A STUDY OF THE EFFECTS OF DAMPING ON NORMAL MODES OF ELECTRICAL AND MECHANICAL SYSTEMS

Thesis by

Stephen Franklin Crumb

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1955

ACKNOWLEDGEMENTS

The writer desires to express his sincere appreciation to the various members of the California Institute of Technology staff with whom he has been associated in the course of his graduate study and research. He wishes particularly to acknowledge the assistance and encourgagement given him by Dr. G. D. McCann, who directed the research, and Dr. R. H. MacNeal, who interested the writer in this project, and whose many suggestions and criticisms were invaluable.

The writer is indebted to the Air Research and Development Command, United States Air Force, for sponsoring the research project of which this study is a part. He is also indebted to the Douglas Aircraft Company, Inc., El Segundo, California, for cooperation in supplying a practical system problem to be used in the experimental work.

The writer is deeply grateful to his wife, LaNelle, for typing the original draft of the thesis, and to Jeanne Shacklett, for her wonderful cooperation in preparing the final copy.

ABSTRACT

This thesis is a general investigation of some of the properties of free and forced vibrations in linear, non-conservative systems. Particular emphasis is placed upon the problems which arise in normal mode studies made on the electric analog computer at the California Institute of Technology.

In Part I the major problems are defined, and limitations of the study are discussed. Part II is a review of the basic theory of normal modes, included primarily to establish familiarity with the notation to be used later. In Part III, modifications of normal mode concepts, as applied to damped systems, are examined. "Small damping" criteria are discussed, and a set of theorems of small damping is presented.

In Part IV a series of normal mode analog circuits for damped systems are developed. Part V is a study of uniform damping, generalizing and extending some of the work of Rayleigh, Bode, and Guillemin. It is shown that for any type of uniform damping, all of the basic normal mode concepts are preserved.

In Part VI the theory of mode separation in uniformly damped systems is considered. Criteria for determining mode frequencies and mode parameters are developed. A multiple drive method of exciting normal modes is proposed. In Part VII, some of the methods of Part VI are extended to non-uniformly damped systems. Equivalent orthogonal systems are proposed to approximate the behavior of systems with moderately non-uniform damping. A quantitative measure of non-uniformity is presented.

In Part VIII, numerical examples and experimental results in support of the theory are presented. Concluding remarks are made in Part IX.

PREFACE

The theory of vibrations is a very old and well developed subject, and the related literature is quite extensive. In the course of this study, a thorough survey of the literature relating to damped vibrations was attempted. It seems inevitable, though, that some important sources of information may have been overlooked.

In order to present a comprehensive treatment of the problem, and to preserve continuity, it has been necessary to include some very basic and well known theory in the text, as well as some published material which is rather obscure. Wherever possible, credit has been given when the results of other investigators are mentioned.

Except for minor novelties in presentation, the material of Part II and the first section of Part III is strictly review. To the best knowledge of the writer, the following portions of the work are presented for the first time, and are either original with him, or were developed by him after suggestions by members of his examining committee:

Part III - (1) Demonstration of the difficulty of expressing T, V, and F in complex normal coordinates. (2) The criterion for suppressing a mode when the system is excited by initial impulses. (3) The treatment of small damping, beginning with Theorem 3.

Part IV - The extension of the normal mode analog to damped systems, and the development of the various forms of these analogs.

Part V - (1) The generalization of some results of Bode and Guillemin, in Theorem 10. (2) Theorems 11 and 12. (3) The loci of Figure 11, with the exception of Figure 11e.

Beginning with Part VI, all of the remaining material is believed to be original, except in portions where direct credit is given.

TABLE OF CONTENTS

I. INTRODUCTION

The concept of normal modes of vibration was developed as a tool for the analysis of linear, conservative mechanical systems, with either continuous or lumped parameters. It continues to be associated primarily with such systems, as well as with electromagnetic systems such as transmission lines, waveguides, and cavity resonators. Though equally valid in the treatment of lumped parameter electric circuits which are purely reactive, the normal mode approach has found less general application for such systems. This is perhaps due to the fact that such circuits usually occur in standardized forms, such as filters or artificial transmission lines, which are more easily treated by other techniques. In the electromagnetic systems, parasitic damping is usually small, and usually uniform; hence it can often be treated by exact solutions of the system differential equations, or by perturbation techniques. Consequently, most of the technical literature relating directly to the effects of damping upon normal modes is concerned with elastic structures and other mechanical systems.

The generalized treatment of vibrating systems was developed to a high degree by early mathematicians and classical physicists, culminating in the comprehensive works of Lord Rayleigh (Refs. 1, 2). In connection with damping, Rayleigh introduced the dissipation function, discussed the special case of uniform damping, and applied perturbation theory to systems with small damping.

Since Rayleigh's time, one of the principal lines of development in normal mode theory has been the search for simpler and more powerful methods, both exact and approximate, for the solution of complex conservative system problems. On the contrary, there has been relatively little parallel development of general non-conservative system techniques. Portions of the literature deal with exact analytical methods, which are seldom applied to many-degree-of-freedom systems because they are so tedious. Much work has been done in investigations of the various types of damping that are encountered in practice, such as structural damping, viscous friction, coulomb friction, etc. Using one or more of these types of damping, inserted in various ways, numerous one, two or three degree of freedom systems have been examined intensively. Higher order damped systems are occasionally investigated; an example is the study of a six degree of freedom geared turbine ship drive in torsional damped vibration by Poritsky and Robinson in 1940 (Ref. 3).

The practical importance of damping has long been recognized in connection with vibrations of such complex structures as bridges (Ref. 4), buildings (Ref. 5), and ship structures (Ref. 6). The need for better understanding and improved techniques became acute with the development of the aircraft industry, and since World War II some generalized approaches to the study of forced damped vibrations have begun to appear (Refs. 7, 8). The development of electric analog circuits to represent complex elastic systems has made possible experimental normal mode studies during early stages of design (Ref. 9). These studies are supplemented by "shake tests" on the completed structure, and the results are used to predict critical flutter speeds of the aircraft, to determine stresses due to landing impact, etc. Thus, in aircraft development work, there are two major problems that arise due to the presence of damping:

(1) If the undamped mode shapes and frequencies of an idealized structure are known, and if the nature and distribution of the damping in various parts of the system are known, then how can the effects of this damping on the transient responses of the system best be predicted?

(2) How can one best determine, experimentally, the idealized normal modes of a structure which contains damping? This applies both to the aircraft shake test and to the electric analog normal mode study, since in both the presence of damping tends to couple the various modes of the driven system. The problem is severe when two or more modes lie close together in frequency. Specifically, how can one excite such a system to minimize the interfering modes, and how can observed data best be corrected to remove the distortion of interfering modes?

This work attempts to consider both of the problems in a very broad manner. At the same time, an effort is made to develop ideas and points of view which it is hoped will result in a better "physical feeling" for the effects of damping on system behavior.

The work is confined to linear, lumped parameter, holonomic systems, which in the absence of damping can be described by Lagrange's equations of motion. The results apply either to the mechanical system or to its electrical counterpart. No restriction is placed upon the number of degrees of freedom of the systems considered, because in practice the finite difference analog of an aircraft structure may contain many degrees of freedom (10 to 100).

The results of the work are applicable to linear continuous systems, elastic or electromagnetic, as a result of the lack of restrictions upon the number of degrees of freedom or the complexity of the system. Such extensions are intuitive, and no attempt is made to give any rigorous justification for them.

The work is further restricted to physically realizable, passive systems. Consequently, all systems considered are bilateral as well as linear (Ref. 10).* This restriction prevents any direct application of the results to many interesting problems, such as feedback control systems, aircraft flutter, and mechanical systems which appear non-bilateral due to the use of a perturbation theory of gyroscopic motion.

An attempt has been made throughout the work to maintain a dual point of view. The mingling of the ideas of electricity and applied mechanics allows the more powerful techniques of each art to be applied to best advantage. In this respect, the advantages of electro-mechanical analogies extend to analytical as well as experimental investigations.

It is hoped that the work will serve three primary purposes:

(1) The presentation of some of the known results of other investigators, in a convenient and unified form.

---

* Unfortunately, many texts on electric circuit theory associate the term "bilateral" with the relation between input impedance and direction of current flow in a two terminal element. As used here, the term has significance only for multi-terminal elements such as amplifiers, control valves, etc. Such elements are non-bilateral if they introduce unsymmetrical coupling terms into the dynamic equations of a system, thereby voiding the reciprocity theorem.

(2) The establishment on a firm basis of some of the methods and ideas which have been used or proposed in the past as a result of intuition and experience.

(3) The introduction of some new ideas, techniques, and points of view.

II. REVIEW OF NORMAL MODE THEORY FOR LINEAR LUMPED PARAMETER, CONSERVATIVE SYSTEMS

A. Normal Modes of Conservative Systems

Consider Lagrange's equations of motion for a holonomic system with n degrees of freedom, the state of the system at any time being completely defined by the values of n independent generalized coordinates, q_i, and their various time derivatives, $\dot{q}_i$, $\ddot{q}_i$, etc. Lagrange's equations of motion for the system take the form (Ref. 11, Ch. 3),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \bar{f}_i \qquad (i = 1, \ldots n) \qquad \text{(II-1)}$$

where T is the system kinetic energy and $\bar{f}_i$ is the generalized force applied to generalized coordinate q_i. We can express $\bar{f}_i$ as the sum of a non-conservative force, f_i, and a conservative force, f_{ci}. In terms of the system potential energy, $V(q_1, q_2, \ldots q_n)$,

$$f_{ci} = -\frac{\partial}{\partial q_i} V(q_1, q_2, \ldots. q_n) \,. \qquad (i = 1, \ldots n) \qquad \text{(II-2)}$$

The conservative forces can be shifted to the left side of (II-1), and since V is a function of position only, the equation can be expressed in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{q_i} = f_i \qquad (i = 1, \ldots n) \qquad \text{(II-3)}$$

where L = (T-V) is the Lagrangian function.

Consider next a conservative, dynamic, holonomic system with n degrees of freedom. Measure the n generalized coordinates $(q_1, q_2, \dots q_n)$ with respect to their equilibrium positions when the system is at rest. Set the potential energy level to zero at equilibrium. Then for small oscillations about equilibrium, the expressions for T and V become linear if higher order terms in q_i and $\dot{q}_i$ are neglected (Ref. 11, Ch. 5), giving

$$T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j \qquad \text{(II-4)}$$

$$V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} q_i q_j \qquad \text{(II-5)}$$

where the m_{ij} are generalized mass coefficients and the k_{ij} are generalized spring coefficients. Inserting (II-4) and (II-5) into Lagrange's equation (II-3), we obtain the system equations,

$$\begin{aligned}
a_{11}(p)q_1 + a_{12}(p)q_2 + \dots + a_{1n}(p)q_n &= f_1 \\
a_{21}(p)q_1 + a_{22}(p)q_2 + \dots + a_{2n}(p)q_n &= f_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\
a_{n1}(p)q_1 + a_{n2}(p)q_2 + \dots + a_{nn}(p)q_n &= f_n
\end{aligned} \qquad \text{(II-6)}$$

where in operational form, $a_{ij}(p) = (m_{ij}p^2 + k_{ij})$, p being the operator, d/dt.

In (II-6), let the non-conservative forces ($f_1 \ldots f_n$) equal zero. To solve the resulting set of homogeneous equations, assume a set of solutions:

$$q_j = \sum_{K=1}^{n} A_{jK}\, u_K \cos\,(\omega_K t + \phi_K) \qquad \text{(II-7)}$$

where u_K and ϕ_K are arbitrary constants of integration, and A_{jK} represents the relative amplitude of oscillation of coordinate q_j in mode K. Substitution of the assumed solution gives the matrix relation

$$\begin{bmatrix} a_{11}(j\omega_K) & \cdots & a_{1n}(j\omega_K) \\ a_{21}(j\omega_K) & \cdots & a_{2n}(j\omega_K) \\ \vdots & & \vdots \\ a_{n1}(j\omega_K) & \cdots & a_{nn}(j\omega_K) \end{bmatrix} \begin{bmatrix} A_{1K} \\ A_{2K} \\ \vdots \\ A_{nK} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{(II-8)}$$

which can be condensed to

$$\left[a_{ij}\,(j\omega_K)\right]\left[A_{iK}\right] = \left[0\right] \quad . \qquad \text{(II-8a)}$$

Non-trivial solutions of (II-8) require the vanishing of the determinant of the matrix $[a_{ij}(j\omega_K)]$, the result being an nth order algebraic equation in ω_K^2 . The n solutions yield the n natural frequencies of vibration of the system. When these are known, the relative amplitude factors A_{jK} can be obtained from (II-8). One way to do this is to move column 1 of (II-8) to the right side and omit row 1:

$$\begin{bmatrix} a_{22}(j\omega_K) & a_{23}(j\omega_K) & \cdots & a_{2n}(j\omega_K) \\ a_{32}(j\omega_K) & a_{33}(j\omega_K) & \cdots & a_{3n}(j\omega_K) \\ \vdots & \vdots & & \vdots \\ a_{n2}(j\omega_K) & a_{n3}(j\omega_K) & \cdots & a_{nn}(j\omega_K) \end{bmatrix} \begin{bmatrix} A_{2K} \\ A_{3K} \\ \vdots \\ A_{nK} \end{bmatrix} = \begin{bmatrix} -a_{21}(j\omega_K)A_{1K} \\ -a_{31}(j\omega_K)A_{1K} \\ \vdots \\ -a_{n1}(j\omega_K)A_{1K} \end{bmatrix} . \qquad \text{(II-9)}$$

From (II-9), the values of $A_{2K} \dots A_{nK}$ can be obtained in terms of A_{1K}, by Cramer's rule.

The n natural frequencies ω_K, and the corresponding sets of amplitude ratios A_{jK}, define the n normal modes of the system. For a conservative system, the A_{jK} must be real numbers, and the ω_K^2 must be real and positive (or zero). For the sake of brevity, the values of ω_K will frequently be referred to as eigenvalues of the system, and the corresponding sets of amplitude ratios will be called eigenvectors.

B. Normal Coordinates of Conservative Systems

The choice of generalized coordinates, q_i, is to a large extent arbitrary. In general, it is necessary only that they be independently variable, and that the actual location of each physical point in the system be expressible as a function of the q_i. In a linearized system, these functions must, in addition, be <u>linear</u> combinations of the q_i. Clearly, then, if one set of suitable generalized coordinates q_i is known, a second set, equally suitable, can be formed from n independent, linear combinations of the q_i. It is often convenient to make this choice so as to obtain the <u>normal</u> coordinates of the system.

The expressions for T and V in (II-4) and (II-5) are positive definite quadratic forms, and a theorem of algebra states that it is always possible, with a pair of expressions of this type, to find a linear coordinate transformation which makes all the m_{ij} and k_{ij} vanish for $i \neq j$. (Ref. 12.) Let the resulting generalized coordinates be designated U_K. Then

$$T = \frac{1}{2} \sum_{K=1}^{n} M_K \dot{U}_K^2 \qquad \text{(II-10)}$$

$$V = \frac{1}{2} \sum_{K=1}^{n} K_K U_K^2 \quad . \qquad \text{(II-11)}$$

Substituting (II-10) and (II-11) into Lagrange's equation, we obtain

$$M_K \ddot{U}_K + K_K U_K = F_K \qquad (K = 1, \ldots n) \qquad \text{(II-12)}$$

where F_K is our designation for the generalized force applied to normal coordinate K. Thus, the set of n simultaneous equations in (II-6) has reduced to n independent equations by the choice of normal coordinates for generalized coordinates.

Taking the non-conservative forces F_K to be zero, the solutions of (II-12) are

$$U_K = u_K \cos\,(\omega_K t + \phi_K) \quad . \qquad \text{(II-13)}$$

Since the physical system has not changed, these are the same ω_K previously determined from solution of (II-8). Combining (II-7) and (II-13), we can express the original coordinates as

$$q_j = \sum_{K=1}^{n} A_{jK} U_K \quad . \tag{II-14}$$

Conversely, the normal coordinates U_K can be expressed as linear combinations of the q_j by inverting the matrix $\left[A_{jK}\right]$ in (II-14), provided it is non-singular.

From (II-12), the eigenvalues of the system are

$$\omega_K^{\ 2} = \frac{K_K}{M_K} \tag{II-15}$$

where K_K and M_K are generalized spring and mass coefficients, respectively, of the Kth normal coordinate. M_K can be evaluated from considerations of total kinetic energy. Substitution of (II-14) into (II-4) gives

$$T = \frac{1}{2} \sum_i \sum_j m_{ij} \sum_K \sum_L A_{iK} A_{jL} \dot{U}_K \dot{U}_L \quad . \tag{II-16}$$

In (II-16) the order of summation can be interchanged:

$$T = \frac{1}{2} \sum_K \sum_L \left(\sum_i \sum_j m_{ij} A_{iK} A_{jL} \right) \dot{U}_K \dot{U}_L \quad . \tag{II-17}$$

The U_K and U_L are independent coordinates when $K \neq L$, and the kinetic energies defined by (II-17) and (II-10) are for the same physical system; therefore,

$$\sum_i \sum_j m_{ij} A_{iK} A_{jK} = M_K \tag{II-18}$$

$$\sum_i \sum_j m_{ij} A_{iK} A_{jL} = 0 \quad , \quad (K \neq L) \quad . \tag{II-19}$$

Relation (II-19) is the very important normal mode orthogonality condition.

Similar manipulation of the potential energy expressions in (II-5) and (II-11) yields values of K_K and a second orthogonality relation:

$$\sum_i \sum_j k_{ij} A_{iK} A_{jK} = K_K \qquad \text{(II-20)}$$

$$\sum_i \sum_j k_{ij} A_{iK} A_{jL} = 0 \quad , \qquad (K \neq L) \; . \qquad \text{(II-21)}$$

To relate the applied forces, assume a set of virtual displacements, δq_i . Equating virtual work in the two coordinate systems,

$$\sum_i f_i \, \delta q_i = \sum_K F_K \, \delta U_K \; . \qquad \text{(II-22)}$$

Substitution of (II-14) in (II-22) and reversal of the order of summation gives

$$F_K = \sum_i A_{iK} f_i \quad . \qquad \text{(II-23)}$$

Equations (II-12), (II-15), (II-18) and (II-23) combine to provide an expression for the normal coordinates in terms of other generalized quantities:

$$\ddot{U}_K + \omega_K^{\,2} U_K = \frac{\sum_i f_i A_{iK}}{\sum_i \sum_j m_{ij} A_{iK} A_{jL}} \quad . \qquad \text{(II-24)}$$

Only the relative values of the A_{iK} coefficients are fixed, their absolute level being arbitrary. The choice of level of the A_{iK} determines the level of M_K, K_K, U_K and F_K. This arbitrariness can be removed by adding some form of normalizing equation. For example, one way to normalize the U_K is to take $M_K = 1$ for all modes, thus simplifying the expressions for T and V. A more convenient normalization for use with damped systems will be mentioned later. Additional types of normalization will be used for special purposes wherever it is found to be desirable.

C. Relation of Normal Modes to Roots of System Characteristic Determinant

It will prove fruitful at this point to examine closely the relations that exist between the normal modes of a conservative system and the complex plane description of the system. One way to approach this is to assume a set of impulsive forcing functions f_i, and apply Laplace transform methods (Ref. 13). Assume that the system is initially in equilibrium, so that initial energy storage terms drop out of the transformed equations. Then in (II-6), let $f_i = H_i\,\delta(t)$, a force impulse of amplitude H_i. Transforming (II-6) to functions of the complex variable s,

$$\begin{bmatrix} a_{11}(s) & a_{21}(s) & \cdots & a_{n1}(s) \\ a_{12}(s) & a_{22}(s) & \cdots & a_{n2}(s) \\ \vdots & \vdots & & \vdots \\ a_{1n}(s) & a_{2n}(s) & \cdots & a_{nn}(s) \end{bmatrix} \begin{bmatrix} q_1(s) \\ q_2(s) \\ \vdots \\ q_n(s) \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{bmatrix} \qquad \text{(II-25)}$$

Solving for $q_j(s)$ by Cramer's rule,

$$q_j(s) = \sum_{i=1}^{n} \frac{H_i \Delta_{ij}(s)}{\Delta(s)} \tag{II-26}$$

where $\Delta(s)$ is the determinant of $\left[a_{ij}(s)\right]$, and $\Delta_{ij}(s)$ is the cofactor of element $a_{ij}(s)$ in $\Delta(s)$.

For the systems with which we are dealing, both $\Delta(s)$ and $\Delta_{ij}(s)$ are polynomials in s. If the system is conservative, then the roots of $\Delta(s)$ are conjugate imaginary roots, $s_K = \pm j\omega_K$. These roots, which lie along the imaginary axis of the s plane, provide part of the definition of the system normal modes. The remaining quantities to be defined are the amplitude ratios, A_{iK}. Taking the inverse transform of (II-26) we obtain:

$$q_j(t) = \sum_i \sum_K H_i \left[\frac{\Delta_{ij}(j\omega_K)}{j\omega_K \underline{\Delta}(j\omega_K)} \; e^{j\omega_K t} \right]_{R.P.} \tag{II-27}$$

where $\underline{\Delta}(j\omega_K) = \left(\frac{\Delta(s)}{s^2 + \omega_K^2} \right)_{s = j\omega_K}$, and R.P. signifies real part.

Interchanging the order of summation, and recalling that the sum of real parts equals the real part of a sum, it is clear that frequencies ω_K will be missing from the system response provided that

$$\sum_i H_i \Delta_{ij}(j\omega_K) = 0 \qquad (j = 1, \dots n) \; . \tag{II-28}$$

From (II-13) and (II-14), $q_j(t)$ can also be expressed as

$$q_j(t) = \sum_K A_{jK}\, u_K \left[e^{j(\omega_K t + \phi_K)} \right]_{R.P.} \qquad \text{(II-29)}$$

Comparison of (II-27) and (II-29) shows that

$$u_K \left| A_{jK} \right| = \left| \sum_i \frac{H_i\, \Delta_{ij}(j\omega_K)}{j\omega_K\, \underline{\Delta}(j\omega_K)} \right| . \qquad \text{(II-30)}$$

But, since $\dfrac{|A_{jK}|}{|A_{hK}|}$ is a property of the system, it must be independent of the forcing impulses, H_i. This requires that the ratio,

$$\frac{\left| \sum_i H_i\, \Delta_{ij}(j\omega_K) \right|}{\left| \sum_i H_i\, \Delta_{ih}(j\omega_K) \right|}$$

also be independent of the values of H_i, a situation which can only exist if

$$\frac{\Delta_{ij}(j\omega_K)}{\Delta_{ih}(j\omega_K)} = \text{Constant for any row } i \;. \qquad \text{(II-31)}$$

The relation (II-31) is, in fact, a valid and necessary one for <u>any</u> vanishing determinant (Refs. 14, 15). Making use of this relation, the rather cumbersome expression given by (II-9) can now be replaced by the following:

$$\frac{A_{jK}}{A_{hK}} = \frac{\Delta_{ij}(j\omega_K)}{\Delta_{ih}(j\omega_K)} \qquad \text{(for any row } i \text{ and any mode } K) \;. \qquad \text{(II-32)}$$

The relations above are not violated if we choose to normalize the A_{jK} such that they are defined by

$$A_{jK} = \sum_i \Delta_{ij}(j\omega_K) \;. \qquad \text{(II-33)}$$

In equation (II-33) we have a direct, compact relation between the components of the eigenvector of mode K and the eigenvalue, $j\omega_K$, which is a root of the characteristic equation, $\Delta(j\omega_K) = 0$.

It should be noted in passing that the impulsive force on normal coordinate U_K is, from (II-23):

$$F_K = \sum_j A_{jK} H_j \quad . \tag{II-34}$$

Hence, using (II-33),

$$F_K = \sum_j \sum_i \Delta_{ij}(j\omega_K) H_j \quad . \tag{II-35}$$

Since the system is bilateral, $\Delta_{ij} = \Delta_{ji}$, and we can interchange dummy indices to give

$$F_K = \sum_j \sum_i H_i \Delta_{ij}(j\omega_K) \tag{II-36}$$

which is in accord with (II-28).

D. The Degenerate Case of Equal Roots

If a passive, conservative system happens to have two equal roots $j\omega_K$ in its system equation, $\Delta(s) = 0$, then it is clear from physical considerations that solutions of the type $(K_1 + K_2 t)e^{j\omega_K t}$ cannot exist for such a system. Instead, one has actually two arbitrary choices of amplitude coefficients for the oscillations of frequency ω_K.

It has been shown by Routh (Ref. 16, Ch. 6), that for a conservative, passive system it is both necessary and sufficient that all first minors $\Delta_{ij}(j\omega_K)$ vanish, in order that two equal roots $j\omega_K$ can exist. For three equal roots, all second minors must vanish, and so on. Thus, for a singly degenerate system, the ratio A_{jK}/A_{hK} given by (II-32) becomes indeterminate. However, suppose we assume that $A_{1K} \equiv 0$. Then ω_K is still a root of the equation $\Delta_{11}(s) = 0$, which is the system one would obtain by constraint of coordinate q_1. Taking second minors,

$$\frac{A_{jK_1}}{A_{hK_1}} = \frac{\Delta_{11ij}(j\omega_K)}{\Delta_{11ih}(j\omega_K)} \quad ; \quad A_{1K_1} \equiv 0 \quad . \tag{11.37}$$

Let the mode defined by (II-37) be designated mode K_1. Then there exists a second mode K_2 at frequency ω_K which is independent of, and orthogonal to, mode K_1. To determine K_2, let some other coordinate, say $\bar{A}_{3K}$, be constrained to zero. Then $\Delta_{33}(s) = 0$ has a root $j\omega_K$ and a set of amplitudes given by

$$\frac{\bar{A}_{jK}}{\bar{A}_{hK}} = \frac{\Delta_{33ij}(j\omega_K)}{\Delta_{33ih}(j\omega_K)} \quad ; \quad \bar{A}_{3K} = 0 \quad . \tag{II-38}$$

The mode shape defined by (II-38) provides an additional solution with arbitrary constants of integration, and the pair of solutions (II-37) and (II-38) constitute a complete solution of the system as far as frequency ω_K is concerned. In general, mode $\bar{K}$, defined by (II-38), is not orthogonal to mode K_1, but rather is a linear combination of K_1 and K_2. To determine A_{jK_2}, set $\bar{A}_{jK} = A_{jK_2} + \beta A_{jK_1}$. Then, using the orthogonality relation,

$$\sum_i \sum_j m_{ij} A_{iK_1} (A_{jK_2} + \beta A_{jK_1}) = \beta M_{K_1}$$

Therefore,

$$\beta = \frac{1}{M_{K_1}} \sum_i \sum_j m_{ij} A_{iK_1} \bar{A}_{jK} \qquad \text{(II-39)}$$

and

$$A_{jK_2} = \bar{A}_{jK} - \beta A_{jK_1} \quad . \qquad \text{(II-40)}$$

Obviously, the modes K_1 and K_2 defined above do not constitute a unique orthogonal pair, but any other valid orthogonal pair can be formed by linear combinations of the A_{jK_1} and A_{jK_2} .

Good physical examples of simply degenerate systems are spherical pendulums, square drumheads, and other systems with identical symmetry in two dimensions. In these cases, the orthogonality of the degenerate modes is geometrical as well as algebraic.

III. EXTENSION OF NORMAL MODE CONCEPTS TO NON-CONSERVATIVE SYSTEMS

A. Some Contributions of Rayleigh

(1) The Rayleigh Dissipation Function

Let viscous friction forces be introduced into an otherwise conservative system of n degrees of freedom, so as to retard the absolute motion of coordinates q_i and their motion with respect to one another. The total viscous friction force exerted on q_i can be expressed as

$$f_{iF} = - \sum_j b_{ij} \dot{q}_j \quad . \tag{III-1}$$

The total work per unit time done on the system by these forces is

$$P = \sum_i f_{iF} \dot{q}_i = - \sum_i \sum_j b_{ij} \dot{q}_i \dot{q}_j \quad . \tag{III-2}$$

Therefore,

$$\frac{\partial P}{\partial \dot{q}_i} = - 2 \sum_j b_{ij} \dot{q}_j = - 2 f_{iF} \quad . \tag{III-3}$$

By definition, the Rayleigh dissipation function is one half the rate of energy dissipation,

$$F \triangleq \frac{1}{2} \sum_i \sum_j b_{ij} \dot{q}_i \dot{q}_j \quad . \tag{III-4}$$

Consequently,

$$f_{iF} = - \frac{\partial F}{\partial \dot{q}_i} \quad . \tag{III-5}$$

Moving the frictional forces to the left, and designating the remaining applied non-conservative forces as f_i, Lagrange's equation becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = f_i \quad . \tag{III-6}$$

(2) Rayleigh Treatment of Small Dissipation Forces

The dissipation function, F, is a positive definite quadratic form, as are T and V. Assume now that in a system with viscous damping, T and V are reduced to sums of squares by a transformation to the normal coordinates of the undamped system. In general, F will not simultaneously reduce to sums of squares under this same coordinate transformation, and the resulting three functions will take the form

$$T = \frac{1}{2} \sum_{K} M_K \dot{U}_K^{2}$$

$$V = \frac{1}{2} \sum_{K} K_K U_K^{2} \tag{III-7}$$

$$F = \frac{1}{2} \sum_{K} \sum_{L} B_{KL} \dot{U}_K \dot{U}_L \quad .$$

Substitution into Lagrange's Equation (III-6) now gives

$$M_K \ddot{U}_K + \sum_{L} B_{KL} \dot{U}_L + K_K U_K = F_K \quad . \tag{III-8}$$

Often the damping coefficients B_{KL} will be so small that they can be assumed to induce only minor perturbations into the system behavior. Under these circumstances, assume that mode K of the undamped system still describes the principal mode of free vibration of the damped system. To a first order approximation, then,

$$(M_K\,p^2 + B_{KK}\,p + K_K)\,U_K \cong 0 \qquad \text{(III-9)}$$

which gives a damped oscillation

$$U_K \cong u_K\,e^{-\alpha_K t}\cos(\beta_K t + \phi_K) = \left[u_K\,e^{\Gamma_K t + j\phi_K}\right]_{R.P.} \qquad \text{(III-10)}$$

where $\Gamma_K = -\alpha_K + j\,\beta_K$ and $\alpha_K = \dfrac{B_{KK}}{2M_K}$; $\beta_K = \sqrt{\omega_K^{\,2} - \alpha_K^{\,2}} \cong \omega_K$.

The response of any other mode U_S, $S \neq K$, will be induced primarily by the coupling force, $B_{KS}U_K$, such that, neglecting second order effects, and assuming variation as $e^{\Gamma_K t}$,

$$(M_S\,p^2 + K_S)\,U_S + j\omega_K\,B_{KS}\,U_K \cong 0 \;. \qquad \text{(III-11)}$$

Taking only the particular solution for U_S, we obtain,

$$\frac{U_S}{U_K} \cong \frac{j\omega_K\,B_{KS}}{M_S\,(\omega_K^{\,2} - \omega_S^{\,2})} \;. \qquad \text{(III-12)}$$

Thus, in the presence of free vibration in one dominant mode, a lightly damped system exhibits an accompanying response of all other coupled modes, these vibrating at the frequency of mode K, but with approximately 90° phase shift.

Rayleigh continues this line of reasoning to get a second order approximation to the response of the dominant mode.

(3) Rayleigh Treatment of Uniform Damping

Let us introduce the type of damping which will be designated henceforth in this work as uniform damping, by a direct quotation from "Theory of Sound" (Ref. 1, Ch. 5, p. 130):

> "There is, however, a not unimportant class of cases in which the reduction of all three functions may be effected; and the theory then assumes an exceptional simplicity. Under this head the most important are probably those when F is of the same form as T or V. The first case occurs frequently, in books at any rate, when the motion of each part of the system is resisted by a retarding force, proportional both to the mass and velocity of the part. The same exceptional reduction is possible when F is a linear function of T and V, or when T is itself of the same form as V. In any of these cases the equations of motion are of the same form as for a system of one degree of freedom, and the theory possesses certain peculiarities which make it worthy of separate consideration."

Clearly, a sufficient condition that the various modes of oscillation remain totally uncoupled in a vibrating system is that

$$b_{ij} = C_m m_{ij} + C_k k_{ij} \qquad \text{(III-13)}$$

where C_m and C_k are constants for all i, j. The transformation to normal coordinates then results in n separate equations of the form

$$M_K \ddot{U}_K + B_K \dot{U}_K + K_K U_K = F_K \quad . \qquad \text{(III-14)}$$

B. *The Meaning of "Normal Mode" and "Normal Coordinate" for a Non-Conservative System*

Consider now any system where T, V, and F are given in terms of generalized coordinates by equations (II-4), (II-5), and (III-4), respectively. Substitution into Lagrange's equation (III-6), gives the following system equations, expressed in p operator notation:

$$\begin{bmatrix} a_{11}(p) & a_{21}(p) & \cdots & a_{n1}(p) \\ a_{12}(p) & a_{22}(p) & \cdots & a_{n2}(p) \\ \vdots & \vdots & & \vdots \\ a_{1n}(p) & a_{2n}(p) & \cdots & a_{nn}(p) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \qquad \text{(III-15)}$$

where $a_{ij}(p) = (m_{ij}\, p^2 + b_{ij}\, p + k_{ij})$.

The possible free vibrations for the damped system are obtained by setting the forces f_i to zero, and assuming solutions with a time variation $e^{\Gamma_K t}$, resulting in

$$\left[a_{ij}\,(\Gamma_K)\right]\left[\tilde{q}_j\right] = \left[0\right] \qquad \text{(III-16)}$$

where $\tilde{q}_j$ represents a complex quantity whose real part is q_j. For existence of non-trivial solutions, the determinant of the matrix $a_{ij}(\Gamma_K)$ must vanish, resulting in an algebraic equation in Γ_K of order 2n. This equation has real coefficients and represents a passive system; hence, it can have only negative real roots, or complex conjugate roots in the left half of the complex plane. In the remainder of the discussion it will be assumed that the system is primarily an oscillatory one, and that all roots will occur in complex conjugate pairs. This is simply a matter of convenience in dealing directly with the systems of greatest interest; it will not necessarily restrict all the results to such systems.

Let the roots of the system equation be designated as

$$\Gamma_K = -\alpha_K + j\,\beta_K$$
$$\Gamma_K^{*} = -\alpha_K - j\,\beta_K \quad . \qquad \text{(III-17)}$$

Solutions of (III-16) can be expressed as

$$q_j = \sum_{K=1}^{n} G_{jK}\, u_K\, e^{-\alpha_K t} \cos\,(\beta_K t + \phi_K + \theta_{jK}) \qquad \text{(III-18)}$$

where u_K and ϕ_K are arbitrary constants of integration, G_{jK} is a relative amplitude factor (real), and θ_{jK} is a relative phase factor. The latter two can be combined as

$$D_{jK} = G_{jK}\, e^{j\theta_{jK}} \qquad \text{(III-19)}$$

where D_{jK} is now a complex relative amplitude factor. Solutions (III-18) can be expressed in a more compact form:

$$q_j = \sum_K u_K \left[D_{jK}\, e^{\Gamma_K t + j\phi_K} \right]_{R.P.} \qquad \text{(III-20)}$$

Substitution of the solution into (III-16) shows that if the system equations are to be satisfied for all values of time, it is necessary that

$$\left[a_{ij}\,(\Gamma_K) \right] \left[D_{jK} \right] = \left[0 \right] \qquad (K = 1, \ldots n) \;. \qquad \text{(III-21)}$$

The complex amplitude ratios, D_{jK}, correspond here to the real amplitude ratios, A_{jK}, for the undamped system.

