SOME INVESTIGATIONS OF HETEROGENEOUS FLOW IN A ROCKET NOZZLE

Thesis by

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ABSTRACT

In the heterogeneous flow of finely divided particles and gas in a rocket nozzle, the assumptions that the flow enters the nozzle with no lags, that the radiative transport of energy between particles is negligible, and that there are no temperature gradients within the particles, are commonly made. In this paper, each of these assumptions is investigated in detail, and they are shown to be reasonable for typical rocket nozzles.

I. INTRODUCTION

In recent years, it has become common to add metals to the propellant in rocket engines to increase the chamber temperature. Unfortunately, most of these additives form some sort of a solid particle in the exhaust, and consequently can cause a depreciation in the thrust and specific impulse from the values based on gaseous products. That is, instead of an expected 20 per cent increase in impulse, say, an increase of 12 per cent is noted.

A very common mixture is a propellant of ammonium perchlorate oxidizer and a polysulfide fuel with aluminum added. For the numerical work in this thesis, this propellant is assumed with pure aluminum added to it.

The amount of metal that can be added is limited, since there is a requirement for the fuel to act as a binder for the oxidizer and metal. Usually, the designer will add as much metal as he can; thus a method is needed whereby he can correlate test engine performance with the predicted performance.

Such a method is presented by Rannie in reference 1. In reference 1, Rannie used a one-dimensional hydraulic approximation, which is known to be valid for homogeneous flow, for the heterogeneous flow of gas and solid particles. In addition, he assumed (i) that the flow entered the convergent section of the nozzle with no lag between the gas and solid particles; (ii) that the radiative transport of energy between particles was negligible; and (iii) that the thermal conductivity of the solid particles was sufficiently high compared to that of the gas that there were no temperature gradients within the particles.

Assumption (i) is investigated in Part II by solving the equations of motion in a cylindrical combustion chamber and determining the velocity lags at the entrance to the nozzle. Axially symmetric flow is assumed in the chamber, and the mixture is approximated by a heterogeneous mixture of gas and solid spherical particles. A constant source distribution of gas and particles is assumed along the cylindrical wall, and viscous effects in the main flow are neglected. Turbulence and temperature gradients in the chamber are also neglected. It is shown that the particle lag at the entrance to the nozzle is indeed very small compared to the gas velocity, and thus the assumption that the lag is zero is a valid one.

The magnitude of the effect of radiative transport of energy between particles is estimated by adding a radiation term to the equation for the heat balance of a particle and investigating the size of this additional term. The ratio of the absorption cross section to the geometric cross section is a necessary parameter in this term, and a value is not available for aluminum oxide particles of these small sizes and high temperatures. However, for any reasonable value of this parameter, the radiative transport of energy is shown to be negligible compared to the heat transfer by convection and conduction.

The temperature distribution within a spherical particle is found for unsteady heat flux over the surface of the particle in terms of the Laplace transform. By expansion of the transform, the solution for uniform temperature within the particle is recovered and the magnitude of the correction term is determined. The correction term is shown to be negligible for practical cases.

II. TWO-DIMENSIONAL ANALYSIS OF THE FLOW IN A COMBUSTION CHAMBER

Following the notation of reference 1, let α be the mass fraction of particles in the heterogeneous mixture, ρ_g the gas density, and ρ_s the density of the solid particles. Then the density of the mixture is given by

$$\frac{1}{\rho} = \frac{1-\alpha}{\rho_g} + \frac{\alpha}{\rho_s} \qquad (1)$$

Let A be the cross-sectional area of the cylindrical combustion chamber and $\dot{m}$ the constant mass flow rate of the mixture. Applying the condition that the net flow rate into any volume must be zero, the continuity equationsfor the solids and gases separately are:

$$\frac{\partial}{\partial x}\left[(1-\alpha)\rho U_g\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[(1-\alpha)\rho V_g r\right] = 0 \qquad (2)$$

$$\frac{\partial}{\partial x}\left[\alpha \rho U_s\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[\alpha \rho V_s r\right] = 0 \qquad (3)$$

where U_g and U_s are the axial velocity components of the gas and solids respectively, and V_g and V_s are the radial velocity components of the gas and solids.

By considering the momentum transport through a small element and neglecting friction forces, the momentum equation in the axial direction can be written

$$\frac{\partial}{\partial x}\left[(1-\alpha)\rho U_g^2\right] r + \frac{\partial}{\partial r}\left[(1-\alpha)\rho U_g V_g r\right] + \frac{\partial}{\partial x}\left[\alpha \rho U_s^2\right] r \qquad (4)$$

$$+ \frac{\partial}{\partial r}\left[\alpha \rho U_s V_s r\right] = -\frac{\partial p}{\partial x} r$$

and the radial momentum equation is

$$\frac{\partial}{\partial x}\left[\alpha \rho U_s V_s\right] r + \frac{\partial}{\partial r}\left[\alpha \rho V_s^2 r\right] + \frac{\partial}{\partial x}\left[(1-\alpha)\rho U_g V_g\right] r + \frac{\partial}{\partial r}\left[(1-\alpha)\rho V_g^2 r\right] = -\frac{\partial(pr)}{\partial r} + p \tag{5}$$

Approximating the particles by spheres of radius a , and assuming that a modified Stokes drag law holds with a correction factor f_d to allow for deviations from Stokes' flow, the force balance in the axial direction on a particle is

$$\frac{4}{3}\pi a^3 \rho_s\left[U_s\frac{\partial U_s}{\partial x} + V_s\frac{\partial U_s}{\partial r}\right] = 6\pi a\mu\left(U_g - U_s\right)\frac{1}{f_d} - \frac{4}{3}\pi a^3\frac{\partial p}{\partial x} \tag{6}$$

The steady state drag law is assumed valid. μ is the viscosity of the gas and is constant since the temperature is assumed constant, and the force due to the pressure gradient is added. The coefficient f_d is a function of the Reynolds number and the Mach number of the gas relative to the particle, and is equal to 1.0 in the Stokes flow regime.

