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ON DYNAMICAL SPACE-TIMES WHICH CONTAIN A CONFORMAL EUCLIDEAN 3-SPACE

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ON DYNAMICAL SPACE-TIMES WHICH CONTAIN A CONFORMAL EUCLIDEAN 3-SPACE.

Einstein's theory of gravitation requires that physical space-time be a four dimensional manifold whose line element is determined by an invariant quadratic form

$$(0.1) \qquad ds^2 = \sum_{\mu\nu} g_{\mu\nu}\, dx^\mu dx^\nu \qquad \mu, \nu = 1 \cdots 4$$

whose coefficients $g_{\mu\nu}$ satisfy the ten differential equations

$$(0.2) \qquad G_{\mu\nu} = \lambda g_{\mu\nu}$$

in points outside of matter. In these equations λ is the so-called cosmological constant and $G_{\mu\nu}$ the contracted Riemann-Christoffel tensor

$$\sum_\epsilon B^\epsilon_{\mu\nu\epsilon} \qquad \epsilon = 1 \cdots 4$$

where

$$B^\epsilon_{\mu\nu\sigma} = \sum_\alpha \left[\{\mu\sigma,\alpha\}\{\alpha\nu,\epsilon\} - \{\mu\nu,\alpha\}\{\alpha\sigma,\epsilon\} \right] + \frac{\partial}{\partial x^\nu}\{\mu\sigma,\epsilon\} - \frac{\partial}{\partial x^\sigma}\{\mu\nu,\epsilon\}$$

$\{\mu\nu,\alpha\}$ being the three index Christoffel symbol of the second kind. Using this value of $B^\epsilon_{\mu\nu\sigma}$, and introducing g, the determinant whose elements are the $g_{\mu\nu}$, we may write

$$G_{\mu\nu} = \sum_{\alpha\beta} \{\mu\alpha,\beta\}\{\nu\beta,\alpha\} - \sum_\alpha \left[\frac{\partial\{\mu\nu,\alpha\}}{\partial x^\alpha} + \{\mu\nu,\alpha\} \frac{\partial \log\sqrt{g}}{\partial x^\alpha} \right] + \frac{\partial^2 \log\sqrt{g}}{\partial x^\mu \partial x^\nu}$$

The functions

$$\frac{B_{\mu\nu\sigma\rho}}{\begin{vmatrix} g_{\mu\sigma} & g_{\mu\nu} \\ g_{\rho\sigma} & g_{\rho\nu} \end{vmatrix}}$$

are called the components of Riemann curvature; if they are constant and equal to each other the manifold in question is a hyper-sphere.

A material particle in a field defined by equations (0.2) is assumed to move along a non-minimal geodesic, the equations of which are

$$(0.3) \qquad \frac{d^2 x^\mu}{ds^2} + \sum_{\alpha\beta} \{\alpha\beta, \mu\} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \qquad \alpha, \beta = 1 \cdots 4$$

A beam of light follows a minimal geodesic, i.e. a geodesic along which

$$(0.31) \qquad ds = 0$$

It is the purpose of this thesis to restrict the functions $g_{\mu\nu}$ a priori, to solve the equations (0.2) under such restrictions, and to give a geometrical or, if possible, a physical interpretation of the solutions. In particular, we examine dynamical orthogonal manifolds which contain a conformal euclidean 3-space. The line element of such a world must be of the form

$$ds^2 = f_1(x^1 x^2 x^3 x^4)\left[dx^{1\,2} + dx^{2\,2} + dx^{3\,2}\right] + f_2(x^1 x^2 x^3 x^4)\, dx^{4\,2}$$

where f_1 is really a function of the variable x^4. If we interpret x^4 as the time coordinate, we see from (0.31) that the velocity of light in this manifold is independent of its direction at a point. Physically, then, we propose to find all dynamical space-times which admit of orthogonal coordinates in which the velocity of light is isotropic.

I The Equations.

The problem may be stated: To determine the functions $\alpha(xyzt)$ and $\beta(xyzt)$ in

$$(1.1)' \qquad ds^2 = e^{2\alpha}\left[dx^2+dy^2+dz^2\right]+e^{2\beta}dt^2$$

so they satisfy the cosmological equations (0.2). For convenience we have here written

$$f_1 = e^{2\alpha} \qquad f_2 = e^{2\beta}; \qquad x^1 = x \qquad x^2 = y \qquad x^3 = z \qquad x^4 = t$$

Letting $\alpha_1 = \dfrac{\partial\alpha}{\partial x}$ $\quad \alpha_{14} = \dfrac{\partial\alpha}{\partial x\partial t}$ etc., we may write the components of the tensor $G_{\mu\nu}$

$$G_{14} = 2(\alpha_{14} - \alpha_4\beta_1)$$

$$G_{23} = \alpha_{23} - \alpha_2\alpha_3 - \alpha_2\beta_3 - \alpha_3\beta_2 + \beta_2\beta_3 + \beta_{23}$$

$$G_{11} = 2\alpha_{11} + \alpha_{22} + \alpha_{33} + \alpha_2^2 + \alpha_3^2 - \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \beta_{11} + \beta_1^2 + e^{2(\alpha-\beta)}\left\{\alpha_{44} + \alpha_4(3\alpha_4 - \beta_4)\right\}$$

$$G_{44} = 3\left\{\alpha_{44} + \alpha_4(\alpha_4 - \beta_4)\right\} + e^{2(\beta-\alpha)}\left\{\beta_{11} + \beta_{22} + \beta_{33} + \beta_1(\alpha_1+\beta_1) + \beta_2(\alpha_2+\beta_2) + \beta_3(\alpha_3+\beta_3)\right\}$$

The other six components are obtained from the above by the cyclic permutation (123).

The equations (0.2) then become

$$\text{A} \qquad G_{14} = G_{24} = G_{34} = 0$$

$$\text{B} \qquad G_{23} = G_{31} = G_{12} = 0$$

$$\text{C} \qquad G_{11} = G_{22} = G_{33}$$

$$\text{D} \qquad G_{11} + G_{22} + G_{33} - G_{44}\,e^{2(\alpha-\beta)} = 2\lambda e^{2\alpha}$$

$$\text{E} \qquad G_{44} = \lambda e^{2\beta}$$

or

A $$\alpha_{14} - \alpha_4\beta_1 = 0 \qquad \text{etc.}$$

B $$\alpha_{23} - \alpha_2\alpha_3 - \alpha_2\beta_3 - \alpha_3\beta_2 + \beta_2\beta_3 + \beta_{23} = 0 \qquad \text{etc.}$$

C $$\alpha_{11} - \alpha_1^2 - 2\alpha_1\beta_1 + \beta_1^2 + \beta_{11} = \alpha_{22} - \alpha_2^2 - 2\alpha_2\beta_2 + \beta_2^2 + \beta_{22} = \text{etc.}$$

D $$2(\alpha_{11} + \alpha_{22} + \alpha_{33}) + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 3e^{2(\alpha-\beta)}\alpha_4^2 = \lambda e^{2\alpha}$$

E $$\beta_{11} + \beta_{22} + \beta_{33} + \beta_1(\alpha_1+\beta_1) + \beta_2(\alpha_2+\beta_2) + \beta_3(\alpha_3+\beta_3) + e^{2(\alpha-\beta)}\{\alpha_{44} + \alpha_4(\alpha_4-\beta_4)\} = \lambda e^{2\alpha}$$

On solving set A we find

(1.3) $$\beta = \log\alpha_4 - \log f(t)$$

where f is an arbitrary function of t. Substituting this value of β in the remaining equations, and letting $\alpha = -\log\rho$

B $$\frac{\rho_{234}}{\rho_4} - 2\frac{\rho_{23}}{\rho} = 0 \qquad \text{etc}$$

C $$\frac{\rho_{114}}{\rho_4} - \frac{2\rho_{11}}{\rho} = \frac{\rho_{224}}{\rho_4} - \frac{2\rho_{22}}{\rho} = \text{etc}$$

D $$-2\rho(\rho_{11} + \rho_{22} + \rho_{33}) + 3(\rho_1^2 + \rho_2^2 + \rho_3^2) = \lambda - 3f^2(t)$$

Equation E is not independent, for it is a third order differential equation of which D is the first integral.

Equations B and C give on integration

$$\rho_{23} = \rho^2 a(xyz) \qquad \text{etc.}$$

$$\rho_{22} - \rho_{33} = \rho^2 l(xyz) \qquad \text{etc.}$$

etc., where a, l etc. are arbitrary functions of x, y and z.

