HEAT TRANSFER IN COMPRESSIBLE LAMINAR BOUNDARY-LAYERS

Thesis by

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To

My Wife

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ABSTRACT

This report is concerned with the investigation of skin-friction and heat-transfer in the two-dimensional flow of a viscous compressible fluid.

The boundary-layer equations are first transformed by the Howarth-Stewartson transformation and then it is shown that for fluids of Prandtl Number unity, if the Chapman viscosity law be assumed to hold, then any boundary-layer problem with the free stream Mach Number different from zero can be formally reduced to a problem for which the free stream Mach Number is equal to zero.

The momentum method is then used to solve the boundary-layer equations in the Howarth-Stewartson form, for the case when the free stream Mach Number is zero. The basic equations developed are first used to solve the case of those specific pressure gradients which lead to "similarity flows". Other investigators have solved the exact equation for these flows on the differential analyser. The results obtained in this report, with the aid of very simple methods, agree to within a few percent with these more exact but laborious computations.

The use of the method for the case of arbitrary pressure gradients is then developed. Three ways of solving the resulting equations are discussed. In particular, an integral solution for the square of the momentum thickness, analogous to the one existing for incompressible fluids but with different exponents, is given. The application of the method is demonstrated by solving an illustrative example.

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LIST OF SYMBOLS.

(Eq. Nos. indicate equation which defines the symbol)

A	constant in $u_e = A x^{\ell}$ also in Eq. 5-33
$\left(\frac{1}{a}\right)$	heat-transfer, Eqs. 3-13 and 4-7c
a	also speed of sound
B	constant, Eq. 5-33
b	wall temperature function, Eq. 3-6a and 4-5c
C	constant of viscosity law, Eq. 2-7 also constant in Eq. 5-26
c_p	specific heat at constant pressure
f	function, Eqs. 3-1 and 4-2a
f_1, f_2, g_1, g_2, g	functions of λ and Δ, Eqs. 3-16
h	enthalpy
$\tilde{h}$	non-dimensional enthalpy Eqs. 3-5 and 4-2b
j	total enthalpy
K	heat transfer, Eq. 2-20b
k	coefficient of heat conduction, Eq. 2-20b
L, M, N, P, Q, R	coefficients, Eqs. 4-13
M	also Mach Number
P_R.	Prandtl Number
l, m, n, p, q, r	functions of λ and Δ, Eqs. 3-19
p	also pressure

ℓ	also exponent in $u_e = Ax^{\ell}$
T	absolute temperature
u, v	components of velocity on x-y plane
u^*, v^*	components of velocity on x-η plane
X, Y	coordinate axes, Eqs. 2-5
x, y	coordinate axes along and normal to body surface
α	skin friction, Eqs. 3-11 and 4-7a
α_i	(i = 0, 1, ...4), coefficients, Eq. 3-1
β	displacement thickness, Eq. 4-7b
β_i	(i = 0, 1, ...4) coefficients, Eq. 3-5
γ	ratio $\frac{c_p}{c_v}$
Δ	ratio of boundary-layer thicknesses, Eq. 3-10
δ_v, δ_T	velocity and temperature boundary-layer thickness respectively
δ^*	displacement thickness Eq. 3-15a
δ^{**}	momentum thickness Eq. 3-15b
ζ	ratio $\frac{\eta}{\delta_T}$
η	distorted y-coordinate, Eq. 2-12b
	similarity variable, Eq. 4-1
ϑ^*	"enthalpy thickness", Eq. 3-15c
ϑ^{**}	"enthalpy flux thickness", Eq. 3-15d
λ	defined by $-2\alpha_2$, Eq. 3-3
λ^*	pressure gradient parameter in similarity flow, Eq. 4-4
μ	coefficient of viscosity
ν	coefficient of kinematic viscosity

ξ ratio $\frac{\eta}{\delta_v}$

ρ density

τ skin friction, Eq. 2-20a

ψ stream function, Eqs. 2-6 and 4-2a

SUBSCRIPTS

e in "external" flow

m at velocity maximum point

o at stagnation point

s at separation point

w at wall

X, Y, x, y, η derivatives

∞ at upstream infinity

I. INTRODUCTION.

The techniques for solving the Prandtl boundary-layer equations for incompressible fluids are highly developed. Exact solutions for the velocity boundary-layer exist for zero pressure gradient and for certain other specific pressure gradients. The Karman-Pohlhausen method gives reasonably good estimates of the skin-friction for arbitrary favourable pressure gradients. The temperature boundary-layer equation has also been solved, exactly for the case of zero pressure gradient and by approximate methods for arbitrary pressure gradients. An account of these methods is given in Reference [1].

In the case of compressible fluids, however, a comparatively fewer number of solutions exist. Here, the problem is complicated by the existence of too many parameters like pressure gradient, Mach number and Prandtl number, and also by the fact that the velocity and temperature boundary-layer equations are now coupled and have to be solved simultaneously. Most of the existing solutions are limited in application because of restrictive assumptions placed on the parameter involved. Thus, either the pressure gradient is assumed equal to zero, or the Mach number is assumed to be negligible, or the Prandtl number is assumed to be equal to unity. All or any of these assumptions simplify the equations considerably, but limit the applicability of the solutions.

In this report the problem studied is that of heat-transfer and skin-friction in two dimensional compressible laminar boundary-layers. The usual Prandtl boundary-layer assumptions are made to derive the basic equations pertaining to the problem. These equations are then transformed by the Howarth transformation as modified by Stewartson (hereafter referred to as the Howarth-Stewartson transformation) and using the Chapman viscosity relation

$$\frac{\mu}{\mu_\infty} = C \frac{T}{T_\infty}$$

where, in the initial formulation of the problem, it is not necessary to assume C to be a constant. In general C will be a function of the enthalpy h i.e. $C = C(h)$. In this report C is assumed to be a constant; but it should be noted that the value of the constant is not restricted to unity. It can take on any constant value depending on where the above linear relation is matched to the actual $\mu - T$ curve.

These boundary-layer equations, in the Howarth-Stewartson form, with C = constant, exhibit an interesting feature. If the special forms of the equations for the two cases of $M = 0$, arbitrary P_R. and $P_R. = 1$, arbitrary M be compared, it is found that a formal correspondence exists between the special forms if the P_R. is assumed equal to unity in both cases. This correspondence then states that for $P_R. = 1$, any boundary-layer problem with $M \neq 0$ can, in principle, be formally reduced to a problem with $M = 0$, provided C be assumed to be constant. A special case of this correspondence is the correlation of flows at $M \neq 0$ and $P_R. = 1$ past insulated bodies with incompressible flow. These questions are discussed in Chapter II.

The present report is concerned only with flow at $M = 0$ and arbitrary $Pr.$. However, with the aid of the correlation just mentioned, the results obtained for the special case of $Pr. = 1$, may be applied to the case where $M \neq 0$. As mentioned earlier, it is always assumed that C = constant, although not necessarily unity.

