# EXACT AND APPROXIMATE SOLUTIONS TO THE PRESSURE-LOADED PLATE STRIP WITH TEMPERATURE DISTRIBUTION

Thesis by

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## ABSTRACT

The large deflection analysis for a plate strip under uniform pressure and temperature loading is extended to include spanwise variation of temperature. An exact temperature distribution is derived and stress and deflection equations are developed for the fundamental mode thereof. A parabolic approximation to the fundamental mode is shown to be reasonably accurate. Using this approximation, a direct analogy to the case of uniform temperature distribution can be demonstrated in terms of "effective" pressure, temperature moment, and average temperature. The equilibrium equations are formally identical, permitting the use of design charts based on spanwise constant loadings.

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## NOTATION (Cont'd)

| | |
|---|---|
| $\alpha$ | coefficient of linear thermal expansion |
| $\eta$ | dimensionless plate semi-span, $\eta = \frac{2y}{b}$ |
| $\theta$ | dimensionless average temperature |
| $\theta_1$ | dimensionless temperature moment |
| $\Lambda^2$ | dimensionless mid-plane stress |
| $\nu$ | Poisson's ratio |
| $\Pi$ | dimensionless normal pressure |
| $\Sigma$ | summation |
| $\sigma$ | stress |

**Subscripts**

| | |
|---|---|
| o | reference value |
| e | effective |
| h | referred to hyperbolic variation |

## NOTATION

| | |
|---|---|
| b | width of plate strip, inches |
| f | functional variation of average temperature |
| g | functional variation of normal pressure |
| h | thickness of plate strip, inches |
| k | constant associated with temperature moment |
| m | parameter for spanwise temperature distribution |
| p | applied normal pressure, $lb/in^2$ |
| r | functional variation of temperature moment |
| s | functional variation of elastic modulus |
| v | in-plane deformation, inches |
| $\bar{v}$ | dimensionless deformation, $\bar{v} = \frac{v}{b}$ |
| w | normal deflection, inches |
| $\bar{w}$ | dimensionless deflection, $\bar{w} = \frac{w}{b}$ |
| x, y, z | three-dimensional coordinates |
| $\bar{z}$ | dimensionless plate thickness, $\bar{z} = \frac{2z}{h}$ |
| | |
| D | flexural rigidity, $\frac{Eh^3}{12(1 - \nu)}$ |
| E | modulus of elasticity |
| F | numerical values of hyperbolic functions of mid-plane stress |
| N | constant of integration associated with mid-plane stress |
| T | temperature |
| $\bar{T}$ | average temperature |
| $T^{(1)}$ | temperature moment |

# I. INTRODUCTION AND SUMMARY

In an earlier paper Williams (Ref. 1) derived the equations governing the large deflection of a plate strip subjected to combined normal pressure and heating for the situation wherein the plate was in plane strain with respect to its (infinite) length. The equations for the in-plane deformation v(y) and normal deflection w(y) are non-linear but can be uncoupled in such a way that under certain restrictions an exact solution can be obtained. These conditions are that the material constants are independent of the temperature, that the pressure normal to the plate is uniformly distributed, and that the temperature distribution, while arbitrary through the plate thickness, does not vary across the finite width (span) of the plate. Design charts were developed, giving the central deflection and critical stresses for two sets of boundary conditions, (i) clamped edges (ii) simply-supported edges, as functions of the applied pressure, average temperature through the plate, and first temperature moment through the plate. It is therefore natural to inquire into the extension of this work; in particular, to remove the restriction to spanwise constant temperature distributions.

The equations for deflection and deformation contain terms which describe the variation of the average temperature and of the first temperature moment. These are integrals with respect to the plate thickness and are thus variables of the spanwise dimension only. Assuming a heat flux into the (upper) surface of a plate strip, diffusion through the plate and heat flow out of the lower surface, including losses through stringers, a general expression for the temperature distribution within the plate can be written. For the particular steady state conditions to be considered in the following analysis, the tempara-ture distribution must, of course, satisfy the Laplace condition as well as the prescribed boundary conditions. The general form of such a temperature distribution is a sum of products of hyperbolic and trigono-metric functions, but examination of the physical conditions of a somewhat idealized plate strip problem leads to consideration of only the fundamental mode of the general temperature distribution by which certain cases of practical interest can be approximated.

Finally, a parabolic approximation to the fundamental (cosine) mode of the general temperature distribution leads to a significant simplification. Parabolic distributions, both spanwise and through the thickness, are fairly good approximations to reality and are quite easy to work with. Their primary value, however, lies in the fact that their mathematical nature allows an algebraic combination of the pressure and temperature terms in the deflection equations. These combinations can be expressed as very simple "effective" pressure and temperature parameters, which permit a direct analogy to the case of uniform span-wise temperature distribution. Hence the extensive calculations for the previous case can be generalized for further design use.

# II. MATHEMATICAL FORMULATION

## A. Displacements

The governing equations for the normal deflection, $\overline{w}$, and in-plane displacement, $\overline{v}$, for a plate of infinite extent in the x-direction and of width b in the y-direction are,

$$\frac{d^2}{d\eta^2}\left[\frac{4E_o h^3}{3(1-\nu^2)b^3}\, s(\eta)\,\frac{d^2\overline{w}}{d\eta^2}\right] - \frac{4N}{b}\frac{d^2\overline{w}}{d\eta^2} = p_o g(\eta) - \frac{4kE_o\alpha}{1-\nu}\left(\frac{h}{b}\right)^2 T_o^{(1)}\frac{d^2 r}{d\eta^2} \quad (1)$$

$$\frac{2E_o h}{1-\nu^2}\, g(\eta)\left[\frac{dv}{d\eta} + \left(\frac{d^2 w}{d\eta^2}\right)^2\right] - \frac{E_o h\alpha\overline{T}_o}{1-\nu}\, f(\eta)\, s(\eta) = N \quad (2)$$

where

$$\eta = \frac{2y}{b} \qquad \overline{z} = \frac{2z}{h} \qquad p(\eta) = p_o\, g(\eta)$$

$$\overline{v} = \frac{v}{b} \qquad \overline{w} = \frac{w}{b} \qquad E(\eta) = E_o\, s(\eta)$$

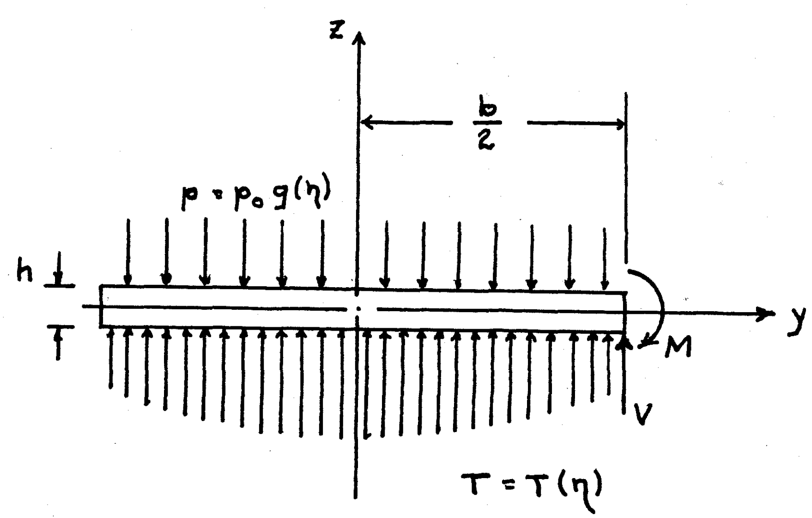



Fig. 1 - Geometry and Loadings

The temperature moments are defined as

$$\bar{T}(\eta) = \bar{T}_o\, f(\eta) = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} T(z, \eta)\, dz = \frac{1}{2} \int_{-1}^{1} T(\bar{z}, \eta)\, d\bar{z} \tag{3}$$

$$kT^{(1)}(\eta) = kT_o^{(1)}\, r(\eta) = \frac{1}{h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} zT(z, \eta)\, dz$$

$$= \frac{1}{4} \int_{-1}^{1} \bar{z}T(\bar{z}, \eta)\, d\bar{z} \tag{4}$$

The (unknown) constant N can be physically identified with the average membrane stress per unit length of plate.

The boundary conditions at the edges ($\eta = \pm 1$), including the temperature effect, are, for clamped edges:

$$\bar{w}\,(\pm 1) = 0$$

$$\frac{d\bar{w}}{d\eta}\,(\pm 1) = 0 \tag{5}$$

$$\bar{v}\,(\pm 1) = 0$$

and for simply-supported edges:

$$\bar{w}\,(\pm 1) = 0$$

$$\frac{d^2\bar{w}}{d\eta^2}(\pm 1) = 8\,\frac{h}{b}\,\frac{r(\pm 1)}{s(\pm 1)}\left[\frac{3(1+\nu)}{8}\left(\frac{b}{h}\right)^2 \alpha\, T_o^{(1)}\,(-k)\right] \tag{6}$$

It will be convenient to combine the loading terms and the material and geometric constants of the plate into dimensionless quantities. Thus, we define:

Average membrane stress

$$\Lambda^2 = 6\left(\frac{b}{h}\right)^2 \frac{N(1-\nu^2)}{2E_o h} = \left(\frac{b}{2}\right)^2 \frac{N}{D} \tag{7}$$

Normal pressure

$$\Pi = \frac{3(1-\nu^2)}{64}\;\frac{P_o b^4}{E_o h^4} = \frac{1}{256}\;\frac{P_o b^4}{Dh} \tag{8}$$

Average temperature (zero temperature moment)

$$\theta = \frac{3(1+\nu)}{8}\left(\frac{b}{h}\right)^2 \alpha\, T_o \tag{9}$$

First temperature moment

$$\theta_1 = \frac{3(1+\nu)}{8}\left(\frac{b}{h}\right)^2 \alpha\, kT_o^{(1)} \tag{10}$$

Equation (1) can be integrated twice and, after substituting the dimensionless loading quantities and rearranging, the governing equations become:

$$s(\eta)\frac{d^2\overline{w}}{d\eta^2} - \Lambda^2\overline{w} = 8\frac{h}{b}\theta_1 r(\eta) + 16\frac{h}{b}\Pi\iint g(\eta)\, d\eta\, d\eta \tag{1a}$$

$$\frac{3}{4}\left(\frac{b}{h}\right)^2 s(\eta)\left[\frac{d\overline{v}}{d\eta} + \left(\frac{dw}{d\eta}\right)^2\right] = \frac{1}{8}\Lambda^2 + \theta f(\eta)\, s(\eta) \tag{2a}$$

If Young's modulus is taken to be independent of the temperature (i.e.: $s(\eta) = 1$), and if the normal pressure is constant (i.e.: $g(\eta) = 1$), one obtains the two equations which are to be considered in this analysis.

$$\frac{d^2\overline{w}}{d\eta^2} - \Lambda^2\overline{w} = 8\frac{h}{b}\left[\theta_1 r(\eta) + \Pi\eta^2 + k_1\eta + k_2\right] \tag{11}$$

$$\frac{d\overline{v}}{d\eta} + \left(\frac{d\overline{w}}{d\eta}\right)^2 = \frac{4}{3}\left(\frac{h}{b}\right)^2\left[\frac{1}{8}\Lambda^2 + \theta f(\eta)\right] \tag{12}$$

It will be observed that if there is no spanwise variation of temperature, $f(\eta) = r(\eta) = 1$, which was the case treated in Reference 1, the two equations reduce immediately to the situation covered therein.