We may now state: A manifold whose metrical relations are determined by

(1.1) $$ds^2 = \frac{dx^2 + dy^2 + dz^2}{\rho^2(xyzt)} + \gamma^2(xyzt)\,dt^2$$

is an Einstein field (i.e. satisfies the equations (0.2)) of the required type, provided γ and ρ satisfy

$$\text{A} \qquad \tau = \frac{\rho_4}{\rho} \cdot \frac{1}{f(t)} \qquad \rho_4 \neq 0$$

$$(1.4) \quad \text{B} \qquad \rho_{23} = \rho^2 a(xyz) \quad \rho_{31} = \rho^2 b(xyz) \quad \rho_{12} = \rho^2 c(xyz)$$

$$\text{C} \qquad \rho_{22} - \rho_{33} = \rho^2 l(xyz) \quad \rho_{33} - \rho_{11} = \rho^2 m(xyz) \quad \rho_{11} - \rho_{22} = \rho^2 n(xyz)$$

$$\text{D} \qquad -2\rho(\rho_{11} + \rho_{22} + \rho_{33}) + 3(\rho_1^2 + \rho_2^2 + \rho_3^2) = \lambda - 3f^2(t)$$

where a, b, c, l, m and n are so far arbitrary functions of x, y and z, except for the identity $l + m + n = 0$.

II The Conditions of Integrability.

We must now find the conditions of integrability which insure the existence of an integral of equations B, C and D above.

The first three sets of conditions are found by differentiating the equations. First, since

$$\frac{\partial \rho_{23}}{\partial x} = \frac{\partial \rho_{31}}{\partial y} = \frac{\partial \rho_{12}}{\partial z}$$

set B requires that

$$(2.1) \qquad \rho_1 a + \tfrac{1}{2}\rho a_1 = \rho_2 b + \tfrac{1}{2}\rho b_2 = \rho_3 c + \tfrac{1}{2}\rho c_3$$

Next,

$$\frac{\partial}{\partial x}(\rho_{22} - \rho_{33}) = \frac{\partial \rho_{12}}{\partial y} - \frac{\partial \rho_{31}}{\partial z}$$

whence from B and C

$$\frac{\partial \rho^2 l}{\partial x} = \frac{\partial^2 \rho^2 c}{\partial y} - \frac{\partial \rho^2 b}{\partial z}$$

The two other equations of C yield similar conditions; this set may be written

$$\rho_1 l - \rho_2 c + \rho_3 b + \tfrac{1}{2}\rho(l_1 - c_2 + b_3) = 0$$

$$(2.2) \qquad \rho_2 m - \rho_3 a + \rho_1 c + \tfrac{1}{2}\rho(m_2 - a_3 + c_1) = 0$$

$$\rho_3 n - \rho_1 b + \rho_2 a + \tfrac{1}{2}\rho(n_3 - b_1 + a_2) = 0$$

The third set is obtained from equation D in conjunction with B and C. Since

$$2\rho_{11} = \rho_{22} + \rho_{33} - \rho^2(m - n)$$

D may be written

$$-\rho(\rho_{22} + \rho_{33}) + \tfrac{1}{3}\rho^3(m - n) + \rho_1^2 + \rho_2^2 + \rho_3^2 = \tfrac{\lambda}{3} - f^2(t)$$

Differentiating this equation with respect to x and substituting

$$\rho_{12} = \rho^2 c \qquad \rho_{13} = \rho^2 b$$

we find

$$\rho^3\left\{-b_3 - c_2 + \tfrac{1}{3}(m_1 - n_1)\right\} = 0$$

The cyclic permutation (abc)(lmn)(123) gives two other conditions, corresponding to differentiation with respect to y and z, so

$$b_3 + c_2 = \tfrac{1}{3}(m_1 - n_1)$$

$$(2.3) \qquad c_1 + a_3 = \tfrac{1}{3}(n_2 - l_2)$$

$$a_2 + b_1 = \tfrac{1}{3}(l_3 - m_3)$$

The conditions (2.1) and (2.2) represent five equations between the logarithmic derivatives $\frac{\rho_1}{\rho}$, $\frac{\rho_2}{\rho}$ and $\frac{\rho_3}{\rho}$ of ρ, which requires that the functions a, l etc. satisfy the two eliminants of the equations. We may then solve for the derivatives of ρ, and demand that this set of differential equations be consistent with the original ones. It will be found convenient to vary the procedure by which this is accomplished, according to the form of a, b and c. We here develop these additional conditions for the case $abc \neq 0$; the corresponding conditions for the alternative cases are of a modified form, and it is of advantage to develop them at a later stage.

In the case $abc \neq 0$, then, we may solve (2.1) for any two of the logarithmic derivatives of ρ (with respect to x, y and z) in terms of the other one; thus

$$\rho_2 = \rho_1 \frac{a}{b} + \rho \frac{a_1 - b_2}{2b}$$

$$\rho_3 = \rho_1 \frac{a}{c} + \rho \frac{a_1 - c_3}{2c}$$

Substituting these into the first equation of (2.2) and collecting coefficients of ρ_1 and ρ we find

(2.21) $$A_1' \rho_1 + A_1 \rho = 0$$

where

$$A_1' = bcl - a(c^2 - b^2)$$

$$A_1 = \tfrac{1}{2}\left\{ bc(l_1 - c_2 + b_3) - a_1(c^2 - b^2) + b_2 c^2 - c_3 b^2 \right\}$$

Proceeding in this way with the other two equations of (2.2) we may throw them into the form

$$A_2' \rho_2 + A_2 \rho = 0$$

$$A_3' \rho_3 + A_3 \rho = 0$$

where A_2' and A_2 etc. are obtained from A_1' and A_1 by the cyclic permutation (abc)(lmn)(123).

In order that our conditions be consistent with equations B we must have

$$\rho_{23} = \frac{\partial}{\partial z}\left[\rho_1 \frac{a}{b} + \rho \frac{a_1 - b_2}{2b}\right] = \rho^2 a$$

Eliminating ρ_3 this becomes

$$B_1' \rho_1 + B_1 \rho = 0 \tag{2.4}$$

where

$$B_1' = a\left(\frac{a_3}{a} - \frac{b_3}{b} + \frac{a_1 - b_2}{2c}\right)$$

$$B_1 = \frac{1}{2}\left\{(a_1 - b_2)_3 - \frac{b_3}{b}(a_1 - b_2) + \frac{1}{2c}(a_1 - c_3)(a_1 - b_2)\right\}$$

Also

$$B_2' \rho_2 + B_2 \rho = 0$$

$$B_3' \rho_3 + B_3 \rho = 0$$

where the new B's are again obtained by cyclic permutation.

Finally we ask that

$$\rho_{22} - \rho_{33} = \frac{\partial}{\partial y}\left[\rho_1 \frac{a}{b} + \rho \frac{a_1 - b_2}{2b}\right] - \frac{\partial}{\partial z}\left[\rho_1 \frac{a}{c} + \rho \frac{a_1 - c_3}{2c}\right] = \rho^2 l$$

Transforming this equation as in the previous set, we may write it

$$C_1'' \rho^2 + C_1' \rho_1 + C_1 \rho = 0 \tag{2.5}$$

where

$$C_1'' = \frac{a}{bc}(c^2 - b^2) - l$$

$$C_1' = \frac{a_2}{b} - \frac{a_3}{c} + \frac{a}{2b^2}(a_1 - 3b_2) - \frac{a}{2c^2}(a_1 - 3c_3)$$

$$C_1 = \frac{1}{2}\left\{\frac{1}{b}(a_1 - b_2)_2 + \frac{1}{2b^2}(a_1 - b_2)(a_1 - 3b_2) - \frac{1}{c}(a_1 - c_3)_3 - \frac{1}{2c^2}(a_1 - c_3)(a_1 - 3c_3)\right\}$$

Correspondingly

$$C_2'' \rho^2 + C_2' \rho_2 + C_2 \rho = 0$$

$$C_3'' \rho^2 + C_3' \rho_3 + C_3 \rho = 0$$

The 14 conditions which have been obtained above are not independent, but we have written them in this form for the sake of symmetry.

III Solution of the Equations.

