LINEARIZED TRANSONIC FLOW ABOUT NON-LIFTING, THIN SYMMETRIC AIRFOILS

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ABSTRACT

Transonic flow about symmetric, non-lifting airfoils is treated by solving an approximate linear differential equation of mixed type in place of the exact small-perturbation equations. The pressure distribution and drag coefficient are obtained in closed form for power series airfoils. The technique of local linearization is also applied to improve the accuracy of the results, particularly near the leading edge where the linearizing approximation is found to be invalid. Numerical results are obtained for the parabolic arc and single wedge airfoils and are found to compare favorably with experimental data and with previous theoretical results.

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I. INTRODUCTION

The analysis of transonic flows about obstacles is severely hindered by the non-linearity of the appropriate small-perturbation equations as derived by means of systematic expansion procedures, such as those detailed by Messiter[1]. Some progress has been made with the exact small perturbation equations for two-dimensional transonic flows by use of the hodograph transformation, since the resulting equations are linear and reducible to the Tricomi equation, which has a fairly extensive literature. However, the hodograph method has several disadvantages: The boundary conditions in all but the simplest cases are difficult, if not impossible, to formulate in the hodograph plane and in the supersonic region of the flow field, the Jacobian of the transformation can, and frequently does, vanish, thus locally invalidating the procedure. None the less, despite these disadvantages, the hodograph method has proven useful, particularly for simple profiles (wedge, flat plate, etc.) and for the inverse problem. An extensive survey of the hodograph method is found in Reference (2).

More recently, transonic flows have been subjected to analysis by replacing the appropriate exact small perturbation equations with linear equations which conform approximately with the exact equations in the region of interest (e. g. on the surface of the obstacle).

Oswatitsch and Keune[3] suggested replacing the exact equation

(1) $$\phi_{rr} + \frac{1}{r}\phi_r = (\gamma+1)\phi_x\phi_{xx}$$

for axisymmetric flow at $M_\infty = 1$ with the linear equation

$$\phi_{rr} + \frac{1}{r}\phi_r = a^2\phi_x ,$$

by reasoning that the flow over most of a body of revolution (in particular, a parabolic arc body) is accelerating and that near the body the term $\frac{1}{r}\phi_r$ is dominant and the form of the term on the right-hand side of the equation is not as important as its existence. The validity of this reasoning is born out by the fact that the surface pressures obtained are not strongly dependent on the choice of the constant a.

Maeder and Thommen[4] proposed the analogous method for two-dimensional flows. They replaced the exact equation

(2) $$(1-M_\infty^2)\phi_{xx} + \phi_{yy} = (\gamma+1)M_\infty^2\phi_x\phi_{xx}$$

with the linear equation

$$(1-M_\infty^2)\phi_{xx} + \phi_{yy} = K\phi_x .$$

The details of this method are given in Reference (5). In this case, however, the validity of the linearizing assumption is more questionable than for axisymmetric flows, primarily because the surface pressures so obtained are strongly dependent on the choice of K. In addition, at sonic velocity most interesting airfoils have, in the small perturbation theory, infinite surface pressure at the leading edge, so that serious error results near the leading edge of the airfoil if the linear equation is used.

The method of local linearization developed by Spreiter and Alksne[(6), (7), (8)] for transonic flows yields considerable improvement of accuracy over the preceding, particularly if the acceleration ϕ_{xx} is not nearly constant in the region of interest. For near sonic velocities, equations 1 and 2 are replaced by

$$\phi_{rr} + \frac{1}{r}\phi_r - \lambda_p \phi_x = f_p ,$$

$$\phi_{yy} - \lambda_p \phi_x = f_p ,$$

respectively, where

$$\lambda_p = (\gamma+1) M_\infty^2 \phi_{xx} ,$$

$$f_p = (M_\infty^2 - 1)\phi_{xx} .$$

After the appropriate equation is solved assuming λ_p to be constant, λ_p is replaced by $(\gamma+1)M_\infty^2\phi_{xx}$ in the expression for ϕ_x, resulting in an ordinary differential equation of the form

$$\frac{d\phi_x}{dx} = F(x, \phi_x).$$

For two-dimensional flow this equation can be integrated directly, but numerical methods must be used for the axisymmetric case.

The above choices of linear equations to approximate the exact small perturbation equations have the disadvantage that the mixed elliptic-hyperbolic form of the exact equations is lost. While the accuracy of the results indicates that the effect of the loss of mixed form is not great, it is still desirable to choose approximate linear equations which retain the mixed form of the exact equations. Royce[9] has done this for axisymmetric flow at $M_\infty = 1$ by replacing equation 1 by

$$\phi_{rr} + \frac{1}{r}\phi_r = a^2(x-\bar{x})\,\phi_{xx},$$

where $\bar{x}$ is the location of the sonic point on the body. The drag given by the solution of this equation is independent of a and the surface pressures are in good agreement with experiments on the forward part of the body. The present paper investigates the approximation of the two-dimensional small perturbation equations by a suitable linear equation of mixed form.

II. METHOD OF SOLUTION

Formulation of the Problem

The small perturbation equations for two-dimensional transonic flow, expressed in terms of the dimensionless velocity perturbation components and coordinates, are

$$\left[1-M_\infty^2-(\gamma+1)M_\infty^2 u\right]u_x + v_y = 0\ ,$$

(3) $$u_y - v_x = 0$$

or, upon eliminating v,

(4) $$\left(\left[1-M_\infty^2-(\gamma+1)M_\infty^2 u\right]u_x\right)_x + u_{yy} = 0\ .$$

(The problem is expressed in terms of u and v, rather than the perturbation potential ϕ, in order to avoid a singularity similar to that found in (9). This singularity will be discussed later.)

The boundary conditions require that the velocity be uniform infinitely far upstream of the airfoil and that the flow be tangent to the airfoil surface. In the thin airfoil limit the condition of tangent flow on the airfoil becomes*

(5) $$v(x, 0\pm) = \pm T'(x)$$

*Here and elsewhere primes indicate differentiation with respect to the argument, unless otherwise noted.

for a symmetric non-lifting airfoil whose surface is described by

$$y = \pm T(x) .$$

Eliminating v between equations 3 and 5 we have

$$u_y(x, 0\pm) = \pm T''(x) .$$

The pressure coefficient based on free stream conditions is given by

$$C_p = -2u$$

in the thin airfoil limit, and the two-dimensional drag coefficient by

$$C_D = 2\int_0^1 C_p \, T'(x) \, dx$$

for an airfoil of unit chord.

Approximate Linear Equation

In equation 4 the term

$$1 - M_\infty^2 - (\gamma+1) M_\infty^2 u$$

becomes $1 - M^2$ in the thin airfoil limit, where M is the local mach number. Thus, if we define

(6) $$-a^2(x - \bar{x}) = 1 - M_\infty^2 - (\gamma+1) M_\infty^2 u ,$$

then $x = \bar{x}$ is the sonic line. The actual sonic line will not necessarily be vertical, so $\bar{x}$ will be a function of y in general. With 6, equation 4 becomes

(7) $$-\left[a^2(x-\bar{x})u_x\right]_x + u_{yy} = 0 .$$

Now to obtain the desired linear equation we assume that a and $\bar{x}$ are constant, or at least vary sufficiently slowly that they can be considered constant in the above equation. The only real justification for this assumption is that it yields a simple linear equation of mixed form, which is an approximation to the exact equation near the airfoil. It can also be looked upon as the first step in an interation procedure.

In general, the values of a and $\bar{x}$ are not given a priori, so suitable means of evaluating them must be found. Since we are primarily interested in the solution at the airfoil surface, $\bar{x}$ may be found by requiring sonic velocity on the airfoil at $x = \bar{x}$:

$$u(x=\bar{x}, y=0) = \frac{1-M_\infty^2}{(\gamma+1)M_\infty^2} .$$

This is reasonable since equation 7 changes form at $x = \bar{x}$. In principle, a^2 should be proportional to some representative acceleration on the airfoil. The problem of deciding what acceleration is representative and the evaluation of a will be deferred until later.

For convenience we will introduce the coordinate transformation $z = x - \bar{x}$, thus placing the sonic point on the airfoil at the origin $z = y = 0$ and the leading edge at $z = -\bar{x}$, $y = 0$ as shown in Figure 1. The governing equations are then

(8) $$-a^2 \left[z u_z \right]_z + u_{yy} = 0 ,$$

(9) $$u_y (z, 0\pm) = \pm S''(z) ,$$

(10) $$u(0,0) = \frac{1 - M_\infty^2}{(\gamma+1) M_\infty^2} ,$$

(11) $$C_p(z,y) = -2 u(z,y) ,$$

(12) $$C_D = \int_{-\bar{x}}^{1-\bar{x}} C_p(z,0) S'(z) \, dz ,$$

with a yet to be determined. Here $y = \pm S(z)$ is the airfoil shape in the (z, y) coordinate system.

Characteristics in the Hyperbolic Region[10]

In order to solve equation 8, it is necessary to know the region of influence of a point in the hyperbolic region; hence, to know the shape of the characteristic curves, since they bound the region of influence.

The differential equation of the characteristics is

$$\left(\frac{dy}{dz}\right)^2 = \frac{1}{a^2 z} ,$$

the solution of which gives the equations of the characteristic curves

$$\frac{a}{2} y \pm \sqrt{z} = \text{constant}.$$

Thus, the characteristics take the form of a one-parameter family of parabolae whose major axes are parallel to the z-axis, as shown in Figure 2.

Green's Theorem for the Linearized Equation[11]

If we define the operator L by

$$(13) \qquad Lf = -a^2[zf_z]_z + f_{yy}$$

and define L^* to be the adjoint operator of L, then we can find a vector $\vec{F}$ such that

$$\nabla \cdot \vec{F} = fLg - gL^*f .$$

Since L as defined by 13 is self-adjoint, $L = L^*$ and the vector $\vec{F}$ has the components

$$\left\{ -a^2 z(fg_z - gf_z), \ (fg_y - gf_y) \right\} .$$

The divergence theorem in two dimensions is

$$\iint_R \nabla\cdot\vec{F}\,d\tau = \int_\Sigma \vec{F}\cdot\vec{n}\,d\sigma , \tag{14}$$

where R is any region in which $\nabla\cdot\vec{F}$ is non-singular, Σ is its boundary on which $\vec{F}$ is non-singular, and $\vec{n}$ is the unit outward normal vector to Σ. Thus we have, substituting the value of $\vec{F}$ into equation 14,

$$\iint_R [\,f L g - g L^* f\,]\,d\tau = \int_\Sigma [\,-\alpha^2 z\,(f g_z - g f_z)\,n_z + (f g_y - g f_y)\,n_y\,]\,d\sigma \tag{15}$$

as the form of Green's theorem for equation 8, where n_z and n_y are the components of $\vec{n}$.

If $\mathbb{K}(z,y;\zeta,\eta)$ is the solution of the pseudo-equation

$$L\,\mathbb{K} = \delta(z-\zeta)\,\delta(y-\eta)$$

and $\mathbb{K}^*(z,y;\zeta,\eta)$ is the solution of

$$L^*\mathbb{K}^* = \delta(z-\zeta)\,\delta(y-\eta) ,$$

both subject to the usual condition that they tend to zero far upstream, then it is well-known that

$$\mathbb{K}^*(z,y;\zeta,\eta) = \mathbb{K}(\zeta,\eta;z,y) .$$

Now we make the following stipulations:

a) $f = \mathbb{K}^*$, $g = u$

b) R is chosen so that $L\mathbb{K} = L^*\mathbb{K}^* = 0$ in R.

The second of these requires that the airfoil and the field point (ζ, η) be excluded from R. Then equation 15 becomes

$$
(16) \qquad \int_{\Sigma'} \Big\{ -a^2 \zeta \left[\mathbb{K}(z,y;\zeta,\eta)\, u_\zeta(\zeta,\eta) - \mathbb{K}_\zeta(z,y;\zeta,\eta)\, u(\zeta,\eta) \right] n_\zeta + \\
+ \left[\mathbb{K}(z,y;\zeta,\eta)\, u_\eta(\zeta,\eta) - \mathbb{K}_\eta(z,y;\zeta,\eta)\, u(\zeta,\eta) \right] n_\eta \Big\}\, d\sigma' \\
= 0 ,
$$

where the airfoil, the field point (z,y), and any singularities of $\mathbb{K}(z,y;\zeta,\eta)$ or $u(\zeta,\eta)$ are excluded from R'. Here we have interchanged the variables ζ, η and z, y and primed R, Σ, and σ to indicate that they are taken in the ζ, η plane. Equation 16 is the form of Green's theorem to be used in this paper.

Fundamental Solution of the Linear Equation

The fundamental solution $\mathbb{K}(z,y;\zeta,\eta)$ of the linear operator L, which corresponds to a mathematical source, is obtained by solving the pseudo-equation

$$
(17) \qquad L\,\mathbb{K} = \delta(z-\zeta)\,\delta(y-\eta) ,
$$

where (ζ,η) is the mathematical source location and δ is the Dirac delta function defined in the usual way. It should be noted that here the "mathematical source" is not equivalent to a "fluid source". In fact, if we write

$$L u = -a^2 [z u_z]_z + [v_y]_z = \delta(z-\zeta)\,\delta(y-\eta)$$

and integrate with respect to z from $-\infty$ to z, we obtain

$$-a^2 z u_z + v_y = \delta(y-\eta) \int_{-\infty}^{z} \delta(z-\zeta)\, dz$$

which shows that the unit mathematical source is equivalent to a linear fluid source, of unit strength per unit length and parallel to the z-axis, which stretches from ζ to $+\infty$. So, to avoid confusion, the word "source" will be taken to mean "mathematical source" hence forth.

By making the change of variable $y_1 = y - \eta$, equation 17 can be rewritten as

$$-a^2 [z \mathbb{K}_z]_z + \mathbb{K}_{y_1 y_1} = \delta(z-\zeta)\,\delta(y_1)\,. \tag{18}$$

The symmetry of $\mathbb{K}$ about $y_1 = 0$ and the form of equation 18 suggest the use of the infinite Fourier cosine transform pair

$$\tilde{f}(\omega) = \int_0^\infty f(y_1) \cos \omega y_1 \, dy_1 \,,$$

$$f(y_1) = \frac{2}{\pi} \int_0^\infty \tilde{f}(\omega) \cos \omega y_1 \, d\omega$$

to find $\mathbb{K}$. With this, equation 18 is, in the transform plane,

(19) $$a^2 [z \tilde{\mathbb{K}}_z]_z + \omega^2 \tilde{\mathbb{K}} = -\frac{1}{2} \delta(z-\zeta) .$$

The solution of this equation is obtained by use of the general solution of the homogeneous equation subject to the conditions

a) $\tilde{\mathbb{K}}$ continuous at $z = \zeta$

b) $\tilde{\mathbb{K}}$ finite for $z \to -\infty$

c) $\tilde{\mathbb{K}}_z$ finite at $z = 0$

d) $\tilde{\mathbb{K}} = 0$ for $z < \zeta$, $\zeta > 0$

and a suitable "jump" condition on $\tilde{\mathbb{K}}_z$ across $z = \zeta$. This jump condition is obtained formally by integrating equation 19 across the point ζ :

$$a^2 [z \tilde{\mathbb{K}}_z] \Big|_{\zeta-\varepsilon}^{\zeta+\varepsilon} = -\frac{1}{2}$$

or

e) $$\tilde{\mathbb{K}}_z \Big]_{\zeta-}^{\zeta+} = -\frac{1}{2a^2 \zeta} .$$

The homogeneous equation is a form of Bessel's equation and has the general solution (12)

(20) $$\tilde{\mathbb{K}} = A\, J_0 \left(\frac{2\omega}{a}\sqrt{z}\right) + B\, Y_0 \left(\frac{2\omega}{a}\sqrt{z}\right) ,$$

or, in a form more convenient for the case $z < 0$,

(21) $$\tilde{\mathbb{K}} = C\, I_0 \left(\frac{2\omega}{a}\sqrt{-z}\right) + D\, K_0 \left(\frac{2\omega}{a}\sqrt{-z}\right) .$$

Here J_o and Y_o are Bessel's functions of the first and second kinds, respectively, and I_o and K_o, modified Bessel's functions of the first and second kinds.