## B. Stresses

The bending moment equation for a pressure - and temperature - loaded plate strip is

$$M(\eta) = -\frac{b}{4}\left[\frac{2E_o h^3}{3(1-\nu^2)b^2}\, 2\frac{d^2}{d\eta^2}\overline{w}(\eta) + \frac{4kE_o h^2 \alpha T_o^{(1)}}{(1-\nu)b} r(\eta)\right] \tag{13}$$

In terms of the dimensionless loading parameters, this becomes

$$M(\eta) = - D\left[\frac{4}{b}\ \frac{d^2}{d\eta^2}\ \bar{w}(\eta) + \frac{32h}{b^2}\quad \theta_1\ r(\eta)\right]$$

The bending stress is related to the bending moment by $M = \frac{h^2}{6}\ \sigma_{bending}$ so that

$$\sigma_{bending} = \frac{4D}{b^2h}\ (48) \left[\theta_1\ r(\eta) - \frac{1}{8}\ \frac{b}{h}\ \frac{d^2}{d\eta^2}\ \bar{w}(\eta)\right] \tag{14}$$

The constant of integration $N$ in Equation 2 is just the membrane stress in the plate, which has been defined (Equation 7) as

$$\Lambda^2 = (\frac{b}{2})^2\ \ \frac{N}{D} = (\frac{b}{2})^2\ \ \frac{\sigma_{membrane}}{D}$$

and one has

$$\sigma_{membrane} = \frac{4D}{b^2h}\ \ \Lambda^2 \tag{15}$$

The total stress, evaluated at any point $\eta$, is then

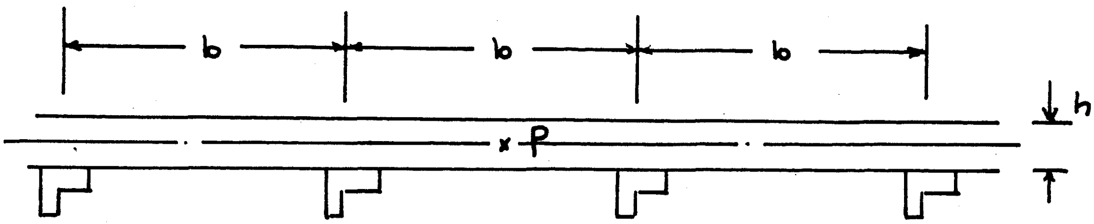
$$\frac{b^2h}{4D}\ \sigma\ =\ \Lambda^2 \pm\ 48 \left[\theta_1\ r(\eta) - \frac{1}{8}\frac{b}{h}\ \frac{d^2}{d\eta^2}\ \bar{w}(\eta)\right] \tag{16}$$

From a similar development, the shear stress is

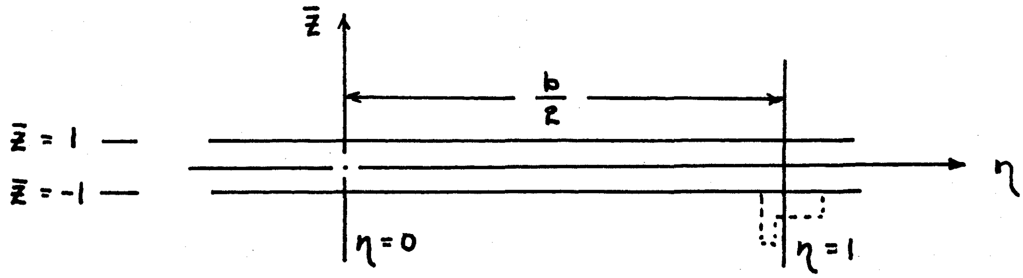
$$2V\ =\ 4\ \sigma_{membrane}\ \frac{d}{d\eta}\ \ \bar{w}(\eta) + \frac{d}{d\eta}\ \ M(\eta) \tag{17}$$

## III. TEMPERATURE DISTRIBUTION

Consider the section of an aircraft surface as shown.




Using the obvious geometrical symmetry, the sketch can be generalized to a non-dimensional coordinate system with the origin at P.




The abcissae $\eta = \pm 1$ pass through the first stringers on either side of the plate and $\bar{z} = 2z/h$.

It has been shown that the transient state of heat flow is characterized by an increase in the value of the spanwise gradient of the temperature from zero initially to some maximum value for the steady state. The transient stresses are therefore intermediate between those for a spanwise constant temperature distribution and those for a fully developed heat flow through the plate. The critical stresses are either (a) those for time $t = 0$, corresponding to a large but spanwise constant temperature moment or (b) those for the steady state. The first case was treated completely in the references. The second case is that for which the Laplace equation

$$\nabla^2 T = 0 \tag{18}$$

is satisfied,

Solutions of the Laplace equation are:

a. $T = T_c$, a constant

b. $T = (a + b\bar{z})(c + d\eta)$, a linear variation in either or both directions

c. $$T = \sum_{n=1}^{\infty} \left[ A_n^{(1)} \cosh \alpha \bar{z} + B_n^{(1)} \sinh \alpha \bar{z} \right] \left[ A_n^{(2)} \cos \alpha \eta + B_n^{(2)} \sin \alpha \eta \right]$$

or $$\sum_{n=1}^{\infty} \left[ A_n^{(1)} \cos \alpha \bar{z} + B_n^{(1)} \sin \alpha \bar{z} \right] \left[ A_n^{(2)} \cosh \alpha \eta + B_n^{(2)} \sinh \alpha \eta \right]$$

d. Any linear combination of these.

Note: The solution $T = c_1(\bar{z}^2 - \eta^2)$ is not considered, since for this geometry it presupposes heat sources or sinks within the plate. The most general solution is the sum of all particular solutions, subject to the physical and geometric boundary conditions of the problem.

Before forming this sum, let us consider solution (c.) in detail. Two arbitrary restrictions are imposed, based on physical considerations, to select the most reasonable set of products. First, temperature distributions are required to be symmetrical about the $\eta = 0$ axis. This is a tremendous mathematical simplification whose physical justification depends primarily on the efficiency of the stringers as heat sinks. Second, the temperature gradient with respect to $\eta$ must vanish at $\eta = \pm 1$. These two conditions are sufficient to restrict the solution to

$$T(\bar{z}, \eta) = \sum_{n=1}^{\infty} \left[ a_n \cosh n\pi\bar{z} + b_n \sinh n\pi\bar{z} \right] \cos n\pi\eta \tag{19}$$

The Fourier coefficients $a_n$ and $b_n$ may now be evaluated in terms of the temperature on the surfaces $\bar{z} = \pm 1$ where it is a function of $\eta$ only.

$$T(1, \eta) = \sum_{n=1}^{\infty} \left[ a_n \cosh n\pi + b_n \sinh n\pi \right] \cos n\pi\eta$$

$$T(-1, \eta) = \sum_{n=1}^{\infty} \left[ a_n \cosh n\pi - b_n \sinh n\pi \right] \cos n\pi\eta$$

Now, multiplying both sides of each equation by cos $m\pi\eta$ and integrating over the interval $-1 < \eta < 1$, gives

$$\int_{-1}^{1} T(1, \eta) \cos m\pi\eta \, d\eta \equiv C_m^{(1)} = a_m \cosh m\pi + b_m \sinh m\pi$$

$$\int_{-1}^{1} T(-1, \eta) \cos m\pi\eta \, d\eta \equiv C_m^{(2)} = a_m \cosh m\pi - b_m \sinh m\pi$$

From these relations it is seen that any surface temperature distribution which meets the two arbitrary restrictions can be represented in terms of

$$a_m = \frac{C_m^{(1)} + C_m^{(2)}}{2\cosh m\pi} \qquad b_m = \frac{C_m^{(1)} - C_m^{(2)}}{2\sinh m\pi}$$

Furthermore, the resulting expression is still a solution of the steady-state Laplace equation. The temperature loading parameters $\overline{T}(\eta)$ and $T^{(1)}(\eta)$ can be derived and the equilibrium equations can be solved in terms of this distribution, for any $m$, if the data for a specific problem should so require. It is believed that the majority of plate strip problems can be prescribed adequately by the fundamental Fourier component, $m = 1$; viz. a single cosine variation. Accordingly, the solution

$$T(\bar{z}, \eta) = \left[ a_1 \cosh \pi\bar{z} + b_1 \sinh \pi\bar{z} \right] \cos \pi\eta$$

will be used.

The sum of individual solutions is then

$$T(\bar{z}, \eta) = T_c + (a + b\bar{z})(c + d\eta)$$

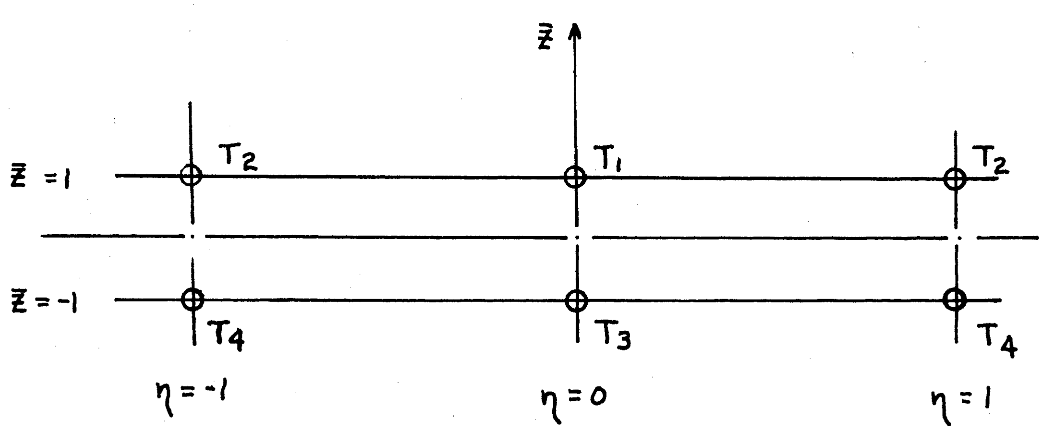
$$+ \left[ a_1 \cosh \pi\bar{z} + b_1 \sinh \pi\bar{z} \right] \cos \pi\eta$$

The symmetry conditions require that $d = 0$, so that

$$T(\bar{z}, \eta) = T_c + c'\bar{z} + \left[ a_1 \cosh \pi\bar{z} + b_1 \sinh \pi\bar{z} \right] \cos \pi\eta \qquad (20)$$

In the discussion of the transcendental part of this solution, an expression for the temperature distribution on the plate surfaces was used. These boundary conditions in the z-direction could just as well have been expressed as the distribution of the normal gradients along the plate surface. Specifying these gradients (which define the heat flow through the surfaces) would result in a Neumann problem, since gradients have already been selected for the boundary conditions on $\eta$. Such conditions would, however, be very difficult to express in physical terms. The heat transport phenomena within the boundary layer are complicated functions of Mach number, fluid viscosity and surface emissivity. The heat losses from the inner surface would be equally complicated functions of the geometry of construction and loading and could, for example, vary with the use of fuel. Solutions in such terms would seem to be of limited usefulness.

