THE MOTION OF A CURRENT ELEMENT THROUGH A FLUID OF LOW ELECTRICAL CONDUCTIVITY

Thesis by

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ABSTRACT

Two-dimensional flow of an incompressible, viscous, electrically conducting fluid past a current element is studied. A solution in the form of an asymptotic development is obtained, valid as a certain dimensionless parameter (essentially the product of the electrical conductivity and the current) tends to zero. An expression for the drag on the current element is computed, and is found to be independent of viscosity.

LIST OF PRINCIPAL SYMBOLS

1. Dimensional Variables and Parameters

$x, y; \xi, \eta$	Cartesian space coordinates, meters
$r, \theta; \rho, \phi$	Polar space coordinates
$\vec{r}\,; \vec{\rho}$	Space vectors
$\vec{i}_x, \vec{i}_y, \vec{i}_z$	Cartesian unit vectors
$\vec{q} = \vec{i}_x u + \vec{i}_y v$	Flow velocity vector, meters/second
p	Pressure, newtons/meter^2
$\vec{B} = \vec{i}_x B_x + \vec{i}_y B_y$	Magnetic induction vector, webers/meter^2
$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$	Vorticity
$U\vec{i}_x$	Velocity at infinity
p_∞	Pressure at infinity
ρ	Density (constant)
ν	Kinematic viscosity (constant)
μ	Magnetic permeability (constant)
I	Current in current element (constant)
σ	Electrical conductivity (constant), mhos/meter
F_{resist}	See equation 6-1
F_{surf}	See equation 6-2

2. Dimensionless Variables

a. Outer variables

$$x^* = x\sigma\mu U, \quad y^* = y\sigma\mu U, \quad \xi^* = \xi\sigma\mu U, \quad \eta^* = \eta\sigma\mu U,$$

$$r^* = r\sigma\mu U, \quad \rho^* = \rho\sigma\mu U,$$

$$\vec{r}^* = \vec{r}\,\sigma\mu U, \quad \vec{\rho}^* = \vec{\rho}\,\sigma\mu U;$$

$$\vec{q}^* = \vec{i}_x u^* + \vec{i}_y v^* = \vec{i}_x \frac{u}{U} + \vec{i}_y \frac{v}{U},$$

$$p^* = \frac{p - p_\infty}{\rho U^2} ,$$

$$\vec{B}^* = \vec{B}/\sigma I U \mu^2$$

b. Inner variables

$$x^+ = x^*/\epsilon, \quad y^+ = y^*/\epsilon , \quad \xi^+ = \xi^*/\epsilon, \quad \eta^+ = \eta^*/\epsilon ,$$

$$r^+ = r^*/\epsilon, \quad \rho^+ = \rho^*/\epsilon,$$

$$\vec{r}^+ = \vec{r}^*/\epsilon, \quad \vec{\rho}^+ = \vec{\rho}^*/\epsilon ;$$

$$p^+ = \epsilon p^*$$

$$\vec{B}^+ = \epsilon \vec{B}^*$$

3. <u>Dimensionless Parameters</u>

$$\alpha = \sigma \nu \mu$$

$$\epsilon = \sigma I \sqrt{\mu^3/\rho}$$

Re_v	Reynolds number (see equation 6-7)
Re	Reynolds number
S_i, S_v	"Cores" for inviscid and viscous fluids
g_i, g_v	Characteristic size of cores
u_i, u_v	Velocity inside core

I. INTRODUCTION

In all but a few problems of magnetohydrodynamics, it has been found necessary to introduce various simplifications in order to facilitate a solution. Part of the difficulty is undoubtedly due to the large number of variables and differential equations involved, but a somewhat more basic difficulty lies in the inherent non-linear character of the equations. Many investigators have circumvented this difficulty by choosing problems for which the non-linearity could be relaxed or even neglected. This thesis investigates a problem in which these non-linear effects are not neglected.

The problem which we shall consider turns out to have two characteristic parameters; an ϵ which depends linearly on both the electrical conductivity and a certain current, and an α which depends linearly on the viscosity of the fluid and on the electrical conductivity. Since the solution we shall obtain is valid for ϵ small and α fixed, two distinct situations present themselves:

(i) we can make ϵ small by keeping the electrical conductivity fixed and letting the current be small, in which case α is constant if the viscosity is kept fixed;

(ii) we can make ϵ small by keeping the current constant and letting the electrical conductivity be small, in which case α is constant only if the viscosity tends to infinity like the reciprocal of electrical conductivity.

Thus, in order to consider fluids of low electrical conductivity, it seems that the fluid must possess very large viscosity. This would not be necessary, however, if our solution were uniformly valid in α; in particular, if α is very small. Two reasons which indicate the solutions might indeed be uniformly valid as $\alpha \rightarrow 0$ are presented in Section X.

The approach to the problem is based largely on the researches of S. Kaplun and P. A. Lagerstrom on low Reynolds number flow. (1)

II. STATEMENT OF THE PROBLEM

We consider the two-dimensional steady flow of an incompressible fluid possessing kinematic viscosity ν and electrical conductivity σ. At the origin of coordinates (see fig. 1), there is a current element carrying I amperes in a direction normal to the flow (toward the viewer in fig. 1). The flow velocity $\vec{q}$ and the fluid dynamic pressure p are constant at infinity.

We denote the position vector by $\vec{r} = \vec{i}_x x + \vec{i}_y y = \vec{i}_r r + \vec{i}_\theta \theta$, the magnetic induction vector by $\vec{B} = \vec{i}_x B_x + \vec{i}_y B_y$, the velocity vector by $\vec{q} = \vec{i}_x u + \vec{i}_y v$, the gradient operator by $\nabla = \vec{i}_x \frac{\partial}{\partial x} + \vec{i}_y \frac{\partial}{\partial y}$, the Laplacian operator ∇^2 by $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and the constant magnetic permeability by μ.

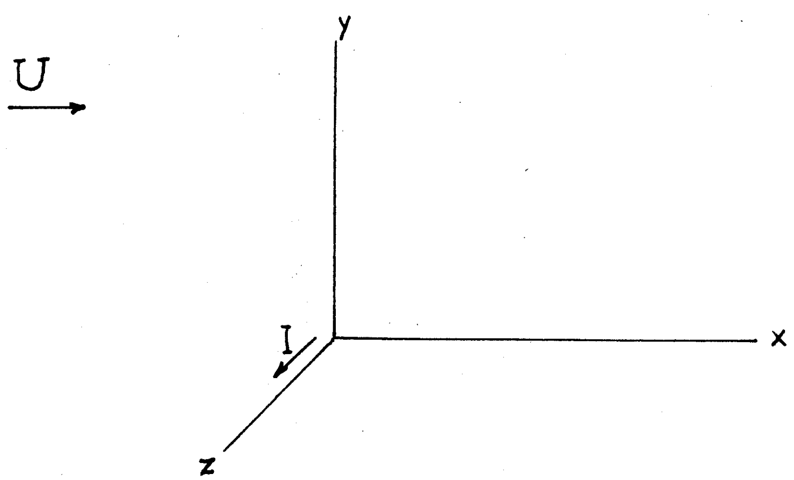

Figure 1

The flow is assumed to be described by the following differential equations and boundary conditions (the rationalized MKSQ system is used throughout) :

$$\nabla \cdot \vec{q} = 0 \tag{2-1}$$

$$(\vec{q} \cdot \nabla)\vec{q} + \frac{1}{\rho}\nabla p = \frac{1}{\rho\mu}(\nabla \times \vec{B}) \times \vec{B} + \nu\nabla^2\vec{q} \tag{2-2}$$

$$\nabla^2\vec{B} + \sigma\mu\nabla \times (\vec{q} \times \vec{B}) = 0 \tag{2-3}$$

$$\nabla \cdot \vec{B} = 0 \tag{2-4}$$

$$p(\infty) = p_\infty, \quad \vec{q}(\infty) = \vec{i}_x U, \quad \vec{B}(\infty) = 0 \tag{2-5}$$

$$\lim_{r \to 0} \oint \vec{B} \cdot d\vec{r} = \mu I \tag{2-6}$$

Before proceeding to non-dimensionalize the above equations, we shall discuss the characteristic lengths and parameters of the problem.

There are three basic characteristic lengths appearing, namely

(i) a length based on the current, $L_1 = \frac{I}{U}\sqrt{\frac{\mu}{\rho}}$;

(ii) a magnetic diffusion length, $L_2 = 1/\sigma\mu U$;

(iii) a length based on viscous diffusion, $L_3 = \nu/U$.

Intuitively, one would expect that close to the origin, where the current is situated, the length over which significant changes take place would be L_1. At greater distances from the origin both L_2 and L_3 would be expected to be important, since the vorticity and magnetic field tend to diffuse away from the streamlines.

From the three characteristic lengths we can obtain the two characteristic dimensionless parameters of the problem, ϵ and a. Thus, the ratio $L_1 : L_2$ gives

$$L_1 : L_2 = I\sigma \sqrt{\frac{\mu^3}{\rho}} = \epsilon$$

and the ratio $L_3 : L_2$ gives

$$L_3 : L_2 = \nu\sigma\mu = a .$$

Thus ϵ small means that the current length (which can in a vague sense be considered a body length) is small compared with the magnetic diffusion length, while a constant means we are keeping the ratio of the two diffusion lengths constant. It will turn out that these two diffusion lengths characterize two different types of wake far downstream. Firstly, there is a magnetic wake whose width depends only on the magnetic diffusion length, L_2, and secondly, a viscous wake whose width depends only on the viscous diffusion length, L_3.

Let us now proceed to write the equations in dimensionless form. Non-dimensional quantities are introduced as follows:

outer variables:

$$\vec{q}^{\,*} = \frac{\vec{q}}{U}, \quad u^* = \frac{u}{U}, \quad v^* = \frac{v}{U};$$

$$p^* = \frac{p - p_\infty}{\rho U^2};$$

$$\vec{B}^* = \frac{\vec{B}}{\sigma I U \mu^2}, \quad B_x^* = \frac{B_x}{\sigma I U \mu^2}, \quad B_y^* = \frac{B_y}{\sigma I U \mu^2};$$

$$\vec{r}^{\,*} = \vec{r}\,\sigma\mu U, \quad x^* = x\sigma\mu U, \quad y^* = y\sigma\mu U .$$

inner variables:

$$\vec{q}^{+} = \frac{\vec{q}}{U}, \quad u^{+} = \frac{u}{U}, \quad v^{+} = \frac{v}{U};$$

$$p^{+} = (p - p_{\infty}) \frac{\sigma I}{U^2} \sqrt{\frac{\mu^3}{\rho^3}} = \epsilon p^{*}$$

$$\vec{B}^{+} = \frac{\vec{B}}{U\sqrt{\rho\mu}} = \epsilon \vec{B}^{*}, \quad B_x^{+} = \frac{B_x}{U\sqrt{\rho\mu}} = \epsilon B_x^{*}, \quad \vec{B}_y^{+} = \frac{B_y}{U\sqrt{\rho\mu}} = \epsilon B_y^{*};$$

$$\vec{r}^{+} = \frac{\vec{r}}{\frac{I}{U}\sqrt{\frac{\mu}{\rho}}}, \quad x^{+} = \frac{x}{\frac{I}{U}\sqrt{\frac{\mu}{\rho}}}, \quad y^{+} = \frac{y}{\frac{I}{U}\sqrt{\frac{\mu}{\rho}}}.$$

(The motivation for the terminology "inner" and "outer" will become apparent later, and the method of non-dimensionalizing the dependent variables is discussed in Section IV.)

Written in outer coordinates, with $\nabla^{*} = \vec{i}_x \frac{\partial}{\partial x^{*}} + \vec{i}_y \frac{\partial}{\partial y^{*}}$, equations 2-1 to 2-6 become

$$\nabla^{*} \cdot \vec{q}^{*} = 0 \tag{2-7}$$

$$(\vec{q}^{*} \cdot \nabla^{*})\vec{q}^{*} + \nabla^{*} p^{*} = \epsilon^2 (\nabla^{*} \times \vec{B}^{*}) \times \vec{B}^{*} + \alpha \nabla^{*2} \vec{q}^{*} \tag{2-8}$$

$$\nabla^{*2} \vec{B}^{*} + \nabla^{*} \times (\vec{q}^{*} \times \vec{B}^{*}) = 0 \tag{2-9}$$

$$\nabla^{*} \cdot \vec{B}^{*} = 0 \tag{2-10}$$

$$\vec{q}^{*}(\infty) = \vec{i}_x, \quad p^{*}(\infty) = 0, \quad \vec{B}^{*}(\infty) = 0 \tag{2-11}$$

$$\lim_{r^{*} \to 0} \oint \vec{B}^{*} \cdot d\vec{r}^{*} = 1 \tag{2-12}$$

where $\alpha = \nu\sigma\mu$ and $\epsilon = \sigma I \sqrt{\frac{\mu^3}{\rho}}$.

Written in inner coordinates, with $\nabla^+ = \vec{i}_x \frac{\partial}{\partial x^+} + \vec{i}_y \frac{\partial}{\partial y^+}$, equations 2-1 to 2-6 become

$$\nabla^+ \cdot \vec{q}^+ = 0 \tag{2-13}$$

$$\epsilon(\vec{q}^+ \cdot \nabla^+)\vec{q}^+ + \nabla^+ p^+ = \epsilon(\nabla^+ \times \vec{B}^+) \times \vec{B}^+ + a\nabla^{+2}\vec{q}^+ \tag{2-14}$$

$$\nabla^{+2}\vec{B}^+ + \epsilon\nabla^+ \times (\vec{q}^+ \times \vec{B}^+) = 0 \tag{2-15}$$

$$\nabla^+ \cdot \vec{B}^+ = 0 \tag{2-16}$$

$$\vec{q}^+(\infty) = \vec{i}_x, \quad p^+(\infty) = 0, \quad \vec{B}^+(\infty) = 0 \tag{2-17}$$

$$\lim_{r^+ \to 0} \oint \vec{B}^+ \cdot d\vec{r}^+ = 1 \tag{2-18}$$

III. ASYMPTOTIC DEVELOPMENTS

(In the following sections, it is assumed that the reader is familiar with reference 1.)

From the non-dimensional form of the equations, we observe that two dimensionless parameters are present, namely ϵ and a. Under the assumption that our problem has a unique solution, we attempt to find an approximate solution valid for ϵ sufficiently small, with a held fixed. Such an approximate solution might be checked by an experiment in which a given fluid flows past a thin current-carrying wire -- just how thin is discussed in section X. The value of ϵ is made small by decreasing the current in the wire. Since a depends only on the properties of the fluid, it remains fixed.

Our approximate solution will be in the form of an asymptotic development. As mentioned in the introduction, the general approach is based on the work of Kaplun and Lagerstrom; however, their methods are aimed at finding a "composite" development uniformly valid in the entire region of space in which the problem is defined. For example, in reference 1 a solution for the region of flow exterior to the sphere is found. There, one has an outer (Oseen) limit, valid near infinity, and an inner (Stokes) limit, valid near the body, which can be obtained by physical reasoning. Then one finds a solution to some approximate differential equations which in a sense completes the determination of the solution to be perturbed by describing what is happening between the vicinity of the body and the vicinity of infinity. In other words, a great deal is known about the solution to be perturbed, based largely on physical reasoning.

