# OPTIMAL DECISIONS WITH APPLICATIONS TO TACTICAL PROBLEMS

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ABSTRACT

The minmax criterion for optimal decisions is applied to a special class of infinite games. The existence of a solution is established, and some optimal strategies are described completely. The games considered include the so-called "silent duels".

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## 1. Introduction.

A zero-sum, two-person game can be defined formally by means of a triplet $(X, Y, \Psi)$ where $X$ and $Y$ are two closed convex sets and $\Psi$ is a real-valued function whose domain is $X \times Y$. The elements $\bar{x} \varepsilon X$ (or $\bar{y} \varepsilon Y$) are called pure strategies, and $\Psi$ is called the pay-off or utility function.

The game is said to have a solution if there exist two positive measures $F^*(\bar{x})$ and $G^*(\bar{y})$ (defined on $X$ and $Y$ respectively and normalized so that each has total measure 1), such that

$$\int \Psi(\bar{x},\bar{y})dF^*(\bar{x}) \geq v, \qquad \text{all } \bar{y} \varepsilon Y,$$

and

$$\int \Psi(\bar{x},\bar{y})dG^*(\bar{y}) \leq v, \qquad \text{all } \bar{x} \varepsilon X,$$

for some number $v$. The measures $F^*$ and $G^*$ are called optimal strategies, and $v$ is called the value of the game.

Each of the games that we consider below can be described intuitively as a contest in which each player is trying to achieve a certain fixed objective, but is allowed to try it only a certain number of times. If at time $t$ contestant I attempts to achieve his goal, he may succeed, or he may fail. (The probability of success is given by a function $P(t)$ and the probability of failure by $1 - P(t)$.) If he succeeds, he wins one unit from his opponent, and the contest is over. If he fails, then the contest continues but the other player is not informed about the unsuccessful attempt. Any attempt made by the other contestant is handled in a similar way, but the probability of success is given by a function $Q(t)$ which need not be the same as $P(t)$. In the problems that we consider the pay-off function corresponds to the expected gain for the first contestant. He is allowed $n$ attempts, and his opponent is allowed $m$ attempts.

A special example consists of a combat between two enemy airplanes. The number of attempts corresponds to the amount of ammunition that they carry, while the functions $P$ and $Q$ correspond to the accuracy of the firing machinery. In this example it is also assumed that a pilot is not aware of the number of times that his enemy has fired and missed. He knows, however, how much ammunition can be carried by each plane, and he also knows the form of the functions $P$ and $Q$. Then the function $\Psi$ represents the probability of survival.

The type of problem just described is often called a silent duel. A special example has been considered by L. Shapley [1], who took $P(t) = Q(t) = t$ and allowed one attempt for one player and two attempts for the other. The pay-off function is

$$\Psi(\bar{x},\bar{y}) = \begin{cases} x + [1-x]\left\{-y_1 + [1-y_1](-y_2)\right\} & 0 \leq x < y_1 < y_2 \leq 1 \\ -y_1 + [1-y_1]\left\{x + [1-x](-y_2)\right\} & 0 \leq y_1 < x < y_2 \leq 1 \\ -y_1 + [1-y_1]\left\{-y_2 + [1-y_2](x)\right\} & 0 \leq y_1 < y_2 < x \leq 1 \end{cases}$$

Here $x$ represents the time of the attempt by contestant I, and $y_1$, $y_2$ represent the attempts by contestant II.

A slightly different problem (called a noisy duel) has been considered by D. Blackwell and M.A. Girshick [2]. Despite the similarity of the problems, it appears that the recursive method that they used cannot be applied to the problems that we consider. Finally, it may be pointed out that the games discussed below are related to a large class of action games that have been considered by M. Shiffman [3] and S. Karlin [4]. Their problems include a large class of utility functions but allow only one action by each player.

## 2. Definitions of X, Y, Ψ.

1) Let $P(t)$ and $Q(t)$ be two real-valued functions defined on the interval $0 \leq t \leq 1$. We assume that $P$ and $Q$ are continuously differentiable, and that they satisfy the following conditions:

$$P(0) = Q(0) = 0$$
$$P(1) = Q(1) = 1$$

$$P'(t) > 0 \,, \qquad 0 < t < 1$$
$$Q'(t) > 0 \,, \qquad 0 < t < 1 \,.$$

2) Let

$$X = \left\{ \bar{x} \in E^n \,|\, 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1 \right\} \,,$$

and

$$Y = \left\{ \bar{y} \in E^m \,|\, 0 \leq y_1 \leq y_2 \leq \cdots \leq y_m \leq 1 \right\} \,.$$

In these definitions, $E^n$ and $E^m$ denote the $n$ and $m$-dimensional Euclidean spaces.

3) Let $\bar{x} \in X$ and $\bar{y} \in Y$ be two vectors such that each component of $\bar{x}$ is different from each component of $\bar{y}$. A new vector $\bar{z} \in E^{n+m}$ is defined by rearranging the numbers $x_1, \ldots, x_n, y_1, \ldots, y_m$ in

increasing order. Then, for each component $z_k$ of $\bar{z}$, two functions $r(z_k)$ and $s(z_k)$ are defined as follows

$$r(z_k) = \begin{cases} P(x_i) & \text{if } z_k = x_i \\ -Q(y_j) & \text{if } z_k = y_j \end{cases}$$

and

$$s(z_k) = \begin{cases} P(x_i) & \text{if } z_k = x_i \\ Q(y_j) & \text{if } z_k = y_j \,. \end{cases}$$

We may point out that this definition can be applied to a single vector $\bar{x}$ (or $\bar{y}$), provided that we know that it is associated with the space X (or Y). In this case we shall say that the other vector has no components.

4) A function $\Psi(\bar{z})$ is defined as follows: if $\bar{z}$ has only one component, then

$$\Psi(\bar{z}) = r(z_1) \,.$$

While if $\bar{z} = (z_1, z_2, \dots, z_k)$, then

$$\Psi(\bar{z}) = r(z_1) + [1 - s(z_1)]\, \Psi(z_2, \dots, z_k) \,.$$

5) The pay-off function $\Psi(\bar{x},\bar{y})$ is defined as follows: if $\bar{x}$ and $\bar{y}$ satisfy the conditions of definition 3), then $\bar{z}$ and $\Psi(\bar{z})$ are constructed as indicated, and $\Psi(\bar{x},\bar{y})$ is defined by

$$\Psi(\bar{x},\bar{y}) \equiv \Psi(\bar{z}) \,.$$

On the other hand, if some component of $\bar{x}$ equals some component of $\bar{y}$, $\Psi(\bar{x},\bar{y})$ is defined by

$$\Psi(\bar{x},\bar{y}) = \lim_{\varepsilon\to 0} \frac{1}{2}\left\{ \Psi(\overline{x+\varepsilon}, \overline{y-\varepsilon}) + \Psi(\overline{x-\varepsilon}, \overline{y+\varepsilon}) \right\} .$$

In this definition $\overline{x+\varepsilon}$ is the vector whose $i^{th}$ component is the minimum of $x_i + \varepsilon$ and 1; similarly, the $i^{th}$ component of $x_i - \varepsilon$ is the maximum of $x_i - \varepsilon$ and 0.

## 3. Description of the Optimal Strategies.

It will be shown below that it is always possible to find optimal strategies of the form

$$F^*(\bar{x}) = \prod_{i=1}^{n} F_i^*(x_i) ,$$

$$G^*(\bar{y}) = \prod_{j=1}^{m} G_j^*(y_j) .$$

Furthermore, all these measures are continuous except at $x_n = 1$ or $y_m = 1$. The support of each measure $F_i^*(x_i)$ is a non-degenerate interval $[a_i, a_{i+1}]$, and in the interior of this interval $F_i^*$ is made up of a finite number of absolutely continuous measures with densities

$$F_i^{*\prime}(x_i) = h_{ij} \frac{Q'(x_i)}{Q^2(x_i)\, P(x_i)} , \qquad j = 1,2, \ldots , r_i .$$

A more detailed description of the optimal strategies will be given in Theorem 1 and in Theorem 4. The location of discontinuities in the densities is established, and some simple equations between $h_{ij}$ and $h_{ij+1}$ are established.

4. Some Properties of $\Psi(\bar{x},\bar{y})$.

The definition of $\Psi$ shows that this function is continuous as long as the relative order of the components of $\bar{x}$ and $\bar{y}$ is not changed. Furthermore, it is easy to verify that this function is also skew-symmetric in the two vector variables $\bar{x}$ and $\bar{y}$ if the roles of $\bar{x}$, $\bar{y}$, X and Y are completely interchanged. Other simple properties of $\Psi$ are given by the following lemmas.

Lemma 1: Let $\bar{z} = (z_1, \dots, z_t, z_{t+1}, \dots, z_k)$. Then

$$\Psi(\bar{z}) = \Psi(z_1, \dots, z_t) + \prod_{i=1}^{t} [1 - s(z_i)]\ \Psi(z_{t+1}, \dots, z_k) \ .$$

Intuitively, the lemma asserts that the total probability of success equals the probability of success in the first $t$ attempts, plus the probability that the contest has not yet ended, multiplied by the probability of success in the remaining $k - t$ attempts. The formal proof, involving a simple induction, is omitted.