In order to solve the equations (1.4) we divide the problem into four mutually exclusive and exhaustive parts, according to the form of the three logarithmic derivatives

$$\frac{\rho_1}{\rho} \qquad \frac{\rho_2}{\rho} \qquad \frac{\rho_3}{\rho}$$

of ρ. The cases which arise are:

(a) <u>none</u> of the three expressions $\frac{\rho_1}{\rho}$, $\frac{\rho_2}{\rho}$, $\frac{\rho_3}{\rho}$ are functions of t

(b) <u>one</u> and <u>one only</u> is a function of t

(c) <u>two</u> and <u>two only</u> are functions of t

(d) <u>all three</u> are functions of t

We now consider these cases separately.

a

In the first case must be of the form

(3.1) a $$\rho = R(t)\,\sigma(x y z)$$

in order that its logarithmic derivatives with respect to x, y and z be independent of t.

From A

$$\tau = \frac{k'(t)}{k(t)} \cdot \frac{1}{f(t)}$$

so τ is a function of t alone.

Substituting the value of ρ given by (3.1)a above into B we find

$$\sigma_{23} = \sigma^2 a k$$

But k is really a function of t, whereas σ and a cannot contain this variable; hence σ_{23} and a must vanish. Applying this reasoning to the remaining equations B and C we see that b, c, l and m must also vanish. The conditions of integrability are satisfied in this case.

Thus the equations B and C become

$$\text{B} \qquad \sigma_{23} = \sigma_{31} = \sigma_{12} = 0$$

$$\text{C} \qquad \sigma_{11} = \sigma_{22} = \sigma_{33}$$

the solution of which is

$$\sigma = d(x^2+y^2+z^2) + d_1 x + d_2 y + d_3 z + d_4$$

where the d's are arbitrary constants. Equation D requires that

$$k^2(t)\, D^2 + f^2(t) = \frac{\lambda}{3}$$

where

$$D^2 = d_1^2 + d_2^2 + d_3^2 - 4 d d_4$$

For case (a), then, the functions τ and ρ are

$$\tau = \frac{k}{k\sqrt{\frac{2}{3} - \sigma^2 k^2}}$$

(3.2) a

$$\rho = k(t) \left\{ d(x^2+y^2+z^2) + d_1 x + d_2 y + d_3 z + d_4 \right\}$$

where k is an arbitrary function of t, and the d's are arbitrary constants.

b

We next consider the case in which only one of the logarithmic derivatives, say $\frac{\rho_1}{\rho}$, is a function of t. Hence

(3.1) b $$\rho = k(x,t)\,\sigma(x,y,z)$$

where $\frac{k_1}{k}$ is really a function of t.

Since $\frac{\rho_2}{\rho}$ and $\frac{\rho_3}{\rho}$ as well as a, l etc. are functions of x, y and z alone, it must be impossible to solve equations (2.1) or (2.2) for $\frac{\rho_1}{\rho}$ in terms of them. Hence a, b, c and l must vanish. But then

$$\rho_{12} = k_1 \sigma_2 + k \sigma_{12} = 0$$

or

$$\frac{k_1}{k} \sigma_2 + \sigma_{12} = 0$$

whence $\sigma_2 = 0$. Similarly $\sigma_3 = 0$. We conclude, therefore, that ρ is a function of x and t alone.

$m = -m$ is now a function of x alone. The conditions of integrability require that

$$m_1 = m_1$$

i.e. $2m_1 = 0$. But then n is a constant.

The line element is thus defined for this case by

$$(3.2)\,b \qquad \Upsilon = \frac{\varrho_4}{\varrho}\,\frac{1}{f(t)}$$

where ϱ is a Weierstrasse $\wp$-function (whose periods are dependent on t) satisfying

$$\left(\frac{\partial\varrho}{\partial x}\right)^2 = \frac{2}{3}m\varrho^3 + \frac{\lambda}{3} - f^2(t)$$

c

We now let $\frac{\varrho_2}{\varrho}$ and $\frac{\varrho_3}{\varrho}$ be functions of x, y, z and t, and $\frac{\varrho_1}{\varrho}$ a function of x, y and z alone. Consequently

$$\varrho = h(y\,z\,t)\,\sigma(x\,y\,z)$$

(2.1) requires that b and c be zero, so since we must have

$$\frac{h_2}{h}\sigma_1 + \sigma_{12} = 0$$

whence $\sigma_1 = 0$. Hence here

$$\varrho = \varrho\,(y\,z\,t)$$

The conditions of integrability (2.2) are

$$(3.1)\,c \qquad \begin{aligned} \varrho_2 m - \varrho_3 a &= -\tfrac{1}{2}\varrho\,(m_2 - a_3) \\ \varrho_3 n + \varrho_2 a &= -\tfrac{1}{2}\varrho\,(n_3 + a_2) \end{aligned}$$

Since it must be impossible to solve these equations for $\frac{\rho_2}{\rho}$ and $\frac{\rho_3}{\rho}$,

$$\left\{ \begin{matrix} m & -a & m_2 - a_3 \\ a & n & n_3 + a_2 \end{matrix} \right\} = 0$$

or

$$a^2 + mn = 0$$

(3.2)c

$$m\,(a_2 + n_3) + a\,(a_3 - m_2) = 0$$

To these may be added (2.3), i.e.

$$m_2 + 2n_2 = 3a_3$$

(3.3)c

$$2m_3 + n_3 = -3a_2$$

In order to integrate the above equations on a, m and n, we write the second of set (3.2)c

$$\left(\frac{m}{a}\right)_2 \left(\frac{m}{a}\right) = \left(\frac{m}{a}\right)_3$$

on eliminating n by means of the first, assuming for the moment $amn \neq 0$. The general integral of this equation is

$$y = F\left(\frac{m}{a}\right) - \frac{m}{a}\,z$$

where F is an arbitrary function of $\frac{m}{a}$.

To evaluate F and complete the integration of (3.3)c it is convenient to interchange the dependent and independent variables m, n and y, z, assuming that the Jacobian

$$\frac{\partial(m\ n)}{\partial(y\ z)} \neq 0$$

Our equations then become

$$y_m - 2y_n = 3\,(a_n z_m - a_m z_n)$$

$$2z_m - z_n = 3\,(-a_n y_m + a_m y_n)$$

where $y_m = \frac{\partial y}{\partial m}$ etc. Eliminating a and y from this set by (3.2)c and its integral, and solving for the two remaining derivatives z_m and z_n we find

$$z_m = \frac{5}{6} (F' - z) \frac{1}{m - n}$$

$$z_n = \frac{1}{6} (F' - z) \frac{2n + 3m}{n(n - m)}$$

The condition

$$\frac{\partial z_m}{\partial n} = \frac{\partial z_n}{\partial m}$$

requires that

$$F''\left(\frac{m}{a}\right) = 0$$

i.e.

$$F\left(\frac{m}{a}\right) = \frac{m}{a} z_0 + y_0$$

We may either integrate the last set of equations or return to the original with y and z as independent variables; the latter proceedure is perhaps preferable. Since

$$y - y_0 = -\frac{m}{a}(z - z_0)$$

no loss of generality is incurred by letting $y_0 = z_0 = 0$. Equations (3.2)c are now

$$m = -a\frac{y}{z} \qquad n = a\frac{z}{y}$$

Using these values of m and n we may solve (3.3)c for $\frac{a_2}{a}$ and $\frac{a_3}{a}$

$$\frac{a_2}{a} = \frac{1}{y}\frac{z^2 - 4y^2}{z^2 + y^2} \qquad \frac{a_3}{a} = \frac{1}{z}\frac{y^2 - 4z^2}{y^2 + z^2}$$

On integration

$$a = \frac{Cyz}{(y^2 + z^2)^{5/2}}$$

whence

$$m = \frac{-Cy^2}{(y^2 + z^2)^{5/2}} \qquad n = \frac{Cz^2}{(y^2 + z^2)^{5/2}}$$

The condition (3.1)c requires that

$$\rho = \rho_2 y + \rho_3 z$$

i.e. ρ is a homogeneous function of y and z, and of first degree. We take

$$\rho = y\, g(\xi, t) \qquad \xi = \frac{z}{y}$$

Substituting the above values of ρ , a, m and n into (1.4) we find that $g(\xi,t)$ must satisfy

$$\left(\frac{\partial g}{\partial \xi}\right)^2 - \frac{2\xi}{1+\xi^2}\, g\, \frac{\partial g}{\partial \xi} + \frac{1}{1+\xi^2}\, g^2 + \frac{2}{3}\, C\, \frac{g^3}{(1+\xi^2)^{5/2}} = \frac{\frac{\lambda}{3} - f^2(t)}{1+\xi^2}$$

where C is an arbitrary constant (not zero). To integrate this equation let

$$g(\xi, t) = \sec\varphi\; U(\varphi, t)$$
$$\xi = \tan\varphi$$

$U(\varphi, t)$ is then defined by

$$\left(\frac{\partial U}{\partial \varphi}\right)^2 = -\frac{2}{3}\, C U^3 - U^2 + \frac{\lambda}{3} - f^2(t)$$

The solution of this equation is

$$U = -\frac{1}{2C} + \wp\left(\sqrt{\frac{-C}{6}}\,\varphi + \pi(t)\right)$$

where $\wp$ is a Weierstrasse $\wp$-function whose invariants are

$$g_2 = \frac{3}{C^2} \qquad g_3 = -\frac{1}{C^3} + \frac{6}{C}\left(\frac{\lambda}{3} - f^2(t)\right)$$

We now return to ρ and express it in cylindrical coordinates $x = x \quad y = \varpi\cos\varphi \quad z = \varpi\sin\varphi$

$$\rho = \varpi\left\{-\frac{1}{2C} + \wp\left(\sqrt{\frac{-C}{6}}\,\varphi + \pi(t)\right)\right\} \tag{3.4)c}$$

where $\pi(t)$ is an arbitrary function of t.