In the problem of the present study, however, the author finds himself unable to state with confidence what the limiting solution is in a region very close to the origin; consequently, the solution found can claim validity only in a region away from the origin. This region will be precisely defined in Section IV; for the present, we shall refer to it as the "exterior region." Moreover, simply knowing what solution in the exterior region to perturb will not guarantee that the terms obtained by the perturbation scheme include all the terms that should be present. In fact, since the magnetic field becomes infinite near the origin, it is conceivable that a disturbance originating very near the origin could yield the largest term in the "exterior" region. However, certain plausability arguments will be given later which provide us with enough information about the flow very near the origin to assure us that terms arising from such disturbances near the origin are small compared with the first and second perturbation terms found by a straightforward perturbation scheme applied to the "exterior" region. Thus, considerable information (including first and second approximations to the drag!) can be derived, even though finding a uniformly valid composite development appears hopelessly complicated. Furthermore, our "exterior" region actually covers much of the flow region, since it includes everything outside a radius of the order $\frac{I}{U}\sqrt{\frac{\mu}{\rho}}$. Mercury, for example, using $U = 10^{-1}$ meter/second, and current of one ampere, gives $\frac{I}{U}\sqrt{\frac{\mu}{\rho}} \sim 10^{-4}$ meters.

The terms of the asymptotic development valid in the "exterior" region will be derived from two "principal" developments; an "inner" development valid in a region of order $\frac{1}{U}\sqrt{\frac{\mu}{\rho}}$ meters from the origin, and an "outer" development, valid at infinity. These "inner" and "outer" developments will in turn be constructed from the differential equations.

The relationship of these asymptotic developments to the exact solution is based on the assumption that asymptotic developments (relative to a suitable sequence) of the exact solution can be constructed by certain limit processes. We shall presently define an "inner" limit process and an "outer" limit process, and the developments which in principle could be obtained from the exact solution by means of these two limit processes are then identified with the inner and outer developments found from the differential equations.

In general, neither the inner nor the outer development can by itself be expected to be uniformly valid in the exterior region. Nevertheless they must be related, since they are assumed to be asymptotic developments of the same function. This relationship is of fundamental importance when discussing so-called "matching conditions." These matching conditions arise as follows: when we come to construct the developments from the differential equations, the inner solution, which is valid near the origin, will not satisfy the boundary condition at infinity. Similarly, the outer development, which does satisfy the boundary condition at infinity, will not in general satisfy the inner boundary conditions, if such conditions exist. The role of the matching condition is to replace the boundary condition at infinity for the inner solution and the boundary condition at the origin for the outer solution. The method of matching adopted in the present problem is based on the methods of reference 1.

Having made a few comments on our proposed asymptotic developments, we proceed now to a somewhat more detailed description.

IV. LIMIT PROCESSES, INNER AND OUTER LIMITS, INNER AND OUTER VARIABLES

We now define in precise terms the limit processes mentioned in the last section. First (for the sake of easy comparison with reference 1), we non-dimensionalize $\vec{r}$ with a magnetic "viscous" length $1/\sigma\mu U$, so that $\vec{r} = \vec{r}^{\,*}/\sigma\mu U$. Recall the characteristic parameter

$$\epsilon = \sigma I \sqrt{\frac{\mu^3}{\rho}} \tag{4-1}$$

We then define the limit "$\lim_f$" of a function $fn(\vec{r}^{\,*};\epsilon)$ as

$$\lim_f \{fn(\vec{r}^{\,*};\epsilon)\} = \lim_{\epsilon \downarrow 0} fn(\vec{r}^{\,*};\epsilon) \tag{4-2}$$

keeping $\vec{r}^{\,(f)} = \dfrac{\vec{r}^{\,*}}{f(\epsilon)}$ constant.

Given a sequence of functions $\{\epsilon_j(\epsilon)\}$ for which

$$\lim_{\epsilon \downarrow 0} \frac{\epsilon_{j+1}(\epsilon)}{\epsilon_j(\epsilon)} = 0 \tag{4-3}$$

we use the above limit process to construct an asymptotic development

$$F(\vec{r}^{\,*};\epsilon) \sim \sum_{k=0}^{N} \epsilon_k(\epsilon)\varphi(\vec{r}^{\,(f)};\epsilon) + O(\epsilon_{N+1}(\epsilon)) \tag{4-4}$$

of a flow quantity $F(\vec{r}^{\,*};\epsilon)$, as follows:

$$\varphi_o(\vec{r}^{\,(f)};\epsilon) = \lim_f F(\vec{r}^{\,*};\epsilon) \tag{4-5}$$

$$\varphi_{n+1}(\vec{r}^{\,(f)};\epsilon) = \lim_f \frac{F(\vec{r}^{\,*};\epsilon) - \sum_{k=0}^{n} \epsilon_k(\epsilon)\varphi_k(\vec{r}^{\,(f)};\epsilon)}{\epsilon_{n+1}(\epsilon)} \tag{4-6}$$

Now the inner and outer independent variables are simply two different forms of $\vec{r}^{(f)}$, and the inner (outer) developments are obtained by performing the above limit process keeping the inner (outer) independent variables fixed. To find the two forms of $f(\epsilon)$ defining the inner and outer variables, we proceed as follows: the differential equations to be used in the perturbation scheme are found by writing equations 2-1 to 2-4 using $\vec{r}^{(f)}$ as independent variable, and

$$\vec{q}^{(f)} = \vec{q}/U \tag{4-7}$$

$$\vec{B}^{(f)} = \frac{\vec{B}}{\sigma I U \mu^2} f(\epsilon) \tag{4-8}$$

and

$$p^{(f)} = \frac{p - p_\infty}{\rho U^2} f(\epsilon) \tag{4-9}$$

as dependent variables.

We non-dimensionalize $\vec{q}$ according to equation 4-7 because (as will be discussed in section VI), in this form the leading velocity term is $O(1)$, in the set $\text{ord}\, \epsilon \leq \text{ord}\, f(\epsilon) \leq \text{ord}\, 1$. The choice of the $\vec{B}^{(f)}$ is derived from the fact that the first term in the magnetic induction development will be obtained from the induction equation, and the choice of $\vec{B}^{(f)}$ in equation 4-8 leads to a development with a leading term $O(1)$. The choice of equation 4-9 for the pressure $p^{(f)}$ comes from the fact that unless the pressure term appears in the approximate momentum equation, the system of approximate equations will be over determined. This is easily seen as follows: the approximate momentum equation can be written

$$a\nabla^{(f)2}\vec{q}^{(f)} - f(\epsilon)\nabla^{(f)}p^{*} = -\frac{\epsilon^2}{f(\epsilon)}(\nabla^{(f)} \times \vec{B}^{(f)}) \times \vec{B}^{(f)} \tag{4-10}$$

where $p^{*} = \frac{p-p_{\infty}}{\rho U^2}$, and the right side of equation 4-10 is $o(1)$ for ord $\epsilon \leq f(\epsilon) \leq$ ord 1. Operating on equation 4-10 with the divergence operator, and noting that $\nabla^{(f)} . \vec{q}^{(f)} = \nabla^{(f)} . \vec{B}^{(f)} = 0$, we obtain

$$-f(\epsilon)\nabla^{(f)2}p^{*} = -\frac{\epsilon^2}{f(\epsilon)}(B_x^{(f)}\nabla^{(f)2}B_y^{(f)} + B_y^{(f)}\nabla^{(f)2}B_x^{(f)}) \tag{4-11}$$

Thus, unless we keep the pressure term as an unknown, we shall have a contradiction if the right side of equation 4-11 is not zero at each stage of the development. It will turn out that the right side is <u>not</u> identically zero at each stage, so we are forced to non-dimensionalize the pressure in such a way that it does not drop out of the approximate differential equation. This is accomplished by the choice given by equation 4-9.

Using the above $\vec{r}^{(f)}$, $\vec{q}^{(f)}$, $\vec{B}^{(f)}$, and $p^{(f)}$ in the differential equations, and setting $\epsilon \equiv 0$, we obtain for the case $\epsilon \leq$ ord $f(\epsilon) <$ ord 1

$$\nabla^{(f)} . \vec{q}^{(f)} = 0 \tag{4-12}$$

$$a\nabla^{(f)2}\vec{q}^{(f)} - \nabla^{(f)}p^{(f)} = 0 \tag{4-13}$$

$$\nabla^{(f)2}\vec{B}^{(f)} = 0 \tag{4-14}$$

$$\nabla^{(f)} . \vec{B}^{(f)} = 0 \tag{4-15}$$

where

$$\nabla^{(f)} = \vec{i}_x \frac{\partial}{\partial x^{(f)}} + \vec{i}_y \frac{\partial}{\partial y^{(f)}}$$

and

$$\nabla^{(f)2} = \frac{\partial^2}{\partial x^{(f)2}} + \frac{\partial^2}{\partial y^{(f)2}}$$

The asymptotic developments resulting from these equations are not valid at infinity. (This will be seen explicitly when we compute the inner development.) However, if we put

$$\text{ord } f(\epsilon) = \text{ord } 1 \qquad (4\text{-}16)$$

then the differential equations to be used in the perturbation scheme are found by writing equations 2-1 to 2-4 in $\vec{r}^{(f_o)}$ independent variables and dependent variables given by equations 4-7, 4-8, 4-9 in which $f(\epsilon) = 1$ (the resulting variables are the same as those defined after equation 2-6), and letting $\epsilon \equiv 0$. The resulting equations are easily seen to be equations 2-7, 2-10,

$$(\vec{q}^* \cdot \nabla^*)\vec{q}^* + \nabla^* p^* = a\nabla^{*2}\vec{q}^* \qquad (4\text{-}17)$$

and

$$\nabla^{*2}\vec{B}^* + \nabla^* \times (\vec{q}^* \times \vec{B}^*) = 0 \qquad (4\text{-}18)$$

Using these equations, we can obtain an asymptotic development satisfying the boundary conditions at infinity. Thus we use

$$r^* = \frac{\vec{r}^*}{1} \qquad (4\text{-}19)$$

as our outer variables, and the outer development will be obtained by perturbing equations 2-7, 2-10, 2-17 and 2-18.

The choice of $f(\epsilon)$ to be used for the inner variables is in a sense more flexible, since the lower limit of the order classes is zero. (This should be compared with the low Reynolds number flow around a sphere, reference 1, in which the lower limit is ord Re.) Furthermore, as we found above, equations 4-12 to 4-15 result for ord 0 < ord $f(\epsilon)$ < ord 1. Actually our choice is based on our information concerning the solution to be perturbed. The choice $f(\epsilon) = \epsilon$ gives us a set of equations valid in a region where the author has considerable confidence in his "guess" of the solution to be perturbed. Thus our choice for inner independent variables is

$$\vec{r}^{\,+} = \frac{\vec{r}^{\,*}}{\epsilon} \tag{4-20}$$

The resulting dependent variables are identical with the "inner" variables defined after equation 2-6.

V. REGIONS OF VALIDITY, MATCHING CONDITIONS

We continue the discussion by stating the following definition from reference 1: $F_1(\vec{r}^*;\epsilon)$ constitutes a uniform approximation of $F(\vec{r};\epsilon)$ to order ϵ_j in a convex set S of equivalence classes if

$$\lim_{\epsilon \downarrow 0} \frac{F-F_1}{\epsilon_j(\epsilon)} = 0 \quad \text{uniformly for } f_1(\epsilon) \leq r^* \leq f_2(\epsilon) \tag{5-1}$$

whenever the equivalence classes $f_1(\epsilon)$ and $f_2(\epsilon)$ are in the set S.

Now from the "extension principle" of reference 1, we know that the inner limit is uniformly valid in the set

$$\text{ord } \epsilon \leq f(\epsilon) \leq \text{ord } \eta_1(\epsilon) \tag{5-2}$$

for some order class $\eta_1(\epsilon)$ satisfying

$$\eta_1(\epsilon) > \text{ord } \epsilon \tag{5-3}$$

Similarly, the outer limit is uniformly valid in the set

$$\eta_2(\epsilon) \leq f(\epsilon) \leq \text{ord } 1 \tag{5-4}$$

for some order class $\eta_2(\epsilon)$ satisfying

$$\eta_2(\epsilon) < \text{ord } 1 \tag{5-5}$$

If it turns out that $\eta_2(\epsilon) < \eta_1(\epsilon)$, then there is a limit process using

$$\tilde{r} = \frac{r^*}{\eta(\epsilon)}, \qquad \eta_1(\epsilon) \leq \eta(\epsilon) \leq \eta_2(\epsilon) \tag{5-6}$$

which, when applied to either the inner or the outer limit gives the same result. (If $\eta_1 < \eta_2$, then it is sometimes possible to construct an "intermediate development" which is used in a more general matching procedure. This is discussed in reference 1.)

For the problem considered in this thesis, it turns out that $\eta_2(\epsilon)$ can be ord ϵ, so that $\eta(\epsilon)$ can be simply ord ϵ. That is, the inner limit process applied to the outer development yields the inner development. Thus, in order to determine whether the outer development matches with the inner development, we write the outer expression in inner independent variables, i.e. $\vec{r}^{\,+}$ variables, and let $\epsilon \downarrow 0$. The result should be the inner development.

The very simple matching condition described above implies that the outer solution will be uniformly valid in the set

$$\text{ord}\ \epsilon \leq \text{ord}\ f(\epsilon) \leq \text{ord}\ 1 \qquad (5\text{-}7)$$

so that once we have the outer development computed (with the help of the inner development when necessary) we can dispense with the inner development.

VI. DISCUSSION OF THE ZEROTH ORDER SOLUTION

As mentioned in Section II, we construct the inner and outer asymptotic developments by using the inner and outer differential equations in a perturbation scheme, with the result being subject to the matching principle stated above. Before proceeding to solve these equations, we must first discover as much as possible about the zeroth order solution. That is, the limit of the exact solution as $\epsilon \downarrow 0$. The arguments to be presented are not rigorous, but rather heuristic and intuitive. However, the author feels that intuition must be founded (perhaps unknowingly) on experimental fact, and since (at least to the author's knowledge) an experiment describing our problem has not been performed, the intuitive arguments are open to criticism. For this reason, an experiment designed to test the theory of this paper would be most welcome. Let us now proceed with our discussion.

Our main concern will be to acquire some assurance that the flow in the region closer to the origin than the region of validity of our inner development does not give rise to terms in the "exterior" region (i. e. the region of validity of the inner and outer developments) larger than the terms we will find by considering only the inner and outer developments. (If such terms do exist, they will be solutions to the homogeneous equations 4-7 to 4-10, and equations 2-7, 2-10, 4-17 and 4-18.) A rather pessimistic attitude is taken in our arguments, so that the actual situation might be better than described.

First, let us determine the outer and inner limits of the velocity field for both a non-viscous and a viscous fluid. The outer limit is found by keeping $\vec{r}^*$ fixed and letting $\epsilon \downarrow 0$. We can fix ideas by keeping σ constant (hence $\vec{r} = \vec{r}^*/\sigma\mu U$ remains constant) and letting ϵ go to zero by letting the current go to zero. It seems obvious that for both the viscous and non-viscous cases, the result is simply the free stream value.