A similar equation for the radial force on a particle is

$$\frac{4}{3}\pi a^3 \rho_s\left[U_s\frac{\partial V_s}{\partial x} + V_s\frac{\partial V_s}{\partial r}\right] = 6\pi a\mu\left(V_g - V_s\right)\frac{1}{f_d} - \frac{4}{3}\pi a^3\frac{\partial p}{\partial r} \tag{7}$$

Equations 6 and 7 are expressed more conveniently in the following form:

$$U_g - U_s = \frac{2}{9}\frac{a^2\rho_s f_d}{\mu}\left\{U_s\frac{\partial U_s}{\partial x} + V_s\frac{\partial U_s}{\partial r} + \frac{1}{\rho_s}\frac{\partial p}{\partial x}\right\} \tag{8a}$$

and

$$V_g - V_s = \frac{2}{9}\frac{a^2 \rho_s f_d}{\mu}\left\{U_s \frac{\partial V_s}{\partial x} + V_s \frac{\partial V_s}{\partial r} + \frac{1}{\rho_s}\frac{\partial p}{\partial r}\right\} \tag{8b}$$

Letting $U_s = U_g + (U_s - U_g)$ and $V_s = V_g + (V_s - V_g$ in equation 3 and subtracting it from equation 2

$$\frac{\partial}{\partial x}\left[\rho U_g\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[\rho V_g r\right] - \frac{\partial}{\partial x}\left[\alpha \rho (U_g - U_s)\right] - \frac{1}{r}\frac{\partial}{\partial r}\left[\alpha \rho (V_g - V_s) r\right] = 0 \tag{9}$$

Introducing the following non-dimensional variables $\phi = \frac{U}{V_o}$, $\psi = \frac{V}{V_o}$, $\xi = \frac{x}{b}$, $\eta = \frac{r}{b}$, where V_o is the radial velocity component of the gas at $\eta = 1$ just outside of the boundary layer and b is the radius of the burning surface. To simplify the equations, also let

$$\epsilon = \frac{2}{9}\frac{a^2 \rho_s V_o f_d}{\mu b}$$

Introducing these in the above equations, we obtain for the combined continuity equation

$$\frac{\partial}{\partial \xi}\left[\rho \phi_g\right] + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left[\rho \psi_g \eta\right] - \frac{\partial}{\partial \xi}\left[\alpha \rho (\phi_g - \phi_s)\right] - \frac{1}{\eta}\frac{\partial}{\partial \eta}\left[\alpha \rho (\psi_g - \psi_s)\eta\right] = 0 \tag{10}$$

and for the momentum equations

$$\frac{\partial}{\partial \xi}\left[(1-\alpha)\rho \phi_g^2\right]\eta + \frac{\partial}{\partial \eta}\left[(1-\alpha)\rho \phi_g \psi_g \eta\right] + \frac{\partial}{\partial \xi}\left[\alpha \rho \phi_s^2\right]\eta + \frac{\partial}{\partial \eta}\left[\alpha \rho \phi_s \psi_s \eta\right] = -\frac{\eta}{V_o^2}\frac{\partial p}{\partial \xi} \tag{11}$$

$$\frac{\partial}{\partial \xi}\left[\alpha \rho \phi_s \psi_s\right]\eta + \frac{\partial}{\partial \eta}\left[\alpha \rho \psi_s^2 \eta\right] + \frac{\partial}{\partial \xi}\left[(1-\alpha)\rho \phi_g \psi_g\right]\eta + \frac{\partial}{\partial \eta}\left[(1-\alpha)\rho \psi_g^2 \eta\right] = \frac{-1}{V_o^2}\frac{\partial(p\eta)}{\partial \eta} + \frac{p}{V_o^2} \qquad (12)$$

and for the force balance equations,

$$\phi_g - \phi_s = \epsilon\left\{\phi_s \frac{\partial \phi_s}{\partial \xi} + \psi_s \frac{\partial \phi_s}{\partial \eta} + \frac{1}{\rho_s V_o^2}\frac{\partial p}{\partial \xi}\right\} \qquad (13)$$

$$\psi_g - \psi_s = \epsilon\left\{\phi_s \frac{\partial \psi_s}{\partial \xi} + \psi_s \frac{\partial \psi_s}{\partial \eta} + \frac{1}{\rho_s V_o^2}\frac{\partial p}{\partial \eta}\right\} \qquad (14)$$

With the help of equations 2 and 3, equations 11 and 12 can be further simplified by rearranging so the dependent variables are ϕ_g, ψ_g, $\phi_s - \phi_g$, and $\psi_s - \psi_g$. Thus equations 11 and 12 become

$$\rho \phi_g \frac{\partial \phi_g}{\partial \xi} + \rho \psi_g \frac{\partial \phi_g}{\partial \eta} + \alpha\rho\left\{\phi_g \frac{\partial(\phi_s-\phi_g)}{\partial \xi} + (\phi_s-\phi_g)\frac{\partial \phi_g}{\partial \xi} + (\phi_s-\phi_g)\frac{\partial(\phi_s-\phi_g)}{\partial \xi} + \psi_g \frac{\partial(\phi_s-\phi_g)}{\partial \eta} + (\psi_s-\psi_g)\frac{\partial \phi_g}{\partial \eta} + (\psi_s-\psi_g)\frac{\partial(\phi_s-\phi_g)}{\partial \eta}\right\} = \frac{-1}{V_o^2}\frac{\partial p}{\partial \xi} \qquad (15)$$

$$\rho \phi_g \frac{\partial \psi_g}{\partial \xi} + \rho \psi_g \frac{\partial \psi_g}{\partial \eta} + \alpha\rho\left\{\phi_g \frac{\partial(\psi_s-\psi_g)}{\partial \xi} + (\phi_s-\phi_g)\frac{\partial \psi_g}{\partial \xi} + (\phi_s-\phi_g)\frac{\partial(\psi_s-\psi_g)}{\partial \xi} + \psi_g \frac{\partial(\psi_s-\psi_g)}{\partial \eta} + (\psi_s-\psi_g)\frac{\partial \psi_g}{\partial \eta} + (\psi_s-\psi_g)\frac{\partial(\psi_s-\psi_g)}{\partial \eta}\right\} = \frac{-1}{V_o^2}\frac{\partial p}{\partial \eta} \qquad (16)$$

First Approximation, Zero Velocity Lags.

When the velocity of the particles approaches that of the gas, (i.e., as $a \rightarrow 0$), $\epsilon \rightarrow 0$ and $\phi_s \rightarrow \phi_g$ and $\psi_s \rightarrow \psi_g$.

Then

$$\phi_s = \phi_g = \phi_o \qquad \text{and} \qquad \psi_s = \psi_g = \psi_o \tag{17}$$

where the subscript zero indicates no lag.

Then equation 10 reduces to

$$\frac{\partial \phi_o}{\partial \xi} + \frac{\partial \psi_o}{\partial \eta} + \frac{\psi_o}{\eta} = 0 \tag{18}$$

and equations 15 and 16 reduce to

$$\rho_o \left[\phi_o \frac{\partial \phi_o}{\partial \xi} + \psi_o \frac{\partial \phi_o}{\partial \eta} \right] = -\frac{1}{V_o^2} \frac{\partial p}{\partial \xi} \tag{19}$$

$$\rho_o \left[\phi_o \frac{\partial \psi_o}{\partial \xi} + \psi_o \frac{\partial \psi_o}{\partial \eta} \right] = -\frac{1}{V_o^2} \frac{\partial p}{\partial \eta} \tag{20}$$

It is a simple matter to verify that the solution to these equations, satisfying the boundary conditions $\phi_o = 0$ at $\xi = 0$ and $\psi_o = -1$ at $\eta = 1$, is

$$\begin{aligned} \phi_o &= 2\xi \\ \psi_o &= -\eta \\ \frac{p}{\rho_o V_o^2} &= -2\xi^2 - \frac{1}{2}\eta^2 + \text{const.} \end{aligned} \tag{21}$$