If a set of impulsive forcing functions, $H_i\ \delta(t)$, are assumed to be applied to the damped system, a Laplace transform analysis similar to that in Part II-C gives a set of solutions

$$q_j(t) = \sum_i \sum_j H_i \left[\frac{\Delta_{ij}(\Gamma_K)\ e^{\Gamma_K t}}{j\ \beta_K\ \underline{\Delta}\ (\Gamma_K)} \right]_{R.P.} \qquad \text{(III-22)}$$

where

$$\underline{\Delta}\ (\Gamma_K) = \left(\frac{\Delta(s)}{(s - \Gamma_K)(s - \Gamma_K^*)} \right)_{s = \Gamma_K} .$$

Here we observe that the cofactor $\Delta_{ij}(\Gamma_K)$ plays the same role for the damped system as $\Delta_{ij}(j\omega_K)$ for the conservative system. By arguments similar to those in Part II-C, the ratio of complex amplitude factors must be

$$\frac{D_{jK}}{D_{hK}} = \frac{\Delta_{ij}(\Gamma_K)}{\Delta_{ih}(\Gamma_K)} \qquad \text{(for any row } i \text{ and any mode } K). \qquad \text{(III-23)}$$

The normalizing condition can be chosen so that the D_{jK} are defined explicitly:

$$D_{jK} = \sum_i \Delta_{ij}(\Gamma_K) \qquad \text{(III-24)}$$

We have now reached a point where it is possible to give a logical definition to the phrase, "normal mode of a damped system". The normal modes of a conservative system are simply the solutions of the homogeneous system equations. They are specified by a set of conjugate imaginary roots of the system characteristic equation, and by a set of real amplitude factors, A_{jK}. As damping is added, the conjugate roots move into the left half complex plane, and the amplitude ratios become complex quantities, D_{jK}. We wish to define normal modes in such a way that they (1) give a complete description of any possible free motion of the system, and (2) retain their identity when damping is added to or removed from the system. Consequently, it is quite natural to propose the following

> <u>Definition</u>: The normal modes of an oscillatory non-conservative system will be taken to be the solutions of the homogeneous system equations. A normal mode will be completely specified by an eigenvalue, Γ_K, of the characteristic equation, and a set of complex relative amplitudes, D_{jK}.*

Such a definition, in itself, reveals nothing new. It simply reflects a point of view which will be worthwhile only insofar as it permits clearer expression and better understanding with respect to the behavior of the damped system.

The question now arises whether or not a useful set of generalized coordinates can be formed which will correspond to the normal coordinates of the conservative system. We know, in advance, that no choice of coordinates can reduce T, V, and F simultaneously to sums of squares, in general, except in the case of uniform damping as discussed in (III-C). However, we also know that in free vibration the system is capable of independent response in any of the normal modes defined above, because these modes represent

---

* The term "normal mode" for an oscillatory damped system has been used by Guillemin in this sense (Ref. 17; Ref. 18, Ch. VII).

independent solutions of the system homogeneous equations. Each normal coordinate should describe one of these modes of free motion, and, in addition, should transform readily into the set of generalized coordinates q_j.

One possibility which suggests itself is to let the normal coordinates U_K be defined by the real part of a complex displacement, $\tilde{U}_K$. During free vibration of the system,

$$\tilde{U}_K = u_K \, e^{\Gamma_K t + j\phi_K} \quad . \qquad \text{(III-25)}$$

These complex normal coordinates can then be related to the complex generalized coordinates by the transformation:

$$\tilde{q}_j = \sum_K D_{jK} \tilde{U}_K \quad . \qquad \text{(III-26)}$$

At this point, a fundamental difficulty arises, in that the transformation (III-26) does not, in general, transform T and V into sums of squares. Thus, the normal modes of our damped system are not orthogonal. As an example of the difficulty let us attempt to relate total system kinetic energy in the two sets of coordinates. The instantaneous total kinetic energy is

$$T = \frac{1}{2} \sum_i \sum_j m_{ij} \dot{q}_i \dot{q}_j \quad .$$

Since $d/dt \left[R.P.(Z)\right] = R.P. \left[dZ/dt\right]$, we can substitute from (III-26):

$$T = \frac{1}{2} \sum_i \sum_j \left\{ m_{ij} \left(\sum_K \left[D_{iK} \dot{\tilde{U}}_K\right]_{R.P.} \right) \left(\sum_L \left[D_{jL} \dot{\tilde{U}}_L\right]_{R.P.} \right) \right\}$$

$$T = \frac{1}{2} \sum_K \sum_L \left\{ \sum_i \sum_j m_{ij} \left[D_{iK} \dot{\tilde{U}}_K\right]_{R.P.} \left[D_{jL} \dot{\tilde{U}}_L\right]_{R.P.} \right\} .$$

The amplitude factors D_{iK} and D_{jL}, being complex, cannot be separated from the normal coordinates in the above expression, because the real part of a product does not equal the product of real parts. There is at least one exception, that of uniform damping, for which the amplitude factors D_{iK} have the <u>real</u> values A_{iK}. For this case, these quantities factor out of the brackets, and the orthogonality relation, together with the value of M_K, can be obtained as was done in (II-B) for a conservative system.

Despite the difficulty in relating T, V, and F to the complex normal coordinates, the transformation defined by (III-26) does possess some useful properties for the general case. For one thing, the matrix $[D_{jK}]$ can normally be inverted to give a unique set of $\tilde{U}_K$ for a given set of $\tilde{q}_j$. We can also demonstrate at least a formal correspondence with some of the force relationships for conservative systems.

For example, (III-22) shows that a set of impulses, $H_i\,\delta(t)$, will fail to excite mode K of the damped system, initially at rest, only if

$$\sum_i H_i\,\Delta_{ij}\,(\Gamma_K) = 0 \qquad (j=1,\ \dots\ n). \qquad \text{(III-27)}$$

Summation on j, substitution of Δ_{ji} for Δ_{ij}, and use of (III-24) changes this condition to the form

$$\sum_i D_{iK}\,H_i = 0\ . \qquad \text{(III-28)}$$

This summation represents a generalized impulsive force on normal coordinate K, and corresponds to (II-28) for the conservative system.

Since the D_{iK} are complex, (III-28) is equivalent to two real equations. Thus, in general it would require impulses at two points in the system to eliminate the real part of the sum, and a set of three impulses to suppress mode K completely.

C. The Effects of Small Damping Upon Normal Modes

Suppose we start with a conservative system of many degrees of freedom. Let viscous damping be added to this system, in such a manner that no new degrees of freedom are created. Assuming that the damping coefficients are small, we wish to examine the following system properties:

Displacement of the system eigenvalues into the left half of the complex plane.

Variation in amplitude factors $|D_{iK}|$.

Variation in phase of D_{iK}/D_{jK}.

Behavior of the orthogonality equations.

The term "small damping" must first be explained. From (III-22), the motion of any coordinate of the system, following a set of initial impulses H_i, can be expressed as

$$q_j(t) = \sum_K \sum_i H_i \left[\frac{\Delta_{ij}(\Gamma_K) e^{\Gamma_K t}}{j\,\beta_K\,\underline{\Delta}\,(\Gamma_K)} \right]_{R.P.}$$

This response depends upon evaluation of terms from the system determinant, $\Delta(s)$, at the various values of Γ_K. Therefore, at present we will consider the damping coefficients, b_{ij}, to be "small" for a mode K if

$$|b_{ij}| << \left| \frac{k_{ij}}{\Gamma_K} \right| \quad \text{and} \quad |b_{ij}| << |m_{ij}\ \Gamma_K| \,. \qquad \text{(III-29)}$$

We will assume that, on the average, the ratio

$$\frac{|b_{ij}|}{\sqrt{|m_{ij}\, k_{ij}|}}$$

is of order of magnitude ϵ . For $i = j$ this ratio has a simple physical significance. With all coordinates constrained except q_i, the system represents a one degree of freedom spring, mass, dashpot combination. Then $1/b_{ii}$ is a measure of the displacement response to a unit force impulse on the dashpot alone. Similarly, $|m_{ii}k_{ii}|^{-1/2}$ is the maximum displacement of the spring-mass combination alone in response to a unit impulse.

(1) Displacement of System Eigenvalues

Let $\Delta_0(s)$ be the determinant of the matrix $[a_{ij}(s)]$ for the original conservative system, and let $\Delta_1(s)$ be the same determinant after the damping terms have been added. Then the former is composed of products of terms $(m_{ij}s^2 + k_{ij})$, while the latter is composed of products of terms $(m_{ij}s^2 + b_{ij}s + k_{ij})$.

Theorem 1. If any of the eigenvalues change their absolute magnitude, then some must increase while others decrease, the product of all remaining constant.

This is easily seen to be true, because the constant term in $\Delta(s) = 0$, which is equal to the product of all the roots, depends only upon the k_{ij}. This theorem holds regardless of the amount of damping.

Now, Rayleigh's treatment of small dissipation forces, which is reviewed in (III-B), shows that to a first order approximation each mode of the system behaves essentially like a one degree of freedom oscillator when damping is introduced. It is well known that for the latter, the root locus is a circular arc, centered at the origin. Thus we have

Theorem 2. As damping is introduced, all of the eigenvalues must depart from the imaginary axis perpendicularly as they move into the left half plane. They move, on the average, along circular arcs centered at the origin. Thus, the best approximation to ω_K is $|\Gamma_K|$.

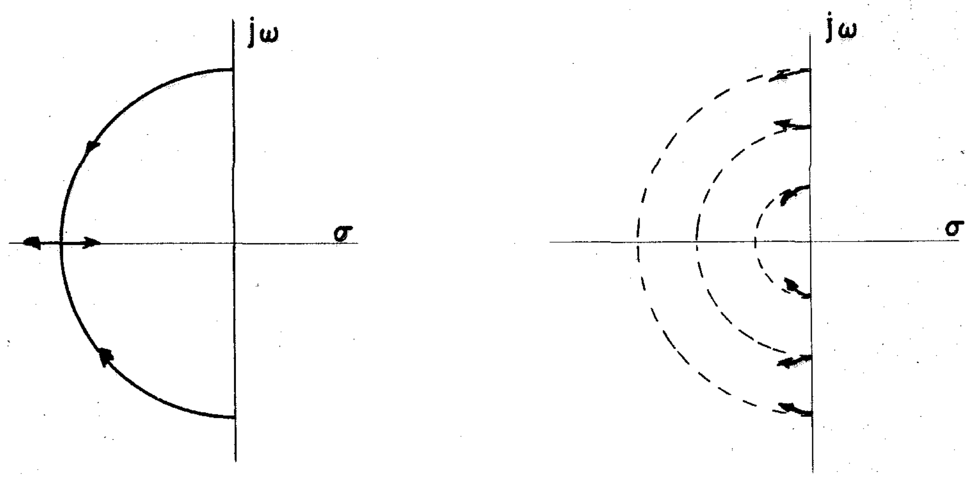


Figure 1. Locus of Displacement of System Eigenvalues Upon Introduction of Damping, (a) for a One Degree of Freedom System, (b) for a Many Degree of Freedom System.

This same result was derived by Routh (Ref. 16, Ch. 7), who reasons somewhat as follows: Clearly $\Delta_o(s)$ can contain only even powers of s. If the damping terms are added, and terms of second order and higher in ϵ neglected, then

$$\Delta_1(s) \cong \Delta_o(s) + s \sum_r \sum_i b_{ri} \Delta_{o_{ri}}(s) \qquad \text{(III-30)}$$

where the first term on the right now contains only even powers of s, and the second term only odd powers of s. Let the roots of $\Delta_1(s) = 0$ be of the form $\Gamma_K = j\omega_K + \rho_K$, where $j\omega_K$ is the corresponding root of $\Delta_o(s) = 0$, and ρ_K is a small deviation of undetermined magnitude and phase. Using a Taylor's series expansion for an analytic function,

$$\Delta_o(j\omega_K + \rho_K) = \Delta_o(j\omega_K) + \rho_K \Delta'_o(j\omega_K) + \frac{\rho_K^2}{2!} \Delta''_o(j\omega_K) + \ldots$$

$$= 0 + \rho_K \Delta'_o(j\omega_K) + \text{higher order terms in } \rho_K.$$

Thus, letting $s = \Gamma_K$ in (III-30),

$$0 \cong \rho_K \Delta'_o(j\omega_K) + (j\omega_K + \rho_K) \sum_r \sum_i b_{ri} \Delta_{ori} (\Gamma_K) .$$

Dropping second order terms in ρ_K and ϵ,

$$0 \cong \rho_K \Delta'_o (j\omega_K) + j\omega_K \sum_r \sum_i b_{ri} \Delta_{ori} (j\omega_K). \qquad \text{(III-31)}$$

Since $\Delta'_o(j\omega_K)$ is pure imaginary, and the second term is likewise pure imaginary, it follows that ρ_K must be real, thus proving Theorem 2. Note that $\rho_K \approx -\alpha_K$; hence the "smallness" of ρ_K is proportional to α_K/ω_K.

In 1928, Guillemin (Ref. 19) used this method to obtain the same result for an electrical system treated on a mesh basis. He went into greater detail than Routh, and presented a number of additional results. By considering the Taylor's series expansion of $\Delta(j\omega_K + \rho_K)$ and the binomial expansion of $(j\omega_K + \rho_K)^{2n}$ Guillemin obtains as conditions for small damping:

$$\frac{|b_{ij}|}{\omega_K} << 1 \qquad \text{(III-32)}$$

and

$$\frac{2n-1}{2}\,\frac{\alpha_K}{\omega_K} << 1 \quad . \tag{III-33}$$

These relations are undoubtedly rigorous and general, and they are presumably the proper ones to use for extreme cases of non-uniform damping, such as might be produced by inserting one or two strong damping elements into a large system. The author feels that they are too strong for systems where the damping has any significant degree of uniformity. For example, (III-32) is dimensionally incorrect and fails to account for the vast variation which may be found between the largest b_{ii} and the smallest b_{ij} in a complex system. Using (III-33), small damping becomes practically impossible for a large degree of freedom system*, and, in the limit, for a continuous system.

For uniformly damped systems, condition (III-29) reduces to

$$\frac{|b_{ij}|}{\sqrt{|m_{ij}\,k_{ij}|}} = \frac{|C_m\,m_{ij} + C_k\,k_{ij}|}{\sqrt{|m_{ij}\,k_{ij}|}}$$

while

$$\frac{\alpha_K}{\omega_K} = \frac{C_m\,M_K + C_k\,K_K}{2M_K\,\omega_K} = \frac{1}{2}\left(\frac{C_m}{\omega_K} + C_k\,\omega_K\right) \quad .$$

Each mode of such a system behaves as a single degree of freedom oscillator; consequently the eigenvalues move precisely along

---

* For a 100 degree of freedom finite difference representation of an airframe, and an accuracy of 4%, this would require a per-unit critical damping of .002, a "Q" of 250, or an equivalent structural damping "g" factor of .004.

circular arcs. For a single degree of freedom oscillator,

$$\frac{b}{\sqrt{mk}} = \frac{2\alpha}{\omega} .$$

Therefore, the following theorem is proposed:

Theorem 3. In approximating an undamped normal mode from a pure damped mode of a freely vibrating system, the order of magnitude of the terms ϵ and δ whose squares are neglected will be taken as

$$\frac{|b_{ij}|}{\sqrt{|m_{ij}\, k_{ij}|}} = \epsilon \quad ; \qquad \frac{2\alpha_K}{\omega_K} = \delta$$

for systems where the damping is highly uniform, and as

$$\frac{|b_{ij}|}{\omega_K} = \epsilon \quad ; \qquad \frac{2n-1}{2}\,\frac{\alpha_K}{\omega_K} = \delta$$

for systems where the damping is highly non-uniform.

A measure of the non-uniformity of damping will be discussed later.

(2) Variation of Amplitude Factors $|D_{iK}|$

We can extend the technique used by Routh to investigate the behavior of the amplitude factors of a normal mode. From (II-33) and (III-24),

$$A_{iK} = \sum_r \Delta_{ori}(j\omega_K); \qquad D_{iK} = \sum_r \Delta_{1ri}(\Gamma_K) . \qquad \text{(III-34)}$$

Expanding $\Delta_{1ri}(\Gamma_K)$ exactly as $\Delta_1(\Gamma_K)$ was expanded above, neglecting second order terms in ϵ and ρ_K, and finally summing on r, we obtain,

$$D_{iK} = A_{iK} + \rho_K \sum_r \Delta'_{ori}(j\omega_K) + j\omega_K \sum_r \sum_m \sum_j b_{mj}\, \Delta_{orimj}(j\omega_K) . \qquad \text{(III-35)}$$

where Δ_{orimj} designates cofactor mj of cofactor ri. The second and third terms on the right are pure imaginary and are of first order in ρ_K and ϵ , thus establishing

Theorem 4. For small damping of order ϵ , δ the variation in the real part of the amplitude factors D_{iK} is of second order in ϵ and δ. The imaginary parts of the D_{iK} are of first order in ϵ and δ.

The theorem shows that large variations in the undamped A_{iK} are not to be expected except near node points of mode K, where the A_{iK} are small. Even near node points, the variations are minor when compared with the larger displacements of the mode. Thus, it is possible to state a slightly different theorem:

Theorem 5. For small damping of order ϵ , δ the variation of the ratios of magnitudes of amplitude factors, $\frac{|D_{iK}|}{|D_{jK}|}$, is of second order in ϵ and δ, provided neither q_i nor q_j is near a node point of mode K.

(3) Variation in Phase of D_{iK}/D_{jK}

From (III-35) and Theorem 4, it is clear that the phase shift from A_{iK} to D_{iK} is of first order in ϵ and δ, provided A_{iK} is not near a node point. Hence, we can state

Theorem 6. The relative phase angle of the ratio (D_{iK}/D_{jK}) will be of first order in ϵ and δ, provided neither q_i nor q_j is near a node of mode K. In the vicinity of a node point, large phase shifts may be produced by small damping.

(4) Behavior of Orthogonality Equations

Using results of (2) let us represent D_{iK} for small damping as

$$D_{iK} = \overline{D}_{iK} + j\,\underline{D}_{iK}$$

where $\underline{D}_{iK}$ is of order ϵ , δ and $\overline{D}_{iK} \cong A_{iK}$, neglecting second order terms. Consider the effect of introduction of damping upon the orthogonality relations (II-19) and (II-21).

$$\sum_i \sum_j m_{ij}\, D_{iK}\, D_{jL} = \sum_i \sum_j m_{ij} \Big[(\overline{D}_{iK}\, \overline{D}_{jL} - \underline{D}_{iK}\, \underline{D}_{jL})$$

$$+ j\,(\overline{D}_{iK}\, \underline{D}_{jL} + \underline{D}_{iK}\, \overline{D}_{jL})\Big].$$

The first term vanishes by (II-19), and the second term is of second order in ϵ , δ . Hence,

$$\Big[\sum_i \sum_j m_{ij}\, D_{iK}\, D_{jL}\Big]_{R.P.} = \text{Terms of order } \epsilon^2, \delta^2. \qquad (K \neq L) \qquad \text{(III-36)}$$

From Theorem 5, it can be seen that another possibility is the use of the expression

$$\sum_i \sum_j m_{ij}\, |D_{iK}|\, |D_{jL}|$$

as a substitute for (II-19). This is the more accurate expression for the large values of $|D_{iK}|$, but less accurate for the small $|D_{iK}|$. This second expression gives justification to the practical expedient of taking the "real part" in (III-36) with respect to a reference phase axis which is an average phase position of the larger amplitude factors D_{iK}. While this is not truly the real axis, the resulting expression will in reality be more accurate than (III-36) if the damping is to any extent uniform, for in this case the roots move approximately along circular arcs. Thus we have

Theorem 7. For small damping,

$$\sum_i \sum_j m_{ij} \bar{D}_{iK} \bar{D}_{jL} \cong \begin{cases} 0 \ , & (K \neq L) \\ M_K \ , & (K = L) \end{cases}$$

where $\bar{D}_{iK}$ is the component of D_{iK} taken along a reference phase which is an average phase angle for the larger values of D_{iK}.

A similar theorem could, of course, be stated involving the k_{ij}. Note, however, that for non-uniform damping

$$\sum_i \sum_j b_{ij} \bar{D}_{iK} \bar{D}_{jL} \neq 0 \quad , \quad (K \neq L) \ .$$

In fact, this quantity will be used subsequently as one measure of the non-uniformity of the damping.

These results can be summarized as a final theorem.

Theorem 8. If a pure mode can be excited in a system with small, arbitrary damping, then the corresponding normal mode of the undamped system can be described accurately by taking the average in phase components $\bar{D}_{iK}$, and taking $\omega_K = |\Gamma_K|$.

IV. NORMAL MODE ANALOGS OF LUMPED PARAMETER ELECTRICAL AND MECHANICAL SYSTEMS

A. Normal Mode Analogs of Conservative Systems

The existence of analogous electrical and mechanical systems, resulting from the identical form of the differential equations which describe the behavior of the systems, is long established and well known. The electrical circuit equations can be set up in terms of the energy expressions T, V, and F as are the Lagrangian equations, on either a mesh basis or a node basis (Ref. 20, Ch. 6). The far greater flexibility of analogous electrical systems has led to an increasing use of the electric analog technique. This has been particularly true in the study of stress and vibration of complex elastic structures, for which practical finite difference electric analogs have been developed (Refs. 9, 21). It is this type of system with which we are primarily concerned.

In general, the nodal electric analog has been found to be more convenient than the mesh analog for studies of mechanical and elastic systems. Hence, we will concentrate attention on the nodal analog. In this analog, the corresponding electrical and mechanical quantities are current - force; voltage - velocity; capacitance - mass; inductance - spring compliance; conductance - damping coefficient; and ideal transformer - lever (Ref. 9).

The true dynamic analog of a conservative mechanical system must, of course, be a purely reactive electrical system of the same degree of freedom. Once this reactive circuit has been obtained, in any form, then as far as measurements made between any two terminals of the network are concerned, it can be replaced by <u>either</u> of Foster's canonic forms (Ref. 22). Either of Foster's canonic forms can be considered as normal mode analogs of the original electrical or mechanical system, the second form being the proper one to use in the nodal analogy. Each parallel resonant tank circuit of Figure 2(b) represents a normal mode of the parent system. If one of the terminals is chosen as a ground terminal (mechanically, the absolute reference frame), then the voltage across each tank circuit will represent the corresponding normal coordinate of velocity, $A_{iK}\dot{U}_K$, and the terminal voltage, by (II-14), is the generalized physical coordinate $\dot{q}_i$ of the parent system.

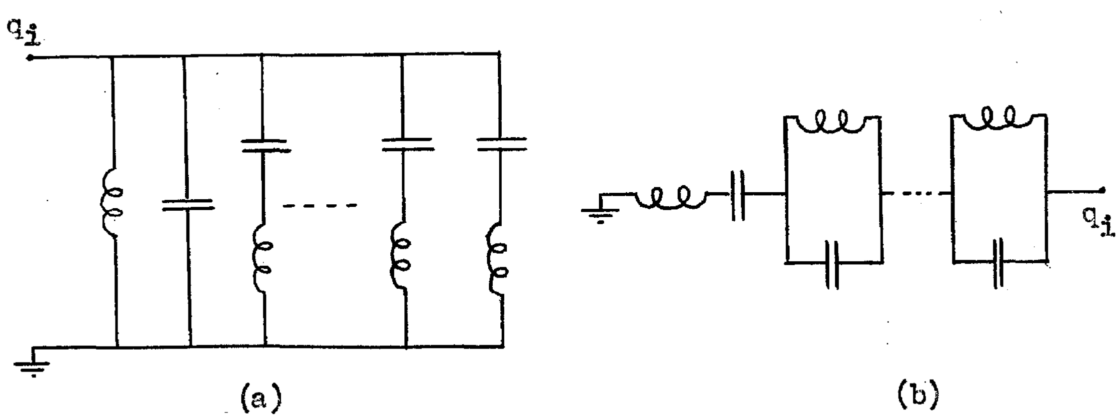
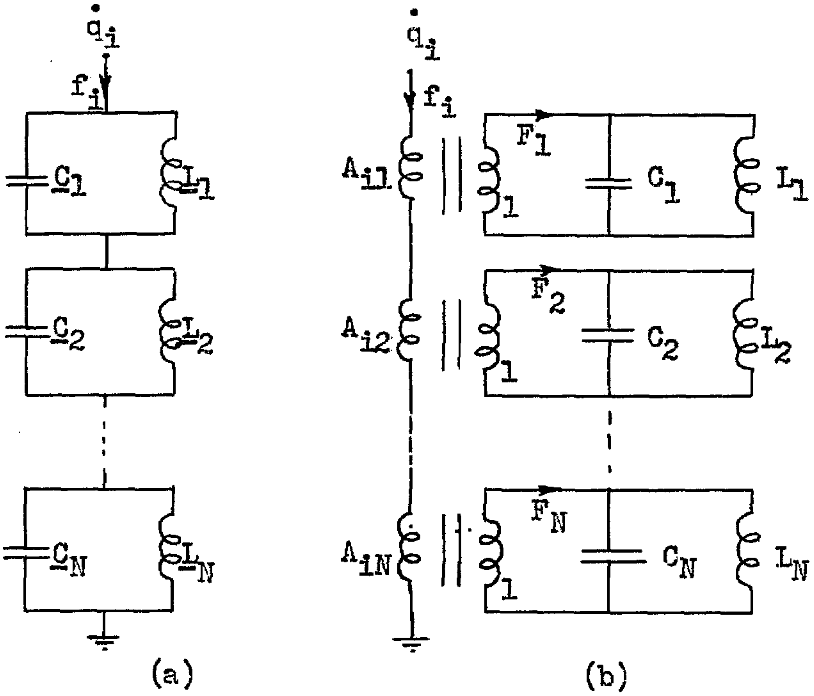


Figure 2. (a) Foster's First Canonic Form
(b) Foster's Second Canonic Form

By the use of ideal transformers, it is possible to extend the usefulness of the normal mode analog represented by Foster's second canonic form. Let the various tank circuits be isolated conductively from one another as shown in Figure 3, such that the reflected impedance of each tank is unchanged when viewed from q_i. The transformer turns ratios are arbitrary, but a convenient choice is A_{iK}:1. To maintain the same operational impedance, take $A_{iK}^2 L_K = \underline{L}_K$

and $A_{iK}^2\underline{C}_K = C_K$, thereby keeping the normal mode frequency constant. The voltage across the isolated tank K is now $\dot{U}_K$.* If a single generalized force, f_i, is applied to generalized coordinate q_i, this is represented by injecting a current, $I_i = f_i$, at q_i. The current injected into tank K, then, is $f_i/A_{iK} = F_K$. Note that this is not the circulating current in tank K. The circulating currents determine the energy stored in the inductors of each mode, which is analogous to strain energy. Thus,



$$V = \frac{1}{2}\sum_K K_K U_K^2$$

$$= \frac{1}{2}\sum_K L_K I_K^2$$

$$= \frac{1}{2}\sum_K \left(\frac{1}{L_K}\right) U_K^2 .$$

Hence, $C_K = M_K$ and $\left(\frac{1}{L_K}\right) = K_K$.

Figure 3. Isolation of tank circuits in Foster's circuit (a) by use of ideal transformers (b).

* In actual physical analogs there will usually be only a proportionality, rather than an equality, between analogous electrical and mechanical quantities. This proportionality may change within the system due to impedance base changes. We will ignore these scale changes in the analytical work and assume direct equality. Henceforth, electrical symbols and their analogous mechanical symbols will be used interchangably, since it is felt that use of mechanical symbols on electrical diagrams leads to a better "physical picture" of the correspondence.

Clearly, the choice of coordinate q_i in Figure 3(b) is arbitrary, and if coordinate q_j were brought out instead, then the only change would be in the transformer turns ratios. Hence, we can make available any number of generalized coordinates by simply adding a new set of transformers for each one. If a complete set of n independent generalized coordinates is thus created, then any set of applied currents, f_i, will induce the correct generalized currents, F_K, into the various tank circuits. In fact, it can be shown, by considering each set of transformers as a constraint upon the system, that whenever new coordinates are created in any electric analog by use of ideal transformers, the resulting forces will automatically be correct. (Ref. 23.)

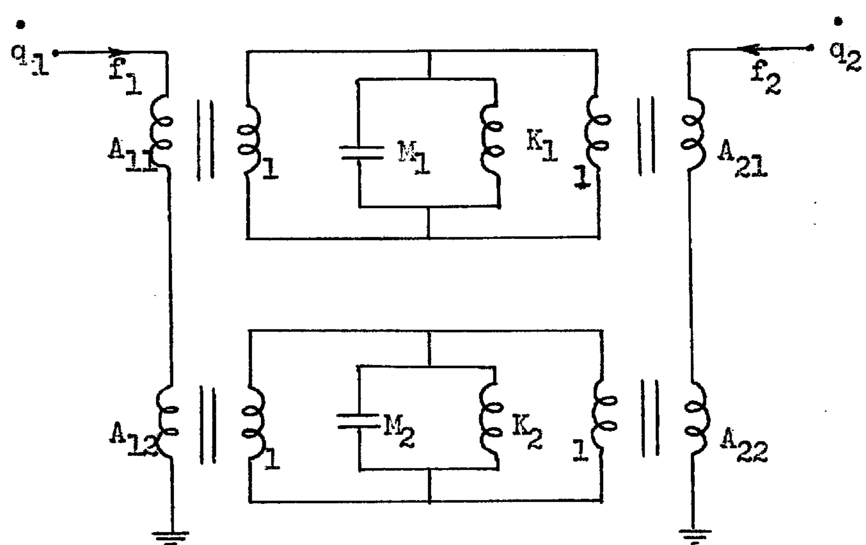


Figure 4. Complete Normal Mode Analog of Two Degree of Freedom System, with Generalized Coordinates Brought Out.

The complete normal mode analog as shown in Figure 4 requires n^2 transformers for an n degree of freedom system. This number can be reduced to n(n-1) by using the original Foster circuit for one generalized coordinate. Since all primary windings shunting a given tank are identical, all of their cores carry identical magnetic fluxes. Consequently, only one core need be retained, and if suitable multi-winding transformers are available, only n of these are required. For completely free systems with zero frequency modes, some "tanks" will degenerate to simple capacitors.

Normal mode analogs of this type have been used in the past on the Analysis Laboratory analog computer for representing a few of the more important modes of a substructure attached to a main structure at a few points. Only the coordinates necessary for attachment are brought out. All modes above the frequency range of interest are approximated by omitting the capacitors and lumping the remaining inductors into a single inductor. Similarly, if there are modes below the frequency range of interest they can be approximated by a single capacitor. Thus the number of transformers required is kept to a minimum.

B. Normal Mode Analogs for Uniformly Damped Systems

Suppose now that our n degree of freedom system has viscous damping introduced between existing coordinates, such that the damping throughout the system is uniform in the sense mentioned by Rayleigh, which was discussed in (III-C). Such a system is completely defined by its energy functions:

$$T = \frac{1}{2} \sum_i \sum_j m_{ij} \dot{q}_i \dot{q}_j$$

$$V = \frac{1}{2} \sum_i \sum_j k_{ij} q_i q_j \qquad \text{(IV-1)}$$

$$F = \frac{1}{2} \sum_i \sum_j b_{ij} \dot{q}_i \dot{q}_j$$

where

$$b_{ij} = C_m m_{ij} + C_k k_{ij} \quad . \qquad \text{(IV-2)}$$

Using the same real coordinate transformation which produced normal coordinates in the corresponding conservative system,

$$q_j = \sum_K A_{jK} U_K \qquad \text{(IV-3)}$$

we obtain sums of squares for T, V, and F, and the resulting uncoupled equations

$$M_K \ddot{U}_K + B_K \dot{U}_K + K_K U_K = F_K \qquad \text{(IV-4)}$$

where

$$B_K = C_m M_K + C_k K_K \quad . \qquad \text{(IV-5)}$$

The solution for each normal coordinate during free oscillation is now

$$U_K = \left[u_K \; e^{\Gamma_K t} \right]_{R.P.} \qquad \text{(IV-6)}$$

where

$$\Gamma_K = - \alpha_K + j\,\beta_K; \quad \alpha_K = \frac{-B_K}{2M_K}; \quad \beta_K = \sqrt{\omega_K^2 - \alpha_K^2}\,.$$

In the direct electric analog of such a system, the damping appears as a conductance element shunting each inductor and/or each capacitor. We will hereafter designate this particular type of damping as uniform shunt damping.

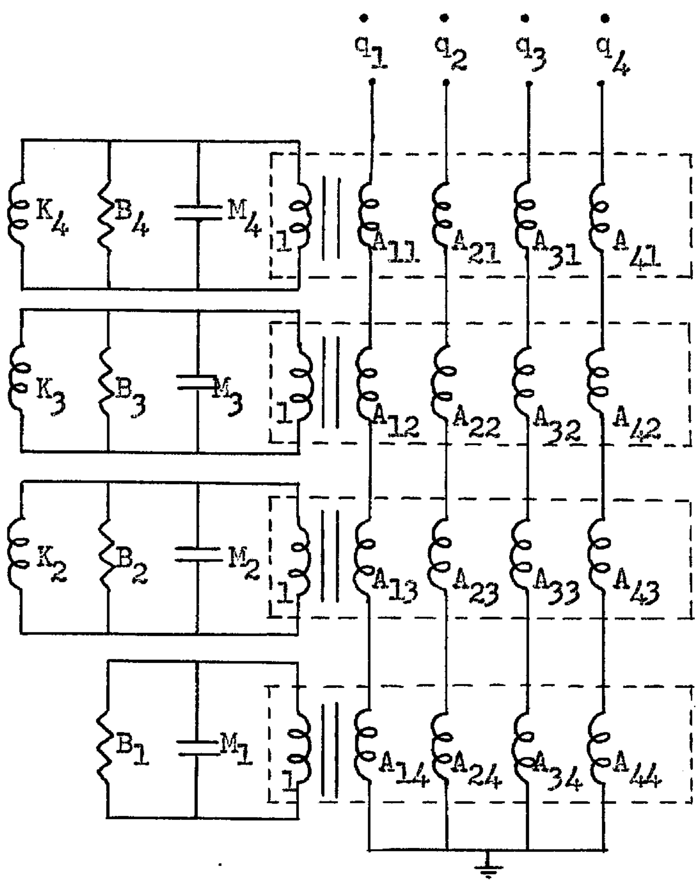


Figure 5. Complete Normal Mode Analog for Four Degrees of Freedom and Uniform Shunt Damping. Mode 1 is a "zero frequency" mode.

The above relations can be reproduced exactly in the normal mode analogs of Figure 3 or Figure 4, by shunting each capacitance M_K by a conductance $C_m M_K$, and shunting each inverse inductance K_K by a conductance $C_k K_K$. Combining the two parallel conductances gives a net conductance $G_K = B_K$ shunting each tank K. The resulting normal mode analog is exact, being described by a set of equations identical with those of the system it replaces.

Figure 5 has been shown with multi-winding transformers, which results in a more compact circuit diagram for a large degree of freedom system.

C. Normal Mode Analogs for Non-Uniformly Damped Systems

Suppose now that the damping coefficients b_{ij} depart from the condition of uniformity defined by (IV-2), and we wish to further modify the basic analogs of Figure 3 and Figure 4 to account for this non-uniform damping.

