QUARK MODELS
AS REPRESENTATIONS OF CURRENT ALGEBRA

Thesis by

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ABSTRACT

The equal time U(12) algebra of scalar, pseudoscalar, vector, axial and tensor currents abstracted from Lagrangian quark field theory is studied. The attempt is made to represent the "good" part of this algebra at infinite momentum on nonexotic states, i.e., on hadron states of conventional nonrelativistic quark models. Relativistic constraints embodied in the angular condition must also be met.

Previous work has shown that the unintegrated algebra cannot be represented on nonexotic states. In this study, the much less restrictive problem of the once and twice integrated algebra is considered. It is found that even the twice integrated algebra cannot be satisfied within nonexotics. This strongly suggests that exotics are an essential part of the hadron spectrum.

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I. Introduction

We shall be concerned here with two ideas: the quark model for the hadron spectrum and current algebra. In particular, we wish to see if the quark model can be a solution to the equal time current algebra.

1. It is well known that mesons can be described approximately as $q\bar{q}$ and baryons as qqq.[1,2] Moreover very naive non-relativistic dynamics, e.g. "springs" betweeen the quarks, seem to give a good approximate description of the spectrum of stable particles and low-lying resonances.[3] Here, the quark model will be taken simply as a specification of the quantum numbers: intrinsic quantum numbers (baryon number, isospin and strangeness), spin ($\mathcal{J}$), mass (M), parity ($\mathcal{P}$), charge conjugation ($\mathcal{C}$) etc.

Among the features of the meson spectrum implied or "explained" by such a description are :

(1) Mesons can be classified into SU(3) $\underset{\sim}{1}$'s and $\underset{\sim}{8}$'s

(2) The spin J can be thought of as made of the quark spin S (= 0, 1 for mesons), and the orbital angular momentum L; and in first approximation, M^2 is linear in L and very approximately independent of S. Mesons can therefore be classified into U(6) $\underset{\sim}{36}$'s.

(3) The parity and charge conjugation satisfy

$\mathcal{P} = (-1)^{L+1}$, $\mathcal{C} = (-1)^{L+S}$.

States violating (1) and (3) are said to be exotic, respectively of the first and second kind.

2. Vector and axial vector currents of hadrons can be defined through their participation in the electromagnetic and weak interactions[4]. Gell-Mann proposed for these currents a local algebra of equal-time commutators[5], which can be abstracted from the currents defined in a field theory of quarks:

$$D(\tfrac{1}{2}\lambda_a\Gamma, x) = \psi^\dagger(x)\ \tfrac{1}{2}\lambda_a\Gamma\ \psi(x)$$

$$= \bar{\psi}(x)\ \tfrac{1}{2}\lambda_a\gamma_0\Gamma\ \psi(x) \qquad (1)$$

where $\psi(x)$ is the 12 component quark field (SU(3) $\otimes$ Dirac indices), λ_a the usual matrices of SU(3) and Γ any one of the 16 Dirac matrices. (By $D(\tfrac{1}{2}\lambda_a\Gamma, x)$ we simply mean a local operator at x, with the Lorentz properties of $\gamma_0\Gamma$ and commuting as the expression on the RHS of (1)). For example, the well known isotopic charges are

$$I_a = \int d^3x\ D(\tfrac{1}{2}\tau_a\ , x) \qquad a = 1, 2, 3 \quad (2)$$

and obey the equal time algebra

$$[I_a\ , I_b] = i\epsilon_{abc}\ I_c \qquad (3)$$

3. There is no a priori reason for any connection between the quark fields in the currents and the quarks in the spectrum - for example a spectrum where mesons occur only in SU(3) $\underset{\sim}{8}$'s can be compatible with current algebra. Whatever connection there might be must be nontrivial, since they have different relativistic properties. In other words, a nonrelativistic "charge distribution" inside a hadron and relativistic form factors are related in nontrivial ways. The possibility of a unitary transformation linking the "current quarks" with the "spectrum quarks" has been emphasized by Gell-Mann.

(Real quarks, should they exist, are a third thing altogether. For example, the negative results of accelerator searches[(6)] indicate that real quarks are rather heavy - several Gev - whereas a "spectrum quark" is usually thought of as having ≈ 1/3 of a proton mass. We ignore completely the question of real quarks.)

There is, however, much phenomenological evidence that "current quarks" and "spectrum quarks" are closely related. Simple quark models of the hadrons seem to yield the correct pattern of electromagnetic, weak and pion-emission amplitudes. Unfortunately these models are typically non-relativistic, or relativistic but infested with ghosts (states with negative norm), or fail to respect current algebra sum rules. For one recent example, see Feynman et al[7]. Perhaps the best known success in calculating electromagnetic matrix elements from a quark picture is the result for the magnetic moments[8]:

$$\mu_p = 3\,\frac{e}{2M} \qquad\qquad \mu_n = -2\,\frac{e}{2M}$$

4. Our starting point is the hypothesis that the spectrum generated by the quark model saturates the equal time commutators abstracted from a field theory of quarks. That is, we try to represent the local current algebra on the space of quark model states. The important constraint will be relativity. The problem of saturating current algebra sum rules was first proposed by Dashen and Gell-Mann[9].

Should the program succeed, i.e. should we find a representation of current algebra on the space of quark model states – perhaps an almost unique one – we would have all electromagnetic and weak matrix elements between hadron states, and by PCAC, soft pion emission amplitudes. One might even find that a particular form of $q\bar{q}$ interaction (e.g. "springs") or that particular values of parameters (e.g. $\alpha_\rho(0) = \frac{1}{2}$) are required – such has been the case with the study of dual resonance models[10].

The program has been known to have difficulties for many years. By considering the subproblem of the isotopic subalgebra restricted to strange meson states, (the case of a single "charge carrier"), Dashen et al[11] showed that tachyons (states with space-like momenta) are required. Moreover, for the full problem of two "charge carriers", Weyers[12] and Hill[13] found that it was impossible to satisfy relativity and current algebra simultaneously.

5. In this study, we loosen the hypothesis from the saturation by nonexotics (i.e. quark model states) of the entire local algebra of densities to saturation of just the integrated algebra of the charges. In doing so, we evade the proof for the necessity of tachyons and also the difficulty found by Weyers and Hill. We find, however, that new difficulties arise and even this relaxed problem has no solution.

We are led to believe that one of the following is true:

(1) Exotics (i.e. states outside the quark model) are an essential part of the real world. This may occur either because of the breakdown of resonance saturation at high energies - the continuum will surely have exotics; or by the coupling of exotic meson resonances at high mass. An exotic meson is one that lies outside **36** of

U(6) or has $\mathcal{C} \neq \mathcal{P}(-1)^{S+1}$.

By comparison, note that duality also forces the existence of exotics, at least in the baryon-antibaryon channel[14].

(2) Exotics can be neglected but the quark mass is fixed (in relation to the slope of the Regge trajectory), i.e. $\alpha(0)$ for the leading trajectory is fixed. Dual resonance model also requires a definite $\alpha(0)$ to yield a bootstrap[10].

(3) Solutions exist for a range of parameters (e.g. quark mass), but are singular in the limit where all mesons are degenerate ($\alpha' \to 0$, $\alpha'^{-1}(0)$ fixed) or the limit where the quark mass is infinite ($\alpha(0) \to -\infty$, α' fixed), rendering simple expansion procedures invalid.

Mathematical techniques employed here — that of expansion in mass splitting or in inverse quark mass — are incapable of distinguishing among these possibilities. (2) is especially intriguing; it would be interesting either to construct an example of this type or to prove its impossibility.

A very similar no-go theorem for the Born term (i.e. no unitarity imposed) of dual models has been proved by Finkelstein[15]. He shows the mutual incompatibility of

(1) The imaginary part of meson-meson scattering amplitudes is saturated by nonexotic intermediate states

($\underset{\sim}{1}$ and $\underset{\sim}{8}$ of SU(3) and $C = (-1)^{L+S}$, $P = (-1)^{L+1}$).

(2) Nonexotic Regge poles control nondiffractive high energy scattering.

(3) Absence of parity-doublets at $t = 0$.

6. Chapter 2 descibes the equal time current algebra and the infinite momentum limit. Chapter 3 defines a class of quark models and summarizes previous work and difficulties. In Chapters 4 and 5, expansions around a degenerate limit and about an infinite quark mass limit will be considered. We shall show that solutions to the problem exist in neither case.

II. Current Algebra and the Infinite Momentum Limit

In this chapter, the essential features of the $p_z \to \infty$ limit, including the angular condition, will be outlined. Sections 1 and 2 follow closely the lectures of Gell-Mann(16) and Gell-Mann and Dashen(17); sections 3 and 4 follow Chang, Dashen and O'Raifeartaigh(18), where the angular condition is derived in great detail.

1. From a field theory of quarks one can construct scalar, pseudoscalar, vector, axial vector and anti-symmetrical tensor currents:

$$\begin{aligned}
S_a(x) &= D(\tfrac{1}{2}\lambda_a\gamma_0 , x) \\
P_a(x) &= D(\tfrac{1}{2}\lambda_a\gamma_0\gamma_5 , x) \\
V_{a\mu}(x) &= D(\tfrac{1}{2}\lambda_a\gamma_0\gamma_\mu , x) \\
A_{a\mu}(x) &= D(\tfrac{1}{2}\lambda_a i\gamma_0\gamma_5\gamma_\mu , x) \\
T_{a\mu\nu}(x) &= D(\tfrac{1}{2}\lambda_a\gamma_0\sigma_{\mu\nu} , x)
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
D(\tfrac{1}{2}\lambda_a\Gamma, x) &= \psi^\dagger(x)\tfrac{1}{2}\lambda_a\Gamma\psi(x) \\
&= \bar{\psi}(x)\tfrac{1}{2}\lambda_a\gamma_0\Gamma\psi(x)
\end{aligned}$$

and $a = 0, \ldots, 8$ (U(3) index); $\mu, \nu = 0, \ldots, 3$ (Lorentz) index; $\psi(x)$ is the 12 component (SU(3) $\otimes$ Dirac) quark field. Our notation is such that

$$\{\gamma_\mu , \gamma_\nu\} = 2g_{\mu\nu} \qquad [\gamma_\mu , \gamma_\nu] = -2i\sigma_{\mu\nu}$$

$$g_{\mu\nu} = \mathrm{diag}\,(1; -1 , -1 , -1)$$

$$\gamma_5 = -\gamma_0\gamma_1\gamma_2\gamma_3$$

The equal time commutation relations are simply

$$[D(M,x),D(M',y)] = iD(M'',x)\delta^3(x-y) + \text{possible gradient terms} \quad (2)$$

$$(x_o = y_o)$$

if $[M , M'] = iM''$. For example

$$[V_{ao}(x) , V_{bo}(y)] = if_{abc}V_{co}(x)\delta^3(x-y) + \ldots$$

$$(x_o = y_o)$$

since $[\tfrac{1}{2}\lambda_a , \tfrac{1}{2}\lambda_b] = if_{abc}\tfrac{1}{2}\lambda_c$

Thus these currents form a local U(12) algebra.

It is now common to postulate that currents can be defined in the real world with these Lorentz properties and these equal-time commutators, exact as far as the strong interactions are concerned(19). The matrix elements of the V and A currents are accessible through electromagnetic and weak interactions. For these, the current algebra hypothesis has received confirmation: the Adler-Weisberger sum rule for $[A_{1+i2,0}(x), A_{1-i2,0}(y)]$ [20] and the Cabibbo-Radicatti sum rule for $[V_{1+i2,0}(x), V_{1-i2,0}(y)]$.[21] We do not know how the other currents are used in nature, if at all.