In Chapter III, an extension of the Karman-Pohlhausen method is applied to the boundary-layer equation in the Howarth-Stewartson form. This gives us the two basic differential equations of the problem. Chapter IV deals with the application of this method to the Falkner-Skan similarity flows. The two ordinary differential equations derived in the previous chapter, in the case of similarity flows, reduce to two algebraic equations. These algebraic equations are then solved for various cases and numerical results obtained for the skin-friction, heat-transfer and the ratio of the two boundary-layer thicknesses. A few of the cases studied here have been previously treated by other investigators who solved the exact boundary-layer equations for the same similarity flow, on the differential analyser. The values of skin-friction and heat transfer obtained in this report, with the aid of very simple methods, agree to within a few percent with these more exact but laborious computations.

The case of arbitrary pressure gradients is dealt with in Chapter V. In this case the basic differential equations may be numerically integrated to obtain a solution. This is, in general, tedious and two approximate solutions to the equations are also suggested. One is a polynomial solution, which will give reasonably good results for cases when the wall temperature is not very different from the free stream temperature. The other involves rewriting the basic differential equations in a simpler form and then making certain linearizing approximations, similar to those made by Holstein and Bohlen for incompressible fluids. This yields a solution, for the momentum thickness in the form of an integral, the integrand being a function of the arbitrary pressure gradient raised to some power. The value of the exponent can be easily obtained for any given case. The distribution of skin-friction and heat-transfer along the wall can then be computed very rapidly and fairly accurately by using the integral solution. A hypothetical problem is then solved by each of the three methods and the numerical results compared to show their relative accuracies.

It is believed that the case of $M \neq 0$ and $Pr. \neq 1$ can also be treated by the methods employed in this report. However, no calculations have been carried out.

II. BASIC EQUATIONS.

2.1. The Boundary-Layer Equations.

The boundary-layer equations for the steady two-dimensional flow of a viscous compressible fluid are:

Continuity

$$(\rho u)_x + (\rho v)_y = 0 \tag{2-1}$$

Momentum

$$\rho\left(u u_x + v u_y\right) + p_x = 0 \tag{2-2a}$$

$$p_y = 0 \tag{2-2b}$$

Energy

$$\rho\left(u j_x + v j_y\right) = \left(\mu j_y\right)_y + \frac{1-P_R}{P_R}\left(\mu h_y\right)_y \tag{2-3}$$

Boundary Conditions

$$u = v = 0 \quad \text{at} \quad y = 0 \tag{2-4a}$$

$$u = u_e(x) \quad \text{at} \quad y = \infty \tag{2-4b}$$

$$h = j = h_w(x) \quad \text{at} \quad y = 0 \tag{2-4c}$$

$$h = h_e(x) \; ; \; j = j_e(x) \quad \text{at} \quad y = \infty \tag{2-4d}$$

where:

$$h = c_p T \quad = \text{enthalpy}$$

$$j = h + \frac{u^2}{2} \quad = \text{total enthalpy}$$

and the other notation is standard. The subscript "w" refers to conditions at the wall or body surface, and the subscript "e" to external conditions i.e. conditions at the outer edge of the boundary-layer. The only assumptions made in deriving the above equations are the usual Prandtl boundary-layer assumptions and that the curvature of the body surface is negligible.

2.2. Howarth-Stewartson Transformation.

We now introduce the Howarth-Stewartson transformation to distort the x and y co-ordinates (along and normal to the wall respectively) to some new co-ordinates X and Y. This transformation, as given in Reference [2] is:

$$X = \int_0^x \left(\frac{p_e}{p_\infty}\right)\left(\frac{a_e}{a_\infty}\right) dx = \int_0^x \left(\frac{a_e}{a_\infty}\right)^{\frac{3\gamma-1}{\gamma-1}} \tag{2-5a}$$

$$Y = \left(\frac{a_e}{a_\infty}\right) \eta = \left(\frac{a_e}{a_\infty}\right) \int_0^y \left(\frac{\rho}{\rho_\infty}\right) dy \tag{2-5b}$$

where the subscript " ∞ " refers to conditions at upstream infinity. The stream function ψ on the x-y plane is defined by:

$$u = \frac{\rho_\infty}{\rho} \left(\psi_y\right)_{x = \text{CONST.}} \tag{2-6a}$$

$$v = -\frac{\rho_\infty}{\rho} \left(\psi_x\right)_{y = \text{CONST.}} \tag{2-6b}$$

If the Chapman-Rubesin(3) viscosity law

$$\frac{\mu}{\mu_\infty} = C \frac{T}{T_\infty} \tag{2-7}$$

where $C = C(k)$, be assumed to hold, then by using the transformation of Eqs. 2-5 and the relation of Eqs. 2-6, the boundary layer equations, Eqs.2-1, 2-2 and 2-3, can be transformed to two partial differential equations in ψ and j on the $X-Y$ plane. These equations, together with the boundary conditions, are given below:

Momentum Equation

$$\Psi_Y \Psi_{XY} - \Psi_X \Psi_{YY} - \frac{j}{j_\infty}\left(a_\infty M_e\right)\frac{d(a_\infty M_e)}{dX} = \nu_\infty \left(C\Psi_{YY}\right)_Y \quad (2\text{-}8a)$$

Energy Equation

$$\Psi_Y j_X - \Psi_X j_Y = \frac{\nu_\infty}{P_R}\frac{\partial}{\partial Y}\left[C\left\{j_Y + (1-P_R)\left(\frac{a_e}{a_\infty}\right)^2 \frac{1}{2}\left(\Psi_Y\right)^2_Y\right\}\right] \quad (2\text{-}8b)$$

Boundary Conditions

$$\Psi = \left(\Psi_Y\right)_{X=\text{CONST.}} = 0 \quad \text{at} \quad Y = 0 \quad (2\text{-}9a)$$

$$\left(\Psi_Y\right)_{X=\text{CONST.}} = a_\infty M_e(X) \quad \text{at} \quad Y = \infty \quad (2\text{-}9b)$$

$$j = h_w(X) \quad \text{at} \quad Y = 0 \quad (2\text{-}9c)$$

$$j = j_e(X) \quad \text{at} \quad Y = \infty \quad (2\text{-}9d)$$

The boundary functions $u_e(X)$, $h_w(X)$ etc. are related in a simple manner to the corresponding functions $u_e(x)$, $h_w(x)$ etc. on the x-y plane. M_e is the Mach Number of the external flow and is given by $M_e = \frac{u_e}{a_e}$.

2.3. Special Forms of the Equations.

Some special forms of Eqs. 2-8 will now be discussed.

