

A NEW APPROACH TO THE ANALYSIS OF
LARGE DEFLECTIONS OF PLATES

Thesis by
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To my wife

Barbara

whose patience and understanding
made this possible

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ABSTRACT

As a result of the assumption that the strain energy due to the second invariant of the middle surface strains can be neglected when deriving the differential equations for a flat plate with large deflections, simplified equations are derived that can be solved readily. Computations using the solution of these simplified equations are carried out for the deflection of uniformly loaded circular and rectangular plates with various boundary conditions. Comparisons are made with available numerical solutions of the exact equations. The deflections found by this approach are then used to obtain the stresses, and the resulting stresses are compared with existing solutions. In all the cases where comparisons could be made, the deflections and stresses agree with the exact solutions within the accuracy required for engineering purposes.

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I. INTRODUCTION AND SUMMARY

If the deflection of a thin plate is small compared to the thickness, the displacements in the plane of the plate can be neglected. In that case the deflection is determined by a fourth-order linear partial differential equation which is referred to as the plate equation. Many solutions to this equation for various loadings and boundary conditions have been worked out. Indeed, the majority of the literature dealing with deflections of plates is based on solutions of the plate equation.

When the deflection becomes very large compared to the thickness of the plate, three non-linear equations are used to find the deflection and the two displacements. However, in this case the bending rigidity of the plate can be neglected, thereby reducing the highest order equation to a second order non-linear differential equation. In this case, the governing equations are referred to as the membrane equations. Fairly complete solutions of these equations have been computed for a few special cases of loading and boundary conditions. However, the literature available on solutions of these equations is considerably smaller than that available on solutions of the plate equations.

If the deflection is of the order of magnitude of the thickness of the plate, the governing equations when written in terms of the deflection and the displacements are one fourth order and two second order non-linear partial differential equations. These equations are referred to as the equations for plates with large deflections. The three equations are coupled together, and solutions of them

are very difficult to obtain. Several techniques have been used to solve them. For example, S. Levy (Refs. 1, 2, and 3) substitutes a double Fourier series solution into the equations for rectangular plates and evaluates the coefficients. Chi-Teh Wang (Refs. 4 and 5) writes the equations for rectangular plates in a finite difference form and solves them by the method of successive approximations. S. Way solves the circular plate equations by substitution of a power series into the differential equations (Ref. 6), and he solves the rectangular plate equations by substitution of a finite power series solution into the energy expression, determining the coefficients by setting the first variation of the strain energy equal to the first variation of the potential energy due to the external loading for any variation of each of the coefficients (Ref. 7).

The main problem studied in this thesis is the determination of the deflection of plates when that deflection is of the order of magnitude of the thickness. The purpose of the present investigation is to develop a simple and yet sufficiently accurate method for solving this problem. This is difficult because the non-linearity of the mathematical problem must be considered if the solution is to resemble reality. Except for those few problems where exact solutions can be found in closed form the previous approaches to this problem have been those that are outlined above. The approach used in the following analysis is to investigate the effect of an approximation to the differential equations, still retaining their essential non-linearity.

The approximation used to find deflections arises from neglecting the strain energy due to the second invariant of the strains

in the middle surface of the plate when deriving the differential equations by energy methods. The resulting differential equations are still non-linear, but they can be decoupled in such a manner that they may be solved readily. The assumption that the portion of the strain energy that is neglected will have little effect on the solution for the deflection is based on physical consideration of known solutions of the exact equations.

It can be reasoned that, in general, this approach will give a good approximation to the deflection in the case of symmetrical loadings. In many problems one need find only the deflection. For example, the aerodynamicist is interested only in deflections when computing the interaction between non-rigid structural elements and aerodynamic forces. However, in most problems a knowledge of the stresses is needed. If the deflection is small, the membrane stresses are very small, and the total stresses can be approximated quite well by the bending stresses alone. This is true even for deflections that are too large to be approximated by a solution of the linear plate equations. Since the bending stresses are computed solely from the bending deflection, a knowledge of the deflection in this case is sufficient for a complete solution of the problem.

As the deflection increases, the membrane stresses become an increasingly greater part of the total stress. Consequently, to solve the problem of large deflections of plates completely, an estimate of the membrane stresses must be made. This can be done by assuming that the deflection is equal to that given by the solution

to the approximate equations and substituting this deflection into the strain energy integral. The strain energy is then a function of the displacements only, and by the principle of virtual displacements differential equations can be derived for these displacements. These differential equations are linear and can be solved readily. Knowing these displacements as well as the deflection enables one to compute the membrane stresses in the plate.

The first part of this thesis is devoted to the derivation of the approximate differential equations that are used to find the deflection. These equations are derived in the manner previously discussed.

The second part gives solutions for circular plates with uniform loading. Clamped, simply supported, and elastically built-in edges are considered. The deflections for these cases are compared with available solutions of the exact equations, and the deflection is shown to approach the exact solution as the deflection decreases.

In the third part of this thesis, deflections are found for rectangular plates with all edges simply supported and with two edges simply supported and two edges clamped. Aspect ratios of 1, 1.5, 2, and infinity are considered. The agreement with results obtained by considering the complete equations is shown to be increasingly better both as the deflection decreases and as the aspect ratio increases.

The fourth part of this thesis outlines the technique for determining stresses once the deflection is given. Calculations are carried out for the clamped circular plate and the errors are shown

to be greater than those for the deflection when given as functions of the loading, but are very small when given as functions of the maximum deflection.

The comparison of the results with available exact solutions shows good agreement for all cases that were considered. The amount of computational effort necessary to find deflections and stresses is considerably less than that required by other methods that have been used in the past. However, no computations were carried out for non-uniform loadings or for oddly shaped plates. One should exercise extreme caution in using the techniques of this investigation for the solution of any plate problems of this nature without first substantiating that the approximations made herein are applicable to the case in question.

II. NOTATION

A, B, C	Constants
$A_n, B_n, C_{kmn}, D_k, E_k$	Constants
D	Bending rigidity $= \frac{Eh^3}{12(1-\nu^2)}$
E	Modulus of elasticity
F(x, y)	Stress function defined in (6.37)
G	Shear modulus (Lamé's Constant) $= \frac{E}{2(1+\nu)}$
$G_k(y)$	Function defined in (6.47)
$H_k(y)$	Function defined in (6.48)
J_n	Bessel function of the first kind of nth order
M_r	Bending moment per unit length along circumferential sections of the plate
R	Radius of the circular plate
U	Strain energy
V	Potential energy
W	Applied work
Y_n	Bessel function of the second kind of nth order
a	Half-width of the rectangular plate
b	Half-length of the rectangular plate
\bar{e}	First strain invariant in three dimensions $= \bar{\epsilon}_x + \bar{\epsilon}_y + \bar{\epsilon}_z$
e	First invariant of middle surface strains $= \epsilon_x + \epsilon_y \quad \text{in rectangular coordinates}$ $= \epsilon_r + \epsilon_\theta \quad \text{in cylindrical coordinates}$

\bar{e}_2	Second strain invariant in three dimensions $= \bar{\epsilon}_x \bar{\epsilon}_y + \bar{\epsilon}_y \bar{\epsilon}_z + \bar{\epsilon}_x \bar{\epsilon}_z - \frac{1}{4} [\bar{\gamma}_{xy}^2 + \bar{\gamma}_{yz}^2 + \bar{\gamma}_{xz}^2]$
e_2	Second invariant of middle surface strains $= \epsilon_x \epsilon_y - \frac{1}{4} \gamma_{xy}^2 \quad \text{in rectangular coordinates}$ $= \epsilon_r \epsilon_\theta \quad \text{in cylindrical coordinates when there is circular symmetry}$
$f_n(y)$	Function defined in (5.1)
$g_k(y)$	Function defined in (5.12)
h	Thickness of the plate
k, m, n	Summation variables
$\ell_k(y)$	Function defined in (5.13)
q	Intensity of the uniform load
r, θ, z	Cylindrical coordinates
u, v	Displacements in the plane of the plate (extensions)
w	Deflection of the plate in the z-direction
x, y, z	Rectangular coordinates
$\theta_k(y)$	Function defined in (6.52)
$\Phi_k(y)$	Function defined in (6.50)
α	Constant of integration
β_n	$= \frac{2n+1}{2\alpha} \pi$
$\bar{\gamma}_{xy}, \bar{\gamma}_{yz}, \bar{\gamma}_{xz}$	Three dimensional shearing strains in rectangular coordinates
γ_{xy}	Shearing strains of the middle surface in rectangular coordinates $= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}$
δ_n	$= \left[\left(\frac{2n+1}{2\alpha} \pi \right)^2 + \alpha^2 \right]^{\frac{1}{2}}$
$\bar{\epsilon}_x, \bar{\epsilon}_y, \bar{\epsilon}_z$	Three-dimensional unit elongations in the x-, y-, and z-directions

$\varepsilon_x, \varepsilon_y$	Unit elongation of the middle surface in the x- and y-directions $\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \quad ; \quad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2$
$\varepsilon_r, \varepsilon_\theta$	Radial and tangential unit elongations of the middle surface $\varepsilon_r = \frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \quad ; \quad \varepsilon_\theta = \frac{u}{r}$
κ	Proportionality constant for elastically built-in edges
λ	Lamé's constant $= \frac{\nu E}{(1+\nu)(1-2\nu)}$
λ_{kmn}	Constant
ν	Poisson's ratio
$\bar{\sigma}_x, \bar{\sigma}_y, \bar{\sigma}_z$	Normal components of the stress in three-dimensional rectangular coordinates
σ_x, σ_y	Normal components of the membrane stresses in rectangular coordinates
σ'_x, σ'_y	Normal components of the bending stresses in rectangular coordinates
σ_r, σ_θ	Normal components of the membrane stresses in cylindrical coordinates
$\sigma'_r, \sigma'_\theta$	Normal components of the bending stresses in cylindrical coordinates
$\bar{\tau}_{xy}, \bar{\tau}_{yz}, \bar{\tau}_{xz}$	Three-dimensional shearing stresses in rectangular coordinates
τ'_{xy}	Bending shearing stress in rectangular coordinates
τ_{xy}	Membrane shearing stress in rectangular coordinates

III. DERIVATION OF EQUATIONS

The usual thin plate assumptions are that all stresses normal to the plane of the plate are zero and that all sections normal to the plane of the plate that are plane before the load is applied remain plane after the load is applied. As a result of these assumptions, one can write

$$\bar{\tau}_{xz} = \bar{\tau}_{yz} = 0 \quad (3.1)$$

$$\bar{\sigma}_z = 0 = \lambda \bar{e} + 2G \bar{\epsilon}_z \quad (3.2)$$

$$\bar{\epsilon}_x = \epsilon_x + z \frac{\partial^2 w}{\partial x^2} \quad (3.3)$$

$$\bar{\epsilon}_y = \epsilon_y + z \frac{\partial^2 w}{\partial y^2} \quad (3.4)$$

$$\bar{\gamma}_{xy} = \gamma_{xy} + 2z \frac{\partial^2 w}{\partial x \partial y} \quad (3.5)$$

A sketch showing the notation for rectangular coordinates is given in Fig. 1. Combining (3.2) with (3.3) and (3.4) gives

$$\bar{\epsilon}_z = -\frac{\nu}{1-\nu} [e + z \nabla^2 w] \quad (3.6)$$

The three-dimensional strain energy, U , expressed in terms of the strains is given by (Ref. 8, p. 148)

$$U = \frac{1}{2} \iiint [(\lambda + 2G) \bar{e}^2 - 4G \bar{\epsilon}_z] dx dy dz \quad (3.7)$$

Substitution of (3.3) through (3.6) into (3.7) gives

$$U = \frac{1}{2} \iiint \left[\frac{E}{1-\nu^2} \left\{ e + z \nabla^2 w \right\}^2 - 4G \left\{ e_2 + z \left[\epsilon_y \frac{\partial^2 w}{\partial x^2} + \epsilon_x \frac{\partial^2 w}{\partial y^2} - \gamma_{xy} \frac{\partial^2 w}{\partial x \partial y} \right] \right. \right. \quad (3.8)$$

$$\left. \left. + z^2 \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} \right] dx dy dz$$

Integrating in the z-direction, one gets

$$U = \frac{D}{2} \iint \left\{ \left[(\nabla^2 w)^2 + \frac{12}{h^2} e^2 \right] - 2(1-\nu) \left[\frac{12}{h^2} e_2 + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy \quad (3.9)$$

This is the standard energy expression for plates with large deflections written in terms of the deflection and the strain of the middle surface (Ref. 9, p. 95, p. 345).

If one writes the strains in (3.9) in terms of the deflection, w , and the displacements, u and v , and then sets the first variation of the potential energy (strain energy minus the "applied work") equal to zero, one obtains the differential equations for u , v , and w . These equations are non-linear and, except for a few special cases, have been solved only by approximate techniques.

It is the intent of this investigation to develop a simplified analysis for finding the deflection of plates when that deflection is large enough so that the strain of the middle surface cannot be neglected. To do this we would like to neglect terms in the differential equations that do not appreciably affect the solution for the deflection with the hope that neglecting these terms will materially simplify the ensuing analysis.

Examination of some available exact solutions for the deflection of uniformly loaded plates, such as S. Way's solution for the clamped circular plate (Ref. 6) leads us to surmise that the terms in the differential equations arising from the second strain invariant of the middle surface strains, e_2 , in the strain energy integral do not appreciably influence the solution for the deflection.

Consequently, as a working hypothesis, the first variation of e_2 is neglected when deriving the differential equations. There appears to be no simple physical justification for this approximation, so its justification must be based on comparisons of the resulting approximate solutions with available exact solutions for the deflection.

A feeling for the nature of the approximation can be obtained by noting that the variation with Poisson's Ratio, ν , in the resulting differential equations is confined to the bending rigidity, D . Thus, the approximation can be interpreted as neglecting part of the variation of the deflection caused by a change in ν . However, any variation with ν arising from the bending rigidity and the boundary conditions will still be present in the proposed solution for the deflection. This will be demonstrated later when studying simply supported circular plates.

The principle of virtual work states that, for a system in equilibrium, the first variation of the strain energy is equal to the first variation of the applied work for any virtual displacement. Thus,

$$\delta V \equiv \delta(U - W) \equiv \delta\left(U - \iint q w dx dy\right) \quad (3.10)$$

which holds for any variation of u , v , and w . Using the aforementioned assumption, the strain energy written in terms of the displacements and the deflection is

$$U = \frac{D}{2} \iint \left\{ (\nabla^2 w)^2 + \frac{12}{h^2} \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right]^2 - 2(1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy \quad (3.11)$$

Using (3.10) and (3.11), the Calculus of Variations gives the following equations (Ref. 10, p. 191)

$$\frac{\partial e}{\partial x} = 0 \quad (3.12)$$

$$\frac{\partial e}{\partial y} = 0 \quad (3.13)$$

$$\nabla^4 w - \frac{12}{h^2} \left[\frac{\partial}{\partial x} \left(e \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(e \frac{\partial w}{\partial y} \right) \right] = \frac{q}{D} \quad (3.14)$$

Integration of (3.12) and (3.13) gives

$$e \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 = \frac{\alpha^2 h^2}{12} \quad (3.15)$$

where α is a constant of integration. Substituting (3.15) into (3.14) gives

$$\nabla^4 w - \alpha^2 \nabla^2 w = \frac{q}{D} \quad (3.16)$$

Equations (3.15) and (3.16) are the differential equations that will be used to determine the deflection of the rectangular plate.

Due to the constant of integration, α , the equations retain their essential non-linearity, but they have been decoupled so that it is possible to solve the latter for w since it is linear in w , and then use this solution in the former which is linear in u and v to determine the constant of integration, α .