Lemma 2: Let $\bar{z} = (z_1, \dots, z_{t-1}, z_t, z_{t+1}, \dots, z_k)$. Then

$$\Psi(\bar{z}) = \Psi(z_1, \dots, z_{t-1}, z_{t+1}, \dots, z_k)$$
$$+ \prod_{i=1}^{t-1} [1 - s(z_i)] \left\{ r(z_t) - s(z_t)\ \Psi(z_{t+1}, \dots, z_k) \right\} .$$

Proof: By Lemma 1,

$$\Psi(\bar{z}) = \Psi(z_1, \dots, z_{t-1}) + \prod_{i=1}^{t-1} [1 - s(z_i)]\ \Psi(z_t, \dots, z_k) ,$$

and

$$\Psi(z_t, z_{t+1}, \dots, z_k) = r(z_t) + [1 - s(z_t)]\, \Psi(z_{t+1}, \dots, z_k).$$

These two equations can be combined, collecting at the same time all the terms that involve $z_t$. Then

$$\Psi(\bar{z}) = \Psi(z_1, \dots, z_{t-1}) + \prod_{i=1}^{t-1} [1 - s(z_i)]\, \Psi(z_{t+1}, \dots, z_k)$$

$$+ \prod_{i=1}^{t-1} [1 - s(z_i)] \left\{ r(z_t) - s(z_t)\, \Psi(z_{t+1}, \dots, z_k) \right\} .$$

The first two terms can be combined by Lemma 1 to give the desired result.

<u>Lemma 3</u>: For any fixed $\bar{y}$, $\Psi(\bar{x},\bar{y})$ is a monotone increasing function of each component $x_i$ of $\bar{x}$ as long as this component ranges over an open interval bounded by two successive components of $\bar{y}$. Similarly if $\bar{x}$ is fixed, $\Psi(\bar{x},\bar{y})$ is a monotone decreasing function of each component $y_j$ of $\bar{y}$ as long as $y_j$ varies between two consecutive components of $\bar{x}$.

<u>Proof</u>: If $\bar{x}$ has one component and $\bar{y}$ has none, then

$$\Psi(\bar{x},\bar{y}) = P(x).$$

This function is monotone increasing because $P' > 0$. Similarly, if $\bar{y}$ has one component and $\bar{x}$ has none, then

$$\Psi(\bar{x},\bar{y}) = -Q(y).$$

This function is monotone decreasing because $Q' > 0$. We can proceed by induction and assume that the lemma is valid when the total number of components in $\bar{x}$ and $\bar{y}$ is $k - 1$. Finally,

$$\Psi(z_1, \dots, z_k) = r(z_1) + [1 - s(z_1)]\, \Psi(z_2, \dots, z_k).$$

In this equation the factor $1 - s(z_1)$ is non-negative and vanishes only when $z_1 = 1$; furthermore, $\Psi(z_2, \dots, z_k)$ depends only on $k - 1$ components so that the lemma is valid when applied to the last $k - 1$ components. Thus we only have to show that the result is valid when it is applied to the component $z_1$. If $z_1$ is a component of $\bar{x}$ the last equation can be written in the form

$$\begin{aligned}\Psi(z_1, \dots, z_k) &= P(x_1) + [1 - P(x_1)]\, \Psi(z_2, \dots, z_k) \\ &= P(x_1)\, [1 - \Psi(z_2, \dots, z_k)] + \Psi(z_2, \dots, z_k).\end{aligned}$$

But if $z_1$ is a component of $\bar{y}$, then

$$\begin{aligned}\Psi(z_1, \dots, z_k) &= - Q(y_1) + [1 - Q(y_1)]\, \Psi(z_2, \dots, z_k) \\ &= - Q(y_1)\, [1 + \Psi(z_2, \dots, z_k)] + \Psi(z_2, \dots, z_k).\end{aligned}$$

In the first case we have a monotone increasing function of $x_1$, and in the second case a monotone decreasing function of $y_1$. (It is easy to verify that $|\Psi(\bar{x},\bar{y})| \leq 1$ always.) Finally, we may point out that the first function is strictly increasing unless $\Psi(z_2, \dots, z_k) = 1$. This requires that $\bar{y}$ be missing and that $x_2 \leq x_3 \leq \dots \leq x_k = 1$.

## 5. Strategies of the Class Q.

The expected value of $\Psi(\bar{x},\bar{y})$ with respect to a measure $F(\bar{x})$ is a function $R(\bar{y})$ defined by

$$R(\bar{y}) = \int \Psi(\bar{x},\bar{y})\, dF(\bar{x}) .$$

The function $R(\bar{y})$ depends on $F(\bar{x})$, and its explicit form may be difficult to find. However, it can be computed explicitly for a special class of measures which includes a particular pair of optimal strategies. These optimal strategies will be exhibited explicitly later.

Any positive measure, with total measure 1, is called a strategy. A strategy $F(\bar{x})$ is said to belong to the class O if it satisfies the following conditions:

1) $F$ is separable, i.e.,

$$F(\bar{x}) = \prod_{i=1}^{n} F_i(x_i),$$

where each $F_i$ is a positive measure with total measure 1.

2) The support of the measure $F_i$ is a non-degenerate interval $[a_i, a_{i+1}]$; also $a_1 > 0$ and $a_{n+1} = 1$.

3) Each measure $F_i(x_i)$ is continuous, except for $F_n(x_n)$; this may be discontinuous only at $x_n = 1$.

<u>Notation</u>: The expected value of $P(t)$, taken with respect to $F_i(t)$, is denoted by $D_i$. Thus

$$D_i = \int_{a_i}^{a_{i+1}} P(t)\, dF_i(t) , \qquad i = 1, \dots , n.$$

A vector $\bar{D}$ (analogous to $\bar{x}$) and a function $\phi$ (analogous to $\Psi$) are defined as follows:

$$\bar{D} = (D_1, \dots , D_n)$$

$$\phi(D_n) = D_n$$

$$\phi(D_k, D_{k+1}, \dots , D_n) = D_k + [1 - D_k]\, \phi(D_{k+1}, \dots , D_n).$$

<u>Lemma 4</u>: If $F(\bar{x})$ belongs to the class O, then

$$\int \Psi(\bar{x})\, dF(\bar{x}) = \emptyset(\bar{D}) .$$

In this lemma it is assumed that $\bar{y}$ has no components.

<u>Proof</u>: If $n = 1$, the lemma reduces to the definition of $D_1$. The rest of the proof is by induction, assuming that the result is valid for all vectors $\bar{x}$ and strategies $F(\bar{x})$ of dimension $n - 1$. Then, if $\bar{x} = (x_1, \ldots , x_n)$,

$$\begin{aligned}
\int \Psi(\bar{x})\, dF(\bar{x}) &= \int \left\{ P(x_1) + [1 - P(x_1)]\, \Psi(x_2, \ldots , x_n) \right\} dF(\bar{x}) \\
&= D_1 + [1 - D_1] \int \Psi(x_2, \ldots , x_n) dF_2(x_2), \ldots, dF_n(x_n) \\
&= D_1 + [1 - D_1]\, \emptyset(D_2, \ldots , D_n) \\
&= \emptyset(\bar{D}) .
\end{aligned}$$

<u>Lemma 5</u>: Let $F(\bar{x})$ be a strategy in the class O, and let $\bar{y} = (y_1, \ldots , y_m)$ be any vector in Y, with the restriction that $y_m$ lies in the open interval $(a_k, a_{k+1})$. Then

$$\begin{aligned}
&R(y_1, \ldots , y_{m-1}, y_m) - R(y_1, \ldots , y_{m-1}) \\
&= \prod_{i=1}^{k-1} [1 - D_i] \prod_{j=1}^{m-1} [1 - Q(y_j)]\, [-Q(y_m)] \cdot \\
&\quad \cdot \left\{ 2 \int_{y_m}^{a_{k+1}} P(x_k)\, dF_k(x_k) + [1 - D_k][1 + \emptyset(\bar{D}^k)] \right\} .
\end{aligned}$$

In this equation $\bar{D}^k$ denotes the vector whose components are the last $n - k$ components of $\bar{D}$; the function $\emptyset(\bar{D}^k)$ is defined like $\emptyset(\bar{D})$.

Proof: Let $\bar{y}$ be the given vector, and let $\bar{x}$ be any vector in the support of the measure $F(\bar{x})$. Let $\bar{z}$ be defined in the usual way so that Lemma 2 can be applied, selecting for $z_t$ the component $y_m$. Then, if $a_k \leq x_k < y_m$

$$\Psi((x_1, \ldots , x_n),(y_1, \ldots , y_m)) - \Psi((x_1, \ldots , x_n),(y_1, \ldots , y_{m-1}))$$

$$= \prod_{i=1}^{k} [1 - P(x_i)] \prod_{j=1}^{m-1} [1 - Q(y_j)][-Q(y_m)][1 + \Psi(x_{k+1}, \ldots , x_n)]$$

$$= \prod_{i=1}^{k-1} [1 - P(x_i)] \prod_{j=1}^{m-1} [1 - Q(y_j)] [-Q(y_m)] \cdot$$

$$\cdot [1 - P(x_k)] \left\{1 + \Psi(x_{k+1}, \ldots , x_n)\right\} .$$

But if $y_m < x_k \leq a_{k+1}$,

$$\Psi((x_1, \ldots , x_n),(y_1, \ldots , y_m)) - \Psi((x_1, \ldots , x_n),(y_1, \ldots , y_{m-1}))$$

$$= \prod_{i=1}^{k-1} [1 - P(x_i)] \prod_{j=1}^{m-1} [1 - Q(y_j)] [-Q(y_m)] [1 + \Psi(x_k, \ldots , x_n)]$$

$$= \prod_{i=1}^{k-1} [1 - P(x_i)] \prod_{j=1}^{m-1} [1 - Q(y_j)] [-Q(y_m)] \cdot$$

$$\cdot \left\{ 1 + P(x_k) + [1 - P(x_k)] \Psi(x_{k+1}, \ldots , x_n)\right\}$$

$$= \prod_{i=1}^{k-1} [1 - P(x_i)] \prod_{j=1}^{m-1} [1 - Q(y_j)] [-Q(y_m)] \cdot$$

$$\cdot \left\{ 2P(x_k) + [1 - P(x_k)] [1 + \Psi(x_{k+1}, \ldots , x_n)]\right\} .$$