The type of damping that we are considering must be introduced, electrically, as shunt conductances between established nodes of the undamped circuit, the reference (ground) node included. These nodes are preserved in the complete analog (Figs. 4, 5). Thus, we can approach the problem by adding the individual b_{ij} conductances directly to the undamped normal mode analog. Figure 6 shows such an analog for a three degree of freedom system.*

---

* It should be noted at this point that the mechanical system may have positive mass coupling coefficients, m_{ij}. As a result, both the direct electric analog and the normal mode analog proposed above may have negative coupling condensers and, consequently, negative coupling conductances. These "unrealizable" circuit elements can often be eliminated from the analog by proper choice of scale factors (Ref. 23). In any event, they can be simulated by using feedback amplifiers (Ref. 24). For purposes of analysis, the signs of the coupling terms are immaterial.

The analog of Figure 6, as it stands, provides us with a vehicle for the study of damped systems which will lead to a number of significant facts. Before going into these applications, however, let us investigate whether there are other forms of the analog which may be equally valuable.

Clearly, all of the damping elements in Figure 6 cannot be brought into the normal mode tanks as was done in Figure 5, because this would result in decoupling the normal modes from one another, violating a well established property of non-uniformly damped systems. However, a possibility which suggests itself is that of bringing a portion of the damping inside the tanks, leaving the remainder external to them. There are at least two ways in which this can be accomplished:

(1) Let b_{ij} consist of two parts, one a uniform part and the other a non-uniform part. The uniform part of b_{ij} can then be brought into the tank circuits directly, leaving only the residual, non-uniform damping to couple the external terminals. The uniform damping should be fixed in form and level such that the residual, non-uniform terms are small, on the average.

(2) Use the real coordinate transformation (IV-3) on the b_{ij}, thus obtaining a set of B_{KL} which can be brought inside the transformers. Proper interconnection of the normal coordinates will then result in an exact analog.

Consider method (1) first. We can express each damping coefficient as

$$b_{ij} = C_m M_{ij} + C_k k_{ij} + g_{ij} \tag{IV-7}$$

where g_{ij} is the residual term, its value being arbitrary. Note that

g_{ij} may be either positive or negative now for any i, j. The first two terms can be brought directly into the tank circuits in the form of shunt conductances (IV-5),

$$B_K = C_m M_K + C_k K_K \quad .$$

The residual terms, g_{ij}, remain to couple the external coordinates, q_j. It is obviously to our advantage to make a choice of C_m and C_k such that the g_{ij} are small, on the average. The equivalent circuit in this second form is shown in Figure 7.

Now consider the second approach. The normal coordinates must satisfy the relations previously considered in (III-B); namely,

$$M_K \ddot{U}_K + \sum_L B_{KL} \dot{U}_L + K_K U_K = F_K \qquad \text{(IV-8)}$$

where

$$B_{KL} = \sum_i \sum_j b_{ij} A_{iK} A_{jL} \qquad \text{(IV-9)}$$

and where no one mode is now considered dominant, so that all n equations of type (IV-8) are to be satisfied simultaneously, in contrast with the treatment in (III-B).

Ignoring the external coordinates and transformers for the moment, equations (IV-8) can be satisfied by grounding one side of each normal mode tank in Figure 4, and considering $\dot{U}_K$ as the voltage to ground from the free terminal. Coordinates U_K are then coupled directly by the B_{KL}. Replacing the transformers and external coordinates, we obtain the complete analog, Figure 8.

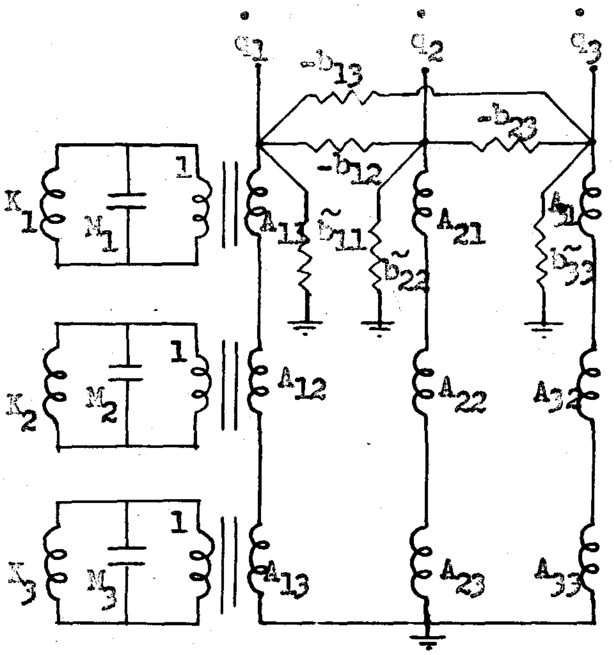


Figure 6. Normal Mode Analog With External Shunt Damping. $\tilde{b}_{ii} = b_{ii} + \sum_{j \neq i} b_{ij}$.

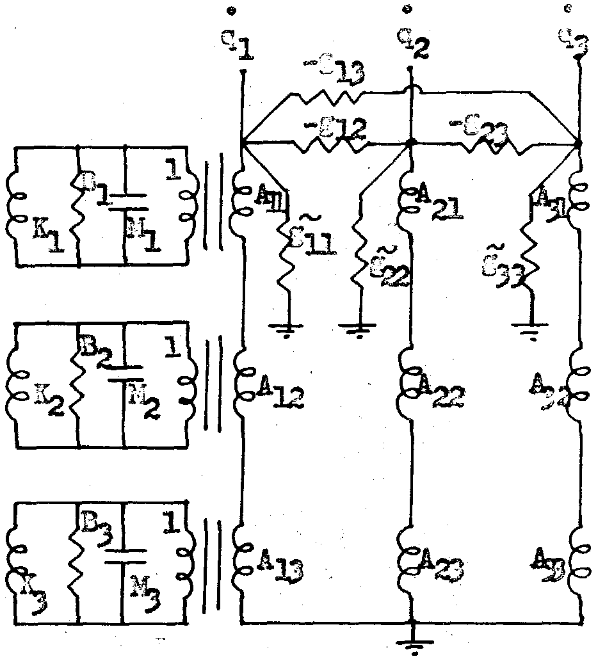


Figure 7. Normal Mode Analog With Part of the Shunt Damping Brought Into the Tanks as Uniform Damping. $\tilde{g}_{ii} = g_{ii} + \sum_{j \neq i} g_{ij}$.

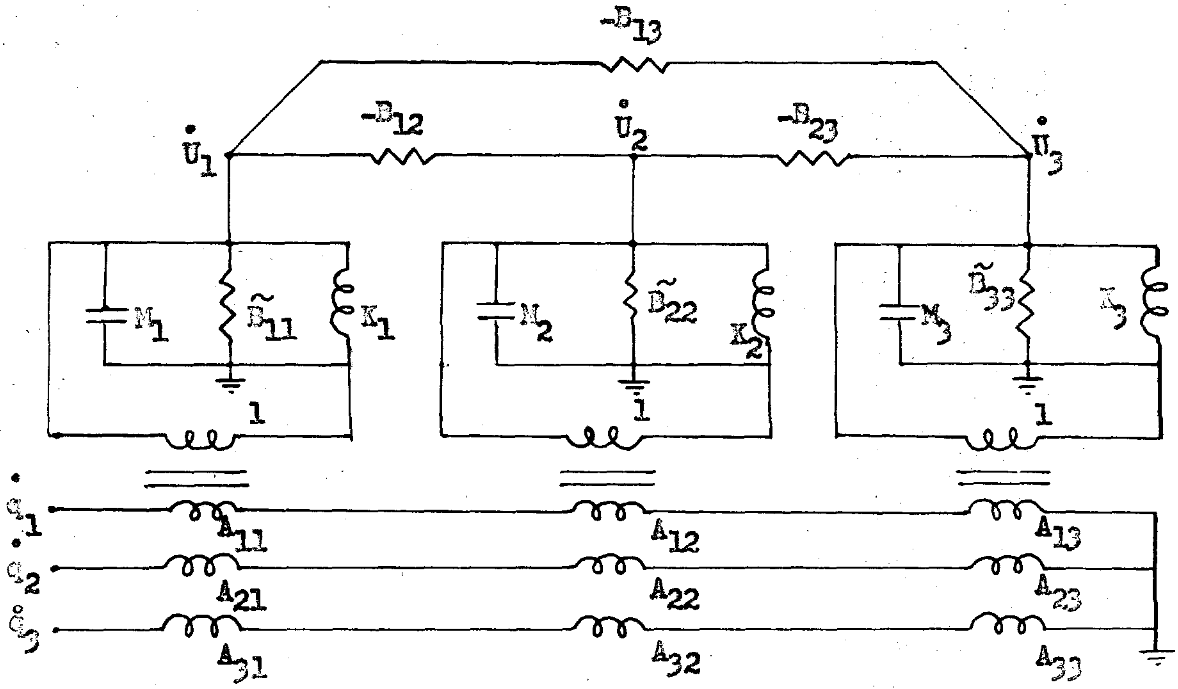


Figure 8. Normal Mode Analog With All Damping Transformed to the Undamped Normal Coordinates. $\tilde{B}_{KK} = B_{KK} + \sum_{L \neq K} B_{KL}$.

V. EXTENSIONS OF THE CONCEPT OF UNIFORM DAMPING

A. Uniform Shunt Damping

This is the type of uniform damping which has already been mentioned in some detail. The excerpt from "Theory of Sound" quoted in (III-C) indicates that the behavior of a complex system with uniform shunt damping was well understood in Rayleigh's time. A survey of more recent literature leaves the impression that this phenomenon is not so well known today. Throughout the literature, scattered references are found to "special cases" of damping where the mode shapes are unaltered. A few refer to Rayleigh's work; others are derived as special cases in studies of particular system types. Practically all of the latter involve the damping - mass relationship. (See, for example, Refs. 25, 26, 27). The significance and usefulness of the damping - stiffness relation appears to have been largely overlooked.

Even Rayleigh places heavy emphasis upon the damping - mass relation, which he says "occurs frequently, in books at any rate". There are, of course, some practical problems where a constant proportionality between b_{ij} and m_{ij} does exist, or can be approximated. An example is that of a beam vibrating in a viscous medium, where distributions of mass and damping reaction are approximately proportional (Ref. 27).

The writer feels that the b_{ij} - k_{ij} relationship is actually of greater significance. It applies, at least to good approximation, to many continuous elastic structures, lumped parameter mechanical systems, electric circuits with parasitic damping, and continuous electrical systems. Perhaps this has not been recognized more generally because the cases of paramount interest involve elastic hysteresis, or parasitic damping in electrical inductors, rather than shunt viscous damping. The relations between these various forms of uniform damping will now be considered.

The analysis of uniform shunt damping was partially carried out in (III-C) and (IV-B). To complete the analysis, it should be noted that the transformation to normal coordinates is a real coordinate transformation. Consequently, in (III-23), as the damping becomes uniform, the ratio of cofactors must become real:

$$\frac{D_{jK}}{D_{hK}} = \frac{\Delta_{ij}(\Gamma_K)}{\Delta_{ih}(\Gamma_K)} = \frac{A_{jK}}{A_{hK}} \quad . \qquad \text{(V-1)}$$

Since (V-1) holds for any row i, this implies that all cofactors of $\Delta(\Gamma_K)$ have a common phase angle for uniform damping; i.e., $\theta_{jK} \rightarrow \theta_K$. Since we are interested only in relative amplitudes, this phase angle can be ignored. The amplitude factors D_{jK} are computed for a simple system with $C_m = C_k = 1/2$, in (VIII-A).

From (IV-6) it is clear that the eigenvalues, Γ_K, move along circular arcs as damping is added uniformly. For damping proportional to mass, $B_K = C_m M_K$; hence,

$$\alpha_K = - \frac{C_m}{2} \quad . \qquad \text{(V-2)}$$

For damping proportional to spring stiffness, $B_K = C_k K_K$; hence,

$$\alpha_K = - C_k \frac{K_K}{2M_K} = - \frac{C_k}{2} \omega_K^2 \quad . \qquad \text{(V-3)}$$

For the mass damping, then, the absolute decay rate is identical for all modes; for spring damping, the decay rate increases rapidly with frequency. These properties are illustrated in Figure 11 a,b.

B. Structural Damping

The literature on this subject is copious, and not always in agreement. In 1912, Hopkinson and Williams (Ref. 28) measured the energy loss per cycle in a vibrating steel rod and found it to be essentially independent of frequency; i.e., an elastic hysteresis effect. Two years later Rowett (Ref. 29) made more sensitive measurements on thin steel tubing and found that the loss per cycle varied as the cube of the stress amplitude. In 1926-27 Kimball and Lovell (Ref. 30) tested a variety of structural materials such as metals, woods, and plastics, using the "whip" induced in a rotating shaft to evaluate the damping loss. The use of large masses of material tended to reduce surface effects, and they found a loss per cycle proportional to the square of the stress amplitude, and independent of frequency except at very low frequencies, where the loss increases.

Designating the energy loss per cycle as $\Delta W = A\ f^m y^n$ where f is frequency and y is vibration amplitude, later investigations have shown (1) that A increases appreciably with temperature, (2) that m is essentially zero, (3) that n is either 2 or 3 or somewhere between, at ordinary temperatures, and (4) that in some cases, particularly at high temperatures, n may go much higher, to around 8 or 10. (Refs. 31, 32, 33, 34.)

The nature of the structure is also an important factor. Kimball points out that built-up steel structures may exhibit a much higher damping than can be attributed to the internal stresses alone, due to friction and impact damping at the joints (Ref. 32). Coleman has measured the damping in various modes of solid and built-up cantilever wing models, and he concludes that for a homogeneous structure the damping coefficient is approximately the same in all modes of vibration (Ref. 25).

About 1940, a complex stiffness factor was introduced as a means of accounting for the effects of linearized strain hysteresis (Refs. 3, 35). The elements of the system matrix can be expressed as:

$$a_{ij}(p) = m_{ij}p^2 + (1 + jg)k_{ij} \ . \qquad \text{(V-4)}$$

By this means, the damping forces are considered to be directly proportional to the spring forces, but 90 degrees out of phase with them. This complex "g-factor" is now widely used in structural analysis. Recently Myklestad (Ref. 36) has pointed out that a more realistic linear representation is

$$a_{ij}(p) = m_{ij}p^2 + e^{jg}\, k_{ij} \ . \qquad \text{(V-5)}$$

For the large class of vibration problems where amplitudes are small and temperatures are moderate, (V-4) or (V-5) are probably accurate representations of the system forces. The approximations involved concern nonlinear effects, and are clearly beyond the scope of this work. Our present purpose is to investigate the effects upon the normal modes of the system, when the damping factor, g, is introduced.

First consider the system equations,

$$\left[a_{ij}(p)\right] \left[q_j\right] = \left[f_i\right] \tag{V-6}$$

where the $a_{ij}(p)$ are given by (V-4). Divide each row by $(1 + jg)$, altering both the a_{ij} and the f_i. Consider an individual term:

$$\frac{a_{ij}(p)}{(1 + jg)} = m_{ij}\left(\frac{p^2}{1 + jg}\right) + k_{ij} \quad . \tag{V-7}$$

It is now clear that the homogeneous equations must be satisfied by a set of eigenvalues Γ_K, where

$$\frac{\Gamma_K^{\,2}}{1 + jg} = -\omega_K^{\,2} \, . \tag{V-8}$$

Consequently,

$$\Gamma_K = j\omega_K \sqrt{1 + jg} \quad . \tag{V-9}$$

This expression defines a rather peculiar root locus in which all roots increase in absolute value, leaving the imaginary axis at right angles and curving to become asymptotic to the radial line at an angle of 135°. If $\sqrt{1 + jg} = a + jb$, then $\Gamma_K = -b\omega_K + ja\omega_K$; hence, for a given g the roots lie along a radial line from the origin, indicating a decay rate proportional to the damped frequency, as they should. This behavior is shown in Figure 11c.

If $(1 + jg)$ is replaced by e^{jg}, we get in place of (V-9),

$$\Gamma_K = j\omega_K \, e^{j\,g/2}$$

$$\alpha_K = \omega_K \sin g/2 \tag{V-10}$$

$$\beta_K = \omega_K \cos g/2 \quad .$$

The roots now move in semicircular arcs, and continue to lie along radial lines, Figure 11d.

The use of the factor e^{jg} corresponds to replacing the actual hysteresis loop by an equivalent ellipse. It is clear, from comparison of Figures 11c and 11d, that the latter is the correct linear approximation, as Myklestad has stated. Physically, the substitution of k_{ij} cos g for the spring stiffness implies that the springs become weaker as the dissipation increases.

The analysis of (V-7) and (V-8) amounts to a complex frequency transformation of the admittance matrix. This is a technique that has been used extensively by Bode (Ref. 37).

It is now clear that introduction of structural damping simply changes all the coefficients $a_{ij}(\Gamma_K)$ by a constant complex damping factor, when (V-10) is combined with (V-5). Then (III-24) gives

$$D_{jK} = e^{j(n-1)g} \sum_i \Delta_{oij}(j\omega_K) \quad . \qquad \text{(V-11)}$$

The constant phase shift drops out in the ratio, so that

$$\frac{D_{iK}}{D_{jK}} = \frac{A_{iK}}{A_{jK}} \quad . \qquad \text{(V-12)}$$

A better point of view is to replace (V-6) by

$$\left[(e^{-jg} m_{ij}p^2 + k_{ij}) \right] \left[q_j \right] = \left[e^{-jg} f_i \right] \quad . \qquad \text{(V-13)}$$

Now for root K the matrix elements on the left are identical in both the damped and undamped case; hence,

$$D_{jK} = A_{jK} \qquad \text{(V-14)}$$

and since all the normal coordinate theory applies, we can modify (II-12) directly:

$$e^{-jg} M_K \ddot{U}_K + K_K U_K = e^{-jg} F_K \qquad \text{(V-15)}$$

where M_K, U_K, K_K and F_K are those of the undamped system.*

The linear electrical analog of a spring with strain hysteresis is a constant "Q" inductor, Q being the quality factor; i.e., 2π times the ratio of maximum stored magnetic energy to total energy loss per cycle, when the inductor is driven sinusoidally. It is easily demonstrated that the "Q" of a spring is $\frac{1}{\tan g}$, or approximately $\frac{1}{g}$ for small g (Appendix B).

The frequency transformation of the admittances must apply regardless of the size or form of the system. Therefore, it applies to the individual tank circuits in the normal mode analog. As a result, the normal mode analog still represents the structurally damped system exactly, if equivalent structural damping is introduced into the tank circuit inductors.

We have thus shown that uniform linear structural damping leaves the normal modes completely uncoupled in free vibration, even as uniform shunt damping. Thus, in a large and important class of damped systems, "small damping" assumptions need not be resorted to.

---

* The Laplace transformed equation would be:

$$(e^{-jg} M_K s^2 + K_K)\, U_K(s) = e^{-jg} F_K(s)$$

hence, the complex frequency s is transformed to $\left(e^{-j\,g/2} s\right)$ in the admittances, but not in the forces.

C. Ignorable Coordinates in Series Damped Systems

Up to this point all of the analysis has been based upon the premise that we are dealing exclusively with dynamic mechanical systems which have n degrees of freedom, either with or without damping, or with the nodal electric analogs of such systems. We now wish to consider the effects of series viscous damping; i.e., the parasitic series ohmic resistance present in the elements of an experimental electric analog.

"Degree of freedom" is a well defined expression in mechanics, being the number of independent coordinates necessary to define completely the state of the system. This concept is based upon the implied choice of displacements as coordinates. Not so in electrical systems, where either currents or voltages are commonly viewed as coordinates. Since the number of independent loops in a circuit is often different from the number of independent nodes, the term "degree of freedom" has some ambiguity for electrical systems.

Since we are investigating normal modes, our primary interest is in the roots and cofactors of the system determinant, $\Delta(p)$. In considering these quantities, the following auxiliary theorem* will be useful.

Let the homogeneous equations of motion of a linear, passive, lumped parameter, electrical or mechanical system be expressed in any suitable set of coordinates. Let these equations be differentiated

---

* This is an elaboration of some ideas communicated to the writer by Prof. A. R. Teasdale, Jr. in a course on Servomechanisms at The University of Texas, 1949. It is essentially the same as an analysis given by Guillemin (Ref. 18, Ch. 5).

(or integrated) until all integral operators ($1/p^r$) have been removed, and all "factorable" differential operators (p^s) have been removed. Then the expanded system determinant will give a system characteristic equation of order n, in the form

$$A_n p^n + A_{n-1} p^{n-1} + \dots + A_1 p + A_o = 0 \qquad \text{(V-16)}$$

where

$$A_n \neq 0; \quad A_o \neq 0 .$$

Theorem 9. The characteristic equation of the system is of order equal to the number of independent energy storage elements in the system.

Corollary 1. The order of the system equation is independent of the type of coordinates chosen. (Loop equations give the same characteristic equation as node equations.)

Corollary 2. If series resistors are added in the loops of an electrical network, they will create new nodes. However, this cannot alter the order of the system equation, unless previously dependent energy storage elements are rendered independent by the additions.

Corollary 3. Adding shunt resistors between established nodes of an electrical network will create new loops. However, this cannot change the order of the system equation, unless previously dependent energy storage elements are rendered independent by the additions.

The theorem can be justified by noting that the homogeneous equations of the system represent an initial value problem. The initial state of the system is completely determined if all initial energy storages (currents in all inductors, charge on all capacitors) are known. Hence, the subsequent state of the system is likewise determinate. But the specification of n independent initial energy storages requires n arbitrary constants in the solution. Consequently, the equation must be of order n.

By reducing the form of the system equation to (V-16), we have factored out all "zero frequency" roots, which represent constant terms in the solution. This means that we must regard ideal capacitors in series, or ideal inductors in parallel, as being dependent storage elements, even though the former can carry a fixed charge, and the latter a fixed circulating current. Figure 9 shows three examples where added resistance destroys the dependence of reactive elements. If the pure reactive network is set up with series and shunt elements of the same type combined wherever possible, and if all transformers in the network remain ideal, then series and shunt resistors associated with the L's and C's cannot possibly change the number of non-zero roots.

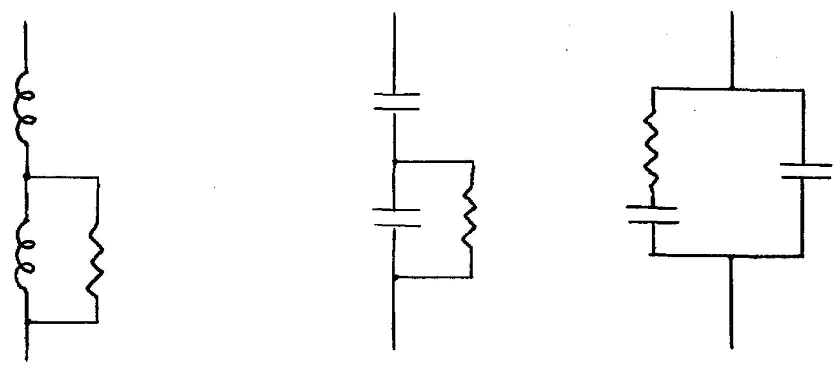
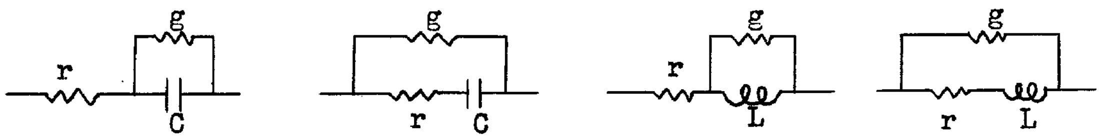


Figure 9. Cases Where Added Resistance Removes Dependency of Energy Storage Elements.



$g = k_1C$	$g = k_1C$	$r = k_3L$	$r = k_3L$
$r = k_2/C$	$r = k_2/C$	$g = k_4/L$	$g = k_4/L$
(a)	(b)	(c)	(d)

Figure 10. Four Basic Cases of Damping Added to Reactive Elements.

In Figure 10 are shown four combinations in which damping (parasitic or otherwise) might be added to reactive elements. For each case we can write an "operational admittance". Simple series or shunt damping occurs as a special case in these more general examples. Table I gives these admittance functions, together with those for the two types of elastic hysteresis damping previously discussed.

Table I. Operational Admittance Functions for Series - Shunt Damping Examples in Figure 10, and for Structural Damping.

Type of Damping	Operational Admittance: General	Operational Admittance: Uniform Damping
Figure 10(a)	$\dfrac{1}{r + \dfrac{1}{g + Cp}}$	$\dfrac{C(p + k_1)}{(k_2p + k_1k_2 + 1)}$
Figure 10(b)	$g + \dfrac{1}{r + \dfrac{1}{Cp}}$	$\dfrac{C(k_1k_2p + p + k_1)}{(k_2p + 1)}$
Figure 10(c)	$\dfrac{1}{r + \dfrac{1}{g + \dfrac{1}{Lp}}}$	$\dfrac{(k_4p + 1)}{L(k_3k_4p + p + k_3)}$

Table I (Continued)

Type of Damping	Operational Admittance: General	Uniform Damping
Figure 10(d)	$g + \frac{1}{r + Lp}$	$\frac{(k_4 p + k_3 k_4 + 1)}{L(p + k_3)}$
Structural	$\frac{1 + jg}{Lp}$	$\frac{1 + jg}{Lp}$
Structural	$\frac{e^{jg}}{Lp}$	$\frac{e^{jg}}{Lp}$

D. A General Theorem of Uniform Damping

Theorem 10. *If linear damping of arbitrary type or types is introduced uniformly into either the inductors or capacitors of a conservative electrical system, or into both capacitors and inductors simultaneously, then the normal modes of the system remain uncoupled; the mode shapes are unaffected; and the system eigenvalues move in a predictable locus, characterized by a one degree of freedom system having damping of the same types. The exact normal mode analog of the system is obtained by introducing damping into both the parent system and the analog according to the same pattern.*

The proof is straightforward, being based upon the use of a complex frequency transformation as in (V-B).* Inspection of Table I shows that any of the types of uniform damping listed there will transform the operational impedances in the form:

---

* Guillemin has discussed this for the specific case of shunt damping proportional to condensers and series damping proportional to inductors (Ref. 20, Ch. 6).

$$C_{ij}\, p \longrightarrow C_{ij}\left(\frac{\alpha p + \beta}{\gamma p + \delta}\right) \tag{V-17}$$

$$\frac{1}{L_{ij}\, p} \longrightarrow \frac{1}{L_{ij}}\left(\frac{\bar{\gamma} p + \bar{\delta}}{\bar{\alpha} p + \bar{\beta}}\right) \tag{V-18}$$

where α, $\bar{\alpha}$, β, $\bar{\beta}$, etc. are constants, real or complex, these transformations having the form of the bilinear transformation (Ref. 38). The equations of the undamped system can be expressed as

$$\left[C_{ij}\, p + \frac{1}{L_{ij}\, p}\right]\left[V_j\right] = \left[I_i\right] \quad . \tag{V-19}$$

After transformation, (V-19) becomes

$$\left[C_{ij}\left(\frac{\alpha p + \beta}{\gamma p + \delta}\right) + \frac{1}{L_{ij}}\left(\frac{\bar{\gamma} p + \bar{\delta}}{\bar{\alpha} p + \bar{\beta}}\right)\right]\left[V_j\right] = \left[I_i\right] . \tag{V-20}$$

Associating the entire transformation with the C_{ij} term,

$$\left[C_{ij}\left(\frac{\alpha p + \beta}{\gamma p + \delta}\right)\left(\frac{\bar{\alpha} p + \bar{\beta}}{\bar{\gamma} p + \bar{\delta}}\right) + \frac{1}{L_{ij}}\right]\left[\ V_j\right] = \left[\left(\frac{\bar{\alpha} p + \bar{\beta}}{\bar{\gamma} p + \bar{\delta}}\right) I_i\right] \tag{V-21}$$

Clearly, the eigenvalues of the damped system must be related to those of the conservative system by

$$\frac{(\alpha\, \Gamma_K + \beta)(\bar{\alpha}\, \Gamma_K + \bar{\beta})}{(\gamma\, \Gamma_K + \delta)(\bar{\gamma}\, \Gamma_K + \bar{\delta})} = -\,\omega_K^{\,2} \tag{V-22}$$

which reduces to

$$\left[\alpha\, \bar{\alpha} + \omega_K^{\,2}\, \gamma\, \bar{\gamma}\right] \Gamma_K^{\,2} + \left[(\beta\, \bar{\alpha} + \alpha\, \bar{\beta}) + \omega_K^{\,2}\ (\delta\, \bar{\gamma} + \gamma\, \bar{\delta})\right]\ \Gamma_K$$

$$+ \left[\beta\, \bar{\beta} + \omega_K^{\,2}\ \delta\, \bar{\delta}\ \right] = 0 \quad . \tag{V-23}$$

When (V-23) holds, the individual elements of $\Delta(\Gamma_K)$ are identical with those of $\Delta_0(j\omega_K)$. As a result, the eigenvectors have the real components A_{iK}, and the first part of the theorem is proved. Since the analysis applies either to the conservative parent system or to its normal mode analog, the second part of the theorem follows. Note that the forcing currents must be operated upon by the same factor in both the parent system and the analog, in the non-homogeneous case.

Experimental confirmation of Theorem 10 is shown in Figure 12. The transient behavior of a system with uniform series inductor damping is compared with that of its normal mode analog, in (a) and (b). In (c) and (d), the series damping has been removed, leaving only computer parasitic damping associated with the iron-cored inductors. For the parent system all inductors were 0.9 henries, but in the normal mode analog they ranged from 2.49 to 0.083 henries. Thus, (c) and (d) of Figure 10 lend support to the assumption that parasitic damping in the computer coils can be assumed to be uniform damping.

Figure 13 illustrates the effects of various types of uniform inductor damping upon the transient response of a system. The relatively greater attenuation of the higher frequencies due to the presence of shunt damping is apparent. The combination of series and shunt damping was chosen to simulate structural damping (constant Q coils) over the frequency range of the system modes.

E. Uniform Series Damping and Bode's "Uniform Dissipation"

Theorem 11. A system with uniform series damping in the inductors has the same eigenvalues as one with uniform shunt damping in the capacitors, if $r/L = g/C$.

Theorem 12. A system with uniform series damping in the capacitors has the same eigenvalues as one with uniform shunt damping in the inductors, if $gL = rC$.

Using the notation of Table I, let $L_{ij}p \rightarrow L_{ij}(p + k_3)$. Then

$$\left[C_{ij}p(p + k_3) + \frac{1}{L_{ij}}\right]\left[V_j\right] = \left[(p + k_3)\, I_i\right] \qquad \text{V-24)}$$

for which

$$\alpha_K = \frac{k_3}{2}\,; \quad \beta_K = \sqrt{\omega_K^2 - \alpha_K^2}\;. \qquad \text{(V-25)}$$

For the corresponding shunt damped capacitors,

$$\alpha_K = \frac{k_1}{2}\,; \quad \beta_K = \sqrt{\omega_K^2 - \alpha_K^2} \qquad \text{(V-26)}$$

and the forcing functions are $\left[p\, I_i\right]$. The proof of Theorem 12 is similar.

Next consider the case where both uniform series damped inductors and uniform shunt damped capacitors are used. We obtain

$$\left[C_{ij}(p + k_1)(p + k_3) + \frac{1}{L_{ij}}\right]\left[V_j\right] = \left[(p + k_3)I_i\right] \qquad \text{(V-27)}$$

from which,

$$\alpha_K = -\frac{(k_1 + k_3)}{2}\;; \quad \beta_K = \sqrt{\omega_K^2 + \frac{(k_1 - k_3)^2}{4}}\;. \qquad \text{(V-28)}$$

For $k_1 = k_3 = k$, we have the "distortionless transmission line" condition, where $r/L = g/C$. All eigenvalues move together along straight lines into the left half plane (Fig. 11e), in agreement with the results of Bode (Ref. 37, Ch. 10), who designates this case as a system with "uniform dissipation". Bode represents the effects of parasitic dissipation in a reactive network by the frequency transformation,

$$p \longrightarrow p + 1/2(r/L + g/C) \qquad \text{(V-29)}$$

which is approximate if $r/L \neq g/C$. From (V-28) the nature of the approximation is easily evaluated. As damping of either type is added, the eigenvalues move along semicircles. Then as the other type of damping is added, the roots move back toward the undamped frequency along a symmetrical arc (Fig. 11f).

With equal, uniform series capacitor damping and shunt inductor damping, the expression for the eigenvalues is

$$\Gamma_K = \frac{-k\omega_K^2}{1 + k^2\omega_K^2} + j\,\frac{\omega_K}{1 + k^2\omega_K^2} \; . \qquad \text{(V-30)}$$

These roots move along semicircles centered at $j\,\frac{\omega_K}{2}$, as shown in Figure 11g.

It should be noted that, since values of the individual a_{ij} are preserved by the transformation (V-22), it can be used on both the poles and the zeros of impedance and admittance functions.

