1}{\varepsilon})$, (ε), or (ε^2) into the solution of equation (3.33). It may be said in general that if the two-variable expansion is applied to differential equations in which the arguments of the trigonometric forcing functions have the form $(1-\varepsilon^n)\varphi$, then the fast variable should be chosen as $\varphi^+=(1-\varepsilon^n)\varphi$.

With the use of familiar trigonometric identities, equation (3.33) may be rewritten as:

$$
\begin{aligned}
S_{11}^{(2)}+S^{(2)} = {} & -2f_0\frac{dg}{d\tilde{\varphi}}\varphi^* + \left[\frac{2dB}{d\tilde{\varphi}}\cos\beta - \left(\frac{2dA}{d\tilde{\varphi}} - 8e_0a_0^2 + 6e_0g_0a_0 + 2\alpha e_0a_0^{\frac{1}{2}}\right)\sin\beta\right]\sin\varphi^* \\
& - \left[\frac{2dB}{d\tilde{\varphi}}\sin\beta + \left(\frac{2dA}{d\tilde{\varphi}} + 7e_0a_0^2 + 6e_0g_0a_0 + 2\alpha e_0a_0^{\frac{1}{2}}\right)\cos\beta\right]\cos\varphi^* - 5e_0a_0^2\cos(3\varphi^*-\beta) \\
& - 6e_0a_0^{\frac{3}{2}}A\sin 2(\varphi^*-\beta) - 6Be_0a_0^{\frac{3}{2}}\cos 2(\varphi^*-\beta) - 3a_0^2\cos 2\varphi^* + 2f_0k \\
& + g^2 - \frac{a_0^2}{2} - 2Be_0a_0^{\frac{3}{2}}
\end{aligned}
\tag{3.34}
$$

Since $S^{(2)}$ must be a bounded function of φ^*, the following conditions are imposed on equation (3.34):

$$\frac{dg}{d\tilde{\varphi}} = 0 \qquad \text{so that } g = g_0 = \text{constant} \tag{3.35}$$

and

$$\frac{2dB}{d\tilde{\varphi}}\cos\beta - \left(\frac{2dA}{d\tilde{\varphi}} - 8e_0a_0^2 + 6e_0g_0a_0 + 2\alpha e_0a_0^{\frac{1}{2}}\right)\sin\beta = 0 \tag{3.36a}$$

$$\frac{2dB}{d\tilde{\varphi}}\sin\beta + \left(\frac{2dA}{d\tilde{\varphi}} + 7e_0a_0^2 + 6e_0g_0a_0 + 2\alpha e_0a_0^{\frac{1}{2}}\right)\cos\beta = 0 \tag{3.36b}$$

or

$$\frac{dB}{d\tilde{\varphi}}\cos\beta - \frac{dA}{d\tilde{\varphi}}\sin\beta = \left(-4e_0a_0^{2} + 3e_0g_0a_0 + \alpha e_0a_0^{\frac{1}{2}}\right)\sin\beta \tag{3.37a}$$

$$\frac{dB}{d\tilde{\varphi}}\sin\beta + \frac{dA}{d\tilde{\varphi}}\cos\beta = -\left(\frac{7}{2}e_0a_0^{2} + 3e_0g_0a_0 + \alpha e_0a_0^{\frac{1}{2}}\right)\cos\beta \tag{3.37b}$$

The solution of the above linear system for A and B is:

$$A = \left(\frac{e_0a_0^{2}}{4} - 3e_0g_0a_0 - \alpha e_0a_0^{\frac{1}{2}}\right)\tilde{\varphi} + \frac{15}{8}e_0a_0^{\frac{1}{2}}\sin 2\beta + A_0 \tag{3.38a}$$

$$B = -\frac{15}{8}e_0a_0^{\frac{1}{2}}\cos 2\beta + B_0 \tag{3.38b}$$

where A_0 and B_0 are constants.

The motion in this problem must be completely bounded. Since the term in equation (3.38a) which is linear in $\tilde{\varphi}$ is unbounded, it must be eliminated by requiring that

$$\alpha = \frac{a_0^{3/2}}{4} - 3g_0a_0^{1/2} \tag{3.39}$$

From the initial conditions (3.14),

$$g_0 = 0 \tag{3.40a}$$

$$A_0 = 0 \tag{3.40b}$$

$$B_0 = \frac{15}{8}e_0a_0^{1/2} \tag{3.40c}$$

$$\tau_0 = 0 \tag{3.40d}$$

Hence

$$\alpha = \frac{1}{4}a_0^{3/2} \tag{3.41}$$

3.8 Comparison of the Results with Reference 2

The final results may then be summarized as follows:

$$S^{(0)} = \frac{1}{a_0(1-e_0^2)}\left[1-e_0\cos(1+\epsilon a_0^{3/2}+\frac{1}{4}\epsilon^2 a_0^3)\varphi^*\right] \tag{3.42a}$$

$$S^{(1)} = \frac{15}{8} e_0 a_0^{\frac{1}{2}}\left[\cos(1+\epsilon a_0^{\frac{3}{2}}+\frac{1}{4}\epsilon^2 a_0^3)\varphi^* - \cos(1-\epsilon a_0^{\frac{3}{2}}-\frac{1}{4}\epsilon^2 a_0^3)\varphi^*\right] \tag{3.42b}$$

$$t^{(0)} = a_0^{\frac{3}{2}}\left[\frac{e_0(1-e_0^2)^{\frac{1}{2}}\sin(1+\epsilon a_0^{\frac{3}{2}}+\frac{1}{4}\epsilon^2 a_0^3)\varphi^*}{1-e_0\cos(1+\epsilon a_0^{\frac{3}{2}}+\frac{1}{4}\epsilon^2 a_0^3)\varphi^*}\right.$$