For the hyperbolic case $(\zeta > 0)$ we use the solution of the form 20, subject to conditions a), d), and e). The first two of these conditions requires that

$$\tilde{\mathbb{K}}(z=\zeta) = A\,J_o\left(\frac{2\omega}{a}\sqrt{\zeta}\right) + B\,Y_o\left(\frac{2\omega}{a}\sqrt{\zeta}\right) = 0, \tag{22}$$

and the third that

$$\left.\tilde{\mathbb{K}}_z\right]_{\zeta-}^{\zeta+} = -\frac{\omega}{a\sqrt{\zeta}}\left[A\,J_1\left(\frac{2\omega}{a}\sqrt{\zeta}\right) + B\,Y_1\left(\frac{2\omega}{a}\sqrt{\zeta}\right)\right] = -\frac{1}{2a^2\zeta}. \tag{23}$$

By use of the Wronskian[12] of $J_o(x)$ and $Y_o(x)$,

$$W\left\{J_o(x), Y_o(x)\right\} = \frac{2}{\pi x},$$

we can solve equations 22 and 23 for A and B :

$$A = \frac{\pi}{2a^2}\,Y_o\left(\frac{2\omega}{a}\sqrt{\zeta}\right),$$

$$B = -\frac{\pi}{2a^2}\,J_o\left(\frac{2\omega}{a}\sqrt{\zeta}\right).$$

Then the transformed solution is

$$\tilde{\mathbb{K}}(z;\zeta) = \begin{cases} 0 & z < \zeta \\ \frac{\pi}{2a}\left[Y_o\left(\frac{2\omega}{a}\sqrt{\zeta}\right)J_o\left(\frac{2\omega}{a}\sqrt{z}\right) - J_o\left(\frac{2\omega}{a}\sqrt{\zeta}\right)Y_o\left(\frac{2\omega}{a}\sqrt{z}\right)\right] & z > \zeta \end{cases} \tag{24}$$

for $\zeta > 0$.

For the elliptic case $(\zeta<0)$ we use the solution of the form 21, subject to conditions a), b), c), and e). The first and third of these conditions require that

$$C(z<\zeta) = 0,$$

$$D(z>\zeta) = 0,$$

the second condition that

$$\tilde{\mathbb{K}}(z=\zeta) = C(z>\zeta)\, I_0\left(\tfrac{2\omega}{a}\sqrt{-\zeta}\right) = D(z<\zeta)\, K_0\left(\tfrac{2\omega}{a}\sqrt{-\zeta}\right), \tag{25}$$

and the last that

$$\tilde{\mathbb{K}}_z\Big]_{\zeta-}^{\zeta+} = -\frac{\omega}{a\sqrt{-\zeta}}\left[D(z<\zeta)\, K_1\left(\tfrac{2\omega}{a}\sqrt{-\zeta}\right) + C(z>\zeta)\, I_1\left(\tfrac{2\omega}{a}\sqrt{-\zeta}\right)\right] = -\frac{1}{2a^2\zeta}. \tag{26}$$

Then, using the Wronskian of $I_0(x)$ and $K_0(x)$,

$$W\{I_0(x), K_0(x)\} = -\frac{1}{x},$$

we can solve equations 25 and 26 and obtain

$$D(z<\zeta) = -\frac{1}{a^2} I_0\left(\tfrac{2\omega}{a}\sqrt{-\zeta}\right),$$

$$C(z>\zeta) = -\frac{1}{a^2} K_0\left(\tfrac{2\omega}{a}\sqrt{-\zeta}\right).$$

Then the transformed solution is

$$\tilde{\mathbb{K}}(z;\zeta) = \begin{cases} -\frac{1}{a^2} I_0\left(\frac{2\omega}{a}\sqrt{-\zeta}\right) K_0\left(\frac{2\omega}{a}\sqrt{-z}\right) & z<\zeta \\ -\frac{1}{a^2} K_0\left(\frac{2\omega}{a}\sqrt{-\zeta}\right) I_0\left(\frac{2\omega}{a}\sqrt{-z}\right) & z>\zeta \end{cases} \tag{27}$$

for $\zeta<0$.

Now, for $\zeta < 0 < z$, we can put $\tilde{\mathbb{K}}$ into a more suitable form than that of equation 27 by making use of the equality

$$I_p(z) = e^{-ip\frac{\pi}{2}} J_p(z e^{i\frac{\pi}{2}}),$$

from which we obtain

$$\tilde{\mathbb{K}}(z;\zeta) = -\frac{1}{a^2} K_0\left(\frac{2\omega}{a}\sqrt{-\zeta}\right) J_0\left(\frac{2\omega}{a}\sqrt{z}\right) \tag{28}$$

for $\zeta < 0 < z$.

The inverse transforms for the right-hand sides of equations 24, 27, and 28 are tabulated in Reference (13). The fundamental solution is found to be

$$\mathbb{K}(z,y;\zeta,\eta) = \begin{cases} 0 & \zeta > 0,\ z < \zeta \text{ OR } \nu < 1 \\ -\frac{1}{2a} z^{-1/4} \zeta^{-1/4} P_{-\frac{1}{2}}(\nu) & 0 < \zeta < z,\ \nu > 1 \\ -\frac{1}{2\pi a} (-z)^{-1/4} (-\zeta)^{-1/4} Q_{-\frac{1}{2}}(-\nu) & \zeta, z < 0 \\ -\frac{1}{2\sqrt{2}a} z^{-1/4} (-\zeta)^{-1/4} (\nu^2+1)^{-1/4} P_{-1/2}\left(-\frac{\nu}{\sqrt{\nu^2+1}}\right) & \zeta < 0 < z \end{cases} \tag{29}$$

where $P_{-\frac{1}{2}}$ and $Q_{-\frac{1}{2}}$ are associated Legendre functions of the first and second kinds, respectively, and where, for convenience, we have introduced the function

$$\nu = \frac{z + \zeta - \frac{a^2}{4}(y-\eta)^2}{2\sqrt{|z|}\sqrt{|\zeta|}}$$

(the absolute values are used to avoid having to worry about signs).

In the elliptic region $(\zeta, z < 0)$, $-\nu$ is unity only at the mathematical source point $(z = \zeta,\ y = \eta)$; it is greater than one for all other points in the region. We can examine the form of $\mathbb{K}$ near the elliptic source through the use of the asymptotic expansion for $Q_{-\frac{1}{2}}(-\nu)$ for $-\nu \to 1$ (14).

$$Q_{-\frac{1}{2}}(-\nu) = -\frac{1}{2} \log\left(-\frac{\nu+1}{2}\right) + O(1) .$$

Then we have

$$\mathbb{K} = \frac{1}{4\pi a} (-\zeta)^{-1/4} (-z)^{-1/4} \left[\log \frac{(\sqrt{-z} - \sqrt{-\zeta})^2 + \frac{a^2}{4}(y-\eta)^2}{4\sqrt{-z}\sqrt{-\zeta}} + O(1) \right] .$$

Now, since we are near the source, let $z = \zeta \left[1 + \left(\frac{z}{\zeta} - 1\right) \right]$ with $\left| \frac{z}{\zeta} - 1 \right| \ll 1$. Then

(30) $$\mathbb{K} = \frac{1}{4\pi a\sqrt{-z}} \log \left[(z-\zeta)^2 + (-a^2\zeta)(y-\eta)^2 \right] + O\left(\frac{\log\sqrt{-\zeta}}{\sqrt{-\zeta}} \right) .$$

Thus if $|\zeta|$ is of order 1, the leading term near the elliptic source has the proper logarithmic behavior. This might have been expected since, if $|\zeta|$ is of order 1, near the source we may replace the coefficient $-a^2 z$ in equation 17 with $-a^2\zeta\left[1 + \left(\frac{z}{\zeta} - 1\right)\right]$, which would lead directly to equation 30.

In the hyperbolic region $(\zeta > 0)$, ν is unity only on the characteristics passing through the source point (ζ, η), as is easily verified with the equations of the characteristic curves. It is greater than 1 only between these characteristics, so the region of influence of $\mathbb{K}$ for $\zeta > 0$ is that region downstream of the source point and bounded by the characteristics passing through the source point, as expected. Now for ν near 1

$$P_{-\frac{1}{2}}(\nu) = 1 + O(\nu - 1) ,$$

so we have

$$\mathbb{K} = -\frac{1}{2a} z^{-1/4} \zeta^{-1/4} \left[1 + O(\nu - 1) \right] \qquad z > \zeta , \ \nu > 1 \tag{31}$$

near the characteristics defined by $\nu = 1$. Sufficiently near the source point we can put $z = \zeta\left[1 + \left(\frac{z}{\zeta} - 1\right)\right]$ with $\left(\frac{z}{\zeta} - 1\right) << 1$ in 31 and the equation $\nu = 1$ to obtain

$$\mathbb{K} = \begin{cases} 0 & z < \zeta \text{ or } (z - \zeta) < a\sqrt{\zeta}\,|y - \eta| \\ -\frac{1}{2a\sqrt{\zeta}} + O\left(\frac{z}{\zeta} - 1\right) & z > \zeta , \ (z - \zeta) > a\sqrt{\zeta}\,|y - \eta| . \end{cases}$$

Again this is the expected behavior near a hyperbolic source and could have been obtained by substituting $-a^2 \zeta \left[1 + \left(\frac{z}{\zeta} - 1\right)\right]$ for $-a^2 z$ in the differential equation.

The fundamental solution $\mathbb{K}$ can also be expressed in terms of K, the complete elliptic integral of the first kind. The easiest way to obtain this representation is to find suitable integral representations[14] of the Legendre functions and evaluate them in terms of elliptic integrals[15]. For example, a suitable integral for $Q_{-\frac{1}{2}}(-\nu)$ is

$$Q_{-\frac{1}{2}}(-\nu) = \frac{1}{\sqrt{2}} \int_0^{\pi} [-\nu + \cos t]^{-1/2}\, dt$$

which can be evaluated to give

$$Q_{-\frac{1}{2}}(-\nu) = \left[\frac{2}{1-\nu}\right]^{1/2} K\left(\left[\frac{2}{1-\nu}\right]^{1/2}\right).$$

Similar expressions can also be found for $P_{-\frac{1}{2}}(\nu)$ and $P_{-\frac{1}{2}}\left(-\frac{\nu}{\sqrt{\nu^2+1}}\right)$. The resulting expressions for $\mathbb{K}$ are:

$$\text{(32)} \qquad \mathbb{K}(z, y; \zeta, \eta) = \begin{cases} 0 & 0<\zeta,\ z<\zeta \text{ OR } \nu<1 \\ -\dfrac{z^{-1/4}\zeta^{-1/4}}{\pi a\,[\nu+\sqrt{\nu^2-1}]^{1/2}}\, K\left(\left[\dfrac{2\sqrt{\nu^2-1}}{\nu+\sqrt{\nu^2-1}}\right]^{1/2}\right) & 0<\zeta<z,\ \nu>1 \\ -\dfrac{(-z)^{-1/4}(-\zeta)^{-1/4}}{2\pi a}\left[\dfrac{2}{1-\nu}\right]^{1/2} K\left(\left[\dfrac{2}{1-\nu}\right]^{1/2}\right) & \zeta, z<0 \\ -\dfrac{z^{-1/4}(-\zeta)^{-1/4}}{\sqrt{2}\,\pi a\,[\nu^2+1]^{1/4}}\, K\left(\left[\dfrac{\nu+\sqrt{\nu^2+1}}{2\sqrt{\nu^2+1}}\right]^{1/2}\right) & \zeta<0<z. \end{cases}$$

Although these expressions appear to be somewhat clumsy, they will be useful later when we consider the thin airfoil limit because the literature concerning elliptic integrals is relatively more extensive than that concerning Legendre functions of order $-\frac{1}{2}$.

We can now examine the behavior of $\mathbb{K}$ as the source point approaches the origin. If we bound z away from $\frac{a^2}{4}(y-\eta)^2$, then as $\zeta \to 0^{\pm}$, $\nu \to \pm\infty$. And, since

$$K(k) = \log\frac{4}{\sqrt{1-k^2}} + O\left([1-k^2]\log[1-k^2]\right)$$

for $|1-k^2| << 1$, the leading terms in $\mathbb{K}$ are

$$\mathbb{K}(z,y;\varepsilon,\eta) = \begin{cases} \frac{1}{2\pi a}\left[z-\frac{a^2}{4}(y-\eta)^2\right]^{-1/2}\log|\varepsilon| & z>\frac{a^2}{4}(y-\eta)^2 \\ -\frac{1}{2a}\left[-z+\frac{a^2}{4}(y-\eta)^2\right]^{-1/2} & z<\frac{a^2}{4}(y-\eta)^2 \end{cases} \tag{33}$$

for $\varepsilon << 1$. The logarithmic singularity for $z > \frac{a^2}{4}(y-\eta)^2$ indicates that caution may be needed near $\zeta = 0$ when applying the fundamental solution $\mathbb{K}$.

Application of Green's Theorem

Having obtained the fundamental solution $\mathbb{K}$, we are now in a position to apply Green's theorem in the form given by equation 15. We will restrict our attention to the cases where there are no shock waves upstream of the trailing edge of the airfoil, so that the shock relations are not needed. Further, we will assume that the slope of the airfoil surface is continuous.

If we take the airfoil to lie on the ζ-axis and consider an arbitrary field point (z,y), where neither z nor y is zero, then there are three general cases: 1) $z < 0$, 2) $0 < z < \frac{a^2}{4}y^2$, 3) $z > \frac{a^2}{4}y^2$.

Taking the case $z < 0$ first, we consider the two regions R_1' and R_2' shown in Figure 3. The region R_1' is bounded by a small rectangle about the field point (z,y) and the rectangle with sides $(\zeta \to -\infty, \eta \to +\infty, \zeta \to 0-, \eta \to 0+)$, and the region R_2' is bounded by the rectangle with sides $(\zeta \to -\infty, \eta \to -\infty, \zeta \to 0-, \eta \to 0-)$. In these regions equation 16 is valid. We will assume a priori that the parts of the boundaries of R_1' and R_2' which are infinitely distant give no contribution to the integral.