<u>Second Approximation, Small Velocity Lags.</u>

We shall assume for the solution that the variable may be expanded in powers of ϵ ; thus

$$\left.\begin{aligned}
\phi_g &= \phi_o + \epsilon\,\phi_{g_1} + \cdots\cdots \\
\phi_s &= \phi_o + \epsilon\,\phi_{s_1} + \cdots\cdots \\
\psi_g &= \psi_o + \epsilon\,\psi_{g_1} + \cdots\cdots \\
\psi_s &= \psi_o + \epsilon\,\psi_{s_1} + \cdots\cdots \\
\alpha &= \alpha_o + \epsilon\,\alpha_1 + \cdots\cdots \\
\rho &= \rho_o + \epsilon\,\rho_1 + \cdots\cdots
\end{aligned}\right\} \qquad (22)$$

The criterion for these expansions to be valid is that

$$\frac{\phi_g - \phi_o}{\phi_o} = \frac{\epsilon\,\phi_{g_1}}{\phi_o} << 1 \qquad (23)$$

and similar inequalities for the other variables. Here ϵ is merely an indicator of a small quantity. Substituting these expansions into equations 13 and 14, and equating coefficients of ϵ ,

$$\phi_{g_1} - \phi_{s_1} = \phi_o \frac{\partial \phi_o}{\partial \xi} + \psi_o \frac{\partial \phi_o}{\partial \eta} - \frac{\rho_o}{\rho_s} 4\xi$$

and

$$\psi_{g_1} - \psi_{s_1} = \phi_o \frac{\partial \psi_o}{\partial \xi} + \psi_o \frac{\partial \psi_o}{\partial \eta} - \frac{\rho_o}{\rho_s} \eta$$

The density of the mixture, ρ_o , is very much less than the density ρ_s of the solid material, so those terms can be neglected that contain ρ_o/ρ_s ; therefore

$$\phi_{g_1} - \phi_{s_1} = \phi_o \frac{\partial \phi_o}{\partial \xi} + \psi_o \frac{\partial \phi_o}{\partial \eta} \qquad (24)$$

and similarly

$$\psi_{g_1} - \psi_{s_1} = \phi_0 \frac{\partial \psi_0}{\partial \xi} + \psi_0 \frac{\partial \psi_0}{\partial \eta} \tag{25}$$

Now defining the first two terms of equation 10 as F , then

$$F = \frac{\partial}{\partial \xi}\left[\rho \phi_g\right] + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left[\rho \psi_g \eta\right] \tag{26}$$

Equation 2 can be written as

$$F - \frac{\partial}{\partial \xi}\left[\alpha \rho \phi_g\right] - \frac{1}{\eta}\frac{\partial}{\partial \eta}\left[\alpha \rho \psi_g \eta\right] = 0$$

Then differentiating this by parts,

$$F - \alpha\left\{\frac{\partial}{\partial \xi}\left[\rho \phi_g\right] - \frac{1}{\eta}\frac{\partial}{\partial \eta}\left[\rho \psi_g \eta\right]\right\} - \rho \phi_g \frac{\partial \alpha}{\partial \xi} - \rho \psi_g \frac{\partial \alpha}{\partial \eta} = 0$$

or

$$F(1-\alpha) - \rho \phi_g \frac{\partial \alpha}{\partial \xi} - \rho \psi_g \frac{\partial \alpha}{\partial \eta} = 0 \tag{27}$$

Also comparing equations 26 and 10 we see that

$$F = \frac{\partial}{\partial \xi}\left[\alpha \rho (\phi_g - \phi_s)\right] + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left[\alpha \rho (\psi_g - \psi_s)\eta\right] \tag{28}$$

From the first approximation, we may evaluate equations 24 and 25 as

$$\left.\begin{aligned} \phi_{g_1} - \phi_{s_1} &= 4\xi \\ \psi_{g_1} - \psi_{s_1} &= \eta \end{aligned}\right\} \tag{29}$$

Substituting these into equation 28, the solution of F is

$$F = 6\alpha_o P_o \tag{30}$$

where use was made of equation 22 and only carried to order epsilon.

Putting equation 30 back into equation 27 and incorporating equation 22, and rearranging,

$$2\xi \frac{\partial \alpha_1}{\partial \xi} - \eta \frac{\partial \alpha_1}{\partial \eta} = 6\alpha_o(1-\alpha_o) \tag{31}$$

Particular solutions for equation 31 are

$$\alpha_1 = 3\alpha_o(1-\alpha_o)\ln\xi + A \tag{32a}$$

$$\alpha_1 = -6\alpha_o(1-\alpha_o)\ln\eta + B \tag{32b}$$

The boundary condition for α_1 is chosen so the mass flow rates of gas and solids are the same as with no lag. Since

$$\left.\begin{aligned} \frac{\dot{m}_g}{A} &= (1-\alpha_o)P_o V_o(1+\epsilon\,\psi_{g_1}) \\ \frac{\dot{m}_s}{A} &= \alpha_o P_o V_o\left[1+\epsilon\left(\frac{\alpha_1}{\alpha_o}+\frac{P_1}{P_o}+\psi_{s_1}\right)\right] \end{aligned}\right\} \tag{33}$$

the conditions that $\dot{m}_g/A$ and $\dot{m}_s/A$ are the same as with no lag, are

$$\begin{aligned} \psi_{g_1} &= 0 \\ \frac{\alpha_1}{\alpha_o}+\frac{P_1}{P_o}+\psi_{s_1} &= 0 \end{aligned} \tag{34}$$

From equation 1, upon substitution of equation 22, and dropping terms of order ϵ^2,

$$\rho_1 = \frac{\rho_0 \alpha_1}{1-\alpha_0} \tag{35}$$

Substituting equations 35 and 29 into equation 34, we find that

$$\alpha_1 = -\alpha_0(1-\alpha_0) \qquad at \quad \eta = 1 \tag{36}$$

satisfies the condition that the last term of equation 34 vanish. Clearly the solution 32a is not applicable, and applying condition 36 to equation 32b,

$$\alpha_1 = -6\,\alpha_0(1-\alpha_0)\left[\ln\eta + \frac{1}{6}\right] \tag{37}$$

The boundary layer and friction forces along the wall are neglected and the mass fraction of particles to the mixture is assumed to be constant along the length of the chamber since the propellant is presumably uniform in mixture.

This solution does become infinite at the centerline, so the solution is valid for the whole chamber except on a line down the center of the chamber. This singularity does not cause any difficulty because the mass flow rate in the axial direction is finite, as will be shown below.