The inner limit is taken by holding $\vec{r}^+$ fixed and letting $\epsilon \downarrow 0$. For the inviscid case, we can fix ideas by keeping the current constant (hence $\vec{r} = \vec{r}^+ \frac{I}{U}\sqrt{\frac{\mu}{\rho}}$ remains fixed) and letting ϵ go to zero by letting the electrical conductivity go to zero. In this limit, there is no interaction between the magnetic induction field and the flow, so we again have the free stream value.

The inner limit for the viscous case is not so simple, because since a is fixed, and we let $\epsilon \downarrow 0$ by letting $\sigma \downarrow 0$, we must make $\nu = \frac{a}{\mu\sigma}$ tend to infinity. (In the outer limit, ν remained fixed.) We shall conclude from the arguments in the following paragraphs that the inner limit in the viscous case is the free stream velocity.

Let us attempt to find out something about the flow in a region inside a radius $O(\frac{I}{U}\sqrt{\frac{\mu}{\rho}}) = O(\frac{\epsilon}{\sigma\mu U})$. We first discuss the case of zero viscosity in order to gain some feeling of the purely magnetic effects. We search for a region of the flow in which the velocity is $o(1)$. Call this region S_i. To see if there exist flows satisfying this condition, we assume the dimensionless velocity in S_i is $u_i(\epsilon)$, and a dimensionless characteristic size (measured in outer variables) of S_i is $g_i(\epsilon)$. We consider two types of force on such a volume, namely the Lorentz forces (body forces) and the fluid dynamic pressure forces. The body force per unit volume is given by the interaction of a current density, whose magnitude is of the order $\sigma U u_i B$, and the magnetic field. Written in physical variables, and using the fact that the magnetic field behaves like $\mu I/r$, we have

$$F_{resist} = \left\{\begin{matrix}\text{Total body force} \\ \text{on } S_i\end{matrix}\right\} \sim (U u_i \sigma B^2) \cdot (\text{vol. of } S_i)$$

$$\sim \frac{\rho U u_i}{\sigma\mu} \epsilon^2 \text{ newtons} \qquad (6\text{-}1)$$

where for B we have used the value at the surface S_i, i. e. $B \sim \frac{\mu I}{g_i/\sigma\mu U}$.

The pressure terms can be estimated by using the stagnation pressure $\frac{1}{2}\rho U^2$. This gives

$$F_{surf} = \left\{\begin{matrix}\text{Pressure forces on} \\ \text{surface of } S_i\end{matrix}\right\} \sim \frac{g_i(\epsilon)}{\sigma\mu U} \cdot \rho U^2$$

$$\sim \frac{\rho U g_i(\epsilon)}{\sigma\mu} \text{ newtons} \qquad (6\text{-}2)$$

The ratio of $F_{resist} : F_{surf}$ is

$$\frac{F_{resist}}{F_{surf}} \sim \frac{\epsilon^2 u_i}{g_i(\epsilon)} \qquad (6\text{-}3)$$

which is of order unity when $g_i(\epsilon) \sim \epsilon^2 u_i$. Since it was assumed $u_i = o(1)$, this implies

$$g_i(\epsilon) \sim o(\epsilon^2) \qquad (6\text{-}4)$$

Thus, in the non-viscous case, there appears to be a possibility of a "core," defined somewhat vaguely as a region in which the zeroth approximation to the velocity is zero (i. e. $o(1)$). The diameter of the core is

$$o(\epsilon^2/\sigma\mu U) \text{ meters} \tag{6-5}$$

If such a core were present in the problem of this paper, in which viscosity is present, comparison with the low Reynolds number flow past a circular cylinder (reference 2) indicates that the flow in the vicinity of infinity would have a term $O(1/\log \epsilon)$ appearing, whereas the perturbation scheme yields a term $O(\epsilon^2 \log \epsilon)$ as the first term. We shall next argue that the presence of viscosity causes the flow to behave in such a manner that a core of this size cannot exist.

Let us consider a viscous fluid, and discuss the existence of a region S_v in which the dimensionless flow velocity is $o(1)$, say u_v. We again look for a balance between the body forces and the surface forces. Following the computation above, the force on S_v due to Lorentz forces is

$$F_{resist} \sim (\sigma U u_v B^2) \cdot (\text{volume of } S_v)$$

$$\sim \frac{\rho U u_v}{\sigma\mu} \epsilon^2 \text{ newtons.} \tag{6-6}$$

In the viscous case, if we think of the region S_v as a more or less solid cylinder with a flow $O(1)$ around it, then the Reynolds number based on such a cylinder will be

$$Re_v = \frac{g_v(\epsilon)}{\sigma\mu U} \frac{U}{\nu} = \frac{g_v(\epsilon)}{\alpha} . \tag{6-7}$$

Since we are assuming α is constant, if we estimate $g_v = g_i = \epsilon^2$, where g_i is the non-dimensional size of the possible inviscid case core, we see that the Reynolds number is small. From the known result that the drag on a circular cylinder in low Reynolds number flow (see reference 2) is

$$\text{Drag} \sim \rho\nu U\left(-\frac{1}{\log Re_v}\right)$$

we then assume that

$$F_{surf} \sim -\rho\nu U\left(\frac{1}{\log g_v(\epsilon)/a}\right)$$

$$= -\frac{\rho a U}{\mu\sigma} \cdot \frac{1}{\log g_v/a} \tag{6-8}$$

If we again consider the ratio $F_{resist}: F_{surf}$, we obtain

$$\frac{F_{resist}}{F_{surf}} \sim -\frac{u_v\epsilon^2}{a}\log\frac{g_v(\epsilon)}{a} \tag{6-9}$$

This is $O(1)$ if

$$\log\frac{g_v(\epsilon)}{a} \sim -\frac{a}{u_v\epsilon^2} \tag{6-10}$$

or

$$g_v(\epsilon) \sim a e^{-a/u_v\epsilon^2} \tag{6-11}$$

In physical variables, then, the size of the possible core is

$$\frac{g_v(\epsilon)}{\sigma\mu U} \sim \frac{a}{\sigma\mu U} e^{-a/u_v\epsilon^2} \tag{6-12}$$

which is much smaller, as $\epsilon\downarrow 0$, than the inviscid estimate, equation 6-5, if we assume $u_v \sim u_i$. It should be emphasized that the estimates discussed above have implicitly separated magnetic and viscous effects.

We have argued that the inner limit of the velocity in the inviscid case was simply the free stream value, and noted that viscous effects might make this same limit in the viscous case (keeping α constant) different from the free stream value. Now it is known (reference 2) that a cylinder in a uniform low Reynolds number flow has a viscous layer surrounding it extending to a radius

$$\sim \frac{1}{Re(\log Re)} \cdot (\text{Length of body}) , \qquad (6\text{-}13)$$

outside which the velocity is essentially the free stream value. If the core whose size is given by equation 6-12 is placed in such a flow, the viscous layer extends to a radius

$$\sim \frac{1}{(\log \frac{g_v}{\alpha})\frac{g_v}{\alpha}} \cdot \frac{g_v}{\sigma\mu U} = \frac{-\alpha}{(\log e^{-\alpha/u_V\epsilon^2})\mu\sigma U}$$

$$= \frac{u_V\epsilon^2}{\sigma} \cdot \frac{1}{\mu U} \quad \text{meters} . \qquad (6\text{-}14)$$

This radius is transcendentally small compared with the region of validity of the inner limit process. This makes it plausible to assume that the inner limit of the velocity in the viscous case is simply the free stream value, and for our inner perturbation scheme we shall assume this is the case.

It is apparent that the effect of a "core" such as we have described above would create a disturbance in the exterior part of the flow. It has been shown in reference 4 that far (in the present case $\epsilon/\sigma\mu U$ is "far") from such a core the resulting disturbance in the velocity field looks like the fundamental solution to the Oseen equations. Thus the outer development would require a term proportional to this fundamental solution, with the constant of proportionality being determined by the drag, which, from equation 6-7, gives a velocity

$$\sim - \frac{U}{\log Re} \sim \frac{Uu_v\epsilon^2}{a} = o(\epsilon^2)\frac{U}{a} \qquad (6\text{-}15)$$

In our perturbation scheme, we shall carry our computation up to terms $O(\epsilon^2 \log\epsilon)$ and $O(\epsilon^2)$, so that according to equation 6-15, the terms that might arise from a core should appear only as the third perturbation. (Thus an experiment designed to determine the existence or non-existence of a core would have to measure a third order effect.)

The foregoing discussion has been an attempt to justify the use of the free stream velocity as both inner and outer limits for the velocity, and also to try and convince ourselves that the terms resulting from the inner and outer developments actually contain <u>all</u> the terms, at least to the order of approximation which is to be computed. Assuming that these conclusions <u>are</u> justified, we now proceed to construct the inner and outer asymptotic developments.

VII. CONSTRUCTION OF THE PRINCIPAL DEVELOPMENTS

We assume the following asymptotic developments:

(1) Outer developments:

$$\vec{q}^{\,*}(\vec{r}^{\,*};\epsilon) = f_o^*(\epsilon)\vec{q}^{\,*(0)}(\vec{r}^{\,*}) + f_1^*(\epsilon)\vec{q}^{\,*(1)}(\vec{r}^{\,*}) + \ldots \tag{7-1}$$

$$\vec{B}^{*}(\vec{r}^{\,*};\epsilon) = g_o^*(\epsilon)\vec{B}^{*(0)}(\vec{r}^{\,*}) + g_1^*(\epsilon)\vec{B}^{*(1)}(\vec{r}^{\,*}) + \ldots \tag{7-2}$$

$$p^*(\vec{r}^{\,*};\epsilon) = \ell_o^*(\epsilon)p^{*(0)}(\vec{r}^{\,*}) + \ell_1^*(\epsilon)p^{*(1)}(\vec{r}^{\,*}) + \ldots \tag{7-3}$$

(2) Inner developments:

$$\vec{q}^{\,+}(\vec{r}^{\,+};\epsilon) = f_o^+(\epsilon)\vec{q}^{\,+(0)}(\vec{r}^{\,+}) + f_1^+(\epsilon)\vec{q}^{\,+(1)}(\vec{r}^{\,+}) + \ldots \tag{7-4}$$

$$\vec{B}^{+}(\vec{r}^{\,+};\epsilon) = g_o^+(\epsilon)\vec{B}^{+(0)}(\vec{r}^{\,+}) + g_1^+(\epsilon)\vec{B}^{+(1)}(\vec{r}^{\,+}) + \ldots \tag{7-5}$$

$$p^+(\vec{r}^{\,+};\epsilon) = \ell_o^+(\epsilon)p^{+(0)}(\vec{r}^{\,+}) + \ell_1^+(\epsilon)p^{+(1)}(\vec{r}^{\,+}) + \ldots \tag{7-6}$$

Strictly speaking, these are not the same as the final results; there will appear, in both the inner and outer developments, additional terms which must be introduced <u>after</u> terms of higher order have already been found. As we shall see, their presence is necessary to satisfy the matching conditions between the inner and outer developments.

A. Determination of $f_o^*(\epsilon)\vec{q}^{\,*(0)}(\vec{r}^{\,*})$ and $f_o^+(\epsilon)\vec{q}^{\,+(0)}(\vec{r}^{\,+})$.

Part of the discussion of Section VI was concerned with showing that for both the inner and the outer limits, the velocity was simply the free stream velocity. Thus

$$f_o^*(\epsilon) = f_o^+(\epsilon) = 1 \tag{7-7}$$

and

$$\vec{q}^{*(0)}(\vec{r}^*) = \vec{q}^{+(0)}(\vec{r}^+) = \vec{i}_x \tag{7-8}$$

Note that this agrees with the statement of equation 5-7; namely, the outer development is valid in the set

$$\text{ord } \epsilon \leq \text{ord } f(\epsilon) \leq \text{ord } 1. \tag{7-9}$$

B. Determination of $g_o^+(\epsilon)\vec{B}^{+(0)}(\vec{r}^+)$

If we substitute the assumed inner developments into the inner induction equation, equation 2-15, we obtain

$$f_o^+(\epsilon)\nabla^{+2}\vec{B}^{(0)} = 0 \tag{7-10}$$

The solution which satisfies the integral condition, equation 2-18, is easily seen to be

$$f_o^+(\epsilon) = 1 \tag{7-11}$$

$$\vec{B}^{+(0)}(\vec{r}^+) = \vec{i}_x \left(-\frac{y^+}{2\pi r^{+2}}\right) + \vec{i}_y\left(\frac{x^+}{2\pi r^{+2}}\right) \tag{7-12}$$

The field lines corresponding to this solution are simply concentric circles. That is, the field is identical with that due to a current element placed in a fluid at rest. Hence, close to the origin, the magnetic field is undisturbed, in the first approximation, by the motion of the fluid.

C. Determination of $g_o^*(\epsilon)\vec{B}^{*(0)}(\vec{r}^*)$

The next step is to substitute the assumed outer developments, equations 7-1, 7-2, 7-3 into the outer induction equation, equation 2-9. Keeping only the lowest order terms, we obtain

$$g_o^*(\epsilon)\left(\frac{\partial^2 B_x^{*(0)}}{\partial x^{*2}} + \frac{\partial^2 B_x^{*(0)}}{\partial y^{*2}} + \frac{\partial B_y^{*(0)}}{\partial y^*}\right) = 0 \qquad (7\text{-}13a)$$

$$g_o^*(\epsilon)\left(\frac{\partial^2 B_y^{*(0)}}{\partial x^{*2}} + \frac{\partial^2 B_y^{*(0)}}{\partial y^{*2}} - \frac{\partial B_y^{*(0)}}{\partial x^*}\right) = 0 \qquad (7\text{-}13b)$$

Using

$$\frac{\partial B_y^{*(0)}}{\partial y^*} = -\frac{\partial B_x^{*(0)}}{\partial x^*} \qquad (7\text{-}14)$$

equations 7-13a, 7-13b become

$$g_o^*(\epsilon)\left(\nabla^{*2}\vec{B}^{*(0)} - \frac{\partial \vec{B}^{*(0)}}{\partial x^*}\right) = 0 \qquad (7\text{-}15)$$

This equation describes how the magnetic induction field is swept downstream from the origin. There is obviously a close analogy between this phenomenon and the convection of vorticity downstream from a finite body as described by the Oseen approximation. In the latter case, the equation is

$$\frac{1}{Re}\nabla^2\omega - \frac{\partial\omega}{\partial x} = 0 \qquad (7\text{-}15a)$$

where Re is the ordinary fluid dynamic Reynolds number based on the body size, and ω is the vorticity.