$P_R = 1$, arbitrary Mach Number

In this case, the momentum equation (Eq. 2-8a) remains unaltered, but the energy equation (Eq. 2-8b) simplifies since the dissipation term, $(1-P_R)\left(\frac{a_e}{a_\infty}\right)^2 \frac{1}{2}\left(\Psi_Y\right)^2_Y$, is now zero. The total dissipation, however, is not zero and is implicitly taken into account by using j instead of h. The resulting equations for this case are:

$$\Psi_Y \Psi_{XY} - \Psi_X \Psi_{YY} - \frac{j}{j_\infty}(a_\infty M_e)\frac{d(a_\infty M_e)}{dX} = \nu_\infty (C\Psi_{YY})_Y \quad (2\text{-}10a)$$

$$\Psi_Y j_X - \Psi_X j_Y = \nu_\infty (C j_Y)_Y \quad (2\text{-}10b)$$

<u>$M = 0$, arbitrary Prandtl Number</u>

If the Mach Number of the free stream is zero, then the effect of dissipation is negligible. This means that the explicit dissipation term in the energy equation, $(1 - P_R)\left(\frac{a_e}{a_\infty}\right)^2 \frac{1}{2}\left(\Psi_Y\right)^2_Y$, is again zero (as in the case for $P_R = 1$). This can be readily shown to be true by writing the equations in non-dimensional form and letting $M \to 0$.

Since we can write j as

$$\frac{j}{h_\infty} = \frac{h}{h_\infty} + \frac{u^2}{2h_\infty} = \frac{h}{h_\infty} + \frac{\gamma - 1}{2}\left(\frac{u}{U}\right)^2 M^2$$

where $u(-\infty, y) = U$ and $M = \frac{U}{a_\infty}$ is the Mach Number of the free stream, it follows then that for $M = 0$, j may everywhere be replaced by h. This is also a consequence of neglecting dissipation, since dissipation is considered implicitly when j is used instead of h in the equations.

Further, as $M \to 0$, both a_e and $a_\infty \to \infty$, but the ratio $\frac{a_e}{a_\infty} \to 1$. Hence $(a_\infty M_e)$ should be replaced by u_e.

Finally, as $M \to 0$, the transformation relations of Eqs. 2-5 also simplify. Since $\left(\frac{a_e}{a_\infty}\right) = 1$, we see from Eq. 2-5a, that the x-coordinate does not distort and $X = x$. The y-co-ordinate, however, does distort and Eq. 2-5b gives $Y = \eta$ where $\eta = \int_0^y \left(\frac{\rho}{\rho_\infty}\right) dy$

Using the simplification given above Eqs. 2-8 for this case reduce to:

$$\Psi_\eta \Psi_{\eta x} - \Psi_x \Psi_{\eta\eta} - \frac{h}{h_\infty} u_e \frac{du_e}{dx} = \nu_\infty (C \Psi_{\eta\eta})_\eta \quad (2\text{-}11a)$$

$$\Psi_\eta h_x - \Psi_x h_\eta = \frac{\nu_\infty}{Pr.} (C h_\eta)_\eta \quad (2\text{-}11b)$$

2.4. Correlation between flows for $M \neq 0$ and $M = 0$

A comparison of Eqs. 2-10 and Eqs. 2-11 brings out the striking fact that if it be assumed that $Pr. = 1$ and C = constant, then the boundary-layer equations for $M \neq 0$ and for $M = 0$ are formally identical, with the following correspondence:

$M \neq 0$	$Pr. = 1$; C = const.	$M = 0$
X	corresponds to	x
Y	" "	η
j	" "	h
$(a_\infty M_e)$	" "	u_e

Thus, for $Pr. = 1$, any boundary-layer problem with $M \neq 0$ may, in principle, be formally reduced to a problem with $M = 0$ provided C be assumed to be constant.

It should be noted that the correspondence emphasizedby Stewartson[2] between the compressible and incompressible boundary-layer equations, is only a special case of the more general correspondence shown above. He considered the case of no heat transfer, and for that case the condition $M = 0$ implies incompressible flow. Thus, in the case of $Pr. = 1 \cdot 0$, and no heat transfer, the compressible boundary-layer equations (i.e. for $M \neq 0$) can be reduced to the incompressible boundary-layer equations (i.e. $M = 0$). In the correspondence shown above, the condition of no heat transfer is not required.

In this report only those problems will be considered for which $M = 0$. No restrictions will be placed on either the $Pr.$ No. or the pressure gradient in the external flow. The method outlined herein will thus enable us to get the skin-friction and heat-transfer for all problems where the $M = 0$ and also, by virtueof the correspondence shown above, to those problems with $M \neq 0$ for which the $Pr. = 1$. Problems for which $M \neq 0$ and $Pr. \neq 1$ are not dealt with in this report, but it is felt that the method given, with suitable modifications, will apply to these cases also.

2.5. Equations for $M = 0$ and resulting mapping.

In section 2.4 we have seen that for the special case when the Howarth-Stewartson transformation relations of Eqs. 2-5 reduce to the following equations:

$$X = x \tag{2-12a}$$

$$Y = \eta = \int_0^y \left(\frac{\rho}{\rho_\infty}\right) dy \tag{2-12b}$$

and the boundary-layer equations, in terms of ψ and h, for C = constant, are given by

$$\psi_\eta \psi_{\eta x} - \psi_x \psi_{\eta\eta} - \frac{h}{h_\infty} u_e \frac{du_e}{dx} = \nu_\infty' \psi_{\eta\eta\eta} \tag{2-13a}$$

$$\psi_\eta h_x - \psi_x h_\eta = \frac{\nu_\infty'}{P_R.} h_{\eta\eta} \tag{2-13b}$$

where $\nu_\infty' = C\nu_\infty$, is a sort of effective viscosity. For $C = 1$, obviously $\nu_\infty' = \nu_\infty$. In all subsequent work C will be assumed equal to unity, although it should be noted that if $C \neq 1$, one would simply have to replace ν_∞ by ν_∞' to take this into account. The constant C of the Chapman linear viscosity law can be used to match the viscosity with Sutherland value at any one desired point. The two curves, however, will not be tangential at this point. The value $C = 1$ corresponds to matching the values at upstream infinity. If, however, the curves are matched at the wall, so that we get a more correct value of viscosity near the wall where the viscous stresses are largest, then the value of the constant C will be given by

$$C = \left(\frac{T_w}{T_\infty}\right)^{1/2} \left(\frac{T_\infty + S}{T_w + S}\right) \tag{2-14}$$

where S is the Sutherland Constant corresponding to the temperature T_∞. It should be noted, however, that since C is assumed to be a constant, Eq. 2-14 will be valid only if the wall temperature T_w is constant, or alternatively, when T_w is varying some average value of T_w is used.