The above solution has been obtained under the restrictions

$$a m n \neq 0 \qquad \frac{\partial(m\ n)}{\partial(y\ z)} \neq 0$$

we now examine the cases in which these do not hold.

First, assume $amn = 0$. Then it is seen from (3.1)c that $a = m = n = 0$, for otherwise it would be possible to express $\frac{\rho_2}{\rho}$ or $\frac{\rho_3}{\rho}$ in terms of x, y and z alone. Equations (1.4) become

$$\rho_{23} = \rho_{22} = \rho_{33} = 0 \qquad \rho_2^2 + \rho_3^2 = \frac{2}{3} - f^2(t)$$

whence

$$\rho = \alpha_2(t)\, y + \alpha_3(t) z + \alpha_4(t)$$

(3.5) c
$$f^2(t) = \frac{2}{3} - \alpha_2^2 - \alpha_3^2$$

Consider now the case in which

$$\frac{\partial(m\ n)}{\partial(y\ z)} = 0$$

i.e. $\varphi(m\ n) = 0$. For this purpose we express a, m and n in terms of a parameter σ

$$a = a(\sigma) \qquad m = m(\sigma) \qquad n = n(\sigma)$$

The equations (3.2)c and (3.3)c are then

$$a^2 + mn = 0$$

(3.21) c
$$\left(\frac{m}{a}\right)'\left(\sigma_3 - \frac{m}{a}\sigma_2\right) = 0$$

$$(m' + 2n')\,\sigma_2 - 3a'\sigma_3 = 0$$

(3.31) c
$$3a'\sigma_2 + (2m' + n')\,\sigma_3 = 0$$

where ' indicates differentiation with respect to σ .

(3.31)c requires that

$$\begin{vmatrix} m' + 2n' & -3a' \\ 3a' & 2m' + n' \end{vmatrix} = 0$$

for if this were not satisfied σ_2 and σ_3 would vanish. But a, m and n would then be constants and the determinant would be zero, contrary to our assumption. Thus one of the equations (3.31)c may be replaced by the eliminant of the two.

The second of equations (3.21)c can only be satisfied if $\left(\frac{m}{a}\right)' = 0$ or $\sigma_3 = \frac{m}{a}\sigma_2$; it can be shown that both possibilities lead to the same solutions. They are here obtained by taking $\left(\frac{m}{a}\right)' = 0$.

Since $\left(\frac{m}{a}\right)' = 0$ then $m = \mu a \quad n = \nu a$ where μ and ν are constants satisfying the condition $\mu\nu + 1 = 0$. The eliminant of (3.31)c yields

$$a'(\mu^2 + 1) = 0$$

The first alternative, $a' = 0$, leads to the conclusion that a, m and n are constants. By (3.1)c

$$\rho_2 m = \rho_3 a$$

so

$$(3.6)c \qquad \rho = K\left(\frac{y}{m} + \frac{z}{a}, t\right) = K(\xi, t) \qquad \xi = \frac{y}{m} + \frac{z}{a}$$

Equations B and C are satisfied if

$$\left(\frac{\partial K}{\partial \xi}\right)^2 = \frac{2}{3}K^3 a^2 m + \frac{\frac{\lambda}{3} - f^2(t)}{\frac{1}{m^2} + \frac{1}{a^2}}$$

Finally, we may have $\mu^2+1=0$ i.e. $\mu=i$.
Then

$$\mu=\nu=i \qquad \sigma_3=i\sigma_2$$

whence $\sigma=fn\,(y+iz)$. From (3.1)c $\rho_3=i\rho_2$ so

$$\rho=\kappa(y+iz,t)$$

The solution of equations (1.4) is found to be

$$(3.7)c \qquad \rho=\kappa(\xi,t) \qquad \xi=y+iz$$

where κ satisfies any equation of the type

$$\frac{\partial^2\kappa}{\partial\xi^2}=\kappa^2\varphi(\xi)$$

and

$$\tau=\frac{\frac{\partial\kappa}{\partial t}}{\kappa}\sqrt{\frac{3}{\lambda}}$$

Four solutions (3.4-3.7)c of the cosmological equations have thus been found in this case. Of these only the first is peculiar to (c), for the other three are (as are also those of (a) and (b)) special cases of solutions considered in detail at the end of this section.

d

The only remaining case is that in which all the logarithmic derivatives of ρ with respect to x, y and z are functions of t. We first dispose of the case in which one of the quantities a, b or c is zero, in order that we may then deal with the conditions of integrability in the form applicable in the case $abc\neq 0$.

If one of the three quantities a, b or c vanish the conditions (2.1) requires that the other two do likewise, for in the case we are now considering it is not possible to express $\frac{\rho_1}{\rho}$ etc. in terms of x, y and z alone. Similarly the conditions (2.2) demand that $l = m = n = 0$. Equations (1.4) can now be integrated immediately, yielding

$$(3.0)\,d \qquad \rho = d(t)(x^2+y^2+z^2) + d_1(t)\,x + d_2(t)\,y + d_3(t)\,z + d_4(t)$$

$$\Upsilon = \frac{\rho_4}{\rho} \cdot \frac{1}{f(t)}$$

where the d's are arbitrary functions of t, and

$$f^2(t) = \frac{\lambda}{3} - D^2(t) \qquad D^2(t) = d_1^2 + d_2^2 + d_3^2 - 4 d d_4$$

Having disposed of the case $abc = 0$ we may take as our conditions of integrability sets (2.1), (2.21), (2.3), (2.4) and (2.5). Since a, l etc. cannot depend on t, (2.21) and (2.4) can only be satisfied if all the coefficients A, A'; B, B' vanish.

First $A_1' = A_2' = A_3' = 0$ so

$$(3.1)\,d \qquad l = \frac{a}{bc}(c^2 - b^2) \quad m = \frac{b}{ca}(a^2 - c^2) \quad n = \frac{c}{ab}(b^2 - a^2)$$

This makes C" in (2.5) vanish, so the remaining coefficients C and C' must also.

If we now consider those conditions on the functions a, b and c which are of first order in their derivatives, we find that they must satisfy 12 first order differential equations

$$A_1 = A_2 = A_3 = 0$$
$$B_1' = B_2' = B_3' = 0$$
$$C_1' = C_2' = C_3' = 0$$
$$\text{eqs} \quad (2.3)$$

However, of these but 6 are linearly independent. For these 6 we may take two of the first set, all of the second and one of the last; they are

(3.2)d
$$\frac{b_1}{b} + \frac{1}{2}\frac{b_2}{a} = \frac{c_1}{c} + \frac{1}{2}\frac{c_2}{a}$$
$$\frac{c_2}{c} + \frac{1}{2}\frac{c_3}{b} = \frac{a_2}{a} + \frac{1}{2}\frac{a_3}{b}$$
$$\frac{a_3}{a} + \frac{1}{2}\frac{a_1}{c} = \frac{b_3}{b} + \frac{1}{2}\frac{b_1}{c}$$
$$-a_1(c^2 - b^2) + b_2 c^2 - c_3 b^2 + bc(l_1 - c_2 + b_3) = 0$$
$$-b_2(a^2 - c^2) + c_3 a^2 - a_1 c^2 + ca(m_2 - a_3 + c_1) = 0$$
$$c_2 + b_3 = \frac{1}{3}(m_1 - n_1)$$

where l, m and n are given by (3.1)d. It can be shown that the conditions of complete integrability are not satisfied for this set of 6 equations in 3 dependent and 3 independent variables, so in order to find a, b and c we next consider the conditions on them which are of second order in their derivatives.