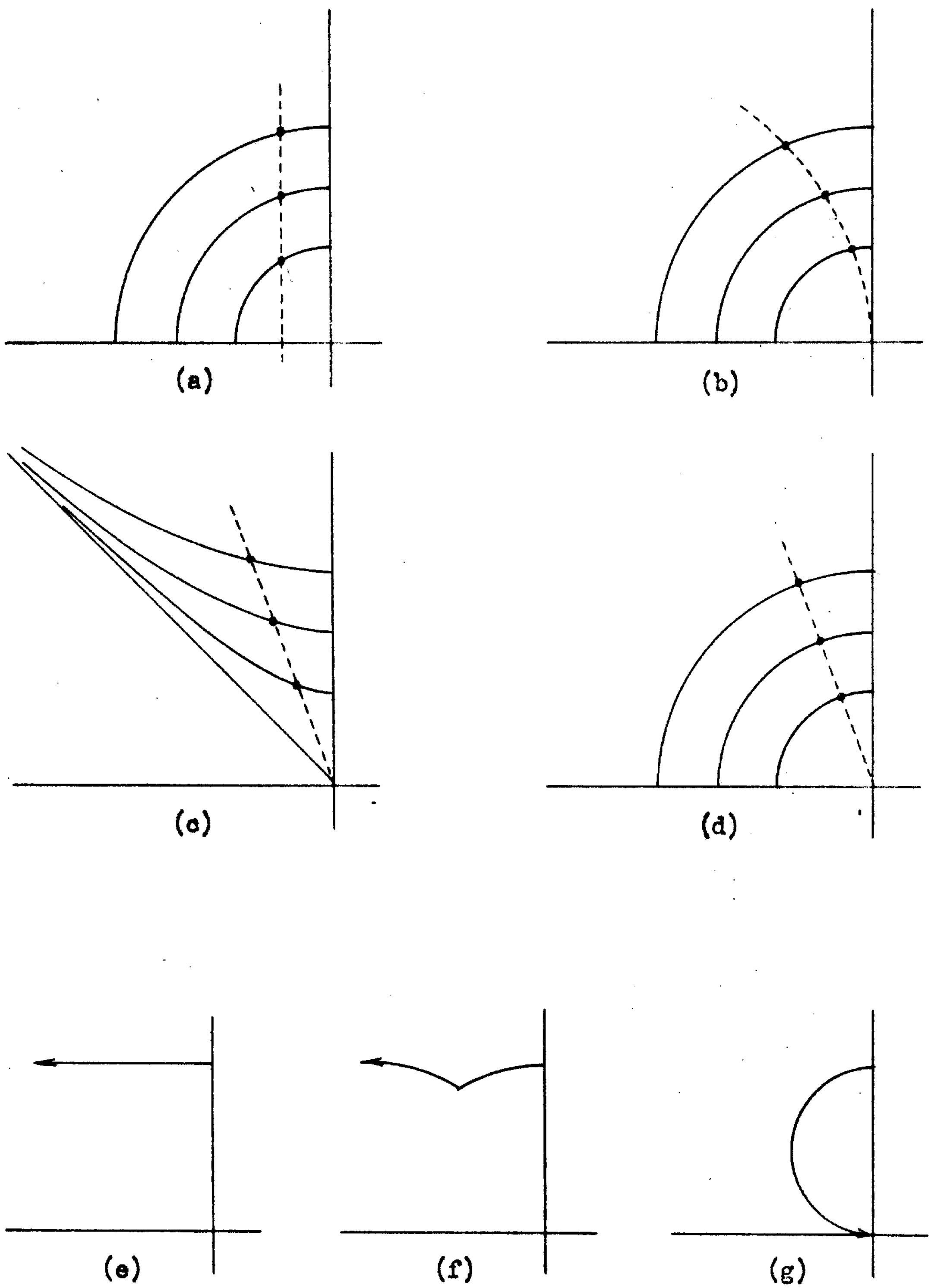
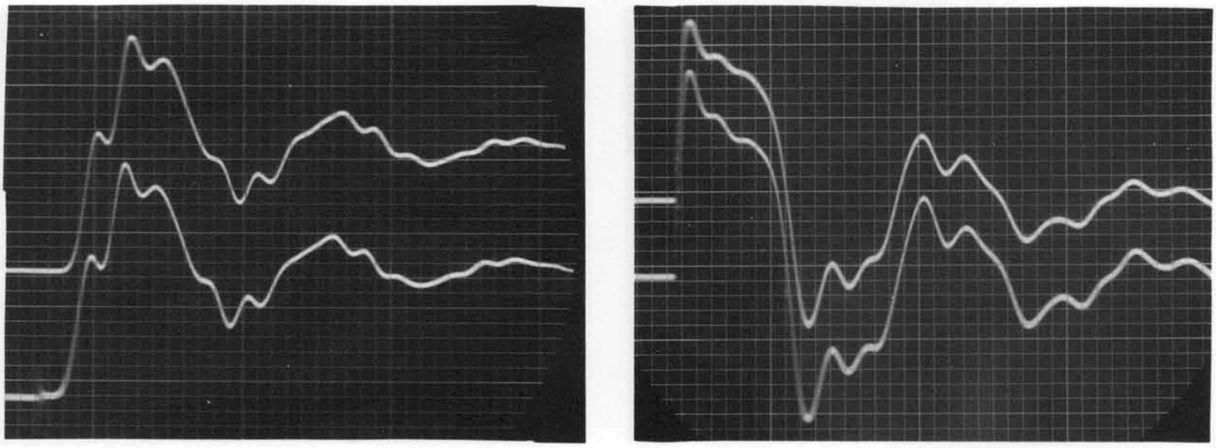
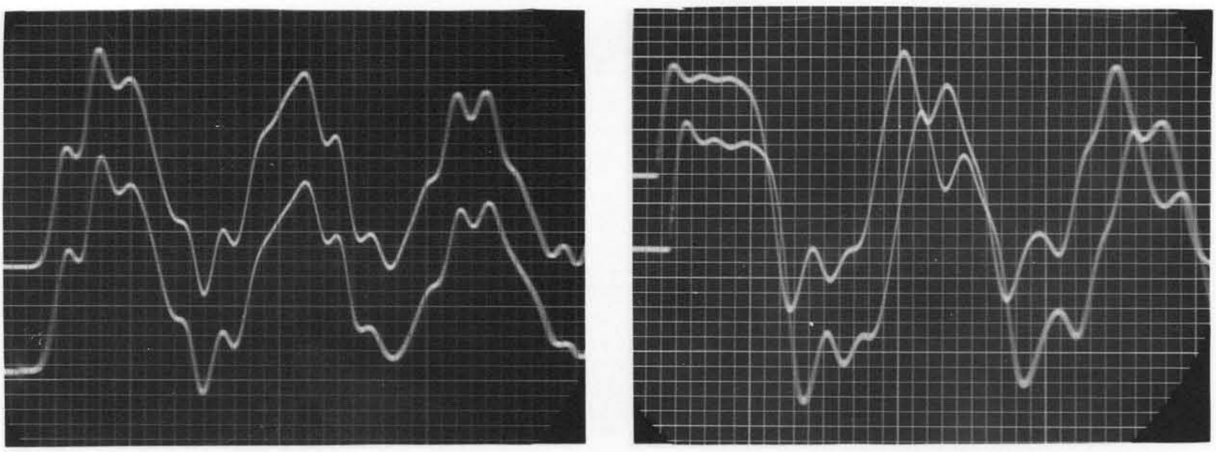


Figure 11. Loci of Eigenvalues for Various Types of Uniform Damping Discussed in Part V.



(a) Velocity at Station 3. (b) Force Applied at Station 7.

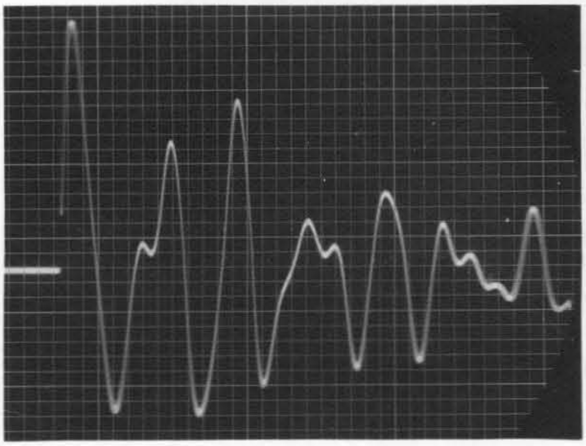
Uniform Series Resistance Added to Reduce Q of Coils to 20 at Highest Mode Frequency.



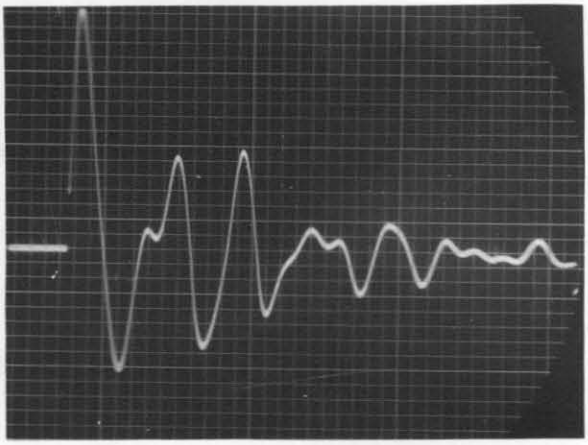
(c) Velocity at Station 3. Parasitic Damping Only.

(d) Force Applied at Station 7. Parasitic Damping Only.

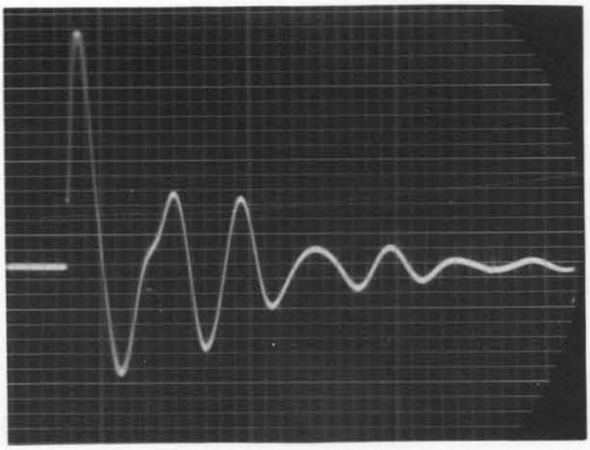
Figure 12. Response of Seven Degree of Freedom Torsion Rod to Step Input Velocity Applied at Station 7. Upper Trace from Direct Electric Analog; Lower Trace from Normal Mode Analog. This is System No. 3 of Part VIII.



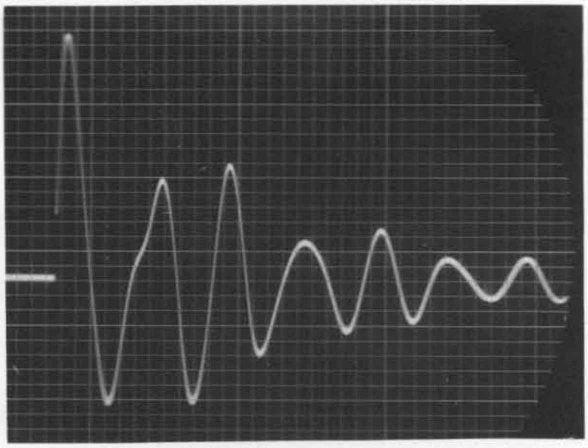
(a) Computer Parasitic Damping



(b) Series Damping Added to Give Coils a Q of 20 at Mean Mode Frequency.



(c) Shunt Damping Added to Series Damping in (b), Reducing Q of Coils to 10.



(d) Series Damping Removed From (c).

Figure 13. Effects of Added Inductor Damping on Reaction Force of Uniformly Damped Torsion System to Step Velocity Input. (System No. 4 of Part VIII.)

VI. SEPARATION OF NORMAL MODES IN SYSTEMS WITH UNIFORM DAMPING

A. The Problem

We now turn to the second major problem mentioned in Part I, that of determining, experimentally, the undamped normal modes of a given physical system which has damping losses. It should be remarked, at the outset, that this cannot be considered a "major" problem as far as electric analog analysis is concerned. Since 1949 the staff of the Analysis Laboratory at California Institute of Technology has made a large number of normal mode studies of complex aircraft structures. In a typical system, there is little difficulty in identifying and separating most of the modes, due to the high quality of the reactive elements and transformers used. However, in cases where two or more modes are close together in frequency, the separation is often difficult and time consuming, and the accuracy of results for such modes may be open to question. In actual shake tests of the elastic structure, the problem is apparently a more serious one, due to the higher damping and the greater inconvenience of making changes in excitation and measurement points.

The usual practice of the Analysis Laboratory is to drive the system sinusoidally with a variable frequency voltage source in the low audio frequency range of the computer. If the losses in the computer circuit were infinitesimal, then it is clear from the normal mode analog that at frequency ω_K the impedance of tank K would be very large relative to impedances of the remaining tanks;

consequently, all of the applied voltage would appear across tank K, and a pure mode shape would be obtained. The presence of a mode would be indicated by a null in driving current, unity power factor of the high input impedance, and an absence of phase shift in the responses of the system coordinates, relative to the driving voltage.

In practice, the losses in computer capacitors are negligible. Computer inductors have both ohmic and magnetic losses, thereby introducing damping that is to a large degree uniform. Computer transformers have both series and shunt losses, which tend to increase the non-uniformity of the damping. In the presence of either uniform or non-uniform damping, all tank circuits in the normal mode analog (Fig. 7) will be excited to some extent if the system is driven at one point. The excitation of these interfering modes will increase the minimum current driving mode K, and shift it in frequency. The unity power factor frequency is also shifted, and relative phase shifts are introduced among the coordinates. If the eigenvalues are well separated, these shifts are small, but when two or more of them are close together the separate modes may be well scrambled.

In shake tests of an airframe, the problem is essentially the same. The system is usually driven by small electro-mechanical shakers, which deliver a force rather than a velocity. A pair of shakers, symmetrically placed, corresponds to a single electrical drive source, because electrically the symmetric and antisymmetric modes of vibration can be separated by suitable constraints on the system. Small pickoffs, usually electrodynamic, measure the accelerations or velocities over the structure. For free - free vibration of an entire airframe, there are suspension problems which do not occur in the electric analog. The damping is primarily structural damping, and as such can be considered highly uniform.

At least two serious studies of this problem of mode separation have been made, both from the mechanical point of view. In 1947, Kennedy and Pancu (Ref. 7) presented an excellent analysis of the problem and a number of techniques for mode separation. Assuming the normal modes to be uncoupled for small damping, the authors proposed to excite the airframe with two pairs of shakers, driven in like or opposite phase, the amplitudes to be adjusted so as to make the resultant force fall at a node of the suppressed mode. Lack of relative phase shifts among coordinates at <u>any</u> frequency in the range of interest would be taken as evidence that the ratio of forces was correct. Where more than two modes are present, a frequency response locus analysis was proposed, based upon the principle that the locus is a sum of circular loci representing the separate modes, and that these individual circles often can be extracted. It was observed that the assumptions based upon small damping were open to question and that a more thorough theoretical study was desirable on this point.

In 1949 and 1950, Lewis and Wrisley (Ref. 8, 39) proposed a method of excitation, based upon a physical argument with respect to the forces involved, whereby each coordinate of the system would be driven by a separate shaker. All shakers would be driven in phase, using forces proportional to the product of mass and amplitude at each coordinate. An iteration procedure was proposed for arriving at this condition, and necessary experimental equipment for carrying out the process was developed. The authors assumed small damping explicitly and uniform damping implicitly. For a complete air-frame, a set of 24 shakers was used. Mutual masses were not considered, nor was the effect of non-uniform damping accounted for in the theory. The method was tested on a simple beam-like structure having lumped masses and shear compliance only. Good results were obtained, both for uniform and non-uniform damping. Tests of three well separated modes of an actual airframe appear to be somewhat less satisfactory, however.

The writer feels that the methods presented in both of these studies are sound and valuable, if intelligently applied. It is felt that the work presented here, in addition to proposing some different techniques and criteria, should serve to substantiate most of the conclusions of the other investigators. Many of the points covered by them will be included in the present analysis, for the sake of continuity.

B. Method of Approach

It is now clear from the preceding work that the one type of many-degree-of-freedom system which lends itself to simplified analysis is the uniformly damped system. Fortunately, the uniform damping can assume a number of forms, and most of the systems with which we are concerned fall approximately into this special category. Therefore, an effort will be made to develop a mode separation technique applicable to uniformly damped systems, with very few restrictions as to the magnitude of the damping. After this has been accomplished, extensions to non-uniformly damped systems can be considered on a more logical basis.

C. Shunt Damped Systems Driven at One Point

(1) Choice of Drive Point; Interpretation of Response

Let us consider first the specific case of a system with uniform shunt damping, being driven sinusoidally at one coordinate. The drive may be either a constant current or a constant voltage, the choice being determined by convenience and accuracy considerations. Electrically, the symmetric and antisymmetric modes will be isolated by applying constraints, so that a single current source corresponds to a symmetrically placed pair of shakers.

The analysis will be made in terms of the normal mode analog discussed in Part IV, but with the impedance level in the tank circuits chosen differently. In (II-B) it was pointed out that the absolute level of amplitude coefficients, A_{1K}, is arbitrary for each mode. A normalizing condition (II-33) was proposed which was convenient for analysis of the undriven system. For the present purpose, it will be more convenient to use a new normalizing condition, such that the resonant admittance of each of the tanks is the same. For shunt damping, choose M_K such that

$$\frac{\omega_K M_K}{Q_K} = B \qquad \text{(VI-1)}$$

making B, an arbitrary shunt conductance, the same for all modes. The corresponding normalizing equation is

$$\sum_i \sum_j m_{ij} A_{iK} A_{jK} = \frac{BQ_K}{\omega_K} \quad . \qquad (VI-2)$$

Figure 14 shows the system to be investigated.

With the excitation restricted to one coordinate, q_j, we are free to choose only the point j and the drive frequency ω. Assume for the moment that the mode frequency can be determined, so that the system is excited at frequency ω_K. Then clearly, the best drive point is one with a reasonably large A_{jK} so that the impedance of tank K will appear large when viewed from q_j. At the same time, we desire that the A_{jL} of the nearby modes be small. Since it will probably be impossible in a large system to choose j such that $A_{jL} << A_{jK}$ for all $L \neq K$, we must consider only those modes which are in a position to contribute significant interference. We can place this matter on a quantitative basis most conveniently by the

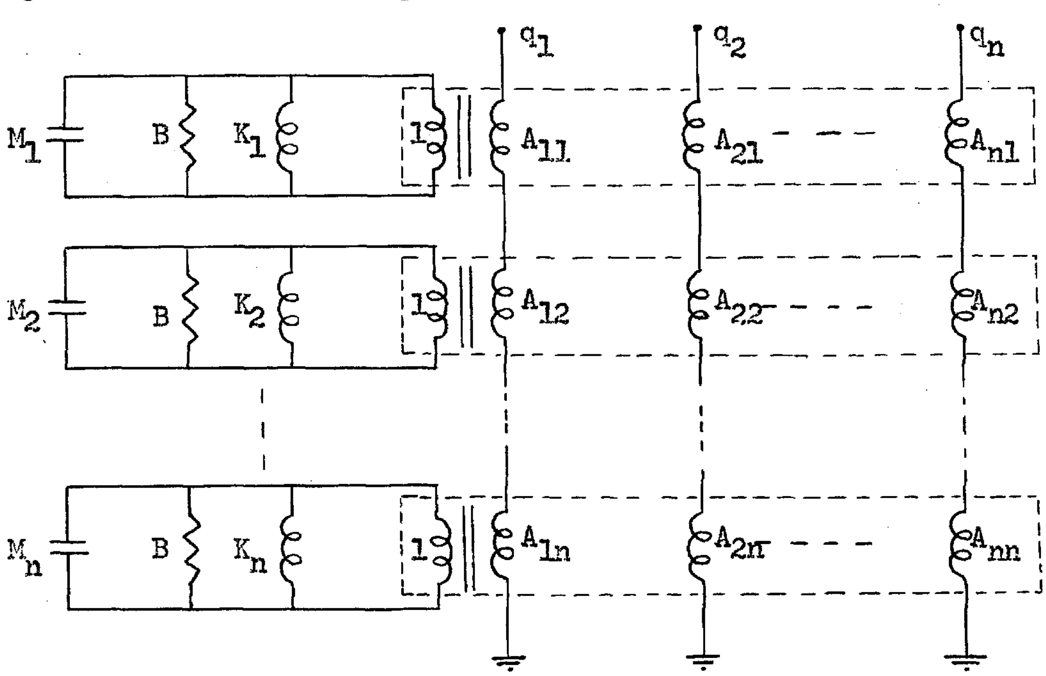


Figure 14. Complete Normal Mode Analog of System With Uniform Shunt Damping, Normalized to Give All Tanks the Same Resonant Admittance.

use of so-called "universal" resonance curves (Ref. 40, Ch. 7 and Appendix A). Figure 15 shows these curves both for entire impedance $|Z|$, and for the real part of the impedance, R, of a parallel resonant circuit. At resonance, $R = |Z| = 1/B$, where B is the conductance common to all tanks in Figure 14. Figure 16 shows a corresponding universal phase shift curve, and Figure 17 a universal circle diagram.

It has been the past practice of the Analysis Laboratory to drive a system with a voltage source at a favorable point, set the frequency to unity power factor or minimum input current (usually the latter), and record the in-phase components of response, referred to the drive voltage. Figure 15 shows clearly that the use of in-phase components is a sound practice; however, the input current should be taken as reference, rather than the input voltage. Since the curve for R approaches zero much faster than $|Z|$, we will assume henceforth that in-phase components of response are always to be taken as the best single measure of the A_{jK}.*

Using Figure 15, it is a simple matter to estimate the relative attenuation of the various modes in the neighborhood of ω_K, due to frequency shift alone. Only the frequencies and the Q's need be known or estimated. For example, if we wish to excite a mode K at 190 cps, and there is a neighboring mode L at 200 cps, then $\Delta\omega/\omega_0$ for mode L is -.05. If mode L has an estimated Q of 20 (g-factor of .05), then the ordinate at -1.0 shows an attenuation of the mode L response to 20% of its resonant value.

---

* Key points on the "universal" curve of R for $Q = \infty$ are at .5, 1.0, and 1.5; the corresponding ordinates of .5, .2, and .1 are easily remembered.

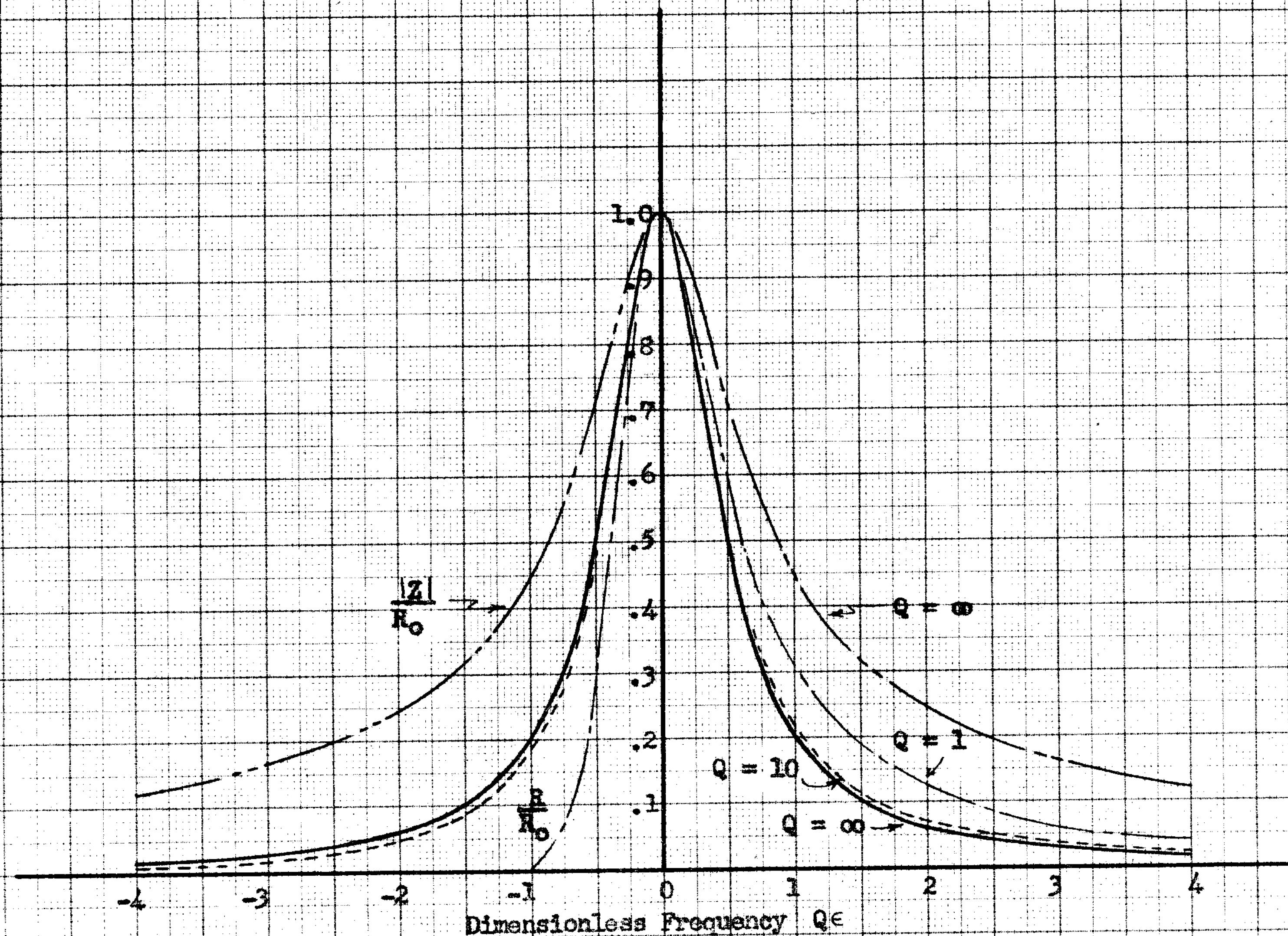


Figure 15. Universal Resonance Curves for Single Degree of Freedom System with Shunt Damping, Showing Real Part, R, and Total Magnitude, |Z|, of Input Impedance. $\epsilon = \Delta\omega/\omega_o$.

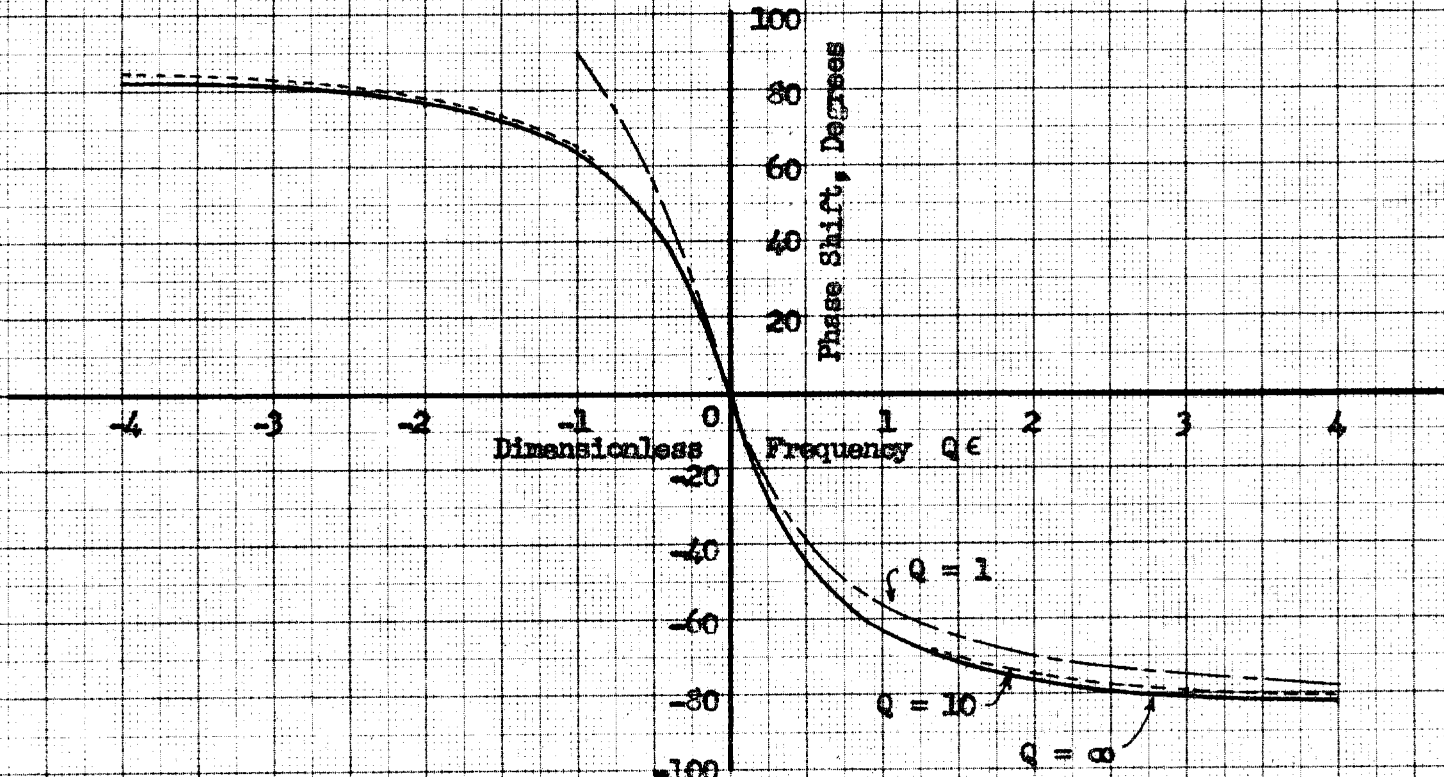


Figure 16. Universal Phase Shift Curves for Input Impedance of Single Degree of Freedom System with Shunt Damping. $\epsilon = \Delta\omega/\omega_0$.

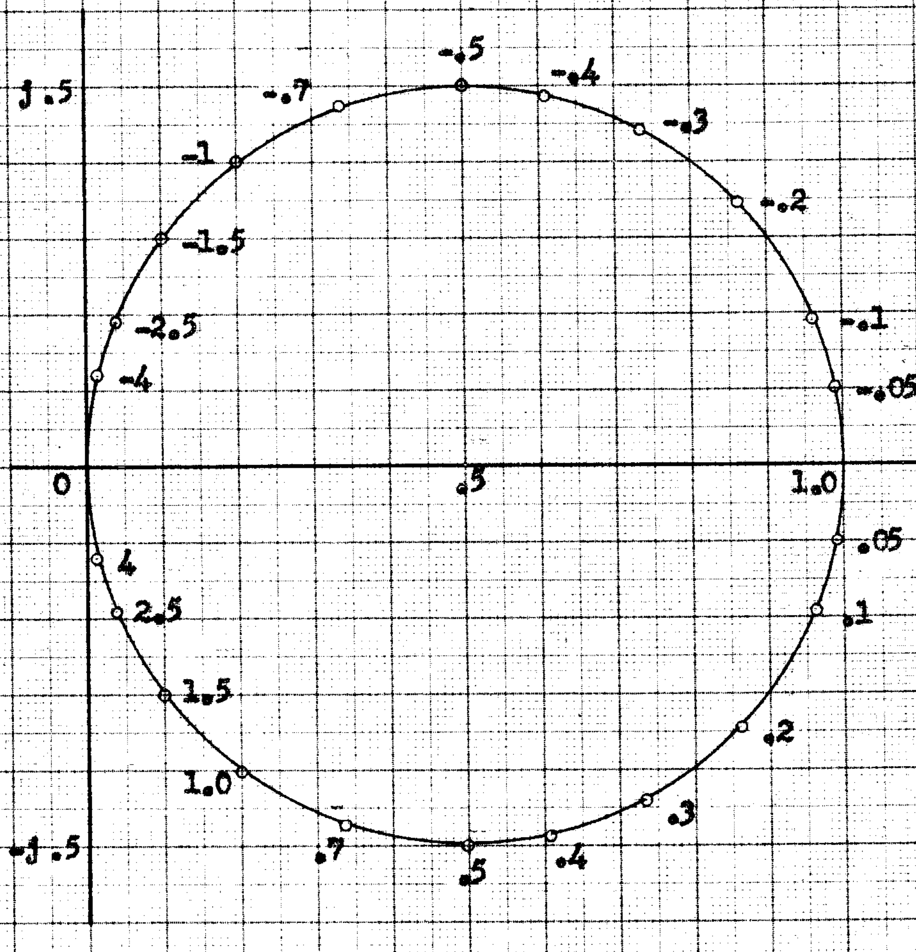


Figure 17. Input Impedance Locus Z/R_o for Single Degree of Freedom System with Shunt Damping. Numerical Values are $Q\epsilon$, for $Q = \infty$. $\epsilon = \Delta\omega/\omega_o$.

Let us designate this as a detuning factor, η_{KL}. Then the ratio of in-phase voltages across tank K and tank L will be:

$$\frac{\dot{U}_L}{\dot{U}_K} = \eta_{KL} \frac{A_{jL}}{A_{jK}} \triangleq \gamma_{jKL} \qquad \text{(VI-3)}$$

where $\dot{U}_L$ is taken here as the component of U_L in phase with the driving current, and where A_{jK} and A_{jL} are normalized by (VI-2). The corresponding in-phase ratio measured at coordinate i would be:

$$\frac{\dot{q}_{iL}}{\dot{q}_{iK}} = \gamma_{jKL} \frac{A_{iL}}{A_{iK}} \triangleq \mu_{ijKL} \quad . \qquad \text{(VI-4)}$$

The drive point should be chosen, then, to give small values of μ_{ijKL} at all points q_i and for all modes L. In practice, one simply hunts for the drive point that produces the least phase shifts in the response. Note, however, that it is possible to have rather large phase shifts and still get good accuracy by taking the in-phase components, if the frequency separation exceeds about 2.0 in Figure 15. Note, also, that it may not always be true that the adjacent modes give the greatest interference. For a given frequency separation, an increase in Q/ω_0 tends to decrease η, from Figure 15, but to increase $A_{jL}A_{iL}$ linearly, from (VI-2); thus, the effects on μ tend to compensate. Suppose, though, that there is a mode in which the heavy masses and springs do not participate to any great extent. The normalization condition (VI-2) then requires <u>large</u> values for the A_{jL}. It may be more desirable to suppress such a mode than to suppress another which is closer in frequency.

The factors defined by (VI-4) are a measure of the accuracy one can expect in the determination of the A_{iK}. For the sake of brevity, let us designate as an <u>interfering</u> mode any nearby mode L which gives sub-standard values of μ_{ijKL} for random choice of j and i. In general, we can hope to achieve partial suppression of one badly interfering mode, in an electrical analog, using a single drive.* Since an absolute node point seldom occurs in a lumped parameter system, complete suppression will usually be impossible, and the best we can do is to choose j for a maximum ratio, $|A_{jK}/A_{jL}|$. If this does not suffice, multiple excitation should be used.

The approximation of the eigenvector components A_{jK} by neglecting the in-phase contributions of the off-resonant modes is essentially equivalent to a converse assumption; namely, that the damping in the off-resonant modes has negligible effect upon the driven system response. Such an assumption has been used in computing the forced response of damped systems (Ref. 3). It is equivalent to dropping the damping elements from all modes except the dominant mode in Figure 14, and working with the simplified system in Figure 18. Clearly, this is the most reasonable simplifying assumption that can be made, and we are justified in proposing:

Theorem 13. In a moderately damped system, driven such that badly interfering modes are suppressed, the best approximation to the shape of a driven mode, based upon a single set of measured velocities, is obtained from the components of velocity in phase with the driving forces, with the drive frequency

---

* Kennedy and Pancu (Ref. 7) have shown that as many as three modes can be suppressed, theoretically, by location and space orientation of a shaker, on a continuous structure.

being the unity power factor frequency of the driven mode. The theorem has not been restricted to uniform shunt damping, because the analysis will be extended later to other types. The term "moderate damping" is intended to be much less restrictive than "small damping".

(2) Determination of Correct Drive Frequency

Next, consider the question of finding the correct drive frequency. Assume that there are no badly interfering modes, or perhaps that there is one such mode which has been effectively suppressed and can be ignored. Then the off-resonant modes which are excited will certainly lie outside of the resonance region ± 1 in Figure 15, and probably beyond ± 2. From Figure 16, the phase shift in the neighboring off-resonant modes will be on the order of 65° or greater.

If only mode K were excited, the impedance locus would be a circle through the origin (Fig. 17). Due to excitation of neighboring modes, this circle will be displaced, the actual locus being the sum of the loci of the various mode responses. Kennedy and Pancu have discussed cases of this kind where the displacement is extreme, due to the presence of badly interfering modes. For the situation we are considering, this displacement should be greatly reduced, at the high amplitude stations of mode K. Furthermore, it is apparent from Figure 17 that $dZ/d\omega$ is much smaller for the off-resonant modes than for the resonant mode, so that the displacement vector is essentially stationary as mode K swings through its resonance peak. This is the basis of the Kennedy-Pancu method.

Consider the driving point impedance at the optimum drive point, j, to be the sum of the impedance of tank K, which describes a circular arc, and the residual impedance of the remaining tanks, which will be represented as a constant displacement of the arc center. Arbitrarily take the sign of A_{jK} as positive. Then the residual impedance at the driving point may be either positive inductive or positive capacitive. If we drive point j and measure point i, taking A_{iK} as positive, then the residual transfer impedance may be either positive or negative. Figure 19 shows these possible displacements, with heavy dots marking the frequencies of zero phase shift, maximum impedance, and true frequency of the desired mode. With a constant current drive, the diagrams may be thought of as response voltage loci rather than impedance loci. Based upon the geometry of Figure 19, we can state:

Theorem 14. In a damped system, driven such that the badly interfering modes are suppressed, the true frequency of the damped mode will normally lie about midway between the maximum impedance frequency and the zero phase shift frequency, as measured at any high amplitude response station.

Theorem 14 has not been restricted to uniform shunt damping because the analysis will be extended later to cover other types. A much more sensitive frequency criterion will be obtained next from the quadrature components of the response voltages:

Theorem 15. In a damped system, driven such that the badly interfering modes are suppressed, the true frequency of the damped mode will be approximately at the point where the in-phase components of response are orthogonal to the quadrature components of response, where phase is measured with respect to the driving force or current.