The boundary layer equations (Eqs. 2-13) for $M = 0$ and $C =$ const. $= 1$, can be written in terms of velocities instead of the stream function ψ. On the $x-y$ plane, the components of velocity u and

v are given by Eqs. 2-6. On the $x-\eta$ plane, the Howarth-Stewartson plane for $M=0$, the components of velocity are given by u^* and v^* where

$$u^* = (\psi_\eta)_{x = \text{CONST.}} = u \qquad (2\text{-}15a)$$

$$v^* = -(\psi_x)_{\eta = \text{CONST.}} = \frac{\rho}{\rho_\infty}(v - y_x u) \qquad (2\text{-}15b)$$

Using these relations, Eqs. 2-13 become

Continuity

$$u_x + v^*_\eta = 0 \qquad (2\text{-}16a)$$

Momentum

$$u u_x + v^* u_\eta - \frac{h}{h_\infty} u_e \frac{du_e}{dx} = \nu_\infty u_{\eta\eta} \qquad (2\text{-}16b)$$

Energy

$$u h_x + v^* h_\eta = \frac{\nu_\infty}{P_R} h_{\eta\eta} \qquad (2\text{-}16c)$$

Boundary conditions

$$u = v^* = 0 \quad \text{at} \quad \eta = 0 \qquad (2\text{-}17a)$$

$$u = u_e(x) \quad \text{at} \quad \eta = \infty \qquad (2\text{-}17b)$$

$$h = h_w(x) \quad \text{at} \quad \eta = 0 \qquad (2\text{-}17c)$$

$$h = h_e = h_\infty = \text{CONST.} \quad \text{at} \quad \eta = \infty \qquad (2\text{-}17d)$$

Boundary condition, Eq. 2-17d, states that since $M = 0$, in the external non-viscous part of the flow, the temperature, and hence the enthalpy is constant (assuming, of course, that c_p does not vary).

2.6. Integral form of the Equations.

The boundary layer equations, Eqs. 2-16 b and c, are two simultaneous partial differential equations. From these equations one may, however, derive two simultaneous ordinary differential equations by integrating them with respect to η. If the exact velocity and temperature boundary layer profiles were known, these would satisfy the integrated equations exactly. One method of using these integral relations for finding approximate solutions is the following. One assumes that in their dependence on η the functions describing the velocity and the temperature can be approximated to by simple functions, say polynomials, which depend only on a finite number of unknown parameters which are functions of x. These parameters may then be determined from certain boundary conditions and matching conditions and from integral relations. This is the Karman-Pohlhausen method. It is known that, at least for incompressible fluids, this method gives good results in regions of favourable pressure gradient.

We first define four thickness parameters on the $x-\eta$ plane. These are

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_e}\right) d\eta \tag{2-18a}$$

$$\delta^{**} = \int_0^\infty \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) d\eta \tag{2-18b}$$

$$\vartheta^{*} = \int_{0}^{\infty}\left(1-\frac{h}{h_e}\right)d\eta \qquad (2\text{-}18c)$$

$$\vartheta^{**} = \int_{0}^{\infty}\frac{u}{u_e}\left(1-\frac{h}{h_e}\right)d\eta \qquad (2\text{-}18d)$$

where, since $M = 0$, in the external flow $h_e = h_\infty$ and $\rho_e = \rho_\infty$.

These thickness parameters have clear physical interpretations. Thus δ^{*} represents the displacement thickness and δ^{**} the momentum thickness on the $x-\eta$ plane. If we use the transformation relation of Eq. 2-12b together with the fact that $\rho_e = \rho_\infty$, we see that δ^{**}, although defined on the $x-\eta$ plane, also represents the value of the momentum thickness on the physical $x-y$ plane. This is, however, not true of δ^{*}. ϑ^{*} may be called an "enthalpy thickness" on the $x-\eta$ plane and ϑ^{**} is an "enthalpy-flux thickness". Like δ^{**}, ϑ^{**} has the same physical interpretation on the $x-y$ plane as it does on the $x-\eta$ plane.

If Eq. 2-16a be used to eliminate v^{*} from Eqs. 2-16b and c, and then the equations be integrated with respect to η from $\eta = 0$ to $\eta = \infty$, then in terms of the thickness parameter defined above, the integrated equations are: :

Integrated Momentum Equation

$$\frac{d(u_e^2\delta^{**})}{dx} + u_e\frac{du_e}{dx}\left(\delta^{*}-\vartheta^{*}\right) = \frac{\tau_w}{\rho_\infty} \qquad (2\text{-}19a)$$

Integrated Energy Equation

$$\frac{d(u_e\vartheta^{**})}{dx} = -\frac{K_w}{\rho_\infty h_\infty} \qquad (2\text{-}19b)$$

where the skin-friction at the wall is:

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = c\mu_\infty\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0} \qquad (2\text{-}20a)$$

and the heat-transfer at the wall is:

$$K_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{c\mu_\infty}{P_R}\left(\frac{\partial h}{\partial \eta}\right)_{\eta=0} \qquad (2\text{-}20b)$$

In obtaining Eqs. 2-19, the Prandtl Number has been assumed to be constant.

For incompressible flows (i.e. ρ = const. and h = const.) $\vartheta^* = \vartheta^{**} = 0$. Hence, for such flows, Eqs. 2-19a and b reduce to the single equation

$$\frac{d(u_e^2\delta^{**})}{dx} + u_e\frac{du_e}{dx}\delta^* = \frac{\tau_w}{\rho_\infty}$$

which is well known. For compressible flows, ϑ^* and ϑ^{**} are not zero and it is these parameters that show the coupling that exists between the momentum and energy equations.

Note: It should be pointed out the integrated equations given in Eqs. 2-19 are very similar to those published by Cohen and Reshotko in NACA TN 3326, very recently in April, 1955. The work for this report was done independently and completed before the publication of the above report. The work of Cohen and Reshotko follows this report rather closely till this stage, but thereafter the two are very different. They do not solve the resulting differential equations but extend the use of Thwaites' technique for incompressible boundary-layers to this compressible case, and attempt to find some semi-empirical relations based on existing exact solutions of the problem. In this report, however, an attempt is made to solve the resulting equations.

III. THE MOMENTUM METHOD OF SOLUTION.

3.1. Introduction.

The momentum method was first used by Karman and Pohlhausen[4] to estimate the skin-friction in incompressible boundary-layers. For compressible heat conducting fluids we have a temperature boundary layer in addition to the velocity boundary layer. The approximations used for the two boundary layers will be the same. We shall assume that the approximating functions satisfy the integrated energy equation as well as the integrated momentum equation.

In this report, quartics are used to represent the η -dependence of the distribution of velocity and temperature in the boundary-layer.

It will be assumed that the velocity reaches its free-stream values at a finite distance from the wall. This distance will depend on x and be denoted by $\delta_V(x)$. $\delta_V(x)$ will be called the thickness of the velocity boundary layer, $\eta = \delta_V(x)$ is then the edge of the velocity boundary layer. Similarly we assume that the temperature boundary layer has a finite thickness $\delta_T(x)$. $\delta_V(x)$ and $\delta_T(x)$ will have to be determined in the course of solving the problem. In general they will not be equal.