Now substitute equation 22 into equation 15, and put $p = p_0 + \epsilon p_1$, then equate the coefficients of ϵ, dropping known zero derivatives (i.e., $\partial\phi_0/\partial\eta$ and $\partial\psi_0/\partial\xi$); the result is

$$\rho_1 \phi_0 \frac{\partial\phi_0}{\partial\xi} + \rho_0 \phi_{g1} \frac{\partial\phi_0}{\partial\xi} + \rho_0\phi_0 \frac{\partial\phi_{g1}}{\partial\xi} + \rho_0 \psi_0 \frac{\partial\phi_{g1}}{\partial\eta}$$

$$-\phi_0\alpha_0\rho_0 \frac{\partial(\phi_{g1}-\phi_{s1})}{\partial\xi} - \alpha_0\rho_0(\phi_{g1}-\phi_{s1})\frac{\partial\phi_0}{\partial\xi} = -\frac{1}{V_0^2}\frac{\partial p_1}{\partial\xi}\,.$$

and upon substituting equations 21, 29, and 35,

$$\frac{4\alpha_1}{1-\alpha_0}\xi + 2\phi_{g_1} + 2\xi\frac{\partial \phi_{g_1}}{\partial \xi} - \eta\frac{\partial \phi_{g_1}}{\partial \eta} - 16\alpha_0\xi = -\frac{1}{\rho_0 V_0^2}\frac{\partial p_1}{\partial \xi}$$

Combining the second and third terms and putting in the value (equation 37) of α_1, we obtain

$$-24\alpha_0\xi\left(\ln\eta + \frac{1}{6}\right) + 2\frac{\partial}{\partial\xi}(\xi\phi_{g_1}) - \eta\frac{\partial \phi_{g_1}}{\partial\eta} - 16\alpha_0\xi = -\frac{1}{\rho_0 V_0^2}\frac{\partial p_1}{\partial\xi} \tag{38}$$

Similarly, equation 16 becomes

$$-6\alpha_0\eta\left(\ln\eta + \frac{1}{6}\right) - \frac{\partial}{\partial\eta}(\eta\psi_{g_1}) + 2\xi\frac{\partial \psi_{g_1}}{\partial\xi} + 2\alpha_0\eta = -\frac{1}{\rho_0 V_0^2}\frac{\partial p_1}{\partial\eta} \tag{39}$$

Let us assume a solution of equation 39 to be

$$\psi_{g_1} = -3\alpha_0\eta\ln\eta \tag{40}$$

Substituting equation 35 into the continuity equation for the gas (equation 2) leads to

$$\frac{\partial \phi_{g_1}}{\partial\xi} = -\frac{1}{\eta}\frac{\partial(\eta\,\psi_{g_1})}{\partial\eta} \tag{41}$$

Upon substitution of ψ_{g_1} into equation 41 and integration,

$$\phi_{g_1} = 6\alpha_0\xi\ln\eta + 3\alpha_0\xi \tag{42}$$

where the constant of integration disappears since ϕ_{g_1} must be zero at $\xi = 0$.

From equation 29 we find

$$\left.\begin{aligned} \phi_{s_1} &= 6\alpha_0\xi\ln\eta + 3\alpha_0\xi - 4\xi \\ \text{and}\quad \psi_{s_1} &= -3\alpha_0\eta\ln\eta - \eta \end{aligned}\right\} \tag{43}$$

To verify the solutions, substitute the values for ψ_{g_1}, ϕ_{g_1}, ψ_{s_1}, and ϕ_{s_1} (equations 40, 42, and 43) into the continuity equations (equations 2 and 3). The mass flow rate of gas for a cylindrical chamber of radius b, with no lags, is

$$\mathring{m}_g = \pi b^2 (1-\alpha_0) \rho_0 V_0 \phi_0 \tag{44}$$

and with velocity lags, is

$$\mathring{m}_{g_\ell} = 2\pi b^2 (1-\alpha_0) \rho_0 V_0 \phi_0 \int_0^1 \left(1 + \epsilon \frac{\phi_{g1}}{\phi_0}\right) \eta \, d\eta \tag{45}$$

Substituting from equation 42 and integrating, equation 45 becomes

$$\mathring{m}_{g_\ell} = \mathring{m}_g \tag{46}$$

The mass flow rate of the particles without a velocity lag is

$$\mathring{m}_s = \pi b^2 \alpha_0 \rho_0 V_0 \phi_0 \tag{47}$$

and with a velocity lag,

$$\mathring{m}_{s_\ell} = 2\pi b^2 \alpha_0 \rho_0 V_0 \phi_0 \int_0^1 \left[1 + \epsilon\left(\frac{\alpha_1}{\alpha_0} + \frac{\rho_1}{\rho_0} + \frac{\phi_{s1}}{\phi_0}\right)\right] \eta \, d\eta \tag{48}$$

Evaluating the integral

$$\mathring{m}_{s_\ell} = \mathring{m}_s \tag{49}$$

showing that the mass flow rate of the particles does not change when velocity lags are introduced. Hence, the assumed solution and constants are consistent with mass conservation.

To obtain some numerical results, a specific chamber must be chosen. The following values are consistent with the numerical example given in reference 1.

$b = 0.366$ ft $\qquad a = 2.5 \times 10^{-4}$ cm.

$\mu = 1.5 \times 10^{-6}$ lb-sec/ft^2 $\qquad r_b = 0.5$ in/sec

$\rho_s = 210$ lb/ft^3 $\qquad p_c = 1000$ lb/in^2

$T_c = 6000$ °R $\qquad \alpha_o = 0.4$

The chamber length is chosen as twice the diameter, and Stokes flow will be assumed $(f_d = 1.0)$. Then

$V_o = -18.7$ ft/sec $\qquad \Phi_o = 74.8$ ft/sec

$\epsilon = 3.43 \times 10^{-3}$ $\qquad \Psi_o = -18.7$ ft/sec.

The magnitude of the velocity lags in relation to the no-lag flow velocity is obtained from equations 29 and 21

$$\frac{\phi_g - \phi_s}{\phi_o} = 2\epsilon \tag{50}$$

which verifies that the lags are indeed small.

The uniformity of the flow across the exit plane was assumed in the analysis in reference 1, and Table 1 is presented as a verification of this assumption. Values of $\epsilon\, \phi_{g_1}/\phi_o$, $\epsilon\, \phi_{s_1}/\phi_o$, and $\epsilon \alpha_1/\alpha_o$ were calculated as functions of η, the non-dimensional radius.

The Reynolds number of the gas relative to the particles was computed at the exit plane using the particle diameter as the characteristic length and found to be 6.5×10^{-2}. The Mach number was also found to be 1.36×10^{-4}. Combining these two values, a value for M/Re of 2.09×10^{-3} is found, which indicates that the particles are in the Stokes flow regime, and thus the correction factor f_d is

TABLE 1.