First, since $B_1 = B_2 = B_3 = 0$ we have

$$(a_1 - b_2)_3 + (a_1 - b_2)\left(\frac{a_1}{2c} - \frac{c_3}{2c} - \frac{b_3}{b}\right) = 0 \quad \text{etc}$$

Now the equations (3.2)d may be solved for all other derivatives in terms of those with respect to z; this being done it is easily shown that

$$\frac{a_1 - c_3}{2c} - \frac{b_3}{b} = -\frac{5}{6}\frac{N_3}{N} - \frac{4}{3}\frac{a_3}{a} - \frac{4}{3}\frac{b_3}{b} - \frac{1}{3}\frac{c_3}{c}$$
$$= -\frac{\partial}{\partial z}\left\{\log N^{5/6} a^{4/3} b^{4/3} c^{1/3}\right\}$$

where

$$N = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

The equations above may now be integrated once, yielding

$$(3.21)\,d \qquad a_1 - b_2 = a^{4/3} b^{4/3} c^{1/3} N^{5/6} \gamma(x\,y)$$

Similarly

$$b_2 - c_3 = b^{4/3} c^{4/3} a^{1/3} N^{5/6} \alpha(y,z)$$

$$c_3 - a_1 = c^{4/3} a^{4/3} b^{1/3} N^{5/6} \beta(z,x)$$

Note that the arbitrary functions α, β and γ are simply connected with a, b and c by the relation

$$\frac{\alpha(y,z)}{a} + \frac{\beta(z\,x)}{b} + \frac{\gamma(x\,y)}{c} = 0$$

Again

$$\frac{c_3 - a_1}{2c} = \frac{5}{6}\frac{N_3}{N} + \frac{4}{3}\frac{a_3}{a} + \frac{1}{3}\frac{b_3}{b} + \frac{1}{3}\frac{c_3}{c} = a^{4/3} b^{1/3} c^{1/3} \frac{\beta}{2}$$

whence

$$(3.22)\,d \qquad \frac{\partial}{\partial z}\left\{\frac{1}{a^{4/3} b^{1/3} c^{1/3} N^{5/6}}\right\} = -\frac{\beta(x,z)}{2}$$

Likewise

$$\frac{\partial}{\partial y}\left\{\frac{1}{a^{4/3} b^{1/3} c^{1/3} N^{5/6}}\right\} = +\frac{\gamma(x,y)}{2}$$

Four other expressions may be obtained from these two by the cyclic permutation (abc)($\alpha\beta\gamma$)(xyz).

In order to determine the so far arbitrary functions $\alpha(y\,z)$, $\beta(z,x)$ and $\gamma(x,y)$ we treat the second set of conditions in an analogous manner; they may be written

$$(3.23)\,d \qquad \frac{1}{b}(a_1 - b_2)\frac{\partial}{\partial y}\left\{\log\left[(a_1 - b_2) N^{-5/6} a^{-4/3} b^{-4/3} c^{-1/3}\right]\right\}$$
$$- \frac{1}{c}(a_1 - c_3)\frac{\partial}{\partial z}\left\{\log\left[(a_1 - c_3) N^{-5/6} a^{-4/3} b^{-1/3} c^{-4/3}\right]\right\} = 0$$

etc.

Using the values of $a_1 - b_2$ etc. given by (3.21)d

$$\frac{\partial \gamma(x,y)}{\partial y} + \frac{\partial \beta(z,x)}{\partial z} = 0 \tag{3.23)d}$$

Correspondingly

$$\frac{\partial \alpha(y,z)}{\partial z} + \frac{\partial \gamma(x,y)}{\partial x} = 0$$

$$\frac{\partial \beta(z,x)}{\partial x} + \frac{\partial \alpha(y,z)}{\partial y} = 0$$

The solutions of these equations are readily found;

$$\begin{aligned} \alpha &= Cy - Bz + A_4 \\ \beta &= Az - Cx + B_4 \\ \gamma &= Bx - Ay + C_4 \end{aligned} \tag{3.24)d}$$

where A, A_4 etc. are arbitrary constants.

Returning now to equations (3.22)d we find with the aid of the above values of α, β and γ

$$\frac{2}{a^{4/3} b^{1/3} c^{1/3} N^{5/6}} = -\frac{A}{2}(y^2+z^2) + x(By+Cz) + C_4 y - B_4 z + X(x) \tag{3.25)d}$$

where X is an arbitrary function of x. Two similar expressions are obtained by the permutation (XYZ)(abc) etc. (3.21)d requires, as we have mentioned

$$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0$$

Applying this condition to equations (3.25)d we find, on equating coefficients of the independent variables, that

$$X = \frac{A}{2}x^2 + \frac{\epsilon^2}{8}A \qquad \epsilon \text{ const}$$

etc. Also

$$AA_4 + BB_4 + CC_4 = 0$$

Hence

$$\frac{1}{a} = \frac{(abc)^{1/3} N^{5/6}}{2}\left\{-\frac{A}{2}(-x^2+y^2+z^2) + x(By+Cz) + C_4 y - B_4 z + \frac{\epsilon^2}{8}A\right\}$$

or

$$(3.26)d \qquad a = \frac{\mu}{u} \qquad b = \frac{\mu}{v} \qquad c = \frac{\mu}{w}$$

where

$$\mu = 2(abc)^{-1/3} N^{-5/6} = \therefore\ 8uvw(u^2+v^2+w^2)^{-5/2}$$

$$u = -\frac{A}{2}(-x^2+y^2+z^2) + x(By+Cz) + C_4 y - B_4 z + \frac{\epsilon^2}{8}A$$

$$AA_4 + BB_4 + CC_4 = 0$$

These values of a, b and c satisfy the first order equations (3.2)d as well as those of second order, and represent the general solution of the conditions of integrability (except for the case $\alpha = \beta = \gamma = 0$ to be considered later) insofar as the conditions do not involve ρ.

We must now return to set (2.1) and substitute the known values of a, b and c. Then

$$(2.1)' \qquad \begin{aligned} v\rho_1 - u\rho_2 + \gamma\rho &= 0 \\ w\rho_2 - v\rho_3 + \alpha\rho &= 0 \\ v\rho_3 - w\rho_1 + \beta\rho &= 0 \end{aligned}$$

In order to effect the integration of these equations we first throw them into vector form

$$\nabla \log \rho \times \bar{d} + \bar{e} = 0$$

where $\quad \bar{d} = (u\ v\ w) \qquad \bar{e} = (\alpha\ \beta\ \gamma)$

Now $\quad \bar{e} = \bar{\pi} \times \bar{A} + \bar{D}$ where

$$\bar{\pi} = (x\ y\ z) \qquad \bar{A} = (A\ B\ C) \qquad \bar{D} = (A_4 B_4 C_4) \qquad \bar{A}\cdot\bar{D} = 0$$

and

$$\bar{d} = -\frac{\bar{A}}{2}\pi^2 + \bar{\pi}\ \bar{A}\cdot\bar{\pi} + \bar{\pi}\times\bar{D} + \frac{\epsilon^2}{8}\bar{A}$$

If $\bar{A} \neq 0$, it is possible to make a transformation of coordinates from $\bar{\pi}$ to $\bar{\pi} + \bar{\chi}$ where $\bar{\chi}$ is defined by

$$\bar{D} = \bar{A} \times \bar{\chi} \qquad \bar{A} \cdot \bar{\chi} = 0$$

in this new system $\bar{e}$ and $\bar{\alpha}$ become

$$\bar{e} = \bar{r} \times \bar{A} \qquad \bar{\alpha} = -\frac{\bar{A}}{2} r^2 + \bar{r}\, \bar{A} \cdot \bar{r} + \frac{\epsilon^2}{8} \bar{A}$$

(2.1)' tells us that $\nabla \log \rho$ lies in the plane determined by $\bar{r}$ and $\bar{A}$ so the surfaces $\rho(x y z t) =$ const. (t fixed) are surfaces of revolution whose **axes** are in the direction A. Changing to polar coordinates $(r \theta \varphi)$ in which the direction $\theta = 0$ is that of $\bar{A}$, the equations (2.1)' are

$$\frac{\partial \log \rho}{\partial \varphi} = 0$$

$$\frac{\partial \log \rho}{\partial r}\left(r^2 - \frac{\epsilon^2}{4}\right) r + \frac{\partial \log \rho}{\partial \log \cos \theta}\left(r^2 + \frac{\epsilon^2}{4}\right) = 2 r^2$$

whence

$$\rho = r \cos \theta \, \mathrm{fn} \left\{ \frac{r^2 - \frac{\epsilon^2}{4}}{r \cos \theta} \right\} = z \, \mathrm{fn} \left\{ \frac{r^2 - \frac{\epsilon^2}{4}}{z} \right\}$$