In either case, the right hand side is continuous and can be integrated with respect to $F(\bar{x})$. Then,

$$R(y_1, \ldots, y_m) - R(y_1, \ldots, y_{m-1})$$

$$= \prod_{i=1}^{k-1} [1 - D_i] \prod_{j=1}^{m-1} [1 - Q(y_j)] \, [-Q(y_m)] \cdot$$

$$\cdot \left\{ \int_{a_k}^{y_m} [1 - P(x_k)] \, dF_k(x_k) \right\} [1 + \phi(\bar{D}^k)]$$

$$+ \prod_{i=1}^{k-1} [1 - D_i] \prod_{j=1}^{m-1} [1 - Q(y_j)] \, [-Q(y_m)]$$

$$\cdot \left\{ \int_{y_m}^{a_{k+1}} \left\{ 2P(x_k) + [1 - P(x_k)][1 + \phi(\bar{D}^k)] \right\} dF_k(x_k) \right\}$$

$$= \prod_{i=1}^{k-1} [1 - D_i] \prod_{j=1}^{m-1} [1 - Q(y_j)][-Q(y_m)] \cdot$$

$$\cdot \left\{ 2 \int_{y_m}^{a_{k+1}} P(x_k) \, dF_k(x_k) + [1 - D_k][1 + \phi(\bar{D}^k)] \right\} .$$

<u>Lemma 6</u>: Let $F(\bar{x})$ be in the class O. Then

$$\int \Psi(\bar{x},\bar{y}) \, dF(\bar{x})$$

is a continuous function of $\bar{y}$, provided that $y_m \neq 1$.

<u>Proof</u>: The result follows from the fact that $\Psi$ is a function with simple discontinuities while $F$ is a continuous measure, except at 1. It is also possible to verify the result by means of Lemma 5; indeed, a simple calculation shows that the right hand side of the last equation varies continuously as $y_m$ increases from $a_k - \varepsilon$ to $a_k + \varepsilon$.

## 6. Corresponding Strategies.

Let $F(\bar{x})$ and $G(\bar{y})$ be two strategies contained in the class O, with

$$F(\bar{x}) = \prod_{i=1}^{n} F_i(x_i), \qquad G(\bar{y}) = \prod_{j=1}^{m} G_j(y_j).$$

We shall say that F and G form a pair of corresponding strategies if both of the following conditions hold:

1) If $\bar{y}$ is in the support of $G(\bar{y})$ and $y_m \neq 1$, then

$$\int \Psi(\bar{x},\bar{y})\, dF(\bar{x}) \equiv \underline{v} \qquad (1)$$

2) If $\bar{x}$ is in the support of $F(\bar{x})$ and $x_n \neq 1$, then

$$\int \Psi(\bar{x},\bar{y})\, dG(\bar{y}) \equiv \bar{v} \qquad (2)$$

The numbers $\underline{v}$ and $\bar{v}$ are of course independent of $\bar{y}$ and $\bar{x}$, respectively.

In the following sections we shall show that it is always possible to find two corresponding strategies which are also optimal for the game. In order to determine the conditions under which equation (1) will hold, we adopt the following standard notation:

Notation: The support of $F_i(x_i)$ is denoted by $[a_i, a_{i+1}]$, and the support of $G_j(y_j)$ is denoted by $[b_j, b_{j+1}]$. The numbers $b_1, \dots, b_m$ are rearranged into subsets, one subset for each of the intervals $[a_i, a_{i+1}]$. The resulting array can be written in the form

$$a_1 \leq b_{11} < b_{12} < \cdots < b_{1r_1} < a_2$$

$$a_2 \leq b_{21} < b_{22} < \cdots < b_{2r_2} < a_3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_n \leq b_{n1} < b_{n2} < \cdots < b_{nr_n} < a_{n+1} = 1.$$

When $\bar{y}$ ranges over the support of $G(\bar{y})$, its last component $y_m$ ranges over the interval $[b_{nr_n}, 1]$, and any other component ranges over an interval bounded by two adjacent b's; then it is possible to identify each component by means of the interval over which it ranges: with this notation $y_m \equiv y_{nr_n}$, and $y_{ik}$ denotes the component of $\bar{y}$ that ranges over the interval $[b_{ik}, b_{ik+1}]$. Whenever two adjacent b's are separated by one of the a's, we may write

$$a_i = b_{io}.$$

In this case $y_{i-1,r_{i-1}}$ and $y_{io}$ denote the <u>same</u> component of $\bar{y}$.

In the following theorem, $\alpha$ denotes the discrete mass that $F_n(x_n)$ may have at $x_n = 1$. The number $D_i$ is defined as on page 9.

<u>Theorem 1</u>: Let F and G be fixed, and let $\bar{y}$ be any vector in the support of $G(\bar{y})$ with $y_m \neq 1$. Then

$$\int \Psi(\bar{x},\bar{y})\, dF(\bar{x}) \equiv \underline{v}$$

if and only if all the following conditions are satisfied:

1) In the interval $b_{ij} < x_i < b_{ij+1}$ the measure $F_i(x_i)$ is absolutely continuous, and

$$F'(x_i) = h_{ij} \frac{Q'(x_i)}{Q^2(x_i)\, P(x_i)}.$$

2) $1 + 2\alpha = D_n + 2h_{nr_n}$

$$h_{ij-1} = [1 - Q(b_{ij})]\, h_{ij}, \qquad j = 1,\ldots,r_i; \quad i = 1,\ldots,n$$

$$h_{ir_i} = [1 - D_i]\, h_{i+1,o}, \qquad i = 1, \ldots , n - 1.$$

<u>Proof</u>: The proof is by induction. In the first step we determine necessary and sufficient conditions for $R(\bar{y})$ to be independent of the last component of $\bar{y}$ when this component ranges over the interval $(b_{nr_n}, 1)$. In order to simplify the notation, a function $K_{n,r_n}(y_{11}, \ldots , y_{nr_n-1})$ is defined by

$$K_{n,r_n} = \prod_{s=1}^{n-1} [1 - D_s] \prod_{y_{st} < y_{nr_n}} [1 - Q(y_{st})],$$

where the second product is to be taken over all components of $\bar{y}$ that precede $y_{nr_n}$. Then, by Lemma 5

$$R(\bar{y}) = R(y_{11}, \ldots , y_{nr_n-1})$$
$$- K_{n,r_n} Q(y_{nr_n}) \left\{ 2 \int_{y_{nr_n}}^{1} P(x_n)\, dF_n(x_n) + 1 - D_n \right\}.$$

It is clear that $R(\bar{y})$ is independent of $y_{nr_n}$ if and only if the coefficient of $K_{n,r_n}$ is a constant; this constant is denoted by $2h_{nr_n}$. Thus $R(\bar{y})$ is independent of $y_{nr_n}$ if and only if

$$Q(y_{nr_n}) \left\{ 2 \int_{y_{nr_n}}^{1} P(x_n)\, dF_n(x_n) + 1 - D_n \right\} \equiv 2h_{nr_n}. \qquad (3)$$

Since $F(\bar{x})$ belongs to the Class O, $F_n$ is the sum of a discrete mass $\alpha$ at 1 and a continuous measure $F^*$. In terms of $F^*$ this equation can be written in the form

$$\frac{2h_{nr_n}}{Q(y_{nr_n})} = 2\int_{y_{nr_n}}^{1} P(x_n)\, dF^*(x_n) + 2\alpha\, P(1) + 1 - D_n\,, \qquad (3a)$$

and it is possible to integrate by parts since $P'$ is continuous; then

$$\frac{2h_{nr_n}}{Q(y_{nr_n})} = 2\,P(1)\,F^*(1) - 2P(y_{nr_n})\,F^*(y_{nr_n}) + 2\alpha + 1 - D_n$$

$$- 2\int_{y_{nr_n}}^{1} F^*(x_n)\,P'(x_n)\,dx_n .$$

In this equation $Q(y_{nr_n})$ and $P(y_{nr_n})$ are bounded below by $Q(b_{nr_n})$ and $P(b_{nr_n})$. Since all other terms are absolutely continuous, $F^*$ must be absolutely continuous too. Then, in equation (3a) we can write

$$dF_n^*(x_n) = F'(x_n)\,dx_n \,.$$

A simple differentiation shows that

$$-2P(y_{nr_n})\,F'(y_{nr_n}) = \frac{-2h_{nr_n}\,Q'(y_{nr_n})}{Q^2(y_{nr_n})}$$

as was asserted in the theorem. Now that the form of $F_n(x_n)$ is known, equation (3) becomes

$$Q(y_{nr_n})\left\{ \frac{2h_{nr_n}}{Q(y_{nr_n})} - 2h_{nr_n} + 2\alpha + 1 - D_n \right\} \equiv 2h_{nr_n} \,.$$

It is clear that this equation is satisfied if and only if

$$2h_{nr_n} = 2\alpha + 1 - D_n \tag{4}$$

as was asserted in Theorem 1. Finally, we notice that if we substitute the result of equation (3) into the expression for $R(\bar{y})$ we obtain

$$R(\bar{y}) = R(y_{11}, \ldots , y_{nr_n-1}) - 2K_{n,r_n} h_{nr_n} .$$

For convenience, the previous results are summarized as follows: $R(\bar{y})$ is independent of $y_{nr_n}$ if and only if the following conditions are all satisfied:

(a) $$Q(y_{nr_n}) \left\{ 2 \int_{y_{nr_n}}^{a_{n+1}} P(x_n)\, dF_n(x_n) + 1 - D_n \right\} \equiv 2h_{nr_n} .$$

(b) $$F_n'(x_n) = h_{nr_n} \frac{Q'(x_n)}{Q^2(x_n)\, P(x_n)} , \qquad b_{nr_n} < x_n < 1.$$

(c) $$2h_{nr_n} = 2\alpha + 1 - D_n .$$

(d) $$R(\bar{y}) = R(y_{11}, \ldots , y_{nr_n-1}) - 2K_{n,r_n} h_{nr_n} .$$

These conditions are not independent since (b) and (c) together determine (a) and (d).