In the case where there is circular symmetry, the same procedure yields the following differential equations

$$e \equiv \frac{du}{dr} + \frac{u}{r} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 = \frac{\alpha^2 h^2}{12} \quad (3.17)$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \left(\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} - \alpha^2 w \right) = \frac{q}{D} \quad (3.18)$$

A sketch showing the notation for the circular plate is given in Fig. 2. The above equations contain all the essential features of those for the rectangular plates, but since they are ordinary differential equations, they can be solved more easily. Consequently, solutions for circular plates are studied before those for rectangular plates so that the essential features of the proposed analysis will be more readily apparent.

IV. CIRCULAR PLATES WITH UNIFORM LOADING

Since the majority of exact solutions are for uniformly loaded plates, all approximate solutions are computed for uniformly loaded plates in order that the two may be compared.

(3.18) can be written as

$$\frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} - \alpha^2 w \right] \right\} = \frac{q}{D} \quad (4.1)$$

Integrating twice, one gets

$$\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} - \alpha^2 w = \frac{qr^2}{4D} + C_1 + C_2 \log r \quad (4.2)$$

But $C_2 = 0$ because w and its second derivative must be finite and its first derivative must be zero at the origin. Since (4.2) is a Bessel Equation of zeroth order, the solution is

$$w = A J_0(i\alpha r) + B - \frac{qr^2}{4D\alpha^2} \quad (4.3)$$

where the coefficient of the Bessel Function of the second kind has been set equal to zero because w must be finite at the origin.

(3.17) can be written as

$$\frac{1}{r} \frac{d}{dr} (ru) = \frac{\alpha^2 h^2}{12} - \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \quad (4.4)$$

Using (4.3) and standard integrals of Bessel Functions (Ref. 11, p. 145), (4.4) can be integrated to get

$$u = \frac{\alpha^2 h^2}{24} r + \frac{A^2 \alpha^2}{4} r \left[J_1^2(i\alpha r) - J_0(i\alpha r) J_2(i\alpha r) \right] - \frac{qAr}{2D\alpha^2} J_2(i\alpha r) - \frac{q^2 r^3}{32D^2\alpha^4} \quad (4.5)$$

where the constant of integration has been set equal to zero, so that u will be finite at the origin.

The equations for w and u contain three arbitrary constants, A , B , and α . A and B are determined from the boundary conditions on w , and then α is determined from the boundary condition on u . Thus, we see that the approach used herein allows one to solve the differential equations readily even with arbitrary boundary conditions. It remains to be seen, of course, whether the solutions given in (4.3) and (4.5) are sufficiently close to the exact solutions to warrant use of the approximation necessary for their derivation.

In order to investigate the accuracy of the approximate solutions as well as to demonstrate the ease with which these solutions can be obtained, three boundary conditions on w are considered in the following analysis--clamped edges, simply supported edges, and edges elastically restrained in rotation. In all three cases u is set equal to zero on the boundary.

If the plate has a hole in the center, the solution can admit singularities at the origin. Consequently, (4.3) would include the terms

$$C Y_0(i\alpha r) - \frac{C_2}{\alpha^2} \log r$$

Similarly, the solution given in (4.5) would include singular terms.

a. Clamped Edges

Assuming that the edges of the plate are restrained from moving and fixed in such a manner that they cannot rotate, the boundary conditions are

$$w(R) = 0 \tag{4.6}$$

$$\left. \frac{dw}{dr} \right|_{r=R} = 0 \quad (4.7)$$

$$u(R) = 0 \quad (4.8)$$

From (4.6) and (4.7) one gets

$$A = \frac{qR^2}{2D\alpha^2} \frac{-1}{i\alpha R J_1(i\alpha R)} \quad (4.9)$$

and

$$B = \frac{qR^2}{4D\alpha^2} \left[1 + \frac{2J_0(i\alpha R)}{i\alpha R J_1(i\alpha R)} \right] \quad (4.10)$$

Substituting (4.9) and (4.10) in (4.3) gives

$$w = \frac{qR^2}{4D\alpha^2} \left[\frac{2[J_0(i\alpha R) - J_0(i\alpha r)]}{i\alpha R J_1(i\alpha R)} + 1 - \left(\frac{r}{R}\right)^2 \right] \quad (4.11)$$

Substituting (4.9) in (4.5) gives

$$u = \frac{\alpha^2 h^2}{24} r - \frac{q^2 R^2 r}{16D^2 \alpha^4} \left[\frac{J_1^2(i\alpha r) - J_0(i\alpha r) J_2(i\alpha r)}{J_1^2(i\alpha R)} - \frac{4J_2(i\alpha r)}{i\alpha R J_1(i\alpha R)} + \frac{1}{2} \left(\frac{r}{R}\right)^2 \right] \quad (4.12)$$

Setting $u(R) = 0$ to determine the constant α , one gets

$$\left(\frac{qR^4}{Dh} \right)^2 = \frac{\frac{1}{3}(\alpha R)^6}{\frac{3}{4} + \frac{4}{(\alpha R)^2} + \frac{J_0(i\alpha R)}{i\alpha R J_1(i\alpha R)} + \frac{1}{2} \frac{J_0^2(i\alpha R)}{J_1^2(i\alpha R)}} \quad (4.13)$$

A plot of the above equation is given in Fig. 3. From this plot,

one finds the proper value of αR to be used in (4.11) and (4.12) for

the given loading. The deflection at the center of the plate is plotted

in Fig. 4, along with the results of S. Way (Ref. 6). When the de-

flection equals the thickness of the plate, a comparison of the two

solutions shows an error of less than two percent for $\nu = 0.25$,

and less than one percent for $\nu = 0.35$. We see that the deflection

at the center of the plate approaches the exact solution as α approaches zero, and that the error increases monotonically as α increases.

Limit $\alpha \rightarrow 0$

From examination of the differential equations, (3.16), we see that the dependence on α disappears as α approaches zero, so we should expect the solution given in (4.3) to be independent of α as it approaches zero, and, therefore, we should expect the solution to approach the exact linear plate solution as α approaches zero. To show that this is the case, we shall take the limit of (4.11) as α approaches zero.

The expansions of $J_0(i\alpha R)$ and $i\alpha R J_1(i\alpha R)$ for small α are (Ref. 11, p. 128)

$$J_0(i\alpha R) = 1 + \left(\frac{\alpha R}{2}\right)^2 + \frac{1}{4} \left(\frac{\alpha R}{2}\right)^4 + \dots \quad (4.14)$$

$$i\alpha R J_1(i\alpha R) = -2 \left(\frac{\alpha R}{2}\right)^2 \left[1 + \frac{1}{2} \left(\frac{\alpha R}{2}\right)^2 + \dots\right] \quad (4.15)$$

Using these expansions in (4.11), the deflection for small values of α is approximately

$$w = \frac{q R^4}{64 D} \left[1 - \left(\frac{r}{R}\right)^2\right]^2 \quad (4.16)$$

As we expected, this is the solution for a clamped circular plate with small deflections (Ref. 9, p. 60).

Limit $\alpha \rightarrow \infty$

The results plotted in Fig. 4 show that the error increases as α increases over the range of values of α for which a comparison could be made. It seems of interest, therefore, to investigate

the error for very large α .

The expansions for $J_0(i\alpha R)$ and $J_1(i\alpha R)$ for large α are (Ref. 11, p. 138)

$$J_0(i\alpha R) = \frac{e^{\alpha R}}{\sqrt{2\pi\alpha R}} \left[1 + \frac{1}{8\alpha R} + \dots \right] \quad (4.17)$$

$$J_1(i\alpha R) = \frac{i e^{\alpha R}}{\sqrt{2\pi\alpha R}} \left[1 - \frac{3}{8\alpha R} - \dots \right] \quad (4.18)$$

Using these expansions in (4.11), the deflection for very large values of α is approximately given by

$$w = \frac{q R^2}{4 D \alpha^2} \left[1 - \left(\frac{r}{R} \right)^2 \right] \quad (4.19)$$

and from (4.13) the loading is approximately given by

$$\left(\frac{q R^4}{D h} \right)^2 = \frac{4}{3} (\alpha R)^6 \quad (4.20)$$

Combining (4.19) and (4.20), one gets the approximate deflection for very large α

$$w = \left[\frac{(1-\nu^2) q R^4}{4 E h} \right]^{\frac{1}{3}} \left[1 - \left(\frac{r}{R} \right)^2 \right] \quad (4.21)$$

For $\nu = 0.3$, the maximum deflection given by (4.21) is

$$w_{max} = 0.610 \left(\frac{q R^4}{E h} \right)^{\frac{1}{3}} \quad (4.22)$$

Hencky's exact result for the circular membrane is (Ref. 12)

$$w_{max} = 0.662 \left(\frac{q R^4}{E h} \right)^{\frac{1}{3}} \quad (4.23)$$

Since the difference between (4.22) and (4.23) is likely to represent the maximum difference that will occur between the exact solution and the solution given in (4.11) for all values of α , the error introduced by the proposed approximation can be estimated to be less than 8 percent for uniformly loaded circular plates where the deflection and displacement are zero on the boundary.

b. Simply Supported Edges

In the case of simply supported edges, it is assumed that the displacement and deflection of the edge of the plate is zero but that it is free to rotate. The boundary conditions in this case are

$$w(R) = 0 \quad (4.24)$$

$$M_r(R) = 0 = \frac{d^2 w}{dr^2} \Big|_{r=R} + \frac{\nu}{R} \frac{dw}{dr} \Big|_{r=R} \quad (4.25)$$

$$u(R) = 0 \quad (4.26)$$

From the first two of the above conditions, the constants in (4.3) are found to be

$$A = \frac{(1+\nu) q}{2D\alpha^4 [J_0(i\alpha R) - \frac{1-\nu}{i\alpha R} J_1(i\alpha R)]} \quad (4.27)$$

and

$$B = \frac{q R^2}{4D\alpha^2} - \frac{(1+\nu) q J_0(i\alpha R)}{2D\alpha^4 [J_0(i\alpha R) - \frac{1-\nu}{i\alpha R} J_1(i\alpha R)]} \quad (4.28)$$

Thus, (4.3) becomes

$$w = \frac{q R^2}{4D\alpha^2} \left\{ 1 - \left(\frac{r}{R}\right)^2 - \frac{2(1+\nu) [J_0(i\alpha R) - J_0(i\alpha r)]}{(\alpha R)^2 J_0(i\alpha R) + (1-\nu) i\alpha R J_1(i\alpha R)} \right\} \quad (4.29)$$

and (4.5) becomes

$$u = \frac{\alpha^2 h^2}{24} r + \frac{q^2 R^2 r}{16D^2 \alpha^4} \left\{ \frac{(1+\nu)^2 [J_1^2(i\alpha r) - J_0(i\alpha r) J_2(i\alpha r)]}{[\alpha R J_0(i\alpha R) - (1-\nu) i J_1(i\alpha R)]^2} - \frac{4(1+\nu) J_2(i\alpha r)}{\alpha R [\alpha R J_0(i\alpha R) - (1-\nu) i J_1(i\alpha R)]} - \frac{1}{2} \left(\frac{r}{R}\right)^2 \right\} \quad (4.30)$$

As in the case of the clamped plate, setting $u(R) = 0$ gives a relation between αR and $\frac{q R^4}{D h}$. A plot of this relation is given in Fig. 3.

A plot of the deflection at the center of the plate is given in Fig. 4 for $\nu = 0.25$. The results of the computations for the simply supported plate are given in Table 2. Unfortunately, there are no available theoretical results for the simply supported plate so the deflection cannot be checked, but some confidence in the results can be obtained by checking the behavior of the solution for small and large values of α .

Limit $\alpha \rightarrow 0$

In the case of the simply supported plate, one find that the deflection for small values of α is approximately given by

$$w = \frac{qR^4}{64D} \left[\frac{5+\nu}{1+\nu} - \left(\frac{r}{R}\right)^2 \right] \left[1 - \left(\frac{r}{R}\right)^2 \right] \quad (4.31)$$

This is the solution for the simply supported circular plate with small deflections (Ref. 9, p. 62) so as in the case of the clamped plate, the deflection given in (4.29) approaches the exact solution as α approaches zero. It should be noted, moreover, that (4.31) has the proper variation with Poisson's Ratio because that variation results from the boundary conditions.

Limit $\alpha \rightarrow \infty$

One sees that the deflection for very large values of α is approximately given by

$$w = \left[\frac{(1-\nu^2) q R^4}{4 E h} \right]^{\frac{1}{3}} \left[1 - \left(\frac{r}{R}\right)^2 \right] \quad (4.32)$$

which, as would be expected, is the same as in the case of the clamped plate.

c. Elastically Built-in Edges

If it is assumed that the edge of the plate in addition to having zero displacement and deflection is restrained in such a manner that the angle through which it rotates is proportional to the bending moment at the edge, the following boundary conditions must be satisfied

$$w(R) = 0 \quad (4.33)$$

$$\left. \frac{dw}{dr} \right|_{r=R} = -\kappa \left\{ \left. \frac{d^2w}{dr^2} \right|_{r=R} + \frac{\nu}{R} \left. \frac{dw}{dr} \right|_{r=R} \right\} \quad (4.34)$$

$$u(R) = 0 \quad (4.35)$$

From (4.33) and (4.34), the constants in (4.3) are found to be

$$A = -\frac{qR^2}{20\alpha^2} \left\{ \frac{1 + \frac{\kappa}{R}(1+\nu)}{\left[1 - \frac{\kappa}{R}(1-\nu)\right] i\alpha R J_1(i\alpha R) - \alpha^2 R \kappa J_0(i\alpha R)} \right\} \quad (4.36)$$

and

$$B = \frac{qR^2}{40\alpha^2} - A J_0(i\alpha R) \quad (4.37)$$

Thus, (4.3) becomes

$$w = \frac{qR^2}{40\alpha^2} \left\{ \frac{2 \left[1 + \frac{\kappa}{R}(1+\nu)\right] [J_0(i\alpha R) - J_0(i\alpha r)]}{\left[1 - \frac{\kappa}{R}(1-\nu)\right] i\alpha R J_1(i\alpha R) - \alpha^2 R \kappa J_0(i\alpha R)} + 1 - \left(\frac{r}{R}\right)^2 \right\} \quad (4.38)$$

One sees that (4.38) approaches the deflection for the simply supported case as κ approaches infinity, and it approaches the deflection for the clamped case as κ approaches zero. For all other values of κ , the deflection given in (4.38) will be between these limiting cases.

As in the case of the clamped and simply supported cases, the displacement, u , is found by substituting (4.36) in (4.5), and the relation between the loading and α is found by satisfying the boundary condition (4.35). The resulting relation would be a curve of αR vs. $\frac{qR^4}{Dh}$ with κ as a parameter.

V. RECTANGULAR PLATES WITH UNIFORM LOADING

The equations that must be solved for the rectangular plate with uniform loading are (3.15) and (3.16). It is not possible to write the solution of these equations in closed form as was done for the circular plate. Therefore, u , v , and w will be expanded in appropriate series. The choice of the type of series that is best suited to the problem will depend on the boundary conditions. Consequently, no general solution for completely arbitrary boundary conditions will be given for the rectangular plate.