$$\left. + \cos^{-1}\frac{\cos(1+\epsilon a_0^{\frac{3}{2}}+\frac{1}{4}\epsilon^2 a_0^3)\varphi^* - e_0}{1-e_0\cos(1+\epsilon a_0^{\frac{3}{2}}+\frac{1}{4}\epsilon^2 a_0^3)\varphi^*}\right] \tag{3.42c}$$

$$t^{(1)} = \frac{15}{4} e_0 a_0^3\left[\sin(1-\epsilon a_0^{\frac{3}{2}}-\frac{1}{4}\epsilon^2 a_0^3)\varphi^* - \frac{7}{15}\sin(1+\epsilon a_0^{\frac{3}{2}}+\frac{1}{4}\epsilon^2 a_0^3)\varphi^*\right] + k(\varphi) \tag{3.42d}$$

The corresponding results obtained in reference 2 are given below

$$S^{(0)} = \frac{1}{a_0(1-e_0^2)}\left[1-e_0\cos\psi\right] \tag{3.43a}$$

$$S^{(1)} = \frac{15}{8} e_0 a_0^{\frac{1}{2}}\left[\cos\psi - \cos(2p_1-1)\psi\right] \tag{3.43b}$$

$$t^{(0)} = a_0^{\frac{3}{2}}\left[\frac{e_0(1-e_0^2)^{\frac{1}{2}}\sin\psi}{1-e_0\cos\psi} + \cos^{-1}\frac{\cos\psi - e_0}{1-e_0\cos\psi}\right] \tag{3.43c}$$

$$t^{(1)} = \frac{15}{4} e_0 a_0^3\left[\sin(2p_1-1)\psi - \sin\psi\right] \tag{3.43d}$$

In order to compare the above results with the solution derived by the present approach, it is necessary to express the quantities p_1 and ψ used in equations (3.43) in terms of φ^*. The angle ψ was defined in reference 2 as

$$\psi = (1+\epsilon^2\omega_1)\overline{\varphi} \tag{3.44}$$

When the expression for $\overline{\varphi}$ given by equation (3.8c) together with values of ω_1, λ_1, and ν_1 given in reference 2 are used, equation (3.44) becomes:

$$\psi = \left[1-\left(\frac{7}{12}-2e_0\right)\epsilon^2 a_0^{\,3}\right]\varphi^* + \left[\epsilon-\left(\frac{1}{6}+2e_0\right)\epsilon^2 a_0^{\,\frac{3}{2}}\right]t \tag{3.45}$$

It is easy to verify that $t = t^{(0)}+\epsilon t^{(1)}$ can be expanded in the form:

$$t = a_0^{\,\frac{3}{2}}(\varphi^*-\beta) + 2a_0^{\,\frac{3}{2}}e_0 \sin(\varphi^*-\beta) + O(e_0^{\,2}) + O(\epsilon) \tag{3.46}$$

Hence ψ becomes

$$\psi = \left[1 + \epsilon a_0^{\,\frac{3}{2}} + \frac{1}{4}\epsilon^2 a_0^{\,3}\right]\varphi^* + 2\epsilon e_0 a_0^{\,\frac{3}{2}} \sin\left(1+\epsilon a_0^{\,\frac{3}{2}} + \frac{1}{4}\epsilon^2 a_0^{\,3}\right)\varphi^* \tag{3.47}$$

Reference 2 also gives

$$p_1 = 1-\epsilon a_0^{\,\frac{3}{2}} + \frac{3}{4}\epsilon^2 a_0^{\,3} \tag{3.48}$$

so that

$$(2p_1-1)\psi = \left(1-\epsilon a_0^{\,\frac{3}{2}} - \frac{1}{4}\epsilon^2 a_0^{\,3}\right)\varphi^* + O(e_0) \tag{3.49}$$

In the above, it is consistent to neglect terms of order e_0 since $S^{(1)}$ and $t^{(1)}$ are multiplied by e_0.

When the results given by equations (3.47) and (3.49) are used to transform (3.43) to the present notation, it is seen that the agreement is exact to the orders retained. It should also be pointed out that the results are in exact agreement with the classical lunar theory of DePontécoulant (c.f., reference 3 and comments in reference 2).

3.9 Summary

In the method of reference 2, the terms of order ε were difficult to derive and arose somewhat artificially as the response to an almost resonant forcing function in the differential equations. In the present formulation these terms appear quite naturally and no difficulties are associated with their evaluation.

Furthermore, the approach of reference 2 fails to give a uniformly valid approximation when applied to orbits with high inclinations. This failure is associated with the presence of almost resonant forcing terms in the differential equation for S. It is anticipated that the two-variable expansion procedure will prove successful in this general case. The solution presented here is to serve as a preliminary step in establishing the choice of variables in the formulation of this problem.

REFERENCES

1. Kevorkian, J., The Uniformly Valid Asymptotic Representation of the Solutions of Certain Non-Linear Differential Equations, Ph.D. thesis, Guggenheim Aeronautical Laboratory, California Institute of Technology, 1961.

2. Kevorkian, J., Uniformly Valid Asymptotic Representation for All Times of the Motion of a Satellite in the Vicinity of the Smaller Body in the Restricted Three-Body Problem, to be published in the Astronomical Journal, May 1962.

3. Brown, E., An Introductory Treatise on the Lunar Theory, Dover Publications, New York, 1960.