2. The use of infinite momentum

Define fourier components

$$D(M,\underset{\sim}{k}) = \int D(M,\underset{\sim}{x})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, d^3x \tag{3}$$

The commutators in (2) then read

$$[D(M,\underset{\sim}{k}), D(M',\underset{\sim}{k}')] = iD(M'',\underset{\sim}{k}+\underset{\sim}{k}') + \ldots \tag{4}$$

where $[M,M'] = iM''$ and ... represents possible gradient terms. We shall sandwich (4) between states $|a\rangle$, $|b\rangle$ with $p_{az} \to \infty$, $p_{bz} \to \infty$ and $p_{az} - p_{bz}$ fixed. The importance of this class of sum rules was first pointed out by Fubini and Furlan[22]. The important features of the $p_z \to \infty$ limit will be illustrated by a simple example:

The matrix element of the $\Delta S = 1$ vector current between K and π states is

$$\langle\pi^-|V_{4-i5,\mu}(k)|K^0\rangle = \tfrac{1}{2}(E_K E_\pi)^{-\frac{1}{2}}\left\{(p+q)_\mu f_+(k^2) + (p-q)_\mu f_-(k^2)\right\}.$$

where

$$\vec{p} = \vec{p}_K = (E_K ; p_x , p_y , p_z)$$

$$\vec{q} = \vec{q}_\pi = (E_\pi ; q_x , q_y , q_z)$$

$$\vec{k} = \vec{q} - \vec{p}$$

Keep $\underset{\sim}{k}$ fixed and send p_z , q_z to ∞, then

$$E_K = p_z + 0(\frac{1}{p_z}) \qquad E_\pi = q_z + 0(\frac{1}{p_z}) = p_z + 0(1)$$

$$\tfrac{1}{2}(E_K E_\pi)^{-\frac{1}{2}}(E_K + E_\pi) = 1 + 0(\frac{1}{p_z})$$

$$\tfrac{1}{2}(E_K E_\pi)^{-\frac{1}{2}}(E_K + E_\pi) = 0(\frac{1}{p_z})$$

$$k^2 = (E_K - E_\pi)^2 - (p_K - p_\pi)^2 = -k_\perp^2 + 0(\frac{1}{p_z})$$

Thus the time component is

$$\langle \pi^- | V_{4-i5,0}(k) | K^0 \rangle \longrightarrow f_+(-k^2)$$

It can be seen that the z-component approaches the same limit, and the transverse components vanish. The following properties are evident:

(1) The $p_z \to \infty$ matrix elements are independent of $\underset{\sim}{p} + \underset{\sim}{q}$, but only on $\underset{\sim}{k}_\perp = (\underset{\sim}{p} - \underset{\sim}{q})_\perp$. So we may as well set $k_z = 0$ from now on and also drop momentum labels other that $\underset{\sim}{k} = (\underset{\sim}{k}_\perp, 0)$.

(2) The t and z components are finite and equal; the x and y components vanish. In general

$$S\,,\,P\,,\,V_x\,,\,V_y\,,\,A_x\,,\,A_y\,,\,T_{xy}\,,\,T_{zt} \;=\; O(\tfrac{1}{p_z})$$

("bad" operators)

$$\begin{matrix} V_t & A_t & T_{tx} & T_{ty} \\ V_z & A_z & T_{zx} & T_{zy} \end{matrix} \;=\; O(1)$$

("good" operators)

and equal in pairs. Note that "good" operators have an even numbers of γ_o or γ_z e.g. $V_t \sim 1$, $V_z \sim \gamma_o \gamma_z$ while "bad" operators contain an odd number e.g. $S \sim \gamma_o$. It follows that

$$G \cdot G = G \qquad G \cdot B = B \qquad B \cdot B = G$$

and commutators fall into three classes, which may be symbolized as

$$[G\,,\,G] = G \qquad \Sigma\, O(1)\; O(1) = O(1)$$

$$[G\,,\,B] = B \qquad \Sigma\, O(1)\; O(\tfrac{1}{p_z}) = O(\tfrac{1}{p_z})$$

$$[B\,,\,B] = G \qquad \Sigma\, O(\tfrac{1}{p_z}) O(\tfrac{1}{p_z}) = O(1)$$

The first class will of course be the most reliable - it is these we shall study here. The second class may not converge, and the last is of course very singular.

This particular feature is most evident in the following example: For the (conserved) isotopic currents $V_{+\alpha}$, $V_{-\beta}$ (+, - are isotopic indices, α,β Lorentz indices), the spin averaged matrix element of the following commutator can be written in terms of the usual structure functions $W_{1,2}$:

$$\frac{1}{2p_z}\int\left\langle p\left|\left[V_{+\alpha}\left(\frac{x}{2}\right), V_{-\beta}\left(-\frac{x}{2}\right)\right]\right|p\right\rangle e^{iq\cdot x}\, d^3x$$

$$= \frac{1}{2p_z^2}\int d\nu\ \left(p_\alpha - \frac{\nu}{q^2}q_\alpha\right)\left(p_\beta - \frac{\nu}{q^2}q_\beta\right) W_2^-(q^2,\nu)$$

$$+ \left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right) W_1^-(q^2,\nu)$$

$$p_z \to \infty,\ \underset{\sim}{p}_\perp = 0,\ \underset{\sim}{q} = (0;\ q\ ,\ 0)$$

where $\nu = p\cdot q$ and commutativity of the $p_z \to \infty$ limit with the integral has been assumed (incorrectly in some cases, as we shall see). For various values of α, β, and ignoring W_1^- for simplicity,

[G , G] e.g. $\alpha = 0$, $\beta = 0$

RHS $\propto \int d\nu\ W_2^-$

[G , B] e.g. $\alpha = 0$, $\beta = 1$

$$\text{RHS} \propto \frac{1}{p_z}\frac{q_x}{q^2}\int d\nu\ \nu W_2^-$$

[B , B] e.g. $\alpha = 1$, $\beta = 1$

$$\text{RHS} \propto \frac{1}{p_z^{\ 2}}\left(\frac{q_x}{q^2}\right)^2\int d\nu\ \nu^2 W_2^-$$

Clearly the three integrals get progressively less convergent. Since W_2^- is crossing-odd, its leading high energy behaviour will be determined by vector exchange, with $\alpha(q^2) = \alpha_\rho(q^2) \approx \frac{1}{2}$:

$$W_2^-(q^2, \nu) \approx \nu^{\alpha(q^2)-2} \approx \nu^{-\frac{3}{2}} \qquad \nu \to \infty$$

so that the integrands behave as $\nu^{-3/2}$, $\nu^{-1/2}$, $\nu^{1/2}$ respectively. Thus the last two integrals are not convergent — and taking $p_z \to \infty$ inside the integral is not valid in those cases.

(3) $k^2 = -k_\perp^2$ is independent of the masses, which is not the case for finite p_z. One is thus able to derive fixed $t = (k + k')^2$ and fixed current mass k^2 , k'^2 sum rules. These involve a dispersion integral over the absorptive part of a fixed t, fixed masses current-hadron amplitude; for $t = 0$ we have sum rules on total cross-sections. The convergence rate can be estimated by Regge theory and comparison with experiment is easier.

For example, the Adler-Weisberger sum rule[20] involves

$$\int \frac{ds}{s-M^2} [\sigma_t(\pi^+ p) - \sigma_t(\pi^- p)]$$

The integrand is asymptotically $s^{\alpha_\rho - 2}$ and convergence is assured.

(4) Since k^2 can be kept small throughout the sum rule, PCAC can be used, as in the Adler-Weisberger sum rule. And since $k^2 < 0$, one never comes across discontinuities in the current channel.

(5) The program of saturation by non-exotics becomes plausible with the elimination of the Z-diagram (II) and the disconnected diagram (III). (See Fig. 1) Diagram II involves pair creation by the current — leading in general to "three particle" intermediate states. But the three particles each has energy $\approx p_z$ and momentum $\approx p_z, -p_z, p_z$, so that the mass square of this intermediate state is

$$s = E^2 - p^2 = (3p_z)^2 - p_z^2 = 8p_z^2 \to \infty$$

But if convergence is assured by Regge theory, then no infinite mass states can contribute. ($\int ds A(s) < \infty \Rightarrow \lim_{s\to\infty} A(s) = 0$), so the Z-diagram must vanish as $p_z \to \infty$.

Figure 1 . Different classes of graphs that contribute to the current commutator

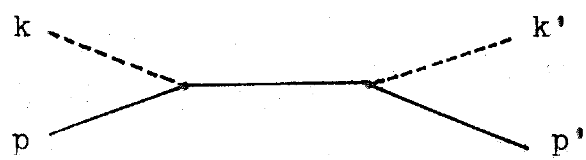
I. Direct graphs

II. Z- graphs

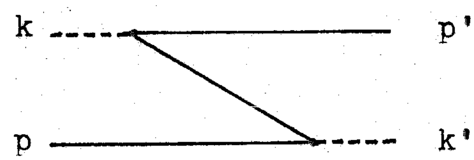
III. Disconnected graphs

In each case the currents are represented by broken lines and the hadrons by solid lines.

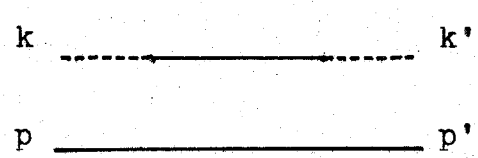
I.



II.



III.



The intermediate state in III has an energy ≈ $p_z + \sqrt{k^2 + m^2}$ and a momentum ≈ $p_z + k$, so that the $(\text{mass})^2$ is

$$s = E^2 - p^2 = 2p_z(\sqrt{k^2 + m^2} - k) + m^2 \to \infty$$

By the same argument as before, the disconnected diagram must also vanish. With only the direct diagram (I) left, it is then at least plausible to assume that the intermediate state is nonexotic.

3. "Reduced" space

Since momentum labels for the states are no longer needed at $p_z \to \infty$, it is sometimes convenient to think of the current operators as defined not on the usual space Ω of (on-shell) hadron states, but on a "reduced" space ($\approx\Omega/R^3$) of states without momentum labels:

$$\langle N'h' | D(M,\underset{\sim}{k}) | Nh \rangle = \lim_{p_z \to \infty} \langle N'h'p' | D(M,\underset{\sim}{k}) | Nhp \rangle \quad (5)$$

$$p' = p + k \qquad k = (k_x, k_y, 0)$$

Here h, h' denote the z-component of spin and N,N' all other "internal" quantum numbers.

4. The angular condition

The angular condition is simply the statement that the currents $D(M,\underset{\sim}{k})$, e.g. $V_{a\mu}(\underset{\sim}{k})$, carry $\Delta\mathcal{J} \leq 1$. One should like to translate this into selection rules on ΔJ, where J is the meson spin. To be exact, J is the "boosted angular momentum" defined by

$$J_i \left|Nhp\right\rangle = \Lambda_p^{-1} \mathcal{J}_i \Lambda_p \left|Nhp\right\rangle$$

where Λ_p is a boost taking $p \to$ rest and $\mathcal{J}$ is the ordinary angular momentum. In other words, J acts only on the spin index h and not on the momentum; thus it plays the role of angular momentum in the reduced space. The components of J and $\mathcal{J}$ along p are of course the same.

Since the initial and final states will in general have different momenta, $\Delta\mathcal{J}$ and ΔJ differ by an orbital part. To get rid of this, we transform the matrix element $\left\langle N'h'p'\right|D(M,k)\left|Nhp\right\rangle$ to the Breit frame, where all momenta are in the x-direction. Then $\mathcal{J}_x = J_x$ and one gets $|\Delta J_x| \leq 1$ in the Breit frame. This, in turn, can be translated into a statement in the infinite momentum frame.

Consider for example the vector current (the SU(3) index will be dropped in this section) and specialize to $\underset{\sim}{k} = k\hat{e}_x$, then the RHS of (5) is

$$\left\langle N'h'p'\right|V_\mu(k)\left|Nhp\right\rangle \qquad \underset{\sim}{p} = (-\tfrac{k_\perp}{2}, p_z) \qquad \underset{\sim}{p}' = (\tfrac{k_\perp}{2}, p_z)$$

where we have arbitrarily chosen $(\underset{\sim}{p}+\underset{\sim}{p}')_{\perp} = 0$. We wish now to take this to the Breit frame.