As approximating functions we assume that for each value of x , $\frac{u}{u_e}$ and $\frac{h}{h_e}$ are quartic polynomials in $\frac{\eta}{\delta_V}$ and $\frac{\eta}{\delta_T}$ respectively. The five coefficients $\alpha_i(x)$ and $\beta_i(x)$ of each of these polynomials are then functions of x . Thus altogether we have to determine twelve unknown functions of x , namely

$$\delta_v(x) \quad ; \quad \delta_T(x) \quad ; \quad \alpha_i(x) \quad , \quad \beta_i(x) \qquad (i = 0, 1, \ldots 4)$$

Twelve conditions must then be imposed on these functions. First we require that the velocity boundary layer joins the free stream so smoothly that at $\eta = \delta_v(x)$, $\frac{u}{u_e}$ as well as its first two derivatives are continuous. This means that $\frac{u}{u_e} = 1$ and its two first derivatives are zero at $\eta = \delta_v(x)$. Analogous conditions are imposed on $\frac{h}{h_e}$ at $\eta = \delta_T(x)$. This gives altogether six conditions. Furthermore, we require that at $\eta = 0$ the polynomials give correctly the value of the functions and their first two derivatives. Since the first derivatives are proportional to the unknown τ_w and K_w resp., this actually represents only four additional conditions. The missing two conditions are then supplied by the requirement that the approximating functions satisfy the integrated momentum and energy equations.

The details of the method described above will now be given.

3.2. Velocity Boundary-Layer.

Letting $\xi = \frac{\eta}{\delta_v}$, we assume a quartic in ξ to represent the shape of the velocity boundary-layer.

Thus, we assume:

$$\frac{u}{u_e} = f'(\eta) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3 + \alpha_4 \xi^4 \qquad (3\text{-}1)$$

The first matching condition at the wall and the three matching concitions at the edge of the boundary layer are then

$$f'(0) = 0 \qquad (3\text{-}2a)$$

$$f'(\delta_v) = 1 \qquad (3\text{-}2b)$$

$$f''(\delta_v) = 0 \qquad (3\text{-}2c)$$

$$f'''(\delta_v) = 0 \qquad (3\text{-}2d)$$

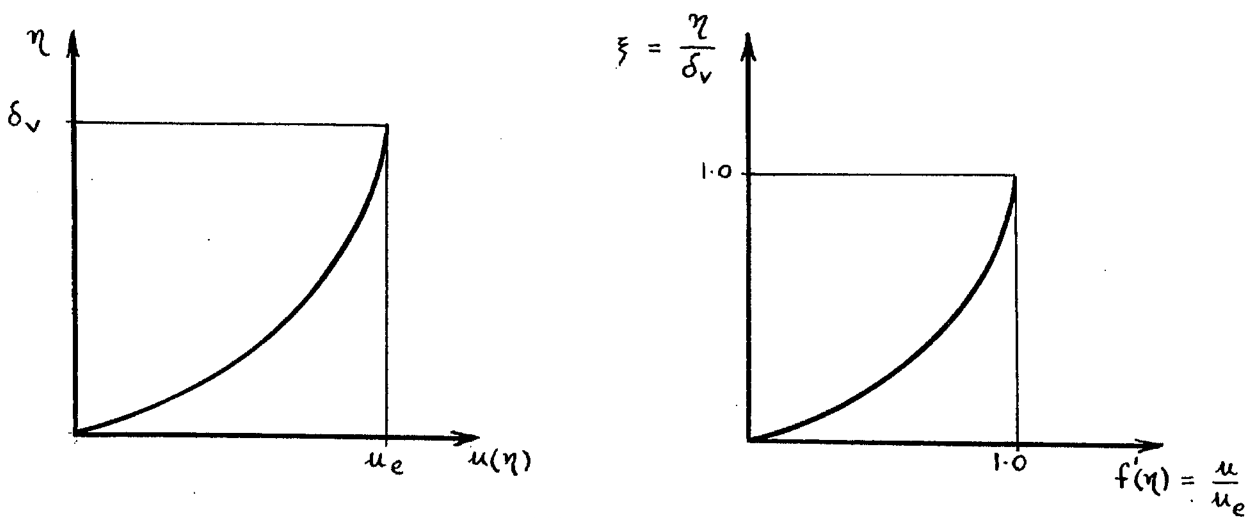

Fig. 1.

These conditions enable us to find all α_i in terms of one of them, say α_2. Carrying this out one finds

$$\frac{u}{u_e} = f'(\eta) = \frac{\lambda+12}{6}\xi - \frac{\lambda}{2}\xi^2 + \frac{\lambda-4}{2}\xi^3 - \frac{\lambda-6}{6}\xi^4 \qquad (3\text{-}3)$$

where a new symbol $\lambda(x) = -2\alpha_2$ has been introduced.

In Eq. 3-3, η is the independent variable. There are also, as yet, two independent parameters λ and δ_v. A relation between them is found by satisfying the equation of motion (Eq. 2-16b) exactly at the wall (i.e. at $\eta = 0$). Since the second η-derivative of u at the wall is determined from the momentum equation, this is simply the matching condition that the quartic of Eq. 3-1 has the correct second derivative at the wall. A simple calculation gives

$$\delta_v^{\,2} = \frac{\nu_\infty}{1+b} \frac{\lambda}{\left(\frac{du_e}{dx}\right)} \tag{3-4}$$

Here $\tilde{h}_w(x) = \frac{h_w(x)}{h_e} = 1 + b(x)$ is the boundary condition on $h(x, \eta)$ at $\eta = 0$. The relation of Eq. 3-4 now reduces Eq. 3-3 to a one parameter family of velocity boundary-layer profiles. Here λ is chosen as that independent parameter. In general, as the shape of the profile changes from point to point along the wall, the value of the parameter λ will also change. Hence, in general, $\lambda = \lambda(x)$

3.3. Temperature Boundary-Layer.

Letting $\zeta = \frac{\eta}{\delta_T}$, we assume a quartic in ζ to represent the shape of the temperature boundary-layer.