Returning to the task of finding $g_o^*(\epsilon)\vec{B}^{*(0)}(\vec{r}^*)$, we note that a solution of equation 7-15 which satisfies 7-14 and vanishes at infinity is

$$g_o^*(\epsilon)\vec{B}^{*(0)}(\vec{r}^*)$$

$$= g_o^*(\epsilon)\left\{\vec{i}_x\left(-\frac{e^{\frac{x^*}{2}}y^*}{4\pi r^*}K_1(\frac{r^*}{2})\right)+\vec{i}_y\left(\frac{e^{\frac{x^*}{2}}x^*}{4\pi r^*}K_1(\frac{r^*}{2})-\frac{e^{\frac{x^*}{2}}}{4\pi}K_o(\frac{r^*}{2})\right)\right\} \quad (7\text{-}16)$$

where K_o, K_1 are the modified Bessel functions of the second kind of the zeroth and first order (8). To verify that equation 7-16 is the correct solution, and to determine $g_o^*(\epsilon)$, we impose the matching condition. We write $g_o^*(\epsilon)\vec{B}^{*(0)}(\vec{r}^*)$ for small $\vec{r}^*$, using

$$K_o(\frac{r^*}{2}) = -\log r^* - \log\frac{\gamma_o}{4} + o(1) \quad (7\text{-}17)$$

where $\gamma_o = e^{\gamma}$, and γ is Euler's constant,

$$K_1(\frac{r^*}{2}) = \frac{2}{r^*} + O(r^*\log r^*) \quad (7\text{-}18)$$

and replace $\vec{r}^*$ by $\epsilon\vec{r}^+$. The result may be written

$$g_o^*(\epsilon)\left(\epsilon\vec{B}^{*(0)}(\vec{r}^+)\right)$$

$$= g_o^*(\epsilon)\left\{\vec{i}_x\left(-\frac{y^+}{2\pi r^{+2}}\right)+\vec{i}_y\left(\frac{x^+}{2\pi r^{+2}}\right)\right\} + \epsilon\log\epsilon\, g_o^*(\epsilon)\left\{\vec{i}_y(\frac{1}{4\pi})\right\}$$

$$+\epsilon g_o^*(\epsilon)\left\{\vec{i}_x\left(-\frac{x^+y^+}{4\pi r^{+2}}\right)+\vec{i}_y\left(\frac{\log r^+}{4\pi}+\frac{\log\frac{\gamma_o}{4}}{4\pi}+\frac{x^{+2}}{4\pi x^{+2}}\right)\right\}+\dots$$

$$(7\text{-}19)$$

Remembering (see definitions before equation 2-7) that $\vec{B}^+ = \epsilon\vec{B}^*$, we see immediately from equations 7-11 and 7-12 that

$$g_o^*(\epsilon) = 1 \tag{7-20}$$

Hence

$$g_o^*(\epsilon)\vec{B}^{*(0)}(\vec{r}^*)$$
$$= \left\{ \vec{i}_x \left(- \frac{e^{\frac{x^*}{2}} y^*}{4\pi r^*} K_1(\frac{r^*}{2}) \right) + \vec{i}_y \left(\frac{e^{\frac{x^*}{2}} x^*}{4\pi r^*} K_1(\frac{r^*}{2}) - \frac{e^{\frac{x^*}{2}}}{4\pi} K_o(\frac{r^*}{2}) \right) \right\}. \tag{7-21}$$

This solution is independent of viscosity, and is actually valid when ν (and hence a) is identically zero. (See Section X for further comments on this point.)

For future convenience, we rewrite equation 7-19:

$$g_o^*(\epsilon)\epsilon \vec{B}^{*(0)}(\vec{r}^*) = \left\{ \vec{i}_x \left(- \frac{y^+}{2\pi r^{+2}} \right) + \vec{i}_y \left(\frac{x^+}{2\pi r^{+2}} \right) \right\} + \epsilon \log \epsilon \left\{ \vec{i}_y (\frac{1}{4\pi}) \right\}$$
$$+ \epsilon \left\{ \vec{i}_x \left(- \frac{x^+ y^+}{4\pi r^{+2}} \right) + \vec{i}_y \left(\frac{x^{+2}}{4\pi r^{+2}} + \frac{\log \frac{\gamma_o}{4}}{4\pi} + \frac{\log r^+}{4\pi} \right) \right\} + \dots \tag{7-22}$$

D. Determination of $g_{1a}^+(\epsilon)\vec{B}^{+(1a)}(\vec{r}^+)$

It can be shown that there is no term $O(\log \epsilon)$ satisfying equations 7-14 and 7-15 which vanishes at infinity and for which the line integral about the origin is zero in the limit as $\vec{r}^* \to 0$. This implies that the term $O(\epsilon \log \epsilon)$ in equation 7-22 must appear in the inner development, if the matching conditions are to be satisfied. If we call this inner term $g_{1a}^+(\epsilon)\vec{B}^{+(1a)}(\vec{r}^+)$, then, as mentioned above,

$$g_{1a}^+(\epsilon) = \epsilon \log \epsilon \tag{7-23}$$

and $\vec{B}^{+(1a)}(\vec{r}^{+})$ must satisfy the homogeneous equation

$$\nabla^{+2}\vec{B}^{+(1a)} = 0 \qquad (7\text{-}24)$$

The solution which matches with the outer development is the constant solution

$$\vec{B}^{+(1a)}(\vec{r}^{+}) = \vec{i}_y\left(\frac{1}{4\pi}\right),$$

so

$$g_{1a}^{+}(\epsilon)\vec{B}^{+(1a)}(\vec{r}^{+}) = \vec{i}_y\,\frac{\epsilon \log \epsilon}{4\pi} \qquad (7\text{-}25)$$

E. Determination of $g_1^{+}(\epsilon)\vec{B}^{+(1)}(\vec{r}^{+})$

The equations for $g_1^{+}(\epsilon)\vec{B}^{+(1)}(\vec{r}^{+})$, found as usual by substituting the assumed developments, equations 7-4 to 7-6, into equation 2-15, are

$$g_1^{+}(\epsilon)\left(\frac{\partial^2 B_x^{+(1)}}{\partial x^{+2}} + \frac{\partial^2 B_x^{+(1)}}{\partial y^{+2}}\right) + \epsilon\,\frac{\partial B_y^{+(0)}}{\partial y^{+}} = 0 \qquad (7\text{-}26)$$

$$g_1^{+}(\epsilon)\left(\frac{\partial^2 B_y^{+(1)}}{\partial x^{+2}} + \frac{\partial^2 B_y^{+(1)}}{\partial y^{+2}}\right) - \epsilon\,\frac{\partial B_y^{+(0)}}{\partial x^{+}} = 0 \qquad (7\text{-}27)$$

Substituting the known values of $\vec{B}^{+(0)}$ from equation 7-12, we then have

$$g_1^{+}(\epsilon)\left(\frac{\partial^2 B_x^{+(1)}}{\partial x^{+2}} + \frac{\partial^2 B_x^{+(1)}}{\partial y^{+2}}\right) + \epsilon\left(-\frac{x^{+}y^{+}}{\pi r^{+4}}\right) = 0 \qquad (7\text{-}28)$$

$$g_1^{+}(\epsilon)\left(\frac{\partial^2 B_y^{+(1)}}{\partial x^{+2}} + \frac{\partial^2 B_y^{+(1)}}{\partial y^{+2}}\right) - \epsilon\left(\frac{1}{2\pi r^{+2}} - \frac{x^{+}}{\pi r^{+4}}\right) = 0 \qquad (7\text{-}29)$$

This implies $g_1^{+}(\epsilon) = \epsilon$, and a particular solution to these equations which satisfies $\nabla^{+}\cdot \vec{B}_p^{+(1)} = 0$ is

$$\epsilon \vec{B}_p^{+(1)}(\vec{r}^{+}) = \epsilon\left\{ \vec{i}_x \left(- \frac{x^{+}y^{+}}{4\pi r^{+2}} \right) + \vec{i}_y \left(\frac{x^{+2}}{4\pi r^{+2}} + \frac{\log r^{+}}{4\pi} + \frac{\log \frac{\gamma_o}{4}}{4\pi} \right) \right\} \quad (7\text{-}30)$$

This, plus a suitable homogeneous solution, must match with the corresponding term in the inner limit of $\vec{B}^{*(0)}(\vec{r}^{*})$, which is given by equation 7-22. Obviously the required homogeneous term is zero, so we have

$$\epsilon \vec{B}^{+(1)}(\vec{r}^{+}) = \epsilon\left\{ \vec{i}_x \left(- \frac{x^{+}y^{+}}{4\pi r^{+2}} \right) + \vec{i}_y \left(\frac{x^{+2}}{4\pi r^{+2}} + \frac{\log r^{+}}{4\pi} + \frac{\log \frac{\gamma_o}{4}}{4\pi} \right) \right\} \quad (7\text{-}31)$$

F. Determination of $\ell_o^{*}(\epsilon)p^{*(0)}(\vec{r}^{*})$

In the outer solution, there is a possible pressure term $O(1)$, corresponding to the velocity term $O(1)$. From the outer momentum equation, equation 2-8, we have simply

$$\nabla^{*} p^{*(0)}(\vec{r}^{*}) = 0 \quad (7\text{-}32)$$

for the largest term. Hence

$$p^{*(0)}(\vec{r}^{*}) = \text{constant} \quad (7\text{-}33)$$

The boundary condition, equation 2-11, requires the constant to be zero:

$$p^{*(0)}(\vec{r}^{*}) = 0 \quad (7\text{-}34)$$

G. Determination of $\ell_o^{+}(\epsilon)p^{+(0)}$

From the inner momentum equation, equation 2-14, we have the largest term

$$\ell_o^+(\epsilon)\nabla^+ p^{+(0)}(\vec{r}^{\,+}) = 0 \quad (7\text{-}35)$$

so that

$$\ell_o^+(\epsilon) p^{+(0)}(\vec{r}^{\,+}) = \text{constant} \quad (7\text{-}36)$$

To match with the outer pressure term, $p^{*(0)}(\vec{r}^{\,*})$, we must have

$$\ell_o^+(\epsilon) p^{+(0)}(\vec{r}^{\,+}) = 0 \quad (7\text{-}37)$$

H. Determination of $f_{1a}^*(\epsilon)\vec{q}^{\,*(1a)}(\vec{r}^{\,*})$

If we substitute the assumed outer asymptotic development into the outer momentum equation, equation 2-8, and keep the lowest order terms, there results

$$\alpha f_1^*(\epsilon)\nabla^{*2} u^{*(1)} - f_1^*(\epsilon)\frac{\partial u^{*(1)}}{\partial x^*} - \ell_1^*(\epsilon)\frac{\partial p^{*(1)}}{\partial x^*} = \epsilon^2 B_y^{*(0)}\left(\frac{\partial B_y^{*(0)}}{\partial x^*} - \frac{\partial B_x^{*(0)}}{\partial y^*}\right) \quad (7\text{-}38)$$

$$\alpha f_1^*(\epsilon)\nabla^{*2} v^{*(1)} - f_1^*(\epsilon)\frac{\partial v^{*(1)}}{\partial x^*} - \ell_1^*(\epsilon)\frac{\partial p^{*(1)}}{\partial y^*} = -\epsilon^2 B_x^{*(0)}\left(\frac{\partial B_y^{*(0)}}{\partial x^*} - \frac{\partial B_x^{*(0)}}{\partial y^*}\right) \quad (7\text{-}39)$$

We see immediately that

$$f_1^*(\epsilon) = \ell_1^*(\epsilon) = \epsilon^2 \quad (7\text{-}40)$$

Using equation 7-21, 7-38 and 7-39 give

$$a\nabla^{*2}u^{*(1)} - \frac{\partial u^{*(1)}}{\partial x^*} - \frac{\partial p^{*(1)}}{\partial x^*} = \frac{e^{x}}{16\pi^2}\frac{x^{*2}}{r^{*2}} K_1(\frac{r^*}{2})K_1(\frac{r^*}{r})$$

$$- \frac{e^{x^*}x^*}{8\pi^2 r^*} K_o(\frac{r^*}{2})K_1(\frac{r^*}{2}) + \frac{e^{x^*}}{16\pi^2} K_o(\frac{r^*}{2})K_o(\frac{r^*}{2}) \quad (7\text{-}41)$$

$$a\nabla^{*2}v^{*(1)} - \frac{\partial v^{*(1)}}{\partial x^*} - \frac{\partial p^{*(1)}}{\partial y^*} = - \frac{e^{x^*}y^*}{16\pi^2 r^*} K_o(\frac{r^*}{2})K_1(\frac{r^*}{2})$$

$$+ \frac{e^{x^*}x^*y^*}{16\pi^2 r^{*2}} K_1(\frac{r^*}{2})K_1(\frac{r^*}{2}) \quad (7\text{-}42)$$

The solution to equations 7-41 and 7-42 will be discussed later (see Section VIII); for the present we use the result that for small values of r^* the vorticity behaves like

$$\epsilon^2\left(\frac{\partial v^{*(1)}}{\partial x^*} - \frac{\partial u^{*(1)}}{\partial y^*}\right) \sim - \frac{\epsilon^2}{8\pi^2 a}\frac{y^*\log r^*}{r^{*2}} + o\left(\frac{\log r^*}{r^*}\right) \quad (7\text{-}43)$$

and the pressure, $p^{*(1)}(\vec{r}^*)$, behaves like

$$\epsilon^2 p^{*(1)}(\vec{r}^*) \sim - \epsilon^2 \frac{x^*\log r^*}{8\pi^2 r^{*2}} + o\left(\frac{\log r^*}{r^*}\right) \quad (7\text{-}44)$$

In order to investigate matching, we replace $\vec{r}^*$ by $\epsilon\vec{r}^+$, so that equation 7-43 becomes

$$\epsilon\left(\frac{\partial v^{*(1)}}{\partial x^+} - \frac{\partial u^{*(1)}}{\partial y^+}\right) \sim -\epsilon \log\epsilon \frac{y^+}{8\pi^2 a r^{+2}} - \epsilon\frac{y^+\log r^+}{8\pi^2 r^{+2}} + \dots \quad (7\text{-}45)$$

and equation 7-44 becomes

$$\epsilon^2 p^{*(1)} \sim -\epsilon \log\epsilon \frac{x^+}{8\pi^2 r^{+2}} - \epsilon\frac{x^+\log r^+}{8\pi^2 r^{+2}} + \dots \quad (7\text{-}46)$$

Now equation 7-45 contains a term $O(\epsilon \log \epsilon)$ and this plus the contribution from a possible term $O(\epsilon^2 \log \epsilon)$ in the outer solution, must match with a term of corresponding order in the inner development. But the $f_1^+(\epsilon)\vec{q}^{+(1)}(\vec{r}^*)$ equation found by the usual substitution of the assumed asymptotic development into the inner momentum equation, (equation (2-14)), has forcing terms $O(\epsilon^2)$, so $f_1^+(\epsilon) = \epsilon^2$, not $\epsilon^2 \log \epsilon$, as required.* Thus a term $O(\epsilon^2 \log \epsilon)$, which we call

$$\epsilon^2 \log \epsilon \, \vec{q}^{+(1a)}(\vec{r}^+) \qquad (7\text{-}47)$$

must satisfy the homogeneous equations

$$\nabla^+ \cdot \vec{q}^{+(1a)} = 0 \qquad (7\text{-}48)$$

$$\epsilon^2 \log \epsilon \, \alpha \nabla^{+2} \vec{q}^{+(1a)} - \epsilon^2 \log \epsilon \nabla^+ p^{+(1a)} = 0 \qquad (7\text{-}49)$$

and have a vorticity which behaves like

$$\epsilon^2 \log \epsilon \left(\frac{\partial v^{+(1a)}}{\partial x^+} - \frac{\partial u^{+(1a)}}{\partial y^+} \right) \sim \epsilon^2 \log \epsilon \, \frac{y^+}{r^{+2}} \qquad (7\text{-}50)$$

To investigate the problem further, we note that the possible term in the outer solution $O(\epsilon^2 \log \epsilon)$, which we call $\vec{q}^{*(1a)}(\vec{r}^*)$, $p^{*(1a)}(\vec{r}^*)$, satisfies the equations

$$\epsilon^2 \log \epsilon \, \nabla^* \cdot \vec{q}^{*(1a)} = 0 \qquad (7\text{-}51)$$

$$\epsilon^2 \log \epsilon \, \alpha \nabla^{*2} \vec{q}^{*(1a)} - \epsilon^2 \log \epsilon \, \frac{\partial \vec{q}^{*(1a)}}{\partial x^*} - \epsilon^2 \log \epsilon \nabla^* p^{*(1a)} = 0 \qquad (7\text{-}52)$$

and possesses a vorticity which for small values of $\vec{r}^*$ behaves like

$$\epsilon^2 \log \epsilon \left(\frac{\partial v^{*(1a)}}{\partial x^*} - \frac{\partial u^{*(1a)}}{\partial y^*} \right) \sim \epsilon^2 \log \epsilon \, \frac{y^*}{r^{*2}} \qquad (7\text{-}53)$$

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* The orders of pressure and vorticity are different for the inner and outer developments. The matching argument is clearer, though more tedious, if physical variables are used.