As an illustration, consider the solution when two sides, $x = \pm a$, are simply supported, and the two other sides, $y = \pm b$, have arbitrary boundary conditions. To do this w is represented by the Fourier Series

$$w = \sum_{n=0}^{\infty} f_n(y) \cos \frac{2n+1}{2a} \pi x \quad (5.1)$$

This series satisfies, term by term, the boundary conditions for the simply supported edges

$$w(\pm a, y) = 0 \quad (5.2)$$

$$\left. \frac{\partial^2 w}{\partial x^2} \right|_{x=\pm a} = 0 \quad (5.3)$$

Substitution of (5.1) into (3.16) gives

$$\sum_{n=0}^{\infty} \cos \beta_n x \left\{ \frac{d^4 f_n}{dy^4} - (\beta_n^2 + \delta_n^2) \frac{d^2 f_n}{dy^2} + \beta_n^2 \delta_n^2 f_n \right\} = \frac{q}{D} \quad (5.4)$$

where

$$\beta_n = \frac{2n+1}{2a} \pi \quad ; \quad \delta_n = \left[\left(\frac{2n+1}{2a} \pi \right)^2 + \alpha^2 \right]^{\frac{1}{2}}$$

Expanding the right-hand side of (5.4) in a Fourier Series in $\cos \beta_n x$ gives

$$\frac{q}{D} = \frac{2q}{D} \sum_{n=0}^{\infty} \frac{(-)^n}{\beta_n} \cos \beta_n x \quad (5.5)$$

Since (5.4) must hold for all values of x , one obtains the following differential equation for $f_n(y)$

$$\frac{d^4 f_n}{dy^4} - (\beta_n^2 + \delta_n^2) \frac{d^2 f_n}{dy^2} + \beta_n^2 \delta_n^2 f_n = \frac{2q(-)^n}{D \beta_n} \quad (5.6)$$

Because of the choice of coordinate system, one is interested only in symmetric solutions for $f_n(y)$. Therefore, the proper solution of (5.6) is

$$f_n(y) = \frac{2q(-)^n}{D \beta_n^3 \delta_n^2} + A_n \cosh \beta_n y + B_n \cosh \delta_n y \quad (5.7)$$

and the solution for w is

$$w = \sum_{n=0}^{\infty} \cos \beta_n x \left\{ \frac{2q(-)^n}{D \beta_n^3 \delta_n^2} + A_n \cosh \beta_n y + B_n \cosh \delta_n y \right\} \quad (5.8)$$

It is convenient to partially sum (5.8). Thus

$$w = \frac{q}{D \alpha^4} \left\{ \frac{\cosh \alpha x}{\cosh \alpha a} - 1 + \frac{(\alpha a)^2}{2} \left[1 - \left(\frac{x}{a} \right)^2 \right] \right\} + \sum_{n=0}^{\infty} \cos \beta_n x \left\{ A_n \cosh \beta_n y + B_n \cosh \delta_n y \right\} \quad (5.9)$$

where the equivalence of (5.8) and (5.9) can be verified by expanding the summed portion of (5.9).

One notes that the first part of (5.9) is the exact solution for the deflection of a simply supported infinite strip given by Shaw et al (Ref. 13), Timoshenko (Ref. 9, p. 5), and others. Since w must be independent of y as b approaches infinity, the second part of (5.9) will approach zero as b approaches infinity, leaving the solution given in (5.9) equal to the exact solution. Thus, the approximate solution

found in this thesis approaches the exact solution in the case of rectangular plates as the aspect ratio, $\frac{b}{a}$, approaches infinity for any a . By the same argument that was used in the case of circular plates, it can be shown that the solution approaches the exact solution as a approaches zero.

Using (5.1) one can write

$$\left(\frac{\partial w}{\partial x}\right)^2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \beta_n \beta_m f_n f_m \sin \beta_n x \sin \beta_m x$$

and this can be written as

$$\left(\frac{\partial w}{\partial x}\right)^2 = \frac{1}{2} \sum_{n=0}^{\infty} \beta_n^2 f_n^2 + \frac{1}{2} \sum_{k=1}^{\infty} \cos \frac{k\pi x}{a} \left\{ \sum_{n=0}^{\infty} \beta_n \beta_{n+k} f_n f_{n+k} - \sum_{n=0}^{k-1} \beta_n \beta_{k-n-1} f_n f_{k-n-1} + \sum_{n=k}^{\infty} \beta_n \beta_{n-k} f_n f_{n-k} \right\} \quad (5.10)$$

Similarly, one can write

$$\left(\frac{\partial w}{\partial y}\right)^2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{df_n}{dy} \frac{df_m}{dy} \cos \beta_n x \cos \beta_m x$$

which can be written as

$$\left(\frac{\partial w}{\partial y}\right)^2 = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{df_n}{dy}\right)^2 + \frac{1}{2} \sum_{k=1}^{\infty} \cos \frac{k\pi x}{a} \left\{ \sum_{n=0}^{\infty} \frac{df_n}{dy} \frac{df_{n+k}}{dy} + \sum_{n=0}^{k-1} \frac{df_n}{dy} \frac{df_{k-n-1}}{dy} + \sum_{n=k}^{\infty} \frac{df_n}{dy} \frac{df_{n-k}}{dy} \right\} \quad (5.11)$$

If one lets

$$u = \sum_{k=1}^{\infty} g_k(y) \sin \frac{k\pi x}{a} \quad (5.12)$$

and

$$v = \sum_{k=0}^{\infty} \ell_k(y) \cos \frac{k\pi x}{a} \quad (5.13)$$

where the above form has been chosen because we know that u is an odd function of x and v is an even function of x .

The boundary condition on u at the edge $x = \pm a$ is

$$u(\pm a, y) = 0 \quad (5.14)$$

which is automatically satisfied by (5.12). If the plate is simply supported so that there is no restraining force to displacement parallel to its edges, the shear must be zero along its edges. Since $\frac{\partial u}{\partial y}$ and $\frac{\partial w}{\partial y}$ are zero along $x = \pm a$, the boundary condition on the shear reduces to the following boundary condition

$$\left. \frac{\partial v}{\partial x} \right|_{x=\pm a} = 0 \quad (5.15)$$

which is automatically satisfied by (5.13). If the plate has restraints preventing motion parallel to its edges, the boundary condition on v would be

$$v(\pm a, y) = 0 \quad \text{or} \quad \sum_{k=0}^{\infty} (-)^k \ell_k(y) = 0 \quad (5.16)$$

If one substitutes (5.10) through (5.13) into (3.15) and equates coefficients, the following set of equations is obtained

$$\frac{\alpha^2 h^2}{12} = \frac{d\ell_0}{dy} + \frac{1}{4} \sum_{n=0}^{\infty} \left\{ \left(\frac{df_n}{dy} \right)^2 + (\beta_n f_n)^2 \right\} \quad (5.17)$$

and

$$\begin{aligned} 0 = & \frac{k\pi}{a} g_k + \frac{d\ell_k}{dy} + \frac{1}{4} \sum_{n=0}^{\infty} \left[\beta_n \beta_{n+k} f_n f_{n+k} + \frac{df_n}{dy} \frac{df_{n+k}}{dy} \right] \\ & - \frac{1}{4} \sum_{n=0}^{k-1} \left[\beta_n \beta_{k-n-1} f_n f_{k-n-1} - \frac{df_n}{dy} \frac{df_{k-n-1}}{dy} \right] \end{aligned} \quad (5.18)$$

$$+ \frac{1}{4} \sum_{n=k}^{\infty} \left[\beta_n \beta_{n-k} f_n f_{n-k} + \frac{df_n}{dy} \frac{df_{n-k}}{dy} \right]$$

$$k = 1, 2, 3, \dots$$

(5.17) can be integrated to get

$$\begin{aligned} \ell_0(y) = & \frac{\alpha^2 h^2}{12} y - \frac{1}{4} \sum_{n=0}^{\infty} \left\{ \left[\frac{2q}{\partial \beta_n^2 \delta_n^2} \right]^2 y - \frac{\alpha^2 B_n^2}{2} y + \beta_n A_n^2 \sinh \beta_n y \cosh \beta_n y \right. \\ & + \frac{\beta_n^2 + \delta_n^2}{2 \delta_n} B_n^2 \sinh \delta_n y \cosh \delta_n y + 2 \beta_n A_n B_n \sinh \beta_n y \cosh \delta_n y \quad (5.19) \\ & \left. + \frac{4q(-)^n}{\partial \beta_n^2 \delta_n^2} \left[A_n \sinh \beta_n y + \frac{\beta_n}{\delta_n} B_n \sinh \delta_n y \right] \right\} \end{aligned}$$

where the constant of integration has been set equal to zero because, for all boundary conditions that are treated here, $\ell_0(y)$ is an odd function. When the boundary conditions at $y = \pm b$ are given, this equation determines the proper relation between α and the loading.

It is not possible to solve for $g_k(y)$ and $\ell_k(y)$ independently from (5.18). It is possible, however, to find the deflection, w , even though one cannot find u and v explicitly from the above formulation of the solution. Also, one sees that it is not necessary to specify the boundary condition on the displacement parallel to the edge in order to find the deflection. This must necessarily be due to the approximation used to derive the differential equations. It appears, therefore, that a part of the error introduced by the simplification made herein must lie in the fact that the deflection determined by this analysis does not vary with the boundary condition on the displacement parallel to the edge. Since the displacement parallel to the edge was necessarily zero because of symmetry in the case of the circular plate, one can expect that this additional source of error in the rectangular plate solution will make the error in the deflection greater in the case of a rectangular plate than it was in the case of a

circular plate. Furthermore, the inability to determine u and v explicitly leads to the conclusion that the method developed so far in this thesis cannot be used to determine accurate membrane stresses since they depend on the displacements. However, in many problems, interest is centered on the deflection, w , and the previous analysis enables one to find this deflection readily.

In the following sections the simply supported and clamped boundary conditions at $y = \pm b$ will be considered, and computations of deflections will be carried out for various aspect ratios.

a. All Edges Simply Supported

When all edges are simply supported, the boundary conditions, in addition to (5.2) and (5.3), are

$$w(x, \pm b) = 0 \quad (5.20)$$

$$\left. \frac{\partial^2 w}{\partial y^2} \right|_{y=\pm b} = 0 \quad (5.21)$$

Using the expansion given in (5.1), these are equivalent to the conditions

$$f_n(\pm b) = 0 \quad (5.22)$$

$$\left. \frac{d^2 f_n}{dy^2} \right|_{y=\pm b} = 0 \quad (5.23)$$

Using the above boundary conditions, the coefficients in (5.7) become

$$A_n = \frac{-2q(-)^n}{\alpha D \alpha^2 \beta_n^3 \cosh \beta_n b} \quad (5.24)$$

and

$$B_n = \frac{2q(-)^n}{aD\alpha^2\beta_n\delta_n^2\cosh\delta_nb} \quad (5.25)$$

Thus, (5.9) can be written as

$$w = \frac{q}{D\alpha^4} \left\{ \frac{\cosh\alpha x}{\cosh\alpha a} - 1 + \frac{(\alpha a)^2}{2} \left[1 - \left(\frac{x}{a}\right)^2 \right] \right\} + \sum_{n=0}^{\infty} \cos\beta_n x \left\{ \frac{2q(-)^n}{aD\alpha^2\beta_n^3\delta_n^2} \left[\beta_n^2 \frac{\cosh\delta_n y}{\cosh\delta_nb} - \delta_n^2 \frac{\cosh\beta_n y}{\cosh\beta_nb} \right] \right\} \quad (5.26)$$

The boundary conditions on u and v for the simply supported plate when the shear is zero along the edges are

$$\begin{aligned} \frac{\partial u}{\partial y} \Big|_{y=\pm b} &= 0 \\ v(x, \pm b) &= 0 \end{aligned} \quad (5.27)$$

and when the plate is restrained from moving parallel to its edges, the boundary conditions are

$$\begin{aligned} u(x, \pm b) &= 0 \\ v(x, \pm b) &= 0 \end{aligned} \quad (5.28)$$

Using the expansions of (5.12) and (5.13), (5.27) becomes

$$\begin{aligned} \frac{dg_k}{dy} \Big|_{y=\pm b} &= 0 \\ \ell_k(\pm b) &= 0 \end{aligned} \quad (5.29)$$

and (5.28) becomes

$$\begin{aligned} g_k(\pm b) &= 0 \\ \ell_k(\pm b) &= 0 \end{aligned} \quad (5.30)$$

The above boundary condition on ℓ_k when substituted into (5.19) gives

$$\frac{\alpha^2 h^2}{12} = \frac{q^2}{a^2 D^2} \sum_{n=0}^{\infty} \frac{1}{\beta_n^4 \delta_n^2} \left\{ 1 - \frac{\beta_n^2}{2\alpha^2 \cosh^2 \delta_n b} - \frac{\delta_n^4}{\alpha^4 \beta_n b} \tanh \beta_n b \right. \\ \left. + \frac{\beta_n^2 (\beta_n^2 + \frac{5}{2} \alpha^2)}{\alpha^4 \delta_n b} \tanh \delta_n b \right\} \quad (5.31)$$

If the relation

$$\sum_{n=0}^{\infty} \frac{1}{a^8 \beta_n^4 \delta_n^4} = \frac{1}{(\alpha a)^7} \left[\frac{5}{4} \tanh \alpha a - \frac{\alpha a}{4} \operatorname{sech}^2 \alpha a - \alpha a + \frac{1}{6} (\alpha a)^3 \right] \quad (5.32)$$

is used, (5.31) can be written as

$$\frac{(\alpha a)^6}{12} = \left(\frac{q a^4}{D h} \right)^2 \left\{ \frac{1}{(\alpha a)^3} \left[\frac{5}{4} \tanh \alpha a - \frac{\alpha a}{4} \operatorname{sech}^2 \alpha a - \alpha a + \frac{1}{6} (\alpha a)^3 \right] \right. \\ \left. - \sum_{n=0}^{\infty} \frac{1}{a^4 \beta_n^4 \delta_n^4} \left[\frac{\alpha^2 \beta_n^2}{2} \operatorname{sech}^2 \delta_n b + \frac{\delta_n^4}{\beta_n b} \tanh \beta_n b \right. \right. \\ \left. \left. - \frac{\beta_n^2 (\beta_n^2 + \frac{5}{2} \alpha^2)}{\delta_n b} \tanh \delta_n b \right] \right\} \quad (5.33)$$

A plot of this equation giving αa as a function of $\frac{q a^4}{D h}$ is given in

Fig. 5 for aspect ratios of 1, 1.5, 2, and ∞ . From this plot one finds the proper value of αa to be used in (5.26) for the desired loading.

A plot of the deflection at the center of the plate is given in Fig. 6 for aspect ratios of 1, 1.5, 2, and ∞ . The results of Levy (Ref. 1) and Wang (Ref. 4 and 5) are also plotted in Fig. 6. The results of the computations for Fig. 5 and 6 are given in Table 3. It should be noted that the series of (5.26) and (5.33) converge so rapidly that never were more than two terms needed. In fact, the first term alone would have yielded sufficient accuracy. Table 4 demonstrates the rapidity of convergence of the series for the worst case of aspect ratio equal to one.

Levy assumed that there was no restraint to displacement parallel to the edges of the plate, while Wang assumed that the displacement parallel to the edge was zero. As was pointed out earlier, the solution given in this thesis is independent of which of the above assumptions is used. We note that for an aspect ratio of one, the solution given in this thesis lies between the solutions of Levy and Wang. We can conclude, therefore, that this solution can be interpreted as representing the deflection for a plate where the displacement parallel to the edge lies somewhere between zero and the value when there is no restraint against such a displacement, but one cannot explain the whole error introduced by neglecting the second strain invariant in the strain energy integral in this manner because, as was shown in the case of a circular plate, there is an error even when both boundary conditions give zero displacement parallel to the edge of the plate.

It can be noticed in Fig. 6 that the curves showing the solution of Wang and the solution given herein cannot be distinguished from each other for an aspect ratio of 1.5. As was mentioned previously, the solution arrived at in this thesis for an infinite aspect ratio is the exact solution, so the deflection plotted in Fig. 6 for $\frac{b}{a} = \infty$ is exact.

Physical intuition leads us to believe that the deflection at the center of the plate for an infinite aspect ratio will be larger than the deflection at the center for any other aspect ratio. The method used in this thesis gives deflections for an aspect ratio of 2 that are greater than those for an infinite aspect ratio. This appears

to be due to the approximate nature of the solution, but it is interesting to note that the calculations of Wang reveal the same result to a lesser degree. This phenomenon will be discussed further when investigating the limit as α approaches infinity.