There is a unique (up to an inconsequential final rotation about $\hat{e}_x$) Lorentz transformation Λ (depending on p_z but approaching a definite limit as $p_z \to \infty$) and a unique q such that

$$\Lambda : \underset{\sim}{p} \to \underset{\sim}{q} = (q, 0, 0)$$

$$\Lambda : \underset{\sim}{p}' \to -\underset{\sim}{q} = (-q, 0, 0)$$

Hence the states $|Nh\underset{\sim}{p}\rangle$ can be expressed in terms of the Breit frame states $|Nh\underset{\sim}{q}\rangle$:

$$|Nh\underset{\sim}{p}\rangle = e^{iJ_y\omega}\Lambda^{-1}|Nh\underset{\sim}{q}\rangle \qquad |N'h'\underset{\sim}{p}'\rangle = e^{iJ_y\omega'}\Lambda^{-1}|Nh-\underset{\sim}{q}\rangle$$

where the rotation $e^{iJ_y\omega}$ accounts for a transformation of the spin index. Since Λ involves boosts in the x-z plane, J_y is invariant, and thus the most general transformation is $e^{iJ_y\omega}$. The angles ω, ω' are straightforward to calculate:

$$\omega = \tan^{-1}\frac{k}{M'+M} + \tan^{-1}\frac{M'-M}{k}$$

$$\omega' = -\tan^{-1}\frac{k}{M'+M} + \tan^{-1}\frac{M'-M}{k} \qquad (6)$$

Therefore the transformation to the Breit frame reads

$$\langle N'h'\underset{\sim}{p}'|e^{iJ_y\omega'}D(M,\underset{\sim}{k})\,e^{-iJ_y\omega}|Nh\underset{\sim}{p}\rangle$$

$$= \langle N'h'-\underset{\sim}{q} | \Lambda D(M,k) \Lambda^{-1} | Nh\underset{\sim}{q} \rangle \tag{7}$$

For $D(M,\underset{\sim}{k}) = V_\mu(\underset{\sim}{k})$ say, $\theta = \Lambda D \Lambda^{-1}$ on the RHS of (7) is just some other fourier component of a combination of V_o, V_x, V_y, V_z and carries $\Delta \mathcal{J} \leq 1$, in particular $|\Delta \mathcal{J}_x| \leq 1$. This holds for the other currents as well. But $J_x = \mathcal{J}_x$ for both the initial and final states (which is of course the reason for going to the Breit frame in the first place), so $\Lambda D \Lambda^{-1}$, and hence $e^{iJ_y\omega'} D(M,k) e^{-iJ_y\omega}$ carries $|\Delta J_x| \leq 1$.

We shall adopt the symbol $\approx$ to mean "equal up to terms with $|\Delta J_x| \leq 1$". Then the angular condition is simply

$$e^{iJ_y\omega'} D(M,\underset{\sim}{k}) e^{-iJ_y\omega} \approx 0 \tag{8}$$

We shall rewrite (8) in the form

$$e^{iQ} D(M,\underset{\sim}{k}) \approx 0 \tag{9}$$

where Q is an operation on an operator

$$Q = (\tan^{-1} \frac{\mathcal{A}_M}{k}) \mathcal{A}_{J_y} - (\tan \frac{k}{\mathcal{B}_M}) \mathcal{B}_{J_y} \tag{10}$$

and $\mathcal{A}_x Y \equiv [X , Y]$; $\mathcal{B}_x Y \equiv \{X , Y\}$.

If X is the LHS of (8), the angular condition can also be written as

$$\Delta J_x = 0, 1$$

or $$(\Delta J_x)^3 = \Delta J_x$$

for matrix elements of X, i.e.

$$[J_x,[J_x,[J_x, X]]] = [J_x, X] \tag{11}$$

This leads, after a few manipulations, to the form of the angular condition used by Dashen et al[18].

It must be emphasized that (9) or (11) expresses nothing more than the $\Delta \mathcal{J} \leq 1$ constraint on the currents. In any manifestly covariant theory (e.g. field theory, infinite component wave equations, 4-dimensional oscillators) there is no need to verify the angular condition. The complexity of the angular condition is the price for using "non-relativistic" models in one particular Lorentz frame (the $p_z \to \infty$ frame).

5. <u>U(6) of currents versus U(6) of spectrum</u>

Recall that the "good" operators at $p_z \to \infty$ are

$$V_{ao}(\underset{\sim}{k}) = V_{az}(\underset{\sim}{k}) = D(\tfrac{1}{2}\lambda_a, \underset{\sim}{k})$$

$$A_{ao}(\underset{\sim}{k}) = A_{az}(\underset{\sim}{k}) = D(\tfrac{1}{2}\lambda_a \sigma_z, \underset{\sim}{k})$$

$$S_{ay}(\underset{\sim}{k}) \equiv T_{aox}(\underset{\sim}{k}) = T_{azx}(\underset{\sim}{k}) = D(\tfrac{1}{2}\lambda_a \sigma_y, \underset{\sim}{k})$$

$$S_{ax}(\underset{\sim}{k}) \equiv -T_{aoy}(\underset{\sim}{k}) = -T_{azy}(\underset{\sim}{k}) = D(\tfrac{1}{2}\lambda_a \sigma_x, \underset{\sim}{k})$$

(Note that $\alpha_z = 1$ in the $p_z \to \infty$ limit, so that $i\gamma_5 = \sigma_z\alpha_z \to \sigma_z$ and so on.) These satisfy a local $U(6)_W$ algebra. It is crucial to note that this is not the same $U(6)_W$ that approximately describes the particle spectrum. For example the axial charge $D(\frac{1}{2}\lambda_a\sigma_z, 0)$ leaks outside any given $U(6)_W$ representation of the spectrum (e.g. $\Gamma_{f\pi\pi} \neq 0 \Rightarrow \langle f|A_{ao}|\pi\rangle \neq 0$). It is precisely the relation

$$U(6)_{W,\text{currents}} \overset{?}{\longleftrightarrow} U(6)_{W,\text{spectrum}}$$

that we study here. This is a more exact statement of the idea

$$\text{"current quarks"} \overset{?}{\longleftrightarrow} \text{"spectrum quarks"}$$

We shall refer to any of these $D(M,\underset{\sim}{k})$ as a "charge" if $\underset{\sim}{k} = 0$ and a "density" otherwise. Commutators are then either charge-charge (twice integrated), density-charge (once integrated) or density-density (unintegrated). Only the last is sensitive to gradient terms.

III. Quark Models

In this chapter we specify a class of quark models that will be studied and write down the general form of the current operators in these models. Previous difficulties are stated in section 6, and section 7 introduces the less restrictive density-charge problem. Only $B = 0$ states (mesons) will be considered.

1. Quark model list of states

To represent the currents we must define a space of states $|Nh\rangle$, where N denotes all "internal" quantum numbers and h the eigenvalue of J_z. The space of all hadron states in the real world (modulo the centre-of-mass momentum) will be such a space. The aim here is to see if a simpler and more manageable space will reflect some of the properties of the real world. Thus the problem is similar to that of finding representations of SU(3); for example — the real world is a representation, but there are simpler (irreducible) ones: $\underset{\sim}{1}$, $\underset{\sim}{8}$, $\underset{\sim}{10}$ etc. The major differences are the relativistic complications embodied in the angular condition and the fact that we now have an (infinite) local algebra, so that any nontrivial solution will require an infinite-dimensional representation.

The use of quark model states is motivated by two experimental features:

(1) It appears that the imaginary part of crossing-odd hadron-hadron scattering amplitudes can be approximated by direct-channel resonance contributions[23]. It is therefore plausible that the same may be true of current-hadron scattering. The amplitude in this case is just the matrix element of a commutator. There is also some direct evidence of resonance saturation from the Adler-Weisberger and the Cabbibo-Radicatti sum rules[20,21].

(2) It also appears that resonances (at least the well-defined low-lying ones) can be described by the quark model. By this we mean principally two limitations on the list of states. First, mesons, being made of $q\bar{q}$ should lie in $\underset{\sim}{3} \times \underset{\sim}{3} = \underset{\sim}{1} + \underset{\sim}{8}$ representations of SU(3). Secondly, the parity and charge conjugation should satisfy $\mathcal{P} = (-1)^{L+1}$, $\mathcal{C} = (-1)^{L+S}$, where $\underset{\sim}{S} = \frac{1}{2}(\underset{\sim}{\sigma} + \underset{\sim}{\sigma}')$ is the quark spin and $\underset{\sim}{L} + \underset{\sim}{S} = \underset{\sim}{J}$. In particular this forbids $NP \equiv (-1)^J \mathcal{P} = +$, $NC \equiv (-1)^J \mathcal{C} = -$, ($J^{PC} = 0^{+-}, 1^{-+}, \ldots$) states. States violating these conditions will be called exotic (of the 1st and 2nd kind respectively).

2. The Simple Harmonic Oscillator (SHO)

The SHO, which pictures the $q\bar{q}$ as being held by a "spring", incorporates the above features, as well as the linear nature of Regge trajectories. A meson is described by the following degrees of freedom:

quark labels	$\lambda_a, \lambda'_a, \underset{\sim}{\sigma}, \underset{\sim}{\sigma}'$
"internal" variables	$\underset{\sim}{x}$ (separation between $q\bar{q}$)
	$\underset{\sim}{p}$ (conjugate to $\underset{\sim}{x}$)

We shall work exclusively in the SU(3) symmetric limit, and take the meson mass to be

$$M^2 = 4[m^2 + (p^2 + cx^2) + m^{-2}(\ldots) + \ldots]$$

$$= M_o^2[1 + \epsilon(p^2 + cx^2) + \ldots \qquad] \qquad (1)$$

where m may be thought of as a quark mass, and ... may involve $\underset{\sim}{x}, \underset{\sim}{p}, \underset{\sim}{\sigma}, \underset{\sim}{\sigma}'$ in any manner. The potential cx^2 may also be replaced by any $U(x^2)$. (Eqn. (1) has been written in two equivalent forms, convenient for expansion in 1/m and around the degenerate limit respectively). The free quark model, which has been solved exactly[13], is a special case of this, with c = 0:

$$M^2 = 4[m^2 + p^2]$$

3. A more general class of quark models

In what follows, a generalization of the SHO will be considered. The spectrum is described by

quark labels	λ_a , λ'_a , $\underset{\sim}{g}$, $\underset{\sim}{g}'$
"internal" variables	"arbitrary" (more will be said as we proceed)

The meson mass is taken to be expansible either in 1/m or around a degenerate limit respectively

$$M^2 = 4[m^2 + K + m^{-2}K' + \dots]$$

$$\text{or} \quad M^2 = M_o^2[1 + \epsilon R + \epsilon^2 R' + \dots] \qquad (2)$$

where K, K'; R, R' are functions of $\underset{\sim}{g}$, $\underset{\sim}{g}'$ and the internal variables. The internal variables must respect $\mathcal{P} = (-1)^{L+1}$, $\mathcal{C} = (-1)^{L+S}$; for example this means that it is impossible to construct a pseudoscalar operator without spin. By allowing for a more general mass spectrum, we no longer assume straight Regge trajectories as in the SHO case.

4. Charge conjugation of the currents

On any quark model state,

$$\mathcal{C} = (\text{reverse charges}) \cdot (-1)^{L+S}$$
$$= (\quad \text{"} \quad) \cdot (-1)^{L+1} (-1)^{S+1}$$
$$= (\quad \text{"} \quad) \cdot \mathcal{P} \cdot (\sigma \leftrightarrow \sigma')$$

Therefore the effect of $\mathcal{P}$ on the current operators D(M,k) are

$$\mathcal{C} = (\lambda_a \leftrightarrow -\lambda'_a) \cdot \mathcal{P} \cdot (\sigma \leftrightarrow \sigma') \qquad (3)$$

which must be respectively $-$, $+$, $-$ on V, A, T. Thus the quark model, in excluding exoticity of the 2nd kind, imposes a severe constraint on the form of the current. <u>The difficulties we encounter disappear if this condition is dropped</u>.

5. <u>General form of currents</u>

We assume that the currents can be represented on this space as*

$$V_a(\underset{\sim}{k}) = D(\tfrac{1}{2}\lambda_a, \underset{\sim}{k}) = \tfrac{1}{2}\lambda_a \; G(\underset{\sim}{k}) + \tfrac{1}{2}\lambda'_a \; G'(\underset{\sim}{k})$$

* We shall not write the time ($\mu = 0$) index on V and A from now on; recall that $S_{ai}(\underset{\sim}{k}) = \frac{1}{2}\varepsilon_{ijk}T_{ajk}(\underset{\sim}{k})$.

$$A_a(\underset{\sim}{k}) = D(\tfrac{1}{2}\lambda_a\sigma_z, \underset{\sim}{k}) = \tfrac{1}{2}\lambda_a \Sigma G(\underset{\sim}{k}) - \tfrac{1}{2}\lambda'_a\Sigma' G'(\underset{\sim}{k})$$

$$S_{ai}(\underset{\sim}{k}) = D(\tfrac{1}{2}\lambda_a\sigma_i, \underset{\sim}{k}) = \tfrac{1}{2}\lambda_a S_i G(\underset{\sim}{k}) + \tfrac{1}{2}\lambda'_a S'_i G'(\underset{\sim}{k}) \qquad (4)$$

$$i = 1, 2$$

where $G(0) = G'(0) = 1$. The same G and G' occur everywhere on account of the density-charge commutation relations. The conditions on $\mathcal{C}$ in the last section means that each of the primed operators (for $\bar{q}$): Σ', S', G' can be obtained from the unprimed ones (for q) by $\mathcal{P}$ and $\sigma \longleftrightarrow \sigma'$.