The proof of Theorem 1 continues by induction: a set of functions $K_{i,j}(y_{11}, \ldots , y_{ij})$ and a set of constants $\gamma_{i,j}$ are defined as follows:

$$K_{i,j} = \prod_{s=1}^{i-1} [1 - D_s] \prod_{y_{st} < y_{ij}} [1 - Q(y_{st})].$$

$$\gamma_{n,r_n} = h_{nr_n} .$$

$$\gamma_{i,j} = h_{ij} + \gamma_{i,j+1} , \qquad j = 0,\dots,r_i - 1; \quad i = n,n-1,\dots, 1.$$

$$\gamma_{i,r_i} = [1 - D_i] \gamma_{i+1,1} + h_{i,r_i} \qquad i = 1,\dots,n-1.$$

In the definition of $K_{i,j}$ the second product is to be taken over all components of $\bar{y}$ that precede $y_{ij}$. The definition of $\gamma_{i,j}$ is applied only when the corresponding $h_{ij}$ has been defined.

Let us assume now that $R(\bar{y})$ has been made independent of all components of $\bar{y}$ that lie beyond $y_{ij}$, and that the function so obtained is given by

$$R(\bar{y}) = R(y_{11}, \dots , y_{ij}) - 2K_{i,j+1} \gamma_{i,j+1} \qquad \text{if} \quad j \neq r_i$$

or

$$R(\bar{y}) = R(y_{11}, \dots , y_{ir_i}) - 2K_{i+1,1} \gamma_{i+1,1} \qquad \text{if} \quad j = r_i.$$

We shall show that we can continue one step further if and only if the analogues of conditions (a), (b), (c),(d) hold for the indices $i$ and $j$. Two cases are considered, depending on the value of $j$.

<u>Case 1:</u> <u>$j \neq r_i$.</u> By assumption

$$R(\bar{y}) = R(y_{11}, \dots , y_{ij}) - 2K_{i,j+1} \gamma_{i,j+1} .$$

Applying Lemma 5 to the first term on the right,

$$R(\bar{y}) = R(y_{11}, \ldots, y_{ij-1}) - 2K_{i,j+1}\,\gamma_{i,j+1}$$

$$- K_{i,j}\, Q(y_{ij}) \left\{ 2 \int_{y_{ij}}^{a_{i+1}} P(x_i)\, dF_i(x_i) + [1 - D_i][1 + \phi(\bar{D}^i)] \right\}$$

Furthermore, by definition,

$$K_{i,j+1} = [1 - Q(y_{ij})]\, K_{i,j} .$$

The last two equations can be combined, collecting the terms that depend on $y_{ij}$. Then,

$$R(\bar{y}) = R(y_{11}, \ldots, y_{ij-1}) - 2K_{i,j}\,\gamma_{i,j+1}$$

$$- K_{i,j}\, Q(y_{ij}) \left\{ 2 \int_{y_{ij}}^{a_{i+1}} P(x_i)\, dF_i(x_i) + [1 - D_i][1 + \phi(\bar{D}^i)] - 2\,\gamma_{i,j+1} \right\}$$

It is clear that $R(\bar{y})$ is also independent of $y_{ij}$ if and only if the coefficient of $K_{i,j}$ in the last term is a constant; this constant may be denoted by $2h_{ij}$ so that

$$\text{(a)} \quad Q(y_{ij}) \left\{ 2 \int_{y_{ij}}^{a_{i+1}} P(x_i) dF_i(x_i) + [1-D_i][1+\phi(\bar{D}^i)] - 2\gamma_{i,j+1} \right\} \equiv 2h_{ij}$$

The argument that was given when we were dealing with the component $y_{nr_n}$ can be used to show that the last equation is satisfied only if

$$\text{(b)} \quad F_i'(x_i) = h_{ij} \frac{Q'(x_i)}{Q^2(x_i)\, P(x_i)} , \qquad b_{ij} < x_i < b_{ij+1} .$$

With this value of $F_i$, the integral given in (a) can be evaluated explicitly over the interval $(b_{ij}, b_{ij+1})$; then

$$2h_{ij} = Q(y_{ij})\left\{ \frac{2h_{ij}}{Q(y_{ij})} - \frac{2h_{ij}}{Q(b_{ij+1})} + 2\int_{b_{ij+1}}^{a_{i+1}} P(x_i)\, dF_i(x_i) \right.$$

$$\left. + [1 - D_i]\,[1 + \phi(\bar{D}^i)] - 2\,\gamma_{i,j+1} \right\} .$$

Thus, equation (a) is satisfied if and only if (b) holds and also

$$\text{(c)} \quad \frac{2h_{ij}}{Q(b_{ij+1})} = 2\int_{b_{ij+1}}^{a_{i+1}} P(x_i)\, dF_i(x_i)$$

$$+ [1 - D_i]\,[1 + \phi(\bar{D}^i)] - 2\,\gamma_{i,j+1} \quad .$$

Finally, when equation (a) is combined with the equation that precedes it, we obtain

$$R(\bar{y}) = R(y_{11}, \ldots , y_{ij-1}) - 2K_{i,j}[\,\gamma_{i,j+1} + h_{ij}]$$

or

$$\text{(d)} \qquad R(y) = R(y_{11}, \ldots , y_{ij-1}) - 2K_{i,j}\,\gamma_{i,j}$$

by definition of $\gamma_{i,j}$. The induction process can then be continued. We point out that in this case conditions (a) and (d) again are implied by (b) and (c).

<u>Case 2.</u> <u>$j = r_i$, and $i < n$.</u> (We recall that in the present notation $y_{ir_i}$ and $y_{i+1,o}$ denote the same component of $\bar{y}$.) In this case the induction hypothesis is given by condition (d), with indices $i + 1$, $1$; explicitly, $R(\bar{y})$ is independent of all the components of

$\bar{y}$ that lie beyond $y_{ir_i}$ only if

$$R(\bar{y}) = R(y_{11}, \dots, y_{ir_i}) - 2K_{i+1,1}\,\gamma_{i+1,1} \;.$$

By means of Lemma 5, this equation can be written in the form

$$R(\bar{y}) = R(y_{11}, \dots, y_{ir_i-1}) - 2K_{i+1,1}\,\gamma_{i+1,1} - K_{i,r_i}\,Q(y_{ir_i})\left\{\;2\int_{y_{ir_i}}^{a_{i+1}} P(x_i)\,dF_i(x_i) + [1-D_i][1+\phi(\bar{D}^i)]\right\} \;.$$

Furthermore,

$$K_{i+1,1} = [1 - D_i]\;[1 - Q(y_{ir_i})]\;K_{i,r_i} \;.$$

The last two equations can be combined, collecting the terms that depend on $y_{ir_i}$. Then

$$R(\bar{y}) = R(y_{11}, \dots, y_{ir_i-1}) - 2K_{i,r_i}[1 - D_i]\,\gamma_{i+1,1} - K_{i,r_i}\,Q(y_{ir_i})\left\{\;2\int_{y_{ir_i}}^{a_{i+1}} P(x_i)\,dF_i(x_i) + [1-D_i][1 + \phi(\bar{D}^i) - 2\gamma_{i+1,1}]\right\}$$

It is clear that the last term is independent of $y_{ir_i}$ if and only if the coefficient of $K_{i,r_i}$ is a constant; this constant is denoted by $2h_{ir_i}$. Once $h_{ir_i}$ is defined, the usual steps show that $R(\bar{y})$ is independent of $y_{ir_i}$ if and only if the following conditions are all satisfied:

(a) $$Q(y_{ir_i})\left\{\;2\int_{y_{ir_i}}^{a_{i+1}} P(x_i)\,dF_i(x_i) + [1-D_i][1+\phi(\bar{D}^i) - 2\gamma_{i+1,1}]\right\} \equiv 2h_{ir_i}$$

(b) $F_i'(x_i) = h_{ir_i} \dfrac{Q'(x_i)}{Q^2(x_i)\ P(x_i)}$ , $\qquad b_{ir_i} < x_i < a_{i+1}$

(c) $\dfrac{2h_{ir_i}}{Q(a_{i+1})} = [1 - D_i]\ [1 + \phi(\bar{D}^i) - 2\,\gamma_{i+1,1}]$

and

$$R(\bar{y}) = R(y_{11}, \ldots , y_{ir_i-1}) - 2K_{i,r_i} \left\{ [1-D_i]\ \gamma_{i+1,1} + h_{ir_i} \right\}$$

or

(d) $R(\bar{y}) = R(y_{11}, \ldots , y_{ir_i-1}) - 2K_{i,r_i}\,\gamma_{i,r_i}$ ,

by definition of $\gamma_{i,r_i}$ . These equations are not independent; indeed, (b) and (c) imply (a) and (d). The induction argument is now complete.