To consider the integrals of equation 16 taken about the small rectangle $(\zeta = z+\varepsilon, \eta = y+\delta, \zeta = z-\varepsilon, \eta = y-\delta)$ enclosing the field point (z, y) as shown in Figure 4, we can use the leading term in the expansion of $\mathbb{K}$ for $-\nu$ near 1, as given in equation 30. From this we have

$$-a^2 \zeta\, \mathbb{K}_\zeta (z,y; z \pm \varepsilon, \eta) = -\frac{a\sqrt{-z}}{2\pi} (\pm\varepsilon) \left[\varepsilon^2 - a^2 z (y-\eta)^2 \right]^{-1} + \cdots ,$$

$$\mathbb{K}_\eta (z,y; \zeta, y \pm \delta) = -\frac{a\sqrt{-z}}{2\pi} (\pm\delta) \left[(\zeta - z)^2 - a^2 z \delta^2 \right]^{-1} + \cdots ,$$

so the integrals are, letting $\bar{\zeta} = \zeta - z$, $\bar{\eta} = \eta - y$,

$$\lim_{\varepsilon, \delta \to 0} \frac{a\sqrt{-z}}{\pi} u(z,y) \left[\int_{-\varepsilon}^{\varepsilon} \frac{\delta}{\bar{\zeta}^2 - a^2 z \delta^2} d\bar{\zeta} + \int_{-\delta}^{\delta} \frac{\varepsilon}{\varepsilon^2 - a^2 z \bar{\eta}^2} d\bar{\eta} \right] = u(z,y) .$$

Finally for the boundary $\zeta \to 0-$, we can use equation 33 for $\mathbb{K}$. Since we have a smooth airfoil, u and u_ζ are finite at $\zeta = 0$. Therefore

$$\lim_{\zeta \to 0-} \zeta u_\zeta \mathbb{K} = \lim_{\zeta \to 0-} \zeta u \mathbb{K}_\zeta = 0$$

and there is no contribution from the boundary $\zeta \to 0-$.

With these results we can now write equation 16 for the regions R_1' and R_2', respectively, as

$$u(z,y) - \int_{-\infty}^{0-} [u_\eta(\zeta,0+)\,\mathbb{K}(z,y;\zeta,0+) - u(\zeta,0+)\,\mathbb{K}_\eta(z,y;\zeta,0+)]\,d\zeta$$

$$= 0 \quad ,$$

$$\int_{-\infty}^{0-} [u_\eta(\zeta,0-)\,\mathbb{K}(z,y;\zeta,0-) - u(\zeta,0-)\,\mathbb{K}_\eta(z,y;\zeta,0-)]\,d\zeta = 0 \quad .$$

But $\mathbb{K}$ and $\mathbb{K}_\eta$ are continuous at $\eta=0$ if $y \neq 0$, and, from equation 5, $u_\eta(\zeta,0\pm) = \pm S''(\zeta)$, so combining the two above equations we have

$$u(z,y) = 2\int_{-\infty}^{0-} S''(\zeta)\,\mathbb{K}(z,y;\zeta,0)\,d\zeta \tag{34}$$

for the streamwise component of the velocity perturbation for $z < 0$.

Now if we let $z \to 0-$, then $-\nu \to \infty$ and

$$\mathbb{K}(z,y;\zeta,0) \to -\frac{1}{2a}\left[-\zeta + \left(\frac{ay}{2}\right)^2\right]^{-1/2}.$$

Therefore from equation 34

$$u(0,y) = u(0-,y) = -\frac{1}{a}\int_{-\infty}^{0-} S''(\zeta)\left[-\zeta + \left(\frac{ay}{2}\right)^2\right]^{-1/2} d\zeta \, , \tag{35}$$

since u is continuous at $\zeta = 0$. As $y \to 0$ we have, from equation 10,

$$-\frac{1}{a}\int_{-\infty}^{0-} S''(\zeta)\,\frac{d\zeta}{\sqrt{-\zeta}} = \frac{1-M_\infty^2}{(\gamma+1)M_\infty^2} \, , \tag{36}$$

which is used to determine $\bar{x}$.

The second case $0 < z < \left(\frac{ay}{2}\right)^2$ is that in which the field point (z, y) is in the hyperbolic region, but outside the region of influence of that part of the airfoil lying in the hyperbolic region. For this case we take the region R_3' shown in Figure 5, whose boundary is formed by the line $\zeta = \varepsilon$ $(\varepsilon \to 0)$ and lines lying just upstream of the characteristics passing through the field point (z, y). For this region, equation 16 is valid.

The equations of the characteristics passing through (z, y) are the solutions of the equation $\nu = 1$, which are

$$(37) \qquad \eta_\pm(\zeta) = y \pm \frac{2}{a}\left[\sqrt{z} - \sqrt{\zeta}\right] ,$$

where the $\pm$ refer to the down-and upstream running characteristics, respectively. From this we find

$$n_\zeta = \frac{1}{\sqrt{1 + a^2 \zeta}} \quad , \quad n_\eta = \pm\frac{a\sqrt{\zeta}}{\sqrt{1 + a^2 \zeta}}$$

to be the components of the unit outward normals to these characteristics. Then that portion of the integral of equation 16 taken along these characteristics is

$$(38) \qquad \int_{(\zeta=\varepsilon)}^{(\zeta = z-)} \left\{ -\frac{a^2\zeta}{\sqrt{1+a^2\zeta}}\left[\mathbb{K}(z,y;\zeta,\eta_\pm)\, u_\zeta(\zeta,\eta_\pm) - \mathbb{K}_\zeta(z,y;\zeta,\eta_\pm)\, u(\zeta,\eta_\pm)\right] \pm \right.$$

$$\left. \pm \frac{a\sqrt{\zeta}}{\sqrt{1+a^2\zeta}}\left[\mathbb{K}(z,y;\zeta,\eta_\pm)\, u_\eta(\zeta,\eta_\pm) - \mathbb{K}_\eta(z,y;\zeta,\eta_\pm)\, u(\zeta,\eta_\pm)\right]\right\} d\sigma_\pm' .$$

The terms in the square brackets can be written as

$$\left[\mathbb{K}^2 \frac{\partial}{\partial \zeta} \left(\frac{u}{\mathbb{K}} \right) \right]_{\eta = \eta_\pm} , \qquad \left[\mathbb{K}^2 \frac{\partial}{\partial \eta} \left(\frac{u}{\mathbb{K}} \right) \right]_{\eta = \eta_\pm} .$$

Also, from equation 37, we have

$$\frac{\partial}{\partial \sigma_\pm'} = \frac{a\sqrt{\zeta}}{\sqrt{1+a^2\zeta}} \frac{\partial}{\partial \zeta} \mp \frac{1}{\sqrt{1+a^2\zeta}} \frac{\partial}{\partial \eta} ,$$

so we can write the integral 38 as

$$(39) \qquad -a \int_{(\zeta=\varepsilon)}^{(\zeta=z-)} \sqrt{\zeta}\; \mathbb{K}^2 \frac{\partial}{\partial \sigma_\pm'} \left(\frac{u}{\mathbb{K}} \right) d\sigma_\pm' .$$

Now, for $\nu = 1$, $P_{-\frac{1}{2}}(\nu) = 1$ and $\mathbb{K} = -\frac{1}{2a} z^{-1/4} \zeta^{-1/4}$. With this and the preceding equation, equation 38 is equal to $\frac{1}{2} u(z,y)$.

The integral along the boundary at $\zeta = \varepsilon$ is

$$(40) \qquad a^2 \varepsilon \int_{\eta_-(\varepsilon)}^{\eta_+(\varepsilon)} \left[\mathbb{K}(z,y;\varepsilon,\eta)\, u_\zeta(\varepsilon,\eta) - \mathbb{K}_\zeta(z,y;\varepsilon,\eta)\, u(\varepsilon,\eta) \right] d\eta .$$

If η is not within a distance of order $\varepsilon^{1/2}$ of $\eta_+(\varepsilon)$ or $\eta_-(\varepsilon)$, then for $\varepsilon \ll 1$ or $z - \left(\frac{ay}{2}\right)^2$ we have $\nu \gg 1$, and from equation 33

$$\mathbb{K}(z,y;\varepsilon,\eta) = -\frac{1}{2\pi a} \left[z - \frac{a^2}{4}(y-\eta)^2 \right]^{-1/2} \log \varepsilon + \cdots .$$

If η is close to $\eta_+(\varepsilon)$ or $\eta_-(\varepsilon)$, $\mathbb{K}$ will be at most of order $\varepsilon^{-1/4}$. Therefore, the first term of the integral 40 is zero in the limit $\varepsilon \to 0$, since $u_\zeta(0,\eta)$ is assumed to be finite. For the second term, if η is not within $O(\varepsilon^{1/2})$ of $\eta_+(\varepsilon)$ or $\eta_-(\varepsilon)$, then

$$\varepsilon\, \mathbb{K}_\zeta(z,y;\varepsilon,\eta) = -\frac{1}{2\pi a}\left[z - \frac{a^2}{4}(y-\eta)^2\right]^{-1/2} + \cdots .$$

While if η is near $\eta_+(\varepsilon)$ or $\eta_-(\varepsilon)$, $\varepsilon\, \mathbb{K}_\zeta(z,y;\varepsilon,\eta)$ is at most of order $\varepsilon^{-1/4}$. However, since this neighborhood of $\eta_+(\varepsilon)$ and $\eta_-(\varepsilon)$ is of order $\varepsilon^{1/2}$, this singularity will contribute nothing in the limit $\varepsilon \to 0$. Thus, in the limit, the integral 40 is

$$-\frac{a}{2\pi}\int_{y-\frac{2}{a}\sqrt{z}}^{y+\frac{2}{a}\sqrt{z}} u(0+,\eta)\left[z - \frac{a^2}{4}(y-\eta)^2\right]^{-1/2} d\eta .$$

Since u is continuous at $\zeta = 0$, $u(0+,\eta)$ is given by equation 35, with which the above integral becomes

$$\frac{1}{4\pi}\int_{y-\frac{2}{a}\sqrt{z}}^{y+\frac{2}{a}\sqrt{z}}\left[z - \frac{a^2}{4}(y-\eta)^2\right]^{-1/2}\int_{-\infty}^{0-} S''(\zeta)\left[-\zeta + \frac{a^2}{4}\eta^2\right]^{-1/2} d\zeta\, d\eta .$$

Interchanging the order of integration, we have

$$-z\int_{-\infty}^{0-} S''(\zeta)\left\{-\frac{1}{2\pi a}\int_{y-\frac{2}{a}\sqrt{z}}^{y+\frac{2}{a}\sqrt{z}}\left[(\eta_+(0)-\eta)(\eta-\eta_-(0))(\eta + i\tfrac{2}{a}\sqrt{-\zeta})(\eta - i\tfrac{2}{a}\sqrt{-\zeta})\right] d\eta\right\} d$$

But the term in the brackets is just[15]

$$-\frac{1}{\pi a}(-\zeta)^{-1/4} z^{-1/4}\left[\nu_0^2+1\right]^{-1/4} K\left(\left[\frac{\nu_0 + \sqrt{\nu_0^2+1}}{2\sqrt{\nu_0^2+1}}\right]^{1/2}\right) ,$$

where $\nu_0 = \nu(\eta = 0)$. This is $\mathbb{K}(z,y;\zeta,0)$ for $\zeta < 0 < z$, so, in the limit $\varepsilon \to 0$, the integral 40 is

$$-z\int_{-\infty}^{0-} S''(\zeta)\, \mathbb{K}(z,y;\zeta,0)\, d\zeta .$$

Then, collecting the above results, equation 16 applied to the region R_3' gives

(41) $$u(x,y) = 2\int_{-\infty}^{0-} S''(\zeta)\, \mathbb{K}(x,y;\zeta,0)\, d\zeta \ .$$

The final case $x > (\frac{ay}{2})^2$ is that in which the field point is influenced by at least a part of the airfoil lying in the hyperbolic region. For this case we consider the two regions R_4' and R_5' as shown in Figure 4. The boundaries are the same as those of R_3' except that the region is cut to exclude the ζ-axis.

We can use the same arguments as used for R_3' to write equation 16 for the regions R_4' and R_5' as, respectively,

$$u(x,y) - \int_{0+}^{[\sqrt{x}-|\frac{ay}{2}|]^2} [\, S''(\zeta)\, \mathbb{K}(x,y;\zeta,0+) - u(\zeta,0+)\, \mathbb{K}_\eta(x,y;\zeta,0+)\,]\, d\zeta -$$
$$- 2\int_{-\infty}^{0-} S''(\zeta)\left\{ -\frac{1}{2\pi a}\int_{0+}^{\eta_+(0)} [(\eta_+(0)-\eta)(\eta-\eta_-(0))(\eta+i\tfrac{2}{a}\sqrt{-\zeta})(\eta-i\tfrac{2}{a}\sqrt{-\zeta})]^{-1/2} d\eta \right\} d\zeta -$$
$$- \frac{1}{2}\, x^{-1/4}\, [\sqrt{x}-|\tfrac{ay}{2}|]^{1/2}\; u([\sqrt{x}-|\tfrac{ay}{2}|]^2, 0+) \quad = \quad 0 \ ,$$

$$-\int_{0+}^{[\sqrt{x}-|\frac{ay}{2}|]^2} [\, S''(\zeta)\, \mathbb{K}(x,y;\zeta,0-) + u(\zeta,0-)\, \mathbb{K}_\eta(x,y;\zeta,0-)\,]\, d\zeta -$$
$$- 2\int_{-\infty}^{0-} S''(\zeta)\left\{ -\frac{1}{2\pi a}\int_{\eta_-(0)}^{0-} [(\eta_+(0)-\eta)(\eta-\eta_-(0))(\eta+i\tfrac{2}{a}\sqrt{-\zeta})(\eta-i\tfrac{2}{a}\sqrt{-\zeta})]^{-1/2} d\eta \right\} d\zeta +$$
$$+ \frac{1}{2}\, x^{-1/4}\, [\sqrt{x}-|\tfrac{ay}{2}|]^{1/2}\; u([\sqrt{x}-|\tfrac{ay}{2}|]^2, 0-) \quad = \quad 0 \ .$$

The integrand of the double integrals is continuous at $\eta = 0$, as are u, $\mathbb{K}$, and $\mathbb{K}_\eta$, so combining these two expressions we have

$$u(x,y) = 2\int_{-\infty}^{[\sqrt{x}-|\frac{ay}{2}|]^2} S''(\zeta)\, \mathbb{K}(x,y;\zeta,0)\, d\zeta \ .$$

Finally equations 34 and 40 have the same form, so we can write

$$(42) \qquad u(z,y) = \begin{cases} z\displaystyle\int_{-\infty}^{0-} S''(\zeta)\, \mathbb{K}(z,y;\zeta,0)\, d\zeta & z < \left(\frac{ay}{2}\right)^2 \\ z\displaystyle\int_{-\infty}^{[\sqrt{z}-|\frac{ay}{2}|]^2} S''(\zeta)\, \mathbb{K}(z,y;\zeta,0)\, d\zeta & z > \left(\frac{ay}{2}\right)^2 . \end{cases}$$

It is of interest at this point to consider the singularity which would have arisen if we had used the perturbation potential ϕ instead of u. The linear equation in terms of ϕ is

$$-a^2 z\, \phi_{zz} + \phi_{yy} = 0.$$

Now let ϕ_1 be the solution of

$$(43) \qquad -a^2 z\, \phi_{1_{zz}} + \phi_{1_{yy}} = \delta(z-\zeta)\,\delta(y-\eta),$$

and ϕ_2, of

$$(44) \qquad -a^2 z\, \phi_{2_{zz}} + \phi_{2_{yy}} = \delta(y-\eta)\int_{-\infty}^{z} \delta(z-\zeta)\, dz ,$$

both subject to the boundary condition that $\nabla\phi \to 0$ far upstream of (ζ, η). Then ϕ_1 is the solution for a unit fluid source and ϕ_2 is the potential corresponding to $\mathbb{K}$. From equations 17, 43, and 44 we have the following relations:

$$(45) \qquad \phi_{2_z} = \mathbb{K} \quad , \quad -\phi_{2_\zeta} = \phi_1 .$$

Now let us assume that the potential ϕ for an airfoil $y = S(z)$ is given by

$$\phi = \begin{cases} 2\int_{-\infty}^{0-} S'(\zeta)\, \phi_1(z,y;\zeta,0)\, d\zeta & z < \left(\frac{ay}{2}\right)^2 \\ 2\int_{-\infty}^{[\sqrt{z}-|\frac{ay}{2}|]^2} S'(\zeta)\, \phi_1(z,y;\zeta,0)\, d\zeta & z > \left(\frac{ay}{2}\right)^2 . \end{cases}$$