A classical Dirichlet problem is also precluded, since there is no choice but to express the spanwise conditions as gradients. We are led, therefore, to a mixed boundary value problem in which the temperature on each surface is prescribed by a cosine curve through two arbitrarily selected points whose designation is as shown.



The coefficients of Equation 20 can be expressed in terms of these points, yielding

$$T(\bar{z}, \eta) = \frac{1}{4}(T_1 + T_2 + T_3 + T_4) + \frac{1}{4}(T_1 + T_2 - T_3 - T_4)\,\bar{z} \tag{21}$$

$$+ \left[\frac{T_1 - T_2 + T_3 - T_4}{4 \cosh \pi} \cosh \pi\bar{z} + \frac{T_1 - T_2 - T_3 + T_4}{4 \sinh \pi} \sinh \pi\bar{z}\right] \cos \pi\eta$$

Equation 21 can now be integrated to find the average temperature and its spanwise variation.

$$\bar{T}(\eta) = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} T(z, \eta)\, dz$$

$$= \frac{1}{2}\int_{-1}^{1} T(\bar{z}, \eta)\, d\bar{z}$$

$$= \bar{T}_o\,(\bar{T}_h + \cos \pi\eta) \tag{22}$$

where

$$\bar{T}_o = \frac{\tanh \pi}{4\pi}(T_1 - T_2 + T_3 - T_4)$$

$$\bar{T}_h = \frac{\pi}{\tanh \pi}\; \frac{T_1 + T_2 + T_3 + T_4}{T_1 - T_2 + T_3 - T_4} \tag{23}$$

Further,

$$\bar{T}(\eta) = \bar{T}_o f(\eta)$$

so that

$$f(\eta) = \bar{T}_h + \cos \pi\eta \tag{24}$$

The definition of the temperature moment in Part I contains a numerical constant k which, for an algebraic temperature expression, relates the integrated moment to the surface and average temperatures. When the temperature distribution is expressed in transcendental functions, k is neither numerical nor constant and its identification is no longer useful. To preserve the form of the original definitions, however, the products $kT^{(1)}(\eta)$ and $kT_o^{(1)}$ can be retained by proceeding as follows:

$$T_o^{(1)} r(\eta) \equiv T^{(1)}(\eta)$$

$$kT_o^{(1)} r(\eta) = kT^{(1)}(\eta)$$

$$= \frac{1}{4} \int_{-1}^{1} \bar{z}\, T(\bar{z}, \eta)\, d\bar{z}$$

$$= \frac{T_1 + T_2 - T_3 - T_4}{24} + \frac{T_1 - T_2 - T_3 + T_4}{8\pi^2}(\pi \coth \pi - 1)\, \pi\eta$$

By now defining

$$kT_o^{(1)} = \frac{\pi \coth \pi - 1}{8\pi^2} (T_1 - T_2 - T_3 + T_4)$$

$$T_h^{(1)} = \frac{\pi^2}{3(\pi \coth \pi - 1)} \quad \frac{T_1 + T_2 - T_3 - T_4}{T_1 - T_2 - T_3 + T_4}, \tag{25}$$

the temperature moment can be written as

$$kT^{(1)}(\eta) = kT_o^{(1)} r(\eta)$$

$$= kT_o^{(1)} (T_h^{(1)} + \cos \pi\eta) \tag{26}$$

whence

$$r(\eta) = T_h^{(1)} + \cos \pi\eta \tag{27}$$

## IV. SOLUTIONS OF THE EQUILIBRIUM EQUATIONS

### A. Normal Deflection

Returning now to Equation 11

$$\frac{d^2\bar{w}}{d\eta^2} - \Lambda^2 \bar{w} = 8\frac{h}{b}\left[\theta_1 \, r(\eta) + \Pi\,\eta^2 + k_1\eta + k_2\right]$$

we substitute Equation 27 for the spanwise variation of the temperature moment.

$$r(\eta) = T_h^{(1)} + \cos \pi\eta$$

The solution of the homogeneous equation is

$$\bar{w}_H = A \sinh \Lambda\eta + B \cosh \Lambda\eta$$

in which the symmetry condition requires that $A = 0$. (Symmetry also requires that $k_1 = 0$ in Equation 11).

Solving the inhomogeneous equation by variation of parameters, we find

$$\bar{w} = B \cosh \Lambda\eta - 8\frac{h}{b}\left[\frac{\theta_1 \cos \pi\eta}{\pi^2 + \Lambda^2} + \frac{\Pi}{\Lambda^2}\eta^2 + \frac{2\Pi}{\Lambda^4} + \frac{\theta_1 T_h^{(1)}}{\Lambda^2} + \frac{k_2}{\Lambda^2}\right] \quad (28)$$

to which the physical boundary conditions can now be applied.

B. In-plane Deformation

In similar fashion, Equation 24

$$f(\eta) = \overline{T}_h + \cos \pi\eta$$

can be substituted into Equation 12 for in-plane deformation

$$\frac{d\overline{v}}{d\eta} + \left(\frac{d\overline{w}}{d\eta}\right)^2 = \frac{4}{3}\left(\frac{h}{b}\right)^2\left[\frac{\Lambda^2}{8} + \theta\, f(\eta)\right] \qquad (12)$$

After rearranging and integrating once, we obtain

$$\frac{3}{4}\left(\frac{b}{h}\right)^2 \overline{v} = -\frac{3}{4}\left(\frac{b}{h}\right)^2 \int_0^{\eta} \left(\frac{d\overline{w}}{d\eta}\right)^2 d\eta + \left(\frac{\Lambda^2}{8} + \theta\overline{T}_h\right)\eta$$

$$+ \frac{\theta}{\pi} \sin \pi\eta + k_4 \qquad (29)$$

Note that the in-plane deformation is a function of the normal deflection and that the latter must be obtained in terms of given edge conditions before Equation 29 can be solved.

C. Clamped-edge Solutions

Applying the boundary conditions

$$\overline{w}(1) = 0 \;\; ; \;\; \frac{d\overline{w}}{d\eta}(1) = 0$$

(these conditions for $\eta = -1$ have already been applied by implication in the symmetry conditions used in Section A) the constants in Equation 28 can be evaluated. The resulting expression for normal deflection is

$$\bar{w}_{CL} = \frac{8}{\Lambda^2}\frac{h}{b}\left\{ \Pi \left[ 1 - \frac{2}{\Lambda}\coth\Lambda - \eta^2 + \frac{2}{\Lambda \sinh\Lambda}\cosh\Lambda\eta \right] - \theta_1 \frac{1+\cos\pi\eta}{1+\pi^2/\Lambda^2} \right\} \tag{30}$$

This expression for $\bar{w}$ can now be substituted into Equation 29. Doing this and using the boundary condition $\bar{v}(1) = 0$ to evaluate the constant $k_4$, we have

$$\begin{aligned} \frac{3}{4}\left(\frac{b}{h}\right)^2 \bar{v}_{CL} = \left(\frac{8\,\Pi}{\Lambda^2}\right)^2 \Bigg\{ & 1 - \eta^3 - F_{CL}^2(1-\eta) + \frac{3\operatorname{csch}^2\Lambda}{4\Lambda}\left[\sinh 2\Lambda - \sinh 2\Lambda\eta - 2\Lambda(1-\eta)\right] \\ & - \frac{6\operatorname{csch}\Lambda}{\Lambda^2}\left[\Lambda\cosh\Lambda - \Lambda\eta\cosh\Lambda\eta + \sinh\Lambda\eta - \sinh\Lambda\right] \Bigg\} + \theta\,\frac{\sin\pi\eta}{\pi} \\ & - \left(\frac{\Lambda^2}{\Lambda^2+\pi^2}\right)\Bigg\{ \theta_1^2\,\frac{\pi}{4}\sin 2\pi\eta - 4\,\Pi\,\theta_1 \Bigg[ \eta + \frac{\Lambda^2+\pi^2}{\Lambda^2}\left(\eta\cos\pi\eta - \frac{1}{\pi}\sin\pi\eta\right) \\ & + \frac{\pi\operatorname{csch}\Lambda}{\Lambda^2}\left(\Lambda\cosh\Lambda\eta\,\sin\pi\eta - \pi\sinh\Lambda\eta\,\cos\pi\eta\right) \Bigg] \Bigg\} \end{aligned} \tag{31}$$

The remaining boundary condition, ($\bar{v}(-1) = 0$ or $\bar{v}(0) = 0$), is now used to determine or evaluate the unknown constant N, or what is equivalent, the membrane stress $\Lambda^2$. Applying this condition to Equation 31 we obtain the relation

$$\frac{\Lambda^2}{8} + \theta\, T_h - \left(\frac{\Lambda^2}{\Lambda^2 + \pi^2}\right)^2 \left(4\, \Pi\, \theta_1 - \frac{\pi^2}{2}\, \theta_1^2\right)$$

$$= \left(\frac{8\,\Pi}{\Lambda^2}\right)^2 \left(1 + \frac{6}{\Lambda^2} - \frac{9}{2\Lambda} \coth \Lambda - \frac{3}{2} \operatorname{csch}^2 \Lambda\right) \tag{32}$$

The total stress, bending plus membrane, for any value of $\eta$ can be found directly from Equation 16.

$$\frac{b^2 h}{4D}\, \sigma_{CL} = \Lambda^2 \pm 48 \left[ \frac{2\,\Pi}{\Lambda^2}\left(1 - \frac{\Lambda \cosh \Lambda \eta}{\sinh \Lambda}\right) + \theta_1 \left(T_h^{(1)} + \frac{\Lambda^2 \cos \pi\eta}{\Lambda^2 + \pi^2}\right) \right] \tag{33}$$

D. <u>Sample Problem</u>

To illustrate the solutions of this Section, consider a plate strip of width b = 10" and thickness h = 0.1", clamped to stringers along each edge and subjected to a uniform normal pressure of 20 psi. The temperatures at (0, $\pm$ 1) and (1, $\pm$ 1) are as shown.