Thus, we assume:

$$\frac{h}{h_e} = \tilde{h}(\eta) = \beta_0 + \beta_1 \zeta + \beta_2 \zeta^2 + \beta_3 \zeta^3 + \beta_4 \zeta^4 \tag{3-5}$$

with the following four matching conditions corresponding to Eqs. 3-2.

$$\tilde{h}(0) = 1 + b(x) \tag{3-6a}$$

$$\tilde{h}(\delta_T) = 1 \tag{3-6b}$$

$$\tilde{h}'(\delta_T) = 0 \tag{3-6c}$$

$$\tilde{h}''(\delta_T) = 0 \tag{3-6d}$$

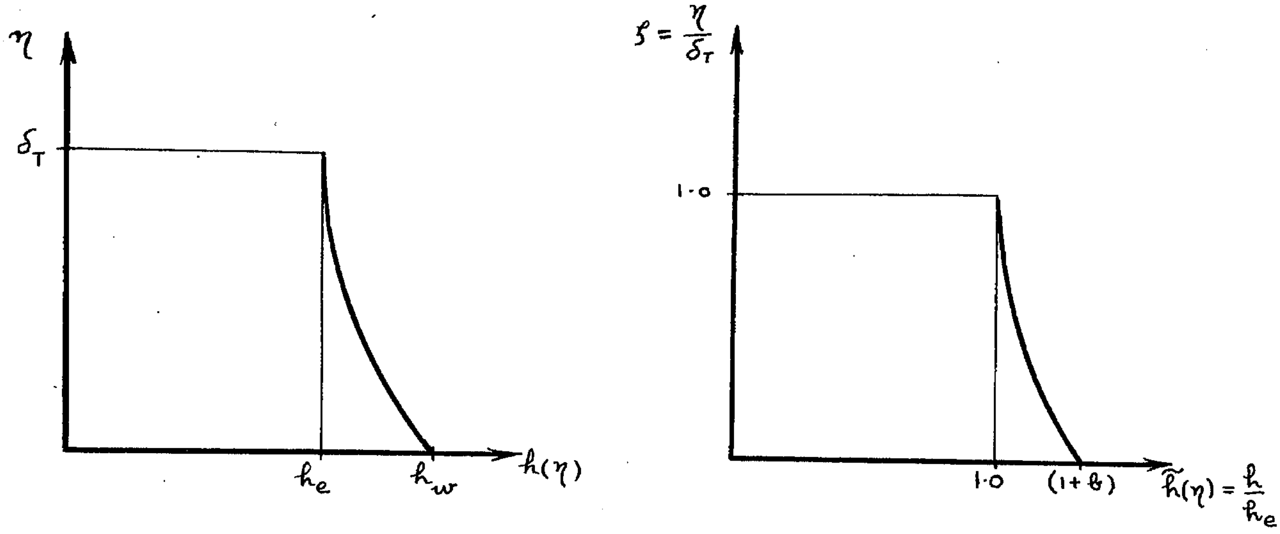

Fig. 2.

It should be noted that the parameter "b" defines the wall temperature, and in general will not be constant. Thus $b = b(x)$, will be a known function of x depending on the given distribution of wall temperature. If, however, the wall temperature is constant then b will be a constant also. The value $b = 0$ will correspond to the case when the temperature at the wall is equal to the temperature at the outer edge of the boundary layer. Since $M = 0$, $b = 0$ will imply incompressible flow. The value $b = -1$ will correspond to the theoretical limiting case when the wall temperature is equal to 0° on the absolute scale.

Using Eqs.3-6 in Eq. 3-5 gives

$$\frac{h}{h_e} = \tilde{h}(\eta) = (1+b) - \left(\frac{6b+\beta_2}{3}\right)\zeta + \beta_2\zeta^2 + (2b-\beta_2)\zeta^3 - \left(\frac{3b-\beta_2}{3}\right)\zeta^4 \qquad (3\text{-}7)$$

In Eq. 3-7, we again have η as the independent variable and two independent parameters β_2 and δ_T . As in the case of the velocity boundary-layer, we can eliminate one of these parameters by satisfying the equation of energy (Eq. 2-16c) exactly at $\eta = 0$. This gives

$$\beta_2 = 0 \tag{3-8}$$

and reduces Eq. 3-7 to a one parameter family of temperature boundary-layer profiles, given by

$$\frac{h}{h_e} = \tilde{h}(\eta) = (1 + b) - 2b\xi + 2b\xi^3 - b\xi^4 \tag{3-9}$$

This independent parameter could be chosen as δ_T. However, in this report a new parameter Δ is chosen. Δ is defined by the relation

$$\Delta = \frac{\delta_v}{\delta_T} \tag{3-10}$$

and represents the ratio of the velocity to the temperature boundary-layer thicknesses. This parameter is of greater physical significance than either δ_v or δ_T whose definitions are rather arbitrary.

3.4. Skin Friction and Heat-Transfer.

We have just seen that Eq. 3-3 together with Eq. 3-4 defines a one parameter family of velocity boundary-layer profiles, and Eq. 3-9 defines a one parameter family of temperature boundary-layer profiles. We can now use these definitions in Eqs. 2-20, and define the skin friction and heat-transfer in terms of these parameters.

If we define α by

$$\alpha = f''(0) = \frac{\lambda + 12}{6\delta_v} \tag{3-11}$$

then the skin friction τ_w is given by

$$\tau_w = \mu_\infty u_e \alpha \qquad (3\text{-}12)$$

If we define $\left(\frac{1}{a}\right)$ by

$$\frac{1}{a} = -\frac{1}{b}\tilde{h}'(0) = \frac{2}{\delta_T} \qquad (3\text{-}13)$$

then the heat-transfer K_w is given by

$$K_w = \frac{\mu_\infty h_e b}{P_R.} \cdot \left(\frac{1}{a}\right) \qquad (3\text{-}14)$$

3. 5. Evaluation of thickness-parameters.

The four thickness parameters defined in Eqs. 2-18 can now be evaluated by using the boundary-layer profiles of Eqs. 3-3 and 3-9. The upper limits of integration in the definitions of Eqs. 2-18 will have to be replaced by the appropriate finite limit δ_V or δ_T. The evaluation of these integrals yields:

$$\delta^* = \int_0^{\delta_V} \left(1 - \frac{u}{u_e}\right) d\eta = \delta_V f_1(\lambda) \qquad (3\text{-}15a)$$

$$\delta^{**} = \int_0^{\delta_V} \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) d\eta = \delta_V f_2(\lambda) \qquad (3\text{-}15b)$$

$$\vartheta^* = \int_0^{\delta_T} \left(1 - \frac{h}{h_e}\right) d\eta = \delta_T\left(-\frac{3}{10} b\right) \qquad (3\text{-}15c)$$

$$\vartheta^{**} = \int_0^{\delta_T} \frac{u}{u_e}\left(1 - \frac{h}{h_e}\right) d\eta = \delta_T\, b\, g(\lambda, \Delta) \qquad (3\text{-}15d)$$

where the functions are defined by

$$f_1(\lambda) = \frac{36-\lambda}{120} \tag{3-16a}$$

$$f_2(\lambda) = -\frac{1}{45,360}\left[5\lambda^2 + 48\lambda - 5,328\right] \tag{3-16b}$$

$$g(\lambda,\Delta) = -\frac{1}{15,120\,\Delta^4}\left[\lambda g_1(\Delta) + g_2(\Delta)\right] \tag{3-16c}$$

$$g_1(\Delta) = \left(168\,\Delta^3 - 180\Delta^2 + 81\,\Delta - 14\right) \tag{3-16d}$$