As in the case of the circular plate, the deflection given in (5.26), for any given aspect ratio, approaches the exact solution (Ref. 9, p. 128) as α approaches zero. To investigate the error as the plate approaches the membrane we take the limit of (5.26) for very large α . In this case the deflection is approximated by

$$w = \frac{q a^2}{2 D \alpha^2} \left[1 - \left(\frac{x}{a} \right)^2 \right] - \frac{2 q}{\alpha D \alpha^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{\beta_n^3} \frac{\cosh \beta_n y}{\cosh \beta_n b} \cos \beta_n x \quad (5.34)$$

The approximate value of $\frac{q a^4}{D h}$ for very large α when obtained from (5.33) is given by

$$\left(\frac{q a^4}{D h} \right)^2 = \frac{\frac{1}{2} (\alpha a)^6}{1 - 6 \frac{a}{b} \sum_{n=0}^{\infty} \frac{1}{(\beta_n a)^5} \tanh \beta_n b} \quad (5.35)$$

Eliminating α from (5.34) and (5.35), one gets

$$\left(\frac{w}{h} \right) = \left(\frac{q a^4}{D h} \right)^{\frac{1}{3}} \left\{ \frac{\frac{1}{2} \left[1 - \left(\frac{x}{a} \right)^2 \right] - \sum_{n=0}^{\infty} \frac{2(-1)^n}{(\beta_n a)^3} \frac{\cosh \beta_n y}{\cosh \beta_n b} \cos \beta_n x}{\left[2 - 12 \frac{a}{b} \sum_{n=0}^{\infty} \frac{1}{(\beta_n a)^5} \tanh \beta_n b \right]^{\frac{1}{3}}} \right\} \quad (5.36)$$

Thus, for $\nu = 0.3$ and $\frac{b}{a} = 1$, the maximum deflection for large α is approximately given by

$$w_{\max} = 0.692 \left(\frac{q a^4}{E h} \right)^{\frac{1}{3}} \quad (5.37)$$

Hencky's exact result for the square membrane, where both u and v are zero at the edges, (Ref. 14) is

$$w_{\max} = 0.665 \left(\frac{q a^4}{E h} \right)^{\frac{1}{3}} \quad (5.38)$$

Thus, the error introduced by the assumption used in this thesis is 4 percent for uniformly loaded square membranes with zero displacements and deflection on the boundary. We note that, contrary to the circular plate case, this is less than the error one gets when the deflection is of the order of magnitude of the thickness.

By examination of (5.36) we see that when finding the deflection at the center of the plate, the difference between the numerators for finite and infinite aspect ratios is of the order of $\operatorname{sech} \frac{2n+1}{2} \pi \frac{b}{a}$, while the difference between the denominators is of the order $\frac{b}{a}$. Since the numerator increases towards its value for the infinite aspect ratio plate at a greater rate than the denominator, the deflection must exceed that of the infinite aspect ratio plate for all aspect ratios greater than the aspect ratio for which the deflection is equal to that of the infinite strip. The maximum deflection given by (5.36) is plotted in Fig. 7 for all aspect ratios. We see that for aspect ratios greater than 1.7, the maximum deflection given by (5.36) is greater than that for the infinite strip.

The approximate solution given by Föppl (Ref. 15, p. 228) is also plotted in Fig. 7. We see that his approximate method gives a deflection that is always too large and that the error increases with the aspect ratio. It is interesting to note that the approximate solution given in (5.36) increases with aspect ratio at approximately the same rate as Föppl's solution for aspect ratios near one, but in contrast with Föppl's solution, the center deflection given by (5.36) starts decreasing for large aspect ratios until it approaches the exact solution for an infinite aspect ratio. The exact solution for an aspect ratio of one is less than 25 percent lower than the exact solution for an

infinite aspect ratio. Consequently, if one were interested in the membrane solution only, he could obtain a much better estimate of the maximum deflection than Föppl's by fairing a simple curve between the exact solutions for aspect ratios of one and infinity.

b. Two Edges Simply Supported and Two Edges Clamped

When the two edges, $x = \pm a$, are simply supported and the two edges, $y = \pm b$, are clamped, we have in addition to (5.2) and (5.3), the boundary conditions

$$w(x, \pm b) = 0 \quad (5.39)$$

$$\frac{\partial w}{\partial y} \Big|_{y=\pm b} = 0 \quad (5.40)$$

Using the expansion of (5.1), these are equivalent to

$$f_n(\pm b) = 0 \quad (5.41)$$

$$\frac{df_n}{dy} \Big|_{y=\pm b} = 0 \quad (5.42)$$

Using the above boundary conditions, the coefficients in (5.7) become

$$A_n = \frac{-2q(-)^n}{\alpha D \beta_n^3 \delta_n^2} \left\{ \frac{1}{\sinh \beta_n b [\coth \beta_n b - \frac{\beta_n}{\delta_n} \coth \delta_n b]} \right\} \quad (5.43)$$

and

$$B_n = \frac{2q(-)^n}{\alpha D \beta_n^3 \delta_n^2} \left\{ \frac{1}{\sinh \delta_n b [\frac{\delta_n}{\beta_n} \coth \beta_n b - \coth \delta_n b]} \right\} \quad (5.44)$$

Thus, (5.9) can be written as

$$w = \frac{q}{D \alpha^4} \left\{ \frac{\cosh \alpha x}{\cosh \alpha a} - 1 + \frac{(\alpha a)^2}{2} \left[1 - \left(\frac{x}{a} \right)^2 \right] \right\} - \frac{2q}{\alpha D} \sum_{n=0}^{\infty} \cos \beta_n x \left\{ \frac{(-)^n}{\beta_n^3 \delta_n^2} \left[\frac{\beta_n \frac{\cosh \delta_n y}{\sinh \delta_n b} - \delta_n \frac{\cosh \beta_n y}{\sinh \beta_n b}}{\beta_n \coth \delta_n b - \delta_n \coth \beta_n b} \right] \right\} \quad (5.45)$$

The boundary conditions on u and v are the same as those given in (5.27) and (5.28). Using the boundary condition on ℓ_0 given in (5.29) or (5.30), (5.19) becomes

$$\frac{(\alpha a)^2}{12} = \left(\frac{q a^4}{D h} \right)^2 \left\{ \frac{1}{(\alpha a)^7} \left[\frac{5}{4} \tanh \alpha a - \frac{\alpha a}{4} \operatorname{sech}^2 \alpha a - \alpha a + \frac{1}{6} (\alpha a)^3 \right] - \sum_{n=0}^{\infty} \frac{\frac{\alpha^2 \beta_n^2}{2} \operatorname{csch}^2 \delta_n b - \frac{\beta_n (\beta_n^2 - \alpha^2)}{b} \coth \beta_n b + \frac{\beta_n^2 (\beta_n^2 - \frac{1}{2} \alpha^2)}{\delta_n b} \coth \delta_n b}{\alpha^8 \beta_n^6 \delta_n^4 [\delta_n \coth \beta_n b - \beta_n \coth \delta_n b]^2} \right\} \quad (5.46)$$

A plot of this equation is given in Fig. 8 for aspect ratios of 1, 1.5, and ∞ . A plot of the deflection at the center of the plate is given in Fig. 9 for the same aspect ratios. The results of the computations used to obtain Figs. 8 and 9 are given in Table 5. For the square plate the deflection lies between that of the fully clamped plate (Refs. 2 and 7) and that of the simply supported plate. As the load increases, the difference in deflection between the semi-clamped plate and the simply supported plate decreases. For an aspect ratio of 1.5, only a slight difference exists between the semi-clamped and simply supported plates. In fact, for large loads the approximate solution for the deflection of the semi-clamped plate is slightly larger than that for the simply supported plate due to the nature of the approximate approach. As the aspect ratio approaches infinity, the solution for the semi-clamped plate approaches that for the simply supported one. Thus, the curve for an infinite aspect ratio plate given in Fig. 9 is the same as that given in Fig. 6. The limit of the deflection given in (5.45) approaches the solution given in (5.34) as a approaches infinity. As a approaches zero, the deflection approaches the exact solution.

VI. DETERMINATION OF STRESSES

The foregoing analysis can be used to determine the deflection of the plate and the bending stresses at the surface of the plate which are given by (Ref. 9, Chap. III) for the circular plate as

$$\sigma_r' = -\frac{6D}{h^2} \left(\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right) \quad (6.1)$$

$$\sigma_\theta' = -\frac{6D}{h^2} \left(\nu \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right) \quad (6.2)$$

and are given by (Ref. 9, Chap. II) for the rectangular plate as

$$\sigma_x' = -\frac{6D}{h^2} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (6.3)$$

$$\sigma_y' = -\frac{6D}{h^2} \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (6.4)$$

$$\tau_{xy}' = -\frac{6D}{h^2} (1-\nu) \frac{\partial^2 w}{\partial x \partial y} \quad (6.5)$$

Since these stresses depend on derivatives of w , they will not be as accurate as w , but they should be sufficiently accurate for engineering use. In many problems one is interested only in the deflection and the bending stresses. However, frequently one needs to know the membrane stresses as well. In such cases the displacements of the middle surface of the plate must be known.

It was possible to determine this displacement for the circular plate in the foregoing analysis, but the membrane stresses computed from this displacement are very inaccurate. Moreover, one should expect that to be true because the deflection and displacement that were determined did not satisfy the equilibrium equations. The error resulting from this had little effect on the deflection, but we

should expect it to have a large effect on the membrane stresses. In the case of the rectangular plate, it was not even possible to uniquely determine the displacements in the plane of the plate, so it was immediately apparent that the technique used to find the deflection could not be used to find the membrane stresses.

It is possible, however, to obtain a good approximation to the membrane stresses by using the previous results for the deflection. If it is assumed that the deflection is a known function (determined by the previous analysis) one can substitute that deflection into the strain energy expression given in (3.9). Then (3.10) will hold for any variations of the displacements u and v . The Calculus of Variations then can be used to derive the appropriate differential equations for the displacements.

In the case of the rectangular plate the equations arrived at in this manner are

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2} \frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{2} \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial W}{\partial x} \left(\frac{\partial^2 W}{\partial x^2} + \frac{1-\nu}{2} \frac{\partial^2 W}{\partial y^2} \right) - \frac{1+\nu}{2} \frac{\partial W}{\partial y} \frac{\partial^2 W}{\partial x \partial y} \quad (6.6)$$

$$\frac{\partial^2 v}{\partial y^2} + \frac{1-\nu}{2} \frac{\partial^2 v}{\partial x^2} + \frac{1+\nu}{2} \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial W}{\partial y} \left(\frac{\partial^2 W}{\partial y^2} + \frac{1-\nu}{2} \frac{\partial^2 W}{\partial x^2} \right) - \frac{1+\nu}{2} \frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial x \partial y} \quad (6.7)$$

where the right-hand side in each case is a known function of x and y . The equations, therefore, are linear equations in u and v and can be solved in a straightforward manner.

For the circular plate, the corresponding equation for the displacement, u , is

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = -\frac{1-\nu}{2r} \left(\frac{dW}{dr} \right)^2 - \frac{1}{2} \frac{d}{dr} \left[\left(\frac{dW}{dr} \right)^2 \right] \quad (6.8)$$

where the right-hand side is a known function.

It can be noted that the above three equations are the usual equilibrium equations in the middle plane of the plate. Consequently, the procedure that has been followed in this thesis can be interpreted as finding a solution for the deflection that satisfies modified equations of equilibrium in the plane of the plate and normal to the plane of the plate, and then using this deflection in the equilibrium equations for the plane of the plate to determine the displacements. With this procedure the potential energy is minimized under the assumption that the deflection is given by the approximate result.

a. Circular Plates with Uniform Loading

For the circular plate, the equation for the displacement, (6.8), can be written as

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (ru) \right] = -\frac{1-\nu}{2r} \left(\frac{dw}{dr} \right)^2 - \frac{1}{2} \frac{d}{dr} \left[\left(\frac{dw}{dr} \right)^2 \right] \quad (6.9)$$

where the deflection, w , is given by (4.3). Integrating (6.9) twice, using the necessary integral relations of the Bessel Functions (Ref. 11, p. 145), the displacement is given by

$$u = Cr - \frac{(3-\nu)q^2 r^3}{64D^2\alpha^4} + \frac{(1+\nu)qA}{2D\alpha^4} i\alpha J_1(i\alpha r) + \frac{qAr}{2D\alpha^2} J_0(i\alpha r) + \frac{\nu}{4} \alpha^2 A^2 r \left[J_0^2(i\alpha r) + J_1^2(i\alpha r) \right] + \frac{1+\nu}{4} A^2 i\alpha J_0(i\alpha r) J_1(i\alpha r) \quad (6.10)$$

where C is a constant of integration. The other constant of integration has been set equal to zero so that u will be finite at the origin.

If u is set equal to zero on the boundary, one can determine C , giving the following expression for u

$$\begin{aligned}
 u = & -\frac{q^2 R^7}{D^2} \left\{ \frac{AD}{q R^4} \frac{(1+\nu)}{2(\alpha R)^3} \left[\frac{r}{R} i J_1(i\alpha R) - i J_1(i\alpha r) \right] \right. \\
 & + \frac{AD}{q R^4} \frac{1}{2(\alpha R)^2} \frac{r}{R} \left[J_0(i\alpha R) - J_0(i\alpha r) \right] - \frac{3-\nu}{64(\alpha R)^4} \frac{r}{R} \left[1 - \left(\frac{r}{R} \right)^2 \right] \\
 & + \frac{A^2 D^2}{q^2 R^8} \frac{\nu}{4} (\alpha R)^2 \frac{r}{R} \left[J_0^2(i\alpha R) + J_1^2(i\alpha R) - J_0^2(i\alpha r) - J_1^2(i\alpha r) \right] \\
 & \left. + \frac{A^2 D^2}{q^2 R^8} \frac{1+\nu}{4} i\alpha R \left[\frac{r}{R} J_0(i\alpha R) J_1(i\alpha R) - J_0(i\alpha r) J_1(i\alpha r) \right] \right\} \quad (6.11)
 \end{aligned}$$

The membrane stresses are given by (Ref. 9, p. 329)

$$\sigma_r = \frac{E}{1-\nu^2} \left[\frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 + \nu \frac{u}{r} \right] \quad (6.12)$$

$$\sigma_\theta = \frac{E}{1-\nu^2} \left[\frac{u}{r} + \nu \frac{du}{dr} + \frac{\nu}{2} \left(\frac{dw}{dr} \right)^2 \right] \quad (6.13)$$

The membrane stresses at the center of the plate are given by

$$(\sigma_r)_{r=0} = (\sigma_\theta)_{r=0} = \frac{E}{1-\nu} \left(\frac{u}{r} \right)_{r=0} \quad (6.14)$$

Using the value of u given in (6.11) one gets

$$\begin{aligned}
 (\sigma_r)_{r=0} = & -\frac{q^2 R^6 E}{(1-\nu) D^2} \left\{ \frac{AD}{q R^4} \frac{1+\nu}{2(\alpha R)^3} \left[i J_1(i\alpha R) + \frac{\alpha R}{2} \right] + \frac{AD}{q R^4} \frac{[J_0(i\alpha R) - 1]}{2(\alpha R)^2} \right. \\
 & + \frac{A^2 D^2}{q^2 R^8} \frac{\nu}{4} (\alpha R)^2 \left[J_0^2(i\alpha R) + J_1^2(i\alpha R) - 1 \right] \\
 & \left. + \frac{A^2 D^2}{q^2 R^8} \frac{1+\nu}{4} \left[i\alpha R J_0(i\alpha R) J_1(i\alpha R) + \frac{(\alpha R)^2}{2} \right] - \frac{3-\nu}{64(\alpha R)^4} \right\} \quad (6.15)
 \end{aligned}$$

At the edge of the plate

$$(\sigma_r)_{r=R} = \frac{1}{\nu} (\sigma_\theta)_{r=R} = \frac{E}{1-\nu^2} \left[\frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \right]_{r=R} \quad (6.16)$$

After performing the indicated differentiations, one gets

$$\begin{aligned}
 (\sigma_r)_{r=R} = & -\frac{E}{1-\nu} \frac{q^2 R^6}{D^2} \left\{ \frac{AD}{q R^4 (\alpha R)^2} \left[\frac{1}{2} J_0(i\alpha R) + \frac{i}{\alpha R} J_1(i\alpha R) \right] - \frac{1}{32(\alpha R)^4} \right. \\
 & \left. + \frac{A^2 D^2}{q^2 R^8} \frac{(\alpha R)^2}{4} \left[J_0^2(i\alpha R) + J_1^2(i\alpha R) + \frac{2}{\alpha R} i J_0(i\alpha R) J_1(i\alpha R) \right] \right\} \quad (6.17)
 \end{aligned}$$

Substituting (4.3) in (6.1) and (6.2), we find the bending stresses at the center to be

$$(\sigma_r')_{r=0} = (\sigma_\theta')_{r=0} = -\frac{6D(1+\nu)}{h^2} \left[\frac{\alpha^2 A}{2} - \frac{q}{2D\alpha^2} \right] \quad (6.18)$$

and the bending stresses at the edge to be

$$(\sigma_r')_{r=R} = -\frac{6D}{h^2} \left[\frac{\alpha}{R} A(1-\nu) iJ_1(i\alpha R) + \alpha^2 A J_0(i\alpha R) - \frac{q(1+\nu)}{2D\alpha^2} \right] \quad (6.19)$$

$$(\sigma_\theta')_{r=R} = -\frac{6D}{h^2} \left[-\frac{\alpha}{R} A(1-\nu) iJ_1(i\alpha R) + \nu\alpha^2 A J_0(i\alpha R) - \frac{q(1+\nu)}{2D\alpha^2} \right] \quad (6.20)$$

The values of A for the clamped, simply supported, and elastically built-in edges are given in (4.9), (4.27), and (4.36), respectively. Substitution of those values into the above equations determines the membrane and bending stresses for those boundary conditions. In what follows, we shall examine the case of the clamped plate in detail and compare the results with available solutions to the exact equations.