Then, for $z < \left(\frac{ay}{2}\right)^2$, we have

$$u = \phi_z = 2\int_{-\infty}^{0-} S'(\zeta)\, \phi_{1_z}(z,y;\zeta,0)\, d\zeta .$$

But by 45 this can be written

$$u = -2\int_{-\infty}^{0-} S'(\zeta)\, \mathbb{K}_\zeta(z,y;\zeta,0)\, d\zeta ,$$

or, upon integrating by parts,

$$u = -2\, S'(\zeta)\, \mathbb{K}(z,y;\zeta,0)\Big|_{-\infty}^{0-} + 2\int_{-\infty}^{0-} S''(\zeta)\, \mathbb{K}(z,y;\zeta,0)\, d\zeta .$$

Then, with equation 33, we obtain

$$(46) \qquad u = \frac{S'(0)}{a}\left[\left(\frac{ay}{2}\right)^2 - z\right]^{-1/2} + 2\int_{-\infty}^{0-} S''(\zeta)\, \mathbb{K}(z,y;\zeta,0)\, d\zeta$$

for $z < \left(\frac{ay}{2}\right)^2$. For $z > \left(\frac{ay}{2}\right)^2$, by the same procedure we have

$$u = \phi_z = 2\, S'\left([\sqrt{z}-|\tfrac{ay}{2}|]^2\right)\Big[\left(1-|\tfrac{ay}{2\sqrt{z}}|\right)\phi_1\left(z,y;[\sqrt{z}-|\tfrac{ay}{2}|]^2,0\right) -$$

$$- \mathbb{K}\left(z,y;[\sqrt{z}-|\tfrac{ay}{2}|]^2,0\right)\Big] + 2\int_{-\infty}^{[\sqrt{z}-|\frac{ay}{2}|]^2} S''(\zeta)\, \mathbb{K}(z,y;\zeta,0)\, d\zeta .$$

We note, without proof, that

$$\phi_1\left(z, y; \left[\sqrt{z} - \left|\tfrac{ay}{2}\right|\right]^2, 0\right) = -\frac{1}{2a} z^{1/4} \left[\sqrt{z} - \left|\tfrac{ay}{2}\right|\right]^{-3/2}$$

(this was found by solving equation 43). Then, with equation 31, we obtain

$$(47) \qquad u = z \int_{-\infty}^{\left[\sqrt{z} - \left|\frac{ay}{2}\right|\right]^2} S''(\zeta)\, \mathbb{K}(z, y; \zeta, 0)\, d\zeta$$

for $z > \left(\frac{ay}{2}\right)^2$. While equation 47 agrees with our results (equation 42), equation 46 has the extra term

$$\frac{1}{a} S'(0) \left[\left(\frac{ay}{2}\right)^2 - z\right]^{-1/2},$$

which is singular at the limiting characteristic $z = \left(\frac{ay}{2}\right)^2$. This singular term is analogous to those found by Royce[9] in his treatment of the axisymmetric problem.

Results on the Airfoil Surface

In general, equation 42 cannot be integrated in closed from for an arbitrary field point (z, y). However, in practice the results on the airfoil surface are of greatest interest. Since the airfoil is thin, the solution on its surface is found by letting $y \to 0$ in equations 32 and 42. The fundamental solution $\mathbb{K}$ for $y = \eta = 0$ is

$$
\mathbb{K}(z,0;\zeta,0) = \begin{cases} 0 & \zeta > 0,\ z < \zeta \\ -\frac{1}{\pi a}\, z^{-1/2}\, K(\sqrt{1-\xi}) & z > \zeta > 0 \\ -\frac{1}{\pi a}(-z)^{-1/2}\,\frac{1}{1+\omega}\, K\left(\frac{2\omega^{1/2}}{1+\omega}\right) & z,\zeta < 0 \\ -\frac{1}{\pi a}\, z^{-1/2}\,\frac{1}{\sqrt{1+\xi}}\, K\left(\frac{1}{\sqrt{1+\xi}}\right) & z > 0 > \zeta \end{cases}
$$

where $\xi = \left|\frac{\zeta}{z}\right|$ and $\omega = \sqrt{\frac{\zeta}{z}}$. Therefore, on the airfoil

$$
(48) \qquad u(z,0) = \begin{cases} -\frac{2}{\pi a}(-z)^{-1/2}\int_{-\infty}^{0-} S''(\zeta)\, K\left(\frac{2\omega^{1/2}}{1+\omega}\right)\frac{d\zeta}{1+\omega} & z < 0 \\ -\frac{2}{\pi a}\, z^{-1/2}\left\{\int_{-\infty}^{0-} S''(\zeta)\, K\left(\frac{1}{\sqrt{1+\xi}}\right)\frac{d\zeta}{\sqrt{1+\xi}} + \right. \\ \left. \qquad + \int_{0+}^{z} S''(\zeta)\, K(\sqrt{1-\xi})\, d\zeta \right\} & z > 0. \end{cases}
$$

This can be put into an interesting form in the following way:

If we define k by

$$
k = \frac{1-\sqrt{1-m^2}}{1+\sqrt{1-m^2}}
$$

for $0 \leq m \leq 1$, then[15]

$$
K(m) = (1+k)\, K(k) .
$$

In the integral in equation 48 for $z < 0$ we put $m = \frac{2\omega^{1/2}}{1+\omega}$, which gives

$$
k = \begin{cases} \omega & \omega \leq 1 \\ \frac{1}{\omega} & \omega \geq 1 \end{cases} .
$$

Then changing the variable of integration to k we obtain

(49) $$u(z,0) = -\frac{4}{\pi a}(-z)^{1/2}\int_0^1 \left[k\, S''(zk^2) + \frac{1}{k^2} S''\left(\frac{z}{k^2}\right)\right] K(k)\, dk$$

for $z < 0$. In the first integral in equation 49 for $z > 0$ we change the variable of integration to $k = \frac{1}{\sqrt{1+\xi}}$ and in the second integral, to $k = \sqrt{1-\xi}$, to obtain

$$u(z,0) = -\frac{4}{\pi a} z^{1/2}\int_0^1 \left[k\, S''(z - zk^2) + \frac{1}{k^2} S''\left(z - \frac{z}{k^2}\right)\right] K(k)\, dk$$

for $z > 0$. Thus, we have expressed the solution on the airfoil in similar forms for $z < 0$ and $z > 0$. The latter equation is more interesting than useful, however.

Power Series Airfoils

If we have an airfoil whose upper surface can be expressed as a power series in z (i. e. all derivatives are continuous on the airfoil surface), then we can write

$$S(z) = \left[H(z+\bar{x}) - H(1-z-\bar{x})\right] \sum_{n=0}^{\infty} \frac{S^{(n)}(0)}{n!}\, z^n ,$$

where $H(x)$ is the Heaviside unit function, $z = -\bar{x}$ is the leading edge of the airfoil, and $S^{(n)}(0) = \left[\frac{d^n}{dz^n} S(z)\right]_{z=0}$. Then

$$S''(z) = S'(-\bar{x})\,\delta(z+\bar{x}) - S'(1-\bar{x})\,\delta(1-z-\bar{x}) +$$
$$+ \left[H(z+\bar{x}) - H(1-z-\bar{x})\right] \sum_{n=0}^{\infty} \frac{S^{(n+2)}(0)}{n!}\, z^n .$$

From equation 49, the pressure coefficient on this airfoil for $z < 0$ is

(50)
$$C_p(z) = \frac{4}{\pi a} S'(-\bar{x}) \frac{1}{\sqrt{\bar{x}}} K\left(\sqrt{-\frac{z}{\bar{x}}}\right) + \frac{8}{\pi a}(-z)^{1/2} \sum_{n=0}^{\infty} \frac{S^{(n+2)}(0)}{n!} z^n \left\{ \int_0^1 k^{2n+1} K(k)\, dk + \int_{\sqrt{-z/\bar{x}}}^1 k^{-2(n+1)} K(k)\, dk \right\}.$$

We now define the function $G_n(x)$ by

$$G_n(x) = \int_0^1 k^{2n+1} K(k)\, dk + \int_x^1 k^{-2(n+1)} K(k)\, dk$$

for $n \geq 0$ and $0 < x \leq 1$. In the Appendix $G_n(x)$ is shown to satisfy the recursion relation

(A. 6)
$$(2n+1)^2 G_n(x) - (2n)^2 G_{n-1}(x) = x^{-(2n+1)} \left[E(x) + 2n(1-x^2) K(x) \right],$$

which gives

$$G_0(x) = \frac{1}{x} E(x) \quad , \quad \text{ETC.}$$

With these definitions, equation 50 can be written

(51)
$$C_p(z) = \frac{4}{\pi a} \frac{S'(-\bar{x})}{\bar{x}^{1/2}} K\left(\sqrt{-z/\bar{x}}\right) + \frac{8}{\pi a}(-z)^{1/2} \sum_{n=0}^{\infty} \frac{S^{(n+2)}(0)}{n!} z^n G_n\left(\sqrt{-z/\bar{x}}\right)$$

for $z < 0$. In the Appendix it is further shown that, for $x < 1$ and

$n = 0, 1, 2, \ldots,$

(A. 8) $$G_n(x) = \frac{\pi}{2} x^{-(2n+1)} \left[\frac{1}{2n+1} + \frac{x^2}{4(2n-1)} + \cdots \right]$$

and

(A. 7) $$K(x) = \frac{\pi}{2} \left[1 + \frac{x^2}{4} + \cdots \right] .$$

Then, if we put $\left|\frac{z}{\bar{x}}\right| << 1$ in equation 51, which is equivalent to letting the field point z approach $0-$, we obtain

(52) $$C_p = \frac{2}{a} \left[\frac{S'(-\bar{x})}{\bar{x}^{1/2}} + \bar{x}^{1/2} \sum_{n=0}^{\infty} \frac{S^{(n+2)}(0)}{n!\,(n+\frac{1}{2})} (-\bar{x})^n \right] - $$
$$- \frac{1}{2a} \frac{z}{\bar{x}} \left[\frac{S'(-\bar{x})}{\bar{x}^{1/2}} + \bar{x}^{1/2} \sum_{n=0}^{\infty} \frac{S^{(n+2)}(0)}{n!\,(n-\frac{1}{2})} (-\bar{x})^n \right] + \cdots .$$

The first term is equivalent to the integral in equation 36, and the second term assures us that u_z is finite for $z \to 0-$.

For the case $z > 0$, we have, from equation 48,

$$C_p(z) = \frac{4}{\pi a} S'(-\bar{x}) \left[z + \bar{x} \right]^{-1/2} K\left(\sqrt{\frac{z}{z+\bar{x}}}\right) +$$
$$+ \frac{4}{\pi a} z^{-1/2} \sum_{n=0}^{\infty} \frac{S^{(n+2)}(0)}{n!} \left\{ \int_{-\bar{x}}^{0} \zeta^n K\left(\frac{1}{\sqrt{1+\xi}}\right) \frac{d\zeta}{\sqrt{1+\xi}} + \right.$$
$$\left. + \int_0^z \zeta^n K(\sqrt{1-\xi})\, d\zeta \right\} ,$$

or, with ξ as the variable of integration,

$$(53) \qquad C_p(z) = \frac{4}{\pi a} S'(-\bar{x}) [z+\bar{x}]^{-1/2} K\left(\sqrt{\frac{z}{z+\bar{x}}}\right) +$$
$$+ \frac{8}{\pi a} z^{1/2} \sum_{n=0}^{\infty} \frac{S^{(n+2)}(0)}{n!} (-z)^n \left\{ \int_0^{\bar{x}/z} \xi^n K\left(\frac{1}{\sqrt{1+\xi}}\right) \frac{d\xi}{\sqrt{1+\xi}} + \right.$$
$$\left. + (-1)^n \int_0^1 \xi^n K(\sqrt{1-\xi}) \, d\xi \right\} .$$

We now define a function $H_n(x)$ by

$$H_n(x) = \begin{cases} -\frac{1}{2} \int_x^{\infty} \xi^n K\left(\frac{1}{\sqrt{1+\xi}}\right) \frac{d\xi}{\sqrt{1+\xi}} & n < -\frac{1}{2} \\ \frac{1}{2} \left\{ \int_0^x \xi^n K\left(\frac{1}{\sqrt{1+\xi}}\right) \frac{d\xi}{\sqrt{1+\xi}} + (-1)^n \int_0^1 \xi^n K(\sqrt{1-\xi}) \, d\xi \right\} & n \geq 0 . \end{cases}$$

In the Appendix it is shown that $H_n(x)$, defined in this way, satisfies the recursion relation

$$\text{(A. 12)} \qquad (2n+1)^2 H_n(x) + (2n)^2 H_{n-1}(x) = x^n \sqrt{1+x} \left[E\left(\frac{1}{\sqrt{1+x}}\right) + 2n \, K\left(\frac{1}{\sqrt{1+x}}\right)\right].$$

Thus

$$H_0(x) = \sqrt{1+x} \, E\left(\frac{1}{\sqrt{1+x}}\right) , \quad H_{-\frac{1}{2}}(x) = x^{-1/2} \sqrt{1+x} \left[E\left(\frac{1}{\sqrt{1+x}}\right) - K\left(\frac{1}{\sqrt{1+x}}\right)\right] , \text{ ETC.}$$

Then we can write equation 53 as

$$(54) \qquad C_p(z) = \frac{4}{\pi a} S'(-\bar{x}) [z+\bar{x}]^{-1/2} K\left(\sqrt{\frac{z}{z+\bar{x}}}\right) + \frac{8}{\pi a} z^{1/2} \sum_{n=0}^{\infty} \frac{S^{(n+2)}(0)}{n!} (-z)^n H_n\left(\frac{\bar{x}}{z}\right)$$

for $z > 0$. In the Appendix it is also shown that, for $n = 0, 1, 2, \ldots$ and $x > 1$,

$$\text{(A. 13)} \qquad H_n(x) = \frac{\pi}{2} x^{n+\frac{1}{2}} \left[\frac{1}{2n+1} - \frac{1}{4(2n-1)} \frac{1}{x} + \cdots \right] ,$$

$$\frac{1}{\sqrt{1+x}} K\left(\frac{1}{\sqrt{1+x}}\right) = \frac{\pi}{2} x^{-1/2} \left[1 - \frac{1}{4x} + \cdots \right] .$$

Therefore

$$C_p = \frac{2}{a}\left[\frac{S'(-\bar{x})}{\bar{x}^{1/2}} + \bar{x}^{1/2}\sum_{n=0}^{\infty}\frac{S^{(n+2)}(0)}{n!\,(n+\frac{1}{2})}(-\bar{x})^n\right] -$$

$$- \frac{1}{2a}\frac{z}{\bar{x}}\left[\frac{S'(-\bar{x})}{\bar{x}^{1/2}} + \bar{x}^{1/2}\sum_{n=0}^{\infty}\frac{S^{(n+2)}(0)}{n!\,(n-\frac{1}{2})}(-\bar{x})^n\right] + \cdots$$

for $\frac{z}{\bar{x}} < 1$, $z > 0$. But these two terms are the same as those in equation 52, so we have ascertained that u and u_z are continuous at the origin (sonic point) for the power series airfoil.