η	$\epsilon \frac{\phi_{s1}}{\phi_o}$	$\epsilon \frac{\phi_{g1}}{\phi_o}$	$\epsilon \frac{\alpha_1}{\alpha_o}$
.1	-7.35×10^{-4}	-3.69×10^{-4}	2.63×10^{-2}
.2	-6.10×10^{-4}	-2.43×10^{-4}	1.78×10^{-2}
.3	-4.75×10^{-4}	-1.55×10^{-4}	1.28×10^{-2}
.4	-4.45×10^{-4}	-9.15×10^{-5}	9.25×10^{-3}
.5	-4.07×10^{-4}	-4.21×10^{-5}	6.49×10^{-3}
.6	-3.69×10^{-4}	-2.29×10^{-6}	4.24×10^{-3}
.7	-3.35×10^{-4}	$+3.14 \times 10^{-5}$	2.35×10^{-3}
.8	-3.05×10^{-4}	6.15×10^{-5}	6.95×10^{-4}
.9	-2.81×10^{-4}	8.41×10^{-5}	-7.58×10^{-4}
1.0	-2.55×10^{-4}	1.10×10^{-4}	-2.06×10^{-4}

approximately equal to one, as taken above. Reference 1 gives a discussion on the calculation of the parameter f_d . Using equation 66 and equation 67 of reference 1,

$$f_d = 1.0 + 4.58 \frac{M}{R_e} = 1.0096 \tag{51}$$

Thus, from the above analysis, the assumption made in reference 1, i.e., that at the entrance to the nozzle the flow was uniform and without lags, is a reasonable assumption.

III. RADIATION BETWEEN PARTICLES

For a one-dimensional steady flow of a homogeneous fluid without heat addition, the energy equation, in terms of internal energy per unit mass, is

$$\rho A u \left(e + \frac{u^2}{2} \right) + A p u = \text{const.} \tag{52}$$

and for the heterogeneous mixture of gas and particles is

$$(1-\alpha)\rho A u_g \left(e_g + \frac{u_g^2}{2}\right) + \alpha \rho A u_s \left(e_s + \frac{u_s^2}{2}\right) + A p \left(u_g \frac{(1-\alpha)\rho}{\rho_g} + u_s \frac{\alpha \rho}{\rho_s} \right) = \text{const.} \tag{53}$$

where α is the mass fraction of particles (subscript s). This equation is general and independent of force interactions and heat transfer processes between the particles and the gas.

If radiative transfer of heat between particles is allowed, an additional term must be added on the left hand side of the equation, since the equation does not take care of interactions and heat transfer processes between particles. The energy transfer between particles due to radiation is

$$\dot{q}_r = -k_r \frac{dT_s}{dx} A \tag{54}$$

where

$$k_r = \frac{16}{3} \sigma_R \ell_R T_s^3 \tag{55}$$

may be regarded as an effective conductivity due to radiant energy transfer. The coefficient σ_R is the Stefan-Boltzmann constant, and ℓ_R is the Rosseland mean free path. Then $\dot{q}_r = -k_r \frac{dT_s}{dx} A$ must be added to the left hand side of equation 53 to account for radiative transfer.

Equations 54 and 55 are applicable only when l_R is very small compared to the length scale appropriate to the problem, for instance the nozzle throat radius, at all important wavelengths of radiation.

The variable mass fraction α in equation 53 can be eliminated with

$$\alpha u_s = (1-\alpha) u_g \frac{\alpha_o}{1-\alpha_o} \tag{56}$$

obtained from equations 2 and 3. In cases of interest, the volume occupied by the particles is negligible when compared to the volume of the gas, and the assumption

$$\rho_g \simeq (1-\alpha)\rho \tag{57}$$

is valid for a particle material density of 210 lb/ft^3 as we have chosen. Since the gas density will seldom be greater than 0.4 lb/ft^3, this approximation is valid over a wide range, but should be checked for any particular problem.

Substituting equations 56 and 57 into equation 53, the energy equation becomes

$$(1-\alpha_o) e_g + \alpha_o e_s + \frac{1}{2} u_g^2 - \frac{1}{2}\alpha_o (u_g^2 - u_s^2) + \frac{(1-\alpha_o)}{\rho_g} p + \frac{\alpha_o p}{\rho_s} - \frac{k_r}{\rho_g u_g}(1-\alpha_o)\frac{dT_s}{dx} = h_c \tag{58}$$

where h_c is the enthalpy in the chamber.

For a perfect gas with constant specific heats

$$e_g + \frac{p}{\rho_g} = C_v T_g + R T_g = C_p T_g \tag{59}$$

and we now introduce a specific heat, C_{p_o}, for the mixture with a particle mass fraction α_o,

$$C_{p_o} = (1-\alpha_o)C_p + \alpha_o C \tag{60}$$

and modify the gas constant to

$$R_o = (1-\alpha_o)R \tag{61}$$

The right hand side of equation 58 is $C_{p_o}T_c$ where T_c is the stagnation temperature of the mixture. Then equation 58 may be rewritten as

$$(1-\alpha_o)C_pT_g + \alpha_o C T_c + \frac{1}{2}(1-\alpha_o)U_g^2 + \frac{1}{2}\alpha_o U_s^2 - \frac{k_r}{\rho_g U_g}(1-\alpha_o)\frac{dT_s}{dx} = C_{p_o}T_c \tag{62}$$

where equations 56 and 59 were used.

Now writing T_s as $T_g + T_s - T_g$ and U_s^2 as $U_g^2 + U_s^2 - U_g^2$ and combining the terms of equation 62,

$$C_{p_o}T_g + \frac{U_g^2}{2} + \alpha_o C(T_s - T_g) + \frac{\alpha_o}{2}(U_s^2 - U_g^2) - \frac{k_r}{\rho_g U_g}(1-\alpha_o)\frac{dT_s}{dx} = C_{p_o}T_c \tag{63}$$

Neglecting radiation, the heat balance for a sphere of radius a and temperature T_s is

$$\frac{4}{3}\pi a^3 \rho_s C U_s \frac{dT_s}{dx} = -\frac{k_g}{a}(T_s - T_g)4\pi a^2 \frac{1}{f_h} \tag{64}$$

where k_g is the conductivity of the gas surrounding the sphere and f_h is a correction factor that takes into account the influence of convection. For the Stokes regime of flow, $f_h = 1.0$, and it decreases at higher Reynolds numbers.

Rearranging equation 64,

$$T_g - T_s = \frac{1}{3} \frac{a^2 \rho_s C f_h}{k_g} u_s \frac{dT_s}{dX} \qquad (65)$$

To express equation 65 in dimensionless form, the following notation from reference 1 is introduced

$$\left.\begin{aligned} u &= \sqrt{R_o T_c}\ \phi \\ T &= T_c\ \tau \\ X &= r_T\ \xi \\ \epsilon &= \frac{2}{9} \frac{a^2 \rho_s \sqrt{R_o T_c}}{r_T \mu_c} \end{aligned}\right\} \qquad (66)$$

giving

$$\tau_g - \tau_s = \epsilon \beta f_h \frac{1}{\tau_g^{\,n}} \phi_s \frac{d\tau_s}{d\xi} \qquad (67)$$

where

$$\beta = \frac{3}{2} \frac{C}{C_p} P_r \qquad (68)$$

has a numerical value not much different from unity. If the particle radius is extremely small, ϵ approaches zero and since the other factors on the right hand side of equation 67 are of order unity, $\tau_g - \tau_s = 0$ and the temperature lag is zero. The velocity lag is also proportional to ϵ so it becomes zero as well. As a increases, ϵ becomes larger, and the principle of the perturbation procedure in reference 1 was to expand τ_g , τ_s , ϕ_g , and ϕ_s in power series in ϵ .