If, then,

$$\rho = z K(\xi, t) \qquad \xi = \frac{r^2 - \frac{\epsilon^2}{4}}{z}$$

the conditions of integrability are satisfied. The equations (1.4) reduce to

$$\left(\frac{\partial K}{\partial \xi}\right)^2 (\xi^2 + \epsilon^2) - 2K \frac{\partial K}{\partial \xi} \xi + K^2 - \frac{128 C}{3} K^3 (\xi^2 + \epsilon^2)^{-3/2} = \frac{2}{3} - f^2(t)$$

which is of the same form as the equation obtained in section (c). If $\epsilon \neq 0$ it may be written

$$\left(\frac{\partial \frac{K}{\sqrt{\xi^2 + \epsilon^2}}}{\partial \tan^{-1} \frac{\xi}{\epsilon}} \right)^2 = \frac{128 C}{3 \epsilon^3} \left(\frac{K}{\sqrt{\xi^2 + \epsilon^2}} \right)^3 - \left(\frac{K}{\sqrt{\xi^2 + \epsilon^2}} \right)^2 + \frac{2}{3} - f^2(t)$$

whence

(3.27) d $$K = \sqrt{\xi^2 + \epsilon^2} \left\{ \frac{\epsilon^3}{128 C} + \wp\left(\sqrt{\frac{32 C}{\epsilon^3}} \tan^{-1} \frac{\xi}{\epsilon} + n(t) \right) \right\}$$

Another type of solution arises if $\epsilon = 0$, for then

$$(3.28)d \qquad \left(\frac{\partial \frac{\kappa}{t}}{\partial \frac{1}{t}} \right)^2 = \frac{128c}{3} \left(\frac{\kappa}{t} \right)^3 + \frac{\lambda}{3} - f^2(t)$$

κ is thus again expressible in terms of a $\wp$-function.

The above solutions were obtained on the assumption $\bar{A} \neq 0$, and are furthermore subject to the restrictions imposed in assuming that no relation exists between a, b and c which renders impossible the algebraic solution of equations (3.2)d as required on p. 20, and to the tacit assumption that α, β or $\gamma \neq 0$, for if this condition were not satisfied it would be impossible to evaluate the functions X, Y and Z as on p. 22. The examination of the cases arising from the relaxation of these conditions leads to 7 additional solutions, 3 of which can be reduced to those encountered in previous sections. (In reducing solutions arising under this case to those of the preceding parts we must of course waive the requirement that all the logarithmic derivatives of ρ with respect to x, y and z are really functions of t. There is, however, no objection to this procedure for we have divided the problem into these four parts merely for convenience in solving it, so if we can show that a solution arising under (d) is reducible to one of (b) or (c) no loss of generality is incurred by so doing.)

We first consider those cases in which the processes leading to equations (3.26)d are justifiable.

If $\bar{A} = 0$ the set (2.1)' may, on shift of origin, be reduced to

$$\nabla \log \rho \times (\bar{\pi} \times \bar{D}) + \bar{D} = 0$$

or

$$\bar{\pi}\,(\nabla \log \rho \cdot \bar{D}) - \bar{D}\,(\bar{\pi} \cdot \nabla \log \rho) + \bar{D} = 0$$

Hence ρ must satisfy

$$\nabla \rho \cdot \bar{D} = 0$$
$$\nabla \rho \cdot \bar{\pi} = \rho$$

Choosing our axes so that $\bar{D} = (A_4, 0, 0)$ the above become

$$A_4 \frac{\partial \rho}{\partial x} = 0$$
$$x \frac{\partial \rho}{\partial x} + y \frac{\partial \rho}{\partial y} + z \frac{\partial \rho}{\partial z} = \rho$$

i.e. ρ is a function of y, z and t which is homogeneous of first degree in y and z; this leads to solution (3.4)c. In this case we have tacitly assumed that $\bar{D}^{\sim} \neq 0$, as well as $\bar{D} \neq 0$ (in this latter case $\alpha\beta\sigma$ would be zero, so we do not entertain this possibility at present.) Similarly we assumed on p. 24 that not only $\bar{A} \neq 0$ but also that $\bar{A}^{\sim} \neq 0$. As long as we are dealing with real vectors these restrictions, i.e. $\bar{D}^{\sim} \neq 0$, $\bar{A}^{\sim} \neq 0$ do not concern us, but if we wish to obtain the complete solution of the problem we must consider the imaginary and complex solutions as well. The value of this completeness will be evident in the next section, in which the relations between our solutions will be shown.

Briefly, if $\bar{A}^{\sim} = 0$ but $\bar{A} \neq 0$ we may take $\bar{A} = (0, B, iB)$. The integration of (2.1) and the subsequent limitation imposed by (1.4) are not as neatly effected here as in the preceding case, but as the reasoning is essentially the same we merely state the result: To this possibility there corresponds the solution

$$(3.3)d \qquad \rho = (y+iz)\,\kappa(\xi, t) \qquad \xi = \frac{r^2 - \frac{\epsilon^2}{4}}{y+iz} \qquad \epsilon \neq 0$$

where

$$\epsilon^2 \kappa'^2 = 4c^2 \kappa^3 + \frac{\lambda}{3} - f^2(t) \qquad \kappa' = \frac{\partial \kappa}{\partial \xi}$$

Again, if $\bar{A} = 0$ and $\bar{D}^2 = 0$ but $\bar{D} \neq 0$ the problem cannot be reduced to that of section (c) by rotation of coordinates as on p. 26, for $\bar{D}$ may only be reduced to the form $(0, B_4, iB_4)$ The solution of (2.1)' is then found to be

$$(3.4)d \qquad \rho = x\,\kappa(\xi, t) \qquad \xi = \frac{y+iz}{x}$$

and the cosmological equations require that κ satisfy

$$\xi^2 \kappa'^2 - 2\xi \kappa \kappa' + \kappa^2 - 4c^2 \frac{\kappa^3}{\xi^3} = \frac{\lambda}{3} - f^2(t)$$

Next we concern ourselves with the case $\alpha\beta\gamma = 0$. X, Y and Z in (3.25)d may no longer be obtained by the device employed above, but we may start with

$$(3.25)'d \qquad \begin{aligned} a &= X(x)\,\psi(xyz) \\ b &= Y(y)\,\psi(xyz) \\ c &= Z(z)\,\psi(xyz) \end{aligned}$$

where X, Y, Z and ψ are to be determined by (3.2)d. It can be shown that the first five of these equations require either (1) X, Y and Z be constant or (2)

$$X = \frac{1}{x} \qquad Y = \frac{1}{y} \qquad Z = \frac{1}{z}$$

To these cases correspond the following solutions of the fundamental equations:

(1)[2] Here it may be shown that rotation of the coordinate system reduces ρ to a function of at most two of the variables x, y and z, and that the solutions then obtained are (3.2)b, (3.6)c and (3.7)c.

(2) In the second case a new solution is obtained. From (3.2)d we find that

$$\psi(xyz) = C\,\frac{xyz}{r^5}$$

whence by (1.4)

$$(3.5)d \qquad \rho = \kappa(r,t)$$

where

$$\kappa'^2 - 2\frac{\kappa}{r}\kappa' - 4c^2\frac{\kappa^3}{r^3} = \frac{a}{3} - f^2(t)$$

We now entertain the last possibility, i.e. that there exists between a, b and c a relation which makes equations (3.2)d linearly independent no longer. Examination shows that if $N = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 0$ we are not justified in concluding (3.21)d. The solution of equations (1.4) incident to this case is of the type

$$(3.6)d \qquad \rho = (y+iz)\,\kappa(\xi,t) \qquad \xi = \frac{r^2}{y+iz}$$

where κ satisfies any equation of the form

$$\frac{\partial^2 \kappa(\xi,t)}{\partial \xi^2} = \kappa^2(\xi,t)\,\varphi(\xi)$$

We have now obtained all solutions of equations (1.4). This has been accomplished by breaking up the problem into four parts and solving the equations under the restrictions imposed in the several sections. It is found, however, that there is to some extent duplication of solutions. In order to avoid this repitition, and to collect our results, we here arrange our 14 solutions into 3 classes, consisting altogether of 10 sub-groups. In collecting we have further simplified the solutions by choosing coordinates from the group of linear orhtogonal transformations which allow us to express ρ in the least number of variables. For all these solutions

$$T = \frac{\frac{\partial\rho}{\partial t}}{\rho}\cdot\frac{1}{\sqrt{\frac{\lambda}{3} - D(t)}}$$

(1) $$\rho = d(t)r^2 + d_1(t)x + d_2(t)y + d_3(t)z + d_4(t)$$

where the d's are arbitrary functions of t and

$$D(t) = d_1^2 + d_2^2 + d_3^2 - 4dd_4$$

(2) a $$\rho = \varrho(\xi, t) \qquad \xi = y + iz$$

satisfying <u>any</u> equation of the type

$$\frac{\partial^2\varrho(\xi,t)}{\partial\xi^2} = \rho^2(\xi,t)\,\varphi(\xi)$$

b $$\rho = (y+iz)\,K(\xi,t) \qquad \xi = \frac{r^2}{y+iz}$$

where $K(\xi,t)$ satisfies the equation in (a) above.