For the clamped plate, (6.15) becomes

$$(\sigma_r)_{r=0} = \frac{q^2 R^6 E}{(1-\nu) D^2} \frac{1}{16(\alpha R)^4} \left\{ \frac{\nu J_0^2(i\alpha R) + \frac{1-\nu}{2}}{J_1^2(i\alpha R)} + \frac{(3-\nu) J_0(i\alpha R) - 2(1-\nu)}{i\alpha R J_1(i\alpha R)} + \frac{3}{4}(1+\nu) + \frac{4}{(\alpha R)^2}(1+\nu) \right\} \quad (6.21)$$

and (6.17) becomes

$$\begin{aligned} (\sigma_r)_{r=R} &= \frac{q^2 R^6 E}{(1-\nu) D^2} \frac{1}{16(\alpha R)^4} \left\{ \frac{J_0^2(i\alpha R)}{J_1^2(i\alpha R)} + \frac{2J_0(i\alpha R)}{i\alpha R J_1(i\alpha R)} + \frac{3}{2} + \frac{8}{(\alpha R)^2} \right\} \\ &= \frac{E h^2 \alpha^2}{24(1-\nu)} \end{aligned} \quad (6.22)$$

The bending stresses are given by

$$(\sigma_r')_{r=0} = (\sigma_\theta')_{r=0} = \frac{qR^2}{h^2} \frac{3(1+\nu)}{(\alpha R)^2} \left[1 + \frac{\alpha R}{2iJ_1(i\alpha R)} \right] \quad (6.23)$$

$$(\sigma_r')_{r=R} = \frac{1}{2} (\sigma_\theta')_{r=R} = \frac{qR^2}{h^2} \frac{6}{(\alpha R)^2} \left[1 + \frac{\alpha R J_0(i\alpha R)}{2iJ_1(i\alpha R)} \right] \quad (6.24)$$

The membrane and bending stresses given in (6.21) through (6.24) are plotted in Fig. 10, along with the results of S. Way (Ref. 6). The results of the computations used in plotting Fig. 10 are given in Table 6. A comparison of the two solutions over the range that was computed by Way shows a maximum error in the bending stresses of less than 6 percent. The curves of Way for the membrane stresses could not be distinguished from the curves for the solution proposed herein over the range that was computed by Way.

Limit $\alpha \rightarrow \infty$

The maximum error in the stresses is likely to occur when the error in the deflection is a maximum which appears to be the case for very large α . The approximate stress at the center given by (6.21) for very large α is

$$(\sigma_r)_{r=0} = \frac{q^2 R^6 E}{D^2} \frac{3-\nu}{64(1-\nu)(\alpha R)^4} \quad (6.25)$$

Substituting (4.20) into the above equation gives

$$(\sigma_r)_{r=0} = \frac{Eh^2}{R^2} \left[\frac{4qR^4}{3Dh} \right]^{\frac{2}{3}} \frac{3-\nu}{64(1-\nu)} \quad (6.26)$$

For very large α the approximate stress at the edge given by (6.22) is

$$(\sigma_r)_{r=R} = \frac{2}{3-\nu} (\sigma_r)_{r=0} \quad (6.27)$$

For $\nu = 0.3$, the approximate stresses given by (6.26) and (6.27) for very large α are

$$(\sigma_r)_{r=0} = 0.359 \left[\frac{q^2 R^2 E}{h^2} \right]^{\frac{1}{3}} = 0.965 \frac{E}{R^2} w_{\max}^2 \quad (6.28)$$

$$(\sigma_r)_{r=R} = 0.266 \left[\frac{q^2 R^2 E}{h^2} \right]^{\frac{1}{3}} = 0.715 \frac{E}{R^2} w_{\max}^2 \quad (6.29)$$

Hencky's exact result for the circular membrane is (Ref. 12)

$$(\sigma_r)_{r=0} = 0.423 \left[\frac{q^2 R^2 E}{h^2} \right]^{\frac{1}{3}} = 0.965 \frac{E}{R^2} w_{\max}^2 \quad (6.30)$$

$$(\sigma_r)_{r=R} = 0.328 \left[\frac{q^2 R^2 E}{h^2} \right]^{\frac{1}{3}} = 0.748 \frac{E}{R^2} w_{\max}^2 \quad (6.31)$$

The stress at the center given in (6.28) is 15 percent lower than that given in (6.30) when the stress is written as a function of the loading. However, when it is written as a function of the deflection, the two results are the same. The stress at the edge given in (6.29) is 19 percent lower than that given in (6.31) when it is written as a function of the loading, but it is less than 5 percent lower when written as a function of the deflection. We see, therefore, that for a uniformly loaded circular plate, the maximum error in the membrane stresses is approximately twice the maximum error in the deflection when both are given as functions of the loading, but when the stresses are given as functions of the deflections, the error is considerably less.

b. Rectangular Plates with Uniform Loading

To find the membrane stresses in a rectangular plate, it is convenient to write (6.6) and (6.7) in terms of those stresses. The relations between the stresses and the deflections are given by (Ref. 9, p. 342)

$$\sigma_x = \frac{E}{1-\nu^2} \left[\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\nu}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] \quad (6.32)$$

$$\sigma_y = \frac{E}{1-\nu^2} \left[\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 + \frac{\nu}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \quad (6.33)$$

$$\tau_{xy} = G \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \quad (6.34)$$

Substitution of these equations into (6.6) and (6.7) gives the equilibrium equations for the stresses in the middle surface of the plate

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (6.35)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \quad (6.36)$$

These equations are identically satisfied by the introduction of a stress function defined by

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \quad (6.37)$$

From the compatibility equation, we get the equation for the stress function

$$\nabla^4 F = E \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \quad (6.38)$$

Using the expansion given in (5.1), the above equation becomes

$$\begin{aligned}
 \frac{1}{E} \nabla^4 F = & \frac{1}{2} \sum_{n=0}^{\infty} \beta_n^2 \left[\left(\frac{df_n}{dy} \right)^2 + f_n \frac{d^2 f_n}{dy^2} \right] \\
 & + \frac{1}{2} \sum_{k=1}^{\infty} \cos \frac{k\pi x}{a} \left\{ \sum_{n=0}^{\infty} \left[\beta_n \beta_{n+k} \frac{df_n}{dy} \frac{df_{n+k}}{dy} + \beta_n^2 f_n \frac{d^2 f_{n+k}}{dy^2} \right] \right. \\
 & - \sum_{n=0}^{k-1} \left[\beta_n \beta_{k-n-1} \frac{df_n}{dy} \frac{df_{k-n-1}}{dy} - \beta_n^2 f_n \frac{d^2 f_{k-n-1}}{dy^2} \right] \\
 & \left. + \sum_{n=k}^{\infty} \left[\beta_n \beta_{n-k} \frac{df_n}{dy} \frac{df_{n-k}}{dy} + \beta_n^2 f_n \frac{d^2 f_{n-k}}{dy^2} \right] \right\} \quad (6.39)
 \end{aligned}$$

Since

$$\begin{aligned}
 2 \frac{df_n}{dy} \frac{df_m}{dy} = & \beta_n \beta_m A_n A_m [\cosh(\beta_n + \beta_m)y - \cosh(\beta_n - \beta_m)y] \\
 & + \delta_n \delta_m B_n B_m [\cosh(\delta_n + \delta_m)y - \cosh(\delta_n - \delta_m)y] \\
 & + \beta_n \delta_m A_n B_m [\cosh(\beta_n + \delta_m)y - \cosh(\beta_n - \delta_m)y] \\
 & + \beta_m \delta_n A_m B_n [\cosh(\beta_m + \delta_n)y - \cosh(\beta_m - \delta_n)y] \quad (6.40)
 \end{aligned}$$

and

$$\begin{aligned}
 2 f_n \frac{d^2 f_m}{dy^2} = & \frac{4g(-)^n}{2D\beta_n^3 \delta_n^2} [\beta_m^2 A_m \cosh \beta_m y + \delta_m^2 B_m \cosh \delta_m y] \\
 & + \beta_m^2 A_n A_m [\cosh(\beta_n + \beta_m)y - \cosh(\beta_n - \beta_m)y] \\
 & + \delta_m^2 B_n B_m [\cosh(\delta_n + \delta_m)y - \cosh(\delta_n - \delta_m)y] \\
 & + \beta_m^2 A_m B_n [\cosh(\beta_m + \delta_n)y - \cosh(\beta_m - \delta_n)y] \\
 & + \delta_m^2 A_n B_m [\cosh(\beta_n + \delta_m)y - \cosh(\beta_n - \delta_m)y] \quad (6.41)
 \end{aligned}$$

One can write (6.39) as

$$\frac{1}{E} \nabla^4 F = \sum_{k=0}^{\infty} \cos \frac{k\pi x}{a} \sum_{m=1}^M \sum_{n=0}^{\infty} C_{kmn} \cosh \lambda_{kmn} y \quad (6.42)$$

where the C_{kmn} 's and the λ_{kmn} 's are defined by substituting (6.40) and (6.41) into (6.39).

If the stress function is expanded in the following series

$$F = \sum_{k=0}^{\infty} F_k(y) \cos \frac{k\pi x}{a} \quad (6.43)$$

and this series is substituted in (6.42), one can solve for $F_k(y)$.

The symmetric solution is

$$F_k = D_k \cosh \frac{k\pi y}{a} + E_k \frac{k\pi y}{a} \sinh \frac{k\pi y}{a} + \sum_{m=1}^M \sum_{\substack{n=0 \\ \neq N_m}}^{\infty} \frac{C_{kmn} \cosh \lambda_{kmn} y}{[\lambda_{kmn}^2 - (\frac{k\pi}{a})^2]^2} + \sum_{m=1}^M \frac{C_{kmn} (\frac{k\pi y}{a})^2 \cosh \frac{k\pi y}{a}}{8 (\frac{k\pi}{a})^4} \quad (6.44)$$

where N_m is defined as that value of n for any given value of m where $\lambda_{kmn}^2 = (\frac{k\pi}{a})^2$

Since the boundary conditions are given on u and v , it is necessary to solve for them in order that one may evaluate the constants B_k and E_k . From (6.32), (6.33), and (6.37) one gets the relations

$$\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 = \frac{1}{E} \left(\frac{\partial^2 F}{\partial y^2} - \nu \frac{\partial^2 F}{\partial x^2} \right) \quad (6.45)$$

$$\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 = \frac{1}{E} \left(\frac{\partial^2 F}{\partial x^2} - \nu \frac{\partial^2 F}{\partial y^2} \right) \quad (6.46)$$

For simplicity, (5.10) and (5.11) can be written as

$$\left(\frac{\partial w}{\partial x} \right)^2 = 2 \sum_{k=0}^{\infty} G_k(y) \cos \frac{k\pi x}{a} \quad (6.47)$$

$$\left(\frac{\partial w}{\partial y}\right)^2 = 2 \sum_{k=0}^{\infty} \frac{d}{dy} \left[H_k(y) \right] \cos \frac{k\pi x}{a} \quad (6.48)$$

Then, using the expansion for u given in (5.12), one gets the following equation from (6.45)

$$\begin{aligned} \frac{k\pi}{a} g_k + \Phi_k &= \left(\frac{k\pi}{a}\right)^2 \left[(1-\nu) D_k + 2 E_k \right] \cosh \frac{k\pi y}{a} \\ &+ (1-\nu) \left(\frac{k\pi}{a}\right)^2 E_k \frac{k\pi y}{a} \sinh \frac{k\pi y}{a} \end{aligned} \quad (6.49)$$

where

$$\begin{aligned} \Phi_k &= G_k - \sum_{m=1}^M \sum_{\substack{n=0 \\ \neq N_m}}^{\infty} C_{kmn} \frac{[\lambda_{kmn}^2 - \nu \left(\frac{k\pi}{a}\right)^2]}{[\lambda_{kmn}^2 - \left(\frac{k\pi}{a}\right)^2]^2} \cosh \lambda_{kmn} y \\ &- \left[\frac{(2-\nu) + \left(\frac{k\pi y}{a}\right)^2}{8 \left(\frac{k\pi}{a}\right)^2} \cosh \frac{k\pi y}{a} + \frac{\frac{k\pi y}{a}}{2 \left(\frac{k\pi}{a}\right)^2} \sinh \frac{k\pi y}{a} \right] \sum_{m=1}^M C_{kmN_m} \end{aligned} \quad (6.50)$$

Using the expansion for v given in (5.13), one gets the following equation from (6.46)

$$\begin{aligned} \frac{d}{dy} [\ell_k + \Theta_k] &= \left(\frac{k\pi}{a}\right)^2 \left[(1-\nu) D_k - 2\nu E_k \right] \cosh \frac{k\pi y}{a} \\ &+ \left(\frac{k\pi}{a}\right)^2 (1-\nu) E_k \frac{k\pi y}{a} \sinh \frac{k\pi y}{a} \end{aligned} \quad (6.51)$$

where

$$\begin{aligned} \frac{d\Theta_k}{dy} &= \frac{dH_k}{dy} - \sum_{m=1}^M \sum_{\substack{n=0 \\ \neq N_m}}^{\infty} C_{kmn} \frac{[\left(\frac{k\pi}{a}\right)^2 - \nu \lambda_{kmn}^2]}{[\lambda_{kmn}^2 - \left(\frac{k\pi}{a}\right)^2]^2} \cosh \lambda_{kmn} y \\ &- \left[\frac{(2\nu-1) + \nu \left(\frac{k\pi y}{a}\right)^2}{8 \left(\frac{k\pi}{a}\right)^2} \cosh \frac{k\pi y}{a} + \frac{\nu \frac{k\pi y}{a}}{2 \left(\frac{k\pi}{a}\right)^2} \sinh \frac{k\pi y}{a} \right] \sum_{m=1}^M C_{kmN_m} \end{aligned} \quad (6.52)$$

Integration of (4.51) gives

$$\ell_k + \Theta_k = \frac{\kappa\pi}{a} \left[(1-\nu) D_k - (1+\nu) E_k \right] \sinh \frac{\kappa\pi y}{a} + \frac{\kappa\pi}{a} (1-\nu) E_k \frac{\kappa\pi y}{a} \cosh \frac{\kappa\pi y}{a} \quad (6.53)$$