The drag coefficient is, from equation 12,

$$C_D = 2\int_{-\bar{x}}^{0} C_p(z)\, S'(z)\, dz + 2\int_{0}^{1-\bar{x}} C_p(z)\, S'(z)\, dz .$$

The first integral is the contribution of the subsonic part of the airfoil and the second integral is that of the supersonic part. If we assume a to be constant, rather than slowly varying, then the above results can be used to find C_D in closed form as follows:

For the subsonic part we have, using equation 51 and the series for $S'(z)$,

$$2\int_{-\bar{x}}^{0} C_p(z)\, S'(z)\, dz = \frac{8}{\pi a}\frac{S'(-\bar{x})}{\bar{x}^{1/2}}\sum_{m=0}^{\infty}\frac{S^{(m+1)}(0)}{m!}\int_{-\bar{x}}^{0} z^n K(\sqrt{-z/\bar{x}})\, dz +$$

$$+ \frac{16}{\pi a}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{S^{(m+1)}(0)\, S^{(n+2)}(0)}{m!\; n!}\int_{-\bar{x}}^{0} z^{m+n}(-z)^{1/2}\, G_n(\sqrt{-z/\bar{x}})\, dz ,$$

or, putting $k = \sqrt{-z/\bar{x}}$,

$$2\int_{-\bar{x}}^{0} C_p(z)\, S'(z)\, dz = \frac{16}{\pi a}\bar{x}^{1/2} S'(-\bar{x})\sum_{m=0}^{\infty}\frac{S^{(m+1)}(0)}{m!}(-\bar{x})^n\int_0^1 k^{2n+1} K(k)\, dk +$$

$$+ \frac{32}{\pi a}\bar{x}^{3/2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{S^{(m+1)}(0)\, S^{(n+2)}(0)}{m!\; n!}(-\bar{x})^{m+n}\int_0^1 k^{2(n+m+1)} G_n(k)\, dk .$$

The integral in the first term is just $G_n(1)$. For the integral in the second term, we integrate by parts to obtain

$$\int_0^1 k^{2(m+n+1)} G_n(k)\,dk = \frac{1}{2m+2n+3}\left[G_n(1) + G_m(1)\right].$$

Thus, the contribution of the subsonic part of the airfoil to the drag coefficient is

$$2\int_{-\bar{x}}^{0} C_p(z)\,S'(z)\,dz = \frac{16}{\pi a}\,\bar{x}^{1/2} S'(-\bar{x}) \sum_{m=0}^{\infty} \frac{S^{(m+1)}(0)}{m!}(-\bar{x})^m G_m(1) + \tag{55}$$

$$+ \frac{16}{\pi a}\,\bar{x}^{3/2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{S^{(m+1)}(0)\,S^{(n+2)}(0)}{m!\,n!\,(m+n+\frac{3}{2})}(-\bar{x})^{m+n}\left[G_n(1) + G_m(1)\right],$$

where $G_n(1)$ is given by

$$(2n+1)^2 G_n(1) - (2n)^2 G_{n-1}(1) = 1.$$

For the supersonic part we have, from equation 54 and the series for $S'(z)$,

$$2\int_0^{1-\bar{x}} C_p(z)\,S'(z)\,dz = \frac{8}{\pi a}\,S'(-\bar{x})\sum_{m=0}^{\infty}\frac{S^{(m+1)}(0)}{m!}\int_0^{1-\bar{x}} z^m K\left(\sqrt{\frac{z}{z+\bar{x}}}\right)\frac{dz}{\sqrt{z+\bar{x}}} +$$

$$+ \frac{16}{\pi a}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{S^{(m+1)}(0)\,S^{(n+2)}(0)}{m!\,n!}(-1)^n\int_0^{1-\bar{x}} z^{n+k+\frac{1}{2}} H_n\left(\frac{\bar{x}}{z}\right)dz,$$

or, putting $\xi = \frac{\bar{x}}{z}$,

$$2\int_0^{1-\bar{x}} C_p(z)\,S'(z)\,dz = \frac{8}{\pi a}\,\bar{x}^{1/2} S'(-\bar{x})\sum_{m=0}^{\infty}\frac{S^{(m+1)}(0)}{m!}\,\bar{x}^m\int_{\frac{\bar{x}}{1-\bar{x}}}^{\infty}\xi^{-(m+\frac{3}{2})} K\left(\frac{1}{\sqrt{1+\xi}}\right)\frac{d\xi}{\sqrt{1+\xi}}$$

$$+ \frac{16}{\pi a}\,\bar{x}^{3/2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{S^{(m+1)}(0)\,S^{(n+2)}(0)}{m!\,n!}(-1)^n\,\bar{x}^{m+n}\int_{\frac{\bar{x}}{1-\bar{x}}}^{\infty}\xi^{-(n+k+\frac{5}{2})} H_n(\xi)\,d\xi.$$

The integral in the first term is $-2\,H_{-(m+\frac{3}{2})}\left(\frac{\bar{x}}{1-\bar{x}}\right)$, and the integral in the second term is found, upon integrating by parts, to be

$$\int_{\frac{\bar{x}}{1-\bar{x}}}^{\infty} \xi^{-(n+m+\frac{5}{2})} H_n(\xi)\,d\xi = \frac{1}{n+m+\frac{3}{2}} \left[\left(\frac{\bar{x}}{1-\bar{x}}\right)^{-(n+m+\frac{3}{2})} H_n\left(\frac{\bar{x}}{1-\bar{x}}\right) - H_{-(m+\frac{3}{2})}\left(\frac{\bar{x}}{1-\bar{x}}\right)\right].$$

Thus, the contribution of the supersonic part of the airfoil is

$$\begin{aligned} 2\int_0^{1-\bar{x}} C_p(z)\,S'(z)\,dz = &-\frac{16}{\pi a}\,\bar{x}^{1/2}\,S'(-\bar{x}) \sum_{m=0}^{\infty} \frac{S^{(m+1)}(0)}{m!}\,\bar{x}^{m}\,H_{-(m+\frac{3}{2})}\left(\frac{\bar{x}}{1-\bar{x}}\right) - \\ &- \frac{16}{\pi a}\,\bar{x}^{3/2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{S^{(m+1)}(0)\,S^{(n+2)}(0)}{m!\,n!\,(n+m+\frac{3}{2})} (-1)^n\,\bar{x}^{n+m} \Big[H_{-(m+\frac{3}{2})}\left(\frac{\bar{x}}{1-\bar{x}}\right) - \\ &- \left(\frac{1-\bar{x}}{\bar{x}}\right)^{n+m+\frac{3}{2}} H_n\left(\frac{\bar{x}}{1-\bar{x}}\right)\Big]. \end{aligned} \tag{56}$$

The total drag coefficient is then the sum of equations 55 and 56:

$$\begin{aligned} C_D = \frac{16}{\pi a}\,\bar{x}^{1/2} \Big\{ & S'(-\bar{x}) \sum_{m=0}^{\infty} \frac{S^{(m+1)}(0)}{m!} (-\bar{x})^m \Big[G_m(1) - (-1)^m H_{-(m+\frac{3}{2})}\left(\frac{\bar{x}}{1-\bar{x}}\right)\Big] + \\ &+ \bar{x} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{S^{(m+1)}(0)\,S^{(n+2)}(0)}{m!\,n!\,(m+n+\frac{3}{2})} (-\bar{x})^{m+n} \Big[G_n(1) + G_m(1) - \\ &- (-1)^m H_{-(m+\frac{3}{2})}\left(\frac{\bar{x}}{1-\bar{x}}\right) + (-1)^m \left(\frac{1-\bar{x}}{\bar{x}}\right)^{n+m+\frac{3}{2}} H_n\left(\frac{\bar{x}}{1-\bar{x}}\right)\Big]\Big\}. \end{aligned} \tag{57}$$

No attempt has been made to prove the convergence of the various series appearing in this section, since, for any practical problem, only a finite number of terms will be used to approximate any smooth airfoil.

Determination of a

Discussion of the methods of determining a has been delayed until now so that the form of the solution of the linear equation would be available. If, in keeping with the linearizing assumptions, a is taken to be constant, then we must have a reasonable method of finding the best value of a. From the viewpoint of simplicity, the suggestion of Reference (4) appears to be best. According to this suggestion, the acceleration u_z, measured at that point on the airfoil which has the minimum pressure in incompressible flow, should be used to find a^2. The reasoning behind this suggestion is that below the lower critical mach number the flow will then be subsonic and will be described by the linear subsonic equation. This method is illustrated in the next section for a parabolic arc airfoil.

If the pressure distribution on the airfoil was reasonably linear, then taking a to be constant would be fairly accurate. However, if $S'(-\bar{x})$ is not zero, then we see from the first term in equation 48 that the pressure is infinite at the leading edge since K has a logarithmic singularity for unit argument. Hence, the pressure distribution is not nearly linear, and therefore no constant value for a can be expected to accurately describe the pressure distribution over the entire airfoil; particularly near the leading edge.

The use of local linearization greatly improves this situation, since a is evaluated locally for each point on the airfoil surface. Since $-a^2 z$ replaced $1 - M_\infty^2 - (\gamma+1) M_\infty^2 u$ in the small perturbation equation, we should replace $-a^2 z$ in the solution (e. g. equation 48) by $1 - M_\infty^2 - (\gamma+1) M_\infty^2 u$ and then solve the resulting equation for u. In general, $\bar{x}$ will be a function of $\frac{1-M_\infty^2}{(\gamma+1)M_\infty^2} a$, so the equation may be difficult to solve. However, if $M_\infty = 1$ or if the sonic point is determined by a shoulder on the airfoil, then $\bar{x}$ is independent of a and the equation for u is easily solved. For example, for $M_\infty = 1$, we see from equation 48 that the solution of the linear equation is proportional to $\frac{1}{a}$ on the airfoil. Therefore, if we take $u_o(z)$ to be this solution, we can write

$$\left| (\gamma+1) u \right|^{1/2} u = a \sqrt{|z|} \; u_o(z) .$$

The right-hand side of this equation is independent of a, hence u is easily obtained once $u_o(z)$ is found.

The above method of local linearization does not exhibit the mach number "freeze" phenomenon (i. e. $dM/dM_\infty = 0$) for $M_\infty = 1$. If we replace $-a^2 z$ by $1 - M_\infty^2 - (\gamma+1) M_\infty^2 u$ in du_o/dz, we have, for $\bar{x}$ fixed,

$$\left| 1 - M_\infty^2 - (\gamma+1) M_\infty^2 u \right|^{1/2} \frac{du}{dz} = a \sqrt{|z|} \; \frac{du_o}{dz} ,$$

the solution of which does exhibit the mach number "freeze". However, this solution is less accurate for $M_\infty = 1$ than the one above, and cannot be evaluated in a closed form, so it is not used in this paper.

III. EXAMPLES

Parabolic Arc Airfoil at $M_\infty = 1$

The parabolic arc airfoil is described by the equation

$$S(z) = 2\tau\,(z+\bar{x})(1-z-\bar{x}) \qquad -\bar{x} < z < 1-\bar{x} \;,$$

where τ is the thickness ratio. Then

$$S'(z) = 2\tau\,(1-2\bar{x}-2z)\left[H(z+\bar{x}) - H(1-z-\bar{x})\right]$$

$$S''(z) = -4\tau\left[H(z+\bar{x}) - H(1-z-\bar{x})\right] + 2\tau\left[\delta(z+\bar{x}) + \delta(1-z-\bar{x})\right].$$

From equation 36 we find

$$\int_{-\infty}^{0} S''(\zeta)\frac{d\zeta}{\sqrt{-\zeta}} = \frac{2\tau}{\bar{x}^{1/2}} - 8\tau\,\bar{x}^{1/2} = 0$$

for $M_\infty = 1$. So $\bar{x} = \frac{1}{4}$ and the sonic point is at the quarter chord line. From equations 51 and 54 we have

$$(58) \qquad C_p(z) = \begin{cases} \dfrac{16\tau}{\pi a}\left[K(2\sqrt{-z}) - E(2\sqrt{-z})\right] & z<0 \\[2ex] \dfrac{16\tau}{\pi a}\left[\dfrac{1}{\sqrt{4z+1}}K\left(\sqrt{\dfrac{4z}{4z+1}}\right) - \sqrt{4z+1}\,E\left(\sqrt{\dfrac{4z}{4z+1}}\right)\right] & z>0 . \end{cases}$$

If we assume a to be constant and follow the suggestion of Reference (4), then we must evaluate u_z at mid-chord since the pressure is minimum there in incompressible flow. Since $C_p = -2u$, we have on the airfoil surface

$$u_z = -\frac{8\tau}{\pi a}\frac{d}{dz}\left[\frac{1}{\sqrt{4z+1}}K\left(\sqrt{\frac{4z}{4z+1}}\right) - \sqrt{4z+1}\,E\left(\sqrt{\frac{4z}{4z+1}}\right)\right] \qquad z>0 .$$

Then, using the relations

$$\frac{d}{dz} K\left(\sqrt{\tfrac{4z}{4z+1}}\right) = \frac{1}{2z}\left[E\left(\sqrt{\tfrac{4z}{4z+1}}\right) - \frac{1}{4z+1} K\left(\sqrt{\tfrac{4z}{4z+1}}\right)\right],$$

$$\frac{d}{dz} E\left(\sqrt{\tfrac{4z}{4z+1}}\right) = \frac{1}{2z}\,\frac{1}{4z+1}\left[E\left(\sqrt{\tfrac{4z}{4z+1}}\right) - K\left(\sqrt{\tfrac{4z}{4z+1}}\right)\right],$$

we find

$$u_z = \frac{16\tau}{\pi a}\left[4z+1\right]^{-1/2} E\left(\sqrt{\tfrac{4z}{4z+1}}\right) \qquad z > 0 \; .$$

Thus, at the mid-chord point $(z = \frac{1}{4})$,

$$u_z(z=\tfrac{1}{4}) = \frac{8\sqrt{2}}{\pi a}\,\tau\, E\left(\tfrac{\sqrt{2}}{2}\right) = \frac{a^2}{\gamma+1} \; ,$$

or

$$a = (\gamma+1)^{1/3}\,\tau^{1/3}\left[\tfrac{8\sqrt{2}}{\pi} E\left(\tfrac{\sqrt{2}}{2}\right)\right]^{1/3} \doteq 1.694\,(\gamma+1)^{1/3}\,\tau^{1/3} \; .$$

Finally, we have

$$(59) \qquad (\gamma+1)^{1/3}\,\tau^{-2/3}\, C_p = \bar{C}_p \doteq \begin{cases} 3.01\left[K(2\sqrt{-z}) - E(2\sqrt{-z})\right] & z < 0 \\ 3.01\left[\frac{1}{\sqrt{4z+1}} K\left(\sqrt{\tfrac{4z}{4z+1}}\right) - \sqrt{4z+1}\, E\left(\sqrt{\tfrac{4z}{4z+1}}\right)\right] & z > 0 \end{cases}$$

on the airfoil at $M_\infty = 1$. Here we see that the solution obeys the transonic similarity rule at $M_\infty = 1$. With a constant we can use equations 55 and 56 to find the drag coefficient. The contribution of the subsonic part of the airfoil is

$$2\int_{-\bar{x}}^{0} C_p(z)\, S'(z)\, dz = \frac{416}{45\pi}\,\frac{\tau^2}{a}$$

$$\doteq 1.74\;\tau^{5/3}\,(\gamma+1)^{-1/3}$$

and the contribution of the supersonic part is

$$2\int_0^{1-\bar{x}} C_p(z)\, S'(z)\,dz = -\frac{16\tau^2}{\pi a}\left[\frac{2}{3} H_{-\frac{3}{2}}\left(\tfrac{1}{3}\right) - \frac{4}{5} H_{-\frac{5}{2}}\left(\tfrac{1}{3}\right) - \frac{4}{5}\sqrt{3}\, H_0\left(\tfrac{1}{3}\right)\right]$$

$$\doteq 3.21\ \tau^{5/3}\,(\gamma+1)^{-1/3} \ .$$

Therefore, the total drag coefficient is

$$(60) \qquad (\gamma+1)^{1/3}\,\tau^{-5/3}\, C_D = \bar{C}_D \doteq 4.95$$

and again transonic similarity is observed.