Proof: Assume that the excitation is such that if the system were driven at frequency ω_K, the in-phase components of the measured $\dot{q}_j$ would be a good approximation of the A_{jK}. Designate the response voltages as

$$\dot{q}_j = \dot{\bar{q}}_j + j\,\underline{\dot{q}}_j \tag{VI-5}$$

the phase to be measured with respect to the driving current. Then at resonance, mode K contributes nothing to the $\underline{\dot{q}}_j$, and the contribution of the remaining modes to the $\dot{\bar{q}}_j$ is small by hypothesis. If the quadrature component of $\dot{U}_L$ with respect to the drive current is $\underline{\dot{U}}_L$, then

$$\underline{\dot{q}}_j = \sum_{L \neq K} A_{jL}\,\underline{\dot{U}}_L \tag{VI-6}$$

and

$$\dot{\bar{q}}_j = A_{jK}\,\dot{U}_K + \sum_{N \neq K} A_{jN}\,\dot{\bar{U}}_N \;. \tag{VI-7}$$

Hence, using the orthogonality relation (II-19),

$$\sum_i \sum_j m_{ij}\,\dot{\bar{q}}_i\,\underline{\dot{q}}_j = \sum_{L \neq K} \dot{U}_K\,\underline{\dot{U}}_L \sum_i \sum_j m_{ij}\,A_{iK}\,A_{jL}$$

$$+ \sum_{N \neq K} \sum_{L \neq K} \underline{\dot{U}}_L \dot{\bar{U}}_N \sum_i \sum_j m_{ij} A_{iN} A_{jL} \;.$$

$$\sum_i \sum_j m_{ij}\,\dot{\bar{q}}_i\,\underline{\dot{q}}_j = \sum_{L \neq K} M_L\,\dot{\bar{U}}_L\,\underline{\dot{U}}_L \cong 0 \;. \tag{VI-8}$$

q.e.d.

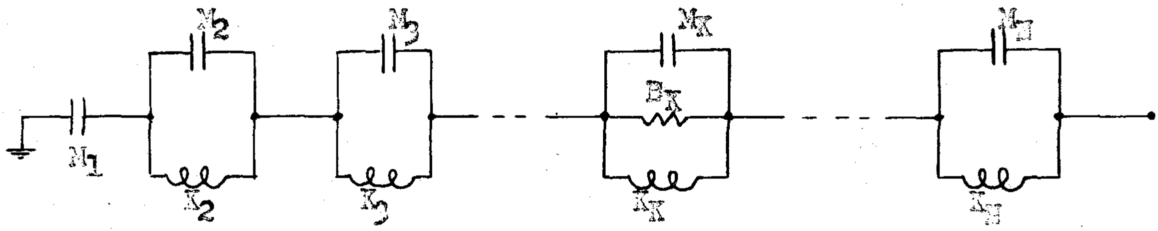


Figure 18. Approximate Normal Mode Analog With Damping Omitted from Off-Resonant Modes.

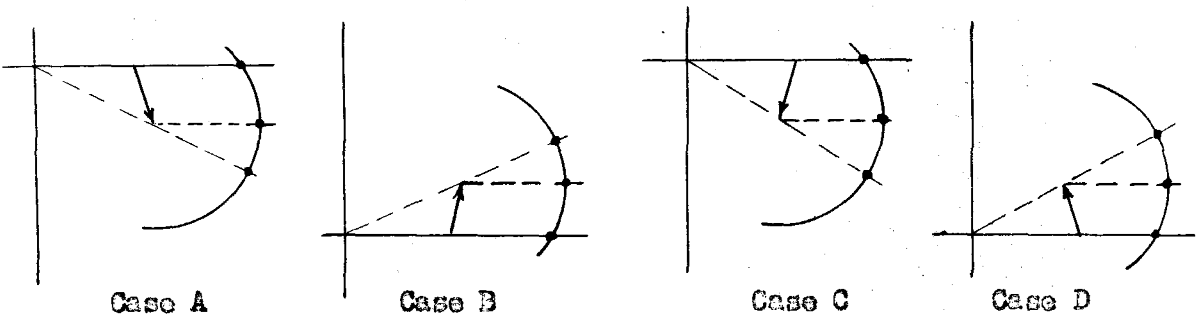


Figure 19. Driving Point and Transfer Impedance Loci For Uniformly Damped System With Badly Interfering Modes Absent or Suppressed. Residual Impedance Capacitive, Positive (Case A); Inductive, Positive (Case B); Inductive, Negative (Case C); and Capacitive, Negative (Case D).

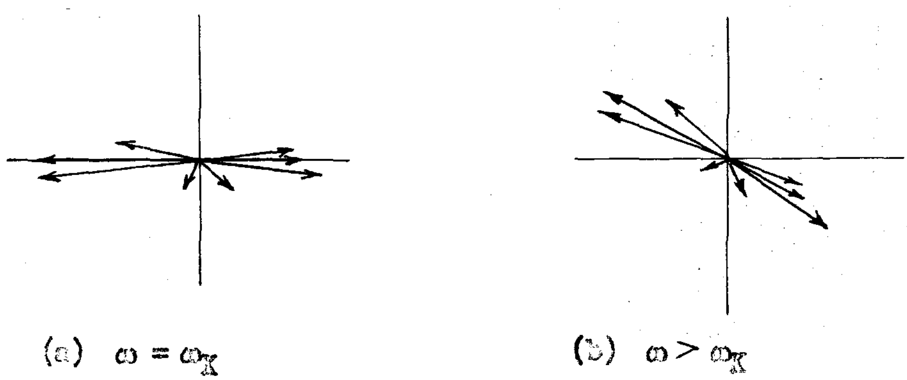


Figure 20. Phase Rotation of Response Voltages as Frequency is Shifted Off Resonance.

Note that the residual term in (VI-8) would be equal to the stored energy in all off-resonant modes, if the phase shifts were all 45°. This is an extremely conservative upper limit.

The frequency criterion of Theorem 15 is extremely sensitive and accurate. In general, if we measured all of the response voltage vectors at frequency ω_K, they would be grouped somewhat symmetrically about the reference axis, as shown in Figure 20a. But from Figure 19, it is clear that a slight increase in frequency tends to rotate all the response vectors clockwise. This rotation is quite rapid in the neighborhood of resonance, so a very small shift can put the vectors in the position of Figure 20b. A resonance indicating factor can be defined as

$$\psi = \frac{\sum_i \sum_j m_{ij} \, \hat{\dot{q}}_i \, \underline{\dot{q}}_j}{\sum_i \sum_j \left| m_{ij} \, \hat{\dot{q}}_i \, \underline{\dot{q}}_j \right|} \quad . \qquad \text{(VI-9)}$$

As the frequency is increased through resonance, ψ makes a rapid transition from the neighborhood of +1 toward -1. The null will usually be so sharp that even if the residual term in (VI-8) were appreciable, the frequency could still be determined with high accuracy. This is demonstrated in Part VIII by numerical example, and by test results from the analog computer. It was found that, in application, the exact calculation of ψ is unnecessary to establish the mode frequency. If a rapid survey of the response voltages across the system capacitors shows that about half are advanced and half retarded in phase, then $|\psi|$ is "in the notch".

(3) Evaluation of Mode Parameters

A complete description of the normal modes of a uniformly damped system requires that all transformer ratios and mode parameters in Figure 14 be evaluated. Methods of determining the relative turns ratios and the frequencies have been discussed. It remains to evaluate M_K, K_K, and B_K. There are a number of possibilities.

If the mass distribution of the system is known, then the kinetic energy in mode K can be computed on the basis of measured in-phase velocities. Moreover, if we follow the usual electrical engineering practice of considering the forces and velocities as root mean square values, then T and V become average energy functions, and the average power input to the system is 2F. If we ignore the energy losses in off-resonant modes, then

$$P \cong 2F_K = B_K \dot{U}_K^2 \qquad \text{(VI-10)}$$

and

$$T_K = 1/2 \, M_K \dot{U}_K^2 \cong 1/2 \sum_i \sum_j m_{ij} \dot{\bar{q}}_i \dot{\bar{q}}_j \, . \qquad \text{(VI-11)}$$

Thus, we can evaluate Q_K:

$$Q_K = \frac{\omega_K M_K}{B_K} \cong \omega_K \frac{\sum_i \sum_j m_{ij} \dot{\bar{q}}_i \dot{\bar{q}}_j}{P} \, . \qquad \text{(VI-12)}$$

All quantities on the right of (VI-12) are easily measured, except the m_{ij} which are presumably known. If a multiple drive is used, P must, of course, be the total average power input to the system from all sources. Once Q_K has been evaluated, the normalizing condition (VI-2) can be applied to determine M_K and the A_{jK}. Normalized values can be obtained directly by making P the same in all modes.

K_K is then obtained from the frequency by (II-15).

If the mass distribution is unknown, or if the interference from other modes is rather severe, then it may be desirable to take a frequency response locus in the vicinity of resonance for a few of the higher amplitude coordinates. Then, following Kennedy and Pancu, a circle may be extrapolated from the circular arc in the neighborhood of resonance. Let the diameter of this circle for the driving point impedance Z_{ii} be designated D. Let $Z_{ii} = R_{ii} + jX_{ii}$; then along the locus we can measure $\Delta X_{ii}/\Delta\omega$ in the neighborhood of ω_K. It is easily demonstrated (Appendix A) that

$$Q_K = -\frac{\omega_K}{2D}\left(\frac{dX}{d\omega}\right)_{\omega_K} \quad . \qquad \text{(VI-13)}$$

Hence, Q_K (or $1/g_K$) is easily obtained from the extrapolated circle. Then M_K can be computed from (VI-1), and K_K from (II-15). The normalization condition is

$$A_{iK} = \sqrt{B\ D} \qquad \text{(VI-14)}$$

where B is the common shunt conductance of all tanks, and D is the impedance circle diameter. Equation (VI-14) is equivalent to (VI-2).

If a transfer impedance locus is used, or if there are multiple drive currents, then it is convenient to refer all drive currents to the point of voltage measurement, and treat the locus as an equivalent driving point impedance. To do this, plot the voltage response at q_j directly, but assume an equivalent current:

$$\tilde{f}_j = \sum_i \frac{A_{iK}}{A_{jK}}\ f_i \quad . \qquad \text{(VI-15)}$$

In general, it is believed that (VI-12) will give a more accurate value for Q_K than (VI-13), due to the magnification of experimental errors in estimating the response circle center, and due to the fact that the vector displacement of the center is actually not stationary. The error in (VI-12) is actually quite small, as can be seen by expanding the numerator in terms of (VI-7), using orthogonality to eliminate some of the terms, and expressing P as the sum of tank circuit losses. The result is

$$Q_K \cong \omega_K \frac{(M_K \dot{U}_K^2 + \sum_{L \neq K} M_L \dot{\bar{U}}_L^2)}{(B_K \dot{U}_K^2 + \sum_{L \neq K} B_L |\dot{U}_L|^2)} \qquad \text{(VI-16)}$$

showing that the errors in numerator and denominator tend to compensate.

In the impedance locus method, the direction of error due to displacement of the circular locus center can be predicted, for the case of an input impedance diagram and a single driving source. Consider the circuit to be in the form shown in Figure 18, with q_i being driven. Then if we neglect damping in the off-resonant tanks, Foster's reactance theorem (Ref. 22) states that their total reactance must be an increasing function of frequency. Figure 21 shows the type of variation to be expected for the quadrature components of the response. Now compare this with the input impedance loci, Figure 19a, b. As the circular locus travels <u>downward</u> in a clockwise arc, the center moves <u>upward</u>. Figure 22 shows the result, a virtual center displaced from the true center. Thus we can state:

Theorem 16. For a system driven at one coordinate, near resonance of one of its modes, the extrapolated "best circle" through the input impedance locus will tend to be too small, due to the variation in impedance of the off-resonant modes.

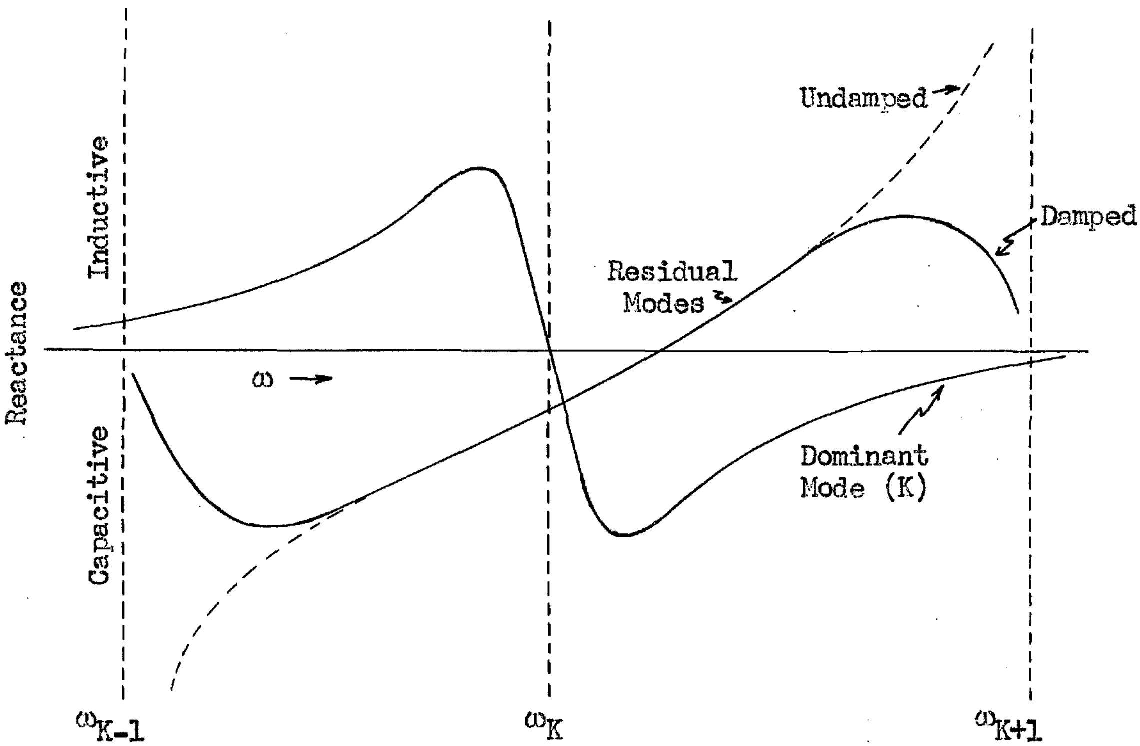


Figure 21. Variation in Reactive Parts of Input Impedance, as Predicted by Foster's Reactance Theorem.

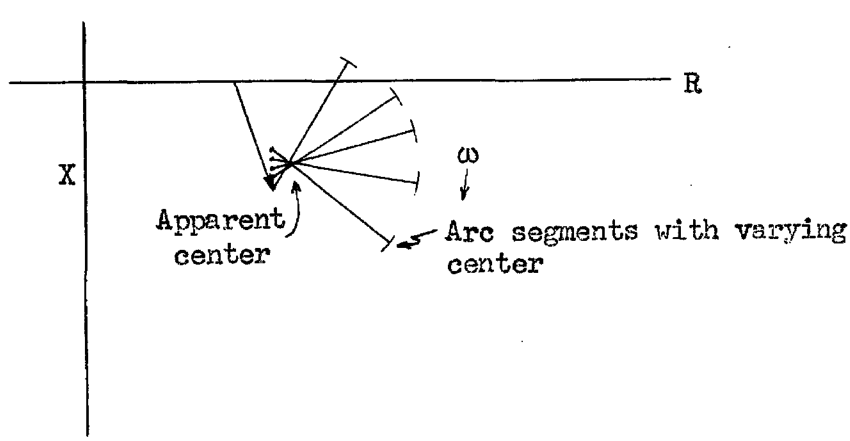


Figure 22. Reduction in Radius of Curvature Due to Shift in the Displacement Vector.

D. Extension To Other Types of Uniform Damping

We will show next that all of the conclusions of (VI-C) are valid for structural damping, with slight modifications. All are valid for small series damping, and some are valid for appreciable series damping. Structural damping and uniform series damping in inductors are the two cases of greatest practical interest, so these will be considered in detail. The analyses will be carried out in Appendix B and C, with only the results stated here. We will work in terms of a damped resonant frequency, $\tilde{\omega}_K$, which is the unity power frequency of tank K.

(1) Uniform Structural Damping

Denoting the "complex stiffness" coefficients of a system with elastic hysteresis as $e^{jg}\, k_{ij}$, the corresponding normal mode spring coefficients are $e^{jg}\, K_K$. The quality factor of mode K is then

$$Q_K = \frac{1}{\tan g} \quad . \qquad \text{(VI-17)}$$

The damped resonant frequency of mode K is

$$\tilde{\omega}_K = \omega_K \sqrt{\cos g} \quad . \qquad \text{(VI-18)}$$

The masses are unaffected, so the normalizing condition equivalent to (VI-1) or (VI-2) is

$$\frac{\tilde{\omega}_K\, M_K}{Q_K} = B \quad . \qquad \text{(VI-19)}$$

where B is an arbitrary constant for all modes, to be considered here as the equivalent shunt conductance of the tank circuit at frequency $\tilde{\omega}_K$.

Universal resonance curves similar to Figures 15 and 17 are shown in Appendix B. The analysis of (VI-C-1) follows, and Theorem 13 holds if the system is driven at frequency $\tilde{\omega}_K$.

The impedance locus, $Z_K(\omega)$, is a distorted circle for structural damping, but in its place we can plot,

$$\tilde{Z}_K = \tilde{R}_K + j\,\tilde{X}_K = \frac{\tilde{\omega}_K}{\omega} Z_K \quad . \tag{VI-20}$$

The locus of $\tilde{Z}_K$ is a true circle, with a diameter

$$D = \frac{Q_K}{C\tilde{\omega}_K} \tag{VI-21}$$

for which

$$\left(\frac{d\tilde{X}_K}{d\omega}\right)_{\tilde{\omega}_K} = -\frac{2D\,Q_K}{\tilde{\omega}_K} \tag{VI-22}$$

Near resonance, $\tilde{Z}$ and Z are approximately equal, so Figure 19 and Theorem 14 are still valid. The proof of Theorem 15 is unchanged, if the drive frequency is now understood to be $\tilde{\omega}_K$. The same is true of the evaluation of Q_K by (VI-12); however, the total power input at frequency $\tilde{\omega}_K$ is

$$P = \tilde{\omega}_K \sin g \sum_L K_L \left|U_L\right|^2 \quad . \tag{VI-23}$$

Hence, (VI-16) is replaced by

$$Q_K \cong \frac{1}{\sin g} \; \frac{\left(M_K \dot{U}_K^{\,2} + \sum_{L\neq K} M_L \dot{U}_L^{\,2}\right)}{\left(K_K|U_K|^2 + \sum_{L\neq K} K_L |U_L|^2\right)} \tag{VI-24}$$

where the error terms are still compensating. Thus, all of the theory of (VI-C) applies to uniform structural damping.

(2) Series Damping in Inductors

Since this type of damping occurs usually in electrical systems, the relations will be stated in analogous electrical terms, with C, L, G, V, and I replacing M, 1/K, B, $\dot{U}$, and F, respectively. The series resistance in L_K will be designated r_K. As before, $\tilde{G}_K$ will be an equivalent shunt conductance which gives the mode exactly the same impedance at its damped resonant frequency $\tilde{\omega}_K$. These quantities are related as follows:

$$\tilde{\omega}_K = \omega_K \sqrt{1 - \frac{1}{Q_K^2}} \qquad \text{(VI-25)}$$

$$\tilde{G}_K = \frac{r_K\, C_K}{L_K} \qquad \text{(VI-26)}$$

$$\tilde{Q}_K = \frac{\tilde{\omega}_K\, L_K}{r_K} = \frac{\tilde{\omega}_K\, C_K}{\tilde{G}_K} \quad . \qquad \text{(VI-27)}$$

The normalizing condition is chosen to make $\tilde{G}_K$ the same for all modes. In Figure 14, series resistors r_K appear in the tanks, replacing the shunt conductances.

The universal resonance curves are shown in Appendix C. For $Q > 10$ it is evident that all of the reasoning leading up to Theorem 13 is still valid, if $\tilde{\omega}_K$ is substituted for ω_K.

The impedance locus, $Z_K(\omega)$, is distorted more severely than was the case for structural damping (Fig. 34, Appendix C). As a result, Theorem 14 may not hold for cases where the residual modes have an inductive impedance. However, the reasoning behind Theorem 15 is still entirely valid, so that ψ in (VI-9) now indicates the proximity to $\tilde{\omega}_K$.

As before, the quality factor can be evaluated from power input and known capacitance distribution, using (VI-12). In evaluating the error in this method, (VI-16) becomes

$$\tilde{Q}_K \cong \tilde{\omega}_K \frac{(C_K V_K^2 + \sum_{L \neq K} C_L \bar{V}_L^2)}{\left(\tilde{G}_K V_K^2 + \sum_{L \neq K} \frac{r_L |V_L|^2}{r_L^2 + \tilde{\omega}_K^2 L_L^2}\right)} \quad . \tag{VI-28}$$

In evaluating mode parameters from the impedance locus, we should now plot the locus of

$$\tilde{Z}_K = \left(\frac{\tilde{\omega}_K}{\omega}\right)^2 Z_K \tag{VI-29}$$

which is approximately circular when $Q^2 \gg \left(\frac{\tilde{\omega}_K}{\omega}\right)^2$. The diameter of the extrapolated circle from $\tilde{Z}_{ii}$ will be

$$D = \frac{\tilde{Q}_K}{C_K \tilde{\omega}_K} \tag{VI-30}$$

and we can normalize to make

$$A_{iK} = \sqrt{G\ D} \quad . \tag{VI-31}$$

The vertical rate of change of the locus of $\tilde{Z}_K$ at $\tilde{\omega}_K$ is

$$\left(\frac{d\tilde{X}_K}{d\omega}\right)_{\tilde{\omega}_K} = \frac{-2\,\tilde{Q}_K^2}{C\,\tilde{\omega}_K^2\,(1 - 1/\tilde{Q}_K^2)} \tag{VI-32}$$

Neglecting $1/\tilde{Q}_K^2$, this becomes

$$\left(\frac{d\tilde{X}_K}{d\omega}\right)_{\tilde{\omega}_K} \cong -\frac{2\,\tilde{Q}_K\,D}{\tilde{\omega}_K} \tag{VI-33}$$

replacing (VI-13).

The reasoning behind Theorem 16 is still valid for high Q systems.

E. A Technique of Multiple Excitation

The analysis just presented is valid only in cases where it is possible to suppress all badly interfering modes. If the preliminary trials show that this is not possible with a single driving source, then multiple excitation must be used. The equivalent circuit of Figure 14 is an aid to the analysis of this problem for uniformly damped systems.

It is clear that in order to achieve a completely pure mode shape, the system must be driven at all of its n external coordinates, as proposed by Wrisley and Lewis (Ref. 8). In a large, complex system this usually will not be necessary, since to the order of accuracy we are interested in, only the interfering modes, as defined in (C), need be suppressed. The trend toward swept wings, delta wings, and pylon mounted nacelles, will increase the probability of near degeneracy and resulting multi-mode interference, but in most practical cases the number of modes contributing significant distortion cannot be expected to exceed two or three.

From Figure 14 and the analysis of Parts II - V, it is evident that any given mode, say mode L, can be left unexcited, provided

$$F_L = \sum_i A_{iL} f_i = 0 \quad . \tag{VI-34}$$

Using r independent forces, it is possible, in general, to eliminate any (r-1) modes by (VI-34). Since the values of the A_{iL} are not known a priori, an iteration procedure is clearly indicated. The procedure suggested is as follows:

(1) Make an initial survey to locate all modes in the frequency spectrum, and to find the best single drive point for each mode.

(2) Take initial mode shapes from best drive points. Set the frequency, by application of Theorem 15, to a point where the response voltages are about half advanced and half retarded in phase. In cases where the distorting modes are remote, and the phase shifts are small, no multiple excitation will be required. For other modes, the best possible estimate of the mode shape should be obtained. In all readings, the phase should be recorded to allow computation of in-phase components, and to give a later check on the purity of the mode and the accuracy of the frequency setting.

(3) Go through the spectrum and refine each distorted mode, using a set of forces (currents) computed from the first trial mode shapes. Use (r+1) driving sources to suppress r interfering modes. As a check on the effectiveness of the multiple drive, vary the frequency in the neighborhood of the undesired modes and observe whether resonant peaks are obtained in the response voltages.

(4) Continue the iteration process until it converges for each mode. Convergence may be indicated by reduced phase shifts, but the phase shifts cannot be eliminated by this method unless all but the most remote modes are suppressed. A repetition in the relative values of the A_{iK} on successive trials may be obtained even though some significant phase shifts remain.

(5) When a satisfactory excitation has been obtained for a mode, complete data should be taken, including phase angles, power input, etc. If the mode parameters are to be evaluated from extrapolated impedance circles, then a frequency survey of response in the

vicinity of resonance should be made.

(6) Use orthogonality relations as a final check on the purity of the various modes. Compute accurate values of ψ in (VI-9) if an accurate check on the frequency is required.

As an example of the process, let us assume that we desire to excite mode 2, and that there are two interfering modes, one at a lower frequency and one at a higher frequency. Designate these as mode 1 and mode 3. Then, choosing three suitable drive points, i, j, and k, and using values from the previous trials,

$$\begin{aligned} A_{i1}\, f_i + A_{j1}\, f_j + A_{k1}\, f_k &= 0 \\ A_{i3}\, f_i + A_{j3}\, f_j + A_{k3}\, f_k &= 0 \quad . \end{aligned} \qquad \text{(VI-35)}$$

These equations are solved for f_i and f_j in terms of f_k, and the proper ratios are then set in. A check should be made to see that

$$A_{i2}\, f_i + A_{j2}\, f_j + A_{k2}\, f_k \neq 0 \; . \qquad \text{(VI-36)}$$

It is very desirable that the terms of (VI-36) all be additive, in order to provide a strong drive on the desired mode.

The three important questions that remain to be answered are (1) will the iteration process converge, (2) can the method be extended to non-uniformly damped systems, and (3) what are the practical problems involved in applying the method. The second and third questions are discussed in subsequent parts. The general question of convergence is best examined by experiment, but we can consider a simple case where only two modes are present. Assume uniform shunt damping, with the normalizing condition (VI-1) applied. Then from (VI-3), our first trial values of A_{iK} for mode 1 would be

$$(A_{11} + \gamma_{j12} A_{12}) \text{ at } q_1 .$$
$$(A_{21} + \gamma_{j12} A_{22}) \text{ at } q_2 . \qquad \text{(VI-37)}$$

To excite mode 2, then, we set

$$(A_{11} + \gamma A_{12}) f_1 + (A_{21} + \gamma A_{22}) f_2 = 0 .$$

Hence,
$$F_1 = A_{11} f_1 + A_{21} f_2 = - \gamma (A_{12} f_1 + A_{22} f_2) .$$

Therefore,
$$\frac{F_1}{F_2} = - \gamma . \qquad \text{(VI-38)}$$

Since the detuning factor again affects the response,

$$\dot{\bar{U}}_1 = - \eta \gamma \dot{U}_2 = - \eta^2 \left(\frac{A_{j1}}{A_{j2}}\right) \dot{U}_2 \qquad \text{(VI-39)}$$

where j is the drive point. Thus, from the iteration we attenuate the undesired mode by the detuning factor, η .

For two or more interfering modes, the analysis is more involved and it is felt that experimental justification is more desirable. In any event, any tendency to diverge in possible unusual cases could be readily detected, and a different set of driving points could be selected.

There is one additional point that might be noted. If only one drive source is available, or if experimentation time is quite limited, then complete data on each mode in the interfering set could be taken at a number of drive points. Using superposition, the iteration process could be carried out later on paper. Such a method has the disadvantages of being tedious, and of allowing the possible cumulation of experimental errors.

VII. EXTENSIONS OF THEORY TO SYSTEMS WITH NON-UNIFORM DAMPING

A. Mode Separation in Shunt Damped Systems

Consider the normal mode analog shown in Figure 7 of Part IV. With uniform shunt damping, the residual conductances g_{ij} are eliminated, and we have shown that to suppress m modes we must use (m+1) driving currents, all of like phase. Now let small non-uniform damping in the form of the g_{ij} be added to the uniformly damped system. Clearly, it is still possible to suppress m of the modes with (m+1) current sources, but these must now have relative phase shifts, if $m < (n-1)$. This will be true because, in general, the excitation of one or more off-resonant tanks will result in small currents of <u>random phase</u> in the coupling conductances, g_{ij}.

Specifically, suppose we desire to make

$$\dot{U}_1 = 0 \; ; \quad \dot{U}_2 = 0 \quad ; \quad \dot{U}_3 = 1 + j0 \quad . \tag{VII-1}$$

while driving the system at frequency ω_3. At steady state, the system equations (III-8) can be expressed as

$$\left[B_{KK} + j(\omega_3 M_K - \frac{K_K}{\omega_3}) \right] \dot{U}_K + \sum_{L \neq K} B_{KL} \dot{U}_L = F_K \tag{VII-2}$$

$$(K = 1, \dots n) \; .$$

Thus, conditions (VII-1) require

$$F_1 = \sum_{L=3}^{n} B_{1L} \dot{U}_L = \underline{F}_1 + j \bar{F}_1$$

$$F_2 = \sum_{L=3}^{n} B_{2L} \dot{U}_L = \underline{F}_2 + j \bar{F}_2 \tag{VII-3}$$

$$F_3 = \sum_{L=3}^{n} B_{3L} \dot{U}_L = \underline{F}_3 + j \bar{F}_3 \quad .$$

If the system is driven from stations x, y, and z, we must require that

$$\begin{bmatrix} A_{x1} & A_{y1} & A_{z1} \\ A_{x2} & A_{y2} & A_{z2} \\ A_{x3} & A_{y3} & A_{z3} \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} \underline{F}_1 + j\overline{F}_1 \\ \underline{F}_2 + j\overline{F}_2 \\ \underline{F}_3 + j\overline{F}_3 \end{bmatrix} \qquad \text{(VII-4)}$$

Letting $f_x = \underline{f}_x + j\overline{f}_x$, etc., the solutions of (VII-4) give the necessary multiple excitation. For n = 3, the $\overline{F_K}$ vanish in (VII-3), and the drive sources are once again in phase. Generalizing this, we can state:

> Theorem 17. In a system of n degrees of freedom with arbitrary shunt damping, it is possible to constrain any m of the undamped normal coordinates to zero, by adjusting (m+1) driving forces in both amplitude and phase. As m approaches (n-1), the relative phase shifts decrease, and when all coordinates are driven it is possible to excite a pure undamped mode with a set of in-phase driving forces.

This theorem generalizes a conclusion reached by Lewis and Wrisley (Ref. 8, 39), who showed that for systems with uniform damping driven at all coordinates, relative phase shifts among the driving sources are not required. For non-uniform damping with all coordinates driven, the drive sources should still be in phase, but the amplitude of f_i is no longer proportional to $\sum_j m_{ij}A_{jK}$, which was the criterion used for the uniformly damped case.

Now, if we wish to suppress two interfering modes in a large system, using the iterative procedure of (VI-E), it appears to be impractical to introduce phase shifts into the drive sources on anything other then a trial and error basis. Such trial and error techniques can be very time consuming where so many variables are involved. If the non-uniform portion of the damping in Figure 7 is small, then the g_{ij} are small, and the phase shifts theoretically required can be neglected. Unfortunately, this has the effect of introducing small quadrature exciting forces, $\bar{F}_L$, into the modes we are attempting to suppress. As a result, the in-phase velocities, $\underline{\dot{q}}_j$, include reactive components excited by the F_L. These do not attenuate very rapidly for small shifts away from resonance, so we should expect the non-uniformity of the damping to reduce the accuracy with which we can determine the modes. Experiment has shown that the iteration process does converge when non-uniform damping is present.

In some systems, there may be highly localized non-uniform damping in one part of the system. In such cases it may be possible to drive all of the coordinates to which the larger g_{ij} are attached, and thus neutralize their effect. This is easily visualized where there is a single large dashpot added somewhere in a uniformly damped system.

B. Best Estimate of System Transient Response; Equivalent Orthogonal System

We should note at this point that the two types of uniform shunt damping which we have been considering are not the only forms of shunt damping for which F, T, and V will reduce simultaneously to sums of squares. Any number of systems possessing this property can be synthesized from the normal mode analog of Figure 8, by simply choosing a set of positive B_{KK} arbitrarily, and setting all coupling terms, B_{KL}, to zero. This results in a physically realizable, passive system for which the normal modes are clearly orthogonal, and for which $B_{KK} \neq C_m M_K + C_k K_K$. If Figure 8 represents the normal mode analog of a conservative physical system, and if positive shunt damping elements B_{KK} are inserted at random, then a transformation back to the coordinates of the original system can be made.* The resulting circuit is always physically realizable with passive damping elements, but the realization may require additional transformers. Let $[E_{Kj}]$ be the inverse of matrix $[A_{jK}]$. Then if coupling conductances B_{KL} are zero in the normal mode tanks,

$$b_{ij} = \sum_K B_{KK} E_{Ki} E_{Kj} \quad . \qquad \text{(VII-5)}$$

Clearly, the self-damping terms b_{ii} are all non-negative. The coupling terms may be either negative or positive, but they are certainly bilateral. Hence, the system can be realized electrically with direct conductive coupling between the q_j if all off-diagonal terms in $[b_{ij}]$ are non-positive, and if the sum of the elements of any row of $[b_{ij}]$ is positive (Ref. 23). In any event, we can admit negative conductances for purposes of analysis, so the question of realizability need not concern us here. The sets of b_{ij} defined by (VII-5) can hardly be regarded as uniform damping, but we can designate them as a type of <u>orthogonal damping</u>, and the resulting system as an <u>orthogonal system</u>; i.e., one with no energy coupling between its normal modes.

The problem now is to represent an actual system having arbitrary shunt damping by an equivalent orthogonal system which best approximates its behavior, both transient and steady state. For the case

---

* As a matter of fact, all tank circuit parameters and all transformer ratios can be picked at random, and a transformation to the generalized coordinates so synthesized can be made, provided the matrix $[A_{jK}]$ is non-singular.

of small damping, it is quite apparent that the desired approximation can be obtained by simply including the entire self conductance B_{KK} as a shunt conductance in the Kth mode, and dropping the coupling elements, B_{KL}. Rayleigh has shown, for small damping, that the <u>second</u> <u>order</u> correction to ω_K is in the nature of a small variation in the ratio M_K/K_K, which affects only the imaginary component of Γ_K (Ref. 1, Ch. 5). This correction is in the nature of an "attraction" of the eigenvalues, with higher modes tending to increase β_K and lower modes tending to reduce β_K.

It has been shown by Guillemin (Ref. 19) that for highly oscillatory systems the first order approximation to α_K is

$$\alpha_K = \frac{\sum_i \sum_j b_{ij} \Delta_{oij} (j\omega_K)}{2\sum_i \sum_j m_{ij} \Delta_{oij} (j\omega_K)} . \qquad \text{(VII-6)}$$

But this result is identically equal to

$$\alpha_K = \frac{\sum_i \sum_j b_{ij} A_{iK} A_{jK}}{2\sum_i \sum_j m_{ij} A_{iK} A_{jK}} = \frac{B_{KK}}{2 M_{KK}} \qquad \text{(VII-7)}$$

because $\Delta_{oij}(j\omega_K)$ is proportional to A_{iK} during the summation on i, and proportional to A_{jK} during the summation on j, from (II-32).

More recently, the same result has been derived by Morduchow for a continuous system, by means of a variational principle (Ref. 27).