With radiative transfer, there is an additional term in the heat balance for the particle. Suppose a particular particle is located at $X = X_o$. The radiative transport of energy in the X - direction per unit area normal to X in the space surrounding the particle and close to the particle is

$$-\left[k_r \frac{dT_s}{dX}\right]_{X_o} - \left[\frac{d}{dX}\left(k_r \frac{dT_s}{dX}\right)\right]_{X_o}(X - X_o)$$

Hence the net radiative transport of energy to the sphere is

$$\int_0^{\pi} 2\pi a \sin\theta \, a \, d\theta \cos\theta \left[-\left(k_r \frac{dT_s}{dx}\right)_{X_o} - \left\{\frac{d}{dx}\left(k_r \frac{dT_s}{dx}\right)\right\}_{X_o}(a\cos\theta)\right]$$

$$= -\frac{4}{3}\pi a^3 \left\{\frac{d}{dx}\left(k_r \frac{dT_s}{dx}\right)\right\}_{X_o}$$

and the heat balance for the particle becomes

$$\frac{4}{3}\pi a^3 \rho_s C u_s \frac{dT_s}{dx} = -\frac{k_g}{a}(T_s - T_g) 4\pi a^2 \frac{1}{f_h} - \frac{4}{3}\pi a^3 \frac{d}{dx}\left(k_r \frac{dT_s}{dx}\right) \tag{69}$$

which replaces equation 64. Rearranging and introducing the dimensionless variables from equation 66

$$\tau_g - \tau_s = \epsilon\left[\beta f_h \frac{1}{\tau_g^{\,n}} \phi_s \frac{d\tau_s}{d\xi} + \frac{3}{2} P_r f_h \frac{1}{\tau_g^{\,n}} \frac{1}{\rho_s C_p \sqrt{R_o T_c}\, r_c} \frac{d}{d\xi}\left(K_r \frac{d\tau_s}{d\xi}\right)\right] \tag{70}$$

The radiative mean free path ℓ_R is given by

$$\ell_R = \frac{1}{n \pi a^2 Q} \tag{71}$$

where n is the number of particles per unit volume and Q is the ratio of absorption cross section to geometrical cross section. From the identity

$$n \frac{4}{3} \pi a^3 \rho_s = \frac{\alpha}{1-\alpha} \rho_g \tag{72}$$

and the equation of state for the gas

$$n \pi a^2 = \frac{3}{4} \frac{\alpha}{1-\alpha} \frac{p}{R T_g \rho_s a} = \frac{3}{4} \frac{p_c}{R T_c \rho_s a} \frac{\alpha}{1-\alpha} \zeta^{1/8} \tag{73}$$

Substituting into equation 71 above

$$l_R = \frac{4}{3} \frac{R T_c}{p_c} \rho_s a \frac{1}{Q} \frac{1-\alpha}{\alpha} \zeta^{-1/8} \tag{74}$$

and hence k_r from equation 55 becomes

$$k_r = \frac{64}{9} \frac{\sigma_R R T_c{}^4}{p_c} \rho_s a \frac{1}{Q} \frac{1-\alpha}{\alpha} \zeta^{-1/8} \tau_s^3 \tag{75}$$

Substituting for k_r into equation 70

$$\tau_g - \tau_s = \epsilon f_n \left[\beta \frac{1}{\tau_g^n} \phi_s \frac{d\tau_s}{d\xi} + \frac{32}{3} P_r \frac{\sigma_R R T_c{}^4 a}{p_c C_p \sqrt{R_o T_c}\, r_c} \frac{1}{Q} \frac{d}{d\xi} \left(\frac{1-\alpha}{\alpha} \zeta^{-1/8} \tau_s^3 \frac{d\tau_s}{d\xi} \right) \right] \tag{76}$$

where Q is assumed to have small variation with ξ and can be taken out of the differential. Since $C_p/R = \gamma/\gamma-1$, equation 76 can be written as

$$\tau_g - \tau_s = \epsilon f_h \left[\beta \frac{1}{\tau_g^n} \phi_s \frac{d\tau_s}{d\xi} + \frac{32}{3} \frac{\gamma-1}{\gamma} P_r f_r \frac{d}{d\xi} \left(\frac{1-\alpha}{\alpha} \zeta^{-1/8} \tau_s^3 \frac{d\tau_s}{d\xi} \right) \right] \tag{77}$$

where

$$f_r = \frac{\sigma T_c{}^4}{p_c \sqrt{R_o T_c}} \frac{a}{r_c} \frac{1}{Q}$$

Since the factors multiplying f_r in equation 77 can be considered to be of order unity, as is the first term in the square brackets, a value of f_r of order unity implies that radiative transport is as effective as the heat transfer by convection and conduction. In the example of reference 1, $f_r \cong 3\times 10^{-8}/Q$; hence, f_r is extremely small even if Q is quite small. Hence the influence of radiative transport on particle lags is negligible.

The zero lag solution is also affected by radiative transport of energy. The energy equation 63 becomes

$$\tau_o + \frac{1}{2}\frac{\gamma_o - 1}{\gamma_o}\phi_o^2 = 1 + \frac{k_r(1-\alpha_o)}{r_t \rho_g \sqrt{R_o T_c}\,\phi_o C_{p_o}}\frac{d\tau_o}{d\xi}$$

Upon substitution of equation 55, we find that the condition for radiation to have negligible effect on the zero lag solution is

$$\frac{\sigma_R T_c^4}{p_c\sqrt{R_o T_c}}\frac{l_R}{r_t} << 1 \qquad (78)$$

since the non-dimensional terms are all approximately of order one.

The concept of an effective conductivity has meaning only if $l_R/r_t << 1$. For chemical rockets $\sigma_R T_c^4/(p_c\sqrt{R_o T_c}) \cong 10^{-3}$. Hence the radiative term gives negligible contribution. The ratio l_R/r_t is

$$\frac{l_R}{r_t} \cong \frac{a}{r_t}\frac{\rho_s R T_c}{p_c}\frac{1-\alpha}{\alpha}\frac{1}{Q}$$

In the example $a/r_t \cong 3\times 10^{-5}$, $\frac{\rho_s R T_c}{p_c} \cong 5\times 10^2$, and $\frac{1-\alpha}{\alpha} = 1.5$. Even if Q is small, $l_R/r_t << 1$.

IV. EFFECT OF PARTICLE CONDUCTIVITY

The effect of finite particle conductivity is investigated by solving the heat conduction equation for the particle with an appropriate boundary condition, then expanding that solution to investigate the size of the correction term.

The heat conduction equation for a spherical, uniform, solid particle with spherically symmetric temperature distribution is

$$\frac{\partial T}{\partial t} = \sigma \nabla^2 T = \sigma \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) \qquad (79)$$

where σ is the diffusivity of the particle material.