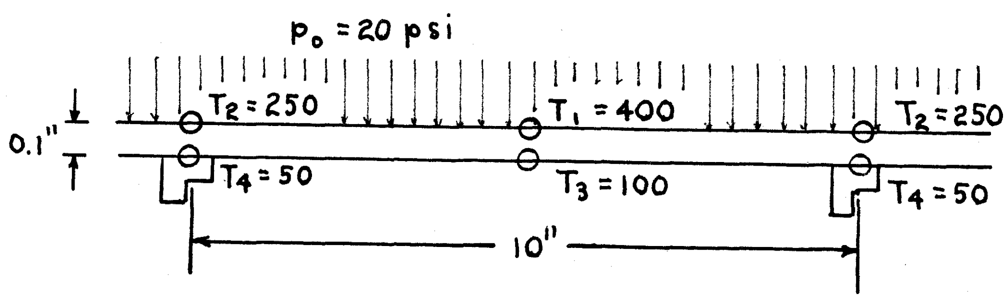



Fig. 2 - Sample Problem

Assume further that the material properties are

$E = 10^7$ psi $\qquad \nu = 0.25 \qquad \alpha = 13 \times 10^{-6}$ in/in/°F

From the definitions of Equations 23 and 25,

$\bar{T}_o = 15.7 \qquad kT_o^{(1)} = 2.73$

$\bar{T}_h = 12.7 \qquad T_h^{(1)} = 7.6$

The dimensionless loading parameters (Equations 8, 9, and 10) are then,

$\Pi = 8.5$

$\theta = 0.96$

$\theta_1 = 0.17$

The membrane stress $\Lambda^2$ can be found by numerical solution of Equation 32.

$\Lambda^2 = 6.25$

All of the constants in the deflection and stress equations (28, 29, and 31) are now known. Plots of normal deflection and stress versus span are shown in Figures 3 and 4.

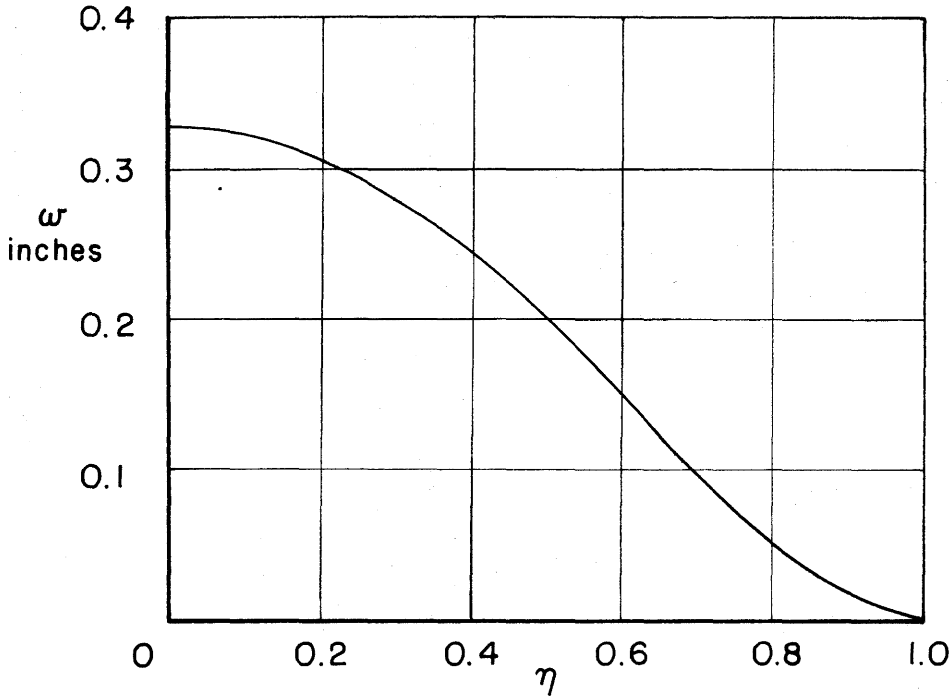



FIG. 3 - SAMPLE PROBLEM, NORMAL DEFLECTION

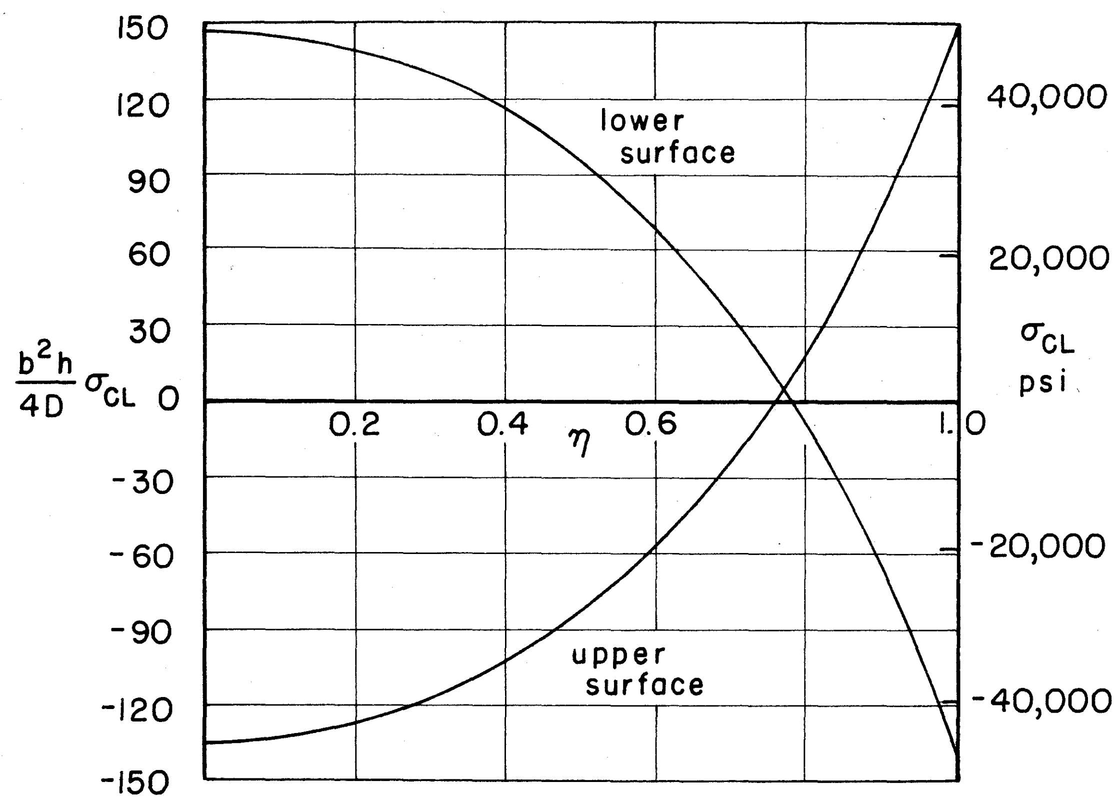



FIG. 4 - SAMPLE PROBLEM, SURFACE STRESS

# V. THE PARABOLIC APPROXIMATION

## A. Formal Considerations

The simplified form of an exact temperature distribution in a plate strip and the resulting expressions for stress and deformation doubtless have some inherent value and, indeed, the method can be applied to distribution modes higher than the first, if the data permit and if such detail is required. Finding the membrane stress (Equation 32) by a numerical method is not difficult, though tedious, and the normal deflection and surface stress equations are relatively simple. The equations do not, however, lend themselves to the development of design charts nor do they exhibit a useful interrelation among the loading parameters. In this section, therefore, we seek an approximation which will permit an explicit grouping of similar terms, with the objectives of combining some of the parameters and of facilitating design computations.

Looking first at Equation 11, we see that the normal deflection is a function of $\eta^2$ and $r(\eta)$, with the dimensionless pressure and temperature moment as parameters on the respective terms. (The coefficient $k_1$ is zero by symmetry).

$$\frac{d^2\overline{w}}{d\eta^2} - \Lambda^2\overline{w} = 8\,\frac{h}{b}\left[\theta_1 r(\eta) + \Pi\,\eta^2 + k_2\right] \tag{11a}$$

It is obvious that by selecting $r(\eta)$ to be the parabolic form

$$r(\eta) = 1 - m\eta^2$$

instead of $r(\eta) = T_h^{(1)} + \cos \pi\eta$, Equation 27, the terms in brackets will become

$$(\Pi - \theta_1 m)\eta^2 + \theta_1 + k_2$$

We see that an "effective" pressure can then be defined

$$\Pi_e = \Pi - \theta_1 m \tag{34}$$

which will serve, formally, at least, as a normal deflection parameter. For the physical situation in which pressure and heat are both applied to the upper surface, the pressure deforms the plate convex downward while the temperature moment causes a convex upward deformation. The net deformation is therefore proportional to the difference in magnitude of the two effects, which is just what the new parameter states.

In attempting to find a new "effective" parameter in Equation 12, two opposing considerations enter.

$$\frac{d\overline{v}}{d\eta} + \left(\frac{d\overline{w}}{d\eta}\right)^2 = \frac{4}{3}\left(\frac{h}{b}\right)^2\left[\frac{1}{8}\Lambda^2 + \theta f(\eta)\right]$$

First is the fact that the spanwise distribution of average temperature, $f(\eta)$, should have the same form as the spanwise distribution of temperature moment, $r(\eta)$. This is seen from the definitions of the two terms, which are thickness integrals of the two-dimensional temperature distribution, leaving the spanwise variation unchanged.

The second consideration is that we might effect some simplification by making the bracketed terms homogeneous in $\eta$, that is, by setting $f(\eta)$ = constant. Although this approximation is physically reasonable (except for the determination of in-plane displacement), it is found that it does not lead to any simplification in either the in-plane displacement or the membrane stress condition. The function $f(\eta)$ carries through the development explicitly and does not combine with or modify any other term.

The role and development of new loading parameters based on the assumption of parabolic temperature functions may be clarified by a mathematical example. Consider a plate strip having the temperature distribution

$$T(z, \eta) = t(z)\,(1 - m\eta^2)$$

then,

$$\bar{T}(\eta) = \bar{T}_o f(\eta) = \left[\frac{1}{h}\int_{-h/2}^{h/2} t(z)dz\right]\left[1 - m\eta^2\right]$$

and

$$T^{(1)}(\eta) = T_o^{(1)}\, r(\eta) = \left[\frac{1}{kh^2}\int_{-h/2}^{h/2} zt(z)dz\right]\left[1 - m\eta^2\right]$$

If we define the reference temperatures $\bar{T}_o$ and $kT_o^{(1)}$ as

$$\overline{T}_o = \frac{1}{h} \int_{-h/2}^{h/2} t(z)\, dz$$

$$kT_o^{(1)} = \frac{1}{h^2} \int_{-h/2}^{h/2} zt(z)\, dz$$

then the spanwise distribution functions become

$$f(\eta) = r(\eta) = 1 - m\eta^2 \tag{35}$$

Substituting in the normal equilibrium Equation 11a,

$$\frac{d^2\overline{w}}{d\eta^2} - \Lambda^2\overline{w} = 8\,\frac{h}{b}\left[\theta_1(1 - m\eta^2) + \Pi\,\eta^2 + k_2\right]$$

$$= 8\,\frac{h}{b}\left[(\Pi - \theta_1 m)\eta^2 + \theta_1 + k_2\right]$$

which may be compared to the equation for normal deflection under spanwise constant temperature moment

$$\frac{d^2\overline{w}}{d\eta^2} - \Lambda^2\overline{w} = 8\,\frac{h}{b}\left[\Pi\,\eta^2 + \theta_1 + k_2\right]$$

Using the newly-defined "effective" pressure, Equation 34,

$$\Pi_e \equiv \Pi - \theta_1 m$$

the differential equation is formally the same.