In order to complete the proof of Theorem 1 it is sufficient to show that the second condition that is given in the theorem is equivalent to the set of conditions that have been denoted by (c) above. This equivalence is established as follows:

1) For the indices $n$, $r_n$, condition (c) coincides with the equation

$$1 + 2\alpha = D_n + 2h_{nr_n} .$$

2) For the indices $i$, $j$ (with $j \neq r_i$) the equation

$$h_{ij} = [1 - Q(b_{ij+1})]\ h_{ij+1}$$

can be obtained by subtracting the equation (c) (with indices $i$, $j+1$) from the corresponding equation with indices $i$, $j$. Indeed, (c) may be written in the form

$$\frac{2h_{ij}}{Q(b_{ij+1})} = 2 \int_{b_{ij+1}}^{a_{i+1}} P(x_i)\, dF_i(x_i) + [1-D_i][1+ \phi(\bar{D}^i)] - 2\, \gamma_{i,j+1}$$

or

$$\frac{2h_{ij}}{Q(b_{ij+1})} = \frac{2h_{ij+1}}{Q(b_{ij+1})} - \frac{2h_{ij+1}}{Q(b_{ij+2})} + 2 \int_{b_{ij+2}}^{a_{i+1}} P(x_i)\, dF_i(x_i)$$

$$+ [1 - D_i][1 + \phi(\bar{D}^i)] - 2\, \gamma_{i,j+1}$$

Also (for the indices i, j+ 1), (c) has the form

$$0 = - \frac{2h_{ij+1}}{Q(b_{ij+2})} + 2 \int_{b_{ij+2}}^{a_{i+1}} P(x_i)\, dF_i(x_i) + [1-D_i][1+\phi(\bar{D}^i)]-2\, \gamma_{i,j+2}$$

Subtracting,

$$\frac{2h_{ij}}{Q(b_{ij+1})} = \frac{2h_{ij+1}}{Q(b_{ij+1})} + 2\, \gamma_{i,j+2} - 2\, \gamma_{i,j+1}$$

$$= \frac{2h_{ij+1}}{Q(b_{ij+1})} - 2h_{ij+1} \quad .$$

Simplifying,

$$h_{ij} = [1 - Q(b_{ij+1})]\, h_{ij+1} \; .$$

3) The equation

$$h_{ir_i} = [1 - D_i]\, h_{i+1,o}$$

can be derived similarly. First, equation (c) (with indices i+1,0) is written in the form

$$0 = -\frac{2h_{i+1,o}}{Q(a_{i+1})} + \frac{2h_{i+1,o}}{Q(a_{i+1})} - \frac{2h_{i+1,o}}{Q(b_{i+1,1})} + 2\int_{b_{i+1,1}}^{a_{i+2}} P(x_{i+1})dF_{i+1}(x_{i+1})$$

$$+ [1 - D_{i+1}][1 + \phi(\bar{D}^{i+1})] - 2\,\gamma_{i+1,1}$$

$$= -\frac{2h_{i+1,o}}{Q(a_{i+1})} + 2D_{i+1} + [1 - D_{i+1}][1 + \phi(\bar{D}^{i+1})] - 2\,\gamma_{i+1,1}$$

$$= -\frac{2h_{i+1,o}}{Q(a_{i+1})} + 1 + \phi(\bar{D}^{i}) - 2\,\gamma_{i+1,1} \text{ .}$$

Next, equation (c) (with indices $i$, $r_i$) is written in the form

$$\frac{2h_{ir_i}}{Q(a_{i+1})} = [1 - D_i]\ [1 + \phi(\bar{D}^{i}) - 2\,\gamma_{i+1,1}] \text{ .}$$

Finally, the last two equations are combined, and the result follows, immediately.

Conversely, it is clear that each of these steps can be reversed. Therefore, the two sets of conditions are completely equivalent.

<u>Some additional remarks</u>. It may happen that one of the b's coincides with one of the a's so that there is no need to introduce the constant $h_{i+1,o}$. In this case it is easy to show that

$$h_{ir_i} = [1 - D_i][1 - Q(a_{i+1})]\ h_{i+1,1},$$

whenever $a_{i+1} = b_{i+1,1}$. This equation is derived like the equation

$$h_{ir_i} = [1 - D_i]\ h_{i+1,o}\ ,$$

but in the first step condition (c) is written for the indices $i + 1$, $1$ instead of $i + 1$, $o$. It is important to notice that $h_{ir_i}$ is a continuous function of the parameter $b_{i+1,1}$. Indeed, if $b_{i+1,1} > a_{i+1}$,

$$h_{ir_i} = [1 - D_i]\, h_{i+1,o}$$

$$= [1 - D_i][1 - Q(b_{i+1,1})]\, h_{i+1,1} .$$

As $b_{i+1,1} \to a_{i+1}$, this expression approaches the value

$$h_{ir_i} = [1 - D_i]\ [1 - Q(a_{i+1})]\ h_{i+1,1}$$

that was derived independently when $b_{i+1,1} = a_{i+1}$.

<u>Theorem 2</u>: Let $F(\bar{x})$ and $G(\bar{y})$ be a pair of corresponding strategies. Then

$$\int \Psi(\bar{x},\bar{y})\ dF(\bar{x}) \geq \underline{v}, \qquad \text{all } \bar{y} \in Y$$

and

$$\int \Psi(\bar{x},\bar{y})\ dG(\bar{y}) \leq \bar{v}, \qquad \text{all } \bar{x} \in X.$$

In this theorem $n$ and $m$ are fixed.

<u>Proof</u>: It is sufficient to show that the first inequality holds. The validity of the second one will follow from the fact that $\Psi$ is skew-symmetric if the roles of $\bar{x}$, $\bar{y}$, $X$ and $Y$ are interchanged. Furthermore, it is necessary to consider only those vectors $\bar{y}$ that have no components smaller than $a_1$. ($a_1$ denotes as usual the first point in the support of $F_1(x_1)$.) Indeed, it follows from Lemma 3 that for every $\bar{x}$ in the support of $F(\bar{x})$, $\Psi(\bar{x},\bar{y})$ is a monotone decreasing function of any component of $\bar{y}$ that is contained in the interval $[0, a_1]$. Hence

$$\int \Psi(\bar{x},\bar{y})\ dF(\bar{x})$$

is also monotone decreasing in this interval. Finally, if $F(\bar{x})$ has a discrete mass $a$ at 1, all vectors $\bar{y}$ with $y_m = 1$ can be neglected; this follows from the fact that

$$\Psi((x_1, \dots, x_{n-1}, 1), (y_1, \dots, y_{m-1}, 1)) \geq$$

$$\geq \Psi(x_1, \dots, x_{n-1}, 1), (y_1, \dots, y_{m-1}, 1 - \varepsilon)).$$

Intuitively, the pay-off function $\Psi$ favors the contestant who anticipates his opponent.

In order to prove the theorem, we shall consider each component of $\bar{y}$ separately, starting with $y_m$. We shall show that we cannot increase $R(\bar{y})$ if we replace $y_m$ by a component $y_m^* \varepsilon [b_{nr_n}, 1)$. The notation of Theorem 1 is also used below.

If $y_m$ is already contained in the interval $[b_{nr_n}, 1)$ we choose $y_m^* = y_m$. But if $y_m$ is contained in some other interval, say $[b_{ij}, b_{ij+1}]$, we apply Lemma 5 to $R(\bar{y})$. Then,

$$R(\bar{y}) = R(y_1, \dots, y_{m-1}) - \prod_{s=1}^{i-1} [1 - D_s] \prod_{t=1}^{m-1} [1 - Q(y_t)] \cdot$$

$$\cdot Q(y_m) \left\{ 2 \int_{y_m}^{a_{i+1}} P(x_i)\, dF_i(x_i) + [1 - D_i][1 + \phi(\bar{D}^i)] \right\}$$

Furthermore, since $F$ and $G$ are a pair of corresponding strategies, $F$ satisfies all the conditions derived in Theorem 1; in particular, if $j < r_i$, condition (a) in Theorem 1 (with indices i, j) implies that

$$Q(y) \left\{ 2 \int_{y}^{a_{i+1}} P(x_i)\, dF_i(x_i) + [1 - D_i][1 + \phi(\bar{D}^i)] - 2\gamma_{i,j+1} \right\} \equiv 2h_{ij}$$

for all $y \in [b_{ij}, b_{ij+1}]$. Thus, by substitution,

$$R(\bar{y}) = R(y_1, \dots, y_{m-1}) - \prod_{s=1}^{i-1} [1 - D_s] \prod_{t=1}^{m-1} [1 - Q(y_t)] \left\{ 2h_{ij} + 2 \gamma_{i,j+1} Q(y_m) \right\} .$$

Similarly, if $j = r_i$

$$R(\bar{y}) = R(y_1, \dots, y_{m-1}) - \prod_{s=1}^{i-1} [1 - D_s] \prod_{t=1}^{m-1} [1 - Q(y_t)] \left\{ 2h_{ir_i} + 2[1-D_i] \gamma_{i+1,1} Q(y_m) \right\} .$$

In either case, $R(\bar{y})$ is a monotone decreasing function of $y_m$, and it achieves its minimum value when $y_m$ is at the right end point of the interval. But this end point is also in the next interval, and the argument can be repeated. In this manner we deduce that $R(\bar{y})$ can only decrease if we replace $y_m$ by $b_{nr_n}$; we write $y_m^* = b_{nr_n}$; the remaining components are kept fixed. Finally, we notice that condition (d) in Theorem 1 is applicable to $y_m^*$. Thus

$$R(\bar{y}) \geq R(y_1, \dots, y_{m-1}, y_m^*) = R(y_1, \dots, y_{m-1}) - 2K_{n,r_n} \gamma_{n,r_n} .$$

The proof continues by induction. We assume that there exists a vector $(y_1, \dots, y_q, y_{q+1}^*, \dots, y_m^*) \in Y$ with the following properties:

1) The first $q$ components of this vector coincide with the first $q$ components of $\bar{y}$.