To guarantee charge-charge commutation relations ($\underset{\sim}{k} = \underset{\sim}{k}' = 0$) one needs

(S_x, S_y, Σ), (S'_x, S'_y, Σ') commute and anticommute as $\underset{\sim}{\sigma}$, $\underset{\sim}{\sigma}'$ (5)

For example, $\Sigma^2 = \Sigma'^2 = 1$ will give the Adler-Weisberger sum rule. To obtain the density-charge commutation relations ($\underset{\sim}{k} \neq 0$, $\underset{\sim}{k}' = 0$) one must have

$$[X, G(\underset{\sim}{k})] = [X, G'(\underset{\sim}{k})] = 0 \qquad (6)$$

where $X = S_i$, Σ, S'_i or Σ'.

To get the density-density commutation relations ($\underset{\sim}{k}$, $\underset{\sim}{k}' \neq 0$), assumed to be free of gradient terms, one needs

$$[G(\underset{\sim}{k}), G(\underset{\sim}{k}')] = [G(\underset{\sim}{k}), G'(\underset{\sim}{k}')] = 0$$

$$G(\underset{\sim}{k})\, G(\underset{\sim}{k}') = G(\underset{\sim}{k} + \underset{\sim}{k}')$$

$$G'(\underset{\sim}{k})G'(\underset{\sim}{k}') = G'(\underset{\sim}{k} + \underset{\sim}{k}') \qquad (7)$$

Since λ_a, λ_a' are assumed to commute with M and J, they factor out of the angular condition (2.9), which then reads

$$e^{iQ} X \approx 0 \qquad (8)$$

where $X = G(\underset{\sim}{k})$, $\Sigma G(\underset{\sim}{k})$, $S_i G(\underset{\sim}{k})$ or their primed counterparts.

A particularly simple subset of constraints can be extracted from (8). Define

$$G(\underset{\sim}{k}) = 1 + ik_i h_i - \tfrac{1}{2}k_i k_j h_{ij} - \tfrac{i}{6}k_l k_i k_j h_{lij} + \ldots$$

then

$$[2MJ_+, h_+] = \tfrac{i}{2}[M^2, h_{++}] \qquad (h_+ = h_x + ih_y \text{ etc,})$$

$$[2MJ_+, h_{++}] = \tfrac{2i}{3}[M^2, h_{+++}] \text{ etc.}$$

$$[2MJ_+, [2MJ_+, \Sigma]] - 2i[2MJ_+, \Sigma h_+]] - [M^2, [M^2, \Sigma h_{++}]] = 0$$

$$[2MJ_+, S_+] = i[M^2, S_+ h_+] \qquad (9)$$

Eqns. (5) – (8) are then necessary conditions for representing the entire current algebra on quark model states.

6. Previous work and difficulties

The system (5) – (8) has been studied extensively. The only known solution, however, is the free quark model. More realistic spectra face two difficulties:

(1) Chang, Dashen and O'Raifeartaigh[11] have argued that (5) – (8) together imply either a free quark spectrum or the existence of negative $(\text{mass})^2$ states (tachyons) coupled to physical states by the current. Since tachyons may not appear in formal power series solutions[24], this problem is difficult to study except in exactly soluble systems.

(2) There is a further difficulty when both q and $\bar{q}$ carry charge. For the SHO, Hill[13] and Weyers[12] found that in an $1/m$ expansion, the condition (7):

$$[G(\underline{k}),G'(\underline{k}')] = 0$$

fails in $O(m^{-3})$.

These difficulties are usually attributed to the presence, presumably at high mass, of exotics in nature. In other words, the currents leak outside the space of nonexotics used. This is expected to be more serious for the higher moments in k, i.e. if

$$V_a(\underset{\sim}{k}) \;=\; A_a + B_{ai}k_i + G_{aij}k_ik_j + \ldots$$

then A_a is just the charge, while B , C , ... connect to increasingly higher masses and are therefore increasingly likely to connect to exotics. One is therefore led to consider the following approximate problems: To allow for leakage in the higher moments, let us require (7) to be satisfied not exactly, but only up to 2nd order in $\underset{\sim}{k}$, $\underset{\sim}{k}'$. For the SHO, however, the same trouble was found to be present: $[G(\underset{\sim}{k}),G'(\underset{\sim}{k}')] \neq 0$ (the nonzero term on the right being $O(kk')$[25]. Details will be found in the appendix.

7. Density-charge algebra

In the rest of this work, a further relazation of the conditions will be studied. We shall only require that the charges: $V_a(0)$, $A_a(0)$, $S_{ai}(0)$ stay within the nonexotics: the densities $V_a(\underset{\sim}{k})$, $A_a(\underset{\sim}{k})$, $S_{ai}(\underset{\sim}{k})$, $\underset{\sim}{k} \neq 0$ may have nonexotic (NEx) to exotic (Ex) matrix elements.

The charge-charge commutators are of course satisfied in the space of nonexotics. Moreover, the density-charge commutators will be valid within nonexotics as well. In obvious notation

$$\langle NEx'|[D, C]|NEx\rangle = \sum_{k} \langle NEx'|D|k\rangle\langle k|C|NEx\rangle - \ldots$$

where the factor $\langle k|C|NEx\rangle$ forces k to be NEx; so the sum can be restricted to the NEx subspace.

The density-density commutators will of course not be valid among nonexotics. Thus we abandon eqn. (7) completely. It should be noted that the restriction of this problem to vector currents alone is now trivial — $G(\underset{\sim}{k})$, $G'(\underset{\sim}{k})$ need only satisfy the angular condition in that case, without any current algebra constraints. Thus the axial and tensor currents will be essential.

Since the problem will be studied in power series expansions, we shall not be able to confront directly the tachyon difficulty. However, we do evade the arguments for their existence given by Chang et al[11], which rely heavily on the density-density commutators. The difficulty found by Hill and Weyers is now irrelevant. Another nice feature is that we are now insensitive to gradient terms in the density-density or unintegrated commutators, which have been the source of some controversy.

The rest of this thesis will be an investigation of this density-charge problem, i.e. eqns. (5), (6), (8). The complexity of the angular condition requires, in practice, an expansion procedure. We study, in Chapter 4, expansions around a degenerate limit and in Chapter 5, expansions in powers of inverse quark mass.

IV. Degenerate Limit

1. We first consider the case where all meson masses are equal and show, under plausible conditions, that no solutions exist. This fact is well known for the density-density problem; we merely extend the result to the density-charge problem.[26,27]

For equal masses, the angular condition (3.9) on Σ and S becomes

$$[J_+ , [J_+ , \Sigma]] = 0 \qquad [J_+ , S_+] = 0 \qquad (1)$$

so that

$$\Sigma = P + A'_z \qquad S_{\substack{x\\y}} = A_{\substack{x\\y}} \pm V_{\substack{y\\x}} \qquad (2)$$

where P, A', A, V are respectively pseudoscalar, axial vector, axial vector and vector. Moreover, Σ, S_x, S_y must behave as a spin under commutation and anticommutation. We shall assume that they are in fact the components of spin.

$$(S_x , S_y , \Sigma) = (\sigma_x , \sigma_y , \sigma_z)$$

$$(S'_x , S'_y , \Sigma') = (\sigma'_x , \sigma'_y , \sigma'_z) \qquad (3)$$

i.e. $P = 0$, $V = 0$, $\underset{\sim}{A}' = \underset{\sim}{A} = \underset{\sim}{\sigma}$. We have not shown that this must be the case, but believe it to be true of all reasonable models. Aside from our inability to construct an alternative to (3), we may make the following remarks to make the uniqueness of (3) plausible:

(1) $S_x^2 = (A_x + V_y)^2 = (A_x^2 + V_y^2) + \{A_x, V_y\} = 1$

means (a) $V_i V_j$ must have no $\Delta J = 2$ piece, (b) V_y has bounded eigenvalues, and (c) $\{A_x, V_y\} = 0$. None of these is easy to arrange without setting $V = 0$. $\Sigma^2 = 1$ also suggests $P = 0$.

(2) If $(S_x, S_y, \Sigma) = e^{i\theta U} \underset{\sim}{\sigma} e^{-i\theta U}$, where U is a hermitian operator, are to have the prescribed angular momentum properties for a range of values of θ, it can be shown that U must be a scalar. So $P = 0$, $V = 0$ and $A' = A$ and we may as well call A the quark spin.

(3) Σ, S close under multiplication, so barring anomolies they must have $\Delta L = 0$ or all ΔL, and the latter case is impossible since $\Delta J \leq 1$. (If A, B have ΔL up to a, b respectively, then "normally" AB and BA have ΔL up to $a + b$). Hence $(S_x, S_y, \Sigma) = \underset{\sim}{\sigma}$ up to a scalar whose square is 1.

(4) $\Sigma = \sigma_z$ yields $-g_A/g_V = 5/3$ in the baryon ground state.

With these considerations in mind, we shall ignore any solution other than eqn. (3).

Since $G(\underset{\sim}{k})$ commutes with Σ, S, Σ', S' (eqn. (3.6)), it cannot involve spin. We can show now that $\Sigma G(\underset{\sim}{k})$ does not satisfy the angular condition (3.8): $e^{iQ}\,\Sigma G(k) \approx 0$.

In the equal mass case, Q becomes

$$Q = (\tan^{-1}\frac{\mathcal{A}_M}{k})\,\mathcal{A}_{J_y} - (\tan^{-1}\frac{k}{\mathcal{B}_M})\,\mathcal{B}_{J_y}$$

$$\longrightarrow -(\tan^{-1}\frac{k}{2M_o})\,\mathcal{B}_J$$

$$e^{iQ}\,X \longrightarrow e^{-i\theta J_y}\,X\,e^{-i\theta J_y} \qquad \theta = \tan^{-1}\frac{k}{2M_o}$$

$$e^{iQ}\,\Sigma G(k) = e^{iQ}\,\sigma_z G(k)$$

$$= e^{-i\theta J_y}\,\sigma_z G(k)\,e^{-i\theta J_y}$$

$$= A \cdot B \cdot C$$

$$A = e^{-\frac{1}{2}i\theta\sigma_y}\,\sigma_z\,e^{-\frac{1}{2}i\theta\sigma_y}$$

$$B = e^{-i\theta\sigma_y'}$$

$$C = e^{-i\theta L_y}G(k)\,e^{-i\theta L_y}$$

The three factors have ΔS and ΔL as follows:

		A	B	C
$\Delta\frac{1}{2}\sigma_x$	=	0 , 1 ;	0 ;	0
$\Delta\frac{1}{2}\sigma'_x$	=	0 ;	0 , 1 ;	0
ΔL_x	=	0 ;	0 ;	?

It is obvious that the product contains $|\Delta J_x| > 1$, violating the angular condition (3.8). This shows that solutions do not exist in the degenerate limit.

Note, however, that if only one quark is charged (e.g. $S = \pm 1$ mesons and isotopic subgroup), there is no need to impose $[\Sigma', G(\underset{\sim}{k})] = [S', G(\underset{\sim}{k})] = 0$, so that $G(\underset{\sim}{k})$ may involve σ' and there is no difficulty.

2. The trouble in the degenerate limit extends immediately to all spectra expansible around that limit. Consider a general meson mass

$$M^2 = M_o^2 (1 + \epsilon R + \epsilon^2 R' + \ldots) \qquad (5)$$

where M_o is a constant and R, R' are arbitrary operators. Assume that the various operators in the problem have a formal power series expression in ϵ:

$$\Sigma = \Sigma_o + \epsilon\Sigma_1 + \ldots$$

$$S = S_o + \epsilon S_1 + \ldots$$

$$G(\underset{\sim}{k}) = G(\underset{\sim}{k})_o + \epsilon G(\underset{\sim}{k})_1 + \ldots$$

and the problem can be studied order by order in ϵ. Then in $O(\epsilon^0)$ we recover the degenerate problem just discussed, which has no solution.

3. If all spectra were expansible around a degenerate limit, this would be the end of the story. Besides the possibility of exotics, realistic spectra may evade the result just obtained by having a solution only for one value of ϵ. This would be an exciting situation — that current algebra determines the parameters of the spectrum. But with the mathematical techniques used here, nothing much can be said about this. A second possibility is that various ϵ's are allowed, but the solution is singular as $\epsilon \to 0$. We now give two examples of this latter kind: the free quark model and the SHO restricted to vector currents, both of which can be solved by other means (exactly for the free quark model, and in 1/m expansion for both.)