Fortuitously, the solution satisfying this requirement is simply proportional to the fundamental solution of the Oseen equations for the case where the concentrated force at the origin has only an x-component. Because of the symmetry of our problem, we shall call this <u>the</u> fundamental solution without risk of confusion. (Of course, a is not a Reynolds number.) The constant of proportionality cannot be determined by matching conditions; one can easily construct different combinations of solutions of equations 7-48, 7-49, 7-51 and 7-52 which satisfy matching conditions. However, we <u>can</u> obtain a unique answer by imposing the requirement that the drag on the current element can be computed either by considering pressure, viscous stresses, and Maxwell stresses at the origin, or by considering the flux of momentum at infinity. We shall verify that the result is (e. g. Ref. 4):

$$\epsilon^2 \log \epsilon \, \vec{q}^{*(1a)}(\vec{r}^*) = -\frac{\epsilon^2 \log \epsilon}{8\pi^2 a} \left\{ a\nabla^* \left[e^{\frac{x^*}{2a}} K_o\left(\frac{r^*}{2a}\right) \right] - e^{\frac{x^*}{2a}} K_o\left(\frac{r^*}{2a}\right) \vec{i}_x + a\nabla^* \log r^* \right\} \qquad (7\text{-}54)$$

with the associated pressure field

$$\epsilon^2 \log \epsilon \; p^{*(1a)}(\vec{r}^*) = \epsilon^2 \log \epsilon \; \frac{x^*}{8\pi^2 r^{*2}} \qquad (7\text{-}55)$$

and

$$\epsilon \log \epsilon \; \vec{q}^{+(1a)}(\vec{r}^+) = \epsilon \log \epsilon \; p^{+(1a)} = 0 \qquad (7\text{-}56)$$

To check the result, we note first that for r^* small, the vorticity from equation 7-54 behaves like

$$\epsilon^2 \log \epsilon \left(\frac{\partial v^{*(1a)}}{\partial x^*} - \frac{\partial u^{*(1a)}}{\partial y^*} \right) \sim \epsilon^2 \log \epsilon \; \frac{y^*}{8\pi^2 a r^{*2}} \qquad (7\text{-}57)$$

or, in inner variables

$$\epsilon^2 \log \epsilon \left(\frac{\partial v^{*(1a)}}{\partial x^*} - \frac{\partial u^{*(1a)}}{\partial y^*} \right) \sim \epsilon \log \epsilon \; \frac{y^+}{8\pi^2 a r^{+2}} \qquad (7\text{-}58)$$

This, plus the $O(\epsilon \log \epsilon)$ term in equation 7-45, gives the contribution from the outer solution. But they exactly cancel, so the contribution is zero. This is consistent with equation 7-56. Next, we note that equation 7-55 written in inner variables is

$$\epsilon^2 \log \epsilon \; p^{*(1a)} = \epsilon \log \epsilon \frac{x^+}{8\pi^2 r^{+2}} \qquad (7\text{-}59)$$

This, added to the contribution $O(\epsilon \log \epsilon)$ from equation 7-46, also gives zero, a result again consistent with equation 7-56. This completes the verification of the matching conditions. The value of drag will be discussed in Section IX.

VIII. DISCUSSION OF $\epsilon^2 \vec{q}^{*(1)}(\vec{r}^*)$ AND $\epsilon \vec{q}^{+(1)}(\vec{r}^+)$

We turn now to the question of solving equations 7-41 and 7-42 (subject to the continuity equation, of course), and the corresponding equations in the inner development.

The velocity and pressure fields, $\epsilon^2 \vec{q}^{*(1)}(\vec{r}^*)$ and $\epsilon^2 p^{*(1)}(\vec{r}^+)$ are brought into being by the forcing terms on the right-hand sides of equations 7-41 and 7-42. These are just the Lorentz body forces. The solution, in integral form, can be constructed formally by use of the fundamental solution tensor of the Oseen equations, reference 5. For example, $\epsilon^2 u^{*(1)}(\vec{r}^*)$ is formally

$$\epsilon^2 u^{*(1)}(\vec{r}^*) =$$

$$\frac{\epsilon^2}{2\pi}\int_{(\text{all } \vec{\rho}^* \text{ space})} \Bigg[\Bigg[\left\{\frac{1}{2a} K_o\left(\frac{|\vec{r}^* - \vec{\rho}^*|}{2a}\right) e^{\frac{x^* - \xi^*}{2a}} - \frac{x^* - \xi^*}{|\vec{r}^* - \vec{\rho}^*|^2} + \frac{1}{2a} K_1\left(\frac{|\vec{r}^* - \vec{\rho}^*|}{2a}\right) \frac{x^* - \xi^*}{|\vec{r}^* - \vec{\rho}^*|} e^{\frac{x^* - \xi^*}{2a}}\right\}.$$

$$\left\{-\frac{e^{\xi^*}}{16\pi^2} \frac{\xi^{*2}}{\rho^{*2}} K_1(\tfrac{\rho^*}{2}) K_1(\tfrac{\rho^*}{2}) + \frac{e^{\xi^*} \xi^*}{8\pi^2 \rho^*} K_o(\tfrac{\rho^*}{2}) K_1(\tfrac{\rho^*}{2}) - \frac{e^{\xi^*}}{16\pi^2} K_o(\tfrac{\rho^*}{2}) K_o(\tfrac{\rho^*}{2})\right\}$$

$$+\left\{-\frac{y^* - \eta^*}{|\vec{r}^* - \vec{\rho}^*|^2} + \frac{1}{2a} K_1\left(\frac{|\vec{r}^* - \vec{\rho}^*|}{2a}\right) \frac{y^* - \eta^*}{|\vec{r}^* - \vec{\rho}^*|} e^{\frac{x^* - \xi^*}{2a}}\right\}.$$

$$\left\{\frac{e^{\xi^*} \eta^*}{16\pi^2 \rho^*} K_o(\tfrac{\rho^*}{2}) K_1(\tfrac{\rho^*}{2}) - \frac{e^{\xi^*} \xi^* \eta^*}{16\pi^2 \rho^{*2}} K_1(\tfrac{\rho^*}{2}) K_1(\tfrac{\rho^*}{2})\right\}\Bigg] d\xi^* d\eta^* \qquad (8\text{-}1)$$

where the variable of integration is $\vec{\rho}^* = \vec{i}_x \xi^* + \vec{i}_y \eta^* = \vec{i}_\rho \rho + \vec{i}_\phi \phi$ in cartesian and polar coordinates respectively. There are two similar expressions for $v^{*(1)}(\vec{r}^*)$ and $p^{*(1)}(\vec{r}^*)$. The fact that the integral is taken over all space is due to the observation that the right-hand side of equations 7-41 and 7-42 are valid in the set

$$\text{ord}\,\epsilon \leqslant \text{ord}\, f(\epsilon) \leqslant \text{ord}\, 1$$

so that from the point of view of the outer development, the non-homogeneous terms are valid right up to the origin.

We see immediately that the integral of equation 8-1, and the corresponding integrals for $\epsilon^2 v^{*(1)}(\vec{r}^*)$ and $\epsilon^2 p^{*(1)}(\vec{r}^*)$, do not exist in the usual sense. However, the so-called "finite part," (see reference 3, page 38), is a solution of the differential equations. (Of course, we can still add a homogeneous solution to this particular solution. Later, we shall attempt to satisfy outselves that this additional solution is zero.) For equation 8-1, the finite part ("Pf") is found by the standard methods to be expressible as

$$\begin{aligned}
&\text{Pf}\,\epsilon^2 u^{*(1)}(\vec{r}^*) \\
&= \frac{\epsilon^2}{2\pi}\int_0^k\int_0^{2\pi} \frac{1}{\rho^*}\left\{F(\vec{r}^*;\rho^*,\phi)\left(-\frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2}\rho^{*2}K_1\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right)\right. \\
&\qquad \left. -4F(\vec{r}^*;0,\phi)\left(-\frac{\cos^2\phi}{16\pi^2}\right)\right\} d\phi\, d\rho^* \\
&\quad + \frac{\epsilon^2}{2\pi}\int_k^\infty\int_0^{2\pi}\frac{1}{\rho^*}\left\{F(\vec{r}^*;\rho^*,\phi)\left(-\frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2}\rho^{*2}K_1\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right)\right\} d\phi\, d\rho^* \\
&\quad + \frac{\epsilon^2}{2\pi}\,\frac{(-\log k)}{4\pi a}\left\{\frac{1}{2}K_o\left(\frac{r^*}{2a}\right)e^{\frac{x^*}{2a}} - a\,\frac{x^*}{r^{*2}} + \frac{1}{2}K_1\left(\frac{r^*}{2a}\right)\frac{x^*}{r^*}e^{\frac{x^*}{2a}}\right\} \\
&\quad + \frac{\epsilon^2}{2\pi}\int_0^\infty\int_0^{2\pi} F(\vec{r}^*;\rho^*,\phi)\left\{\frac{e^{\rho^*\cos\phi}\cos\phi}{8\pi^2}K_o\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right. \\
&\qquad \left. -\frac{e^{\rho^*\cos\phi}}{16\pi^2}K_o\left(\frac{\rho^*}{2}\right)K_o\left(\frac{\rho^*}{2}\right)\right\}\rho^*\, d\phi\, d\rho^*
\end{aligned}$$

$$+\frac{\epsilon^2}{2\pi}\int_0^\infty\int_0^{2\pi}\left\{-\frac{y^*-\rho^*\sin\phi}{|\vec{r}^*-\vec{\rho}^*|^2}+\frac{1}{2a}K_1\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2a}\right)\frac{y^*-\rho^*\sin\phi}{|\vec{r}^*-\vec{\rho}^*|}e^{\frac{x^*-\rho^*\cos\phi}{2a}}\right\}\cdot$$

$$\left\{\frac{e^{\rho^*\cos\phi}\sin\phi}{16\pi^2}K_o\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)-\frac{e^{\rho^*\cos\phi}\cos\phi\sin\phi}{16\pi^2}K_1\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right\}\rho^* d\rho^* d\phi \tag{8-2}$$

where

$$F(\vec{r}^*;\rho^*,\phi)=\left\{\frac{1}{2a}K_o\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2a}\right)e^{\frac{x^*-\rho^*\cos\phi}{2a}}-\frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|^2}\right.$$

$$\left.+\frac{1}{2a}K_1\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2a}\right)\frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|}e^{\frac{x^*-\rho^*\cos\phi}{2a}}\right\} \tag{8-3}$$

Similar expressions will arise in the determination of $\epsilon^2 v^{*(1)}(\vec{r}^*)$ and $\epsilon^2 p^{*(1)}(\vec{r}^*)$. The value of k is to be taken as that value of ρ^* below which we can use the approximate expressions, equations 7-17 and 7-18, for $K_o(\frac{\rho^*}{2})$ and $K_1(\frac{\rho^*}{2})$. The above integral is independent of k, but this choice of k will be used when we consider matching with $\vec{q}^{+(1)}(\vec{r}^+)$ and $\vec{p}^{+(1)}(\vec{r}^+)$. We will find that the outer solutions are valid in the set

$$\text{ord}\,\epsilon \leq f(\epsilon) \leq \text{ord}\,1$$

To investigate the question of matching, we first observe that the equations for $\vec{q}^{+(1)}(\vec{r}^+)$ and $\vec{p}^{+(1)}(\vec{r}^+)$, found from substituting the asymptotic developments into the momentum and continuity equations, 2-13 and 2-14, are

$$f_1^+(\epsilon)\nabla^+ \cdot \vec{q}^{+(1)} = 0 \tag{8-4}$$

$$af_1^+(\epsilon)(\nabla^{+2}u^{+(1)}) - \ell_1^+(\epsilon)\frac{\partial p^{+(1)}}{\partial x^+} = \epsilon^2 \frac{x^{+2}}{4\pi r^{+4}} \tag{8-5}$$

$$af_1^+(\epsilon)(\nabla^{+2}v^{+(1)}) - \ell_1^+(\epsilon)\frac{\partial p^{+(1)}}{\partial y^+} = \epsilon^2 \frac{x^+y^+}{4\pi r^{+2}} \tag{8-6}$$

so that

$$f_1^+(\epsilon) = \epsilon^2 \tag{8-7}$$

and

$$\ell_1^+(\epsilon) = \epsilon^2 \tag{8-8}$$

Before considering solutions of equations 8-4, 8-5 and 8-6, we apply the inner limit process to $\epsilon^2\vec{q}^{*(1)}(\vec{r}^*) + \epsilon^2\log\epsilon\,\vec{q}^{*(1a)}(\vec{r}^*)$ and $\epsilon^2 p^{*(1)}(\vec{r}^*) + \epsilon^2\log\epsilon\, p^{*(1a)}(\vec{r}^*)$. The result for $u^{*(1)}$ is

$$u^{*(1)}(\epsilon\vec{r}^{\,+})$$

$$\sim \frac{\epsilon^2}{2\pi}\int_0^{k/\epsilon}\int_0^{2\pi}\left\{\left(-\frac{1}{2a}\log|\vec{r}^{\,+}-\vec{\rho}^{\,+}| + \frac{(x^+-\rho^+\cos\phi)^2}{2a|\vec{r}^{\,+}-\vec{\rho}^{\,+}|^2} + \frac{1}{2a}\log r^+ - \frac{x^{+2}}{2ar^{+2}}\right)\cdot\left(-\frac{\cos^2\phi}{4\pi^2\rho^{+2}}\right)\right.$$

$$\left. + \left(\frac{(y^+-\rho^+\sin\phi)(x^+-\rho^+\cos\phi)}{2a|\vec{r}^{\,+}-\vec{\rho}^{\,+}|^2}\right)\cdot\left(-\frac{\sin\phi\cos\phi}{4\pi^2\rho^{+2}}\right)\right\}\rho^+ d\phi d\rho^+$$

$$+ \epsilon^2\log\epsilon\,\frac{\log k}{16\pi^2 a} - \epsilon^2\frac{\log k}{16\pi^2 a}\left(-\log r^+ + \frac{x^{+2}}{r^{+2}}\right)$$

$$-\frac{\epsilon^2(\log\epsilon)^2}{16\pi^2 a} - \frac{\epsilon^2\log\epsilon}{16\pi^2 a}\left(\log r^+ - \frac{x^{+2}}{r^{+2}} - \log 4a + \gamma\right)$$

$$+ \epsilon^2(\text{constant}) + o(\epsilon^2) \tag{8-9}$$

We shall next show that equation 8-9 is a solution for the inner development up to $o(\epsilon^2)$. This is done as follows: equations 8-4, 8-5, and 8-6 are simply non-homogeneous Stokes equations (except a is of course not a Reynolds number), with the right-hand side valid in a set

$$f_1(\epsilon) < \text{ord } f(\epsilon) \leq \text{ord } \epsilon \tag{8-10}$$

the existence of $f_1(\epsilon) < \text{ord } \epsilon$ following from the extension principle. Expressed in inner variables, this says the right-hand side is valid for $r^+ \leq \frac{k}{\epsilon}$, where k is a constant, which we can define to be the same as described below equation 8-3. Let us then define the right-hand side to be identically zero for $r^+ > \frac{k}{\epsilon}$. Then, using the fundamental solution of the Stokes equations (see reference 3) in a purely formal manner, we obtain for

$$\epsilon^2 u^{+(1)}(\vec{r}^+) = \frac{1}{2\pi}\int_0^{k/\epsilon}\int_0^{2\pi}\Bigg\{\left[-\frac{1}{2a}\log\left|\vec{r}^+-\vec{\rho}^+\right| - \frac{3}{4a} + \frac{(x^+-\rho^+\cos\phi)^2}{2a\left|\vec{r}^+-\vec{\rho}^+\right|^2}\right]\cdot \left[-\frac{\cos^2\phi}{4\pi^2\rho^{+2}}\right]$$

$$+\left[\frac{(x^+-\rho^+\cos\phi)(y^+-\rho^+\sin\phi)}{2a\left|\vec{r}^+-\vec{\rho}^+\right|^2}\right]\left[-\frac{\sin\phi\cos\phi}{4\pi^2\rho^{+2}}\right]\Bigg\}\rho^+ d\rho^+ d\phi \tag{8-11}$$