2) The last $m - q$ components coincide with the last $m - q$ components of some vector in the support of $G(\bar{y})$.

3) $R(\bar{y}) \geq R(y_1, \dots, y_q, y_{q+1}^*, \dots, y_m^*)$.

Then, in order to prove Theorem 2, it is sufficient to show that we can continue at least one more step, since $R(y_1^*, \ldots, y_m^*)$ equals $\underline{y}$ (by Theorem 1).

For definiteness, let $[b_{kl+1}, b_{kl+2}]$ denote the interval that contains $y_{q+1}^*$. Then, condition (d) in Theorem 1 can be applied to all the starred components, and

$$R(y_1, \ldots, y_q, y_{q+1}^*, \ldots, y_m^*)$$

$$= R(y_1, \ldots, y_q) - 2K_{k,l+1}\,\gamma_{k,l+1} \qquad (5)$$

At this point we must consider three distinct cases:

<u>Case 1</u>: $y_q \in [b_{kl}, b_{kl+1}]$. In this case we take $y_q^* = y_q$.

<u>Case 2</u>: $y_q < b_{kl}$. In this case we repeat the argument that was given for $y_m$: Lemma 5 is applied to equation (5), and the result is simplified by means of the appropriate condition (a) from Theorem 1; if $y_q \in [b_{ij}, b_{ij+1}]$ the term involving $y_q$ will be of the form

$$-\prod_{s=1}^{i-1}[1-D_s]\prod_{t=1}^{q-1}[1-Q(y_t)]\ \left\{2h_{ij} + 2Q(y_q)[\gamma_{i,j+1} - \prod_{\sigma=i}^{k-1}[1-D_\sigma]\,\gamma_{k,l+1}]\right\}$$

The definition of the $\gamma$'s shows that this is a monotone decreasing function of $y_q$. Hence, we may take $y_q^* = b_{kl}$.

<u>Case 3</u>: $y_q > b_{kl+1}$. In this case $y_q$ must be contained in the interval $[b_{kl+1}, b_{kl+2}]$, because $y_q < y_{q+1}^*$, and $y_{q+1}^* \in [b_{kl+1}, b_{kl+2}]$. Since the interval that contains $y_q$ is known, lemma 5 can be applied to equation (5), and the result can be simplified by means of the

appropriate condition (a) from Theorem 1. Then

$$R(y_1, \dots, y_q, y^*_{q+1}, \dots, y^*_m)$$
$$= R(y_1, \dots, y_{q-1})$$
$$- 2 \prod_{s=1}^{k-1} [1-D_s] \prod_{t=1}^{q-1} [1-Q(y_t)] \left\{ \gamma_{k,l+1} + h_{kl+1}[1-Q(y_q)] \right\}$$

It may happen that the component $y_{q-1}$ is also contained in the interval $[b_{kl+1}, b_{kl+2}]$. Then, the previous steps may be repeated, and then

$$R(y_1, \dots, y_q, y^*_{q+1}, \dots, y^*_m)$$
$$= R(y_1, \dots, y_{q-2}) - 2 \prod_{s=1}^{k-1} [1-D_s] \prod_{t=1}^{q-2} [1-Q(y_t)] \cdot$$
$$\cdot \left\{ \gamma_{k,l+1} + h_{kl+1}[1-Q(y_{q-1})] \left\{ 1+[1-Q(y_q)] \right\} \right\}$$

The process can be repeated for all the components of $\bar{y}$ that lie in the interval $[b_{kl+1}, b_{kl+2}]$. It is clear that the result is a monotone increasing function of these components, and the value of $R(\bar{y})$ can only decrease if all of them are replaced by $b_{kl+1}$. In particular we take $y^*_q = b_{kl+1}$ which is contained in the interval $[b_{kl}, b_{kl+1}]$ and hence satisfies the induction hypothesis. However, the other components that were in the interval $[b_{kl+1}, b_{kl+2}]$ no longer agree with the components of $\bar{y}$. But they are now contained in the interval $[b_{kl}, b_{kl+1}]$, and by the previous argument, we only decrease $R(\bar{y})$ if we replace all of them by $b_{kl}$; in particular we take $y^*_{q-1} = b_{kl}$; the process can be continued until we obtain a vector of the form

$$(y_1, \dots, y_{p-1}, y^*_p, \dots, y^*_{q-1}, y^*_q, \dots, y^*_m)$$

that satisfies the induction hypothesis. The proof of the Theorem is now complete.

<u>Corollary</u>. The equation

$$\int \Psi(\bar{x},\bar{y})\, dF(\bar{x}) = \underline{v}$$

is valid only when $\bar{y}$ is in the support of $G(\bar{y})$.

This result follows from the strict monotonicity of the functions that appear in the proof of Theorem 2.

<u>Theorem 3</u>: Suppose that $F(\bar{x})$ and $G(\bar{y})$ are a pair of corresponding strategies, and that at least one of them is continuous at 1. Then $\underline{v} = \bar{v}$, and $F(\bar{x})$ and $G(\bar{y})$ are optimal for the game generated by $\Psi$.

<u>Proof</u>: For definiteness we assume that $G(\bar{y})$ is continuous at 1. Then, by definition of $\underline{v}$,

$$\int \Psi(\bar{x},\bar{y})\, dF(\bar{x}) = \underline{v}$$

for all $\bar{y}$ in the support of $G(\bar{y})$, provided that $y_m \neq 1$; furthermore, the vectors with $y_m = 1$ have G-measure zero, and

$$\int\int \Psi(\bar{x},\bar{y})\, dF(\bar{x})\, dG(\bar{y}) = \int \underline{v}\, dG(\bar{y}) = \underline{v}. \qquad (6)$$

Similarly, by definition of $\bar{v}$,

$$\int \Psi(\bar{x},\bar{y})\, dG(\bar{y}) = \bar{v}$$

for all vectors $\bar{x}$ in the support of $F(\bar{x})$; this result is valid even if $x_n = 1$ because the left hand side is continuous. Then

$$\int\int \Psi(\bar{x},\bar{y})\, dG(\bar{y})\, dF(\bar{x}) = \int \bar{v}\, dF(\bar{x}) = \bar{v}. \qquad (7)$$

Equations (6) and (7) imply that $\underline{v} = \bar{v}$, and this number is denoted by $v$. Then, by Theorem 2,

$$\int \Psi(\bar{x},\bar{y})\, dF(\bar{x}) \geq v, \qquad \text{all } \bar{y} \in Y$$

and

$$\int \Psi(\bar{x},\bar{y})\, dG(\bar{y}) \leq v, \qquad \text{all } \bar{x} \in X.$$

Hence, $F(\bar{x})$ and $G(\bar{y})$ are optimal.

<u>Remark</u>: It is well known that if $F$ and $G$ are optimal for a game, then

$$\int \Psi(\bar{x},\bar{y})\, dF(\bar{x}) = v$$

for all points $\bar{y}$ in the support of $G(y)$, provided that $\bar{y}$ is a point of continuity of the integral. Furthermore, a similar result is valid when $F$ and $G$ are interchanged. In particular, any pair of optimal strategies that belong to the class $O$ must be a pair of corresponding strategies; and it is easy to show that at least one of them must be continuous at 1. This remark may be taken as the converse of Theorem 3.

## 7. <u>Existence of a Solution</u>.

The problem of finding a solution for the given game has been reduced to the problem of finding two strategies $F(\bar{x})$ and $G(\bar{y})$ that satisfy the hypothesis of Theorem 3. The new problem can be simplified considerably if $F(\bar{x})$ and $G(\bar{y})$ are described by means of the equations derived in Theorem 1.

We begin by reviewing the necessary notation, and at the same time we introduce the notation that is needed in order to apply Theorem 1 to $G(\bar{y})$. The strategies $F$ and $G$ are of the form

$$F(\bar{x}) = \prod_{i=1}^{n} F_i(x_i), \qquad G(\bar{y}) = \prod_{j=1}^{m} G_j(y_j)$$

The support of the measure $F_i$ is denoted by $[a_i, a_{i+1}]$, and the support of $G_j$ is denoted by $[b_j, b_{j+1}]$; furthermore, $a_{n+1} = b_{m+1} = 1$, and $F_n(x_n)$ and $G_m(y_m)$ may have discrete masses $\alpha$ and $\beta$ at $x_n = 1$ and $y_m = 1$. Each measure is normalized so that

$$\int_{a_i}^{a_{i+1}} dF_i(x_i) = 1, \qquad \int_{b_j}^{b_{j+1}} dG_j(y_j) = 1. \tag{8}$$

The constants $D_i$ and $E_j$ are defined by the equations

$$D_i = \int_{a_i}^{a_{i+1}} P(x_i)\, dF_i(x_i), \qquad E_j = \int_{b_j}^{b_{j+1}} Q(y_j)\, dG_j(y_j).$$

In order to apply Theorem 1 to $F$, the numbers $b_1, \dots, b_m$ are rearranged into subsets, one subset for each interval $[a_i, a_{i+1}]$. Thus

$$a_1 \leq b_{11} < b_{12} < \cdots < b_{1r_1} < a_2$$

$$a_2 \leq b_{21} < b_{22} < \cdots < b_{2r_2} < a_3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_n \leq b_{n1} < b_{n2} < \cdots < b_{nr_n} < a_{n+1} = 1.$$

In order to apply Theorem 1 to $G$, the numbers $a_1, \dots, a_n$ are rearranged into subsets, one subset for each interval $[b_j, b_{j+1}]$. Thus

$$b_1 \leq a_{11} < a_{12} < \cdots < a_{1s_1} < b_2$$

$$b_2 \leq a_{21} < a_{22} < \cdots < a_{2s_2} < b_3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$b_m \leq a_{m1} < a_{m2} < \cdots < a_{ms_m} < b_{m+1} = 1.$$