$$g_2(\Delta) = \left(2,016\,\Delta^3 - 324\,\Delta + 84\right) \tag{3-16e}$$

3.6. The differential equations for $\lambda(x)$ and $\Delta(x)$.

The values of δ^*, δ^{**}, ϑ^* and ϑ^{**}, as given by Eqs. 3-15 and 3-16, can now be substituted in the integrated equations of motion and energy (Eqs.2-19). The values of τ_w and K_w are given by (cf. Eqs. 3-11, 3-12, 3-13 and 3-14)

$$\tau_w = \frac{\mu_\infty u_e (\lambda + 12)}{6\,\delta_v} \tag{3-17a}$$

$$K_w = \frac{2\mu_\infty h_e\, b\,\Delta}{P_R.\ \delta_v} \tag{3-17b}$$

and Eq. 3-4 gives the value of δ_v . The resulting differential equation for $\lambda(x)$ and $\Delta(x)$ are of the form

$$\ell(\lambda) \frac{d\lambda}{dx} + m(\lambda, \Delta) = 0 \tag{3-18a}$$

$$p(\lambda, \Delta) \frac{d\lambda}{dx} + q(\lambda, \Delta) \frac{d\Delta}{dx} + r(\lambda, \Delta) = 0 \tag{3-18b}$$

The functions ℓ , m , p , q and r will, in general, also depend on the external velocity $u_e(x)$ and its derivatives and the wall temperature function $b(x)$ and its derivatives. These will, however, be known for any given problem.

For the particular case of $b(x)$ = const., i.e. a constant wall temperature, these functions are defined by Eqs. 3-19 below.

$$\ell(\lambda) = \left[2\lambda f_2' + f_2\right] \tag{3-19a}$$

$$m(\lambda,\Delta) = \left[\lambda f_1 + \frac{3}{10} b \frac{\lambda}{\Delta} - \frac{(\lambda+12)(1+b)}{6}\right] - \frac{\lambda}{2} f_2 \left[\frac{u_e'' u_e}{(u_e')^2} - 4\right] \tag{3-19b}$$

$$p(\lambda,\Delta) = -\frac{u_e}{2u_e'}\left[3\lambda g_1 + g_2\right] \tag{3-19c}$$

$$q(\lambda,\Delta) = \frac{u_e}{u_e'} \frac{\lambda}{\Delta}\left[(5g_1 - \Delta g_1')\lambda + (5g_2 - \Delta g_2')\right] \tag{3-19d}$$

$$r(\lambda,\Delta) = \frac{\lambda}{2}\left[\lambda g_1 + g_2\right]\left[\frac{u_e'' u_e}{(u_e')^2} - 2\right] + \frac{30{,}240(1+b)\Delta^6}{P_R} \tag{3-19e}$$

where:

$$u_e' = \frac{du_e}{dx} \; ; \; u_e'' = \frac{d^2u_e}{dx^2} \; ; \; f_1' = \frac{df_1}{d\lambda} \; ; \; g_1' = \frac{dg_1}{d\Delta} \qquad \text{and so on.}$$

For the case of varying wall temperatures, when $b(x) \neq$ const., Eqs. 3-18 will still be valid, although the functions l , m , p , q and r will be different from those given by Eqs. 3-19 and will have to be worked out for each particular distribution of $b(x)$. For the sake of simplicity in illustrating the method, the value of $b(x)$ will be assumed to be a constant from now on.

IV. SIMILARITY SOLUTIONS (FOR $M = 0$)

4.1. Basic Equations.

In some problems, with specific pressure gradients, a "similarity variable" may be formed by a suitable combination of the independent variables (x and η), such that the partial differential equations of Eqs. 2-11 can be reduced to ordinary differential equations.

Classic examples of such similarity flows are the "wedge flows" of Falkner and Skan in which the external velocity u_e is given by $u_e = A x^{\ell}$, (A is a constant). We shall consider the case when the wall temperature is constant (i.e. b = const.) and C = const. In this case we introduce a similarity variable θ , defined by

$$\theta = \sqrt{\frac{(\ell+1) A}{2 C \nu_\infty}} \; x^{\frac{\ell-1}{2}} \eta \tag{4-1}$$

and assume that ψ and $\tilde{h}$ have the following form

$$\psi = \sqrt{\frac{2 A C \nu_\infty}{(\ell+1)}} \; x^{\frac{\ell+1}{2}} f(\theta) \tag{4-2a}$$

$$\tilde{h} = \tilde{h}(\theta) \tag{4-2b}$$

Inserting these values into Eqs. 2-11 one finds that they reduce to a pair of ordinary differential equations for $f(\theta)$ and $\tilde{h}(\theta)$

$$f''' + f f'' + 2\lambda^* \left[\tilde{h} + (f')^2 \right] = 0 \tag{4-3a}$$

$$\tilde{h}'' + P_R . f \tilde{h}' = 0 \tag{4-3b}$$

In the above equations, primes denote differentiation with respect to θ , and

$$\lambda^* = \frac{\ell}{\ell + 1} \tag{4-4}$$

The boundary conditions are:

$$f(0) = f'(0) = 0 \tag{4-5a}$$

$$f'(\infty) = 1 \tag{4-5b}$$

$$\tilde{h}(0) = \tilde{h}_w = 1 + b \tag{4-5c}$$

$$\tilde{h}(\infty) = 1 \tag{4-5d}$$

4.2. The Momentum Method

From the relation of Eq. 4-1 we see that if for any given x, η varies from 0 to ∞, then θ also varies from 0 to ∞. So if the momentum method of solution is to be used the integrals of Eq. 4-3 with respect to θ from $\theta = 0$ to $\theta = \infty$ will have to be satisfied. These integrals are:

$$(1 + 2\lambda^*) \int_0^\infty f f'' d\theta + 2\lambda^* \int_0^\infty (\tilde{h} - 1)\, d\theta = \alpha - 2\lambda^* \beta \tag{4-6a}$$

$$\int_0^\infty f \tilde{h}'\, d\theta = -\frac{b}{a\, P_R} \tag{4-6b}$$

where:

$$\alpha = f''(0) \quad \text{(proportional to skin-friction)} \tag{4-7a}$$

$$\beta = \int_0^\infty (1 - f')\, d\theta \quad \text{(proportional to displacement-thickness)} \tag{4-7b}$$

$$\frac{1}{a} = -\frac{1}{b}\tilde{h}'(0) \quad \text{(proportional to heat-transfer)} \qquad (4\text{-}7c)$$

The integrals involved in the above equations can be evaluated for this special case of similarity flows in the same way it was done in Section 3.5 for the more general case.