We may write the meson mass in the free quark model as:

$$M^2 = 4(m^2 + \epsilon^2 p^2) = 4(m^2 + \hat{p}^2)$$

where $\hat{p} = \epsilon p$, $\hat{x} = \epsilon^{-1} x$. The solution is $G(\underset{\sim}{k}) = e^{i\underline{k}\cdot\underline{h}}$ where

$$h_x = \tfrac{1}{2}\hat{x} + \frac{1}{m}(\ldots) + \ldots$$

$$= \tfrac{1}{2}\epsilon x + \frac{1}{m}(\ldots) + \ldots$$

so $G(\underset{\sim}{k})$ has formally an essential singularity at $\epsilon = 0$, (when expressed in terms of $\underset{\sim}{x}$ and $\underset{\sim}{p}$). Likewise, the SHO mass operator is

$$M^2 = 4[m^2 + \epsilon^2(p^2 + cx^2)] = 4[m^2 + \hat{p}^2 + \epsilon^4 c\hat{x}^2]$$

The solution is again

$$h_x = \tfrac{1}{2}\hat{x} + \frac{1}{m}(\ldots) + \ldots$$

$$= \tfrac{1}{2}\epsilon x + \frac{1}{m}(\ldots) + \ldots$$

(The ... are different from the free quark case due to the "potential" term, but the first term is the same.) The same essential singularity occurs in G(k).

In both of these cases the $\epsilon \neq 0$ solution is not smoothly connected to the $\epsilon = 0$ case, so the degenerate situation is irreleant. It turns out that for these two examples, an expansion in powers of 1/m is possible. This means that as the meson masses are brought together, the common mass must be sent to infinity in order to obtain a smooth limit. Whether or not this is true of the real world is of course a separate question.

In the next two chapters we shall study spectra expansible in powers of $1/m$.

V. Expansion in Inverse Quark Mass

1. In this chapter we shall take the meson mass to be of the form

$$M^2 = 4(m^2 + K + m^{-2}K' + \dots) \qquad (1)$$

For simplicity, we shall assume that K is independent of spin. The proof for the general case is more involved and will not be exhibited here. All operators are assumed to have expansion in in powers of 1/m:

$$\Sigma = \Sigma_o + \frac{1}{m}\Sigma_1 + \dots$$

$$S = S_o + \frac{1}{m}S_1 + \dots$$

$$G(\underset{\sim}{k}) = 1 + ikh_x - \frac{k^2}{2}h_{xx} + \dots \qquad \text{for } \underset{\sim}{k} = k\,\hat{e}_x$$

$$h_x = h_{xo} + \frac{1}{m}h_{x1} + \dots$$

$$h_{xx} = h_{xxo} + \frac{1}{m}h_{xx1} + \dots \qquad \text{etc.} \qquad (2)$$

and we investigate the problem order by order in 1/m. It is convenient to expand the angular condition in powers of 1/m. The expansion for e^{iQ} in (3.8) is given by Hill:

$$e^{iQ} = 1 + \frac{i}{m}Q_1 - \frac{1}{m^2}Q_2 - \frac{i}{m^3}Q_3 + \dots$$

$$Q_1 = \frac{i}{k} \mathcal{A}_K \mathcal{A}_{J_y} - \frac{k}{4} \mathcal{B}_{J_y}$$

$$Q_2 = \frac{1}{2k^2} \mathcal{A}_K^2 \mathcal{A}_{J_y}^2 - \frac{1}{4} \mathcal{A}_K \mathcal{A}_{J_y} \mathcal{B}_{J_y} + \frac{k^2}{32} \mathcal{B}_{J_y}^2$$

$$Q_3 = \frac{1}{6k^3} \mathcal{A}_K^3 \mathcal{A}_{J_y} (\mathcal{A}_{J_y}^2 + 2) -$$

$$\frac{1}{8k} (\mathcal{A}_K^2 \mathcal{A}_{J_y}^2 \mathcal{B}_{J_y} - 2 \mathcal{A}_{K^2} \mathcal{A}_{J_y} + 8 \mathcal{A}_{K'} \mathcal{A}_{J_{J_y}}) + O(k) \qquad (3)$$

(Recall that $\mathcal{A}_X Y = [X , Y]$, $\mathcal{B}_X Y = \{X , Y\}$). One feature worth pointing out is $e^{iQ} = 1$ in $O(1/m^o)$ whereas $e^{iQ} \neq 1$ in the degenerate limit. To see this,

$$Q = (\tan^{-1} \frac{\mathcal{A}_M}{k}) \mathcal{A}_{J_y} - (\tan^{-1} \frac{k}{\mathcal{B}_M}) \mathcal{B}_{J_y}$$

$$\mathcal{A}_M = \Delta M \approx \frac{\Delta M^2}{2M} = 2\frac{\Delta K}{M} = O(\tfrac{1}{m})$$

$$\mathcal{B}_M \approx 2M \qquad\qquad \frac{1}{\mathcal{B}_M} = O(\tfrac{1}{m})$$

$$e^{iQ} = 1 + O(\tfrac{1}{m})$$

Thus the "infinite quark mass limit" is simpler, and as noted in the last chapter, possibly smoother, than the degenerate limit.

The angular condition (3.9) (which are contained in

(3.8) but written separately because they are especially simple) can also be expanded in powers of $1/m$. We write down only the first few terms of the expansion:

$$[J_+ , h_{+o}] = 0$$

$$[J_+ , h_{+1}] = \frac{i}{2}[K , h_{++o}]$$

$$[J_+ , h_{+2}] = \frac{i}{2}[K , h_{++1}] - \frac{1}{2}[KJ_+ , h_{+o}]$$

$$[J_+ , h_{+3}] = \frac{i}{2}[K , h_{++2}] - \frac{1}{2}[KJ_+ , h_{+1}]$$

$$+ \frac{i}{2}[K' , h_{++o}]$$

$$[J_+ , h_{++o}] = 0$$

$$[J_+ , h_{++1}] = \frac{2i}{3}[K , h_{++o}]$$

$$[J_+ , h_{+++o}] = 0 \qquad \text{etc.}$$

$$[J_+ , [J_+ , \Sigma_o]] = 0$$

$$[J_+ , [J_+ , \Sigma_1]] - 2i [J_+ , (\Sigma h_+)_o]] = 0$$

$$[J_+ , [J_+ , \Sigma_2]] - 2i [J_+ , (\Sigma h_+)_1]] + [KJ_+ , \Sigma_o]]$$

$$- [K , [K , (\Sigma h_{++})_o]] = 0$$

$$[J_+ , S_{+o}] = 0$$

$$[J_+ , S_{+1}] = i[K , S_{+o}h_{+o}]$$

$$[J_+ , S_{+2}] = i[K , (S_+h_+)_1] - \tfrac{1}{2}[KJ_+ , S_{+o}] \quad \text{etc.} \qquad (4)$$

These determine the part of X_n with the largest ΔJ in terms of X_{n-1} , Y_{n-1} , ... , X_{n-2} , Y_{n-2} , ... where X , Y are any of Σ , S , h and n is an index of the power in 1/m. Thus in each order in 1/m, the angular condition leaves us with some freedom in the lower ΔJ with which to satisfy the current algebra constraints.

The gist of the calculation below is that by systematically following (3.5), (3.6) and (3.8), a contradiction is found in $0(m^{-3})$. Since there is no trouble in the SHO up to $0(m^{-2})$, even for the full density-density problem, it is clear that nothing much can be learnt without going to $0(m^{-3})$. Consequently the calculation is somewhat tedious.

2. Order m^o

From eqn. (4) we get

$$[J_+ , [J_+ , \Sigma_o]] = 0 \qquad\qquad [J_+ , S_{+o}] = 0$$

which is of course the same as in the degenerate case. We again assume, with the same plausibility arguments as in Chapter 3, that

$$(S_x, S_y, \Sigma) = (\sigma_x, \sigma_y, \sigma_z)$$

$$(S'_x, S'_y, \Sigma') = (\sigma'_x, \sigma'_y, \sigma'_z)$$

so that to maintain $[\Sigma, G(k)] = 0$ etc., $G(k)_o$ must not involve spin. The angular condition $e^{iQ}\,\Sigma G(k) \approx 0$ for $O(m^o)$ is

$$\Sigma_o G(k) \approx 0$$

and since $\Sigma_o = \sigma_z$ has $\Delta J_x = 1$, $G(k)_o = 1 + ikh_{xo} - \frac{k^2}{2}h_{xxo} + \ldots$ must have $\Delta J_x = 0$, hence

$$h_{xo} = \frac{1}{2}t_x \qquad h_{xxo} = \frac{1}{4}t_{xx} \qquad h_{xxxo} = \frac{1}{8}t_{xxx}, \ldots \tag{6}$$

where the t's are arbitrary symmetrical tensors in 3 dimensions. (Recall that the h's are tensors in 2 dimensions, i.e. the indices on the h's only specify transformation properties under J_z.) The $\bar{q}$ operators h'_{xo}, h'_{xxo} etc. are obtained by $\mathcal{P}$. Factors of 2^{-n} are put in for later convenience.

Note that if density-density commutation relations were to be imposed, $G(\underset{\sim}{k})$ would take an exponential form $e^{i\underset{\sim}{k}\cdot\underset{\sim}{h}}$, so that $h_{xx} = h_x^2$, $h_{xxx} = h_x^3$ etc. This is <u>not</u> assumed here; however, as shall be shown later, the system has an inherent inclination towards the exponential form, for the following reason:

To get say $[\Sigma, G(k)] = 0$, one must have

$$[\Sigma, h_x] = [\Sigma, h_{xx}] = [\Sigma, h_{xxx}] = \ldots = 0$$

This infinity of conditions collapse into just the first one if one has $h_{xx} = h_x^2$, $h_{xxx} = h_x^3$ etc. The angular conditions likewise collapse into a finite number of constraints in the exponential case. In other words, deviations from the exponential form are severely constrained. For example, in the next section it will be shown that t_{ij} is "almost" $t_i t_j$.

3. Order m^{-1}

The calculation in this order will be presented in some detail since it exhibits the major steps followed in higher orders and is not overly messy.

(i) Calculate Σ_1

From the angular condition (4)

$$[J_+, [J_+, \Sigma_1]] - 2i[J_+, [K, (\Sigma h_+)_o]] = 0$$

so one finds

$$\Sigma_1 = \dot{t}_\perp \cdot \sigma_\perp + (\text{pseudoscalar}) + (\text{axial vector})_z$$

where for any operator X, $\dot{X}$ is defined by

$$[K, X] = -2i\dot{X} \tag{7}$$

The (pseudoscalar) + (axial vector)$_z$ is the most general object (subject to $e^{i\pi J_y} = +$, $\Delta J_z = 0$) annihilated by two J_+'s.

$$1 = \Sigma^2 = (\Sigma_o + \frac{1}{m}\Sigma_1 + \ldots)^2$$
$$= \Sigma_o^2 + \frac{1}{m}\{\Sigma_o, \Sigma_1\} + \ldots$$

so that Σ_1 must not contain any σ_z or any term independent of $\underset{\sim}{\sigma}$. The most general solution is then

$$\Sigma_1 = \dot{t}_\perp \cdot \sigma_\perp + (\underset{\sim}{u} \times \underset{\sim}{\sigma})_z \tag{8}$$

where $\underset{\sim}{u}$ is an arbitrary axial vector*.

(ii) Calculate S_{x1}, S_{y1}

From the angular condition (4)

$$[J_+, S_{+1}] = i[K, S_{+o}h_{+o}]$$

one finds

$$S_{+1} = -\dot{t}_+\sigma_z + (\Delta J = 1)$$

* In what follows, we shall assume for simplicity that u does not depend on $\underset{\sim}{\sigma}'$. The argument remains valid even if $\underset{\sim}{u}$ depends on $\underset{\sim}{\sigma}'$, but is somewhat messier to write out.