In both of these cases $D(t) = 0$.

(3) a $\qquad \rho = \rho(r, t)$

satisfying

$$\left(\frac{\partial\rho}{\partial r}\right)^2 - 2\frac{\rho}{r}\frac{\partial\rho}{\partial r} - 4C^2\frac{\rho^3}{r^3} = D(t)$$

b $\qquad \rho = z\kappa(\xi, t) \qquad \xi = \frac{y}{z}$

where

$$\left(\frac{\partial\kappa}{\partial\xi}\right)^2(\xi^2+1) - 2\xi\kappa\frac{\partial\kappa}{\partial\xi} + \kappa^2 - 4C^2\frac{\kappa^3}{(\xi^2+1)^{3/2}} = D(t)$$

c $\qquad \rho = z\kappa(\xi, t) \qquad \xi = \frac{r^2-\epsilon^2}{z}$

where

$$\left(\frac{\partial\kappa}{\partial\xi}\right)^2(\xi^2+4\epsilon^2) - 2\xi\kappa\frac{\partial\kappa}{\partial\xi} + \kappa^2 - 4C^2\frac{\kappa^3}{(\xi^2+4\epsilon^2)^{3/2}} = D(t)$$

d This solution is really a special case of (c) above, arising on putting $\epsilon = 0$, but the auxilliary equation for κ is integrated by means of a different substitution (cf. (3.28)d as opposed to (3.27)d).

$$\rho = z\kappa(\xi, t) \qquad \xi = \frac{r^2}{z}$$

where κ satisfies

$$\left(\frac{\partial\kappa}{\partial\xi}\right)^2\xi^2 - 2\xi\kappa\frac{\partial\kappa}{\partial\xi} + \kappa^2 - 4C^2\frac{\kappa^3}{\xi^3} = D(t)$$

e $\qquad \rho = x\kappa(\xi, t) \qquad \xi = \frac{y+iz}{x}$

satisfying the same equation as in (d) above.

f $\qquad \rho = (y+iz)\kappa(\xi, t) \qquad \xi = \frac{r^2-\epsilon^2}{y+iz} \qquad \epsilon \neq 0$

in which

$$4\epsilon^2\left(\frac{\partial\kappa}{\partial\xi}\right)^2 = 4C^2\kappa^3 + D(t)$$

g $\qquad \rho = \rho(x, t)$

where

$$\left(\frac{\partial\rho}{\partial x}\right)^2 = 4C^2\rho^3 + D(t)$$

Of these solutions only (1) has been expressed in integral form. The seven cases arising under (3) have been defined by first order, second degree ordinary differential equations, all of which may be thrown into the form

$$\left(\frac{d\chi}{d\eta}\right)^2 = 4C^2\chi^3 + \delta\chi^2 + D(t) \qquad \delta = \pm 1 \text{ or } 0$$

by one or another of the substitutions suggested. The solution of this equation is readily obtained in terms of a Weierstrass' $\wp$-function whose argument is $C\eta$, but the interpretation to be given in the succeeding section makes it desirable to retain the differential form at present.

Only two of the ten solutions, those of case (2), cannot be generally expressed in integral form. To obtain a solution of this type we must first assign to $\varphi(\xi)$ a definite form, integrate the resulting ordinary differential equation in ξ, and then replace the two constants of integration by arbitrary functions of t.

IV Interpretation.

Now that we have solved the problem proposed at the beginning of this thesis, we turn to the interpretation of the solutions thus obtained. This may be done along either geometrical or physical lines; for this purpose we first set out the appropriate tools and then proceed to the consideration of the solutions themselves.

In all cases $\tau = \dfrac{e_4}{e\,f(t)}$ and the geodesics (0.3) are given by

$$\ddot{x} + \frac{\rho_1}{\rho}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \left(\frac{\rho_{14}}{\rho_4} - \frac{\rho_1}{\rho}\right)\frac{\rho_4^2\,\dot{t}^2}{f^2(t)} - 2\dot{x}\,\frac{\dot{\rho}}{\rho} = 0$$

with two analogous equations for y and z, and

$$\frac{\rho_4^2}{f^2(t)}\ddot{t} + \frac{\rho_4}{\rho}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \left(\frac{\rho_{44}}{\rho_4} - \frac{\rho_4}{\rho} - \frac{f'(t)}{f(t)}\right)\frac{\rho_4^2}{f^2(t)}\dot{t}^2 + 2\frac{\rho_4^2}{f^2(t)}\left(\frac{\dot{\rho}_4}{\rho_4} - \frac{\dot{\rho}}{\rho} - \frac{f'(t)}{f(t)}\dot{t}\right)\dot{t} = 0$$

where the dot indicates differentiation along the curve. One integral of these equations is, of course,

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \frac{\rho_4^2}{f^2(t)} = \rho^2$$

so for our equations we may take this relation together with any three of the four

$$\left(\frac{\dot{x}}{\rho^2}\right)^{\cdot} + \frac{\rho_1}{\rho} - \frac{\rho_{14}\,\rho_4}{\rho^2 f^2(t)}\,\dot{t}^2 = 0$$

$$\left(\frac{\dot{y}}{\rho^2}\right)^{\cdot} + \frac{\rho_2}{\rho} - \frac{\rho_{24}\,\rho_4}{\rho^2 f^2(t)}\,\dot{t}^2 = 0$$

(4.01)
$$\left(\frac{\dot{z}}{\rho^2}\right)^{\cdot} + \frac{\rho_3}{\rho} - \frac{\rho_{34}\,\rho_4}{\rho^2 f^2(t)}\,\dot{t}^2 = 0$$

$$\left(\frac{\rho_4^2\,\dot{t}}{\rho^2 f^2(t)}\right)^{\cdot} + \frac{\rho_4}{\rho} - \frac{\rho_{44}\,\rho_4}{\rho^2 f^2(t)}\,\dot{t}^2 + \frac{\rho_4^2 f'(t)}{\rho^2 f^3(t)}\,\dot{t}^2 = 0$$

From these equations it is seen that if ρ does not contain one of the variables x, y or z, say x, then a particle which is at any time moving in a surface x = const. always moves in that surface.

Next we produce the components of the covariant Riemann-Christoffel tensor $B_{\mu\nu\sigma\rho}$. They are

$$B_{2323} = \frac{1}{\rho^4}\left\{(\rho_{22}+\rho_{33})\rho - (\rho_1^2+\rho_2^2+\rho_3^2+f^2(t))\right\}$$

$$B_{1414} = -\frac{\rho_4}{\rho^3}\left(\frac{\rho_4}{\rho} - \frac{f'(t)}{f(t)}\right) - \frac{\rho_4^2}{\rho^2 f^2(t)}\left(\frac{\rho_{114}}{\rho_4} - \frac{\rho_{11}}{\rho}\right)$$

(4.02)
$$- \frac{\rho_4^2}{\rho^4 f^2(t)}(\rho_1^2+\rho_2^2+\rho_3^2) + \frac{\rho_4}{2\rho^3 f^2(t)}(\rho_1^2+\rho_2^2+\rho_3^2)_4$$

$$B_{1312} = \frac{\rho_{23}}{\rho^3} \qquad B_{2434} = -\frac{\rho_4^2}{\rho^3 f^2(t)}\,\rho_{23}$$

$$B_{1412} = B_{3432} = 0$$

The remaining 8 non-vanishing components may be obtained from the first four above by the cyclic permutation (123).