In terms of this notation, the previous results may be summarized as follows: the given strategies $F(\bar{x})$ and $G(\bar{y})$ are optimal if and only if they satisfy all the following conditions:

1) For each $i$ and $j$,

$$F_i'(t) = h_{ij} \frac{Q'(t)}{Q^2(t)\,P(t)}, \qquad b_{ij} < t < b_{ij+1}\,.$$

For each $j$ and $i$,

$$G_j'(t) = k_{ji} \frac{P'(t)}{P^2(t)\,Q(t)}, \qquad a_{ji} < t < a_{ji+1}\,.$$

2) $1 + 2\alpha = D_n + 2h_{nr_n}$

$1 + 2\beta = E_m + 2k_{ms_m}$

3) $h_{ij} = [1 - Q(b_{ij+1})]\, h_{ij+1}, \qquad j = 0,\dots,r_i-1;\ i = 1,\dots,n$

$k_{ji} = [1 - P(a_{ji+1})]\, k_{ji+1}, \qquad i = 0,\dots,s_j-1;\ j = 1,\dots,m.$

4) $h_{ir_i} = [1 - D_i]\, h_{i+1,o} \qquad i = 1,\dots,n-1$

$k_{js_j} = [1 - E_j]\, k_{j+1,o} \qquad j = 1,\dots,m-1.$

5) $a_1 = b_1$ and $\alpha\beta = 0.$

<u>Note</u>: The equation $\alpha\beta = 0$ is equivalent to the continuity hypothesis of Theorem 3; the equation $a_1 = b_1$ is derived as follows: in order to apply Theorem 1 to $F$ it is necessary that $a_1 \leq b_1$; in order to apply it to $G$, $b_1 \leq a_1$, hence $a_1 = b_1$. It should be pointed out that

if a solution is constructed from this system of equations, it is necessary to have each measure normalized in the usual way. Thus, equation (8) should be added to the system.

In order to show that the given system has a solution, it is convenient to transform it first into an equivalent system in which the number of unknowns and equations is smaller. The numbers $b_1, \dots, b_m$ are regarded as a fixed set of parameters, and a discontinuous function $f^*(x)$ is defined as follows:

$$f^*(x) = \frac{Q'(x)}{Q^2(x)\,P(x)}, \qquad b_m < x \leq 1$$

$$f^*(x) = [1 - Q(b_m)]\,\frac{Q'(x)}{Q^2(x)\,P(x)}, \qquad b_{m-1} < x \leq b_m$$

etc. In general, as $x$ moves to the left of $b_j$, the factor $[1 - Q(b_j)]$ is excluded. Formally,

$$f^*(x) = \prod_{j=k+1}^{m} [1 - Q(b_j)]\,\frac{Q'(x)}{Q^2(x)\,P(x)}, \qquad b_k < x \leq b_{k+1}\,.$$

If the numbers $a_1, \dots, a_n$ are also regarded as parameters, then a discontinuous function $g^*(y)$ can be defined by

$$g^*(y) = \prod_{i=k+1}^{n} [1 - P(a_i)]\,\frac{P'(y)}{P^2(y)\,Q(y)}, \qquad a_k < x \leq a_{k+1}\,.$$

The factors that appear above are precisely those that relate two consecutive h's (or two consecutive k's) in condition (3). Thus a single $h_i$ (or $k_j$) is sufficient to describe each $F_i$ (or $G_j$); then the given condition can be eliminated completely if condition (1) is replaced by the simpler equations

$$F_i'(t) = h_i \, f^*(t), \qquad a_i < t < a_{i+1}$$
$$G_j'(t) = k_j \, g^*(t), \qquad b_j < t < b_{j+1} \, . \tag{9}$$

The coefficients $h_{ir_i}$ and $k_{js_j}$ can be related to the new coefficients $h_i$ and $k_j$ by means of the following equations:

$$h_{nr_n} = h_n$$

$$k_{ms_m} = k_m$$

$$h_{ir_i} = \prod_{t=i+1}^{n} \left[ \prod_{j=1}^{r_t} [1 - Q(b_{tj})] \right] h_i, \qquad i = 1, \ldots, n-1$$

$$k_{js_j} = \prod_{t=j+1}^{m} \left[ \prod_{i=1}^{s_t} [1 - P(a_{ti})] \right] k_j, \qquad j = 1, \ldots, m-1.$$

The equations that appear in condition (4) can be expressed now in terms of the new coefficients; the result can be simplified because the same factors appear in both sides of the equation; then

$$h_i = [1 - D_i] \, h_{i+1},$$

and

$$k_j = [1 - E_j] \, k_{j+1}.$$

Finally, the normalizing equations and the definitions of $D_i$ and $E_j$ can be expressed in terms of $f^*$ and $g^*$. The complete system of equations is as follows:

Definitions of $D_i$ and $E_j$:

$$D_n = h_n \int_{a_n}^{1} P(t)\ f^*(t)\ dt + \alpha \tag{10}$$

$$E_m = k_m \int_{b_m}^{1} Q(t)\ g^*(t)\ dt + \beta \tag{11}$$

$$D_i = h_i \int_{a_i}^{a_{i+1}} P(t)\ f^*(t)\ dt, \qquad i = 1,\ldots,n-1 \tag{12}$$

$$E_j = k_j \int_{b_j}^{b_{j+1}} Q(t)\ g^*(t)\ dt, \qquad j = 1,\ldots,m-1 \tag{13}$$

Normalizing Equations:

$$1 = h_n \int_{a_n}^{1} f^*(t)\ dt + \alpha \tag{14}$$

$$1 = k_m \int_{b_m}^{1} g^*(t)\ dt + \beta \tag{15}$$

$$1 = h_i \int_{a_i}^{a_{i+1}} f^*(t)\ dt, \qquad i = 1,\ldots,n-1 \tag{16}$$

$$1 = k_j \int_{b_j}^{b_{j+1}} g^*(t)\ dt, \qquad j = 1,\ldots,m-1 \tag{17}$$

Equations from Theorem 1 and Theorem 3:

$$1 + \alpha = h_n \left\{ \int_{a_n}^{1} P(t)\ f^*(t)\ dt + 2 \right\} \tag{18}$$

$$1 + \beta = k_m \left\{ \int_{b_m}^{1} Q(t)\ g^*(t)\ dt + 2 \right\} \tag{19}$$

$$h_i = [1 - D_i]\, h_{i+1} \qquad i = 1,\dots,n-1 \tag{20}$$

$$k_j = [1 - E_j]\, k_{j+1} \qquad j = 1,\dots,m-1 \tag{21}$$

$$a_1 = b_1 \tag{22}$$

$$\alpha \beta = 0 \,. \tag{23}$$

<u>Theorem 4</u>: The given system of equations has a unique solution, and this solution determines two strategies $F(\bar{x})$ and $G(\bar{y})$ that are optimal for the game with pay-off $\Psi(\bar{x},\bar{y})$. Furthermore, these are the only optimal strategies that belong to the class O.

In order to prove this theorem it is sufficient to show that there is a unique set of numbers $\alpha, \beta, a_1, \dots, a_n, h_1, \dots, h_n, b_1, \dots, b_m, k_1, \dots, k_m$ that satisfy the given system of equations. The strategies $F(\bar{x})$ and $G(\bar{y})$ are determined by means of equation (9), and then the other assertions of this theorem follow directly from Theorem 1 and Theorem 3.

In order to show that the given system of equations has a solution, it will be necessary to prove a series of lemmas. As a preliminary step, we develop a method for computing $a_1, \dots, a_n$ as functions of $b_1, \dots, b_m$ and $\alpha$. Since the problem is symmetric, this method will also serve to compute $b_1, \dots, b_m$ in terms of $a_1, \dots, a_n$.

The coefficient $h_n$ is eliminated from equations (14) and (18). Then,

$$2(1 - \alpha) = \int_{a_n}^{1} [(1 + \alpha) - (1 - \alpha)\, P(t)]\, f^*(t)\, dt \tag{24}$$

Also, equation (20) can be written in the form

$$\frac{1}{h_i} - \frac{D_i}{h_i} = \frac{1}{h_{i+1}}, \qquad i = 1,\dots,n-1.$$

The left hand side of this equation can be simplified by means of (12) and (16). Then,

$$\int_{a_i}^{a_{i+1}} [1 - P(t)]\ f^*(t)\ dt = \frac{1}{h_{i+1}}, \qquad i = 1,\dots,n-1. \quad (25)$$

Here $h_{i+1}$ is to be determined by equation (14), or (16). It is shown below that these equations can be used to compute $a_n, a_{n-1}, \dots, a_1$. In the following lemmas it is assumed that the parameters $b_1, \dots, b_m$ satisfy the following restriction:

$$0 < b_1 < b_2 < \cdots < b_m < 1.$$

<u>Lemma 7</u>: Let $b_1, \dots, b_m$ be a fixed set of parameters, and let $a$ be any number such that $0 < a \leq 1$. Then

$$\lim_{x \to 0^+} \int_x^a [1 - P(t)]\ f^*(t)\ dt = +\infty.$$

<u>Proof</u>: Let $M = \prod_{j=1}^{m} [1 - Q(b_j)] > 0$. Then, for all $t$,

$$f^*(t) \geq M \frac{Q'(t)}{Q^2(t)\ P(t)}.$$

Furthermore, since $P$ is a continuous function and $P(0) = 0$, there exists a number $c > 0$ such that

$$\frac{1 - P(t)}{P(t)} \geq 1$$

for all $t \leq c$. Then, if $x \leq c$,

$$\int_x^c [1 - P(t)]\ f^*(t)\ dt \geq \int_x^c M \frac{Q'(t)}{Q^2(t)}\ dt = M[\frac{1}{Q(x)} - \frac{1}{Q(c)}] \quad .$$

The last expression clearly tends to $+\infty$ as $x \to 0$. Since the integral is finite over the interval $[a,c]$, the lemma is valid for all $a$.