Now we use the local linearization approach instead of assuming a to be constant. If we take $C_p(z)$ as given in equation 58 to be C_{p_0} , then, as outlined in the last section, we have, for $M_\infty = 1$,

$$(\gamma+1)^{1/2}(-u)^{3/2} = \frac{8\tau}{\pi}\sqrt{-z}\,\left[K\left(2\sqrt{-z}\right) - E\left(2\sqrt{-z}\right)\right] \qquad z<0,$$

$$(\gamma+1)^{1/2}\,u^{3/2} = \frac{8\tau}{\pi}\,z^{1/2}\left[\frac{1}{\sqrt{4z+1}}\,K\left(\sqrt{\frac{4z}{4z+1}}\right) - \sqrt{4z+1}\,E\left(\sqrt{\frac{4z}{4z+1}}\right)\right] \qquad z>0.$$

From which we find

$$(61) \qquad \bar{C}_p(z) = \begin{cases} \dfrac{8}{\pi^{2/3}}(-z)^{1/3}\left[K\left(2\sqrt{-z}\right) - E\left(2\sqrt{-z}\right)\right] & z<0, \\[2ex] -\dfrac{8}{\pi^{2/3}}\,z^{1/3}\left[\dfrac{1}{\sqrt{4z+1}}\,K\left(\sqrt{\dfrac{4z}{4z+1}}\right) - \sqrt{4z+1}\,E\left(\sqrt{\dfrac{4z}{4z+1}}\right)\right] & z>0. \end{cases}$$

The drag coefficient is found by numerical integration to be

$$(62) \qquad \bar{C}_D \doteq 4.58 \ .$$

The pressure coefficients given by equations 59 and 61 are plotted for comparison in Figure 7. The results are in close agreement on the supersonic part of the airfoil, but are considerably different over most of the subsonic part. This discrepancy is directly related to the breakdown of the linearizing assumption at the leading edge of the airfoil. Sufficiently near the leading edge, the leading term of $K(2\sqrt{-z})$ is $\frac{1}{2}\log\frac{1}{4z+1}$, so the leading term in $\bar{C}_p$, as given by equation 59, is

$$\bar{C}_p \doteq 1.5 \log \frac{1}{4z+1} + \cdots = 1.5 \log \frac{1}{x} + \cdots ,$$

while the leading term in $\bar{C}_p$, as given by equation 61, is

$$\bar{C}_p \doteq 3.7 \left[\log \frac{1}{x} \right]^{2/3} + \cdots .$$

In Figure 8, equations 59 and 61 are compared with the results of References (5) and (7) and experimental results from Reference (16). To the right of the sonic point the four theoretical distributions agree fairly well with each other and with the experimental data. The discrepancy between experiment and theory near the trailing edge is probably due to separation of the flow caused by boundary layer-shock wave interaction. Forward of the sonic point the results are in considerable disagreement, except for the distributions of equation 61 and Reference (7), which are very close. The experimental data are probably in considerable error near the leading edge due to the experimental procedure, in which the airfoil was simulated by a bump on the tunnel wall and hence was immersed in the wall boundary layer. A better experimental procedure would probably produce much better agreement between experiment and theory.

The drag coefficients $\bar{C}_D$ as given by equations 60 and 62 compare fairly well with the value $\bar{C}_D \doteq 4.77$ obtained in Reference (7). The corrected experimental results contained in (7) show these three results to lie within the experimental scatter.

Single Wedge Airfoil at $M_\infty = 1$

The upper surface of a single-wedge profile is given by

$$S(z) = \begin{cases} \tau(\bar{x} + z) & -\bar{x} < z < \frac{1}{2} - \bar{x} \\ \frac{\tau}{2} & z > \frac{1}{2} - \bar{x} \end{cases} ,$$

where τ is the wedge semiapex angle, to the order of the thin airfoil theory. Equation 36 places the sonic point at the shoulder, so $\bar{x} = \frac{1}{2}$. The presence of the shoulder invalidates the equation for the pressure distribution for $z > 0$. However, if we are only interested in the pressures on the wedge, we can proceed as before. Then from equation 51, we have

$$(63) \qquad C_p(z) = \frac{4\sqrt{2}\,\tau}{\pi a} K(\sqrt{-2z}) \qquad -\tfrac{1}{2} < z < 0 .$$

Now this is obviously a very poor approximation to the actual pressure distribution since, for $z \to 0-$, it gives $C_p \to 2\sqrt{2}\,\tau/a$, which is non-zero for finite a. This result is due to the fact that the linearizing assumption breaks down near the shoulder.

However, if the local linearization technique used earlier is applied to equation 63, a fairly accurate result is obtained. We have, for $M_\infty = 1$,

$$\left[-(\gamma+1)u\right]^{1/2} u = -\frac{2\sqrt{2}}{\pi}\tau\sqrt{-z}\,K(\sqrt{-2z}),$$

or

$$\bar{C}_p = \left[\frac{8}{\pi}\sqrt{-z}\,K(\sqrt{-2z})\right]^{2/3}. \tag{64}$$

This result is plotted in Figure 9. Also shown in the same figure are the distributions from References (7) and (17) and experimental points from Reference (18). The results due to Guderley and Yoshihara[17] are usually taken to be the exact solution for the wedge at $M_\infty = 1$, although some approximations are made in the numerical calculations. As can be seen, the present results are somewhat too high except near the leading edge but agree very well with those of Reference (7).

IV. APPENDIX

Evaluation of Certain Integrals

The function $G_n(x)$ is defined by

$$G_n(x) = \int_0^1 k^{2n+1} K(k)\, dk + \int_x^1 k^{-2(n+1)} K(k)\, dk \tag{A. 1}$$

for $n \geq 0$, $0 < x \leq 1$. From Reference (15) we have

$$\begin{aligned} \frac{d}{dk} K(k) &= \frac{1}{k k'^2} \left[E(k) - k'^2 K(k) \right], \\ \frac{d}{dk} E(k) &= \frac{1}{k} \left[E(k) - K(k) \right], \end{aligned} \tag{A. 2}$$

where $k' = \sqrt{1-k^2}$ and $E(k)$ is the complete elliptic integral of the second kind. Then we have

$$\frac{d}{dk}\left[k^{2n}(E(k) - k'^2 K(k)) \right] = (2n+1) k^{2n+1} K(k) + 2n\, k^{2n-1} \left[E(k) - K(k) \right],$$

$$\frac{d}{dk}\left[k^{2n} E(k) \right] = (2n+1) k^{2n-1} E(k) - k^{2n-1} K(k),$$

and, upon eliminating $k^{2n-1}E(k)$ from these expressions,

$$\frac{d}{dk}\left[k^{2n}(E(k) - [2n+1] k'^2 K(k)) \right] = (2n+1)^2 k^{2n+1} K(k) - (2n)^2 k^{2n-1} K(k). \tag{A. 3}$$

We integrate this from 0 to 1 to obtain, for $n \geq 0$,

$$1 = (2n+1)^2 \int_0^1 k^{2n+1} K(k)\, dk - (2n)^2 \int_0^1 k^{2n-1} K(k)\, dk. \tag{A. 4}$$

If we substitute $-n-\frac{1}{2}$ for n in equation A. 3, we obtain

$$\frac{d}{dk}\left[k^{-(2n+1)}\left(E(k)+2n\,k'^2K(k)\right)\right] = (2n)^2 k^{-2n}\;K(k) - (2n+1)^2 k^{-2(n+1)}K(k)$$

and, upon integrating this from x to 1,

$$1 - x^{-(2n+1)}\left[E(x) + 2n(1-x^2)K(x)\right] = \tag{A. 5}$$

$$(2n)^2\int_x^1 k^{-2(n-1)}K(k)\,dk \;-\; (2n+1)^2\int_x^1 k^{-2(n+1)}K(k)\,dk$$

for $0<x\leq 1$. Finally, taking the difference of equation A. 4 and A. 5 and using the definition A. 1 of $G_n(x)$,

$$(2n+1)^2 G_n(x) - (2n)^2 G_{n-1}(x) = x^{-(2n+1)}\left[E(x) + 2n(1-x^2)K(x)\right] \tag{A. 6}$$

for $n \geqslant 0$ and $0<x\leq 1$.

For $|x|<1$ we have the following series for $E(x)$ and $K(x)$ (15):

$$\begin{aligned} E(x) &= \frac{\pi}{2}\left[1-\frac{1}{4}x^2-\frac{3}{64}x^4-\cdots\right], \\ K(x) &= \frac{\pi}{2}\left[1+\frac{1}{4}x^2+\frac{9}{64}x^4+\cdots\right]. \end{aligned} \tag{A. 7}$$

Substituting these into equation A. 6, we obtain for $0<x<1$

$$(2n+1)^2 G_n(x) - (2n)^2 G_{n-1}(x) = \frac{\pi}{2}\,x^{-(2n+1)}\left[(2n+1) - \frac{6n+1}{4}x^2 + \cdots\right].$$

Now, if $n = 0, 1, 2, \ldots$, we find by induction that the leading term of $G_n(x)$ for $|x| \ll 1$ is $\frac{\pi}{2}\frac{1}{2n+1}\;x^{-(2n+1)}$. Therefore, we can eliminate $G_{n-1}(x)$ term by term to obtain

(A. 8) $$G_n(x) = \frac{\pi}{2} x^{-(2n+1)} \left[\frac{1}{2n+1} - \frac{1}{4(2n-1)} x^2 + \cdots \right]$$

for $n = 0, 1, 2, \ldots,\ 0 < x < 1$.

The function $H_n(x)$ is defined by

$$H_n(x) = \begin{cases} -\frac{1}{2} \int_x^\infty \xi^n K\left(\frac{1}{\sqrt{1+\xi}}\right) \frac{d\xi}{\sqrt{1+\xi}} & n < -\frac{1}{2} \\ \frac{1}{2} \left\{ \int_0^x \xi^n K\left(\frac{1}{\sqrt{1+\xi}}\right) \frac{d\xi}{\sqrt{1+\xi}} + (-1)^n \int_0^1 \xi^n K(\sqrt{1-\xi})\, d\xi \right\} & n \geq 0 . \end{cases}$$

In the integral for $n < -\frac{1}{2}$ and in the first integral for $n \geq 0$ we make the change of variable $k = \frac{1}{\sqrt{1+\xi}}$, and in the second integral for $n \geq 0$, $k = \sqrt{1-\xi}$. Then we have

(A. 9) $$H_n(x) = \begin{cases} -\int_0^{\frac{1}{\sqrt{1+x}}} \frac{k'^{2n}}{k^{2(n+1)}} K(k)\, dk & n < -\frac{1}{2} \\ \int_{\frac{1}{\sqrt{1+x}}}^1 \frac{k'^{2n}}{k^{2(n+1)}} K(k)\, dk + (-1)^n \int_0^1 k\, k'^{2n} K(k)\, dk & n \geq 0 . \end{cases}$$

Using equations A. 2 we have

$$\frac{d}{dk}\left[k'^{2n} k^{-(2n+1)} E(k) \right] = -2n \frac{k'^{2(n-1)}}{k^{2(n+1)}} E(k) - \frac{k'^{2n}}{k^{2(n+1)}} K(k) ,$$

$$\frac{d}{dk}\left[k'^{2n} k^{-(2n+1)} K(k) \right] = \frac{k'^{2(n-1)}}{k^{2(n+1)}} E(k) - 2(n+1) \frac{k'^{2n}}{k^{2(n+1)}} K(k) -$$

$$- 2n \frac{k'^{2(n-1)}}{k^{2n}} K(k) ,$$

and, upon eliminating $E(k)$,

(A.10) $$-\frac{d}{dk}\left[\frac{k'^{2n}}{k^{2n+1}}\left(E(k)+2nK(k)\right)\right]=(2n+1)^2\frac{k'^{2n}}{k^{2(n+1)}}K(k)+(2n)^2\frac{k'^{2(n-1)}}{k^{2n}}K(k).$$

Also, using equations A. 2,

$$\frac{d}{dk}\left[k'^{2n}E(k)\right]=\left[(2n+1)k'^2-2n\right]\frac{k'^{2(n-1)}}{k}E(k)-\frac{k'^{2n}}{k}K(k),$$

$$\frac{d}{dk}\left[(2n-[2n+1]k'^2)\,k'^{2n}K(k)\right]=-\left[(2n+1)k'^2-2n\right]\frac{k'^{2(n-1)}}{k}E(k)+$$

$$+\frac{k'^{2n}}{k}K(k)+(2n+1)^2kk'^{2n}K(k)-(2n)^2kk'^{2(n-1)}K(k),$$

the sum of which is

$$\frac{d}{dk}\left[k'^{2n}\left(E(k)+[2n-(2n+1)k'^2]K(k)\right)\right]=$$

$$(2n+1)^2kk'^{2n}K(k)-(2n)^2kk'^{2(n-1)}K(k).$$

Integrating this expression from 0 to 1, we obtain

$$\int_0^1 kK(k)=1 \qquad n=0$$

(A.11) $$(2n+1)^2\int_0^1 kk'^{2n}K(k)dk-(2n)^2\int_0^1 kk'^{2(n-1)}K(k)dk=0 \qquad n>0.$$

Now, if we integrate equation A.10 from $\frac{1}{\sqrt{1+x}}$ to 1 for $n\geq 0$, we obtain

$$(2n+1)^2\int_{\frac{1}{\sqrt{1+x}}}^1\frac{k'^{2n}}{k^{2(n+1)}}K(k)dk+(2n)^2\int_{\frac{1}{\sqrt{1+x}}}^1\frac{k'^{2(n-1)}}{k^{2n}}K(k)dk=$$

$$x^n\sqrt{1+x}\left[E\left(\tfrac{1}{\sqrt{1+x}}\right)+2nK\left(\tfrac{1}{\sqrt{1+x}}\right)\right]$$

for $n > 0$, and

$$\int_{\frac{1}{\sqrt{1+x}}}^{1} \frac{1}{k^2} K(k)\, dk = \sqrt{1+x}\; E\left(\frac{1}{\sqrt{1+x}}\right) - 1$$

for $n = 0$. Then finally, adding this to equation A. 11, we have

$$(2n+1)^2 H_n(x) + (2n)^2 H_{n-1}(x) = x^n \sqrt{1+x}\left[E\left(\frac{1}{\sqrt{1+x}}\right) + 2n\, K\left(\frac{1}{\sqrt{1+x}}\right)\right]$$

for $n \geq 0$. If we integrate equation A. 10 from 0 to $\frac{1}{\sqrt{1+x}}$ for $n \leq -\frac{1}{2}$, then we again obtain the above equation. Therefore

(A. 12) $$(2n+1)^2 H_n(x) + (2n)^2 H_{n-1}(x) = x^n \sqrt{1+x}\left[E\left(\frac{1}{\sqrt{1+x}}\right) + 2n\, K\left(\frac{1}{\sqrt{1+x}}\right)\right]$$

for $n \geq 0$ or $n \leq -\frac{1}{2}$.