If (VII-7) is used to evaluate the order of magnitude of the damping in Theorem 3 of Part III, we obtain, for damping distributed at random throughout the system,

$$\delta = \frac{2\alpha_K}{\omega_K} = \frac{B_{KK}}{\omega_K M_K} = \frac{1}{Q_{KK}} , \qquad \text{(VII-8)}$$

or, for highly localized damping,

$$\delta = \left(\frac{2n-1}{4}\right) \frac{1}{Q_{KK}} \quad . \qquad \text{(VII-9)}$$

These relations will be taken as the quantitative evaluation of "smallness" of damping, for a non-uniformly damped system.

Next, consider the situation where the damping is large in the sense of (VII-8), but highly uniform, so that only the coupling terms, B_{KL}, are small. Assume first that all other modes are well separated from mode K in the frequency spectrum, and apply the perturbation technique of Rayleigh. If mode K is the dominant mode, then we can neglect the summation in

$$(M_K\, p^2 + B_{KK}\, p + K_K)\, U_K + \sum_{L \neq K} B_{KL}\, U_L = 0 \qquad \text{(VII-10)}$$

giving

$$U_K \cong u_K\, e^{\Gamma_K t}$$

as in (III-10), where $\alpha_K = -\dfrac{B_{KK}}{2M_K}$.

Then, for any other mode, S, excited by the dominant mode,

$$(M_S\, \Gamma_K^2 + B_{SS}\, \Gamma_K + K_S)\, U_S \cong -\, \Gamma_K\, B_{KS}\, U_K \, . \qquad \text{(VII-11)}$$

Hence,

$$\frac{U_S}{U_K} \cong \frac{-\,\Gamma_K\, B_{KS}}{M_S\, \Gamma_K^2 + B_{SS}\, \Gamma_K + K_S} \quad . \qquad \text{(VII-12)}$$

This can be expressed as

$$\frac{U_S}{U_K} \cong \frac{-\,\Gamma_K\, B_{KS}}{M_S\, (\Gamma_K - \Gamma_S)(\Gamma_K - \Gamma_S^*)} \quad . \qquad \text{(VII-13)}$$

Thus, the neglected terms in (VII-10) are approximately

$$\sum_{L \neq K} \frac{-\Gamma_K^2 \, B_{KL}^2 \, U_K}{M_L \, (\Gamma_K - \Gamma_L)(\Gamma_K - \Gamma_L^*)} \quad . \tag{VII-14}$$

These are of second order in B_{KS}, and we are justified in neglecting them provided Γ_K and Γ_L are well separated.

For two badly interfering modes we can estimate the interaction effect by ignoring all the remaining modes, so that the approximate system equations are

$$\begin{aligned} (M_K \, p^2 + B_{KK} \, p + K_K) \, U_K + (B_{KS} \, p) \, U_S &\cong 0 \\ (B_{SK} \, p) \, U_K + (M_S \, p^2 + B_{SS} \, P + K_S) \, U_S &\cong 0 \quad . \end{aligned} \tag{VII-15}$$

The eigenvalues must satisfy

$$(M_K \, p^2 + B_{KK} \, p + K_K) \cong \frac{B_{KS}^2 \, p^2}{M_S \, p^2 + B_{SS} \, p + K_S} \quad . \tag{VII-16}$$

Assume that the introduction of the coupling shifts the Kth eigenvalue from Γ_K to $(\Gamma_K + \rho)$, and let $(\Gamma_S + \mu) = (\Gamma_K + \rho)$. Then, neglecting ρ^2 and μ^2,

$$\rho(2M_K\Gamma_K + B_{KK}) = \frac{B_{KS}^2(\Gamma_K + \rho)(\Gamma_S + \mu)}{\mu(2 \, M_S \, \Gamma_S + B_{SS})} \quad . \tag{VII-17}$$

Substitution of the unperturbed values of Γ_K and Γ_S gives

$$\rho = \frac{-\,B_{KS}^2 \, (\Gamma_K + \rho)(\Gamma_S + \mu)}{4\mu \, M_K \, M_S \, \beta_K \, \beta_S} \quad . \tag{VII-18}$$

The displacement ρ is of order of magnitude

$$\left(\frac{1}{\mu}\right) \frac{B_{KS}^2}{4M_K M_S} \quad . \tag{VII-19}$$

This implies that a logical quantitative estimate of the uniformity of damping can be expressed in terms of the following factors:

$$\alpha_{KS} \triangleq \frac{|B_{KS}|}{2\sqrt{M_K M_S}} \tag{VII-20}$$

$$\frac{1}{Q_{KS}} \triangleq \frac{2\,\alpha_{KS}}{\sqrt{\omega_K \omega_S}} = \frac{|B_{KS}|}{\sqrt{B_{KK} B_{SS}}} \left(\frac{1}{\sqrt{Q_{KK} Q_{SS}}} \right) \cdot \tag{VII-21}$$

When Q_{KS} is large for mode K with respect to other nearby modes, we can safely apply the approximation that mode K is uncoupled in the circuit of Figure 8. This gives us an equivalent orthogonal system with which to work. Such a substitution is similar to the use of an equivalent linearized system to replace one with parasitic non-linearities.

It is clear that when the system can be driven so as to suppress the badly interfering modes, all of the theory of (VI-C) still applies; however, we should expect the accuracy of the approximations to be somewhat less for the non-uniform case. In Figure 8, the modes which are suppressed, or which are so far off resonance that they show a weak response, can be considered to be essentially at ground potential, so that the measured damping parameter will be approximately that of the equivalent orthogonal system, B_{KK}.

C. Non-Uniform Series and Structural Damping

In considering the effects of non-uniform series or structural damping, we can make a few general observations:

(1) The addition of such damping cannot change the number of non-zero eigenvalues, provided no change is made in the independence of the system energy storage elements.

(2) As linear damping of any sort is added gradually, the eigenvalues must move continuously in the complex plane. Hence, there must exist some criterion by which we can specify "small" perturbations of any type of damping, with the assurance that the corresponding displacement of the eigenvalues will likewise be small.

(3) For any given series or structurally damped element at a particular frequency (real or complex), there exists an equivalent shunt damped element. The equivalent shunt reactance differs from the actual series reactance by terms of order Q^{-2} in the vicinity of the real frequency axis.

The concept of an equivalent shunt damped system shows immediately that all of the theorems of small damping in Part III can be applied to any form of small linear damping. Moreover, a normal mode analog of the type shown in Figure 6 can be used to represent exactly any lightly damped system with steady state sinusoidal excitation. Such an exact representation is not possible for transient analysis, because the shunt conductances are now functions of frequency.

Next consider the case of a system with series or structural damping where the departure from a uniform damping condition is small, even though the total damping may not be small. For such a system, at any given complex frequency, we can replace each damped element by an equivalent uniformly damped element, in shunt with a small damper (positive or negative) which gives the correct energy dissipation to the combination. The uniformly damped elements can then be brought directly into the normal mode tanks. The residual equivalent conductances are frequency dependent, but if they are small we can neglect their variation in the neighborhood of an eigenvalue or a resonant drive frequency, and for some purposes we can neglect the residual conductances altogether.

Series or structurally damped systems in which the damping is orthogonal, but non-uniform, can be synthesized exactly as in (VII-B). Thus it is apparent that all of the theory so far developed can be extended, with slight modifications, to include any forms of linear, non-uniform damping, provided only that the non-uniformity be small. For all such cases, we can work with an equivalent orthogonal system, and apply our theory directly.

D. The Effects of Transformer Parasitic Damping

The finite difference electric analogs of elastic structures involve ideal transformers as well as inductors and capacitors. Experimentally, the use of actual transformers introduces additional losses into the system. Following standard transformer theory, we can represent such losses by adding a small shunt conductance and a small series resistance to each transformer. These may be referred to either side of the transformer. In the equivalent circuit for a beam in bending, the transformer shunt damping can be placed in the bending circuit, where it represents a rotary damping element. The series damping, if placed in the deflection circuit, represents a sort of "shear creep" in the beam, just as series leakage reactance represents a small shear compliance (Ref. 41). Clearly, then, transformer shunt losses have the effect of adding non-uniform shunt damping to existing coordinates of the system. Such damping has already been considered. The transformer series losses can be thought of as additional non-uniform series damping, provided we add some additional degrees of freedom to the system. The series leakage reactance does, in fact, add such degrees of freedom to the laboratory set-up. The additional eigenvalues introduced in this way will normally be far above the frequency range of interest.

VIII. NUMERICAL EXAMPLES AND EXPERIMENTAL RESULTS

The numerical examples have been chosen more or less at random, to illustrate various portions of the theory.

The experimental phase of the research was confined to electrical analogs. It was performed primarily for the purpose of investigating convergence of the iteration technique of mode suppression. There are practical problems involved in applying the technique to complex electrical systems, and these were investigated.

A. Computation of Normal Modes for System With Uniform Shunt Damping - System Number 1

(1) Figure 23 shows a simple spring-mass system. The equations of the undamped system are

$$\begin{bmatrix} (p^2 + 2) & -1 \\ -1 & (2p^2 + 2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The characteristic equation of the system is

$$2\,p^4 + 6\,p^2 + 3 = 0 \quad .$$

The eigenvalues are

$$\omega_1^{\,2} = \frac{\sqrt{3}}{2}\,(\sqrt{3} - 1) \quad ; \quad \omega_2^{\,2} = \frac{\sqrt{3}}{2}\,(\sqrt{3} + 1) \; .$$

Use the normalizing condition, $A_{jK} = \sum_i \Delta_{ij}(j\omega_K)$:

Mode 1: $A_{11} = \sqrt{3}$; $A_{21} = \dfrac{3 + \sqrt{3}}{2}$

Mode 2: $A_{12} = -\sqrt{3}$; $A_{22} = \dfrac{3 - \sqrt{3}}{2}$.

Orthogonality Check:

$$m_1A_{11}A_{12} + m_2A_{21}A_{22} = -3 + \frac{2}{4}(3+\sqrt{3})(3-\sqrt{3}) = 0 \ .$$

(2) Add shunt damping such that $b_{ij} = 1/2\ m_{ij} + 1/2\ k_{ij}$. The equations of the damped system are

$$\begin{bmatrix} (p^2 + 3/2\ p + 2) & (-1/2\ p - 1) \\ (-1/2\ p - 1) & (2p^2 + 2p + 2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives a characteristic equation,

$$2\ p^4 + 5\ p^3 + 8.75\ p^2 + 6\ p + 3 = 0$$

which factors to

$$(p^2 + 1.685\ p + 2.368)(p^2 + .815\ p + .633) = 0 \ .$$

The eigenvalues, to slide rule accuracy, are

$$\Gamma_1 = -.4075 + j\ .683 \ ; \quad \Gamma_2 = -.8425 + j\ 1.287 \ .$$

Use the normalizing condition, $D_{jK} = \sum_i \Delta_{ij}(\Gamma_K)$:

Mode 1: $D_{11} = 1.379 + j\ .595$; $D_{21} = 1.884 + j\ .81$

Mode 2: $D_{12} = -1.004 - j\ 1.12$; $D_{22} = .366 + j\ .405$

$$\frac{D_{21}}{D_{11}} = \frac{2.05\ \angle 23.27^\circ}{1.50\ \angle 23.36^\circ} = 1.368\ \angle -0.09^\circ \ ; \quad \frac{A_{21}}{A_{11}} = 1.366\ \angle 0^\circ \ .$$

$$\frac{D_{12}}{D_{22}} = \frac{-1.505\ \angle 48.1^\circ}{0.546\ \angle 47.9^\circ} = -2.76\ \angle 0.2^\circ \ ; \quad \frac{A_{12}}{A_{22}} = -2.74\ \angle 0^\circ \ .$$

Orthogonality Check:

$$m_1 D_{11} D_{21} + m_2 D_{12} D_{22} = .005 - j\,.023.$$

B. Transient Response of Free-Free Torsion System With Uniform Shunt Damping - System Number 2

(1) A three degree of freedom torsion system is shown in Figure 24. Assume that a unit impulse of force is applied to coordinate θ_3, with the system initially at rest. The response will be obtained by superposition of the responses of the normal modes. For this simple system, the undamped normal modes can be written by inspection, the amplitude factors being normalized to unity:

Table II

Normal Modes of System Number 2

Mode	Type	ω_K	A_{1K}	A_{2K}	A_{3K}
1	Free Rotation	0	1	1	1
2	Antisymmetric Oscillation	$\sqrt{2}$	-1	0	1
3	Symmetric Oscillation	2	1	-1	1

The corresponding normal mode parameters are computed from (II-18) and (II-15) or (II-20):

$$M_1 = 2 \qquad M_2 = 1 \qquad M_3 = 2$$

$$K_1 = 0 \qquad K_2 = 2 \qquad K_3 = 8 \quad .$$

With $f_1 = f_2 = 0$ and $f_3 = \delta(t)$, the generalized forces applied to the normal coordinates are obtained from (II-23):

$$F_1 = \delta(t) \qquad F_2 = \delta(t) \qquad F_3 = \delta(t) \quad .$$

For the undamped system, solutions of (II-12) are

$$U_1 = 1/2\ t \;;\; U_2 = \frac{1}{\sqrt{2}} \sin\sqrt{2}\ t \;;\; U_3 = 1/4 \sin 2\ t \quad .$$

From (II-14),

$$\theta_1 = 1/2\ t - \frac{1}{\sqrt{2}} \sin\sqrt{2}\ t + 1/4 \sin 2\ t$$

$$\theta_2 = 1/2\ t - 1/4 \sin 2\ t$$

$$\theta_3 = 1/2\ t + \frac{1}{\sqrt{2}} \sin\sqrt{2}\ t + 1/4 \sin 2\ t \quad .$$

(2) Next add damping uniformly to both the springs and the masses, such that $b_{ij} = 0.2\ m_{ij} + 0.1\ k_{ij}$, as shown in Figure 24 (c). Then $B_K = 0.2\ M_K + 0.1\ K_K$, giving,

$$B_1 = 0.4 \qquad B_2 = 0.4 \qquad B_3 = 1.2 \quad .$$

The solutions of (IV-4) give the normal coordinate response of the damped system:

$$U_1 = 2.5\ (1 - e^{-.2t}) \;;\quad U_2 = e^{-.2t}\,\frac{\sin 1.4t}{1.4} \;;$$

$$U_3 = e^{-.3t}\frac{\sin\sqrt{3.91}\ t}{2\sqrt{3.91}}$$

and, as in the undamped solution,

$$\theta_1 = U_1 - U_2 + U_3$$

$$\theta_2 = U_1 - U_3$$

$$\theta_3 = U_1 + U_2 + U_3 \quad .$$

These solutions agree with solutions obtained directly in terms of the θ_i .

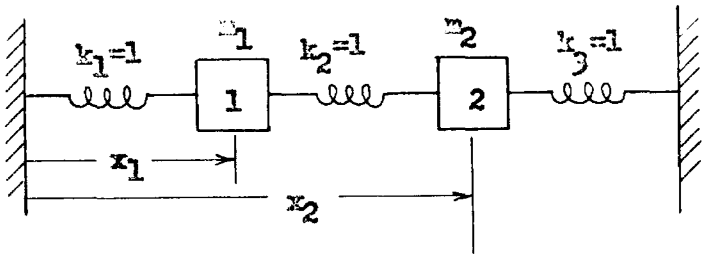


Figure 23. System No. 1 Without Damping.

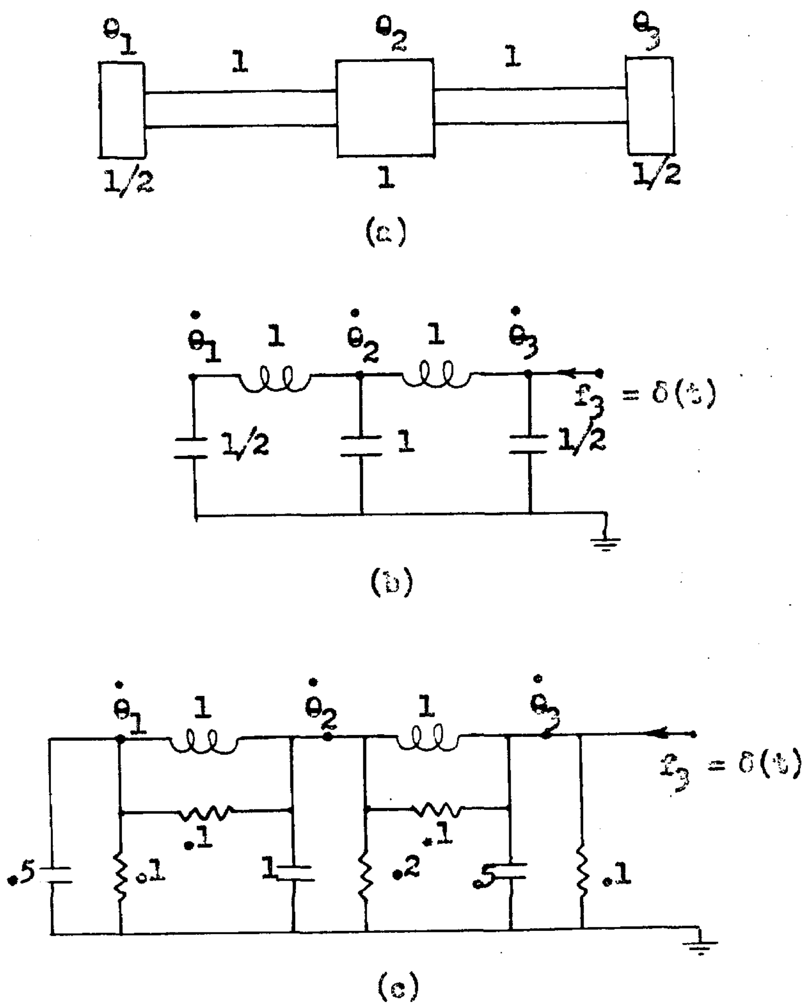


Figure 24. System No. 2.

C. Analysis of Uniform Beam In Torsion; Seven Degree of Freedom, Finite Difference Analog - System No. 3

(1) Consider the finite difference analog of a free-free, uniform beam in pure torsion. The continuous structure is approximated by five full finite difference cells and two half cell terminations, in Figure 25. The normal mode frequencies of this system can be obtained from a solution of the difference equation of the system (Ref. 11, Ch. 11). For the free-free boundary condition, they are

$$\omega_K = \frac{1}{\sqrt{LC}} \sqrt{2\left(1 - \cos\frac{(K-1)\pi}{6}\right)} \quad .$$

Take L = C = 1 for convenience. Then the mode frequencies are 0, .5177, 1, $\sqrt{2}$, $\sqrt{3}$, 1.932, and 2.

The amplitude factors, A_{iK}, are easily computed for each mode frequency by assuming a unit reference voltage, $V_7 = 1 + j0$, and computing the voltages at the other six stations by a step-by-step application of Kirchhoff's laws. The generalized masses M_K and the generalized spring constants K_K are then obtained from (II-18), and (II-15) or (II-20). The results are shown in Table III.

Table III

Normal Modes of System Number 3

Mode	ω_K	M_K	K_K	A_{1K}	A_{2K}	A_{3K}	A_{4K}	A_{5K}	A_{6K}	A_{7K}
1	0	6	0	1	1	1	1	1	1	1
2	.5177	3	$3(2-\sqrt{3})$	-1	-.866	-.5	0	.5	.866	1
3	1.0	3	3	1	.5	-.5	-1	-.5	.5	1
4	$\sqrt{2}$	3	6	-1	0	1	0	-1	0	1
5	$\sqrt{3}$	3	9	1	-.5	-.5	1	-.5	-.5	1
6	1.932	3	$3(2+\sqrt{3})$	-1	.866	-.5	0	.5	-.866	1
7	2.0	6	24	1	-1	1	-1	1	-1	1

(2) Now let us introduce uniform series damping into this system. By adding 0.1 ohm resistance to the unit inductances of Figure 25, we obtain a Q for these elements ranging from 20 at ω_7 to 5.177 at ω_2. Applying Theorem 10 of Part V, we can add series resistance in like proportion to each inductor of the normal mode analog. Figure 26 shows the resulting system, viewed from station 7.

If symmetry is ignored in Figure 25, and the series damped system is driven at station 7, then a progressive phase shift which is characteristic of damped torsion systems can be obtained. From (VI-25), the damped resonant frequency of mode 7 is

$$\tilde{\omega}_7 = \omega_7 \sqrt{1 - 1/Q^2} = 1.9975 \quad .$$

Using this frequency, the response voltages were computed for a unit current input at station 7:

$I_7 = 1.00\ (0^\circ)$ $\quad$ $V_4 = -1.65\ (18.1^\circ)$

$V_7 = 4.32\ (-46.8^\circ)$ $\quad$ $V_3 = 1.41\ (42.1^\circ)$

$V_6 = -3.08\ (-27.7^\circ)$ $\quad$ $V_2 = -1.36\ (58.9^\circ)$

$V_5 = 2.20\ (-6.3^\circ)$ $\quad$ $V_1 = 1.36\ (64.6^\circ)$

It is clear from Table III that the principal cause of this phase shift is the excitation of mode 6, which adds to mode 7 at one end of the beam and subtracts at the other end.

(3) Table III indicates that a reasonably good excitation of the highest mode can be obtained by a symmetrical drive at stations 3 and 5, which eliminates the antisymmetrical modes completely. (Electrically, this is equivalent to splitting the system in half at station 4, and using a single drive at station 5.) For this excitation, the response at $\tilde{\omega}_7$ is

$$I_3 = I_5 = .5 \quad (0^\circ)$$
$$V_1 = V_7 = 1.655\ (12.38^\circ) = 1.616 + j\ .354$$
$$V_2 = V_6 = -1.655\ (6.63^\circ) = -1.645 - j\ .191$$
$$V_3 = V_5 = 1.720\ (-10.12^\circ) = 1.694 - j\ .302$$
$$V_4 = 1.720\ (-4.37^\circ) = -1.715 + j\ .131$$

The response of mode 7 alone would be $\mp$ 1.667 volts. The in-phase components of response range from 3.06% low to 2.88% high. The resonance indication factor given by (VI-9) is

$$\psi = -\ .019\ .$$

The Q for mode 7, evaluated from (VI-12), is

$$Q_7 = 19.72$$

which is 1.45% low.

(4) The system response to symmetrical drive at stations 3 and 5 was computed for a range of frequencies near ω_7, in order that an input impedance locus might be plotted. The equivalent input impedance for the entire system is taken as $V_5/2I_5$. Table IV shows the results, and illustrates the extreme sensitivity of the factor ψ to small frequency changes.

TABLE IV

Computed Frequency Response of System No. 3 With Uniform Series Damping in Inductors. Symmetrical Drive at Stations 3 and 5 to Excite Highest Mode

ω	ψ	Z_{in}	$\tilde{Z}_{in} = \left(\frac{\tilde{\omega}_7}{\omega}\right)^2 Z_{in}$	Comments
1.980		1.502 (7.3°)	1.515 + j .194	
1.990	+.868	1.640 (- 1.9°)	1.651 - j .055	Unity Power Factor
1.9975	-.019	1.720 (-10.12°)	1.693 - j .302	Damped Resonance, $\tilde{\omega}_7$
2.000	-.354	1.738 (-12.78°)	1.690 - j .383	Undamped Resonance
2.005		1.750 (-18.12°)	1.650 - j .540	Maximum $\lvert\tilde{Z}_{in}\rvert$
2.0075	-.985	1.754 (-20.82°)	1.622 - j .617	Maximum $\lvert Z_{in}\rvert$
2.010		1.750 (-23.55°)	1.585 - j .690	
2.020		1.711 (-33.07°)	1.400 - j .912	

The input impedance locus is plotted in Figure 27. Computations of mode 7 parameters from the extrapolated circle give:

$$D = 1.58 \text{ ohms} \qquad \tilde{\omega}_7 = 1.9975 \qquad \left(\frac{\Delta \tilde{X}}{\Delta \omega}\right)_{\tilde{\omega}_7} = -32.35$$

$\tilde{Q}_7 = 20.4$ from (VI-33)

$C_7 = 6.47$ from (VI-30) .

(5) This system was set up experimentally on the analog computer, using L and C values of .9 henries and .704 microfarads in the direct analog (Figure 25). This gave a resonant frequency of 400 cps for mode 7. The normal mode analog was set up on the same scale, and transformers were added to bring out station 3 as well as station 7. The transient responses of the two analogs are compared in Figure 12, page 66, both with and without added series damping. An experimental input impedance locus, taken from the direct analog, is shown in Figure 27.

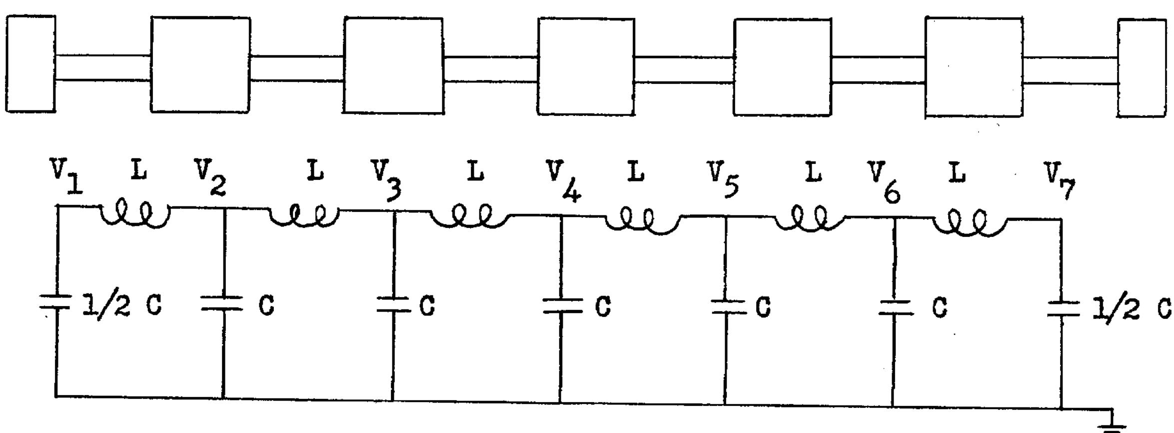


Figure 25. System No. 3. Finite Difference Analog of Uniform Beam in Pure Torsion.

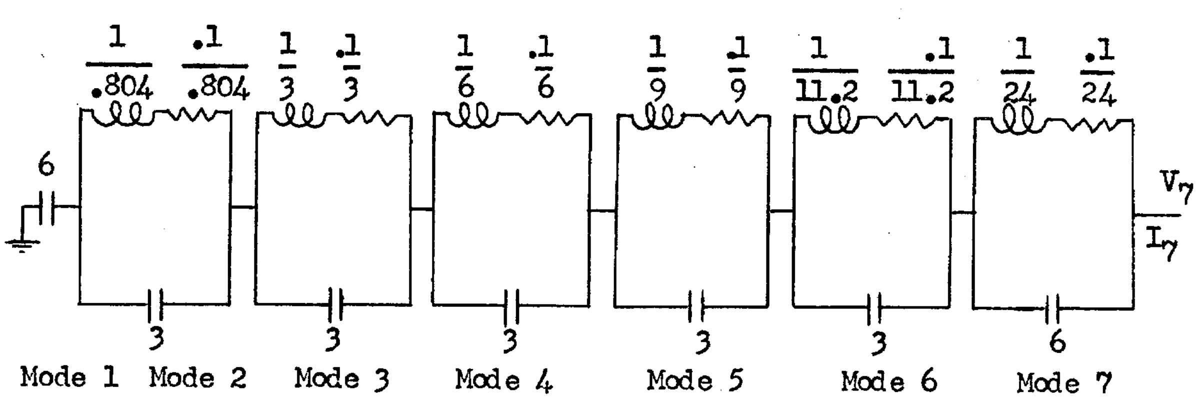


Figure 26. Normal Mode Analog of System No. 3 With Uniform Series Damping in Inductors. The numerical values are R, L, and C.

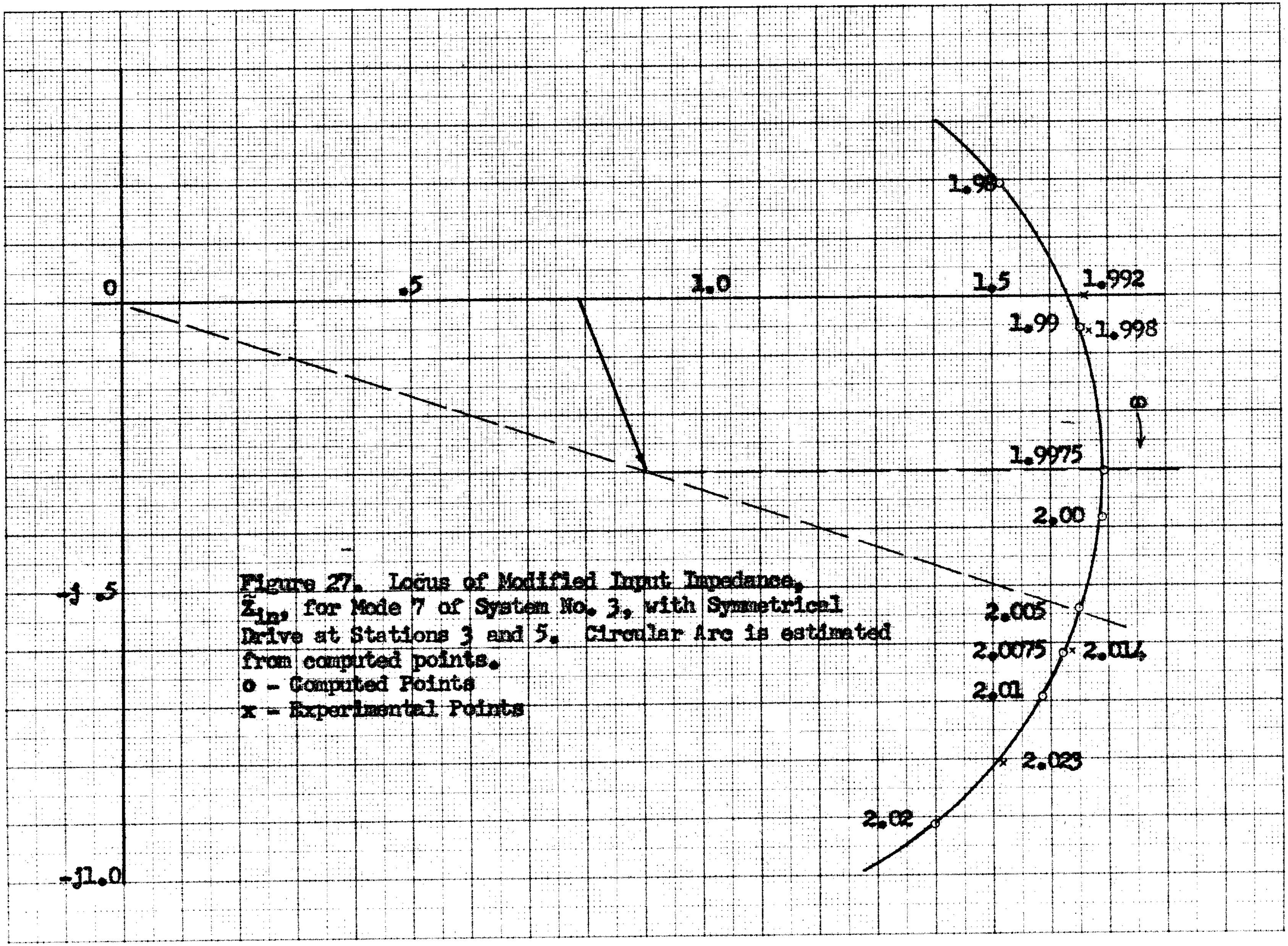


Figure 27. Locus of Modified Input Impedance, $\tilde{Z}_{in}$, for Mode 7 of System No. 3, with Symmetrical Drive at Stations 3 and 5. Circular Arc is estimated from computed points.
o - Computed Points
x - Experimental Points

D. Non-Uniform Torsion System With Simulated Structural Damping - System Number 4

(1) For the initial tests of the multiple excitation technique of mode separation proposed in Part VI, a small system, suitable for rapid testing and easy analysis, was desired. In order to achieve truly uniform damping it was necessary to use a system with no transformers. To make the test a severe one, closely spaced mode frequencies and an excessively high amount of damping were desirable.

The requirements were met by forming a symmetrical torsion system of cascaded stages having the same natural frequency, each cascaded stage having a smaller rotary inertia. A five degree of freedom system was set up in this way. Alterations were then made to destroy the symmetry of the system. The system frequencies were computed in order to establish scale factors that would put the normal modes in the working range of the computer. Then a final alteration was made, in order that the test could be made without prior knowledge of the exact mode frequencies. The resulting conservative system is shown in Figure 28.

It was anticipated that the system frequencies would be in the range from 180 to 300 cps. The uniform damping was introduced by adding series resistance to each inductor until its measured Q at 240 cps was reduced to 20. Then shunt resistance was added across the series combination to give a further reduction to a Q of 10. For one sample inductor, the variation in Q with frequency was checked:

f(cps) :	60	120	180	240	300
Inductor Q :	3.06	8.25	9.75	10.0	9.63

Thus, over the frequency range of interest, the system damping closely approximates the constant Q condition of structural damping.

(2) An initial survey of the frequency spectrum was made, driving each of the five coordinates in turn with a voltage source. Phase shifts were quite high, and at some points only one apparent mode was observed. A summary of the data revealed a well defined mode near 156 cps, another mode in the vicinity of 232 cps, and a probable mode or modes in the interval 312-337 cps. The lower modes were designated "A" and "B" respectively, the zero frequency mode being designated "O". First trial mode shape data were taken for modes A and B, using the most favorable driving point for each. These were designated "A_1" and "B_1".

Next, a set of three input currents was computed for the suppression of A_1 and B_1, as in (VI-35). Stations 1, 3, and 5 were chosen as drive points because these were <u>high amplitude</u> points of the undesired modes, the supposition being that high amplitude points are more accurately defined than low amplitude points. This approach is in direct contrast with the conventional method of seeking a null point on which to drive.

Using the excitation just described, the frequency spectrum was re-surveyed, and a new mode was discovered near 243 cps, which was designated "C_1". Finally, mode "D_1" was excited with a single voltage source, and a rough mode shape obtained, in the presence of large phase shifts, at 320 cps.

Each mode was then revised in succession, beginning with mode A_1, the two adjacent modes being suppressed in each case. The frequencies were set to a value for which positive and negative phase shifts appeared to balance (see page 84). After a third trial, the response in each mode appeared to be satisfactory with regard to phase shifts. The exact normal modes of the conservative system were then computed. The results are tabulated in Table V, with only the in-phase components of the response being listed.

The mode data for the third trials in Table V appear to be rather good, with the exception of mode A. For this mode, the iteration process apparently was divergent. However, a computation of some of the interference factors defined in (VI-4) was made, and it was found that mode C had an appreciable distorting effect upon mode A. To check this, mode A was driven with a set of currents computed from the exact amplitudes of modes O and B. The response, listed as A_4, shows that the iteration process actually was convergent, but that mode C should have been suppressed.

(3) It was recognized that modes B and C contribute an unusual amount of distortion, due to their very low values of generalized mass (see page 78). The relative values of generalized mass were computed to be

$$M_O = 12.99 \qquad M_A = 4.92 \qquad M_B = .343 \qquad M_C = .554 \qquad M_D = 11.595,$$

using the amplitude factors as normalized in Table V.

Since the zero frequency mode "O" contributes only a quadrature component to the response, a new set of currents was computed to excite mode A while suppressing B and C. Also, mode C was excited by eliminating A and B, rather than B and D. The process was started from the second trial values of Table V, and two additional iterations were made. The results, shown in Table VI, seem quite satisfactory for such a highly damped system.