<u>Lemma 8</u>: Let $b_1, \ldots, b_m$ be a fixed set of parameters, and let $\alpha$ be any number such that $0 \leq \alpha < 1$. Then, equation (24) has a unique solution $a_n$ which is contained in the interval $(0,1)$. Furthermore, this solution $a_n$ is a monotone increasing function of $\alpha$, and it approaches 1 as $\alpha$ approaches 1.

<u>Proof</u>: If $0 \leq \alpha \leq 1$, then

$$[(1 + \alpha) - (1 - \alpha)\ P(t)]\ f^*(x) \geq [1 - P(t)]\ f^*(t) \geq 0$$

for all $t$. The right hand side of the equation

$$2(1 - \alpha) = \int_{a_n}^1 [(1 + \alpha) - (1 - \alpha)\ P(t)]\ f^*(t)\ dt$$

is then a continuous, strictly decreasing function of $a_n$, and it achieves the value zero when $a_n = 1$. Furthermore, as $a_n \to 0$, the right hand side approaches $+\infty$ (by lemma 7). Thus, for each $\alpha$, there exists a unique solution. Furthermore, the quantity

$$[(1 + \alpha) - (1 - \alpha)\ P(t)]\ f^*(t)$$

is a monotone increasing function of $\alpha$, while $2(1 - \alpha)$ is a monotone decreasing function. Hence, as $\alpha$ increases, the solution $a_n$ must also increase in order to preserve the equality. Finally, when $\alpha = 1$, the equation becomes

$$0 = 2 \int_{a_n}^{1} f^*(t)\, dt.$$

The only solution is $a_n = 1$.

Lemma 9: Under the assumption of the previous lemma, it is possible to find a unique set of numbers $a_1 < a_2 < \cdots < a_n$ that satisfy equations (24) and (25).

Proof: The number $a_n$ has already been determined in Lemma 8. Then $h_n$ can be computed directly from equation (14) since the parameters $b_1, \ldots, b_m$ and $\alpha$ are given. Then, Lemma 7 implies that the equation

$$\int_{a_{n-1}}^{a_n} [1 - P(t)]\, f^*(t)\, dt = \frac{1}{h_n}$$

has a unique solution $a_{n-1} < a_n$. When $a_{n-1}$ is known, $h_{n-1}$ can be computed from equation (16). The process can be continued until all the numbers $a_1, \ldots, a_n$ are determined.

Lemma 10: For each fixed $\alpha$, the solution $a_n$ of equation (24) is a monotone decreasing function of the parameters $b_1, \ldots, b_m$. (It is a strictly decreasing function of the parameters that lie to the right of $a_n$.) Furthermore, $a_n$ approaches 0 as $b_m$ approaches 1.

Proof: The definition of $Q(t)$ implies that $[1 - Q(b_j)]$ is a decreasing function of $b_j$. Then, for each $t < b_j$, $f^*(t)$ is a monotone decreasing function of the parameter $b_j$; and for $t > b_j$, $f^*(t)$ is independent of $b_j$. Thus, in the equation

$$2(1 - \alpha) = \int_{a_n}^{1} [(1 + \alpha) - (1 - \alpha) P(t)] f^*(t)\, dt$$

the integral decreases as the parameters $b_1, \ldots, b_m$ increase. Then $a_n$ must decrease in order to preserve the equality. Furthermore, for $t < b_m$,

$$f^*(t) \leq [1 - Q(b_m)] \frac{Q'(t)}{Q^2(t)\, P(t)} .$$

Then, as $b_m \to 1$, $f^*(t)$ tends to zero uniformly over any interval of the form $[c, b_m]$ with $c > 0$. Therefore,

$$\int_{c}^{1} [(1 + \alpha) - (1 - \alpha) P(t)] f^*(t)\, dt$$

also tends to zero. Hence, $a_n \to 0$.

Proof of Theorem 4: Let $\alpha$ and $\beta$ be two fixed numbers that satisfy the conditions

$$0 \leq \alpha < 1, \qquad 0 \leq \beta < 1$$

and

$$\alpha \beta = 0.$$

Then a set of numbers $a_1^*, \ldots, a_n^*, b_1^*, \ldots, b_m^*$ is computed as follows: first, $a_n$ is computed from equation (24), without any parameters $b_1, \ldots, b_m$; similarly, $b_m$ is computed from the equation

$$2(1-\beta) = \int_{b_m}^{1} [(1+\beta) - (1-\beta)\, Q(t)]\, g^*(t)\, dt, \qquad (26)$$

and in this equation $g^*$ is defined without any parameters. Next, the numbers $a_n$ and $b_m$ are compared, and the larger one is kept as a parameter; the smaller one is completely neglected. For definiteness we may assume that $a_n$ is the larger one, and define $a_n^* = a_n$. The corresponding $h_n^*$ can be computed from equation (14). In the next step a number $a_{n-1}$ is computed from equation (25) (without parameters), and a new $b_m$ is computed from equation (26) (with the single parameter $a_n^*$). The two numbers $a_{n-1}$ and $b_m$ are compared, and the larger one is kept as a parameter; the smaller one is completely neglected. For definiteness we may assume that $a_{n-1}$ is the larger number, and define $a_{n-1}^* = a_{n-1}$. The process continues in this manner: at each step a new $a_i$ and a new $b_j$ are computed, using as parameters the previously starred a's and b's. The two numbers are compared, and the larger one is kept as a parameter, denoted by $a_i^*$ (or $b_j^*$). When $a_i^*$ (or $b_j^*$) is found, the corresponding $h_i^*$ (or $k_j^*$) is computed by means of the appropriate normalizing equation. In this manner it is possible to construct the numbers $a_1^*, \ldots, a_n^*, b_1^*, \ldots, b_m^*$ and the corresponding $h_i^*, k_j^*$. Furthermore, the construction shows that these numbers satisfy the system of equations (10) through (23) <u>except for equation (22)</u>.

In order to show that there exist values of $\alpha$ and $\beta$ for which equation (22) is also satisfied, one may begin by applying the previous construction with

$$\alpha = \beta = 0.$$

If it happens that $a_1^* = b_1^*$, then all the equations are satisfied. Otherwise, either $a_1^* < b_1^*$ or $b_1^* < a_1^*$. For definiteness, it may be assumed that $b_1^* < a_1^*$. In this case the same construction is applied with $\alpha = 0$, and $\beta = 1 - \varepsilon$, for small $\varepsilon$. By taking $\beta$ sufficiently close to 1, $b_m$ can be made to approach 1; then $a_n < b_m$, and $b_m^* = b_m$. It follows from Lemma 10 that in the next step $a_n \to 0$. All the $b^*$'s are computed without parameters, and they remain bounded away from zero; but $a_n^*$ is computed with the parameter $b_n^*$, so that it must approach zero. Since $a_1^* < a_n^*$, it follows that in this case $a_1^* < b_1^*$. Thus the inequality has been reversed. Finally, since all the $a_i^*$ and $b_j^*$ are computed as limits of integration for the densities $f^*$ and $g^*$, it is easy to see that $a_1^*$ and $b_1^*$ are continuous functions of $\alpha$ and $\beta$. Since the direction of the inequalities is reversed when $\beta$ increases from 0 to 1, there must exist some positive $\beta$ with the property that the previous construction gives $a_1^* = b_1^*$. Thus, in this case, the game has a solution with $\alpha = 0$, $\beta > 0$.

If it happens that the given construction (with $\alpha = \beta = 0$) leads to the inequality $a_1^* < b_1^*$, then the system of equations has a solution with $\beta = 0$, $\alpha > 0$.

Uniqueness. It is easy to see that the solution of the given system of equations is unique. Indeed, suppose that there are two solutions $a_1, \dots, a_n, b_1, \dots, b_m$ and $a_1^*, \dots, a_n^*, b_1^*, \dots, b_m^*$; these solutions give rise to two pairs of optimal strategies $F(\bar{x})$, $G(\bar{y})$ and $F^*(\bar{x}), G^*(\bar{y})$. Then, since $F$ and $G^*$ are optimal, it follows that

$$\int \Psi(\bar{x},\bar{y})\, dF(\bar{x}) \equiv v$$

for all $\bar{y}$ in the support of $G^*(\bar{y})$ (with $y_m \neq 1$). Then the corollary to the Theorem 2 implies that the support of $G^*(\bar{y})$ must be contained in the support of $G(\bar{y})$. Since the argument is symmetric, the supports of $G$ and $G^*$ must coincide; thus $b_j^* = b_j$, for all $j$. A similar argument shows that $a_i^* = a_i$, for all $i$.

8. Example.

It is very easy to compute the solution for the completely symmetric problem in which $n = m$ and $P(t) \equiv Q(t)$. In this case, by symmetry,

$$\alpha = \beta = 0.$$

Then, by equation (24),

$$P(a_n) = P(b_n) = \frac{1}{3} ;$$

and by equation (25)

$$P(a_{n-k}) = P(b_{n-k}) = \frac{1}{2k+3} , \qquad k = 1,\ldots,n-1.$$

The optimal strategies are of the form

$$G'_{n-k}(t) = F'_{n-k}(t) = \frac{1}{4(k+1)} \frac{P'(t)}{P^3(t)} .$$

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