The initial condition is

$$T = T_c \quad \text{at} \quad t = 0 \qquad (80)$$

and the boundary condition at the particle surface is

$$k_s \left(\frac{\partial T}{\partial r}\right)_{r=a-0} = k_g \left(\frac{\partial T}{\partial r}\right)_{r=a+0} \qquad (81)$$

where T_c is the chamber temperature and t is the time measured from the instant the particles enter the convergent portion of the nozzle. The conductivities of the solid and gas are k_s and k_g respectively.

For steady state, with spherical symmetry, the equation for heat conduction in the gas is

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = 0 \qquad (82)$$

which has the solution

$$T = \frac{A}{r} + B$$

At $r = \infty$, $T = T_g$, and thus $B = T_g$ and at $r = a$, $T = T_a$ which gives $A = a(T_a - T_g)$. Thus

$$\left(\frac{\partial T}{\partial r}\right)_{r=a+0} = \frac{1}{a}(T_g - T_a)$$

or

$$k_g\left(\frac{\partial T}{\partial r}\right)_{r=a+0} = \frac{k_g}{a}(T_g - T_a) \tag{83}$$

For quasi-steady heat flow in particle and gas, substitute from equation 83 into 81 to obtain

$$k_s\left(\frac{\partial T}{\partial r}\right)_{r=a-0} = \frac{k_g}{a}(T_g - T_a) \tag{84}$$

This will be used for the approximate boundary condition for the heat conduction equation of the particle. The solution to these equations (79 , 80 , 84) will be obtained by making use of the Laplace transform.

Define

$$p\int_0^\infty e^{-p\tau} F(\tau)\, d\tau = \mathcal{L}\{F(\tau)\} = \overline{F(p)} \tag{85}$$

and

$$r\overline{T} = \overline{\sigma} \tag{86}$$

Then equation 79 becomes

$$\bar{\Theta}'' - \frac{p}{\sigma}\bar{\Theta} + \frac{pr}{\sigma}T_c = 0 \qquad (87)$$

Since there is no source at $r=0$, $\frac{\partial T}{\partial r} = 0$ at $r=0$ and therefore $\bar{T}' = 0$ where $\bar{T}' = \partial \bar{T}/\partial r = -\bar{\Theta}/r^2 + \bar{\Theta}'/r$

which means that

$$r\bar{\Theta}' - \bar{\Theta} \longrightarrow 0 \quad \text{as} \quad r \longrightarrow 0 \qquad (88)$$

The solution of equation 87 is

$$\bar{\Theta} = Ae^{\sqrt{p/\sigma}\, r} + Be^{-\sqrt{p/\sigma}\, r} + T_c r \qquad (89)$$

Substituting this into equation 88

$$r\bar{\Theta}' - \bar{\Theta} = r\left\{\sqrt{\frac{p}{\sigma}}\left(Ae^{\sqrt{p/\sigma}\, r} - Be^{-\sqrt{p/\sigma}\, r}\right) + T_c\right\} - \left\{Ae^{\sqrt{p/\sigma}\, r} + Be^{-\sqrt{p/\sigma}\, r} + T_c r\right\} \qquad (90)$$

and to satisfy condition 88, $B = -A$, thus

$$\bar{T} = \frac{\bar{\Theta}}{r} = \frac{A}{r} 2\sinh qr + T_c \qquad (91)$$

where $q = \sqrt{p/\sigma}$.

Substituting this into the boundary condition (equation 84), we obtain for the left hand side

$$k_s\left(\frac{\partial \bar{T}}{\partial r}\right)_{r=a} = k_s\left\{2A\left[q\frac{\cosh qr}{r} - \frac{\sinh qr}{r^2}\right]\right\}_{r=a}$$

and for the right hand side

$$\mathcal{L}\left\{\frac{k_g}{a}(T_g - T_a)\right\} = \frac{k_g}{a}\left[\overline{T}_g - \frac{2A\sinh qa}{a} - T_c\right]$$

since $\overline{T} = \overline{T}_a$ at $r = a$. Combining these equations, equation 84 becomes

$$k_s\left\{2A\left[\frac{q\cosh qa}{a} - \frac{\sinh qa}{a^2}\right]\right\} = \frac{k_g}{a^2}\left[\overline{T}_g - \frac{2A\sinh qa}{a} - T_c\right] \tag{92}$$

and solving for A

$$A = a\frac{k_g(\overline{T}_g - T_c)}{k_s\left[qa\cosh qa - \sinh qa\left(1 - \frac{k_g}{k_s}\right)\right]} \tag{93}$$

Substituting equation 93 into 91 gives

$$\overline{T} = \frac{a k_g(\overline{T}_g - T_c)}{r\, k_s}\left[\frac{\sinh qr}{qa\cosh qa - \sinh qa\left(1 - \frac{k_g}{k_s}\right)}\right] + T_c \tag{94}$$

Let us now define an average sphere temperature T_s as

$$T_s = \frac{\int_0^a 4\pi r^2 T\,dr}{\frac{4}{3}\pi a^3}$$

and hence

$$\overline{T}_s = \frac{3}{a^3}\int_0^a r^2\overline{T}\,dr \tag{95}$$

Then $\overline{T_s}$ will be compared with the uniform sphere temperature in the heat balance equation for k_s infinite (reference 1).

$$\frac{4}{3}\pi a^3 \rho_s c\, u_s \frac{dT_s}{dx} = -\frac{k_g}{a}\left(T_s - T_g\right) 4\pi a^2 \frac{1}{f_h} \tag{96}$$

Also define:

$$B = a\frac{k_g}{k_s}\,\frac{\left(\overline{T_g} - T_c\right)}{ga\cosh ga - \sinh ga\left(1 - k_g/k_s\right)}$$

so that equation 94 may be written as

$$\overline{T} = B\,\frac{\sinh gr}{r} + T_c \tag{97}$$

Then substituting and evaluating equation 95

$$\overline{T_s} = \frac{3B}{ga^2}\left\{\cosh ga - \frac{\sinh ga}{ga}\right\} + T_c \tag{98}$$

Because large values of time t correspond to small values of g, expand the expression for $\overline{T_s}$ in a power series in g to obtain the form correct for small g, i.e.,

$$\left(\overline{T_s} - T_c\right) = \left\{\frac{1 + \frac{1}{10}(ga)^2 + \frac{1}{280}(ga)^4 + \frac{1}{15120}(ga)^6 + \cdots}{1 + \frac{1}{3}\left(\frac{1}{2} + \frac{k_s}{k_g}\right)(ga)^2 + \frac{1}{30}\left(\frac{1}{4} + \frac{k_s}{k_g}\right)(ga)^4 + \frac{1}{840}\left(\frac{1}{6} + \frac{k_s}{k_g}\right)(ga)^6 + \cdots}\right\}\left(\overline{T_g} - T_c\right) \tag{99}$$