(4) In order to observe the effects of non-uniform damping, the added series resistance was removed from the two inductors on the right in Figure 28, and the series resistance in the inductors on the left was doubled. The shunt damping remained uniform. The procedure described in (2) was repeated, and the results are summarized in Table VII. As might be expected, the accuracy is not as good as in the case of uniform damping, but the value of the multiple drive method is illustrated by the successful excitation of mode C, which again could not be identified using a single voltage drive.

(5) Some transient response records were taken for System No. 4 to illustrate the effects of various types of uniform damping. These are shown in Figure 13, page 67.

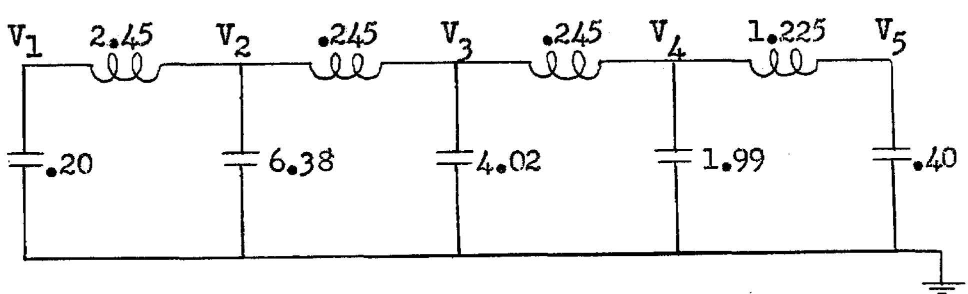


Figure 28. System No. 4. Inductor Values are Henries, Capacitor Values are Microfarads. Added Damping Elements are not Shown.

TABLE V

RESULTS OF FIRST ITERATION TEST ON SYSTEM NO. 4 WITH UNIFORM DAMPING. RESPONSE VOLTAGES ARE IN-PHASE COMPONENTS REFERRED TO INPUT CURRENTS

Driven Mode	Suppressed Modes	f (cps)	V_1	V_2	V_3	V_4	V_5
0	-	0	1	1	1	1	1
A_1	-	156.0	-1	-.523	.330	.839	1.483
A_2	0 ,B_1	155.5	-1	-.522	.287	.931	1.727
A_3	0 ,B_2	156.5	-1	-.480	.275	.795	1.575
A (computed)		155.4	-1	-.534	.308	.861	1.617
*A_4	0 ,B	155.5	-1	-.485	.276	.783	1.518
B_1	-	232.0	1	-.338	-.018	.027	.108
B_2	A_2,C_1	226.0	1	-.016	-.065	.013	.324
B_3	A_3,C_2	226.5	1	-.011	-.083	.010	.400
B (computed)		227.3	1	.001	-.101	.0003	.506
**C_1	A_1,B_1	242.5	-1	.177	-.328	-.110	1.51
C_2	B_2,D_1	238.9	-1	.120	-.184	-.080	.850
C_3	B_3,D_2	239.0	-1	.107	-.157	-.072	.719
C (computed)		238.6	-1	.102	-.146	-.070	.693
D_1	-	320	.187	-.197	1	-1.353	1.047
D_2	B_2,C_2	320	.195	-.202	1	-1.718	1.802
D_3	B_3,C_3	320	.218	-.202	1	-1.675	1.780
D (computed)		319.3	.202	-.197	1	-1.734	1.818

* Drive currents chosen to suppress computed mode B.

** Mode C could not be identified using a single driving source.

TABLE VI

RESULTS OF SECOND ITERATION TEST ON SYSTEM NO. 4
WITH UNIFORM DAMPING. RESPONSE VOLTAGES
ARE IN-PHASE COMPONENTS REFERRED TO INPUT CURRENTS

Driven Mode	Suppressed Modes	f (cps)	V_1	V_2	V_3	V_4	V_5
0	-	0	1	1	1	1	1
A_1	-	156.0	-1	-.523	.330	.839	1.483
A_2	0, B_1	155.5	-1	-.522	.287	.931	1.727
A_3	B_2, C_2	155.5	-1	-.506	.304	.807	1.49
A_4	B_3, C_3	155.5	-1	-.525	.312	.850	1.59
A^4 (computed)		155.4	-1	-.534	.308	.861	1.62
B_1	-	232.0	1	-.338	-.018	.027	.108
B_2	A_2, C_1	226.0	1	-.016	-.065	.013	.324
B_3	A_3, C_2	227.0	1	-.011	-.082	.010	.403
B_4	A_4, C_3	227.0	1	-.008	-.091	.008	.440
B^4 (computed)		227.3	1	.001	-.101	.0003	.506
*C_1	A_1, B_1	242.5	-1	.177	-.328	-.110	1.51
C_2	B_2, D_1	238.9	-1	.120	-.184	-.080	.850
C_3	A_3, B_3	239.5	-1	.108	-.158	-.065	.700
C_4	A_4, B_4	239.0	-1	.106	-.149	-.064	.659
C^4 (computed)		238.6	-1	.102	-.146	-.070	.693
D_1	-	320	.187	-.197	1	-1.353	1.047
D_2	B_2, C_2	320	.195	-.202	1	-1.718	1.802
D_3	B_3, C_3	319	.227	-.203	1	-1.735	1.785
D_4	B_4, C_4	319	.175	-.202	1	-1.730	1.775
D^4 (computed)		319.3	.202	-.197	1	-1.734	1.818

* Mode C could not be identified using a single driving source.

TABLE VII

RESULTS OF ITERATION TEST ON SYSTEM NO. 4
WITH NON-UNIFORM DAMPING. RESPONSE VOLTAGES
ARE IN-PHASE COMPONENTS REFERRED TO INPUT CURRENTS

Driven Mode	Suppressed Modes	f (cps)	V_1	V_2	V_3	V_4	V_5
0	-	0	1	1	1	1	1
A_1	-	164	-1	-.385	.188	.661	1.430
A_2	0 ,B_1	159	-1	-.447	.248	.734	1.470
A_3	0 ,B_2	160	-1	-.447	.238	.739	1.497
A (computed)		155.38	-1	-.534	.308	.861	1.617
B_1	-	225.5	1	-.0052	-.0906	.00595	.481
B_2	A_2,C_1	226.5	1	-.0044	-.0962	.00597	.484
B_3	A_3,C_2	226.5	1	-.0021	-.104	.0048	.516
B (computed)		227.3	1	.0006	-.101	.0003	.506
*C_1	A ,B	238.5	-1	.131	-.197	-.085	.987
C_2	B_2,D_1	238	-1	.120	-.187	-.078	.865
C_3	B_3,D_2	238	-1	.116	-.178	-.074	.820
C (computed)		238.6	-1	.102	-.146	-.070	.693
D_1	-	324.5	.379	-.248	1	-1.647	1.957
D_2	B_2,C_2	320.5	.258	-.247	1	-1.775	2.225
D_3	B_3,C_3	320.5	.258	-.254	1	-1.785	2.308
D (computed)		319.3	.202	-.197	1	-1.734	1.818

* Used the same set of driving currents that was used to excite mode C_3 of Table V. Mode C could not be identified with a single voltage drive.

E. *Separation of Modes in a Complex Airframe Analog - System No. 5.*

(1) It was desirable that the multiple drive iteration technique be tested on a practical problem, preferably one involving a large number of transformers. For this purpose, an airframe analog was obtained from the Douglas Aircraft Company, El Segundo, California. This system had been studied quite thoroughly, and good agreement had been obtained between analog computer results and tests of the elastic structure. However, a great deal of difficulty had been encountered in exciting the third symmetric mode, and the final values used included a number of points with large phase displacement.

The physical outlines of the electrical analog are shown in Figure 29. The basic components of the system are represented as a rigid fuselage, an attached wing structure with coupled bending and torsion, and a pylon mounted engine nacelle. For symmetrical modes, the fuselage is free to move vertically, and to pitch. All anti-symmetric motions of the fuselage are constrained to zero.

(2) A careful survey was made of the system modes, using a single voltage source for excitation. Then data on the first five symmetric modes were taken, with the drive point chosen in each case so as to achieve the least possible phase displacements among the larger response voltages. In addition, the sixth mode was recorded, to be used in computing the excitation for mode 5.

Next, the system was excited by a set of three current sources. The relative values of the input currents were computed in advance from the best previous data on the two adjacent interfering modes.

Each of the lowest five oscillatory modes was excited in this manner, after which the process was repeated. A third iteration was made on mode 3 only.

The final response voltages are compared, in Table VIII, with the initial values obtained using a single voltage drive. In Table IX, the corresponding in-phase components of response are compared. Table X shows the driving currents used in the final runs. In order to compute mode parameters and check orthogonality, the voltage across every capacitor in the system was required. Table X includes those voltages which were omitted from Table VIII and Table IX.

From Table VIII it is evident that the simple voltage excitation gives reasonably low phase shifts in this system, in all modes except the third. The multiple current drive was successful in producing a significant reduction in the phase shifts of the third mode. A large number of trials were made, using various sets of driving points, in an effort to reduce these phase shifts. The values presented represent the best response that could be obtained with three driving sources.

It is interesting to note, in Table IX, that the in-phase components of response do not differ as much as might be expected for the third mode. The basic shape of the mode is well defined by the voltage drive data, despite the large phase displacements. Thus, if high accuracy is not required, mode data containing large phase displacements may be perfectly acceptable. The important factors with regard to accuracy, in such cases, are the ratios of in-phase response of the interfering modes to their quadrature response. Such ratios can be estimated from the universal resonance curves.

It is believed that the phase displacements which remain in the final values of Table VIII are due largely to non-uniformities in the system damping, caused by the large number of transformers. There seems to be no way of eliminating such phase displacements without varying the relative phases of the driving currents, and this does not appear to be a practical procedure.

(3) A complete orthogonality check was made from the final mode data of Tables IX and X. The degree of departure from orthogonality can be measured by the dimensionless ratio

$$E_{KL} = \frac{\sum_i \sum_j m_{ij} A_{iK} A_{jL}}{\sqrt{M_K \; M_L}} \quad . \qquad \text{(VIII-1)}$$

For an electrical system with lumped capacitors, (VIII-1) can be expressed as

$$E_{KL} = \frac{\sum_i c_i A_{iK} A_{iL}}{\sqrt{(\sum_j c_j A_{jK}^2)(\sum_r c_r A_{rK}^2)}} \quad . \qquad \text{(VIII-2)}$$

In the process of computing these ratios, the generalized "mass" of each mode was obtained. From the measured power inputs, the Q's of the modes were computed from (VI-12).

The quantities E_{KL} and Q_K are independent of the normalizing condition used, but the M_K are not. In the data of Tables IX and X, the normalization was arbitrarily chosen for convenient comparison of the various modes. However, it was pointed out in (VI-C) that the interference effects of the modes could best be evaluated by using the normalizing condition (VI-1). From (VI-10), this latter condition can be approximated by adjusting the input power to the same value for each driven mode. Such an adjustment was computed for the in-phase voltages of Tables IX and X, making use of the fact that the input power varies as the square of the voltage level. The necessary normalizing factors are tabulated in Table XI, and the generalized masses are computed on this basis. Table XII shows the matrix of error factors, E_{KL}.

Table XI gives at least a partial explanation for the difficulty in exciting mode 3. The interference from mode 2 is appreciable, due to its low Q, small frequency separation, and higher normalizing factor. When mode 2 is driven, the interference from mode 3 is reduced by the ratio of normalizing factors. This lack of reciprocity between the interactions of two modes upon one another can be quite misleading unless a logical normalization procedure is followed.

The low values of Q in Table XI can be explained by the large number of transformers used in the system. For the first two modes there is good agreement with values obtained from Douglas Aircraft Company decay records. The decay records for the third mode were so full of "beats" that no estimate of Q was given. No comparisons were available for modes 4 and 5. The energy in mode 4 is highly concentrated in one part of the nacelle analog.

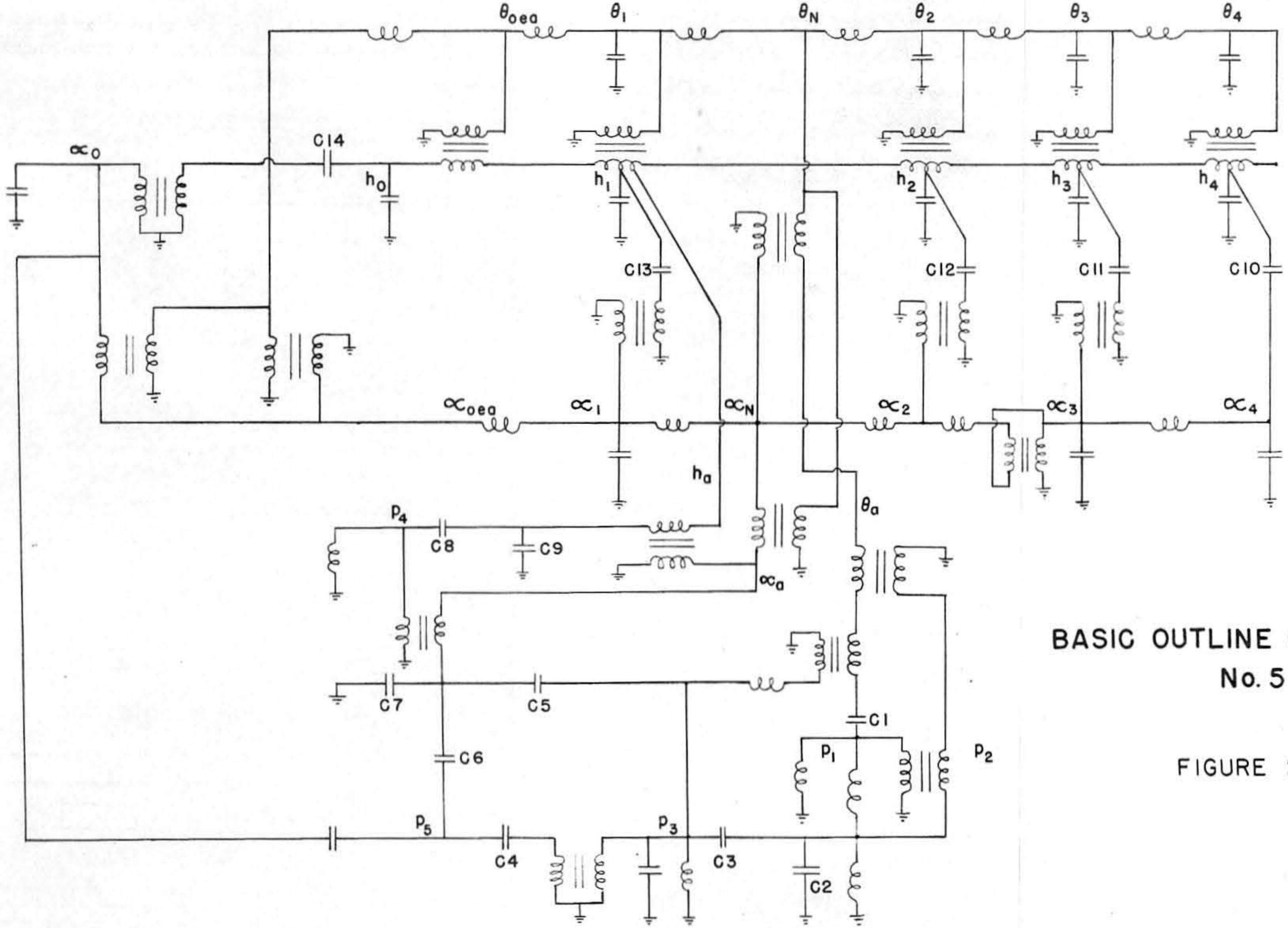
The orthogonality check in Table XII is excellent, and this gives some assurance that the final mode data accurately represent the idealized modes. The only significant lack of orthogonality appears between modes 1 and 5. Due to the wide frequency separation of the modes, this could hardly be due to an interference effect. The discrepancy may possibly be due to an incorrect reading in the fifth mode data.

(4) Using the final set of driving currents for mode 3, the frequency was varied in a region near resonance, and the voltages at coordinate h_0 were recorded. The variation of damping with frequency was assumed to be approximately the same as for structural damping, and the relations of (VI-D-1) were employed. Figure 30 shows the response locus and the extrapolated circle diagram. The algebraic sign of Vh_0 has been reversed to put the circle in the right half plane. The definition of the circle is surprisingly good, and the displacement of the circle is essentially a quadrature voltage. In computing Q_3 from Figure 30, the interval between 246.5 cps and 251.5 cps was used to evaluate $d\tilde{x}/d\omega$ in (VI-22). The smaller interval near resonance was avoided because of the lack of precision in reading frequency from a master oscillator dial. The value of Q_3 computed from Figure 30 is 17.9. This is low in comparison with the value in Table XI, as would be expected, since $d\tilde{x}/d\omega$ is maximum at $\tilde{\omega}_3$.

(5) During the tests on System No. 5, a number of practical problems were encountered which had not appeared in the tests on simpler systems. One of these was the choice of driving points when three current sources were to be used. It is desirable that all of the drive sources be additive in their effect upon the driven mode, and one is tempted to select the three coordinates of highest apparent amplitude. However, these are points where large inaccuracies may exist in the data on the modes to be suppressed. The best set of drive points for suppressing two modes seems to be a set with reasonably high amplitudes in all three modes. If the algebraic signs of the chosen points are satisfactory, it is possible to get a strong

drive, and yet base the suppression on a set of accurately defined values. There are exceptions, of course, and the technique of choosing suitable drive points comes only with experience.

The other problem was a metering problem, which appeared when current sources were used to drive a high Q system. If the current sources are assumed to have an infinite output impedance, then it is clear that the system impedance, viewed from the voltmeter terminals, will be quite high at resonant frequencies. The voltmeter and associated metering circuit may have enough conductive and capacitive loading effect to reduce the amplitude of oscillation of the desired mode. Since this loading effect will be different at different metering points, a false mode shape can be obtained. It was not feasible to change the analog computer metering system, so the difficulty was overcome by paralleling one of the current sources with a voltage source. The voltage source was adjusted in amplitude and phase until it delivered no current to the system. Under these conditions, the system as viewed from the metering point appears to be off resonance due to the constraint imposed by the voltage source. This greatly reduces the loading effect, particularly if the voltage source is located at a high amplitude drive point.



BASIC OUTLINE OF SYSTEM
No. 5

FIGURE 29

TABLE VIII

COMPARISON OF INITIAL MODE DATA, USING SINGLE VOLTAGE DRIVE, WITH FINAL MODE DATA, USING THREE CURRENT DRIVES, IN SYSTEM NO. 5.

	Mode 1		Mode 2	
Excitation	Voltage	Current	Voltage	Current
Drive Points	h_3	h_3, θ_a, p_1	h_2	h_1, α_2, p_1
Frequency (cps)	161	161	224	223.3
Coordinate				
h_4	18.6 (-1)	20.9 (-3)	15.0 (-8)	11.95(0)
h_3	11.5 (0)	13.5 (-1)	8.85(-6)	6.55(.5)
h_2	5.64(0)	6.85(-.6)	4.0 (-6)	2.92(0)
h_1	1.13(-2)	1.39(-3)	1.16(-7)	.83(-4)
h_o	- 1.18(.5)	- 1.44(.5)	.10(-40)	.13(90)
α_4	.25(-3)	.30(-1)	- .83(-1)	- .58(-2)
α_3	.28(-2)	.34(-1)	- .76(-1)	- .53(-2)
α_2	.86(-3)	1.05(-1)	- 2.78(-2)	- 1.91(-2)
α_N	.80(-3)	.99(-.8)	- 2.81(-2)	- 1.94(-3)
α_1	.61(-2)	.74(-.8)	- 1.65(-2)	- 1.15(-4)
α_{oea}	.25(.5)	.30(-1)	.31(.5)	.22(5)
α_o	.30(3)	.36(3)	.41(-.5)	.30(3)
p_1	- 4.98(-2)	- 6.08(4)	16.5(-6)	15.25(.5)
p_2	.64(-1)	.80(0)	- 1.37(2.5)	- 1.01(-3)
p_3	.19(-1)	.24(2)	- .79(3)	- .58(0)
p_4	- .04(-13)	- .04(-17)	.81(-3)	.57(-6)
p_5	- .46(2)	- .56(3)	- 1.38(-5)	- .95(-4)
θ_4	8.43(0)	10.1 (-1)	8.96(-7)	6.67(-1)
θ_3	6.96(.5)	8.38(0)	6.46(-6)	4.74(0)
θ_2	5.08(.5)	6.13(.5)	3.51(-5)	2.57(1)
θ_N	4.45(.5)	5.40(.5)	3.60(-5)	1.92(2)
θ_1	3.99(.5)	4.84(.2)	2.11(-5)	1.57(4)
θ_{oea}	1.41(-1)	1.70(-2)	.51(-4)	.40(10)
h_a	2.73(-.5)	3.33(-1)	2.00(-6)	1.46(-.5)
α_a	- .99(3)	- 1.19(2)	- 3.82(20)	- 2.71(-3)
θ_a	5.10(0)	6.15(0)	.50(-22)	.44(22)

Values shown in parentheses are phase angles in degrees referred to input currents.

TABLE VIII (continued)

	Mode 3		Mode 4	
Excitation	Voltage	Current	Voltage	Current
Drive Points	h_o	h_o, α_N, p_1	p_6	h_4, p_2, p_6
Frequency (cps)	243.5	248.5	387	387
Coordinate				
h_4	3.78(-15)	1.37(-16)	2.42(3)	- 4.12(.8)
h_3	1.78(-9)	.67(-15)	.60(6)	- .99(-.4)
h_2	.84(25)	.22(-11)	- .03(-32)	.07(17)
h_1	- .88(-29)	- .45(-4)	- .06(0)	.11(7)
h_o	- 2.36(5)	- 1.40(-1)	- .12(5)	.21(3)
α_4	2.82(29)	1.30(8)	.02(38)	- .03(39)
α_3	2.77(28.5)	1.28(8)	.04(15)	- .06(15)
α_2	9.30(29)	4.37(8)	.10(12)	- .17(17)
α_N	9.20(29)	4.32(8)	.09(12)	- .16(18)
α_1	5.76(29)	2.67(8)	.06(12)	- .11(18)
α_{oea}	- .43(45)	- .18(23)	.004(-5)	- .007(-14)
α_o	- .62(34)	- .27(13)	.004(0)	- .006(-12)
p_1	27.0 (29)	12.95(3)	- 1.98(-2)	3.88(2)
p_2	- .79(36)	- .22(-16)	-19.1 (-3)	34.3 (-.2)
p_3	- 1.86(27)	- .78(3)	1.50(-1)	- 2.71(0)
p_4	- 3.55(29)	- 1.63(5)	- .24(4)	.41(3)
p_5	6.75(29)	3.05(6)	- .75(-3)	1.37(-1)
θ_4	2.56(-24)	.86(-19)	2.59(-2)	- 4.35(-1)
θ_3	1.84(-24)	.62(-16)	1.19(2)	- 2.03(.8)
θ_2	1.06(-38)	.27(-20)	.05(29)	- .06(5)
θ_N	.92(-55)	.14(-31)	- .24(-4)	.42(0)
θ_1	1.92(13)	1.01(.5)	- .02(-83)	.02(-20)
θ_{oea}	2.18(23)	1.03(3)	.12(7)	- .20(2)
h_a	.87(72)	- .02(-24)	- .05(-7)	.11(10)
α_a	9.28(31)	4.24(9)	.19(5)	- .34(8)
θ_a	6.80(24)	3.43(7)	- .20(-14)	.34(-10)

TABLE VIII (continued)

	Mode 5	
Excitation	Voltage	Current
Drive Points	θ_4	h_2, θ_3, p_5
Frequency (cps)	515	517
Coordinate		
h_4	- 6.50(-1)	14.25(-5)
h_3	1.09(7)	- 2.39(-14)
h_2	2.30(.5)	- 5.01(-7)
h_1	.94(-.6)	- 2.05(-7)
h_o	- .20(0)	.43(-3)
α_4	.26(-2)	- .57(-9)
α_3	.12(0)	- .27(-12)
α_2	.16(-2)	- .38(-11)
α_N	- .02(8)	.03(3)
α_1	.02(-10)	- .05(-14)
α_{oea}	.03(-4)	- .06(-8)
α_o	.02(3)	- .05(-2)
p_1	1.81(0)	- 3.925(-4)
p_2	- .66(7)	1.47(-11)
p_3	.45(-6)	- .97(3)
p_4	.48(-3)	- 1.03(-9)
p_5	- .80(-4)	1.74(-8)
θ_4	-11.65(-.6)	25.75(-7)
θ_3	- 3.43(-3)	7.40(-3)
θ_2	.99(4)	- 2.15(-8)
θ_N	1.41(1)	- 3.10(-7)
θ_1	1.69(.5)	- 3.68(-7)
θ_{oea}	.90(-.6)	- 1.95(-6)
h_a	1.63(-.5)	- 3.55(-7)
α_a	- .59(2)	1.26(-7)
θ_a	1.40(1)	- 3.05(-7)

TABLE IX

IN-PHASE COMPONENTS OF RESPONSE IN SYSTEM NO. 5.
INITIAL VOLTAGE DRIVE COMPARED WITH FINAL CURRENT DRIVES
FOR EACH MODE. VALUES NORMALIZED TO BE EQUAL AT ONE POINT.

	Mode 1		Mode 2		Mode 3	
Excitation	Voltage	Current	Voltage	Current	Voltage	Current
Drive Points	h_3	h_3, θ_a, p_1	h_2	h_1, α_2, p_1	h_o	h_o, α_N, p_1
Frequency(cps)	161	161	224	223.3	243.5	248.5
Coordinate						
h_4	1.00	1.00	.907	.784	.155	.102
h_3	.619	.647	.537	.430	.075	.050
h_2	.303	.328	.243	.191	.032	.017
h_1	.061	.066	.070	.054	- .033	- .035
h_o	- .063	- .069	.005	0	- .100	- .108
α_4	.013	.015	- .059	- .038	.105	.100
α_3	.015	.016	- .046	- .034	.103	.098
α_2	.046	.050	- .167	- .125	.346	.335
α_N	.042	.047	- .171	- .127	.342	.331
α_1	.033	.036	- .101	- .075	.214	.204
α_{oea}	.013	.014	.019	.015	- .013	- .013
α_o	.016	.017	.025	.020	- .022	- .020
p_1	- .268	- .291	1.00	1.00	1.00	1.00
p_2	.034	.038	- .084	- .066	- .027	- .016
p_3	.011	.011	- .048	- .038	- .071	- .061
p_4	- .002	- .002	.049	.037	- .131	- .125
p_5	- .025	- .027	- .084	- .062	.250	.235
θ_4	.453	.485	.542	.438	.099	.063
θ_3	.321	.402	.392	.311	.071	.046
θ_2	.273	.294	.214	.169	.035	.020
θ_N	.239	.259	.157	.125	.022	.009
θ_1	.215	.232	.128	.102	.079	.078
θ_{oea}	.076	.082	.031	.026	.085	.080
h_a	.141	.159	.121	.096	.011	- .001
α_a	- .053	- .057	- .220	- .178	.339	.325
θ_a	.274	.294	.029	.027	.264	.263

TABLE IX (continued)

	Mode 4		Mode 5	
Excitation	Voltage	Current	Voltage	Current
Drive Points	p_6	h_4, p_2, p_6	θ_4	h_2, θ_3, p_5
Frequency (cps)	387	387	515	517
Coordinate				
h_4	.127	- .120	-1.00	1.00
h_3	.031	- .029	.166	- .164
h_2	- .001	.002	.354	- .350
h_1	- .003	.003	.145	- .143
h_o	- .006	.006	- .031	.031
α_4	.001	- .0006	.040	- .040
α_3	.001	- .002	.018	- .019
α_2	.005	- .005	.025	- .025
α_N	.005	- .005	- .004	.002
α_1	.003	- .003	.003	- .004
α_{oea}	.000	- .0002	.004	- .004
α_o	.000	- .0002	.004	- .004
p_1	- .103	.113	.278	- .276
p_2	-1.00	1.00	- .101	.102
p_3	.079	- .079	.069	- .068
p_4	- .012	.012	.073	- .071
p_5	- .039	.040	- .123	.121
θ_4	.136	- .127	-1.79	1.803
θ_3	.062	- .059	- .526	.520
θ_2	.002	- .002	.151	- .150
θ_N	- .012	.012	.217	- .217
θ_1	.000	.0006	.260	- .258
θ_{oea}	.006	- .006	.138	- .137
h_a	- .003	.003	.251	- .248
α_a	.010	- .010	- .091	.088
θ_a	- .010	.010	.215	- .213

TABLE X

DRIVING CURRENTS AND ADDITIONAL CAPACITOR VOLTAGES FOR FINAL MODE DATA OF TABLES VIII AND IX.

	Mode 1		Mode 2	
C1	12.45(.5)	.597	-15.55(-1)	-1.020
C2	- 1.66(7)	- .079	5.19(2)	.340
C3	- 1.89(6)	- .090	5.75(2)	.377
C4	.33(3)	.016	1.53(-3)	.100
C5	1.37(3)	.066	3.14(-5)	.205
C6	- .58(2)	- .028	- 2.72(-4)	- .178
C7	- 1.14(3)	- .054	- 3.67(-3)	- .240
C8	1.88(-5)	.090	- 2.57(5)	- .168
C9	1.84(-5)	.088	- 2.01(5)	- .132
C10	20.8 (-3)	.100	12.5 (0)	.820
C11	13.9 (-1)	.667	6.00(1)	.394
C12	7.45(-.5)	.357	1.78(2)	.116
C13	1.91(-2)	.092	.02(-27)	.001
C14	1.08(-.5)	.052	- .32(-22)	- .020

Mode 1 Excitation: $I_{h_3} = .264$; $I_{\theta_a} = .2416$; $I_{p_1} = -.125$

Mode 2 Excitation: $I_{h_1} = .762$; $I_{\alpha_2} = -.2775$; $I_{p_1} = .125$

First column for each mode shows measured values, with phase angle in degrees, referred to input currents. Second column shows normalized in-phase components. Driving currents are given in milliamperes.

TABLE X (continued)

	Mode 3		Mode 4	
C1	-10.00(1)	- .775	8.75(2)	.255
C2	5.05(4)	.391	36.0(1)	1.050
C3	5.84(4)	.451	38.5 (.5)	1.122
C4	- 2.29(7)	- .176	1.37(1)	.040
C5	- 7.70(5)	- .594	- 1.70(-2)	- .049
C6	4.04(7)	.312	- 2.38(0)	- .069
C7	7.02(7)	.540	- 1.02(4)	- .030
C8	7.13(7)	.548	- .75(3)	- .022
C9	5.58(7)	.430	- .35(3)	- .010
C10	.55(-82)	.006	- 4.12(0)	- .120
C11	1.93(0)	.150	- 1.05(0)	- .031
C12	2.85(6)	.220	- .03(15)	- .0009
C13	1.46(11)	.111	.04(-15)	.001
C14	1.66(.5)	.129	- .20(3)	- .006

Mode 3 Excitation: I_{h_o} = - 1.612; I_{α_N} = .988; I_{p_1} = .125
Mode 4 Excitation: I_{h_4} = -.016; I_{p_2} = .1625; I_{p_6} = .012

	Mode 5	
C1	.86(-5)	.060
C2	- .22(58)	- .008
C3	.85(-8)	.059
C4	- .83(-23)	- .054
C5	- 4.10(-7)	- .287
C6	1.41(-11)	.098
C7	3.18(-9)	.221
C8	- .97(-8)	- .068
C9	- 2.01(-8)	- .140
C10	14.9(-5)	1.046
C11	- 2.69(-14)	- .184
C12	- 5.26(-7)	- .367
C13	- 2.12(-7)	- .148
C14	- .38(-4)	- .027

Mode 5 Excitation: I_{θ_3} = .14; I_{p_5} = .254; I_{h_2} = -.865

TABLE XI

NORMAL MODE PARAMETERS FOR SYSTEM NO. 5.

Mode	1	2	3	4	5
Frequency (cps)	161	223.3	248.5	387	517
Normalizing Factor for Tables IX and X	1.90	1.91	1.00	1.63*	1.30
Generalized "Mass" (μf)	.998	1.27	.675	.427*	.275
Q	20.9	36.9	21.9	21.5*	18.5

* These are minimum possible values. Exact values for Mode 4 cannot be given with certainty, due to a discrepancy in power input data.

TABLE XII

ORTHOGONALITY CHECK OF FINAL MODE DATA IN TABLES IX AND X, AS INDICATED BY THE MATRIX OF RATIOS E_{KL}.

Mode	1	2	3	4	5
1	1.000	- .0128	.0103	.0008	- .0539
2	- .0128	1.000	.0089	.0015	- .0177
3	.0103	.0089	1.000	.0014	- .0046
4	.0008	.0015	.0014	1.000	.0177
5	- .0539	- .0177	- .0046	.0177	1.000

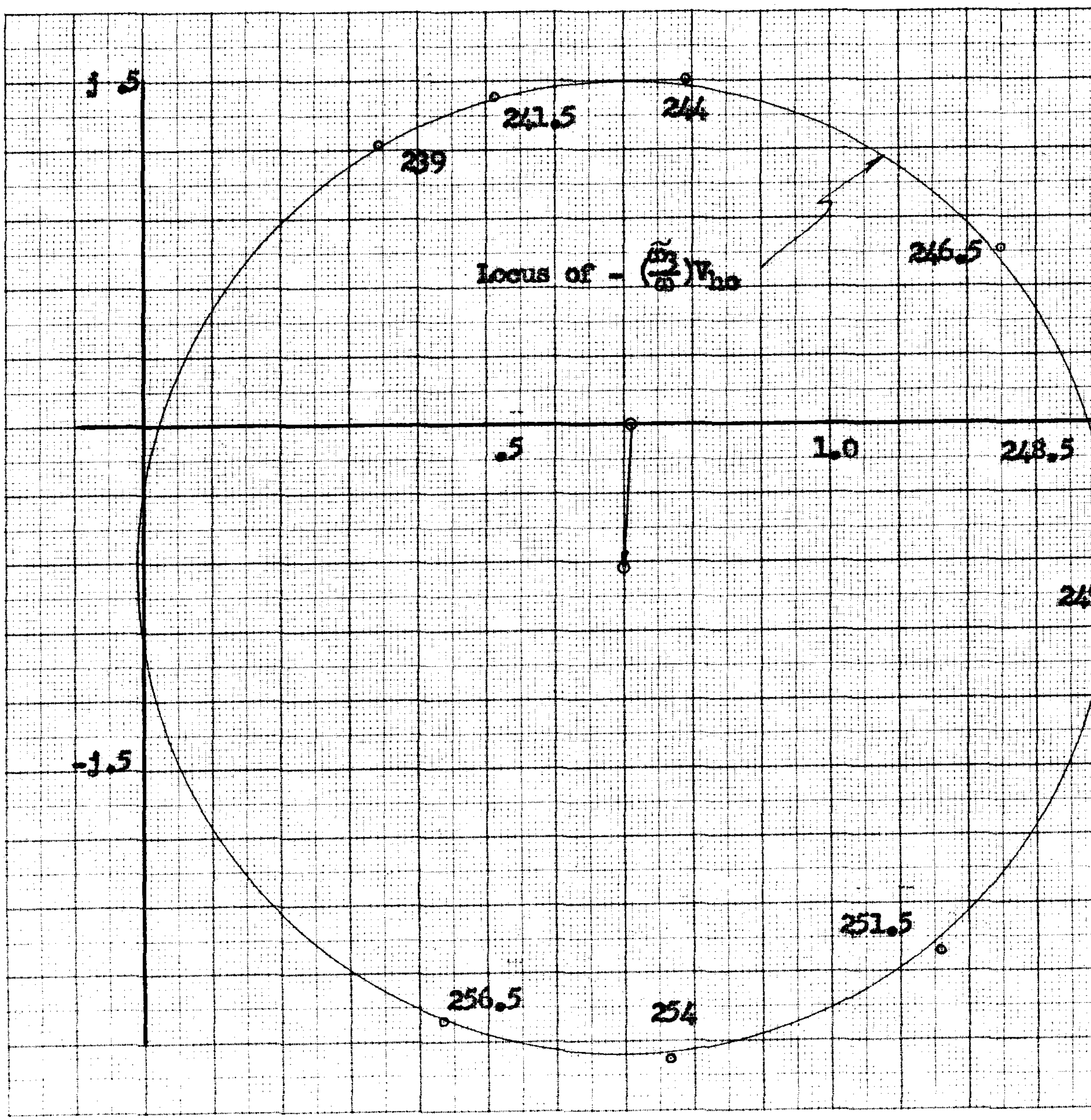


Figure 30. Frequency Response Locus of Mode 3, System No. 5, Using Final Current Drive. Phase Angles are Referred to the Driving Currents.