(Similar formal expressions will be obtained for $\epsilon^2 v^{+(1)}(\vec{r}^+)$ and $\epsilon^2 p^{+(1)}(\vec{r}^+)$.) We note immediately that equation 8-11 does not converge in the usual sense, but the "finite part" satisfies the differential equations. Similar statements are true for $v^{+(1)}(\vec{r}^+)$ and $p^{+(1)}(\vec{r}^+)$. Using standard methods to express the finite part, we obtain

$\epsilon^2 u^{+(1)}(\vec{r}^{\,+}) =$

$$\frac{\epsilon^2}{2\pi} \int_0^{k/\epsilon} \int_0^{2\pi} \left\{ \left[-\frac{1}{2a} \log |\vec{r}^{\,+} - \vec{\rho}^{\,+}| + \frac{(x^+ - \rho^+ \cos\phi)^2}{2a|\vec{r}^{\,+} - \vec{\rho}^{\,+}|^2} + \frac{1}{2a} \log r^+ - \frac{x^{+2}}{2ar^{+2}} \right] \cdot \right.$$

$$\left[-\frac{\cos^2\phi}{4\pi^2 \rho^{+2}} \right]$$

$$\left. + \left[\frac{(x^+ - \rho^+ \cos\phi)(y^+ - \rho^+ \sin\phi)}{2a|\vec{r}^{\,+} - \vec{\rho}^{\,+}|^2} \right] \cdot \left[-\frac{\sin\phi\cos\phi}{4\pi^2 \rho^{+2}} \right] \right\} \rho^+ d\rho^+ d\phi$$

$$- \frac{\epsilon^2 \log k}{16\pi^2 a} \left(-\log r^+ + \frac{x^{+2}}{r^{+2}} - \frac{3}{2} \right) + \frac{\epsilon^2 \log \epsilon}{16\pi^2 a} \left(-\log r^+ + \frac{x^{+2}}{r^{+2}} - \frac{3}{2} \right)$$

$$+ O(\epsilon^3) . \qquad (8\text{-}12)$$

Plainly, equation 8-12 is identical with equation 8-9, except for constant terms. These constant terms can obviously be added in the form of homogeneous solutions. Similar statements hold for $\epsilon^2 v^{+(1)}(\vec{r}^{\,+})$ and $\epsilon^2 p^{+(1)}(\vec{r}^{\,+})$, so we have demonstrated that matching is satisfied, and the outer development, to terms in ϵ^2, is uniformly valid in the set

$$\text{ord}\, \epsilon \leq f(\epsilon) \leq \text{ord}\, 1 .$$

Furthermore, the inner development, to this order, is redundant. Because of this redundancy our further discussion will be concerned with the outer development only. Our next step will be a discussion of the homogeneous solution that can be added to $\vec{q}^{*(1)}(\vec{r}^*)$ and $p^{*(1)}(\vec{r}^*)$.

A. Further discussion of $\epsilon^2\vec{q}^{*(1)}(\vec{r}^*)$ and $\epsilon^2 p^{*(1)}(\vec{r}^*)$.

In section 8, we found a solution to the differential equations satisfied by $\vec{q}^{*(1)}(\vec{r}^*)$ and $p^{*(1)}(\vec{r}^*)$. (For $u^{*(1)}(\vec{r}^*)$ see equation 8-2 ; for $v^{*(1)}(\vec{r}^*)$ and $p^{*(1)}(\vec{r}^*)$ see equations 10-2 and 10-5.) To this solution can be added a term proportional to the fundamental solution; in this section we shall attempt to argue heuristically that the constant of proportionality should be zero.

The argument will be based on the following observation: singularities in the outer flow represent physical disturbances, and, unless forced by some physical cause, do not exist. Now one can look at the formal solution to $u^{*(1)}(\vec{r}^*)$ as a perfectly well-behaved function plus a very bad singularity. We can remove this singularity by subtracting suitable singular solutions; the author feels that of the many possibilities, we should use the one that is just sufficient to remove the singularity -- no more, no less. From this point of view, we can write equation 8-1, and the corresponding equations for $v^{*(1)}$ and $p^{*(1)}$ using as lower limit of radius some constant $R > 0$, and then add to the velocity field the term

$$-\frac{1}{8\pi^2}(\log R)\vec{G}(\vec{r}^*) \tag{8-13}$$

where

$$\vec{G}(\vec{r}^*) = -\nabla^*[\epsilon^{x^*/2a}K_o(\frac{r^*}{2a})] + \frac{1}{a}e^{x^*/2a}K_o(\frac{r^*}{2a})\vec{i}_x - \nabla^* \log r^* ; \tag{8-14}$$

and add to the pressure field the pressure field associated with equation 8-13,

$$-\frac{1}{8\pi^2}(\log R)\frac{x^*}{r^{*2}} \quad . \tag{8-15}$$

This done, we let $R \to 0$.

To illustrate this point in more detail, let us consider the x-component of the velocity. The above statement says to add

$$-\frac{1}{8\pi^2}(\log R)\left\{\frac{1}{2a}K_o\left(\frac{r^*}{2a}\right)e^{x^*/2a} - \frac{x^*}{r^{*2}} + \frac{1}{2a}K_1\left(\frac{r^*}{2a}\right)\frac{x^*}{r^{*2}}e^{x^*/2a}\right\}. \tag{8-16}$$

This can also be written

$$\frac{1}{2\pi}\left\{\frac{1}{2a}K_o\left(\frac{r^*}{2a}\right)e^{x^*/2a} - \frac{x^*}{r^{*2}} + \frac{1}{2a}K_1\left(\frac{r^*}{2a}\right)\frac{x^*}{r^{*2}}e^{x^*/2a}\right\}\int_R^1\int_0^{2\pi}\frac{\cos^2\phi}{4\pi^2\rho^*}d\rho^*d\phi \tag{8-17}$$

$$= \frac{1}{2\pi}\int_R^1\int_0^{2\pi}4F(\vec{r}^*;0,\phi)\frac{\cos^2\phi}{16\pi^2\rho^*}\,d\rho^*d\phi\,, \tag{8-18}$$

where $F(\vec{r}^*;\rho^*,\phi)$ is defined by equation 8-3. It can be seen that adding equation 8-18 to equation 8-1 (using R as lower limit of radius integration) and letting $R \downarrow 0$ gives equation 8-2.

It should be re-emphasized that the above statements are based on rather uncertain assumptions. However, we at least have a particular solution, so that our expression is correct up to an additive term proportional to thefundamental solution of the Oseen equations.

IX. CALCULATION OF DRAG

In principle, the force on the current element may be computed by evaluating forces at the origin of coordinates or by studying the flux of momentum at infinity. The first approximation to the drag can thus be found by considering the momentum flux at infinity from the solution $\epsilon^2 \log \epsilon\, \vec{q}^{*(1a)}(\vec{r}^{*})$, $\epsilon^2 \log \epsilon\, p^{*(1a)}(\vec{r}^{*})$, and any Maxwell stresses of the same order. There are no such Maxwell stresses of this order, and since $\vec{q}^{*(1a)}(\vec{r}^{*})$, $p^{*(1a)}(\vec{r}^{*})$ is proportional to the fundamental solution of the Oseen equations, the drag, expressed in physical variables, is well known to be the product of ρU and the source strength, in physical variables, of the so-called longitudinal component. If the first approximation to the drag is computed by finding the stresses at the origin, we find that the contribution is entirely from Maxwell stresses arising from

$$\vec{B}^{+(0)}(\vec{r}^{+}) + \epsilon \log \epsilon\, \vec{B}^{+(1a)}(\vec{r}^{+}),$$

which can be interpreted as a current element $\vec{i}_z I$ situated in a constant magnetic field

$$\vec{i}_y \frac{\epsilon \log \epsilon}{4\pi} U \sqrt{\rho\mu} .$$

In either case, the result, expressed in physical variables, is

$$\text{Drag/unit length} = -\log\left(\sigma I \sqrt{\frac{\sigma^3}{\rho}}\right) \frac{I^2 \sigma U \mu^2}{4\pi} \quad \frac{\text{newtons}}{\text{meter}} . \qquad (9\text{-}1)$$

The contribution to the drag arising from $\epsilon^2 \vec{q}^{*(1)}(\vec{r}^*)$, $\epsilon^2 p^{*(1)}(\vec{r}^*)$ and $B^{*(a)}(\vec{r}^*)$, can similarly be found by considering either the solution near the origin or at infinity. In this case, the answer is not found so easily as was the first approximation, so we restrict ourselves to considering the flux of momentum at infinity. It is easily demonstrated that the Maxwell stresses from $\vec{B}^{*(0)}(\vec{r}^*)$ die out sufficiently rapidly at infinity to give zero contribution. Hence we need contend with only $\vec{q}^{*(1)}(\vec{r}^*)$ and $p^{*(1)}(\vec{r}^*)$. The actual computation can be reduced to a single integration which must be carried out numerically. The argument leading to this integral can be stated as follows (also see Appendix): we know from symmetry that the force on the current element is in the x-direction, i. e., there is no lift. Far away from the origin, the solution ($\vec{q}^{*(1)}(\vec{r}^*)$ and $p^{*(1)}(\vec{r}^*)$) behaves like the response to a concentrated force situated at the origin and directed along the x-axis, provided the integrands of the integral representations decay sufficiently fast, and the constant of proportionality of the response is given by the integral of the "forcing" terms in the integrand. This constant corresponds to $\sigma\mu/2\pi$ times the source strength of the longitudinal component expressed in physical variables, and so the drag will be simply the product of ρU and this source strength. If the integral is negative, the contribution to the drag will be positive. Let us now carry out the procedure.

The integral of the x-direction forcing terms in the integrand (equation 8-2 for example) is

$$J = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \frac{1}{\rho^*}\left\{-\frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2}\rho^{*2}K_1(\frac{\rho^*}{2})K_1(\frac{\rho^*}{2}) + \frac{4\cos^2\phi}{16\pi^2}\right\}d\phi d\rho^*$$

$$+\frac{1}{2\pi}\int_1^{\infty}\int_0^{2\pi}\left\{-\frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2}K_1(\frac{\rho^*}{2})K_1(\frac{\rho^*}{2})\right\}\rho^* d\phi d\rho^*$$

$$+\frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\left\{\frac{e^{\rho^*\cos\phi}\cos\phi}{8\pi^2}K_o(\frac{\rho^*}{2})K_1(\frac{\rho^*}{2})\right.$$

$$\left. -\frac{e^{\rho^*\cos\phi}}{16\pi^2}K_o(\frac{\rho^*}{2})K_o(\frac{\rho^*}{2})\right\}\rho^* d\phi d\rho^* . \qquad (9\text{-}2)$$

The author carried out the computation on a desk calculator, and obtained

$$J = -\frac{1}{16\pi^2}(1.90). \qquad (9\text{-}3)$$

Thus the contribution to the drag per unit length of current element is

$$\text{Force/unit length} = \frac{1.9}{16\pi^2}\,\sigma I^2\mu^2 U \;\frac{\text{newtons}}{\text{meter}} . \qquad (9\text{-}4)$$

Combining equations 9-1 and 9-4, we obtain for the total drag to the order considered

$$\text{Drag/unit length} = \left\{-\log\left(\sigma I\sqrt{\frac{\mu^3}{\rho}}\right)\frac{1}{4\pi} + \frac{1.9}{16\pi^2}\right\}\sigma I^2\mu^2 U$$

$$\text{newtons per meter} . \qquad (9\text{-}5)$$

When $\epsilon \sim 1$, the contributions to the drag from equation 9-1 and 9-4 are about the same. For $\epsilon \sim \frac{1}{4}$, the contribution from equation 9-1 is about 10 times the contribution from equation 9-4. Thus first order effects can be expected to dominate when $\epsilon \sim \frac{1}{4}$ or less.