If the shear is zero along the edges of the plate, the boundary conditions on $g_k(y)$ and $\ell_k(y)$ are given in (5.29). Substituting them into (6.49) and (6.53) and solving for D_k and E_k , one gets

$$D_k = \frac{\frac{d\bar{\Phi}_k(b)}{dy} \left[\frac{1+\nu}{1-\nu} - \frac{\kappa\pi b}{a} \coth \frac{\kappa\pi b}{a} \right] + \left(\frac{\kappa\pi}{a} \right)^2 \bar{\Theta}_k(b) \left[\frac{3-\nu}{1-\nu} + \frac{\kappa\pi b}{a} \coth \frac{\kappa\pi b}{a} \right]}{4 \left(\frac{\kappa\pi}{a} \right)^3 \sinh \frac{\kappa\pi b}{a}} \quad (6.54)$$

$$E_k = \frac{\frac{d\bar{\Phi}_k(b)}{dy} - \left(\frac{\kappa\pi}{a} \right)^2 \bar{\Theta}_k(b)}{4 \left(\frac{\kappa\pi}{a} \right)^3 \sinh \frac{\kappa\pi b}{a}} \quad (6.55)$$

If the displacements are zero on the boundary, the boundary conditions given in (5.30) can be used. Substituting them into (6.49) and (6.53) and solving for D_k and E_k , one gets

$$D_k = \frac{\bar{\Phi}_k(b) \left[\frac{1+\nu}{1-\nu} - \frac{\kappa\pi b}{a} \coth \frac{\kappa\pi b}{a} \right] + \bar{\Theta}_k(b) \frac{\kappa\pi}{a} \left[\frac{2}{1-\nu} \coth \frac{\kappa\pi b}{a} + \frac{\kappa\pi b}{a} \right]}{\left(\frac{\kappa\pi}{a} \right)^2 \left[(3+\nu) \cosh \frac{\kappa\pi b}{a} - (1-\nu) \frac{\kappa\pi b}{a} \operatorname{csch} \frac{\kappa\pi b}{a} \right]} \quad (6.56)$$

$$E_k = \frac{\bar{\Phi}_k(b) - \bar{\Theta}_k(b) \frac{\kappa\pi}{a} \coth \frac{\kappa\pi b}{a}}{\left(\frac{\kappa\pi}{a} \right)^2 \left[(3+\nu) \cosh \frac{\kappa\pi b}{a} - (1-\nu) \frac{\kappa\pi b}{a} \operatorname{csch} \frac{\kappa\pi b}{a} \right]} \quad (6.57)$$

The stress function has now been completely determined. The membrane stresses may then be found by differentiation according to (6.37).

The bending stresses are found by substituting (5.9) into (6.3), (6.4), and (6.5). This gives

$$\sigma_x' = \frac{6D}{h^2} \left\{ \frac{q}{D\alpha^2} \left[1 - \frac{\cosh \alpha x}{\cosh \alpha d} \right] + \sum_{n=0}^{\infty} \left[(1-\nu)\beta_n^2 A_n \cosh \beta_n y + (\beta_n^2 - \nu\delta_n^2) B_n \cosh \delta_n y \right] \cos \beta_n x \right\} \quad (6.58)$$

$$\sigma_y' = \frac{6D}{h^2} \left\{ \frac{\nu q}{D\alpha^2} \left[1 - \frac{\cosh \alpha x}{\cosh \alpha d} \right] - \sum_{n=0}^{\infty} \left[(1-\nu)\beta_n^2 A_n \cosh \beta_n y + (\delta_n^2 - \nu\beta_n^2) B_n \cosh \delta_n y \right] \cos \beta_n x \right\} \quad (6.59)$$

$$\tau_{xy}' = \frac{6D}{h^2} (1-\nu) \sum_{n=0}^{\infty} \left[\beta_n^2 A_n \sinh \beta_n y + \beta_n \delta_n B_n \sinh \delta_n y \right] \cos \beta_n x \quad (6.60)$$

It is apparent that a considerable amount of computational work is necessary to compute the membrane stresses at a given point on a rectangular plate. However, the fast convergence of the series that are involved makes the actual computations less formidable than might be indicated by the complication of the formulae. On the other hand, the bending stresses are quite simple and can be computed easily. The errors involved in the stresses, as in the case of the circular plate, will be greater than those in the deflection, and the greatest error for any given point on the plate will occur when a approaches infinity.

VII. DISCUSSION OF RESULTS

The approximation based on neglecting the second strain invariant in the strain energy integral when deriving the differential equations has been investigated for uniformly loaded circular and rectangular plates. Maximum deflections were computed for clamped, simply supported, and elastically built-in circular plates and simply supported and semi-clamped rectangular plates. Membrane and bending stresses were computed for clamped circular plates. The results are plotted in Figs. 4, 6, 7, 9, and 10. In all cases where comparisons were possible good agreement with exact solutions is obtained over the complete range from the linear plate to the membrane. The amount of calculation necessary to determine the deflection or the stresses for any given problem is considerably less than that needed when the complete equations are solved. Consequently, this approximation enables one to carry out a simple analysis for plates where the combination of size and loading lies between those that can be analyzed by the linear plate theory and those that can be treated as membranes.

It appears that no simple physical interpretation can be given to the approximation resulting from neglecting the second strain invariant. It is possible, however, to interpret the resulting analysis in terms of the stresses by investigating the assumptions on the stresses that must be made to derive the differential equations that are used in this thesis. The equation for the deflection, (3.16), could have been derived if it had been assumed that

$$\sigma_x = \sigma_y = \frac{D\alpha^2}{h} \quad ; \quad \tau_{xy} = 0 \quad (7.1)$$

That is, (3.16) is the equation for the deflection when the membrane stresses are those of an ideal membrane. The constant, α , which can be interpreted as an average value of the membrane stresses over the plate when (7.1) is not satisfied exactly, is then determined from (3.15) in the case of a rectangular plate. The condition given in (3.15) is equivalent to assuming that the sum of the stresses is given by

$$\sigma_x + \sigma_y = \frac{D\alpha^2}{h} (1+\nu) \quad (7.2)$$

everywhere on the plate. Thus, once the deflection has been found as a function of α , the condition that the normal stresses are constant and equal is relaxed, and the constant, α , is determined using the assumption that the sum of the stresses is given by (7.2). It is important to note that the sum of the stresses given in (7.2) is not equal to the sum of the stresses given in (7.1). This difference may well account for the accuracy of the approximation that has been demonstrated by the calculations carried out in this thesis.

The displacements that can be obtained at this point of the analysis are found by assuming that the deflection is given by the above approximation, and then this approximate deflection is substituted in (3.15) or (7.2) to find the displacements. (Since there is only one equation and there are two displacements in the case of a rectangular plate, only a relation between the displacements can be found. If, for example, it were reasonable to assume, in addition, that τ_{xy} is zero everywhere, one could find the displacements explicitly.)

If the assumption given in (7.2) is not a good approximation, it is necessary to go one step further to determine the stresses. In this case the deflection that is found from the above outlined analysis is used in the exact equilibrium equations in the plane of the plate to find the displacements or the membrane stresses. The stresses determined in this manner will be reliable if the approximation to the deflection is good.

From the foregoing discussion it seems reasonable to assume that the proposed analysis will give good results if the conditions on the stresses given in (7.1) or (7.2) are nearly satisfied. The real test, however, of whether or not the proposed analysis will give a good approximation to the solution of any particular problem must be based on whether the deflection determined in this manner is a good approximation to the true deflection.

In the case of uniformly loaded clamped circular plates, it was possible to check the deflections for values of αR from zero to approximately 3.5 and for very large values of αR . For simply supported circular plates it was possible to check the deflection only for very small and very large values of αR . The deflections for uniformly loaded clamped rectangular plates were checked for values of αa from zero to approximately 5.5 and for very large values of αa . For semi-clamped rectangular plates the deflections were checked only for very small and very large values of αa . For values of αR or αa other than those specified above, no checks were possible. Consequently, the curves that are plotted in this thesis for these unchecked values can only be considered as tentative solutions until checks are available to verify their accuracy. Similarly, if the proposed approach is extended to other problems where no checks are available, the solutions must also be considered tentative.

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TABLE 1
CENTER DEFLECTIONS OF CLAMPED CIRCULAR PLATES

aR	$\frac{qR^4}{Dh}$	$\frac{w_{\max}}{h}$
1.0	24.13	0.353
1.5	39.01	0.527
2.0	57.25	0.700
2.5	79.92	0.870
3.0	108.06	1.037
3.5	142.73	1.202
4.0	184.87	1.364
4.5	235.47	1.523
5.0	295.44	1.680
5.5	365.73	1.835

TABLE 2
CENTER DEFLECTIONS OF
SIMPLY SUPPORTED CIRCULAR PLATES

aR	$\frac{qR^4}{Dh}$	$\frac{w_{\max}}{h}$
1.0	5.91	0.310
1.5	11.05	0.464
2.0	18.80	0.617
2.5	30.05	0.768
3.0	45.66	0.917
3.5	66.50	1.065
4.0	93.43	1.212
4.5	127.33	1.357
5.0	169.04	1.502
5.5	219.44	1.647

TABLE 3

CENTER DEFLECTIONS OF RECTANGULAR PLATES WITH ALL EDGES SIMPLY SUPPORTED

a/a	$\frac{b}{a} = 1$		$\frac{b}{a} = 1.5$		$\frac{b}{a} = 2$		$\frac{b}{a} = \infty$	
	$\frac{qa^4}{Dh}$	$\frac{w}{h} \max$	$\frac{qa^4}{Dh}$	$\frac{w}{h} \max$	$\frac{qa^4}{Dh}$	$\frac{w}{h} \max$	$\frac{qa^4}{Dh}$	$\frac{w}{h} \max$
1.0	6.64	0.358	4.33	0.410	3.60	0.436	2.47	0.366
1.5	12.03	0.533	8.26	0.618	7.03	0.647	5.04	0.548
2.0	19.95	0.707	14.32	0.817	12.43	0.852	9.21	0.729
2.5	31.19	0.877	23.19	1.012	20.38	1.051	15.51	0.909
3.0	46.59	1.046	35.58	1.203	31.66	1.251	24.48	1.088
3.5	66.95	1.211	52.17	1.392	46.78	1.447	36.65	1.266
4.0	93.10	1.375	73.64	1.577	66.41	1.641	52.54	1.444
4.5	125.86	1.538	100.69	1.765	91.19	1.834	72.68	1.621
5.0	166.03	1.697	134.00	1.949	121.77	2.026	97.61	1.798
5.5	214.42	1.858	174.27	2.134	158.76	2.219	127.85	1.975

TABLE 4

CONVERGENCE OF SERIES IN THE SOLUTION FOR SIMPLY SUPPORTED SQUARE PLATES

aa	$\sum_{n=0}^{\infty} \frac{1}{2^4 \beta_n^4 \delta_n^4} \left\{ \alpha^2 \beta_n^2 \operatorname{sech}^2 \delta_n b + \frac{\delta_n^4}{\beta_n b} \tanh \beta_n b - \frac{\beta_n^2 (\beta_n^2 + \frac{5}{2} \alpha^2)}{\delta_n b} \tanh \delta_n b \right\}$			$\sum_{n=0}^{\infty} \frac{2(-1)^n}{2^5 \alpha^2 \beta_n^3 \delta_n^2} \left\{ \delta_n^2 \operatorname{sech} \beta_n b - \beta_n^2 \operatorname{sech} \delta_n b \right\}$		
	1 TERM	2 TERMS	3 TERMS	1 TERM	2 TERMS	3 TERMS
1.0	0.011769	0.011770	0.011770	0.094258	0.094210	0.094210
1.5	0.030826	0.030833	0.030833	0.064414	0.064371	0.064371
2.0	0.049460	0.049478	0.049478	0.043723	0.043686	0.043686
2.5	0.063584	0.063620	0.063620	0.030470	0.030438	0.030438
3.0	0.073298	0.073358	0.073358	0.022017	0.021990	0.021990
3.5	0.079802	0.079891	0.079891	0.016483	0.016460	0.016460
4.0	0.084174	0.084294	0.084294	0.012736	0.012717	0.012717
4.5	0.087162	0.087313	0.087313	0.010109	0.010093	0.010093
5.0	0.089248	0.089429	0.089429	0.008206	0.008193	0.008193
5.5	0.090737	0.090946	0.090946	0.006790	0.006779	0.006779

TABLE 5

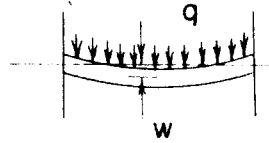
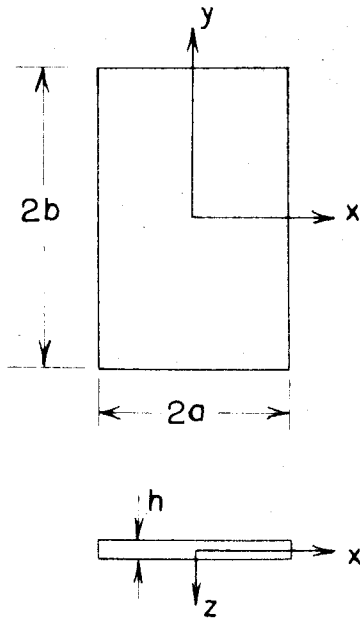
CENTER DEFLECTIONS OF RECTANGULAR PLATES
WITH TWO EDGES SIMPLY SUPPORTED
AND TWO EDGES CLAMPED

a/a	b/a = 1		b/a = 1.5	
	$\frac{qa^4}{Dh}$	$\frac{w_{max}}{h}$	$\frac{qa^4}{Dh}$	$\frac{w_{max}}{h}$
1.0	13.61	0.377	6.33	0.447
1.5	22.61	0.577	11.44	0.664
2.0	34.63	0.740	18.89	0.877
2.5	50.40	0.920	29.43	1.084
3.0	70.88	1.098	43.78	1.287
3.5	96.92	1.271	62.66	1.484
4.0	129.41	1.442	86.76	1.679
4.5	169.19	1.612	116.79	1.871
5.0	217.10	1.778	153.45	2.060
5.5	274.03	1.944	197.43	2.248

TABLE 6

STRESSES OF CLAMPED CIRCULAR PLATES

αR	BENDING STRESSES		MEMBRANE STRESSES	
	AT THE CENTER $\frac{(\sigma'_r)_{r=0} R^2}{E h^2}$	AT THE EDGE $\frac{(\sigma'_r)_{r=R} R^2}{E h^2}$	AT THE CENTER $\frac{(\sigma_r)_{r=0} R^2}{E h^2}$	AT THE EDGE $\frac{(\sigma_r)_{r=R} R^2}{E h^2}$
1.0	0.994	1.593	0.121	0.060
1.5	1.461	2.459	0.270	0.134
2.0	1.898	3.406	0.475	0.238
2.5	2.299	4.455	0.728	0.372
3.0	2.661	5.620	1.030	0.536
3.5	2.988	6.918	1.377	0.729
4.0	3.281	8.355	1.766	0.952
4.5	3.546	9.941	2.196	1.205
5.0	3.787	11.677	2.663	1.488
5.5	4.010	13.570	3.170	1.801