Requiring Σ , S to commute and anticommute as a spin removes the ambiguity in ($\Delta J = 1$) and one gets

$$S_{x1} = -\dot{t}_x \sigma_z + (\underset{\sim}{u} \times \underset{\sim}{\sigma})_x \qquad (9)$$

(iii) Calculate h_{x1}

Information about h_{x1} can be obtained from the angular condition (3.8) and the current algebra requirement (3.6) that $[X , \underset{\sim}{h}] = [X , \underset{\sim}{h}'] = 0$ where $X = \Sigma , \Sigma'$, S or S'. It is convenient to choose first of all the following combination from (3.6):

$$0 = [\Sigma + \Sigma' , h_x + h'_x]$$

which, to $O(m^{-1})$, gives

$$0 = [\Sigma_o + \Sigma'_o , h_{x1} + h'_{x1}] + [\Sigma_1 + \Sigma'_1 , h_{xo} + h'_{xo}]$$

$$= [\sigma^s_z , h_{x1} + h'_{x1}]$$

since $h_{xo} + h'_{xo} = 0$ and we have used the notation $\underset{\sim}{\sigma}^{s,a} = \underset{\sim}{\sigma} \pm \underset{\sim}{\sigma}'$. The last equation is also valid with $\Sigma \rightarrow S$ i.e. with $\sigma^s_z \rightarrow \underset{\sim}{\sigma}^s_\perp$. Thus $h_{x1} + h'_{x1}$ does not involve spin. And given, from charge conjugation considerations, that h_{x1} and h'_{x1} are related by $\mathcal{P}$ and $\sigma \longleftrightarrow \sigma'$, one gets

$$h_{x1} = (\text{spin independent terms}) + (\mathcal{P} = + , a) + (\mathcal{P} = - , s)$$

where s , a denote symmetric and antisymmetric under $\sigma \leftrightarrow \sigma'$. Now from the angular condition (4)

$$[J_+ , h_{+1}] = \frac{i}{2}[K , h_{++o}]$$

one finds, by evaluating the RHS

$$h_{+1} = -\tfrac{1}{2}\dot{t}_{+z} + (\Delta J = 1)$$

$$= -\tfrac{1}{2}\dot{t}_{+z} + (\Delta J = 1, \text{ spin independent}) + \bar{a}^s_x + \bar{b}^a_y \qquad (10)$$

where $\bar{a}^s$ = vector, symmetric
$\bar{b}^a$ = axial vector, antisymmetric under $\sigma \leftrightarrow \sigma'$,

More information on h_{x1} can be obtained from the angular condition on the axial current: $e^{iQ}\,\Sigma G(k) \approx 0$ which to $O(m^{-1})$ gives

$$0 \approx \Sigma_o G_1 + \Sigma_1 G_o + iQ_1 \Sigma_o G_o$$

and choosing $O(k^1)$ gives

$$\sigma_z h_{x1} \approx \sigma_z \; \tfrac{1}{2}\,(\dot{t}_z t_x - \dot{t}_{xz}) + \tfrac{1}{2}L_y + \tfrac{1}{4}\sigma'_y$$

where terms with $\Delta J_x < 1$, e.g. $(u \times \sigma)_z t_x$, have been discarded. Hence

$$h_{x1} = \tfrac{1}{2}(\dot{t}_z t_x - \dot{t}_{xz}) + \tfrac{1}{2}L_y + (\Delta J_x = 0)$$
$$+ \text{(terms depending on } \sigma) \qquad (11)$$

Comparison of (10), (11) gives

$$h_{x1} = -\tfrac{1}{2}t_z \dot{t}_x + \tfrac{1}{2}L_y - \tfrac{1}{4}(\sigma - \sigma')_y + a_x + \bar{a}^s_x \qquad (12)$$

where a = vector, independent of spin

$\bar{a}^s$ = vector, symmetric under $\sigma \longleftrightarrow \sigma'$

and in addition one finds

$$\dot{t}_{ij} = t_i \dot{t}_j + \dot{t}_i t_j \qquad \text{if } i \neq j$$

or $$t_{ij} = t_i t_j + (\ \ldots\ )\delta_{ij} +$$
$$\text{(something that commutes with K)} \qquad (13)$$

(iv) Impose density-charge algebra

These are the conditions $[X, h_x] = [X, h'_x] = 0$ where $X = \Sigma, \Sigma', S$ or S', and can be re-written as

$$[X + X', h_x \pm h'_x] = 0$$

where $X = \Sigma$ or $\underset{\sim}{S}$ (The $X - X'$ commutator will be guaranteed by the symmetry between primed and unprimed operators). The $h_x + h'_x$ commutator has already been exploited — it is particularly simple because $h_{xo} + h'_{xo} = 0$. Now consider the $h_x - h'_x$ commutator to $O(m^{-1})$ and take $X = \Sigma$

first.

$$0 = [\Sigma_o + \Sigma_o' , h_{x1} - h_{x1}'] + [\Sigma_1 + \Sigma_1' , h_{xo} - h_{xo}']$$

$$= [\sigma_z^s , -\tfrac{1}{2}\sigma_y^a + 2a_x + 2\bar{a}_x^s] + [\dot{t}_\perp \cdot \sigma_\perp^a + (\underset{\sim}{u} \times \underset{\sim}{\sigma}^s)_z , t_x] \quad (14)$$

The σ^a terms give

$$0 = -\tfrac{1}{2}[\sigma_z^s , \sigma_y^a] + [\dot{t}_\perp \cdot \sigma_\perp^a , t_x]$$

or

$$[t_i , \dot{t}_j] = i\delta_{ij} \quad (15)$$

This suggests of course that $\underset{\sim}{t} = \underset{\sim}{x}$, $K = p^2 + U(x^2)$, $\dot{\underset{\sim}{t}} = \underset{\sim}{p}$. In any event, the solution is "not far" from an SHO.

The symmetric part of (14) gives

$$[\sigma_z^s , \bar{a}_x^s] = -\tfrac{1}{2}[(\underset{\sim}{u} \times \underset{\sim}{\sigma}^s)_z , t_x]$$

If instead of (14) we consider the analogous condition with $\Sigma \rightarrow \underset{\sim}{S}$, we get

$$[\sigma_i^s , \bar{a}_x^s] = -\tfrac{1}{2}[(u \times \sigma^s)_i , t_x] \quad (16)$$

Thus h_{x1} is almost uniquely determined. Similar steps give h_{xx1}, h_{xxx1} and so on, e.g.

$$h_{xx1} = -\tfrac{1}{2}t_{xz}\dot{t}_x + \tfrac{1}{2}t_x L_y - \tfrac{1}{2}t_x(\sigma - \sigma')_y + a_{xx} + \bar{a}^s_{xx}$$

where a_{xx} = tensor, independent of spin

$\bar{a}^s_{xx}$ = tensor, dependent on σ, σ' and symmetrical under $\sigma \leftrightarrow \sigma'$.

In fact, similar to (16), we have

$$[\sigma^s_i, \bar{a}^s_{xx}] = -\tfrac{1}{2}[(u \times \sigma^s)_i, t_{xx}] \qquad (17)$$

Just as t_{xx} is "almost" t_x^2, (see eqn. (13)), from (16), (17) one sees that $\bar{a}^s_{xx}$ is "almost" $\bar{a}^s_x t_x$ and h_{xx1} is "almost" $2h_{xo}h_{x1} = t_x h_{x1}$.

Several important features are worth emphasizing:

(a) "Almost SHO" ($[t_i, \dot{t}_j] = i\delta_{ij}$)

(b) "Almost exponential" (h_{xx1} "almost" $\{h_{xo}, h_{x1}\}$)

(c) The leading ΔJ is completely known. The "new" arbitrary operators (e.g. $\underset{\sim}{u}$) only modify the subsidiary ΔJ.

4. Order m^{-2}

The calculation will not be shown in detail. In fact, only the result on Σ and $\underset{\sim}{S}$ will be needed:

$$\Sigma_2 = \tfrac{1}{2}\dot{t}_\perp^2\sigma_z - \tfrac{1}{2}\dot{t}_\perp\cdot\sigma_\perp \dot{t}_z - t_\perp\cdot\sigma_\perp \ddot{t}_z$$

$$- \tfrac{1}{2}u_\perp^2\sigma_z + \tfrac{1}{2}u_\perp\cdot\sigma_\perp u_z + (\underset{\sim}{v} \times \underset{\sim}{\sigma})_z + \ldots$$

$$S_{x2} = \dot{t}_z\dot{t}_x\sigma_z - \tfrac{1}{2}\dot{t}_x\dot{t}\cdot\sigma + t_z\ddot{t}_x\sigma_z$$
$$- \tfrac{1}{2}u^2\sigma_x + \tfrac{1}{2}u\cdot\sigma u_x + (\underset{\sim}{v} \times \underset{\sim}{\sigma})_x + \ldots$$

Since we shall only be interested in those terms symmetrical in $\underset{\sim}{\sigma}$, $\underset{\sim}{\sigma}'$ when the combination $\Sigma + \Sigma'$, $S + S'$ are formed, the negative parity terms in Σ_2 and S_2 are irrelevant and have been represented simply by ... . Similar to $O(m^{-1})$, we get an arbitrary axial vector $\underset{\sim}{v}$.

One could, of course, continue the calculation by brute force along the lines of section 3. A contradiction in $O(m^{-3})$ will be found; this was, in fact, how the contradiction was discovered. A somewhat more transparent method of obtaining the same result is given in the next two sections.

5. Order m^{-3} : the angular condition

The following notations will be convenient in this and the next section:

$$\underset{\sim}{\sigma}^{s,a} = \underset{\sim}{\sigma} \pm \underset{\sim}{\sigma}'$$

$$\hat{\Sigma} = \Sigma + \Sigma' , \qquad \hat{h}_x = h_x + h_x'$$

< > = matrix element between degenerate eigenstates of K.

From the angular condition (3.4)

$$[J_+ , h_{+3}] \; = \; \tfrac{i}{2}[K , h_{++2}] - \tfrac{1}{2}[KJ_+ , h_{+1}] + \tfrac{i}{2}[K' , h_{++o}]$$

we get, after adding the analogous equation with h_+' and commuting with J_+

$$[J_+ , [J_+ , \hat{h}_{+3}]] \; = \; \tfrac{i}{2}[K,[J_+ , \hat{h}_{++2}]] - \tfrac{1}{2}[KJ_+,[J_+ , \hat{h}_{+1}]]$$
$$= \; -\tfrac{1}{3}[K,[K , \hat{h}_{+++1}]] - \tfrac{i}{2}[K,[KJ_+ , \hat{h}_{++o}]] \qquad (18)$$

$$[J_+,[J_+,[J_+ , \hat{h}_{+3}]]] \; = \; -\tfrac{1}{3}[K,[J_+ , \hat{h}_{+++1}]]]$$
$$= \; -\tfrac{i}{4}[K,[K,[K,\hat{h}_{++++o}]]] \qquad (19)$$

Equation (18) means $\langle h_{x3} \rangle$ has $\Delta J \leqslant 2$. We may decompose $\hat{h}_{x3}$ into

$$\hat{h}_{x3} = (\text{terms independent of spin}) + (\text{terms} \propto \sigma + \sigma') + (\text{terms} \propto \sigma - \sigma') + (\text{terms} \propto \sigma\sigma')$$

and the $\Delta J \leqslant 2$ condition must be separately satisfied for each group of terms. We shall single out the terms $\propto \sigma + \sigma'$ and define

$$H_x \equiv (\text{terms} \propto \sigma + \sigma' \text{ in } \hat{h}_{x3})$$
$$\equiv \alpha\sigma_x^s + \beta\sigma_y^s + \gamma\sigma_z^s \qquad (20)$$

so that the angular condition (18) for H_x is

$$\langle H_x \rangle \text{ has } \Delta J \leqslant 2 \qquad (21)$$

We also note that the RHS of (19) is independent of spin, hence

$$H_x \text{ has } \Delta J \leq 3 \qquad (22)$$

Equations (21) and (22) are two of the conditions imposed by the angular condition. In the next section we shall calculate H_x by using the constraints of current algebra and find that the result is not compatible with (21) and (22).