Retaining only terms to order $(qa)^4$ in equation 99

$$\bar{T}_g - T_c = (\bar{T}_s - T_c)\left[1 + \frac{1}{3}\left(\frac{1}{2} + \frac{k_s}{k_g}\right)(qa)^2 + \frac{1}{30}\left(\frac{1}{4} + \frac{k_s}{k_g}\right)(qa)^4\right]\left[1 - \frac{1}{10}(qa)^2 - \frac{1}{280}(qa)^4\right]$$

which reduces to

$$\bar{T}_g - T_c = (\bar{T}_s - T_c)\left[1 + \frac{1}{3}\left(\frac{1}{5} + \frac{k_s}{k_g}\right)(qa)^2 - \frac{1}{84}(qa)^4\right] \tag{100}$$

Recalling that $q^2 = \frac{p}{\sigma} = \frac{p \rho_s C}{k_s}$ we find that for k_s equal to infinity, equation 100 reduces to

$$\bar{T}_g - T_c = \left(1 + \frac{1}{3}\frac{\rho_s C a^2}{k_g}\right)(\bar{T}_s - T_c) \tag{101}$$

and taking the inverse Laplace transformation of this leads to

$$\frac{k_g}{a}(T_g - T_s) = \frac{1}{3}\rho_s C a \frac{dT_s}{dt} \tag{102}$$

which is identical with the assumption used in the heat balance of a particle of infinite thermal conductivity, i.e., equation 96.

Reference 2 gives values for the thermal conductivity of aluminum oxide and its variation with temperature. Measured values are given for temperatures up to 1600°C, and one more point (1800°C) is presented from extrapolated data. The thermal conductivity decreases with an increase in temperature up to 1400°C, and then begins to increase again. The value given for 1400°C (the minimum value of k_s for this range) is .0125 $\frac{\text{cal}}{\text{cm}^2\ \text{sec}} \cdot \frac{\text{cm}}{{}^\circ\text{C}}$ or

$36.25 \frac{BTU}{ft^2 hr} \cdot \frac{in}{^oF}$.

To find a reasonable value for the thermal conductivity of the gas, the data from the numerical example in reference 1 was used, that is,

$$C_p = 0.500 \ BTU/lb \ ^oF ,$$

$$\mu_c = 1.5 \times 10^{-6} \ lb. sec/ft^2 ,$$

$$Pr = 0.74 .$$

To be most conservative in evaluating k_s/k_g , a value of μ was used that would give a high k_g . Since μ varies considerably with temperature and the Prandtl number is almost constant over a large temperature range, the thermal conductivity varies with the temperature in the same manner as μ . Over the temperature range of interest, a power law of the following form is satisfactory.

$$k_g = \frac{C_p}{P_r} \mu_c \left(\frac{T_g}{T_c} \right)^{ab} \tag{103}$$

The maximum k_g would be obtained when $T_g = T_c$, thus

$$k_{g\,max} = 1.412 \ \frac{BTU}{ft^2 hr} \cdot \frac{in}{^oF}$$

Thus the value of k_s/k_g in equation 100 will always be greater than 25.7 , and thus one may neglect 1/5 in comparison to k_s/k_g .

With this simplification in equation 100 and substituting $\rho_s C_p/k_s$ for q^2 ;

$$T_g - T_c = (\bar{T}_s - T_c)\left\{1 + \frac{1}{3}\frac{\rho_s C a^2}{k_g} p - \frac{1}{84}\frac{\rho_s^2 C^2 a^4}{k_s^2} p^2\right\} \tag{104}$$

Taking the inverse of the Laplace transform yields

$$\frac{k_g}{a}(T_g - T_s) = \frac{1}{3}\rho_s C a \frac{dT_s}{d\tau} - \frac{1}{84}\frac{k_g}{k_s^2}\rho_s^2 C^2 a^3 \frac{d^2 T_s}{d\tau^2} \tag{105}$$

Without the second term on the right, this equation is identical with equation 101, and the last term is a correction term. To investigate the size of the last term, rewrite equation 102 as

$$\frac{dT_s}{d\tau} = \frac{3 k_g}{\rho_s C a^2}(T_g - T_s) \tag{106}$$

and differentiate to obtain

$$\frac{d^2 T_s}{d\tau^2} = \alpha\left(\frac{dT_g}{d\tau} - \frac{dT_s}{d\tau}\right) \tag{107}$$

where $\alpha = 3 k_g / \rho_s C a^2$. Then equation 105 becomes

$$\frac{k_g}{a}(T_g - T_s) = \frac{1}{3}\rho_s C a\left[\frac{dT_s}{d\tau} - \frac{3}{28}\left(\frac{k_g}{k_s}\right)^2\left\{\frac{dT_g}{d\tau} - \frac{dT_s}{d\tau}\right\}\right] \tag{108}$$

The condition that the approximation used in the heat balance equation of a particle, equation 96 , is valid is

$$\frac{\frac{3}{28}\left(\frac{k_g}{k_s}\right)^2\left\{\frac{dT_g}{d\tau} - \frac{dT_s}{d\tau}\right\}}{\frac{dT_s}{d\tau}} << 1 \tag{109}$$

For a first approximation, one can say that $dT_g/d\tau$ and $dT_s/d\tau$ are of the same order, while still unevaluated. The coefficient of the numerator is 1.621×10^{-4} based on the constants evaluated earlier. Since $dT_g/d\tau - dT_s/d\tau$ can be at most the same order as $dT_s/d\tau$ and will probably be less, we can say that condition 109 is satisfied.

Since this condition is satisfied, equation 96 is a valid approximation which would be in error by much less than one per cent.

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LIST OF SYMBOLS

A	cross section area
C	specific heat
F	$\frac{\partial}{\partial \xi}[\rho \phi_g] + \frac{1}{\eta}\frac{\partial}{\partial \eta}[\rho \psi_g \eta]$
M	Mach number
P_r	Prandtl number
Q	ratio of absorption cross-sectional area/geometrical cross-sectional area
R	gas constant
Re	Reynolds number
T	temperature
u	axial velocity component
V	radial velocity component
a	radius of solid particle
b	radius of combustion chamber
e	internal energy/unit mass
f	correction factor
h	enthalpy
k	thermal conductivity
l_R	Rosseland mean free path
$\dot{m}$	mass flow rate
n	number of particles/unit volume
p	pressure; variable of Laplace transform
$\dot{q}$	rate of heat transfer

q	$\sqrt{p/\rho}$
r	radial distance; burning rate
t	time
x	axial distance
α	mass fraction of particle
γ	ratio of specific heats
ϵ	indicator of a small quantity
ζ	pressure ratio
η	non-dimensional radial distance
θ	angle (radians)
μ	viscosity
ξ	non-dimensional axial distance
ρ	density
σ	diffusivity
σ_R	Stefan-Boltzmann constant
τ	temperature ratio
ϕ	non-dimensional axial velocity component
ψ	non-dimensional radial velocity component

Subscripts

b	burning surface
c	chamber
d	drag
g	gas phase
h	convection
l	with lag

p constant pressure

r radiation

s solid particle

t throat

v constant volume

0 no lag

1 with small lag