Similarly in the second equation of equilibrium, one finds after integration,

$$\bar{v}(\eta) + \int_0^\eta \left(\frac{d\bar{w}}{d\eta}\right)^2 d\eta = \frac{4}{3}\left(\frac{h}{b}\right)^2 \left[\frac{1}{8}\ \Lambda^2 + \theta\left(1 - \frac{m}{3}\eta^2\right)\right]\eta$$

Imposing the boundary condition $\bar{v}(\pm 1) = 0$, we see that if we replace $\theta\,(1 - \frac{m}{3})$ by an effective average temperature

$$\theta_e = \theta\left(1 - \frac{m}{3}\right) \qquad (36)$$

the situation is formally the same as for a spanwise constant temperature, although elsewhere in the plate $\bar{v}(\eta)$ will depend upon the particular temperature distribution. The interpretation of such an effective average temperature is entirely proper however for evaluating the (unknown) membrane stress $\Lambda^2$, under this boundary condition.

If these effective quantities will now formally satisfy the boundary conditions on bending, their utility will be established. Certainly, the temperature terms do not enter the boundary conditions for clamped edges ($\bar{w} = \bar{w}' = 0$ at $\eta = \pm 1$; Equation 5), so this case is immediately verified. On the other hand, for simply-supported edges, one has from Equation 6

$$\frac{d^2\bar{w}}{d\eta^2}(\pm 1) = r(\pm 1)\ 8\,\frac{h}{b}\ \theta_1$$

$$= 8\,\frac{h}{b}\,(1 - m)\ \theta_1$$

A new effective temperature moment

$$\theta_{1e} \equiv (1 - m)\,\theta_1 \tag{37}$$

must therefore be introduced for the analogy to hold.

As stated in Part II, the stresses are

$$\sigma_{membrane} = \frac{4D}{b^2 h}\,\Lambda^2, \text{ psi}$$

and the bending stress at any point $\eta^*$, found from the bending moment equation, is

$$M(\eta^*) = -\frac{b}{4}\left[\frac{2E_o h^3}{3(1-\nu^2)b^2}\,2\overline{w}''(\eta^*) + \frac{4kE_o h^2 \alpha T_o^{(1)}}{(1-\nu)b}\,r(\eta^*)\right]$$

More specifically,

$$M(\eta^*) = -D\left[\frac{4}{b}\,\overline{w}''(\eta^*) - \frac{32h}{b^2}\,\theta_1\,r(\eta^*)\right] = \frac{h^2}{6}\,\sigma_{bend}$$

and therefore

$$\sigma_{bending} = \frac{4D}{b^2 h}(48)\left[\theta_1\,r(\eta^*) - \frac{1}{8}\,\frac{b}{h}\,\overline{w}''(\eta^*)\right] \tag{38}$$

The total stress is found in the form

$$\frac{b^2 h}{4D}\,\sigma = \Lambda^2 \pm 48\left[\theta_1\,r(\eta^*) - \frac{1}{8}\,\frac{b}{h}\,\overline{w}''(\eta^*)\right] \tag{39}$$

We have shown that the assumption of a parabolic approximation to an exact (Laplacian) temperature distribution will satisfy the formal mathematical requirements of the differential equations and the displacement and stress conditions. This conclusion might better be expressed in the negative sense; that the mathematics of the problem does not prohibit the use of this approximation. It remains to develop the substance of the functions $f(\eta)$ and $r(\eta)$ and to show that the approximation is sufficiently accurate.

B. <u>The Approximate Distribution Functions</u>

Given a temperature distribution defined by $T_1$, $T_2$, $T_3$, and $T_4$, the exact spanwise variations of the average temperature and temperature moment are given by Equations 22 and 26:

$$\bar{T}(\eta) = \bar{T}_o (\bar{T}_h + \cos \pi\eta)$$

$$kT^{(1)}(\eta) = kT_o^{(1)}(T_h^{(1)} + \cos \pi\eta)$$

It is now required to find the numbers A, B, and m, so that the functions

$$\bar{T}(\eta) = A (1 - m\eta^2)$$

$$kT^{(1)}(\eta) = B (1 - m\eta^2)$$

best approximate the exact expressions. Considering first the temperature moment, it is seen that the points $\eta = 0$ and $\eta = 1$ are of greatest interest for the deflection and surface stress. Equating the transcendental and parabolic terms for the temperature moment, evaluated at these two points, gives

$$B = kT_o^{(1)}(T_h^{(1)} + 1)$$

$$m = \frac{2}{T_h^{(1)} + 1}$$

The average temperature appears most critically in the membrane stress and in-plane displacement equations. In these cases, one is interested in the value of the parameter integrated across the semi-span. Accordingly, the number $A$ is found by equating the integrals of the exact and approximate expressions.

$$\int_o^1 A(1 - m\eta^2)d\eta = \int_o^1 \bar{T}_o(\bar{T}_h + \cos \pi\eta)d\eta$$

$$A(1 - \frac{m}{3}) = \bar{T}_o \bar{T}_h$$

$$A = \frac{\bar{T}_o \bar{T}_h}{1 - m/3}$$

The numbers $A$ and $B$ are, of course, the parabolic approximations of $\bar{T}_o$ and $kT_o^{(1)}$, respectively.

The parabolic approximations to the exact temperature distribution functions may then be written:

Average temperature

$$\begin{aligned}\bar{T}(\eta) &= \bar{T}_o\, f(\eta) \\ &= \bar{T}_o(\bar{T}_h + \cos \pi\eta) \\ &\simeq \frac{\bar{T}_o\,\bar{T}_h}{1 - m/3}(1 - m\eta^2)\end{aligned} \tag{40}$$

Temperature moment

$$\begin{aligned}kT^{(1)}(\eta) &= kT_o^{(1)}\, f(\eta) \\ &= kT_o^{(1)}(T_h^{(1)} + \cos \pi\eta) \\ &\simeq kT_o^{(1)}(T_h^{(1)} + 1)(1 - m\eta^2)\end{aligned} \tag{41}$$

in both of which

$$m = \frac{2}{T_h^{(1)} + 1} \tag{42}$$

where the exact parameters $\bar{T}_o$, $\bar{T}_h$, $T_o^{(1)}$, and $T_h^{(1)}$ are defined in Equations 23 and 25.

# VI. APPROXIMATE SOLUTIONS

Substituting the approximate expressions for spanwise distribution of the average temperature and temperature moment (Equation 35), the equilibrium equations can readily be solved for $\bar{w}(\eta, \Lambda)$ and $\bar{v}(\eta, \Lambda)$.

$$\bar{w} = B \cosh \Lambda\eta - \frac{8}{\Lambda^2}\frac{h}{b}\left[(\Pi - \theta_1 m)(\eta^2 + \frac{2}{\Lambda^2}) + k_2 \Pi + \theta_1\right] \tag{43}$$

$$\bar{v} = -\int_0^{\eta} (\frac{d\bar{w}}{d\eta})^2 d\eta + \frac{4}{3}(\frac{h}{b})^2 \left[\frac{\Lambda^2}{8} \eta + \theta(1 - \frac{m}{3})\eta^3\right] \tag{44}$$

The physical edge conditions must now be applied to determine the constants for specific cases.

## A. Clamped Edges

Applying the boundary conditions

$$\bar{w}(1) = \frac{d\bar{w}}{d\eta}(1) = 0$$

the constants $B$ and $k_2$ can be determined so that

$$\bar{w}_{CL} = \frac{8}{\Lambda^2}\frac{h}{b}(\Pi - \theta_1 m)\left[1 - \eta^2 - \frac{2}{\Lambda}\coth\Lambda + \frac{2}{\Lambda \sinh\Lambda}\cosh\Lambda\eta\right] \tag{45}$$

After computing and integrating the term $(\frac{d\bar{w}}{d\eta})^2_{CL}$, the in-plane displacement is found to be

$$\frac{3}{4}\left(\frac{b}{h}\right)^2 \bar{v}_{CL} = \frac{1}{3} m\theta(1 - \eta^3) - \left(\frac{\Lambda^2}{8} + \theta\right)(1 - \eta) \tag{46}$$

$$+\left[\frac{8}{\Lambda^2}(\Pi - \theta_1 m)\right]^2 \left\{1 + \frac{3}{4}\frac{\operatorname{csch}^2\Lambda}{\Lambda}\left[\sinh 2\Lambda - \sinh 2\Lambda\eta - 2\Lambda(1-\eta)\right]\right.$$

$$\left. - 6\frac{\operatorname{csch}\Lambda}{\Lambda^2}\left[\Lambda\cosh\Lambda - \Lambda\eta\cosh\Lambda\eta + \sinh\Lambda\eta - \sinh\Lambda\right]\right\}$$

The remaining boundary condition on in-plane displacement, $\bar{v}(0) = 0$, is now applied to Equation 46 to find the load-stress relation

$$\frac{\Lambda^2}{8} + \theta\left(1 - \frac{m}{3}\right) = \left[\frac{8}{\Lambda^2}(\Pi - \theta_1 m)\right]^2 \left[1 + \frac{6}{\Lambda^2} - \frac{9}{2\Lambda}\coth\Lambda - \frac{3}{2}\operatorname{csch}^2\Lambda\right] \tag{47}$$

or, in terms of the "effective" parameters,

$$\frac{\Lambda^2}{8} + \theta_e = \left(\frac{8}{\Lambda^2}\right)^2 \Pi_e^2 \left[1 + \frac{6}{\Lambda^2} - \frac{9}{2\Lambda}\coth\Lambda - \frac{3}{2}\operatorname{csch}^2\Lambda\right] \tag{48}$$

In this form, the membrane stress condition is identical to the condition developed in Reference 2 for spanwise constant temperature distribution. Reference 2 further defines the expression in brackets as $F_{CL}^2$ and tabulates it as a function of $\Lambda$. The resulting expression

$$\frac{\Lambda^2}{8} + \theta_e = \left[\frac{8}{\Lambda^2}\Pi_e\right]^2 F_{CL}^2 \tag{49}$$

can readily be plotted with any one of the three variables as a parameter. $\Lambda^2$ versus $\Pi_e$ for various values of $\theta_e$ is shown as Figure 5.