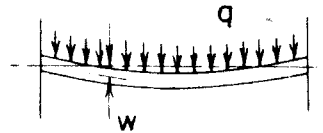
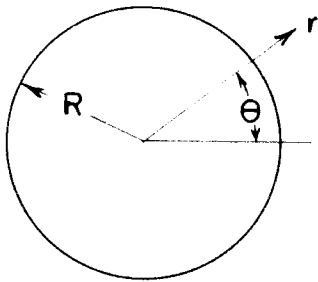
u = displacement of the middle surface in the x -direction

v = displacement of the middle surface in the y -direction

w = deflection of the middle surface in the z -direction

q = intensity of the uniform load

FIG.1 - SKETCH SHOWING NOTATION FOR RECTANGULAR PLATES



u = displacement of the middle surface in radial direction

w = deflection of the middle surface in the z -direction

q = intensity of the uniform load

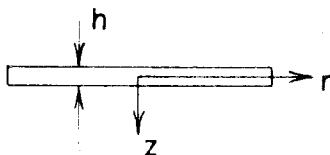


FIG.2 - SKETCH SHOWING NOTATION FOR CIRCULAR PLATES

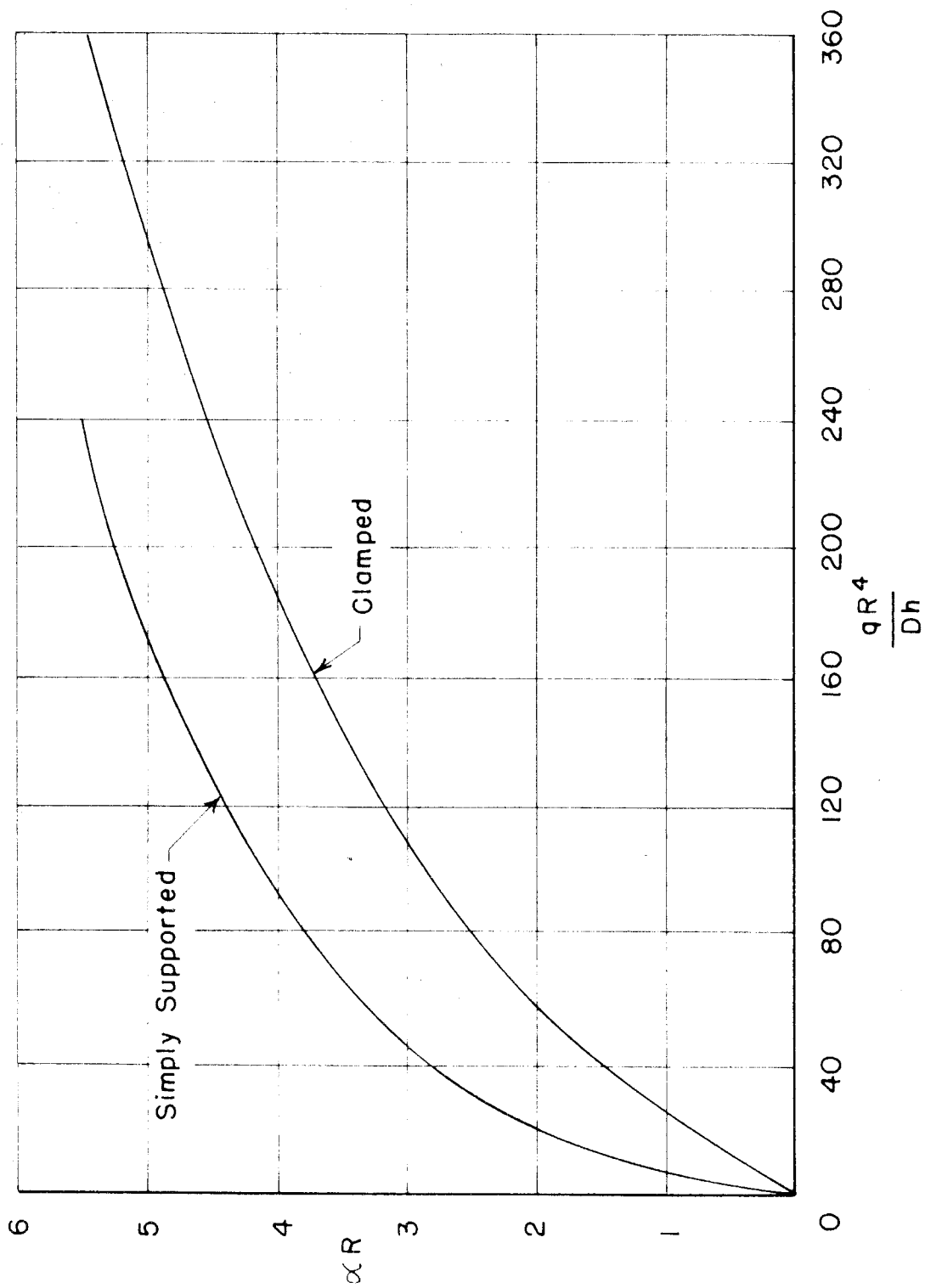


FIG. 3- αR vs. $\frac{q R^4}{D h}$ FOR CIRCULAR PLATES

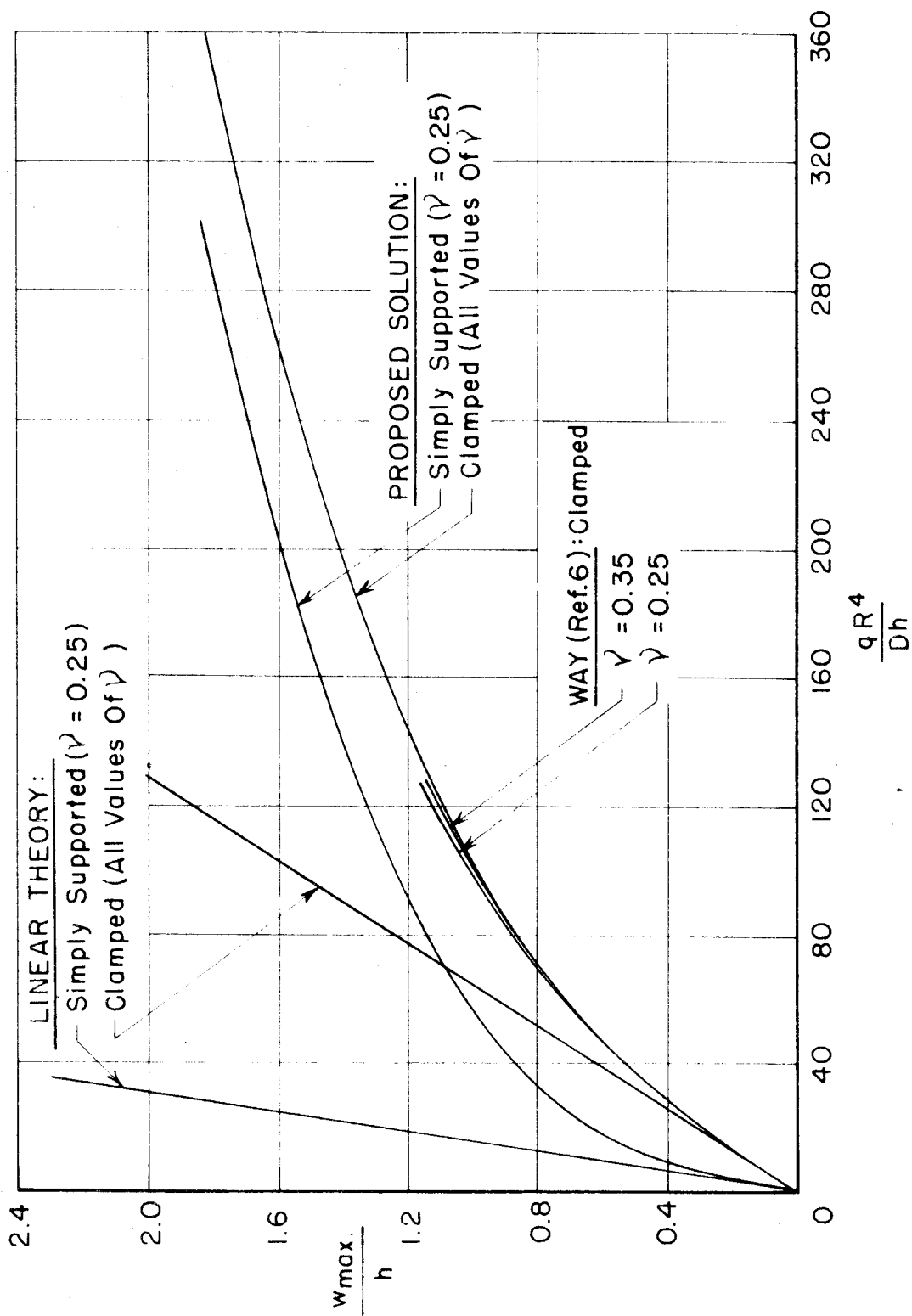


FIG. 4 - CENTER DEFLECTIONS FOR CIRCULAR PLATES