6. Order m^{-3} : current algebra

The density-charge algebra requires (among other things)

$$[\hat{X}_o , \hat{h}_{x3}] + [\hat{X}_1 , \hat{h}_{x2}] + [\hat{X}_2 , \hat{h}_1] = 0 \qquad (23)$$

where $\hat{X} = \hat{\Sigma}$, $\hat{S}_x$ or $\hat{S}_y$ and we have used $\hat{h}_{xo} = h_{xo} + h'_{xo} = 0$. (This explains why we choose $h_x + h'_x$ to work with — the necessity for knowing $\hat{X}_3$ is avoided.) We shall consider only terms $\propto \sigma + \sigma'$ in (23). For $\hat{X} = \hat{\Sigma}$ we get

$$[\hat{\Sigma}_o , \hat{h}_{x3}] = [\sigma^s_z , \hat{h}_{x3}] = [\sigma^s_z , H_x] + \ldots$$
$$= 2i(\alpha\sigma^s_y - \beta\sigma^s_x) + \ldots$$

where ... are terms not $\propto \sigma + \sigma'$ and will therefore be ignored. Hence

$$2i(\alpha\sigma_y^s - \beta\sigma_x^s) = -[\hat{\Sigma}_1 , \hat{h}_{x2}] - [\hat{\Sigma}_2 , \hat{h}_{x1}]$$

(where it is understood that only terms $\propto \sigma + \sigma'$ on the RHS are to be kept). Commute with σ_z^s again

$$4(\alpha\sigma_x^s + \beta\sigma_y^s) = -[\sigma_z^s , [\hat{\Sigma}_1 , \hat{h}_{x2}]] - [\sigma_z^s , [\hat{\Sigma}_2 , \hat{h}_{x1}]] \tag{24a}$$

Similarly, with Σ replaced by S_x or S_y one gets

$$4(\beta\sigma_y^s + \gamma\sigma_z^s) = -[\sigma_x^s , [\hat{S}_{x1} , \hat{h}_{x2}]] - [\sigma_x^s , [\hat{S}_{x2} , \hat{h}_{x1}]] \tag{24b}$$

$$4(\alpha\sigma_x^s + \gamma\sigma_z^s) = -[\sigma_y^s , [\hat{S}_{y1} , \hat{h}_{x2}]] - [\sigma_y^s , [\hat{S}_{y1} , \hat{h}_{x1}]] \tag{24c}$$

Thus if we know everything up to $O(m^{-2})$, we can calculate α, β, γ from (24) — each in two ways, thus obtaining some consistency relations in the bargain. It is however much simpler to add the three equations to obtain a more symmetrical expression:

$$\begin{aligned} 8H_x \\ = & - [\sigma_z^s , [\hat{\Sigma}_1 , \hat{h}_{x2}]] - [\sigma_x^s , [\hat{S}_{x1} , \hat{h}_{x2}]] - [\sigma_y^s , [\hat{S}_{y1} , \hat{h}_{x2}]] \\ & - [\sigma_z^s , [\hat{\Sigma}_2 , \hat{h}_{x1}]] - [\sigma_x^s , [\hat{S}_{x2} , h_{x1}]] - [\sigma_y^s , [\hat{S}_{y2} , h_{x1}]] \end{aligned} \tag{25}$$

Jacobi's identity can now be exploited to give

$$8H_x$$

$$= [\hat{h}_{x2}, [\sigma_z^s, \hat{\Sigma}_1] + [\sigma_x^s, \hat{S}_{x1}] + [\sigma_y^s, \hat{S}_{y1}]]$$

$$+ [\hat{\Sigma}_1, [\hat{h}_{x2}, \sigma_z^s]] + [\hat{S}_{x1}, [\hat{h}_{x2}, \sigma_x^s]] + [\hat{S}_{y1}, [\hat{h}_{x2}, \sigma_y^s]]$$

$$+ [\hat{h}_{x1}, [\sigma_z^s, \hat{\Sigma}_2] + [\sigma_x^s, \hat{S}_{x2}] + [\sigma_y^s, \hat{S}_{y2}]]$$

where we have also used the fact that $\hat{h}_{x1}$ does not depend on spin. The second line can be further reduced by noting

$$0 = [\hat{\Sigma}_o, \hat{h}_{x2}] + [\hat{\Sigma}_1, \hat{h}_{x1}] \qquad (\hat{h}_{xo} = 0)$$

$$= [\sigma_z^s, \hat{h}_{x2}] + [\hat{\Sigma}_1, \hat{h}_{x1}]$$

$$8H_x = [\hat{h}_{x2}, W_1] + [\hat{h}_{x1}, W_2]$$
$$+ [\hat{\Sigma}_1, [\hat{\Sigma}_1, h_{x1}]] + [\hat{S}_{x1}, [\hat{S}_{x1}, h_{x1}]] + [\hat{S}_{y1}, [\hat{S}_{y1}, h_{x1}]] \qquad (26)$$

where W_1, W_2 can be calculated from what is already known:

$$W_1 \equiv [\sigma_z^s, \hat{\Sigma}_1] + [\sigma_x^s, \hat{S}_{x1}] + [\sigma_y^s, \hat{S}_{y1}]$$

$$= 4i(\dot{\underset{\sim}{t}} \times \underset{\sim}{\sigma}^a)_z - 4i\, \underset{\sim}{u} \cdot \underset{\sim}{\sigma}^s$$

$$W_2 \equiv [\sigma_z^s, \hat{\Sigma}_2] + [\sigma_y^s, \hat{S}_{x2}] + [\sigma_y^s, \hat{S}_{y2}]$$

$$= 2i\left\{ -\dot{t}_z(\dot{\underset{\sim}{t}} \times \underset{\sim}{\sigma}^s)_z - 2t_z(\ddot{\underset{\sim}{t}} \times \underset{\sim}{\sigma}^s)_z + \tfrac{1}{2}\underset{\sim}{u} \cdot (\underset{\sim}{u} \times \underset{\sim}{\sigma}^s) - 2\underset{\sim}{v} \cdot \underset{\sim}{\sigma}^s \right\}$$

Notice that since $\hat{h}_{x1}$ does not depend on spin, to get something $\propto \sigma + \sigma'$ in $[h_{x1}, W_2]$ we need only terms $\propto \sigma + \sigma'$ in W_2, and hence only terms $\propto \sigma + \sigma'$ in Σ_2 etc. — which explains why we kept only $\rho = +$ terms in Σ_2 etc. Also

$$
\begin{aligned}
&[\hat{\Sigma}_1, [\hat{\Sigma}_1, \hat{h}_{x1}]] \\
&= [\dot{t}_\perp \cdot \sigma_\perp^a + (\underset{\sim}{u} \times \underset{\sim}{\sigma}^s)_z \;,\; [\dot{t}_\perp \cdot \sigma_\perp + (\underset{\sim}{u} \times \underset{\sim}{\sigma}^s)_z \;,\; \hat{h}_{x1}]] \\
&= [\dot{t}_\perp \cdot \sigma_\perp^a \;,\; [\dot{t}_\perp \cdot \sigma_\perp^a \;,\; \hat{h}_{x1}]] + [(\underset{\sim}{u} \times \underset{\sim}{\sigma}^s)_z \;,\; \hat{h}_{x1}]] + \dots \\
&= \left\{ \begin{matrix} \dot{t}_x[\dot{t}_y \;,\; \hat{h}_{x1}] - \dot{t}_y[\dot{t}_x \;,\; \hat{h}_{x1}] + \\ u_x[u_y \;,\; \hat{h}_{x1}] - u_y[u_x \;,\; \hat{h}_{x1}] \end{matrix} \right\} 2i\sigma_z^s + \dots
\end{aligned}
$$

where ... are terms not $\propto \sigma + \sigma'$. Likewise

$$
\begin{aligned}
&[\hat{S}_{x1}, [\hat{S}_{x1}, \hat{h}_{x1}]] \\
&= \left\{ u_y[u_z, \hat{h}_{x1}] - u_z[u_y, \hat{h}_{x1}] \right\} 2i\sigma_x^s + \dots
\end{aligned}
$$

so that

$$
\begin{aligned}
&[\hat{\Sigma}_1,[\hat{\Sigma}_1, \hat{h}_{x1}]] + [\hat{S}_{x1},[\hat{S}_{x1}, \hat{h}_{x1}]] + [\hat{S}_{y1},[\hat{S}_{y1}, \hat{h}_{x1}]] \\
&= 2i \left\{ \dot{t}_x[\dot{t}_y, \hat{h}_{x1}] - \dot{t}_y[\dot{t}_x, \hat{h}_{x1}] \right\} \sigma_z^s \\
&\quad + 2i\epsilon_{ijk} u_i [u_j, \hat{h}_{x1}] \, \sigma_x^s \, .
\end{aligned}
$$

With these, (26) becomes

$$
\begin{aligned}
&8H_x \\
&= 4i\,[\hat{h}_{x2}\,,\,(\dot{t}\times\sigma^a)_z - \underset{\sim}{u}\cdot\underset{\sim}{\sigma}^s] \\
&+ 2i\,[\hat{h}_{x1}\,,\,-\dot{t}_z(\dot{t}\times\sigma^s)_z - 2t_z(\ddot{t}\times\sigma^s)_z - \tfrac{1}{2}\underset{\sim}{u}\cdot(\underset{\sim}{u}\times\underset{\sim}{\sigma}^s)-2\underset{\sim}{v}\cdot\underset{\sim}{\sigma}^s] \\
&+ 2i\left\{\,\dot{t}_x[\dot{t}_y\,,\,\hat{h}_{x1}] - \dot{t}_y[\dot{t}_x\,,\,\hat{h}_{x1}]\,\right\}\sigma^s_z \\
&+ 2i\epsilon_{ijk}u_i[u_j\,,\,\hat{h}_{x1}]\,\sigma^s_k \quad . \qquad (27)
\end{aligned}
$$

To obtain a contradiction with the angular condition, only $\Delta J > 2$ terms need be kept; it can be verified quite simply that the terms involving $\underset{\sim}{u}$ or $\underset{\sim}{v}$ can be discarded.

$$
\begin{aligned}
&4i\,[\hat{h}_{x1}\,,\,(\dot{t}\times\sigma^a)_z] \\
&= 4i\,[\hat{h}_{x1}\,,\,\sigma^a_x]\dot{t}_y - 4i\,[\hat{h}_{x2}\,,\,\sigma^a_y]\dot{t}_x + \ldots \\
&= 4i\,[\hat{h}_{x2}\,,\,S_{xo} - S'_{xo}]\dot{t}_y - 4i[\hat{h}_{x2}\,,\,S_{yo} - S'_{yo}]\dot{t}_x \\
&= -4i\,[\hat{h}_{x1}\,,\,S_{x1} - S'_{x1}]\dot{t}_y + 4i[\hat{h}_{x1}\,,\,S_{y1} - S'_{y1}]\dot{t}_x \\
&= -4i\,[\hat{h}_{x1}\,,\,-\dot{t}_x\sigma^s_z]\,\dot{t}_y + 4i\,[\hat{h}_{x1}\,,\,-\dot{t}_y\,\sigma^s_z]\dot{t}_x \\
&= +4i\,[\hat{h}_{x1}\,,\,\dot{t}_x]\dot{t}_y\sigma^s_z - 4i\,[\hat{h}_{x1}\,,\,\dot{t}_y]\dot{t}_x\sigma^s_z \\
&= 2\times(\text{3rd line of }(23))
\end{aligned}
$$

where we have used

$$
[\hat{h}_{x2}\,,\,S_{xo} - S'_{xo}] + [\hat{h}_{x1}\,,\,S_{x1} - S'_{x1}] = 0
$$

and the fact $\hat{h}_{x1}$ does not depend on spin. Thus (27) yields

$$8H_x = 2i[\hat{h}_{x1}, -\dot{t}_z(\dot{\underset{\sim}{t}} \times \underset{\sim}{\sigma}^s)_z - 2t_z(\ddot{\underset{\sim}{t}} \times \underset{\sim}{\sigma}^s)_z]$$
$$+ 6i\left\{\dot{t}_x[\dot{t}_y, h_{x1}] - \dot{t}_y[\dot{t}_x, h_{x1}]\right\}\sigma_z^s + (\Delta J \leqslant 2) \qquad (28)$$

where $\hat{h}_{x1} = -t_z\dot{t}_x + L_y$.

7. Contradiction

For an SHO, for which, in suitable units

$$\underset{\sim}{t} = \underset{\sim}{x} \qquad \dot{\underset{\sim}{t}} = \underset{\sim}{p} \qquad \ddot{\underset{\sim}{t}} = -\underset{\sim}{x}$$

(28) reduces to

$$8H_x = 4p_z^2\sigma_y^s + (\Delta J < 2)$$

where the $\Delta J = 3$ piece obviously does not vanish between degenerate states of K, and a contradiction has been shown.

Since the $\Delta J = 3$ piece does not depend on the unknown operators $\underset{\sim}{u}$, $\underset{\sim}{v}$ etc. (which give most of the arbitrariness that distinguishes one model from another) it may be expected that the general case will resemble the SHO very closely. By requiring the $\Delta J = 4$ piece in (28) to vanish (see eqn. (22)) one finds

$$[\ddot{t}_i, \dot{t}_j] = -iB\delta_{ij} \qquad (29)$$

where B is a scalar commuting with t. This is to be compared with $[t_i, \dot{t}_j] = i\delta_{ij}$, from which it is plausible that $\ddot{t} = -Bt$, with $[\dot{t}, B] = 0$. This can in fact be shown to be the case, using the consistency conditions referred to just below eqn. (23). With this in mind, one finds

$$\begin{aligned} 8H_x &= (6\dot{t}_z^2 - 2Bt_z^2)\sigma_y^s + (\Delta J < 2) \\ &= (6\dot{t}_z^2 + 2t_z\ddot{t}_z)\sigma_y^s + (\Delta J < 2) \\ &= 4\dot{t}_z^2\sigma_y^s + 2(\dot{t}_z^2 + t_z\ddot{t}_z)\sigma_y^s + (\Delta J < 2) \\ &= 4\dot{t}_z^2\sigma_y^s + 2(t_z\dot{t}_z)^{\cdot}\ \sigma_y^s + (\Delta J < 2) \end{aligned}$$

$$8\langle H_x\rangle = 4\langle\dot{t}_z^2\rangle\ \sigma_y^s + (\Delta J < 2)$$

Again, the $\Delta J = 3$ part is not zero if the model is to be nontrivial. This is the contradiction we seek.