B. Simply-supported Edges

In a similar manner the boundary conditions for simply-supported edges

$$\overline{w}(1) = 0$$

$$\frac{d^2\overline{w}}{d\eta^2}(1) = 3(1+\nu)\frac{b}{h}k\alpha T_o^{(1)} r(\eta)$$

$$= 8\frac{h}{b}\theta_1(1 - m\eta^2)$$

can be applied to Equations 43 and 44, with the following results:

$$\overline{w}_{SS} = \frac{8}{\Lambda^2}\frac{h}{b}\left\{(\Pi - \theta_1 m)\left[1 - \eta^2 - \frac{2}{\Lambda^2}\left(1 - \frac{\cosh\Lambda\eta}{\cosh\Lambda}\right)\right] - \theta_1(1-m)\left(1 - \frac{\cosh\Lambda\eta}{\cosh\Lambda}\right)\right\} \tag{50}$$

$$= \frac{8}{\Lambda^2}\frac{h}{b}\left\{\Pi_e\left[1 - \eta^2 - \frac{2}{\Lambda^2}\left(1 - \frac{\cosh\Lambda\eta}{\cosh\Lambda}\right)\right] - \theta_{1e}\left(1 - \frac{\cosh\Lambda\eta}{\cosh\Lambda}\right)\right\}$$

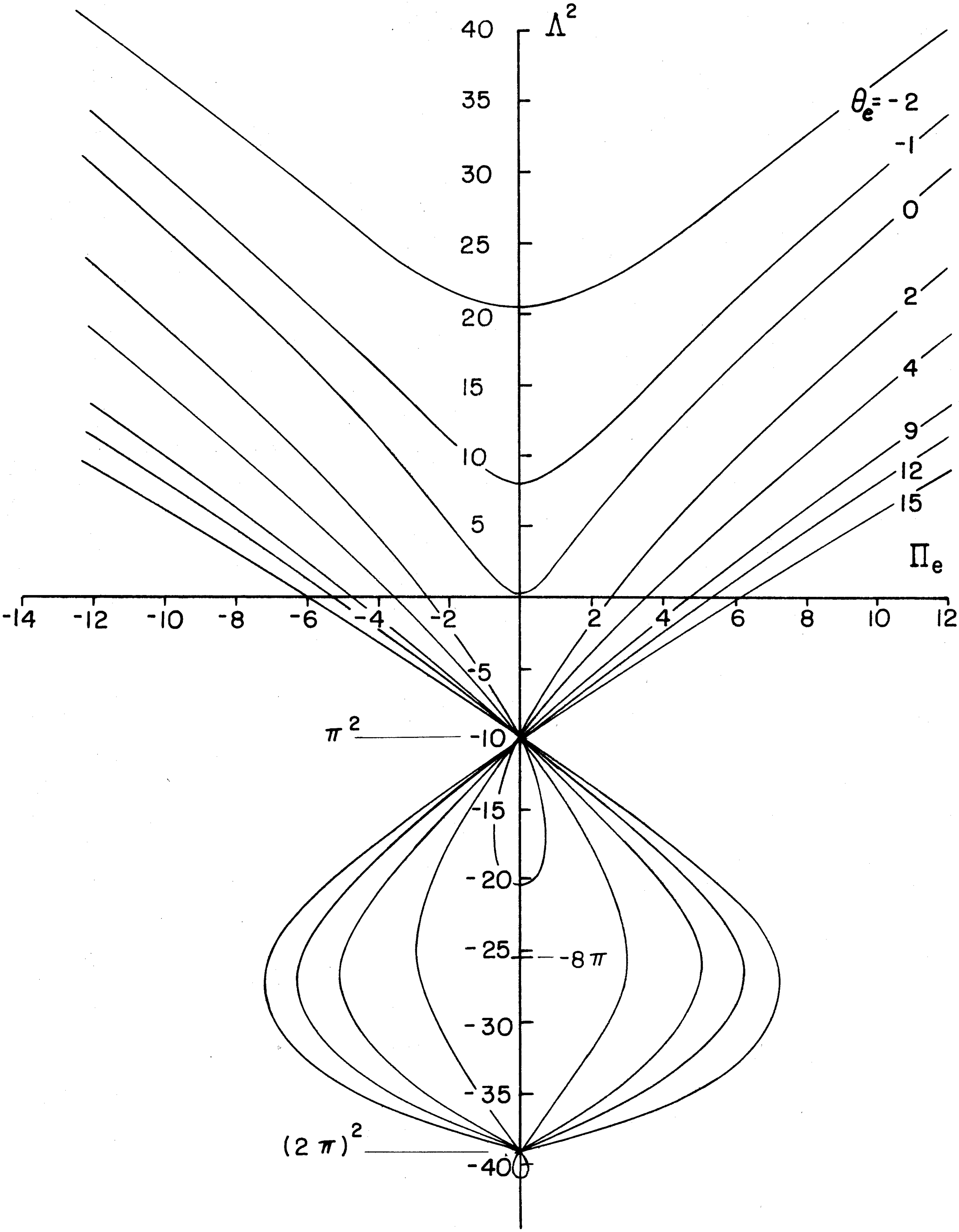



FIG. 5 - EFFECTIVE PRESSURE VS. MEMBRANE STRESS FOR VARIOUS AVERAGE TEMPERATURES (CLAMPED EDGES)

$$\frac{3}{4}\left(\frac{b}{h}\right)^2 \bar{v}_{SS} = \left(\frac{\Lambda^2}{8} + \theta\right)\eta - \frac{1}{3} m\theta\eta^3 - \left(\frac{8}{\Lambda^2}\right)^2 (II - \theta_1 m)^2 \eta^3$$

$$+ \left(\frac{8}{\Lambda^2}\right)^2 \left\{ \frac{3}{\Lambda \cosh \Lambda} \left[ \frac{2}{\Lambda^2} (II - \theta_1 m) + \theta_1 (1 - m) \right] (\Lambda\eta \cosh \Lambda\eta - \sinh \Lambda\eta) \right.$$

$$\left. - \frac{3\Lambda}{16 \cosh^2 \Lambda} \left[ \frac{2}{\Lambda^2} (II - \theta_1 m)^2 + \theta_1 (1-m) \right]^2 (\sinh 2\Lambda\eta - 2\Lambda\eta) \right\}$$

(51)

$$\frac{3}{4}\left(\frac{b}{h}\right)^2 \bar{v}_{SS} = \left(\frac{\Lambda^2}{8} + \theta\right)\eta - \frac{1}{3} m\theta\eta^3 - \left(\frac{8}{\Lambda^2}\right)^2 II_e^2 \eta^3$$

$$+ \left(\frac{8}{\Lambda^2}\right)^2 \left[ \frac{3}{\Lambda \cosh \Lambda} \left( \frac{2}{\Lambda^2} II_e + \theta_{1e} \right) (\Lambda\eta \cosh \Lambda\eta - \sinh \Lambda\eta) \right.$$

$$\left. - \frac{3\Lambda}{16 \cosh^2 \Lambda} \left( \frac{2}{\Lambda^2} II_e^2 + \theta_{1e} \right)^2 (\sinh 2\Lambda\eta - 2\Lambda\eta) \right]$$

The membrane stress condition for simply-supported edges is derived from Equation 51 by requiring $\bar{v}(1) = 0$.

$$\theta(1 - \frac{m}{3}) + \frac{\Lambda^2}{8} = (\frac{8}{\Lambda^2})^2 \Big[ (\Pi - \theta_1 m)^2 F^2(\Lambda)$$

$$+ \frac{3}{2} (\Pi - \theta_1 m)(\theta_1 - \theta_1 m) F_1^2(\Lambda) + \frac{3\Lambda^2}{8} (\theta_1 - \theta_1 m)^2 F_2^2(\Lambda) \Big] \qquad (52)$$

$$\theta_e + \frac{\Lambda^2}{8} = (\frac{8}{\Lambda^2})^2 \Big[ \Pi_e^2 \, F^2(\Lambda) + \frac{3}{2} \Pi_e \theta_{1e} F_1^2(\Lambda) + \frac{3\Lambda^2}{8} \theta_{1e}^2 F_2^2(\Lambda) \Big]$$

where

$$F^2(\Lambda) = 1 - \frac{6}{\Lambda^2} + \frac{15}{2} \frac{\tanh \Lambda}{\Lambda^3} - \frac{3}{2} \frac{\operatorname{sech}^2 \Lambda}{\Lambda^2}$$

$$F_1^2(\Lambda) = \frac{3 \tanh \Lambda}{\Lambda} - 2 - \operatorname{sech}^2 \Lambda$$

$$F_2^2(\Lambda) = \frac{\tanh \Lambda}{\Lambda} - \operatorname{sech}^2 \Lambda$$

These functions are again tabulated in Reference 2 and the stress condition is presented in a series of parametric plots of $\Lambda^2$ versus $\Pi$, for several values of $\theta$ and $\theta_1$.

## C. Total Stress

In Part II, the general expression for surface stress was given (Equation 16) as

$$\frac{b^2 h}{4D} \sigma = \Lambda^2 \pm 48 \left[ \theta_1 r(\eta) - \frac{1}{8} \frac{b}{h} \frac{d^2}{d\eta^2} \overline{w}(\eta) \right]$$

Substituting $f(\eta) = 1 - m\eta^2$ and the two values of $\bar{w}(\eta)$ from Equations 45 and 50, we have for the two cases

$$\frac{b^2h}{4D}\,\sigma_{CL} = \Lambda^2 \pm 48\left[\frac{2}{\Lambda^2}\,\Pi_e\left(1 - \frac{\Lambda\cosh\Lambda\eta}{\sinh\Lambda}\right) + \theta_1(1-m\eta^2)\right] \tag{53}$$

$$\frac{b^2h}{4D}\,\sigma_{SS} = \Lambda^2 \pm 48\left\{\frac{2}{\Lambda^2}\,\Pi_e\left[1 - \left(1 + \frac{\Lambda^2\theta_{1e}}{2\Pi_e}\right)\frac{\cosh\Lambda\eta}{\cosh\Lambda}\right] + \theta_1(1-m\eta^2)\right\} \tag{54}$$

D. Compressive Membrane Stress

As discussed in the references, the character of the analysis changes when the membrane stress $\Lambda^2$ becomes negative. The computations are carried out in the same manner, the only change being that $\Lambda$ becomes $i\Lambda$, $\sinh\Lambda$ becomes $\sinh i\Lambda = i\sin\Lambda$, and $\cosh\Lambda$ becomes $\cosh i\Lambda = \cos\Lambda$.

Physically, this situation results when the linear expansion across the span due to the average temperature rise causes a compression which exceeds the tension due to the pressure load.