For $\frac{1}{\sqrt{1+x}} < 1$, the series A. 7 are convergent, so for $x > 0$ equation A. 12 can be expressed as follows:

$$(2n+1)^2 H_n(x) + (2n)^2 H_{n-1}(x) = \frac{\pi}{2}\, x^n \sqrt{1+x}\left[(2n+1) + \frac{2n-1}{4(1+x)} + \cdots\right].$$

If $x > 1$, then the series

$$\sqrt{1+x} = x^{1/2}\left[1 + \frac{1}{2x} + \cdots\right],$$

$$\frac{1}{1+x} = \frac{1}{x} - \frac{1}{x^2} + \cdots$$

are convergent. Therefore we can write

$$(2n+1)^2 H_n(x) + (2n)^2 H_{n-1}(x) = \frac{\pi}{2}\, x^{n+\frac{1}{2}}\left[(2n+1) + \frac{6n+1}{4x} + \cdots\right]$$

for $x > 1$. The convergence of the series follows from the double series theorem. Now if $n = 0, 1, 2, \ldots$, then by induction the leading term of $H_n(x)$ is $\frac{\pi}{2} \frac{1}{2n+1} x^{n+\frac{1}{2}}$, and therefore

(A.13) $$H_n(x) = \frac{\pi}{2} x^{n+\frac{1}{2}} \left[\frac{1}{2n+1} + \frac{1}{4(2n-1)} \frac{1}{x} + \cdots \right]$$

for $x > 1$, $n = 0, 1, 2, \ldots$.

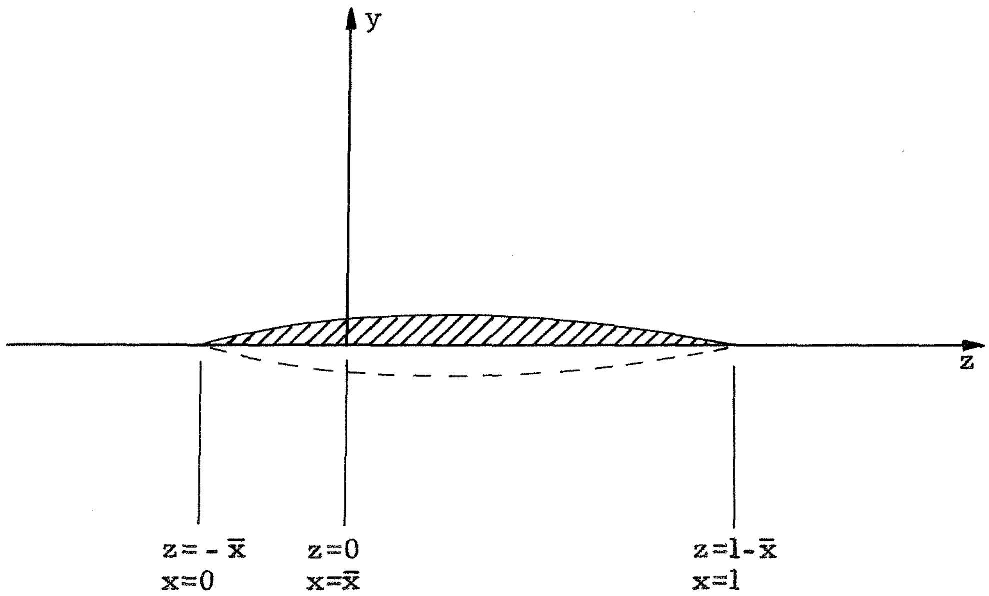

Figure 1. Coordinate System of the Linearized Problem.

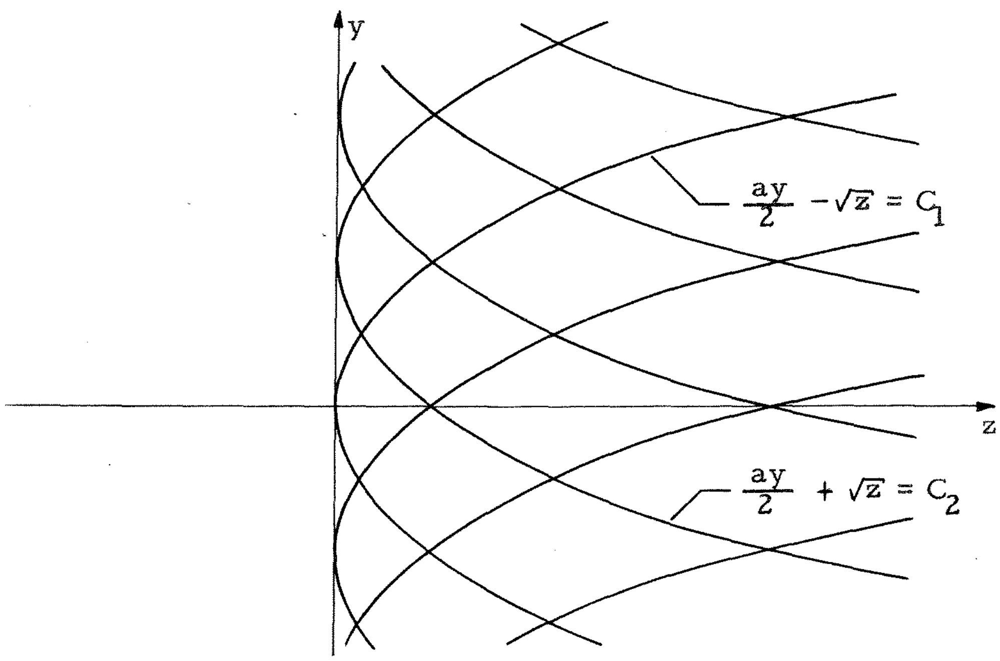

Figure 2. Characteristics of the Linear Equation.

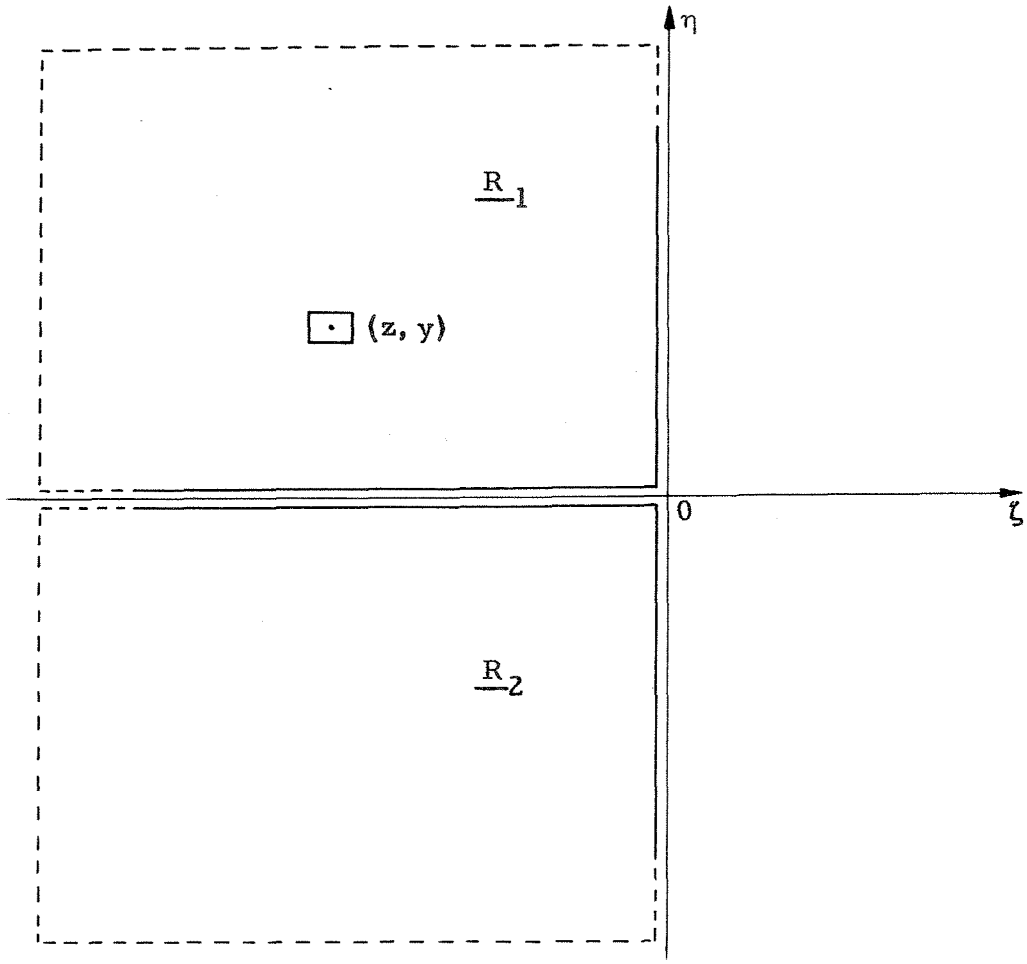

Figure 3. Regions of Integration for $z < 0$.

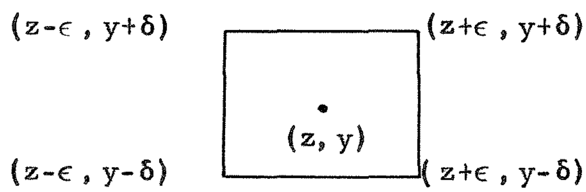

Figure 4. Path of Integration About (z, y) for $z < 0$.

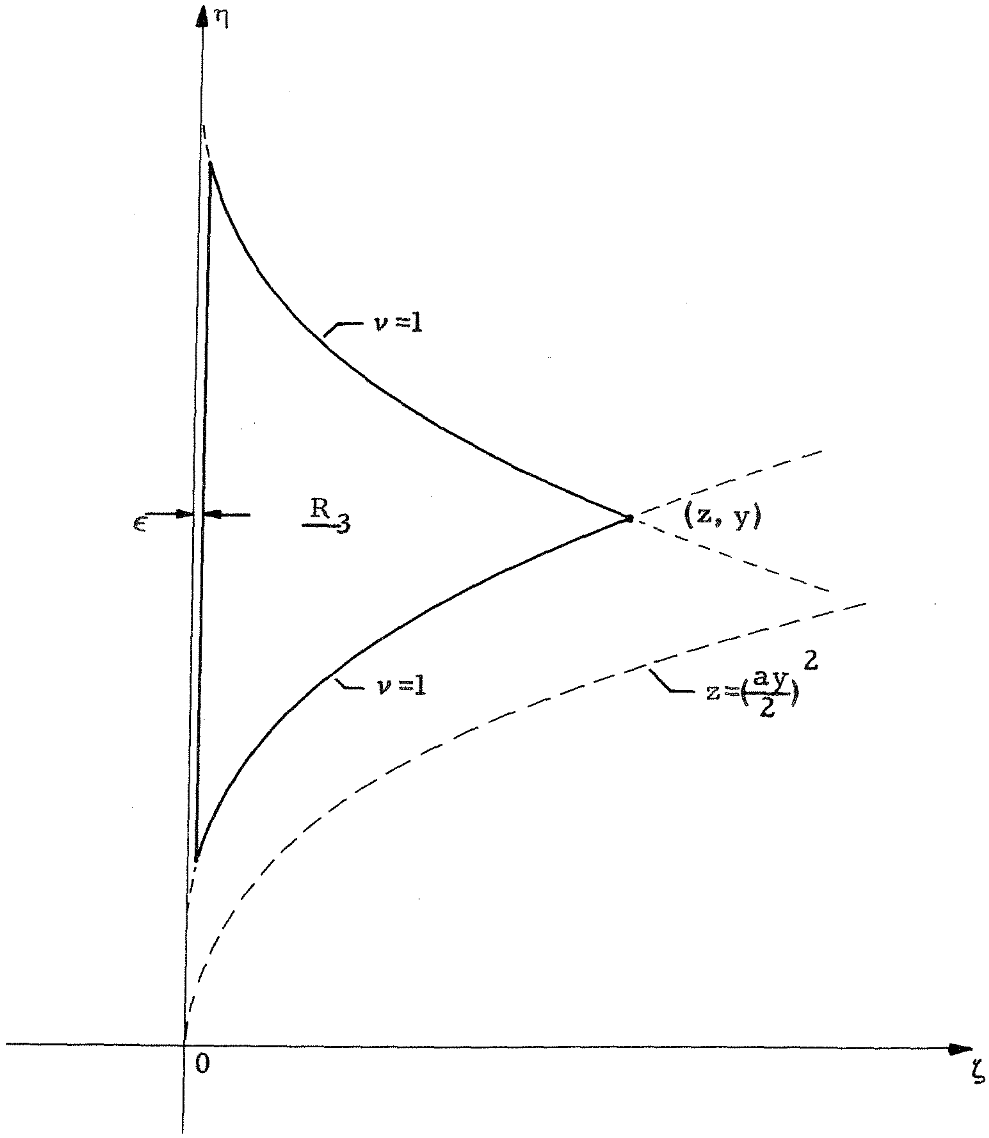

Figure 5. Region of Integration for $0 < z < (\frac{ay}{2})^2$.

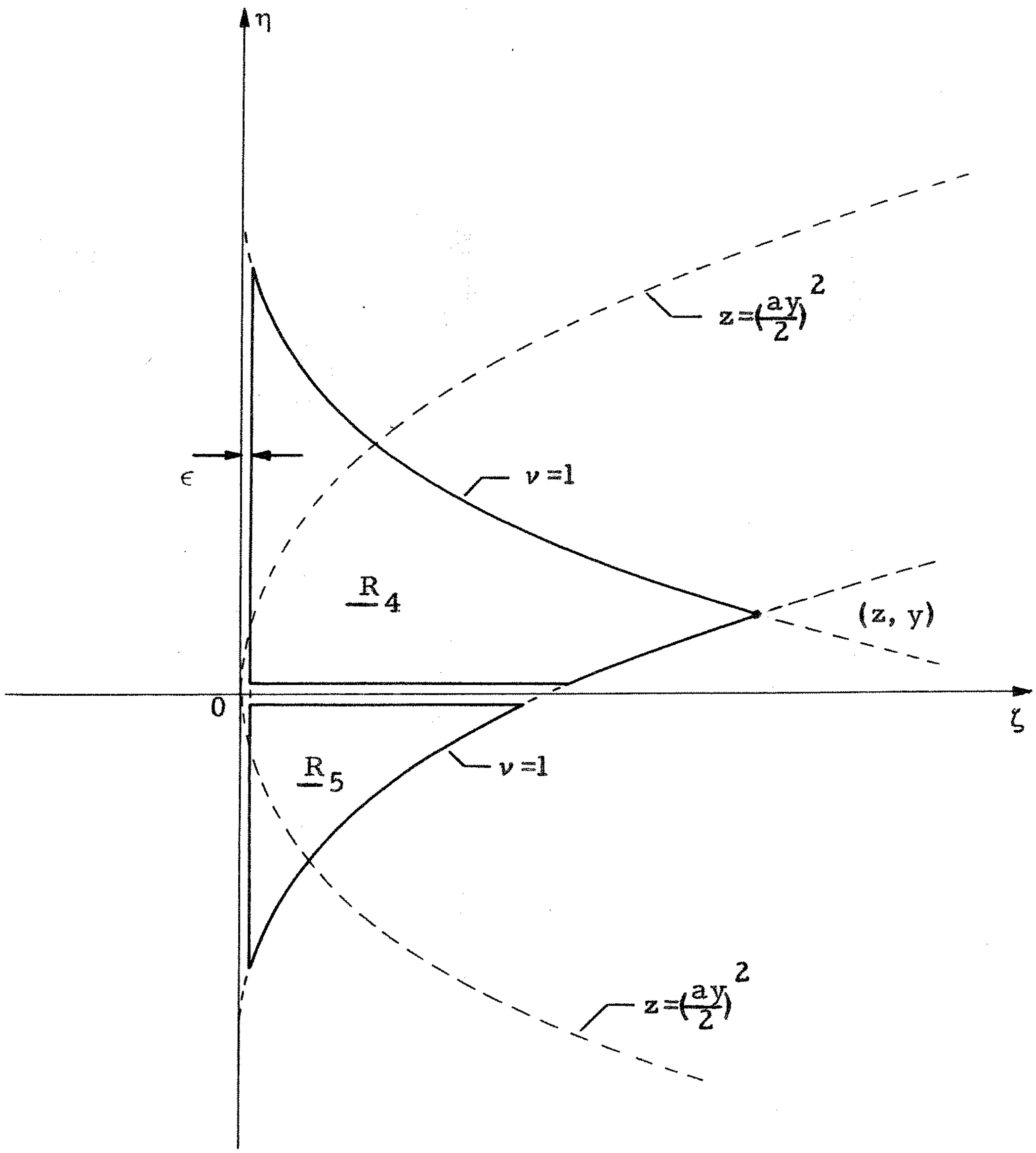

Figure 6. Regions of Integration for $z > (\frac{ay}{2})^2$.