IX. DISCUSSION AND CONCLUSIONS

In this work, an effort has been made to find some practical solutions to the two major problems discussed in Part I. The nature of the problems are so general that no coverage of the subject could possibly be regarded as complete. The work has been directed primarily toward an investigation of approximate solutions which are simple enough to be of use in industrial problems.

The writer feels that the contributions of greatest practical value are contained in Part VI and its extensions. The theory developed there is based upon the assumption that a system is uniformly damped. Admittedly, such an assumption is restrictive, but some property of the damping must be specified in order to simplify the analysis. It has been demonstrated that the condition of uniform damping applies approximately to a number of important types of physical systems. The theme of the entire investigation has been that the degree of uniformity of damping is an important system property, together with the smallness of damping.

Multiple driving methods and iteration techniques are in general use in the aircraft industry at present. A method commonly employed is to drive the system at apparent nodal points or lines of an undesired mode. The disadvantage of this approach lies in the fact that the low amplitude coordinates are usually the least accurately defined coordinates in a mode, and the true nodes may be quite different from the apparent nodes. Moreover, the nodes may be inconveniently located for driving purposes, in a continuous system, and they may not be physically available in a lumped parameter system.

It is believed that the iteration procedure proposed in Part VI will be very valuable in systems where three or more modes are badly interfering. During the early stages of the investigation, an opportunity was presented to test the method on a commercial problem in the Analysis Laboratory. The structure being studied had two modes closely adjacent to a third mode which was being driven. Many attempts were made to obtain a suitable excitation, and finally a reasonably good result was obtained with a differential voltage drive. To test the proposed current drive method, the best previous data on adjacent modes were used to compute a set of drive currents. This excitation gave essentially the same mode shape as the differential voltage drive, but the phase shifts were somewhat less. The data gave an excellent orthogonality check with other modes, and were used in the commercial report. Valuable time might have been saved had the current drive method been used in the beginning.

The problem of exciting a system which has degenerate modes was not considered in detail. For cases of degeneracy or near degeneracy, the proposed iteration procedure becomes impossible or impractical. The accidental occurrence of modes which are completely degenerate is not likely in an aircraft structure, but near degeneracy is sometimes encountered. It is believed that no single procedure can be applied to all such cases. There are a few rather obvious techniques that might be tried, such as the following:

(1) A trial and error procedure with two current sources might be used, in an attempt to eliminate relative phase shifts. This would not be practical in the presence of additional interfering modes.

(2) The system might be altered slightly by adding a variable mass at some point, in an effort to spread the two modes apart. Once the separate identities of the modes were established, a suitable drive could be computed for each. The perturbing mass could then be reduced slowly to zero, with small adjustments in the drive, if necessary, to maintain low phase shifts.

(3) If there is a small, but detectable, difference in the two mode frequencies, then some advantage might be gained by driving each mode slightly off its resonant frequency in the direction away from the interfering mode. In this way one could take advantage of the maximum slope region of the resonance curve.

A second problem which has not been considered is that of finding suitable excitations for taking decay records of pure modes, and of interpreting such records when they are badly distorted. This problem, and the problem of nearly degenerate damped modes, are suitable topics for further investigation.

Although a quantitative measure of the uniformity of damping has been proposed in Part VII, no studies of specific cases or specific system types have been made. Additional work should be done along the lines of correlating the quantities Q_{KS}, defined in (VII-21), with the errors resulting from the use of equivalent orthogonal systems. A method of measuring B_{KS} experimentally would be quite valuable.

In summary the following conclusions can be stated:

(1) Exact normal mode analogs exist for systems with shunt damping or for systems with any type of uniform damping.

(2) The degree of uniformity of damping throughout a system is an important consideration in simplifying the analysis of the system.

(3) Application of the normal mode orthogonality property provides a sensitive criterion for estimating normal mode frequencies.

(4) Practical approximate methods of evaluating normal mode parameters are available.

(5) A multiple drive iteration technique has been used successfully for the suppression of interfering modes under steady state conditions.

(6) Moderate phase shifts in the steady state response of a system do not necessarily result in inaccurate mode data, if the system is driven at the proper frequency and the in-phase components of response are used.

REFERENCES

1. Strutt, J. W., Baron Rayleigh: "The Theory of Sound," Vol. 1, Macmillan and Co., London, 1894.

2. Strutt, J. W., Baron Rayleigh: "Scientific Papers," Vol. 1-6, Cambridge University Press, London, 1899-1920.

3. Poritsky, H., and C. S. L. Robinson: "Torsional Vibration in Geared-Turbine Propulsion Equipment," A.S.M.E. Transactions, V. 62, p. A117, 1940.

4. Inglis, C. E.: "A Mathematical Treatise on Vibrations in Railway Bridges," Cambridge University Press, London, 1934.

5. Crookes, S. I.: "Structural Design of Earthquake-Resistant Buildings," Leightons, Ltd., Aukland, New Zealand, 1940, (Chapter II).

6. Robb, A. M.: "Theory of Naval Architecture," Charles Griffin and Co., Ltd., London, 1952.

7. Kennedy, C. C., and C. D. P. Pancu: "Use of Vectors in Vibration Measurement and Analysis," Jour. of Aero. Sciences, V. 14, p. 603, Nov. 1947.

8. Lewis, R. C., and D. L. Wrisley: "A System for the Excitation of Pure Natural Modes of Complex Structure," Jour. of Aero. Sciences, V. 17, p. 705, Nov. 1950.

9. MacNeal, R. H., G. D. McCann, and C. H. Wilts: "The Solution of Aeroelastic Problems by Means of Electrical Analogies," Jour. of Aero. Sciences, V. 18, p. 777, Dec. 1951.

10. Everitt, W. L.: "Communication Engineering," 2nd Ed., Ch. 1, McGraw-Hill, New York, 1937.

11. Von Karman, T., and M. A. Biot: "Mathematical Methods in Engineering," McGraw-Hill, New York, 1940.

12. Guillemin, E. A.: "The Mathematics of Circuit Analysis," Ch. 6, John Wiley and Sons, New York, 1949.

13. Gardner, M. F., and J. L. Barnes: "Transients in Linear Systems," Vol. 1, John Wiley and Sons, New York, 1942.

14. Thomson, W. (Lord Kelvin), and P. G. Tait: "Treatise on Natural Philosophy," Vol. 1, No. 343b, Cambridge University Press, London, 1879.

15. Scott, R. F., and G. B. Matthews: "The Theory of Determinants," 2nd Ed., Ch. 6, Cambridge University Press, London, 1904.

16. Routh, E. J.: "Advanced Dynamics of a System of Rigid Bodies," Vol. 2, 6th Ed., Macmillan and Co., London, 1905.

17. Guillemin, E. A.: "Making Normal Coordinates Coincide With the Meshes of an Electrical Network," Proc. of I.R.E., V. 15, p. 935, 1927.

18. Guillemin, E. A.: "Communication Networks," Vol. 1, John Wiley and Sons, New York, 1931.

19. Guillemin, E. A.: "Approximate Solutions for Electrical Networks When These Are Highly Oscillatory," Trans. A.I.E.E., V. 47, p. 361, 1928.

20. Guillemin, E. A.: "Communication Networks," Vol. 2, John Wiley and Sons, New York, 1935.

21. McCann, G. D. and R. H. MacNeal: "Beam Vibration Analysis with the Electric Analog Computer," Jour. of App. Mechanics, V. 17, p. 13, March 1950.

22. Foster, R. N.: "A Reactance Theorem," Bell System Technical Journal, V. 3, p. 259, April 1924.

23. MacNeal, R. H.: "Transformation Theory of Analogies," from course notes, "Theory and Operation of the Electric Analog Computer," Calif. Institute of Technology, Summer, 1951.

24. Wilts, C. H.: "Application of Amplifiers to Analog Computation," from course notes, "Methods of Machine Computation in Engineering Analysis," Calif. Institute of Technology, 1953-54.

25. Coleman, R. P.: "Damping Formulas and Experimental Values of Damping in Flutter Models," N.A.C.A. Technical Note No. 751, Washington, Feb. 1940.

26. Traill-Nash, R. W.: "The Response of a Damped Continuous Linear Elastic System to Transient Force," Report S.M. 132, Div. of Aeronautics, Dept. of Supply and Development, Melbourne, Australia.

27. Morduchow, M.: "On Application of a Quasi-Static Variational Principle to a System With Damping," Jour. of App. Mechanics, V. 21, p.8, March 1954.

28. Hopkinson, B., and G. T. Williams: "The Elastic Hysteresis of Steel," Proc. Royal Society of London, Series A, V. 87, p. 502, 1912.

29. Rowett, F. E.: "Elastic Hysteresis in Steel," Proc. Royal Society of London, Series A, V. 89, p. 528, 1914.

30. Kimball, A. L. and D. E. Lovell: "Internal Friction in Solids," Phys. Review, V. 30, p. 948, 1927.

31. Schabtach, C., and R. O. Fehr: "Measurement of the Damping of Engineering Materials During Flexural Vibration at Elevated Temperatures," A.S.M.E. Transactions, V. 66, p. A-86, 1944.

32. Kimball, A. L.: "Friction and Damping in Vibrations," A.S.M.E. Transactions, pp. A-37, A-135, 1941.

33. Robertson, J. M., and A. J. Yorgiadis: "Internal Friction in Engineering Materials," A.S.M.E. Transactions, V. 68, p. A-173, 1946.

34. Lazan, B. J.: "Effect of Damping Constants and Stress Distribution on the Resonance Response of Members," Jour. of App. Mechanics, V. 20, p. 201, 1953.

35. Theodorsen, T., and I. E. Garrick: "Mechanism of Flutter - A Theoretical and Experimental Investigation of the Flutter Problem," N.A.C.A. Report No. 685, Washington, 1940.

36. Myklestad, N. O.: "The Concept of Complex Damping," Jour. of App. Mechanics, V. 19, p. 284, 1952.

37. Bode, H. W.: "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Co., New York, 1945.

38. Churchill, R. V.: "Introduction to Complex Variables and Applications," McGraw-Hill, New York, 1948.

39. Lewis, R. C., and Wrisley, D. L.: "Report on Excitation of Pure Natural Modes of Complex Structure; Phase A, Experimental Development," Mass. Inst. of Tech., Aero-Elastic and Structures Research, Contract No. N156s-25797, 1949.

40. LePage, W. R., and S. Seely: "General Network Analysis," McGraw-Hill, New York, 1952.

41. Russell, W. T.: "Lumped Parameter Analogies for Continuous Mechanical Systems," Ph.D. Thesis, Calif. Inst. of Tech., 1950.

APPENDIX

A. Resonance Properties of Single Degree of Freedom System with Shunt Damping.

For a simple tank circuit with shunt damping, the input impedance is

$$Z = \frac{1}{G + j(\omega C - \frac{1}{\omega L})} = R + jX$$

where

$$R = \frac{G}{G^2 + (\omega C - 1/\omega L)^2} \quad ; \quad X = \frac{-(\omega C - 1/\omega L)}{G^2 + (\omega C - 1/\omega L)^2} \quad .$$

The resonant frequency and the Q are given by

$$\omega_o = \frac{1}{\sqrt{LC}} \quad ; \quad Q_o = \frac{\omega_o C}{G} = \frac{1}{G\,\omega_o L} \quad .$$

Near frequency ω_o,

$$X \cong \frac{(1/\omega L - \omega C)}{G^2} = R_o^{\,2}(1/\omega L - \omega C)$$

and

$$\left(\frac{dX}{d\omega}\right)_{\omega_o} = -R_o^{\,2}\left(C + \frac{1}{L\omega_o^{\,2}}\right) = -2\,R_o^{\,2}C \quad .$$

Hence,

$$Q_o = -\frac{\omega_o}{2R_o}\left(\frac{dX}{d\omega}\right)_{\omega_o} \quad .$$

On the circle diagram of input impedance, the diameter of the circle is equal to R_o, the shunt resistance.

The expressions for R and X, in terms of Q_o and ω_o, are

$$R = R_o\left[1 + Q_o^{\,2}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2\right]^{-1}$$

$$X = -Q_o\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)R \quad .$$

If we let $\omega = \omega_o(1 + \epsilon)$, then $\epsilon = \Delta\omega/\omega_o$, and

$$\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right) = \epsilon\left(\frac{2+\epsilon}{1+\epsilon}\right) .$$

Thus,

$$R = R_o \left[1 + Q_o^{\,2}\epsilon^2 \left(\frac{2+\epsilon}{1+\epsilon}\right)^2\right]^{-1}$$

$$X = -\,Q_o\,\epsilon\left(\frac{2+\epsilon}{1+\epsilon}\right) R \ .$$

For $\epsilon << 1$, R and X are approximately functions of the single parameter $(Q_o\epsilon)$:

$$\frac{R}{R_o} \cong \frac{1}{1 + 4(Q_o\epsilon)^2} \ .$$

$$\frac{X}{R_o} \cong \frac{-\,2\,Q_o\,\epsilon}{1 + 4(Q_o\epsilon)^2} \ .$$

The "universal" resonance curves (for $Q = \infty$) are plotted from these expressions.

B. Resonance Properties of Single Degree of Freedom System with Structural Damping.

Using the structural damping approximation of (V-5), the equation of a single degree of freedom oscillator with such damping is

$$(Mp^2 + e^{jg} K)\, U = F \quad .$$

If the system is forced sinusoidally at frequency ω, the mechanical "admittance" of the system is

$$Y = \frac{F}{U} = \left(\frac{K}{\omega} \sin g\right) + j\left(\omega M - \frac{K}{\omega} \cos g\right) \quad .$$

At the damped resonant frequency, $\tilde{\omega}$, the imaginary part of Y must vanish; hence,

$$\tilde{\omega} = \sqrt{\frac{K}{M} \cos g} = \omega_o \sqrt{\cos g} \quad ,$$

where ω_o is the resonant frequency of the corresponding undamped system. The "Q" of the spring element is the ratio of the imaginary partto the real part of the spring "admittance", or

$$Q = \frac{\cos g}{\sin g} = \frac{1}{\tan g} \quad .$$

For structural damping, Q does not vary with frequency.

Consider the expression

$$\tilde{Y} = \frac{\omega}{\tilde{\omega}} Y = \frac{K \sin g}{\tilde{\omega}} + j \frac{M}{\tilde{\omega}} \left(\omega^2 - \tilde{\omega}^2\right)$$

When plotted as a function of ω, this is a vertical line in the complex plane. The inverse quantity,

$$\tilde{Z} = \frac{1}{\tilde{Y}} = \tilde{R} + j\tilde{X}$$

has a circular locus in the complex plane, with a diameter

$$\tilde{R}_o = \frac{\tilde{\omega}}{K \sin g} ,$$

which can be expressed as

$$\tilde{R}_o = \frac{Q}{M \tilde{\omega}} .$$

Inversion of $\tilde{Y}$, and differentiation of the resulting $\tilde{X}$ with respect to ω, gives

$$\left(\frac{d\tilde{X}}{d\omega}\right)_{\tilde{\omega}} = - \frac{2 \tilde{R}_o Q}{\tilde{\omega}} .$$

These relations can be expressed in terms of the electric analog by replacing M by C and K by 1/L .

In order to plot universal resonance curves, we can invert Y, and express the resulting R and X in dimensionless form, in terms of Q, $\tilde{R}_o$, and ϵ , as was done for shunt damping in Appendix A. The resulting expressions are

$$\frac{R}{\tilde{R}_o} = \frac{1 + \epsilon}{1 + Q^2\epsilon^2(2+\epsilon)^2}$$

$$X = - Q \epsilon (2 + \epsilon) R .$$

For $\epsilon << 1$, the resonance curves are essentially the same as those for shunt damping, the curves for "$Q = \infty$" being identical.

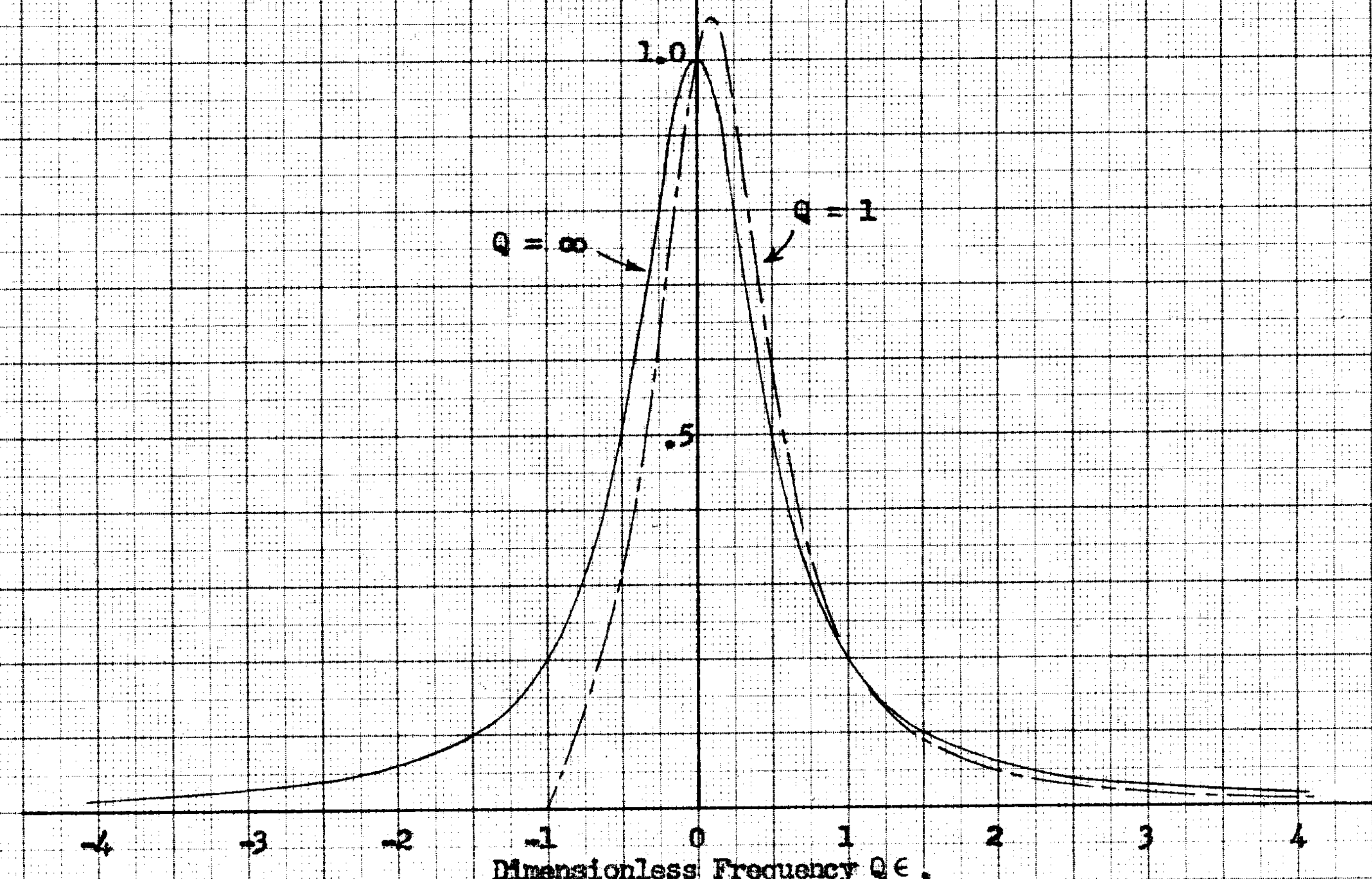


Figure 31. Universal Resonance Curves Showing Real Part of Input Impedance R/R_o for Single Degree of Freedom System with Structural Damping. $\epsilon = \Delta\omega/\bar{\omega}$.

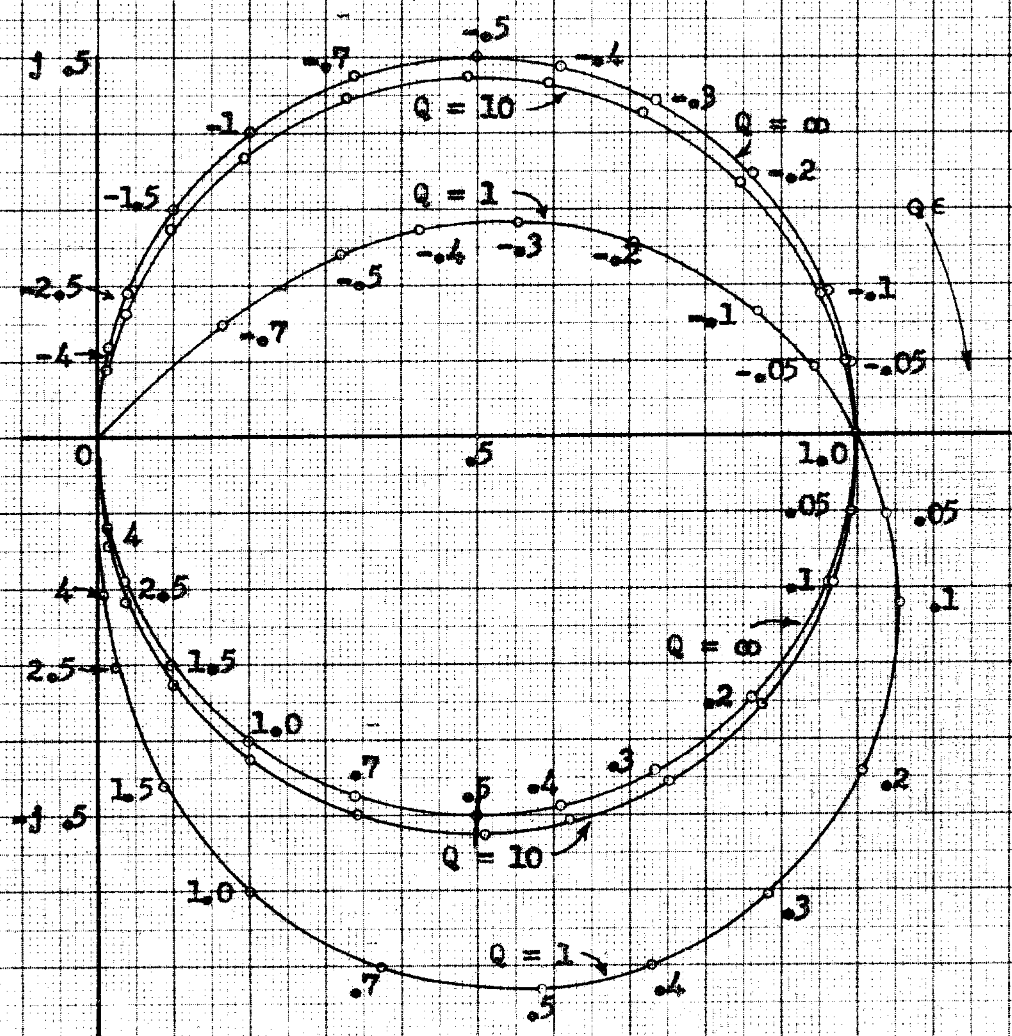


Figure 32. Impedance Loci Z/R_o for Single Degree of Freedom System with Structural Damping.

C. Resonance Properties of Single Degree of Freedom System with Series Damping.

In a resonant tank circuit with series resistance, r, in the inductor, the admittance is

$$Y = \left(\frac{r}{r^2 + \omega^2 L^2}\right) + j\omega\left(C - \frac{L}{r^2 + \omega^2 L^2}\right) .$$

At the damped resonant frequency, $\tilde{\omega}$, Y must be real; hence,

$$\tilde{\omega}^2 = \frac{1}{L^2}\left(\frac{L}{C} - r^2\right) = \omega_o{}^2\left(1 - \frac{1}{Q_o{}^2}\right)$$

where ω_o is the resonant frequency of the undamped system, and Q_o is the Q of the inductor at frequency ω_o. At frequency $\tilde{\omega}$, the inductor has a Q defined as

$$\tilde{Q} = \frac{\tilde{\omega}L}{r} = \sqrt{Q_o{}^2 - 1} .$$

At the damped resonant frequency, the real part of Y is

$$\tilde{G}_o = \frac{rC}{L} = \frac{\omega_o C}{Q_o} = \frac{\tilde{\omega}C}{\tilde{Q}} .$$

The real part of Y varies with frequency, so the locus of Z = 1/Y is not circular. However, we can plot the reciprocal of

$$\tilde{Y} = \left(\frac{\omega}{\tilde{\omega}}\right)^2 Y = \frac{1}{r\left[\tilde{Q}^2 + \left(\frac{\tilde{\omega}}{\omega}\right)^2\right]} + j\omega\left[\left(\frac{\omega}{\tilde{\omega}}\right)^2 C - \frac{L}{r^2\left[\tilde{Q}^2 + \left(\frac{\tilde{\omega}}{\omega}\right)^2\right]}\right] .$$

For $\tilde{Q} >> 1$, the real part of $\tilde{Y}$ is essentially constant near resonance, and the locus of the reciprocal, $\tilde{Z}$, is essentially a circular arc with a diameter

$$\tilde{R}_o = r(1 + \tilde{Q}^2) = \frac{\tilde{Q}}{\tilde{\omega}C} .$$

The expression for $\tilde{Y}$ can be inverted, and if the imaginary part of $\tilde{Z}$ is $\tilde{X}$, differentiation gives

$$\left(\frac{d\tilde{X}}{d\omega}\right)_{\tilde{\omega}} = \frac{-2\,\tilde{Q}^2}{C\tilde{\omega}^2\,(1 + 1/\tilde{Q}^2)} \quad .$$

If $\tilde{Q}^2 >> 1$, this expression becomes

$$\left(\frac{d\tilde{X}}{d\omega}\right)_{\tilde{\omega}} \cong \frac{-2\,\tilde{Q}\,\tilde{R}_o}{\tilde{\omega}} \quad .$$

which is used in (VI-33), with $\tilde{R}_o$ replaced by D.

In order to plot universal resonance curves, we can invert Y, and express the resulting impedance components in dimensionless form. The desired expressions are

$$\frac{R}{\tilde{R}_o} = \frac{\epsilon^2 + \tilde{Q}^2\epsilon^2(1+\epsilon)^2}{(\epsilon^2 + \tilde{Q}^2\epsilon^2)\left[1 + \dfrac{(1+\epsilon)^2(2+\epsilon)^2\,\tilde{Q}^6\epsilon^6}{(\epsilon^2 + \tilde{Q}^2\epsilon^2)^2}\right]}$$

$$X = \frac{-(1+\epsilon)(2+\epsilon)\,\tilde{Q}^3\epsilon^3}{(\epsilon^2 + \tilde{Q}^2\epsilon^2)}\;R$$

where $\epsilon = \dfrac{\Delta\omega}{\tilde{\omega}}$. For $\epsilon << 1$, these functions approach those for shunt damping.

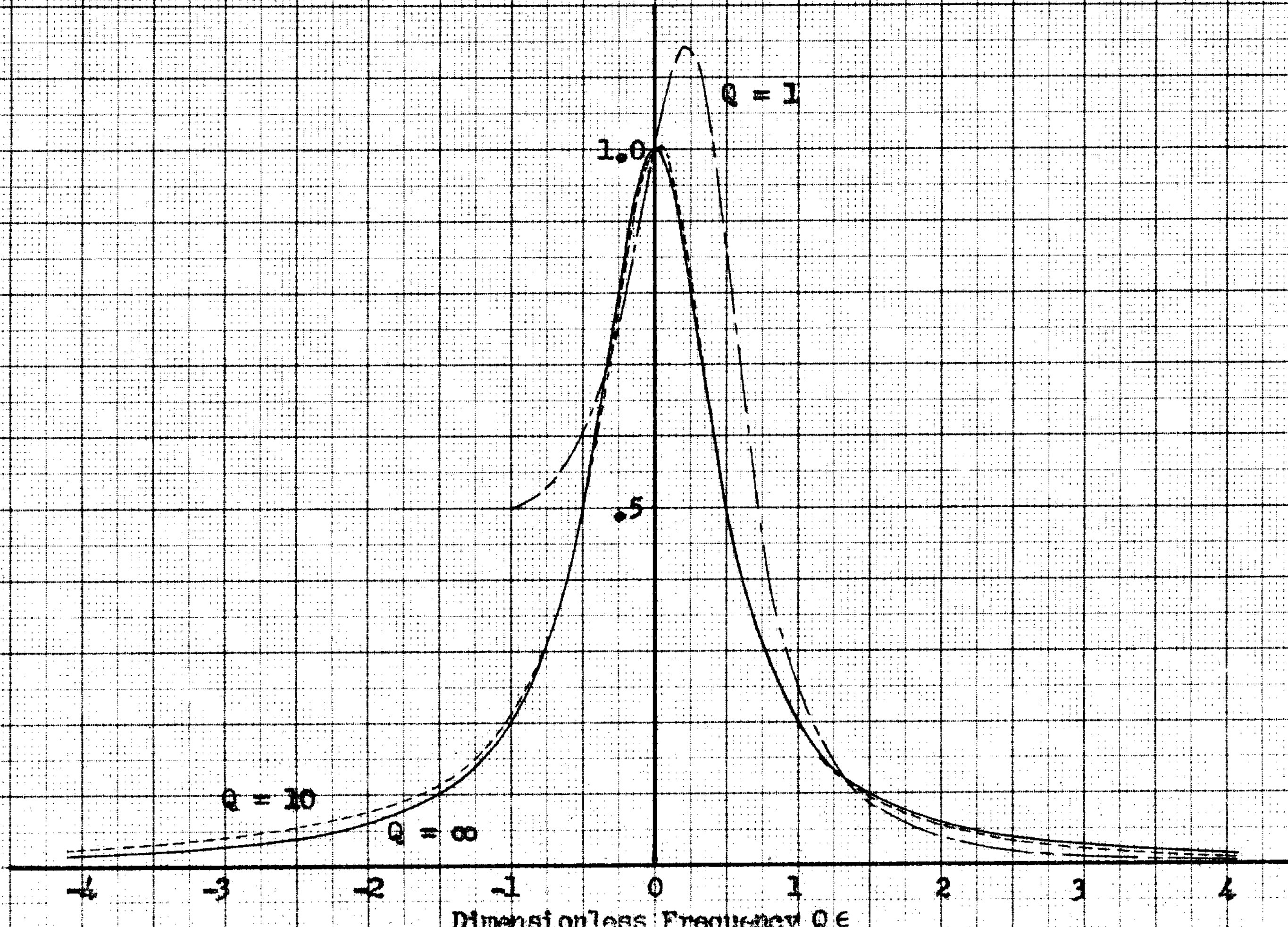


Figure 33. Universal Resonance Curves Showing Real Part of Input Impedance R/R_o for Single Degree of Freedom System with Series Damping. $\epsilon = \Delta\omega/\bar{\omega}$.

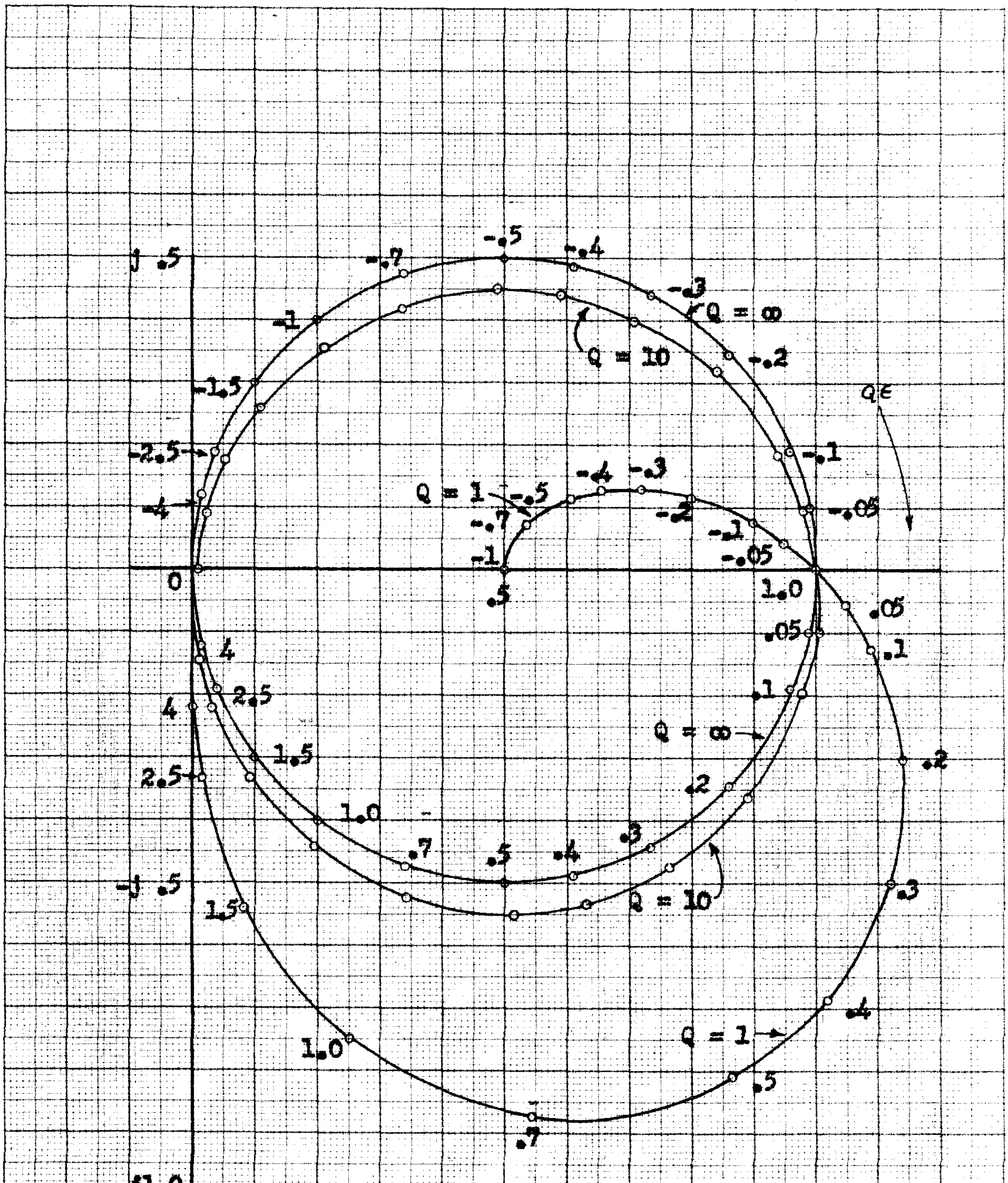


Figure 34. Impedance Loci Z/R_o for Single Degree of Freedom System with Series Damping.