Two points are raised which require special attention. To illustrate the first we write Equation 53 as

$$\frac{b^2h}{192D}\,\sigma_{CL} = \frac{\Lambda^2}{48} \pm \left[\frac{2}{\Lambda^2}\,\Pi_e\left(1 - \frac{\Lambda\cosh\Lambda\eta}{\sinh\Lambda}\right) + \theta_1(1-m\eta^2)\right]$$

For $\Lambda^2 < 0$, this becomes

$$\frac{b^2 h}{192D}\sigma_{CL} = \frac{(i\Lambda)^2}{48} \pm \left[\frac{2}{(i\Lambda)^2}\Pi_e\left(1 - \frac{i\Lambda\cosh i\Lambda\eta}{\sinh i\Lambda}\right) + \theta_1(1-m\eta^2)\right]$$

$$= -\frac{\Lambda^2}{48} \pm \left[-\frac{2}{\Lambda^2}\Pi_e\left(1 - \frac{\Lambda\cos\Lambda\eta}{\sin\Lambda}\right) + \theta_1(1-m\eta^2)\right]$$

where $\Lambda$ is the positive square root of $\Lambda^2$. The second term within the brackets is seen to be cyclic, yielding multiple solutions as different values of $(-\Lambda^2)$ are read from Figure 5. These higher energy configurations will generally be unstable and the smallest absolute value of $\Lambda^2$ will be the correct one. The discontinuities arising from the $\sin\Lambda$ term in the denominator are of second order, existing only when $\Pi_e$ equals zero, and can usually be ignored*.

---

* By substituting $i\Lambda = \Lambda$ in the membrane stress condition (Equation 48), one obtains

$$-\frac{\Lambda^2}{8} + \theta_e = \left[-\frac{8}{\Lambda^2}\Pi_e\right]^2\left[1 - \frac{6}{\Lambda^2} + \frac{9\cot\Lambda}{2\Lambda} + \frac{3}{2}\csc^2\Lambda\right]$$

Multiplying through by $\sin^2\Lambda$,

$$\left(-\frac{\Lambda^2}{8} + \theta_e\right)\sin^2\Lambda = \left[-\frac{8}{\Lambda^2}\Pi_e\right]^2\left[\sin^2\Lambda - \frac{6\sin^2\Lambda}{\Lambda^2} + \frac{9\sin\Lambda\cos\Lambda}{2\Lambda} + \frac{3}{2}\right]$$

The discontinuities will occur for values of $\Lambda = n\pi$, for which

$$0 = \left[-\frac{8}{\Lambda^2}\Pi_e\right]^2\left[\frac{3}{2}\right]$$

$$\Pi_e = 0$$

The second point of interest under the subject of compressive membrane stress is our basic assumption that the edges $\eta = \pm 1$ remain fixed under the load. It seems reasonable that a continuous structure which contains ribs or bulkheads as well as stringers would allow only negligible lateral motion of the plate edges under pressure loading alone. In the case of thermal loading, however, it is certain that the plate edges would move apart, due to expansion of the bulkheads. This movement would relieve the compressive stress in the plate itself. Thus the general expansion has the nature of a superimposed tensile stress which must be added algebraically to the (negative) membrane stress. In other words, the compressive effect of the average temperature across the plate with fixed edges would, for practical purposes, be opposed by the same general expansion throughout the structure. An analysis based on fixed edges, such as the present one, therefore errs in favor of compressive stresses. Any specific structure should be analyzed to determine its overall thermal expansion characteristics before attempting a detailed analysis of its components. The present study will yield unconservative tensile stresses if the edges of the strip are allowed to move apart.

E. Temperature Moment Greater than Pressure Moment

The case for $\Pi < \theta_1 m$, that is, when the moment caused by pressure loading is less than the moment caused by uneven heating, results in nothing more than a change of sign. Figure 5 is symmetrical about the $\Pi = 0$ axis and the same values of $\Lambda^2$ will be used (for the same $\theta$). In computing normal deflection, the sign of $\bar{w}$ must be changed if negative values of $(\Pi - \theta_1 m)$ are used. In computing the bending stress, such negative values will give the result that the two bending stress terms now have the same sign and so yield greater absolute stress values.

F. Pressure and Temperature Moments of Like Sense

If the (outer) surface is pressure loaded and also chilled, the average temperature and the temperature moment will be smaller than their respective references and $\theta$ and $\theta_1$ will be negative. The effective pressure, for example, becomes $\Pi_e = \Pi - (-\theta_1 m)$ or just $\Pi + \theta_1 m$ and the design charts are entered at the resulting value. The charts include curves for $\theta_e = -1$, $-2.25$, and $-4$, which are used in exactly the same manner as their positive counterparts.

# VII. COMPARISON OF EXACT AND APPROXIMATE SOLUTIONS

The sample problem of Part IV D can now be set up with the parabolic approximations and the two sets of solutions can be compared directly.

Material and Geometry:

Clamped edges; $b = 10$ in; $h = 0.1$ in

$E = 10^7$ psi; $\nu = 0.25$; $\alpha = 13 \times 10^{-6}$ in/in/$^\circ$F

Applied Pressure and Temperature Loads:

$p_o = 20$ psi $\quad T_1 = 400^\circ$F $\quad T_2 = 250^\circ$F

$T_3 = 100^\circ$F $\quad T_4 = 50^\circ$F

The exact definitions of the average temperature and temperature moment distributions were found to be

$$\overline{T}(\eta) = \overline{T}_o(\overline{T}_h + \cos \pi\eta)$$

$$= 15.7(12.7 + \cos \pi\eta)$$

$$kT^{(1)}(\eta) = kT_o^{(1)}(T_h^{(1)} + \cos \pi\eta)$$

$$= 2.73\,(7.6 + \cos \pi\eta)$$

The corresponding values to be used in the parabolic approximation are

$$m = \frac{2}{1 + T_h^{(1)}} = \frac{2}{1 + 7.6} = 0.23$$

$$\overline{T}(\eta) = \frac{\overline{T}_o \, \overline{T}_h}{1 - m/3}(1 - m\eta^2)$$

$$= \frac{(15.7)(12.7)}{1 - \frac{1}{3}(0.23)}(1 - 0.23\,\eta^2)$$

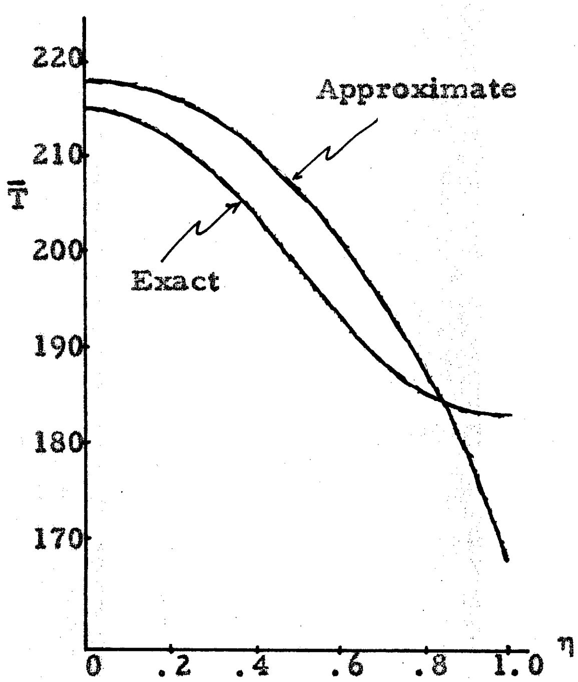
$$= 217(1 - 0.23\,\eta^2)$$

$$kT^{(1)}(\eta) = kT_o^{(1)}(T_h^{(1)} + 1)(1 - m\eta^2)$$

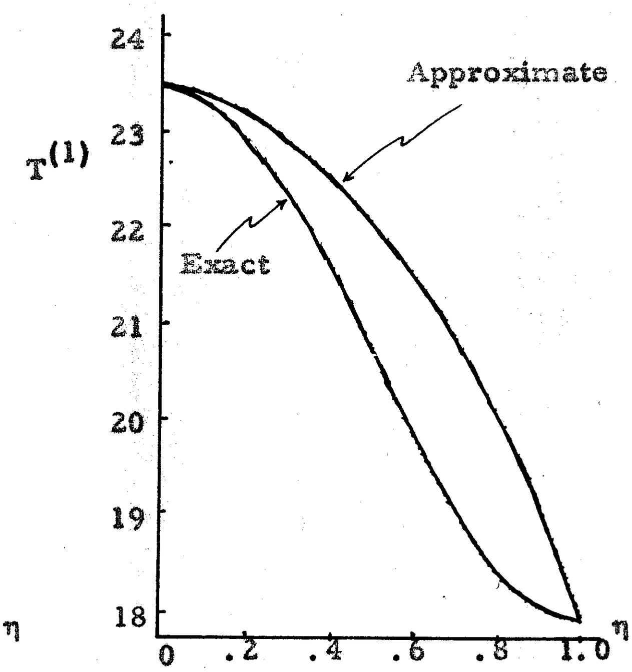
$$= 2.73(7.6 + 1)(1 - 0.23\,\eta^2)$$

$$= 23.5(1 - 0.23\,\eta^2)$$

These distribution functions are compared graphically below.




Average Temperature




Temperature Moment

The approximate dimensionless loading parameters are

$$\Pi = 8.5$$

$$\theta = 13.4$$

$$\theta_1 = 1.43$$

and the effective parameters are

$$\Pi_e = \Pi - m\theta_1 = 8.5 - (0.23)(1.43) = 8.2$$

$$\theta_e = \theta(1 - \frac{m}{3}) = 13.4\ (1 - \frac{0.23}{3}) = 12.3$$

$$\theta_{1e} = \theta_1(1 - m) = 1.43\ (1 - 0.23) = 1.11$$

Enter Figure 5 at $\Pi_e = 8.2$ and move up to the line for $\theta_e = 12$. Read $\Lambda^2 = 4.5$ corresponding to $\theta_e = 12.3$. (This value of $\Lambda^2$ compares to $\Lambda^2 = 6.25$ which was found by the exact equations. This error in membrane stress is not an accurate measure of the approximation, since the parabolic distributions were selected to equate the bending stresses and deflections, rather than the membrane stress.) Figure 5 is, of course, a plot of the solutions of Equation 49, from which the membrane stress could also have been found.

The normal deflection (Equation 45) and the surface stresses (Equation 53) can now be found as functions of $\eta$. These curves are plotted in dashed lines on Figures 6 and 7. For comparison, the exact solutions as found in Part IV D are shown in solid lines.