VI. Conclusion

We have shown, for spectra expansible (a) around a degenerate limit or (b) in powers of inverse quark mass, that nonexotics cannot saturate the density-charge sum rules of $U(6)_W$ current algebra. For (a), exotic SU(6) representations (i.e. other than $\underset{\sim}{1}$ and $\underset{\sim}{35}$ for mesons) will be required. For (b), either exotic SU(6) representations or $\mathcal{C} \neq \mathcal{P}\,(-1)^{S+1}$ is necessary.

In terms of Regge parameters, the two expansions correspond to extrapolation from the limits (a) $\alpha'^{-1}(0)$ = constant, $\alpha' \to \infty$ (b) $\alpha(0) \to -\infty$, α' = constant, where $\alpha(s)$ is the leading meson trajectory. The real world may evade our result by not having a smooth extrapolation from either limit. However, all known, exactly soluble models of simpler problems (free quarks(12), "factored" case(24,27)) are expansible in at least one of the above ways.

An important question that remains to be answered is the role played by exotics. For example, can we find a solution by allowing a small number of exotic states (e.g. $qq\bar{q}\bar{q}$ for mesons) ?

It may be that there is no answer unless the spectrum is allowed to contain states of arbitrary "exoticity" (e.g. $nqn\bar{q}$ for mesons) — in which case one might be back to the Lagrangian field theories, from which the equal time commutators were abstracted in the first place.

APPENDIX

Here we show that considering the first few moments of the vector currents alone leads to similar difficulties. Indeed, it is the discovery of this fact that led us to consider the even less restrictive problem of the zeroth moments (twice integrated algebra) described in the body of this thesis.

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LOW MOMENTS OF CURRENT ALGEBRA AT INFINITE MOMENTUM *

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It is shown that even the first few moments of the vector and axial vector charge density algebra at infinite momentum cannot be represented realistically in the space of the non-exotic states.

It has been known for some time that the problem of saturating the SU(3)×SU(3) algebra of vector and axial vector charge densities at infinite momentum with non-exotic states admits only trivial and unrealistic solutions: that of degenerate masses and that of the free quark model. Two difficulties were discovered in attempts to construct more realistic solutions. In the simplified problem where only one quark carries charge, the spectrum contains a spacelike part coupled to the timelike part by the current. Secondly, when more than one quark carries charge, the operators for the different quarks fail to commute. We wish to consider here the possibility of circumventing the second difficulty, if, instead of the full set of commutation relations, we impose only those involving up to the second moments of the charge densities.

Let us first summarize the assumptions and results of the exact problem. Detailed accounts can be found in many places [1-10].

It is assumed that the vector and axial vector charge densities, $F_a(x)$ and $F^5_a(x)$, under equal time commutation, obey the local SU(3)×SU(3) algebra without gradient terms. The Fourier transforms then satisfy

$$[F_a(\boldsymbol{k}),\ F_b(\boldsymbol{k}')] = \mathrm{i}\ f_{abc}\ F_c(\boldsymbol{k}+\boldsymbol{k}'),\ \text{etc.} \quad (1)$$

The resulting sum rules are assumed to be saturated by finite mass intermediate states; or equivalently, the corresponding dispersion integrals are assumed to require no subtractions. In particular, if, following Fubini and Furlan [11], eq. (1) is sandwiched between states with $p_z = \infty$, pair states will not contribute, since these correspond to intermediate states of infinite mass. (The pair state is due to the current creating a pair and the initial state propagating freely.) With pair states eliminated, it is then at least plausible to assume, as an approximation to the real world, that non-exotic states saturate the sum rules. Another feature of the infinite momentum limit is that $\langle f|F_a(\boldsymbol{k})|i\rangle$ and $\langle f|F^5_a(\boldsymbol{k})|i\rangle$ no longer depends on $p_i + p_f$, so that $F_a(\boldsymbol{k})$ and $F^5_a(\boldsymbol{k})$ may be represented as matrices acting only on the internal variables of the states $|i\rangle$ and $|f\rangle$ [1].

We try then to represent on the space of non-exotics the charge densities $F_a(\boldsymbol{k})$, $F^5_a(\boldsymbol{k})$, the mass M, the transformed angular momentum $\boldsymbol{J}$† and the parity P. Relativity imposes a severe constraint among these operators. In the Breit frame, where initial and final particles have opposite momenta, say along the x-direction, the current can change J_x by at most one unit. This is the angular condition. Because particles of different masses require different Lorentz transformations to be brought from $p_z = \infty$ to the Breit frame, the angular condition in the $p_z = \infty$ frame becomes rather complicated and reads [9]:

$$I^3_k(F_a(\boldsymbol{k})) = I_k K_k\ [F_a(\boldsymbol{k})]\ , \quad (2)$$

and similarly for $F^5_a(\boldsymbol{k})$. The operations I_k, K_k are defined by

$$I_k(\theta) = [M^2, [J_3, \theta]] - 2[M\boldsymbol{k}\cdot\boldsymbol{J}, \theta] - k^2\{J_3, \theta\}\ , \quad (3a)$$

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† $\boldsymbol{J}$ is the angular momentum boosted to $p_z = \infty$. Specifically, $\boldsymbol{J}$ acts only on the helicity of the state and not on its momentum.

$$K_k(\theta) = [M^2,[M^2,\theta]] + 2k^2\{M^2,\theta\} + k^4\theta \ . \quad (3b)$$

Consider first the simplified problem of the SU(2)×SU(2) subalgebra of isotopic spin, restricted to states with one "active" quark (the others being in an SU(2) singlet); for example the K's in the meson case and the Ξ's in the baryon case. This simplified problem has been solved exactly [4, 9], but the solution suffers from the first difficulty mentioned earlier: except for degenerate masses, the spectrum contains a spacelike part, and except in the free quark model, where $M = 2\sqrt{m^2+p^2}$, the spacelike and timelike parts are coupled by the current [8, 9]. However, if the solution is formally expanded in the mass splitting, the spacelike part does not appear in any finite order [8].

In the full SU(3)×SU(3) algebra with more than one active quark, the charge densities may be represented as [3]

$$F_a(\boldsymbol{k}) = \sum_i \tfrac{1}{2}\lambda_a^{(i)} \exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{h}^{(i)}] \ , \quad (4a)$$

$$F_a^5(\boldsymbol{k}) = \sum_i \tfrac{1}{2}\lambda_a^{(i)}\,\omega^{(i)} \exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{h}^{(i)}] \ , \quad (4b)$$

where $\lambda_a^{(i)}$ are the usual SU(3) matrices for each quark i. We shall consider the meson problem, so $i = 1, 2$. In order to satisfy the commutation relations (1), $h^{(i)}, \omega^{(j)}$ must commute with one another, and $\omega^{(j)2} = 1$. The angular momentum is represented by

$$\boldsymbol{J} = \boldsymbol{x}\times\boldsymbol{p} + \tfrac{1}{2}\boldsymbol{\sigma}^{(1)} + \tfrac{1}{2}\boldsymbol{\sigma}^{(2)} \ . \quad (5)$$

Furthermore, $h^{(i)} = (h_x^{(i)}, h_y^{(i)})$ must behave as a two vector under J_3 and $\omega^{(i)}$ as a scalar. Properties under parity and time reversal are given in ref. [3].

It remains to determine $\boldsymbol{h}^{(i)}$ and $\omega^{(i)}$ so as to satisfy the angular condition. The case of all masses equal and infinite gives the static quark model, with $\boldsymbol{h}^{(1)} = \frac{1}{2}x$, $\omega^{(1)} = \sigma_z(1)$, etc. [3]. The more general problem where $M = 2\sqrt{m^2+p^2+U(x)}$ can be solved by expanding around the static solution, i.e., in powers of $1/m$ [3, 10, 12]. For the free quark model $[U(x) = 0]$, the solution has also been obtained in closed form by covariant methods [3]. However, by carrying out the calculation to third order in $1/m$, it was found that for any $U(x) \neq 0$, $\boldsymbol{h}^{(1)}$, $\boldsymbol{h}^{(2)}$ fail to commute [10, 12]. In other words, there are no solutions corresponding to bound quarks.

We wish to see whether keeping only the constraints on the low moments of the charge densities will permit a solution where $[\boldsymbol{h}^{(1)}, \boldsymbol{h}^{(2)}] = 0$ for some non-zero potential $U(x)$. Expanding eq. (2) in powers of k yields a sequence of k-independent equations (of which only a finite number are independent) [9]. The one involving the lowest moments is

$$[2MJ_\pm, h_\pm^{(i)}] = \pm\tfrac{1}{2}\mathrm{i}[M^2, h_\pm^{(i)2}] \ , \quad (6)$$

where $J_\pm = J_x \pm \mathrm{i}\,J_y$, $h_\pm^{(i)} = h_x^{(i)} \pm \mathrm{i}\,h_y^{(i)}$. Another equation is, for instance,

$$[h_j^{(i)}, [h_k^{(i)}, [h_l^{(i)}, M^2]]] = 0 \quad (7)$$

and there are somewhat more complicated conditions on $\omega^{(i)}$.

We impose the weaker condition (6) instead of the exact condition (2) on the vector charge densities. This is equivalent to imposing the exact angular condition but requiring the charge densities to satisfy only those current commutation relations involving no moments higher than the second, i.e., requiring eq. (1) to be true only up to second order in k, k'. We then expand around the static solution in powers of $1/m$ $h_+^{(i)} = \sum_n h_{+n}^{(i)} m^{-n}$. Starting with $h_{+0}^{(i)} = \frac{1}{2}x_+$,

eq. (6) is used to determine succesively $h_{+1}^{(1)}, h_{+2}^{(1)}, \ldots$. The analogous operators for the second quark can be obtained by letting $x \to -x$, $p \to -p$, $\sigma^{(1)} \leftrightarrow \sigma^{(2)}$.

Owing the weaker condition, there is a limited amount of extra freedom. For example, eq. (6) expanded in powers of $1/m$ gives

$$[J_+, h_{+1}^{(1)}] = \tfrac{1}{2}\mathrm{i}[\{p^2 + U(x),\ h_{+0}^{(1)}\}h_{+0}^{(1)}] \ , \quad (7a)$$

$$[J_+, h_{+2}^{(1)}] = \tfrac{1}{2}\mathrm{i}[\{p^2 + U(x),\ h_{+0}^{(1)}\}h_{+1}^{(1)}] + \tfrac{1}{2}\mathrm{i}[\{p^2 + U(x),\ h_{+1}^{(1)}\}, h_{+0}^{(1)}] - \tfrac{1}{2}[(p^2 + U(x))\,J_+,\ h_{+0}^{(1)}] \ , \quad (7b)$$

and so on. Putting the initial condition $h_{+0}^{(1)} = \frac{1}{2}x_+$ into (7a), the right-hand side becomes $\frac{1}{2}x_+p_+$. Therefore we have

$$h_{+1}^{(1)} = a(-\tfrac{1}{2}x_+p_z) + b(-\tfrac{1}{2}zp_+) + c(\hat{z}\times\boldsymbol{\sigma}^{(1)})_+ + d(\hat{z}\times\boldsymbol{\sigma}^{(2)})_+ \ , \quad (8)$$

where $a+b=1$ in order to satisfy (7a). Demanding $[h^{(1)}, h^{(2)}] = 0$ to order $1/m$ gives some constraints ($b=0$). We then put (8) into (7b) to calculate $h^{(1)}_{+2}$, with still more undetermined coefficients. Carrying this out to third order in $1/m$, we find that $h^{(1)}$, $h^{(2)}$ again fail to commute. The trouble here, as in the exact problem, comes from terms in $h^{(1)}_x$, $h^{(2)}_x$ with the maximum allowed number of z indices.

The same difficulty of $[h^{(1)}, h^{(2)}] \neq 0$ is encountered if x and p are generalized to four-dimensional variables.

Thus it seems that even the approximate problem of representing the zeroth, first, and second moments of the charge densities requires the introduction of exotics.

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