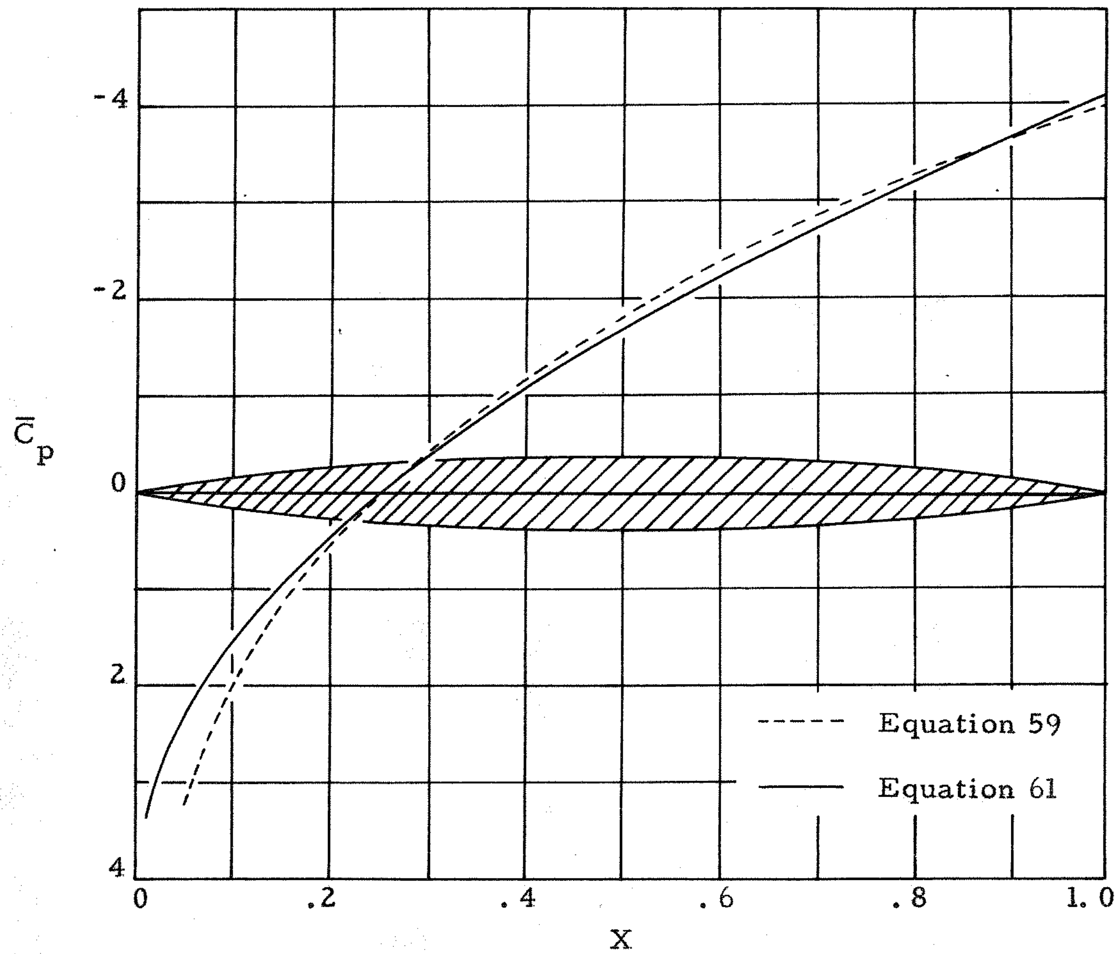

Figure 7. Pressure Distribution on Parabolic Arc Airfoil at $M_\infty = 1$.

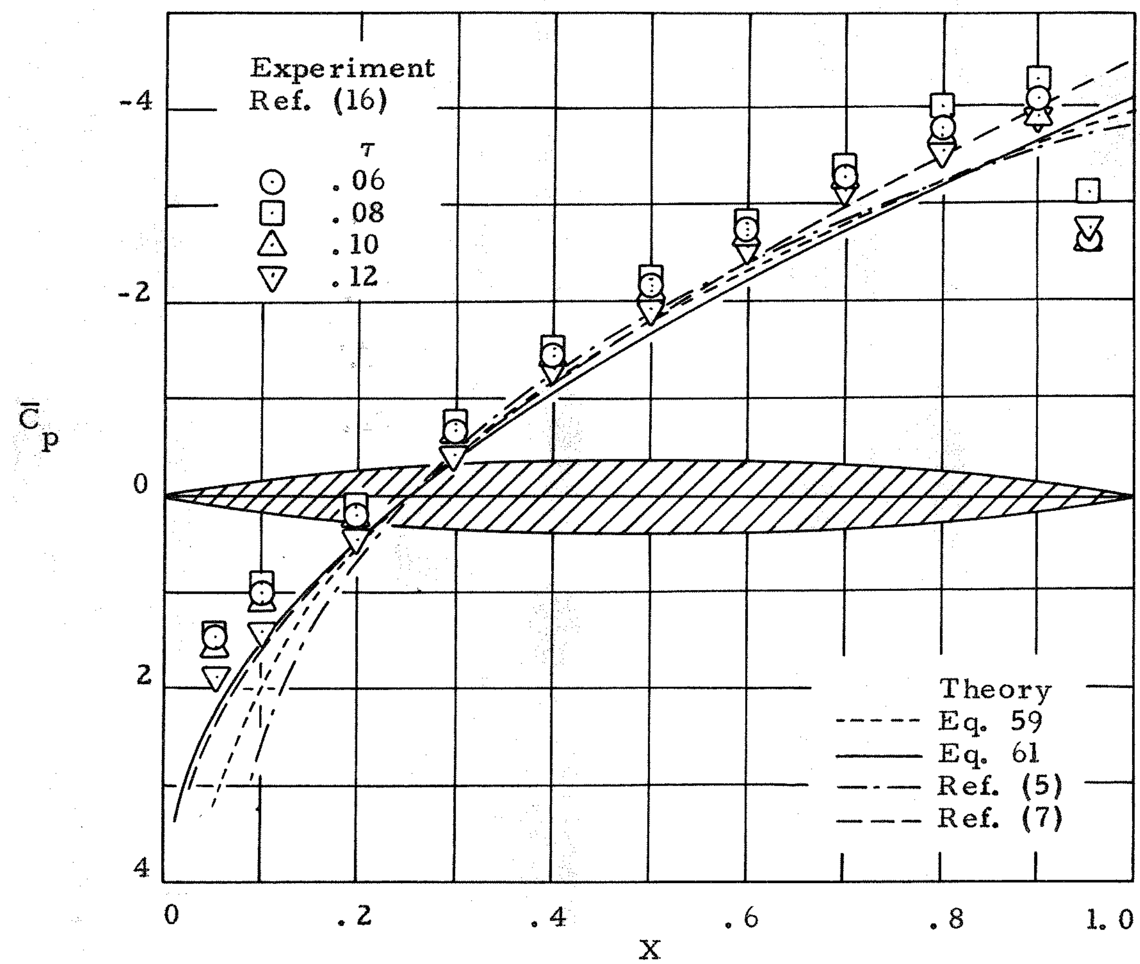

Figure 8. Comparison of Theoretical and Experimental Results for Parabolic Arc Airfoil at $M_\infty = 1$.

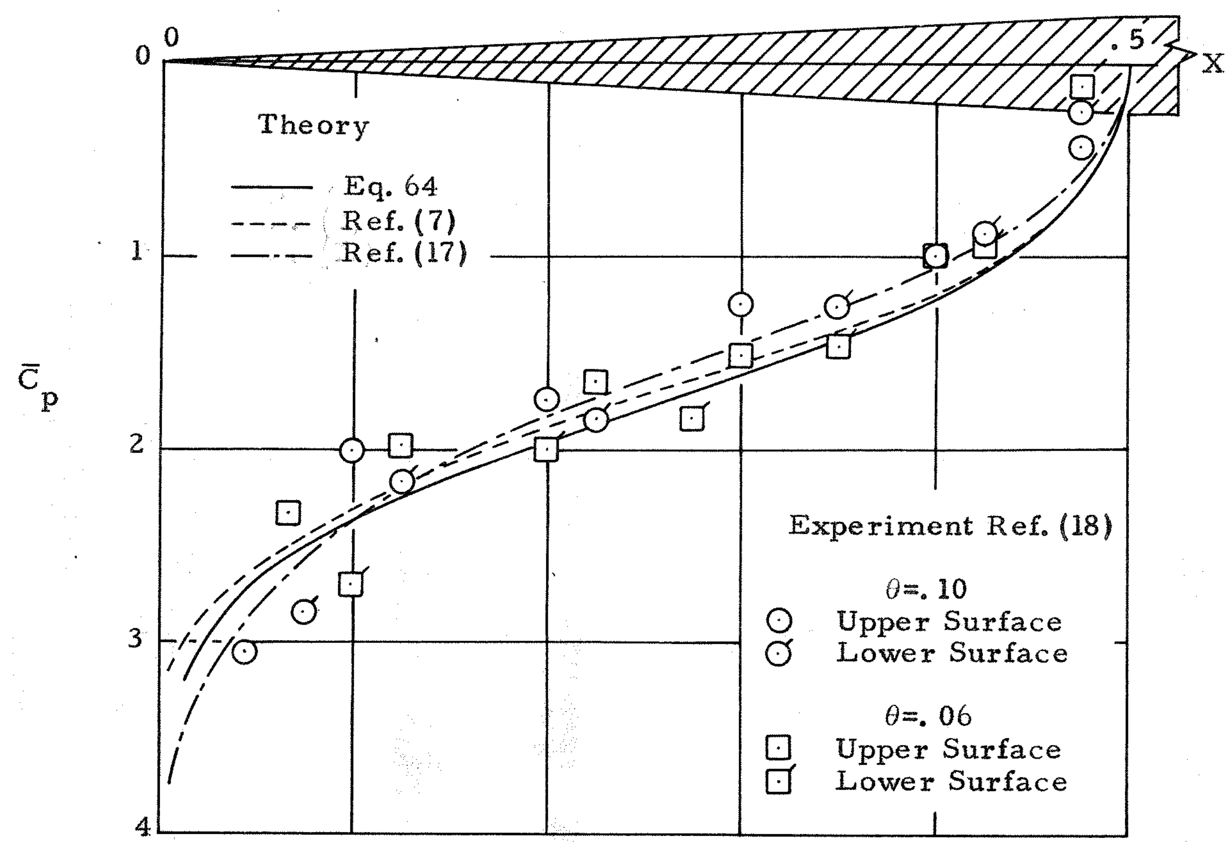

Figure 9. Comparison of Results for Wedge at $M_\infty = 1$.

VI. NOTATION

Some of the symbols have been used in several different contexts. In this case, the page is given on which a particular meaning is first used or best illustrated.

A	constant of integration
a	streamwise acceleration constant as used by Oswatitsch p. 2 as used in the present case p. 6
B	constant of integration
C	constant of integration
C_D	drag coefficient based on freestream conditions
$\bar{C}_D$	$\frac{[M_\infty^2(\gamma+1)]^{1/3}}{\tau^{5/3}} C_D$
C_p	pressure coefficient based on freestream conditions
$\bar{C}_p$	$\frac{[M_\infty^2(\gamma+1)]^{1/3}}{\tau^{2/3}} C_p$
D	constant of integration
E	complete elliptic integral of the second kind
e	2.71828...
F	function p. 3
$\vec{F}$	vector
f	arbitrary function
f_p	$(M_\infty^2-1)\phi_{xx}$
G_n	function defined by Eq. A.1
g	arbitrary function
H	Heaviside unit function

H_n	function defined on p. 34
I_p	modified Bessel's function of the first kind
J_p	Bessel's function of the first kind
$\mathbb{K}$	solution of the linear equation for a unit mathematical source
$\tilde{\mathbb{K}}$	infinite Fourier cosine transform of $\mathbb{K}$
$\mathbb{K}^*$	solution of the adjoint equation for a unit mathematical source
K_o	modified Bessel's function of the second kind
K	streamwise acceleration constant of Maeder and Thommen p. 2
K	complete elliptic integral of the first kind p. 18
k	variable (always defined as the argument of K or E)
L	operator $-\alpha^2[z(\ )_z]_z + (\ )_{yy}$
L^*	adjoint operator to L
M	local mach number
M_∞	freestream mach number
m	summation index (0, 1, 2, ...) p. 35
m	variable p. 30
n	summation index p. 31
n	arbitrary real constant p. 32
$\vec{n}$	unit outward normal vector to Σ
n_z, n_y	components of $\vec{n}$
$P_{-1/2}$	Legendre function of the first kind
$Q_{-\frac{1}{2}}$	Legendre function of the second kind
R	region in the (z,y) plane

r	radial coordinate in axisymmetric flow
$S(z)$	airfoil shape in (z, y) coordinate system
$S^{n}(0)$	$\left[\frac{d^n}{dz^n} S(z) \right]_{z=0}$
$T(x)$	airfoil shape in (x, y) coordinate system
u	z (or x) - component of the dimensionless perturbation velocity
v	y - component of the dimensionless perturbation velocity
W	Wronskian
x	dimensionless coordinate oriented along the free stream velocity vector ($x = 0$ at airfoil leading edge, $x = 1$ at trailing edge)
x	variable p. 31
$\bar{x}$	sonic line in (x, y) coordinate system
Y_0	Bessel's function of the second kind
y	dimensionless coordinate normal to x-axis
y_1	$y - \eta$
z	$x - \bar{x}$
γ	ratio of specific heats
δ	Dirac delta function p. 11
δ	arbitrary small positive number p. 21
ε	arbitrary small positive number
ζ	z - coordinate of source point
η	y - coordinate of source point

λ_p $(\gamma+1) M_\infty^2 \phi_{xx}$

ν $\dfrac{z+\zeta-\frac{a^2}{4}(y-\eta)^2}{2\sqrt{|z|}\sqrt{|\zeta|}}$

ν_0 $\nu(\eta=0)$

ξ variable

$\left|\frac{\zeta}{z}\right|$ p. 30

$\frac{\bar{x}}{z}$ p. 36

π 3.14159...

Σ boundary of R

τ airfoil thickness ratio

ϕ perturbation velocity potential

ϕ_1 fluid source potential

ϕ_2 potential corresponding to $\mathbb{K}$

ω infinite Fourier transformation variable p. 12

ω $\sqrt{\frac{\zeta}{z}}$ p. 30

∇ operator $\vec{i}\frac{\partial}{\partial z} + \vec{j}\frac{\partial}{\partial y}$

Subscripts

$(\ )_\pm$ refers to down- and up-stream running characteristics

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