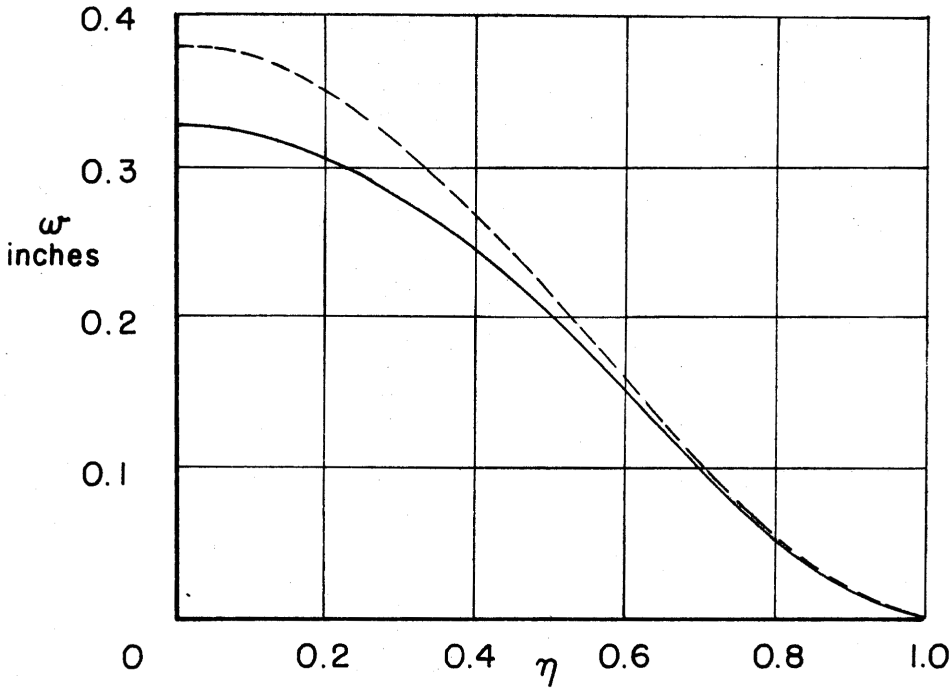



FIG. 6 – NORMAL DEFLECTION, EXACT (SOLID) AND APPROX. (DASHED)

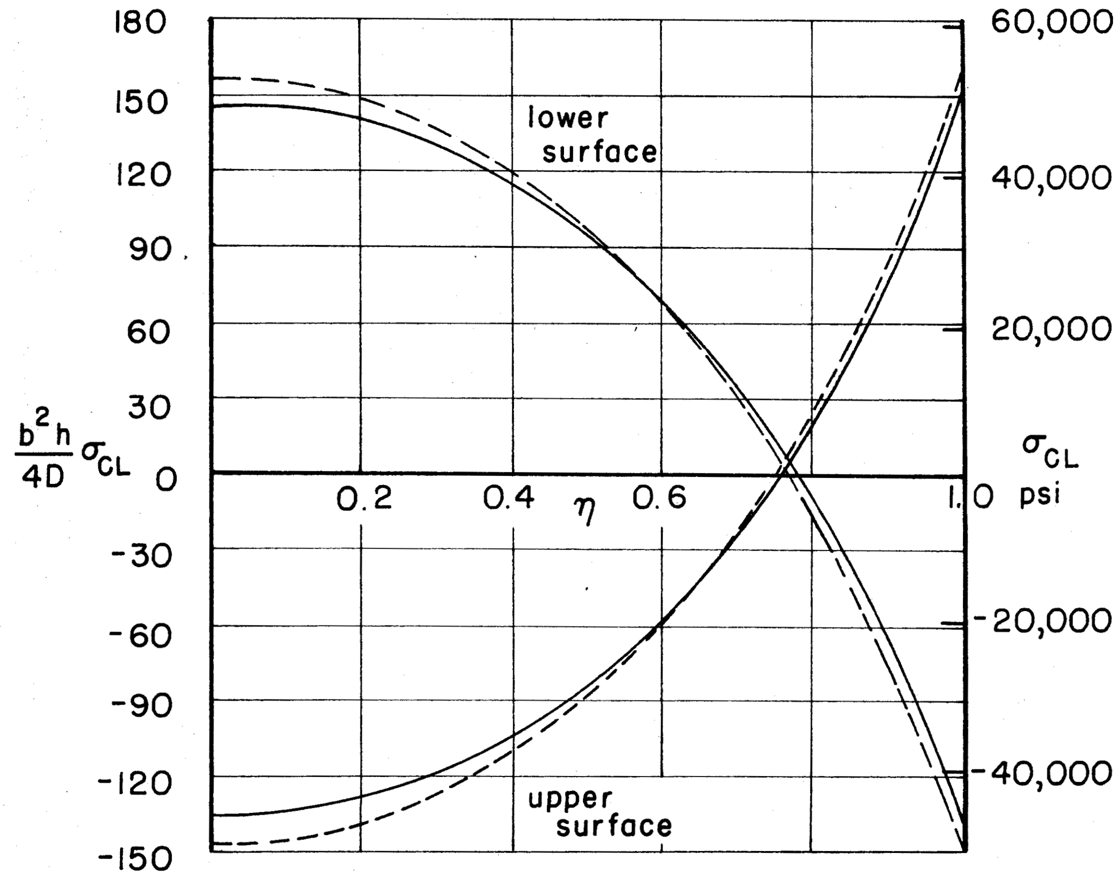



FIG. 7 – SURFACE STRESS, EXACT (SOLID) AND APPROX. (DASHED)

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1. Williams, M. L., "Large Deflection Analysis for a Plate Strip Subjected to Normal Pressure and Heating", Journal of Applied Mechanics (December 1955), Vol. 22, No. 4, pp. 458-464.

2. Williams, M. L., "Further Large Deflection Analysis for a Plate Strip Subjected to Normal Pressure and Heating", Journal of Applied Mechanics (June 1958), Vol. 25, No. 2, pp. 251-258.

3. Lindquist, D. M. W., "Large Deflection Analysis for a Plate Strip under Combined Pressure and Spanwise Varying Temperature", (June 1959), California Institute of Technology Southern California Cooperative Wind Tunnel Report No. P-20.

# APPENDIX A

## A SOLUTION OF DIFFERENTIAL EQUATIONS OF GENERAL ORDER

In developing the parabolic temperature distribution function $1 - m\eta^2$, several higher order functions of the form

$$1 - m\eta^n$$

were considered. Using this function in the normal equilibrium equation gives an equation which may be written as

$$\frac{dw}{d\eta^2} - aw = b\eta^2 + c + d\eta^n$$

Since there are an infinite number of linearly independent derivatives of the last term, a closed - form solution cannot be written. A general solution can be obtained, however, by the following procedure.

The solution of the homogeneous equation is

$$w_H = A_1 e^{a\eta} + B_1 e^{-a\eta}$$

$$= A_2 \sinh a\eta + B_2 \cosh a\eta$$

Particular solutions to the inhomogeneous equation can be found by assigning successive integral values to n. Such solutions are tabulated in Table A. 1.

We note that the $\eta^2$ and $\eta^0$ terms in the inhomogeneous equation give rise to recurring terms in the coefficients of $\eta^2$ and $\eta^0$, independent of the value of n. Thus $w_{p_1} = -\frac{b}{a}\eta^2 - \frac{2b}{a^2} - \frac{c}{a}$ will appear as part of each solution. Rewriting the table, omitting these terms, we obtain the coefficients given in Table A. 2.

| | $\eta^6$ | $\eta^5$ | $\eta^4$ | $\eta^3$ | $\eta^2$ | $\eta^1$ | $\eta^0$ |
|---|---|---|---|---|---|---|---|
| n = 0 | | | | | $-\frac{b}{a}$ | 0 | $-\frac{2b}{a^2}-\frac{c}{a}-\frac{d}{a}$ |
| 1 | | | | | $-\frac{b}{a}$ | $-\frac{d}{a}$ | $-\frac{2b}{a^2}-\frac{c}{a}$ |
| 2 | | | | | $-\frac{b}{a}-\frac{d}{a}$ | 0 | $-\frac{2b}{a^2}-\frac{c}{a}-\frac{2d}{a^2}$ |
| 3 | | | | $-\frac{d}{a}$ | $-\frac{b}{a}$ | $-\frac{6d}{a^2}$ | $-\frac{2b}{a^2}-\frac{c}{a}$ |
| 4 | | | $-\frac{d}{a}$ | 0 | $-\frac{b}{a}-\frac{12d}{a^2}$ | 0 | $-\frac{2b}{a^2}-\frac{c}{a}-\frac{24d}{a^3}$ |
| 5 | | $-\frac{d}{a}$ | 0 | $-\frac{20d}{a^2}$ | $-\frac{b}{a}$ | $-\frac{120d}{a^3}$ | $-\frac{2b}{a^2}-\frac{c}{a}$ |
| 6 | $-\frac{d}{a}$ | 0 | $-\frac{30d}{a^2}$ | 0 | $-\frac{b}{a}-\frac{360d}{a^3}$ | 0 | $-\frac{2b}{a^2}-\frac{c}{a}-\frac{720d}{a^4}$ |

Table A. 1

| | $\eta^6$ | $\eta^5$ | $\eta^4$ | $\eta^3$ | $\eta^2$ | $\eta^1$ | $\eta^0$ |
|---|---|---|---|---|---|---|---|
| n = 0 | | | | | | | $-\frac{d}{a}$ |
| 1 | | | | | | $-\frac{d}{a}$ | |
| 2 | | | | | $-\frac{d}{a}$ | | $-\frac{2d}{a^2}$ |
| 3 | | | | $-\frac{d}{a}$ | | $-\frac{6d}{a^2}$ | |
| 4 | | | $-\frac{d}{a}$ | | $-\frac{12d}{a^2}$ | | $-\frac{24d}{a^3}$ |
| 5 | | $-\frac{d}{a}$ | | $-\frac{20d}{a^2}$ | | $-\frac{120d}{a^3}$ | |
| 6 | $-\frac{d}{a}$ | | $-\frac{30d}{a^2}$ | | $-\frac{360d}{a^3}$ | | $-\frac{720d}{a^4}$ |

Table A.2

The pattern followed by the coefficients now becomes obvious. We can write

$$w_{p_2} = -\frac{n!}{n!}\frac{d}{a}\eta^n - \frac{n!}{(n-2)!}\frac{d}{a^2}\eta^{n-2} - \frac{n!}{(n-4)!}\frac{d}{a^3}\eta^{n-4} - - -$$

This sequence can be written in summation form as

$$w_{p_2} = -\sum_{p=0,\,2,\,4,}^{n} \frac{n!}{(n-p)!} a^{-\frac{1}{2}(p+2)} d\eta^{n-p}$$

The complete solution is then

$$w = w_H + w_{p_1} + w_{p_2}$$

$$w = A_2 \sinh a\eta + B_2 \cosh a\eta - \frac{b}{a}\left(\eta^2 + \frac{2}{a} + \frac{c}{b}\right)$$

$$- \sum_{p=0}^{n} \frac{n!}{(n-p)!} \; a^{-\frac{1}{2}(p+2)} d\eta^{n-p}$$