In the previous section we used freely any linear transformation of coordinates x, y and z which left the euclidean 3-space element $d\sigma^2 = dx^2 + dy^2 + dz^2$ invariant, i.e. any such that

$$d\sigma^2 = d\Sigma^2 \qquad d\Sigma^2 = dX^2 + dY^2 + dZ^2$$

But we may use a much more general class of transformations which sends our line element ds (1.1) into one of the same form. In particular, this is accomplished if

$$d\sigma^2 = \chi^2(XYZ)\, d\Sigma^2$$

for ds^2 is then transformed into

$$dS^2 = \frac{d\Sigma^2}{P^2} + T^2 dt^2$$

In fact

$$P(XYZt) = \frac{1}{\chi(XYZ)}\, \rho(x(XYZ), y(XYZ), z(XYZ), t)$$

That T(XYZt) is of the correct form, i.e.

$$T(XYZt) = \frac{\partial P(XYZt)}{\partial t} \cdot \frac{1}{P(XYZt)} \cdot \frac{1}{F(t)}$$

is a consequence of (4.03).

We consider the class of transformations due to Kelvin (cf. Bateman "Electrical and Optical Wave Motion" p.31)

$$x = \frac{aX}{Y+iZ} \qquad y = \frac{R^2 - a^2}{2(Y+iZ)} \qquad z = \frac{R^2 + a^2}{2i(Y+iZ)}$$

whence

$$d\sigma^2 = \frac{a^2}{(Y+iZ)^2}\, d\Sigma^2$$

Inversion, for which

$$x = \frac{a^2 X}{R^2} \qquad y = \frac{a^2 Y}{R^2} \qquad z = \frac{a^2 Z}{R^2} \qquad \chi = \frac{a^2}{R^2}$$

may be obtained by the double application of a transformation of the previous form.

Hence we may state the important theorem: If $\rho(xyzt)$ is a solution of the cosmological equations (1.4), then

$$(4.04) \qquad P(XYZt) = \frac{Y+iZ}{a}\,\rho\left(\frac{aX}{Y+iZ}, \frac{R^2-a^2}{2(Y+iZ)}, \frac{R^2+a^2}{2i(Y+iZ)}, t\right)$$

is also. Similarly, inversion leads to a solution

$$(4.041) \qquad P(XYZt) = \frac{R^2}{a^2}\,\rho\left(\frac{a^2X}{R^2}, \frac{a^2Y}{R^2}, \frac{a^2Z}{R^2}, t\right)$$

We now proceed to the consideration of the various types of solutions.

1

If we substitute the values of ρ and τ given by solution (1) p.29 into the components of the Riemann-Christoffel tensor $B_{\mu\nu\sigma\rho}$, we find that the manifold which they define is a hypersphere of Riemann curvature $-\frac{\lambda}{3}$. This solution contains as a special case that of E. Kasner, in which $\rho\tau = 1$. (American Journal of Mathematics, Vol. XLIII (1922)).

An interesting theorem may be deduced from the results of part (a) section III above. Here τ is, as we have mentioned a function of t alone, and a slight generalization of the work of that section allows us to state: If "stationary" observers (i.e. observers at a point x, y, z constant) in an orthogonal dynamical space-time find that their fundamental interval is independent of their position in space, and that the velocity of light is isotropic at a point, then space-time is a hypersphere. If the cosmological constant λ is zero, then a space time of this type is Galilean.

If we transform this solution by (4.04) above we find

$$P(XYZ\,t) = D(t)R^2 + D_1(t)X + D_2(t)Y + D_3(t)Z + D_4(t)$$

where

$$d_1^2 + d_2^2 + d_3^2 - 4dd_4 = D_1^2 + D_2^2 + D_3^2 - 4DD_4$$

Hence all transformations of type (4.04) or (4.05) send this solution into itself. Further, this is the only solution of our problem for which all the components of what we have called the Riemann curvature are finite.

2

We do not believe that the two solutions (2) p. 29 can have any physical significance, for they are essentially complex. Again, if the cosmological constant λ is zero the line element ds degenerates.

In order to illustrate the method of applying the transformation (4.04) for this and the succeeding section, we here carry it out for the solution (2)b. Here

$$\rho = (y+iz)\,\kappa\left(\frac{r^2}{y+iz}, t\right) \qquad \frac{\partial^2 \kappa}{\partial \xi^2} = \kappa^2 \varphi(\xi) \qquad \xi = \frac{r^2}{y+iz}$$

so there must exist a solution

$$P = \frac{R^2}{a}\,\kappa\left(-a^2\,\frac{Y-iZ}{R^2}, t\right)$$

Writing this $\quad P = (Y-iZ)\,\Pi(\Xi, t) \qquad \Xi = \frac{R^2}{Y-iZ}$

$$\kappa\left(-\frac{a^2}{\Xi}, t\right) = \frac{a}{\Xi}\,\Pi(\Xi, t)$$

and

$$\frac{\partial^2 K(\xi,t)}{\partial \xi^2} = \frac{\Xi^3}{a^3}\,\frac{\partial^2 \bar{K}(\Xi,t)}{\partial \Xi^2}$$

Hence $\bar{K}(\Xi,t)$ satisfies

$$\frac{\partial^2 \bar{K}}{\partial \Xi^2} = \bar{K}^2\,\varphi\left(-\frac{a^2}{\Xi}\right)\,\frac{a^5}{\Xi^5}$$

But this equation is of the form

$$\frac{\partial^2 \bar{K}}{\partial \Xi^2} = \bar{K}^2\,\Phi(\Xi)$$

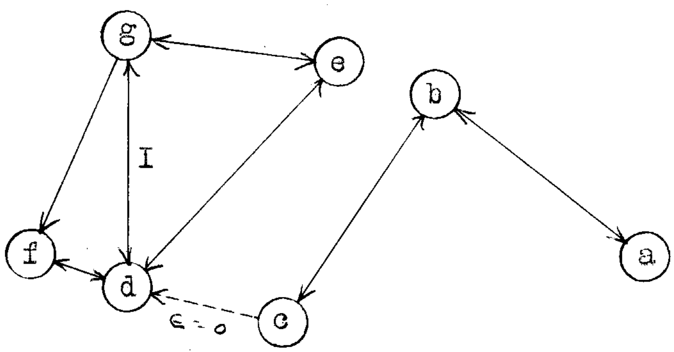
whence (2)b goes into itself on applying (4.04). On performing various transformations of Kelvin's type on (2)a and (2)b we find they go into themselves or into each other, but not into a solution of any other type.

3

The seven solutions of group (3) offer the most interesting possibilities. We first examine the relations existing between the various solutions (a) - (g).

As already mentioned, solution (d) may be derived from (c) by letting $e \to 0$. We have applied to all these solutions the transformations (4.04) and (4.041), and have found that in each case we get a solution which has already been placed in this group. Further, the solutions naturally divide up into two groups (a), (b), (c) and (d), (e), (f), (g) such that starting with any solution in either group, the remaining solutions of that group may be obtained from it by means of (4.04). These relations are graphically shown on the accompanying diagram: here a plain arrow $(i) \longrightarrow (j)$ means that i gives j on transforming by (4.04), and $(i) \xrightarrow{I} (j)$ that j may be obtained from i on inversion (4.041).

All solutions, with the exception of (d) and (g), go into themselves on inversion.



In order to discuss this group it is only necessary to consider (c), keeping in mind the possibility $\epsilon \to 0$. It is, however, more convenient to discuss two separate solutions, one from each of the sub-groups previously mentioned. In particular (a) and (g) are chosen, for they are of comparatively simple form. (a) represents a spherically symmetric dynamical solution, ρ being expressible in terms of a $\wp$-function of r and t alone. This being the case, there exist at any time an infinite number of spheres each point of which is a singular point, and these spheres are not stationary in time. Similarly with (g), in which ρ is a $\wp$-function of x and t, there exist at any time an infinity of surfaces of singular points. We do not see at present any physical significance in these solutions.

In conclusion, then, we have found that there exist but three distinct orthogonal dynamical 4-spaces which contain a conformal euclidean 3-space and satisfy the cosmological equations. The first of these is a hypersphere and another is determined by functions whose arguments are essentially complex. The only case in which we may hope to find a physical content is seen to represent fields of a so far unrecognized type.

I desire to here express my appreciation of the interest which Dr. H. Bateman and Dr. Paul Epstein of the Institute faculty have taken in this problem, and especially for the constructive criticism which Dr. Bateman has offered during the course of the work. These acknowledgements would be incomplete did I not mention Dr. E. T. Bell of the University of Washington, who first interested me in the Theory of Relativity.