X. SUMMARY AND CONCLUSIONS

We can conveniently summarize the results by stating the outer asymptotic developments. The non-dimensional velocity is

$$\vec{q}^*(\vec{r}^*) \sim \vec{i}_x + \epsilon^2 \log \epsilon\, \vec{q}^{*(1a)}(\vec{r}^*) + \epsilon^2 \vec{q}^{*(1)}(\vec{r}^*) + \ldots \qquad (10\text{-}1)$$

where $\vec{q}^{*(1a)}(\vec{r}^*)$ is given by equation 7-54, $u^{*(1)}(\vec{r}^*)$ is given by equation 8-2, and $v^{*(1)}(\vec{r}^*)$ is given by

$$v^{*(1)}(\vec{r}^*)$$

$$= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \frac{1}{\rho^*}\Big\{F_v(\vec{r}^*;\rho^*,\phi)\Big(-\frac{e^{\rho^*\cos\phi}\cos^2\phi\,\rho^{*2}}{16\pi^2} K_1(\tfrac{\rho^*}{2})K_1(\tfrac{\rho^*}{2})\Big)$$

$$-4F_v(\vec{r}^*;0,\phi)\Big(-\frac{\cos^2\phi}{16\pi^2}\Big)\Big\}d\phi d\rho^*$$

$$+\frac{1}{2\pi}\int_1^{\infty}\int_0^{2\pi} F_v(\vec{r}^*;\rho^*,\phi)\Big\{-\frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2}K_1(\tfrac{\rho^*}{2})K_1(\tfrac{\rho^*}{2})\Big\}\rho^* d\phi d\rho^*$$

$$+\frac{1}{2\pi}\int_0^{\infty}\int_0^{2}\Big\{F_v(\vec{r}^*;\rho^*,\phi)\Big\}\cdot\Big\{\frac{e^{\rho^*\cos\phi}\cos\phi}{8\pi^2}K_o(\tfrac{\rho^*}{2})K_1(\tfrac{\rho^*}{2})$$

$$-\frac{e^{\rho^*\cos\phi}}{16\pi^2}K_o(\tfrac{\rho^*}{2})K_o(\tfrac{\rho^*}{2})\Big\}\rho^* d\phi d\rho$$

$$+\frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\Big\{-\frac{1}{2a}K_o\Big(\frac{|\vec{r}^*-\vec{\rho}^*|}{2a}\Big)e^{\frac{x^*-\rho^*\cos\phi}{2a}}\;-\frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|^2}$$

$$+\frac{x^*-\rho^*\cos\phi}{2a|\vec{r}^*-\vec{\rho}^*|}K_1\Big(\frac{|\vec{r}^*-\vec{\rho}^*|}{2a}\Big)e^{\frac{x^*-\rho^*\cos\phi}{2a}}\Big\}\cdot$$

$$\Big\{\frac{e^{\rho^*\cos\phi}\sin\phi}{16\pi^2}K_o(\tfrac{\rho^*}{2})K_1(\tfrac{\rho^*}{2}) - \frac{e^{\rho^*\cos\phi}\sin\phi\cos\phi}{16\pi^2}K_1(\tfrac{\rho^*}{2})K_1(\tfrac{\rho^*}{2})\Big\}\rho^* d\phi d\rho^*$$

(10-2)

where

$$F_v(\vec{r}^*;\rho^*,\phi)$$

$$= -\frac{y^*-\rho^*\sin\phi}{|\vec{r}^*-\vec{\rho}^*|^2} + \frac{y^*-\rho^*\sin\phi}{2a|\vec{r}^*-\vec{\rho}^*|} K_1\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2a}\right)e^{\frac{x^*-\rho^*\cos\phi}{2a}} . \qquad (10\text{-}3)$$

The result for the non-dimensional pressure is

$$p^*(\vec{r}^*) \sim \epsilon^2\log\epsilon \frac{x^*}{8\pi^2 r^{*2}} + \epsilon^2 p^{*(1)}(\vec{r}^*) + \ldots \qquad (10\text{-}4)$$

where

$$p^{*(1)}(\vec{r}^*)$$

$$= \frac{1}{2\pi}\int_0^1\int_0^{2\pi}\frac{1}{\rho^*}\left\{\frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|^2}\left(-\frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2}\rho^{*2}K_1\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right)\right.$$
$$\left. - \frac{4x^*}{r^{*2}}\left(-\frac{\cos^2\phi}{16\pi^2}\right)\right\} d\rho^* d\phi$$
$$+ \frac{1}{2\pi}\int_1^{\infty}\int_0^{2\pi}\left\{\frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|^2}\right\}\cdot\left\{-\frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2}K_1\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right\}\rho^* d\phi d\rho^*$$
$$+ \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\left\{\frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|^2}\right\}\cdot\left\{\frac{e^{\rho^*\cos\phi}\cos\phi}{8\pi^2}K_o\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right.$$
$$\left. - \frac{e^{\rho^*\cos\phi}}{16\pi^2}K_o\left(\frac{\rho^*}{2}\right)K_o\left(\frac{\rho^*}{2}\right)\right\}\rho^* d\phi d\rho^*$$
$$+ \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\left\{\frac{y^*-\rho^*\sin\phi}{|\vec{r}^*-\vec{\rho}^*|^2}\right\}\cdot\left\{\frac{e^{\rho^*\cos\phi}\sin\phi}{16\pi^2}K_o\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right.$$
$$\left. - \frac{e^{\rho^*\cos\phi}\cos\phi\sin\phi}{16\pi^2}K_1\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right\}\rho^* d\phi d\rho^* . \qquad (10\text{-}5)$$

The result for the non-dimensional magnetic induction field is

$$\overline{B}^*(\overline{r}^*) = \left\{\overline{i}_x\left(-\frac{e^{x^*/2}y^*}{4\pi r^*}K_1\left(\frac{r^*}{2}\right)\right) + \overline{i}_y\left(\frac{e^{x^*/2}x^*}{4\pi r^*}K_1\left(\frac{r^*}{2}\right) - \frac{e^{x^*/2}}{4\pi}K_o\left(\frac{r^*}{2}\right)\right)\right\} + \ldots \qquad (10\text{-}6)$$

Finally, our expression for the drag is

$$\frac{\text{Drag}}{\text{unit length}} \sim \left\{-\frac{1}{4\pi}\log\left(\sigma l\sqrt{\frac{\mu^3}{\rho}}\right) + \frac{1.9}{16\pi^2}\right\}\sigma l^2\mu^2 U \ \frac{\text{newtons}}{\text{meter}}. \qquad (10\text{-}7)$$

Some of the more interesting features of our solution may now be discussed. First of all, we note that the fluid acquires vorticity by the action of the non-conservative electromagnetic forces. (This was pointed out by Ludford and Murray, reference 6.) This can be most easily seen if we take the curl of the vector equation formed from equations 7-41 and 7-42. The result is

$$a\nabla^{*2}\omega^* - \frac{\partial\omega^{*(1)}}{\partial x^*} = F(x^*, y^*) \neq 0 \qquad (10\text{-}8)$$

where F is made up of terms which for large r^* have the factor

$$e^{x^* - \rho^*} \qquad (10\text{-}9)$$

appearing. Thus, at least for r^* <u>not too large,</u> $\omega^{*(1)}$ should have larger values in a parabolic wake region whose shape is outlined by lines of $x^* - \rho^*$ = constant. This is just one half the size of the parabolic wake possessed by the magnetic field, in which (from equation 10-6) the factor

$$e^{\frac{x^* - \rho^*}{2}} \qquad (10\text{-}10)$$

appears. This is similar to the result of Ludford and Murray. (Note that the magnetic wake is independent of ν.)

Far from the origin, however, all the perturbation terms we have computed behave like the fundamental solution of the Oseen equations. These solutions have a parabolic wake which depends only on the kinematic viscosity, and not on the electrical properites of the fluid. This can be seen by noting that the wake is outlined by lines where

$$\frac{x^* - r^*}{2\alpha} = \text{constant}, \tag{10-11}$$

which, when written in physical variables, becomes

$$\frac{x - r}{2\nu/U} = \text{constant} \tag{10-11a}$$

This result is consistent with the work of I-dee Chang (ref. 4), which states that the flow far from the finite region where disturbances are present (in our case, these disturbances come from electromagnetic forces which are essentially in a finite region) has a general characteristic behavior independent of the details of the disturbances. In particular, if the only force on the body is a drag, the flow at large distances behaves like the fundamental solution of Oseen's equations when the concentrated force is in the x-direction. This is obviously precisely what we have found.

The following figure (fig. 2) shows the general outline of the two types of wake, and representative field lines of the magnetic induction field.

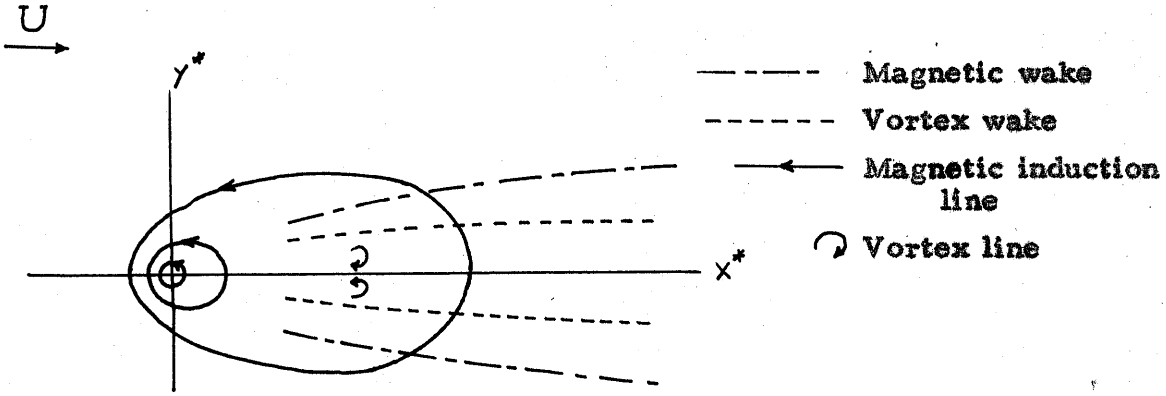

Fig. 2

Another interesting feature of our result is that the drag is independent of viscosity, to the approximation considered. This makes it very tempting to hypothesize that the drag and the flow quantities we have computed are uniformly valid in α as $\alpha \to 0$. Note that the viscous wake would then collapse onto the x-axis.

There is another reason why our solution might be uniformly valid in space and in α as $\alpha \to 0$: before studying the problem presented here, the author attempted (with only limited success) to solve the same problem with zero kinematic viscosity. The magnetic field turned out to be identical with that given by equation 10-6, and far from the origin, the first perturbation term of the velocity was a simple source flow with a line singularity along the positive x-axis which carried an influx of mass just equal to the outflux from the source. This velocity term was $O(\epsilon^2 \log \epsilon)$. This solution is precisely what is obtained from $\vec{q}^{*(1a)}(\vec{r}^*)$ in the limit as $\nu \to 0$. In particular the first term of the drag is the same in both the viscous and non-viscous cases.

A natural extension of the above hypothesis to three dimensions would be the study of inviscid flow past a sphere in which is embedded a magnetic dipole source. From the above conjecture in the two-dimensional inviscid case, one could suggest that if the sphere were sufficiently small compared with some characteristic size which increases or decreases as the strength of the dipole increases or decreases, the first approximation to the drag would be independent of the body size. A similar problem is that studied by Ludford and Murray, reference 6; however, they consider the opposite case in which the size of the sphere is large compared to a length based on the dipole strength. As would be expected, their drag depends on the size of the sphere.

XI. EXPERIMENTAL POSSIBILITIES

In section VI, we presented arguments to support the statement that the leading terms in the asymptotic development are given by equations 10-1, 10-4 and 10-7. According to that discussion, the maximum size of a "core" around the origin should be given by equation 6-12. Thus, an experiment designed to test our theory might be set up by studying the flow of a conducting fluid past a current-carrying wire. In order to avoid purely viscous effects, the size of this wire should be at least as small as the possible core, namely

$$\text{wire diameter} \sim \alpha e^{-\alpha/u_v \epsilon^2} / \sigma\mu U \qquad (11\text{-}1)$$

where $u_v = o(1)$.

Two fluids which might be considered for this experiment are sulphuric acid and mercury. For these fluids we have the following approximate values for ϵ and α:

Sulphuric acid at 18°C, 400 gms/liter of solution:

$$\epsilon = (3 \times 10^{-9}) \times (\text{number of amperes}) \qquad (11\text{-}2a)$$

$$\alpha = 1.7 \times 10^{-10}. \qquad (11\text{-}2b)$$

Mercury at 20°C:

$$\epsilon = (1.3 \times 10^{-5}) \times (\text{number of amperes}) \qquad (11\text{-}3a)$$

$$\alpha = 1.5 \times 10^{-7}.$$

These numbers are extremely small, and while ϵ can be made <u>smaller</u> by choosing small current, from the standpoint of measurements it would be more desirable if ϵ were not so small. For example, for mercury, if $I = 100$ amperes, the magnetohydrodynamic drag, $Drag_m$, is only

$$Drag_m \sim 10^{-3} \text{ newtons/meter} . \qquad (11\text{-}4)$$

(Note that since a is only 1.5×10^{-7}, equation 11-4 would have to be based on the assumption that our solution is uniformly valid as $a \to 0$.) Even if the wire carrying the current were as large as $\epsilon/\sigma\mu U$, this would still be 10^{-5} meters diameter, hardly a reasonable size for carrying 100 amperes.

A possible alternative is to use a wire large enough to carry a very high current. If we use mercury, take the wire to be 0.01 meter in diameter, and $U = 0.1$ meter/sec, then since the fluid dynamic drag coefficient is in this case of the order unity, the fluid dynamic drag, $Drag_{f.d.}$, is

$$Drag_{f.d.} \sim 1 \text{ newton/meter} . \qquad (11\text{-}5)$$

The magnetohydrodynamic drag when the current is 100 amperes is given by equation 11-4. The ratio $Drag_m : Drag_{f.d.}$ is

$$\frac{Drag_m}{Drag_{f.d.}} \sim 10^{-3} . \qquad (11\text{-}6)$$

Thus the main force is the ordinary fluid dynamic drag. It might be possible to look for the smaller contribution from magnetohydrodynamic effects by measuring the drag on the wire with and without the current flowing.

The above discussion suggests that we consider the possibilities of a plasma. Let us assume we use Argon at 10,000 °Kelvin, U ~ 100 meters/second, wire diameter ~ 10^{-2} meters, and current in the wire ~ 100 amperes. We must of course make certain that the gas behaves like a continuum, if the theory is to apply. We shall assume we do have continuum behavior if the current length, L_1, is larger than the mean free path between electron-neutral molecule collisions. Using $s \sim 10^{-20}$ meter^2 as the cross section for this type of collision, and assuming the neutral gas particles obey the perfect gas law, $p = N_n kT$, we find that the density N_n of neutral particles must satisfy

$$N_n >> 6 \times 10^{20} \text{ particles/meter}^3 . \qquad (11\text{-}7)$$

To obtain estimates, let us assume

$$N_n \sim 6 \times 10^{20} \text{ particles/meter}^3. \qquad (11\text{-}8)$$

Since at 10,000°Kelvin the degree of ionization is of the order 10^{-3}, the electron density, N_e, is

$$N_e \sim 6 \times 10^{17} \text{ electrons/meter}^3. \qquad (11\text{-}9)$$

Using the theory of Spitzer, (7), Chapter 5, we obtain for the conductivity

$$\sigma \sim 2 \times 10^3 \text{ mhos/meter} \qquad (11\text{-}10)$$

For a current of 100 amperes,

$$\epsilon \sim 6 \times 10^{-2}, \qquad (11\text{-}11)$$

and the magnetohydrodynamic drag, $Drag_m$, is

$$Drag_m \sim 10^{-3} \text{ newtons/meter} . \qquad (11\text{-}12)$$

For these flow conditions, the ordinary fluid dynamic drag coefficient is again approximately unity, so that the fluid dynamic drag, $Drag_{f.\,d.}$, is

$$Drag_{f.\,d.} \sim L\rho U^2 \sim 10^{-3} \text{ newtons/meter} . \qquad (11\text{-}13)$$

Comparison of equations 11-12 and 11-13 shows that the magnetohydrodynamic drag is comparable to the fluid dynamic drag, so there is a possibility of finding the magnetohydrodynamic drag by measuring the force on the wire with and without the current flowing.