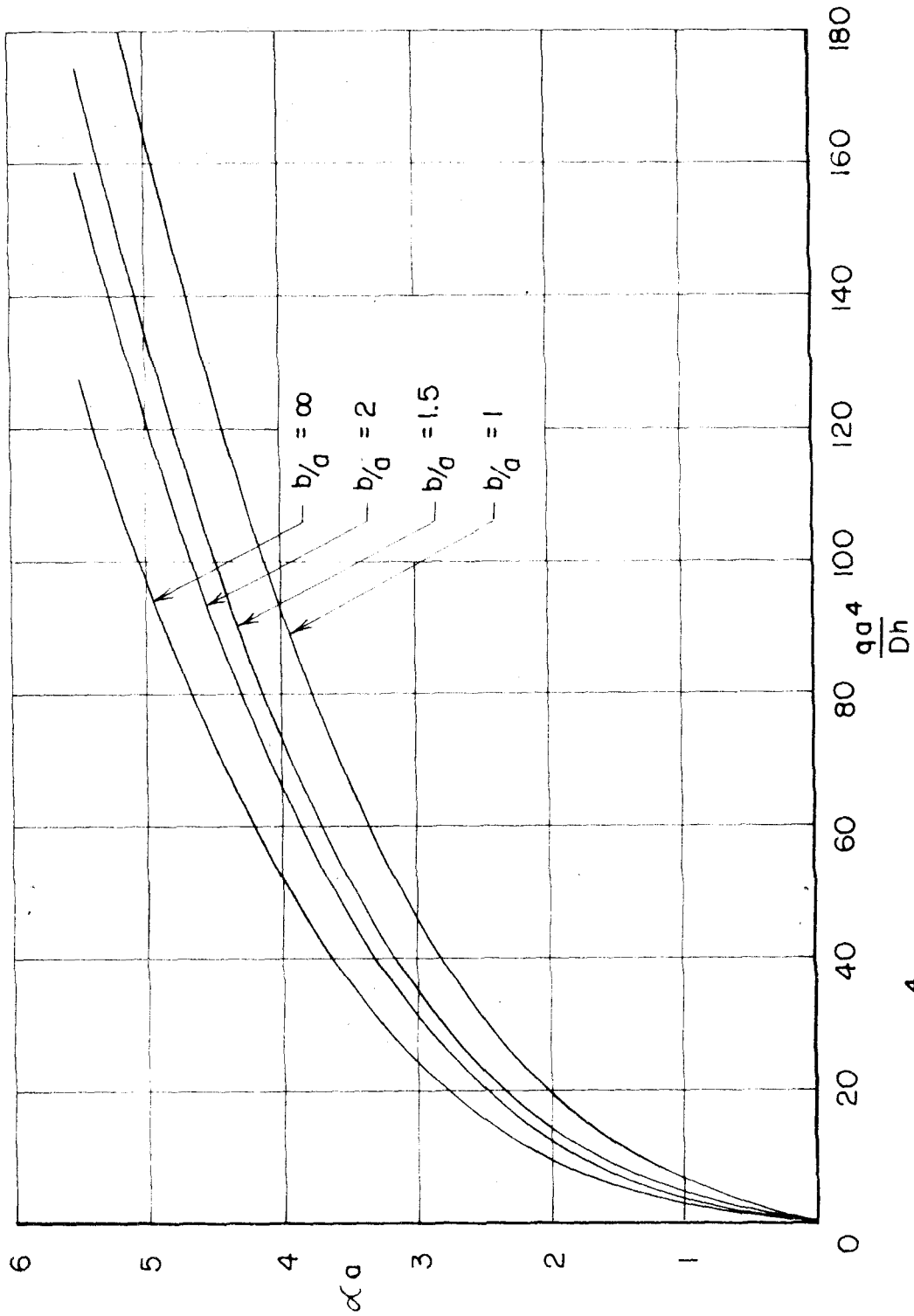


FIG. 5 - α_a vs. $\frac{q a^4}{D h}$ FOR RECTANGULAR PLATES WITH ALL EDGES SIMPLY SUPPORTED

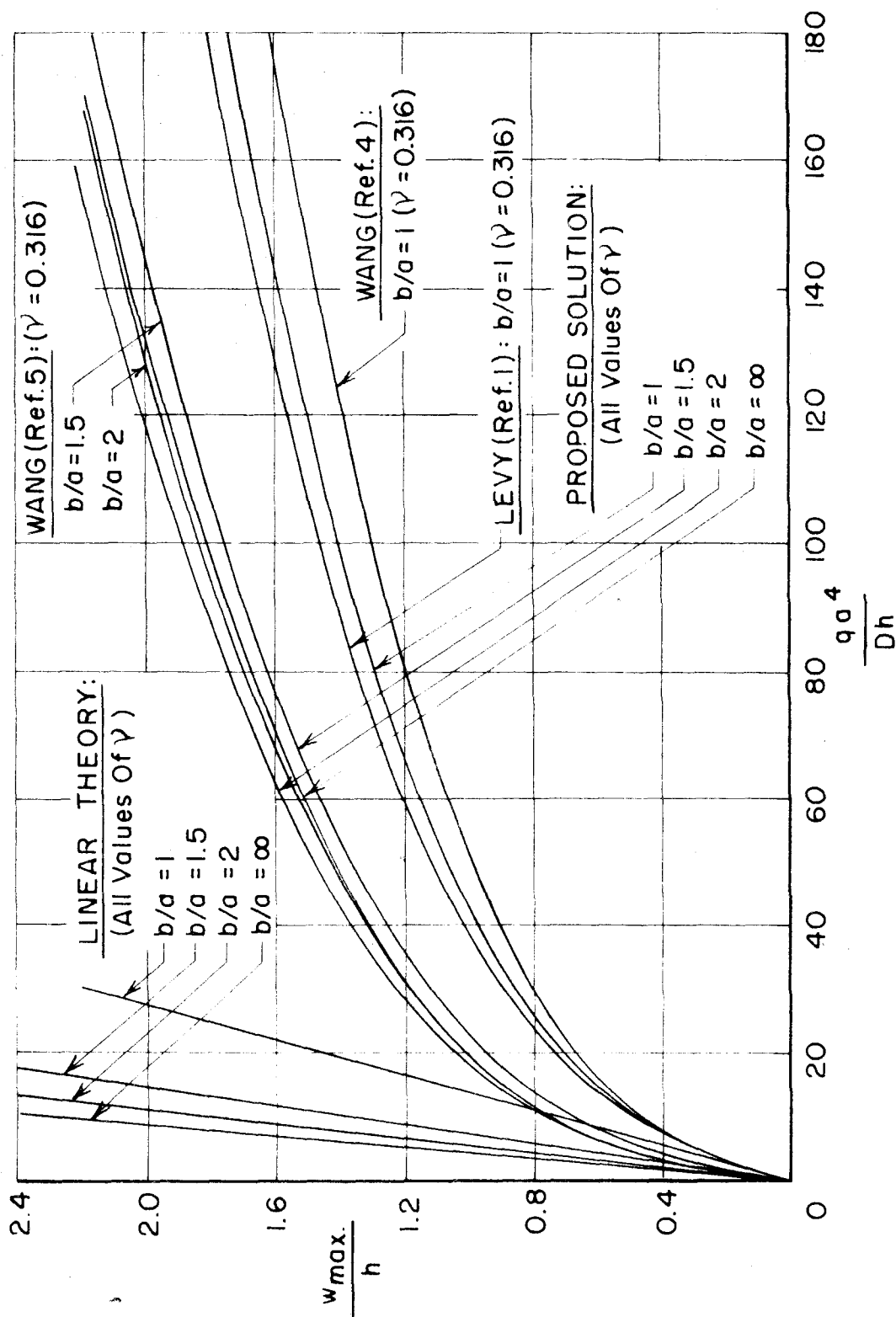


FIG. 6 — CENTER DEFLECTIONS FOR RECTANGULAR PLATES WITH ALL EDGES SIMPLY SUPPORTED

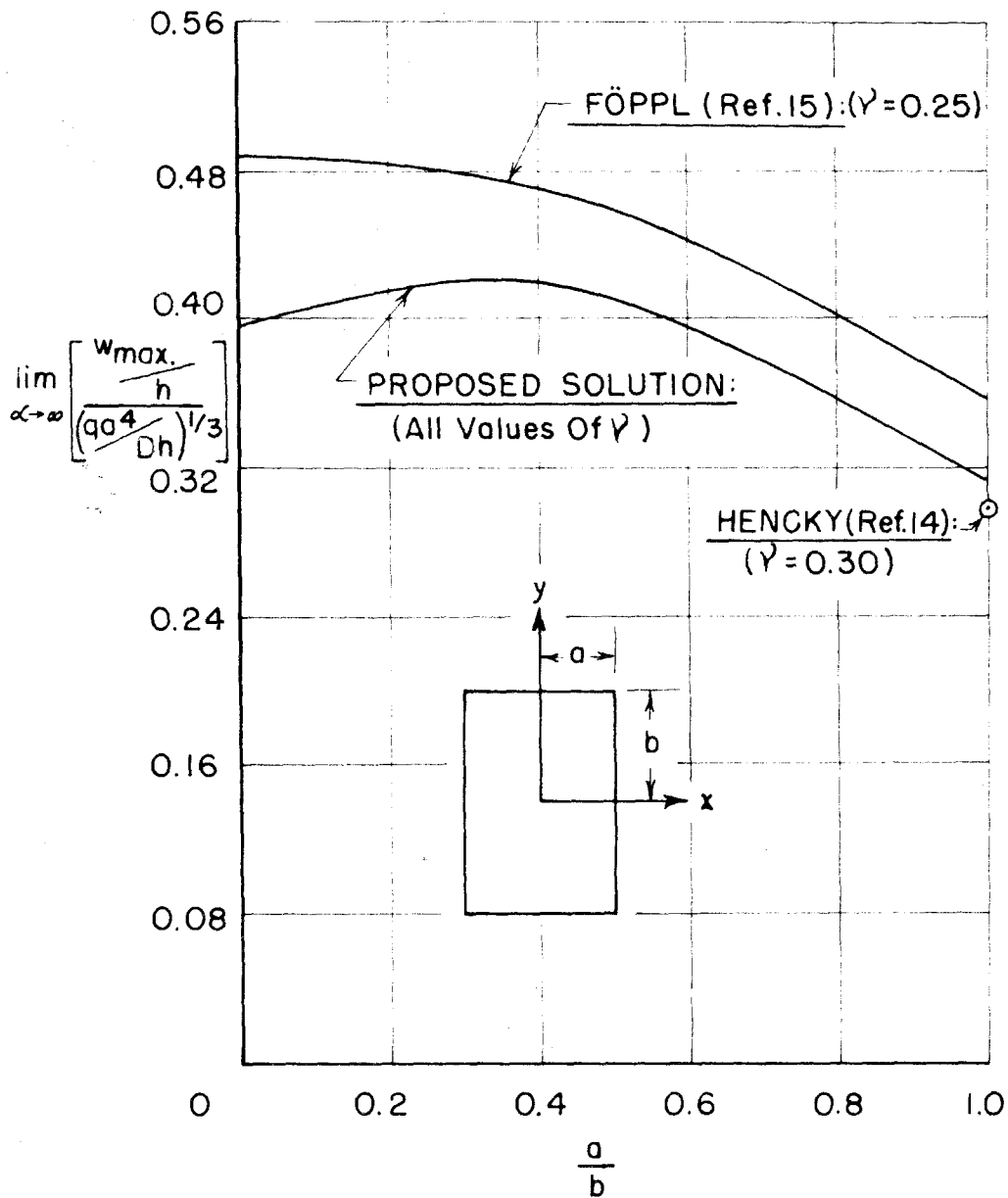


FIG. 7 - CENTER DEFLECTIONS FOR RECTANGULAR MEMBRANES

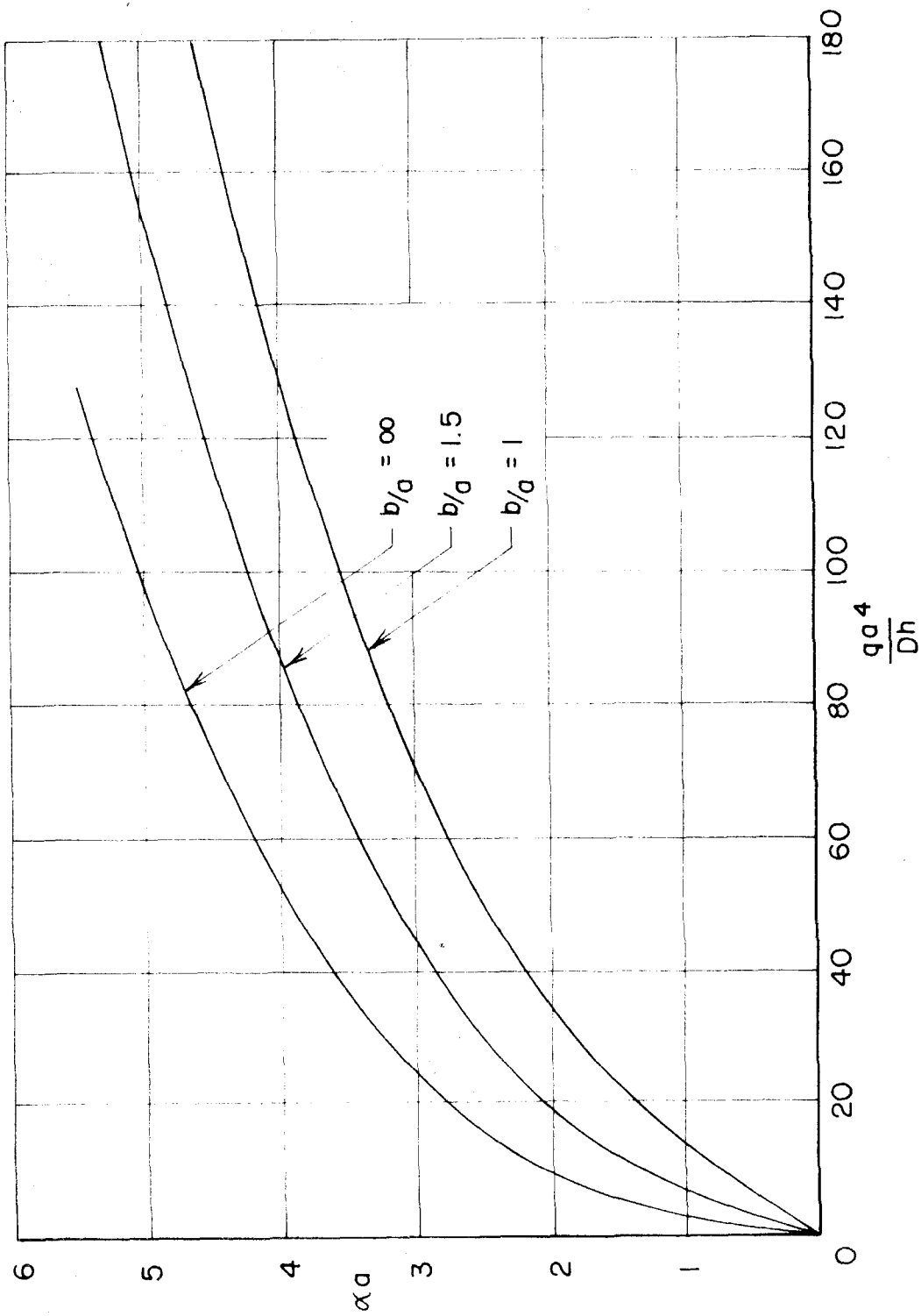


FIG. 8 - αa vs. $\frac{qa^4}{Dh}$ FOR RECTANGULAR PLATES WITH TWO EDGES
SIMPLY SUPPORTED AND TWO EDGES CLAMPED

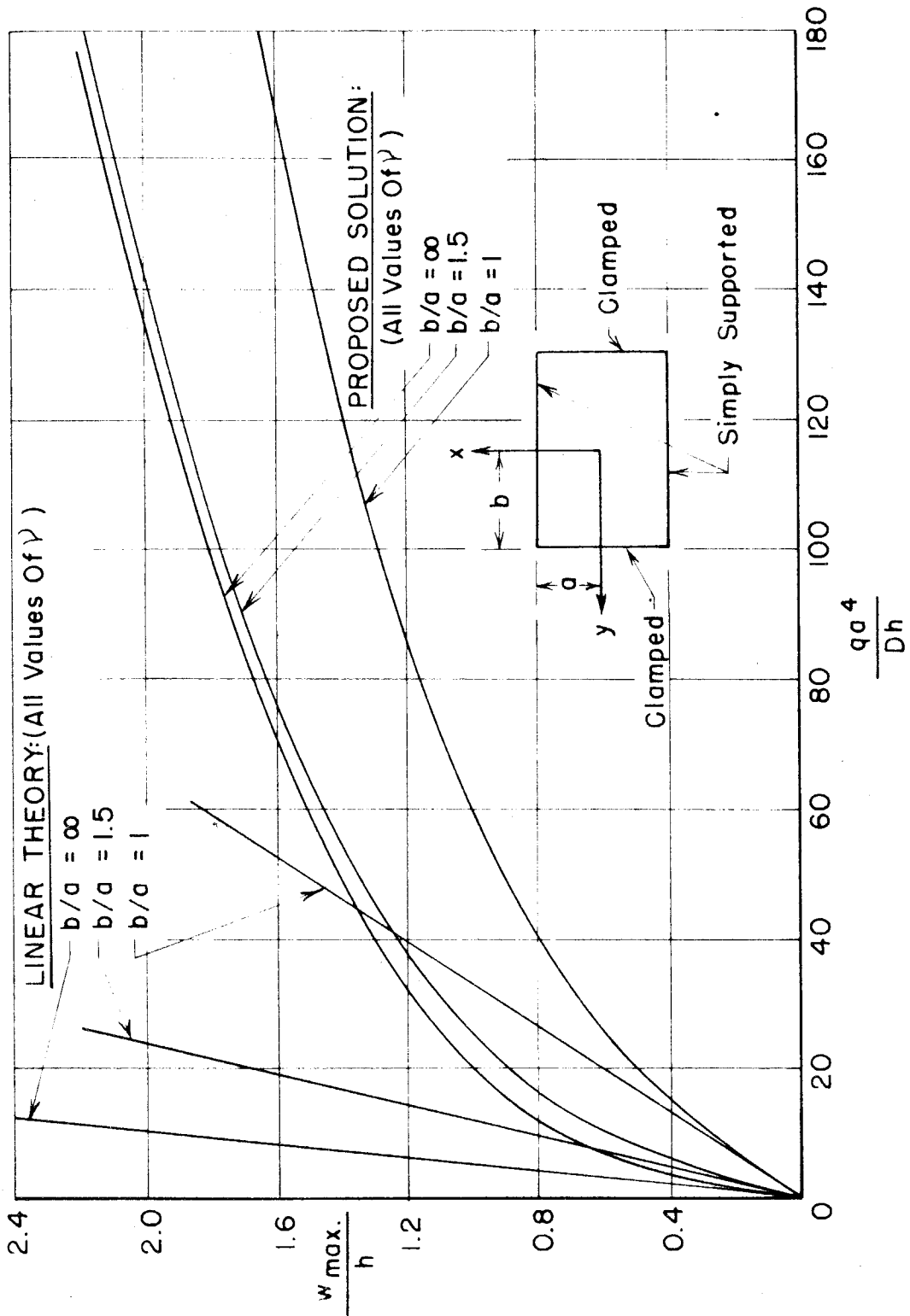


FIG. 9 — CENTER DEFLECTIONS FOR RECTANGULAR PLATES WITH TWO EDGES SIMPLY SUPPORTED AND TWO EDGES CLAMPED

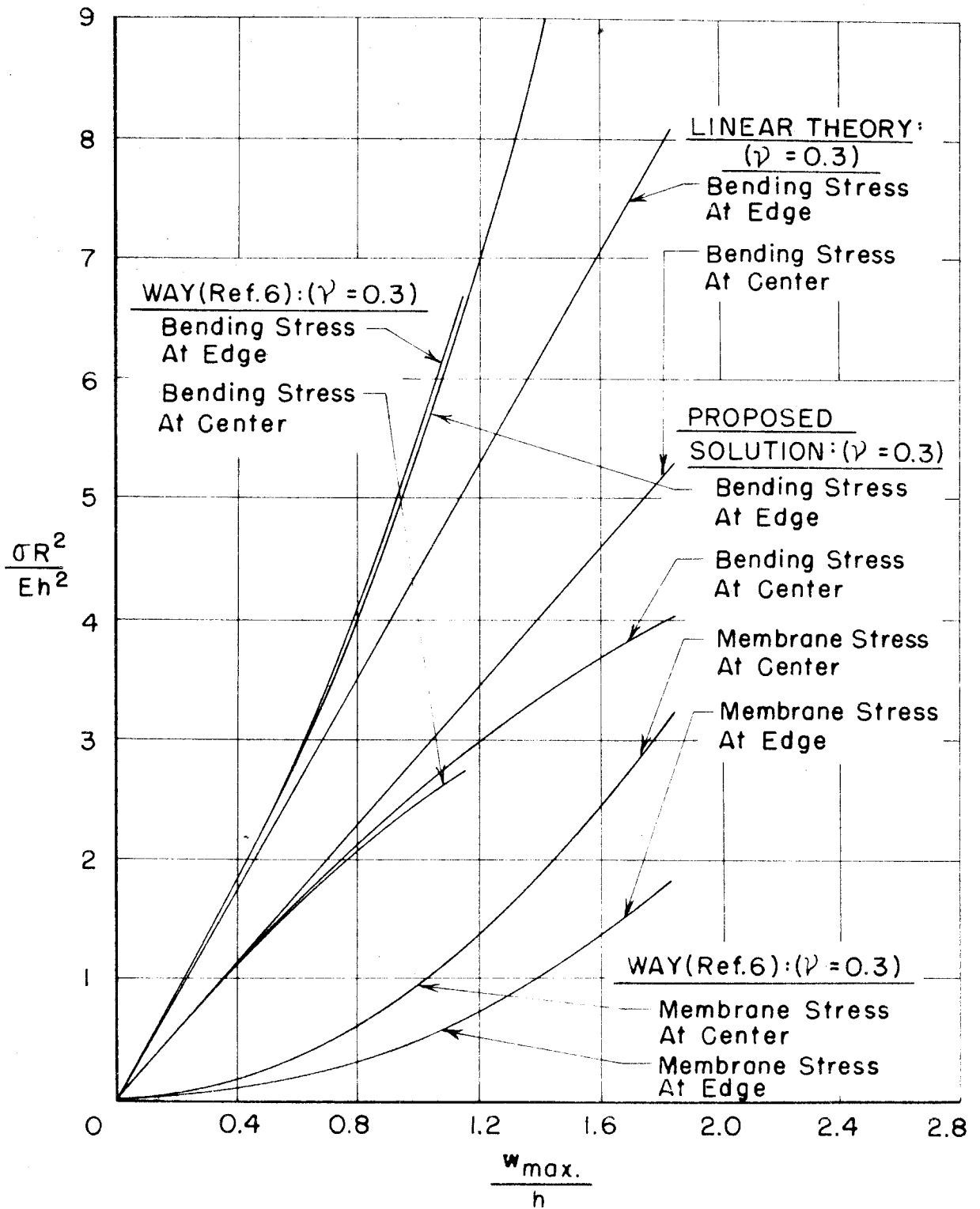


FIG.10 - CENTER AND EDGE STRESSES FOR CLAMPED CIRCULAR PLATES