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APPENDIX

In the text, it was stated that far away from the origin, the velocity and pressure field behaved like the fundamental solution of the Oseen equations, with a magnitude proportional to the integral of the forcing terms. The author has been unable to prove this statement. However, the following weaker statement can be proved: one can find an R such that the drag is determined to any accuracy by considering the integral of the forcing terms to radius R. Furthermore, the contribution from this part of the integral gives a velocity and pressure field at infinity which does behave like the fundamental solution. The proof is as follows:

We consider the volume bounded by a small circle at the origin and a circle of radius unity. The integral form of the momentum equation (for the x-direction momentum) states that a certain combination of surface integrals and volume integrals is zero, and the drag will be found by considering the flux of momentum on the small circle at the origin taken as its radius tends to zero. (Since we allow no sources, this momentum input will be in the form of viscous, pressure, and Maxwell stresses.) At the surface on the circle of unit radius the influx of momentum is due to viscous stresses, pressure, Maxwell stresses, and in addition transported momentum. Now the pressure, velocities and derivatives of the velocities evaluated on the unit circle are given in terms of integrals. We can choose an R_1 such that the contribution to these quantities (and hence to momentum flux across the unit circle) from the forcing terms outside R_1 is as small as we please. As for the

Maxwell stresses, we know that they are expressible in terms of the $\vec{j} \times \vec{B}$ volume forces outside the unit circle and the Maxwell stresses at infinity. At infinity, the Maxwell stresses consist of terms like

$$e^{x}K_i\left(\frac{r}{2}\right)K_j\left(\frac{r}{2}\right) \sim \frac{e^{x-r}}{r}, \quad i, j = 0, 1 \ ,$$

which is easily shown to give a zero contribution to the surface integral at infinity. Thus the Maxwell stresses on the unit circle are given by the volume integral of the $\vec{j} \times \vec{B}$ terms (which are the forcing terms in our equations) from one to infinity. But by choosing an R_2 large enough, we can make their contribution to the Maxwell stresses from the terms outside R_2 as small as we please. Hence we may choose $R = \max(R_1, R_2)$, and make the total influx of momentum at the unit circle arising from forcing terms outside R as small as we please, or, equivalently, we can find the forces on the small circle at the origin as accurately as we please by restricting ourselves to the forcing terms up to some large but finite radius R.

We shall next show that the forcing terms inside the radius R give velocity and pressure fields which behave like the fundamental solution to the Oseen equations, using the expressions for $u^{*(1)}$ as an example; the computations for $p^{*(1)}$ and $v^{*(1)}$ are exactly similar.

The contribution (call it $u_R^{*(1)}(\vec{r}^{\,*})$) to the $u^{*(1)}$ due to forcing terms inside R is given by

$$u_R^{*(1)}(\vec{r}^*) =$$

$$\frac{1}{2\pi}\int_0^R\int_0^{2\pi}\left\{\frac{1}{2\alpha}K_0\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2\alpha}\right)e^{\frac{x^*-\rho^*\cos\phi}{2\alpha}} - \frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|^2} + \frac{1}{2\alpha}K_1\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2\alpha}\right)\frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|}e^{\frac{x^*-\rho^*\cos\phi}{2\alpha}}\right\}\cdot$$

$$\cdot\left\{\frac{e^{\rho^*\cos\phi}\cos\phi}{8\pi^2}K_0\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right) - \frac{e^{\rho^*\cos\phi}}{16\pi^2}K_0\left(\frac{\rho^*}{2}\right)K_0\left(\frac{\rho^*}{2}\right)\right\}\rho^* d\phi\, d\rho^*$$

$$+\frac{1}{2\pi}\int_1^R\int_0^{2\pi}\left\{\frac{1}{2\alpha}K_0\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2\alpha}\right)e^{\frac{x^*-\rho^*\cos\phi}{2\alpha}} - \frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|^2} + \frac{1}{2\alpha}K_1\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2\alpha}\right)\frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|}e^{\frac{x^*-\rho^*\cos\phi}{2\alpha}}\right\}\cdot$$

$$\cdot\left\{-\frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2}K_1\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right\}\rho^* d\phi\, d\rho^*$$

$$+\frac{1}{2\pi}\int_0^1\int_0^{2\pi}\left[\left\{\frac{1}{2\alpha}K_0\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2\alpha}\right)e^{\frac{x^*-\rho^*\cos\phi}{2\alpha}} - \frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|^2} + \frac{1}{2\alpha}K_1\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2\alpha}\right)\frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|}e^{\frac{x^*-\rho^*\cos\phi}{2\alpha}}\right\}\cdot\right.$$

$$\cdot\left\{-\frac{e^{\rho^*\cos\phi}\cos^2\phi\,\rho^*}{16\pi^2}K_1\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right\}$$

$$-\left\{\frac{1}{2\alpha}K_0\left(\frac{|\vec{r}^*|}{2\alpha}\right)e^{\frac{x^*}{2\alpha}} - \frac{x^*}{|\vec{r}^*|^2} + \frac{1}{2\alpha}K_1\left(\frac{|\vec{r}^*|}{2\alpha}\right)\frac{x^*}{|\vec{r}^*|}e^{\frac{x^*}{2\alpha}}\right\}\cdot$$

$$\left.\cdot\left\{-\frac{4\cos^2\phi}{16\pi^2\rho^*}\right\}\right]d\rho^* d\phi. \qquad \text{(A-1)}$$

The proof that the first two integrals behave like the fundamental solution with coefficient given by the integral of the forcing terms is quite easy, but the last integral may offer some difficulty. To avoid being pedantic we shall consider only this last integral. The technique for treating the first two will then be obvious. Thus we are saying that the difference

$$\Bigg| \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \Bigg[\Bigg\{ \frac{1}{2\alpha} K_0\left(\frac{|\vec{r}^* - \vec{\rho}^*|}{2\alpha}\right) e^{\frac{x^* - \rho^* \cos\phi}{2\alpha}} - \frac{x^* - \rho^* \cos\phi}{|\vec{r}^* - \vec{\rho}^*|^2} + \frac{1}{2\alpha} K_1\left(\frac{|\vec{r}^* - \vec{\rho}^*|}{2\alpha}\right) \frac{x^* - \rho^* \cos\phi}{|\vec{r}^* - \vec{\rho}^*|} e^{\frac{x^* - \rho^* \cos\phi}{2\alpha}}$$

$$- \frac{1}{2\alpha} K_0\left(\frac{|\vec{r}^*|}{2\alpha}\right) e^{\frac{x^*}{2\alpha}} + \frac{x^*}{|\vec{r}^*|^2} - \frac{1}{2\alpha} K_1\left(\frac{|\vec{r}^*|}{2\alpha}\right) \frac{x^*}{|\vec{r}^*|} e^{\frac{x^*}{2\alpha}} \Bigg\} \cdot$$

$$\cdot \Bigg\{ - \frac{e^{\rho^* \cos\phi} \cos^2\phi \; \rho^*}{16\pi^2} K_1\left(\frac{\rho^*}{2}\right) K_1\left(\frac{\rho^*}{2}\right) \Bigg\}$$

$$- \Bigg\{ \frac{1}{2\alpha} K_0\left(\frac{|\vec{r}^*|}{2\alpha}\right) e^{\frac{x^*}{2\alpha}} - \frac{x^*}{|\vec{r}^*|^2} + \frac{1}{2\alpha} K_1\left(\frac{|\vec{r}^*|}{2\alpha}\right) \frac{x^*}{|\vec{r}^*|} e^{\frac{x^*}{2\alpha}}$$

$$- \frac{1}{2\alpha} K_0\left(\frac{|\vec{r}^*|}{2\alpha}\right) e^{\frac{x^*}{2\alpha}} + \frac{x^*}{|\vec{r}^*|^2} - \frac{1}{2\alpha} K_1\left(\frac{|\vec{r}^*|}{2\alpha}\right) \frac{x^*}{|\vec{r}^*|} e^{\frac{x^*}{2\alpha}} \Bigg\} \cdot$$

$$\cdot \Bigg\{ - \frac{4\cos^2\phi}{16\pi^2 \rho^*} \Bigg\} \Bigg] d\phi \, d\rho^* \Bigg|$$

$$= 0 \Bigg\{ \frac{1}{2\alpha} K_0\left(\frac{|\vec{r}^*|}{2\alpha}\right) e^{\frac{x^*}{2\alpha}} - \frac{x^*}{|\vec{r}^*|^2} + \frac{1}{2\alpha} K_1\left(\frac{|\vec{r}^*|}{2\alpha}\right) \frac{x^*}{|\vec{r}^*|} e^{\frac{x^*}{2\alpha}} \Bigg\} \cdot$$

(A-2)

The left-hand side can be written as

$$\left| \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \left\{ \frac{1}{2\alpha}K_0\left(\frac{|\vec{r}^* - \vec{\rho}^*|}{2\alpha}\right) e^{\frac{x^* - \rho^*\cos\phi}{2\alpha}} - \frac{x^* - \rho^*\cos\phi}{|\vec{r}^* - \vec{\rho}^*|^2} + \frac{1}{2\alpha}K_1\left(\frac{|\vec{r}^* - \vec{\rho}^*|}{2\alpha}\right) \frac{x^* - \rho^*\cos\phi}{|\vec{r}^* - \vec{\rho}^*|} e^{\frac{x^* - \rho^*\cos\phi}{2\alpha}} \right.\right.$$

$$\left. - \frac{1}{2\alpha}K_0\left(\frac{|\vec{r}^*|}{2\alpha}\right) e^{\frac{x^*}{2\alpha}} + \frac{x^*}{|\vec{r}^*|^2} - \frac{1}{2\alpha}K_1\left(\frac{|\vec{r}^*|}{2\alpha}\right) \frac{x^*}{|\vec{r}^*|} e^{\frac{x^*}{2\alpha}} \right\} \cdot$$

$$\left. \cdot \left\{ - \frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2} \rho^* K_1\left(\frac{\rho^*}{2}\right) K_1\left(\frac{\rho^*}{2}\right) \right\} d\phi\, d\rho^* \right|$$

$$= \left| \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \left[\frac{1}{2\alpha}K_0\left(\frac{|\vec{r}^*|}{2\alpha}\right) e^{\frac{x^*}{2\alpha}} - \frac{x^*}{|\vec{r}^*|^2} + \frac{1}{2\alpha}K_1\left(\frac{|\vec{r}^*|}{2\alpha}\right) \frac{x^*}{|\vec{r}^*|} e^{\frac{x^*}{2\alpha}} \right.\right.$$

$$+ (-\rho^*\cos\phi)\left\{ \frac{\partial}{\partial(\rho^*\cos\phi)} \left(\frac{1}{2\alpha}K_0\left(\frac{|\vec{r}^* - \vec{\rho}^*|}{2\alpha}\right) e^{\frac{x^* - \rho^*\cos\phi}{2\alpha}} - \frac{x^* - \rho^*\cos\phi}{|\vec{r}^* - \vec{\rho}^*|^2} \right.\right.$$

$$\left.\left. + \frac{1}{2\alpha}K_1\left(\frac{|\vec{r}^* - \vec{\rho}^*|}{2\alpha}\right) \frac{x^* - \rho^*\cos\phi}{|\vec{r}^* - \vec{\rho}^*|} e^{\frac{x^* - \rho^*\cos\phi}{2\alpha}} \right) \right\}_{\text{evaluated at some } \vec{\rho}_1^*.}$$

$$+ (-\rho^*\sin\phi)\left\{ \frac{\partial}{\partial(\rho^*\sin\phi)} \left(\frac{1}{2\alpha}K_0\left(\frac{|\vec{r}^* - \vec{\rho}^*|}{2\alpha}\right) e^{\frac{x^* - \rho^*\cos\phi}{2\alpha}} - \frac{x^* - \rho^*\cos\phi}{|\vec{r}^* - \vec{\rho}^*|^2} \right.\right.$$

$$\left.\left. + \frac{1}{2\alpha}K_1\left(\frac{|\vec{r}^* - \vec{\rho}^*|}{2\alpha}\right) \frac{x^* - \rho^*\cos\phi}{|\vec{r}^* - \vec{\rho}^*|} e^{\frac{x^* - \rho^*\cos\phi}{2\alpha}} \right) \right\}_{\text{evaluated at some } \vec{\rho}_1^*.}$$

$$\left. - \frac{1}{2\alpha}K_0\left(\frac{|\vec{r}^*|}{2\alpha}\right) e^{\frac{x^*}{2\alpha}} + \frac{x^*}{|\vec{r}^*|^2} - \frac{1}{2\alpha}K_1\left(\frac{|\vec{r}^*|}{2\alpha}\right) \frac{x^*}{|\vec{r}^*|} e^{\frac{x^*}{2\alpha}} \right] \cdot$$

$$\left. \cdot \left[- \frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2} \rho^* K_1\left(\frac{\rho^*}{2}\right) K_1\left(\frac{\rho^*}{2}\right) \right] d\phi\, d\rho^* \right|$$

(A-3)

where, according to the mean value theorem, $\left|\vec{\rho}_1^{\,*}\right| < 1$. Now the last expression satisfies

$$\left[\text{Equation A-3}\right] \le \frac{1}{2\pi} \max_{|\vec{\rho}^*|\le 1} \left| \frac{\partial}{\partial(\rho^*\cos\phi)} \left(\frac{1}{2\alpha} K_0\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2\alpha}\right) e^{\frac{x^*-\rho^*\cos\phi}{2\alpha}} \right.\right.$$

$$\left.\left. - \frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|^2} + \frac{1}{2\alpha} K_1\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2\alpha}\right) \frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|} e^{\frac{x^*-\rho^*\cos\phi}{2\alpha}} \right)\right| \cdot$$

$$\cdot \int_0^1\int_0^{2\pi} \left| -\rho^*\cos\phi \left(- \frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2} \rho^* K_1\left(\frac{\rho^*}{2}\right) K_1\left(\frac{\rho^*}{2}\right)\right) \right| d\phi\, d\rho^*$$

$$+ \frac{1}{2\pi} \max_{|\vec{\rho}^*|\le 1} \left| \frac{\partial}{\partial(\rho^*\sin\phi)} \left(\frac{1}{2\alpha} K_0\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2\alpha}\right) e^{\frac{x^*-\rho^*\cos\phi}{2\alpha}} - \frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|^2} \right.\right.$$

$$\left.\left. + \frac{1}{2\alpha} K_1\left(\frac{|\vec{r}^*-\vec{\rho}^*|}{2\alpha}\right) \frac{x^*-\rho^*\cos\phi}{|\vec{r}^*-\vec{\rho}^*|} e^{\frac{x^*-\rho^*\cos\phi}{2\alpha}} \right)\right| \cdot$$

$$\cdot \int_0^1\int_0^{2\pi} \left| -\rho^*\sin\phi \left(- \frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2} \rho^* K_1\left(\frac{\rho^*}{2}\right) K_1\left(\frac{\rho^*}{2}\right)\right) \right| d\phi\, d\rho^* . \qquad \text{(A-4)}$$

The integrals exist and are independent of $\vec{r}^*$. The derivatives go to zero, as $\vec{r}^*$ approaches infinity, at least as fast as $\vec{r}^{*\,-3/2}$, which is a faster decay than the decay of the fundamental solution. This completes the proof of equation A-2. As stated above, the first two integrals of equation A-1 can be shown to have a similar asymptotic behavior.

Now that we know the behavior at infinity of u_R, v_R, p_R, it is a simple matter to compute their contribution to the drag. But we showed above that the resulting drag can be made as close to the real drag as we please by choosing R large enough. This proves that the method used in section IX is correct.