Let $\theta = \delta_V$ and $\theta = \delta_T$ represent the edges of the two boundary layers. Note that δ_V and δ_T are now constants. Furthermore the meaning of these symbols is not the same as in the preceding Chapter. Let $(\delta_V)_3$ and $(\delta_V)_4$ be the two parameters as defined in Chapters III and IV respectively. Putting $\eta = (\delta_V)_3$ and $\theta = (\delta_V)_4$ in Eq. 4-1 then shows that

$$(\delta_V)_3 = \sqrt{\frac{2c\,\nu_\infty}{(\ell+1)A}}\; x^{\frac{1-\ell}{2}}\,(\delta_V)_4$$

A similar equation applies to δ_T. On the other hand, if the ratio $\Delta = \frac{\delta_V}{\delta_T}$ is formed the similarity factors will cancel, so that the definition of Δ in the present Chapter agrees with that of the other Chapters.

We assume polynomials of the fourth degree in $\frac{\theta}{\delta_V}$ for the function $f'(\theta)$, which represents the velocity profile, and in $\frac{\theta}{\delta_T}$ for the function $\tilde{h}(\theta)$, which represents the temperature profile. These profiles are then made to match four boundary conditions similar to those of Eqs. 3-2 and 3-6, and are finally reduced to a one parameter family by satisfying Eqs. 4-3 exactly at the wall (i.e. $\theta = 0$). The resulting profiles are:

$$\frac{u}{u_e} = f'(\theta) = \frac{\lambda+12}{6}\left(\frac{\theta}{\delta_V}\right) - \frac{\lambda}{2}\left(\frac{\theta}{\delta_V}\right)^2 + \frac{\lambda-4}{2}\left(\frac{\theta}{\delta_V}\right)^3 - \frac{\lambda-6}{6}\left(\frac{\theta}{\delta_V}\right)^4 \qquad (4\text{-}8a)$$

$$\frac{h}{h_e} = \tilde{h}(\theta) = (1+b) - 2b\left(\frac{\theta}{\delta_T}\right) + 2b\left(\frac{\theta}{\delta_T}\right)^3 - b\left(\frac{\theta}{\delta_T}\right)^4 \qquad (4\text{-}8b)$$

where

$$\delta_v^{\ 2} = \frac{\lambda}{2\lambda^*(1+b)} \tag{4-9}$$

The skin-friction and heat-transfer are given by

$$\frac{\tau_w}{\rho_\infty u_e^2} = \sqrt{\frac{\ell+1}{2}}\sqrt{\frac{c\,\nu_\infty}{x u_e}}\;\alpha \tag{4-10a}$$

$$\frac{K_w}{\rho_\infty u_e h_e} = \sqrt{\frac{\ell+1}{2}}\sqrt{\frac{c\,\nu_\infty}{x u_e}}\;\frac{b}{a\,P_R} \tag{4-10b}$$

where, as before,

$$\alpha = \frac{\lambda + 12}{6\,\delta_v} \tag{4-11a}$$

$$\frac{1}{a} = \frac{2\Delta}{\delta_v} \tag{4-11b}$$

We can now evaluate the integrals involved in Eqs. 4-6 by changing the upper limit of integration from "∞" to the appropriate finite limit δ_v or δ_T , and using the expressions for $f'(\theta)$ and $\tilde{h}(\theta)$ as given by Eqs. 4-8. In this manner Eqs. 4-6 can be reduced to two simultaneous algebraic equations in λ and Δ , where Δ , as before, is defined by $\Delta = \frac{\delta_v}{\delta_T}$. The resulting equations are of the form:

$$\lambda^3 + P\lambda^2 + Q\lambda + R = 0 \tag{4-12a}$$

$$L\lambda^2 + M\lambda + N = 0 \tag{4-12b}$$

where the coefficients P, Q, R and L, M, N are defined below:

$$P = 9.60 + 151.2\left(\frac{\lambda^*}{1+2\lambda^*}\right) \tag{4-13a}$$

$$Q = \left[3{,}024\, b - 2419.2 - \frac{5{,}443.2}{\Delta}\, b\right]\left(\frac{\lambda^*}{1+2\lambda^*}\right) - 1{,}065.6 \tag{4-13b}$$

$$R = 36{,}288\,(1+b)\left(\frac{\lambda^*}{1+2\lambda^*}\right) \tag{4-13c}$$

$$L = \left(168\,\Delta^3 - 180\,\Delta^2 + 81\,\Delta - 14\right) = g_1(\Delta) \tag{4-13d}$$

$$M = \left(2{,}016\,\Delta^3 - 324\,\Delta + 84\right) = g_2(\Delta) \tag{4-13e}$$

$$N = -\frac{60{,}480}{P_R}\,(1+b)\,\lambda^*\,\Delta^6 \tag{4-13f}$$

4.3. Similarity flows as a special case of general flows.

The differential equations (Eqs. 3-18) derived in Section 3.6 are completely general as regards pressure gradients, and hence should include the similarity flows as a special case. It is an interesting check, therefore, to deduce the simultaneous algebraic equations of Eqs. 4-12, that result in the similarity case, from Eqs. 3-18.

Since, in the similarity case, the shape of the boundary-layer profiles does not change with x, the parameters λ and Δ which define these profiles, are constants. Thus

$$\frac{d\lambda}{dx} = \frac{d\Delta}{dx} = 0 \tag{4-14}$$

and Eqs. 3-18 reduce to the following

$$m(\lambda, \Delta) = 0 \tag{4-15a}$$

$$r(\lambda, \Delta) = 0 \tag{4-15b}$$

Assuming the velocity in the external flow to be given by $u_e = A x^{\ell}$, and noting that $\ell = \frac{\lambda^*}{1-\lambda^*}$, we have

$$\left[\frac{u_e'' u_e}{(u_e')^2} - 4 \right] = -\left(\frac{3\ell + 1}{\ell} \right) = -\left(\frac{1 + 2\lambda^*}{\lambda^*} \right) \tag{4-16a}$$

$$\left[\frac{u_e'' u_e}{(u_e')^2} - 2 \right] = -\left(\frac{\ell + 1}{\ell} \right) = -\left(\frac{1}{\lambda^*} \right) \tag{4-16b}$$

Substituting these and the values of $m(\lambda, \Delta)$ and $r(\lambda, \Delta)$ from Eqs. 3-19b, e in Eqs. 4-15 above gives us two simultaneous algebraic equations of the form of Eqs. 4-12 with the coefficients identical with those given in Eqs. 4-13.

We can thus derive the equations governing similarity flows from the more general equations of the previous Chapter.

4.4. Method of solution.

Since λ^* and b are known constants for any given case (λ^* represents the pressure gradient and b the constant wall temperature), Eqs. 4-12 can now be solved simultaneously for the two unknowns λ and Δ . A graphical solution is easily obtained by solving each equation in turn for the unknown λ for assumed values of Δ . Two curves, one corresponding to each equation, are thus obtained on the $\lambda - \Delta$ plane. One of these, the one corresponding to Eq. 4-12a depends on the value of Pr. while the other does not. The point of intersection of these curves gives us the required solution.

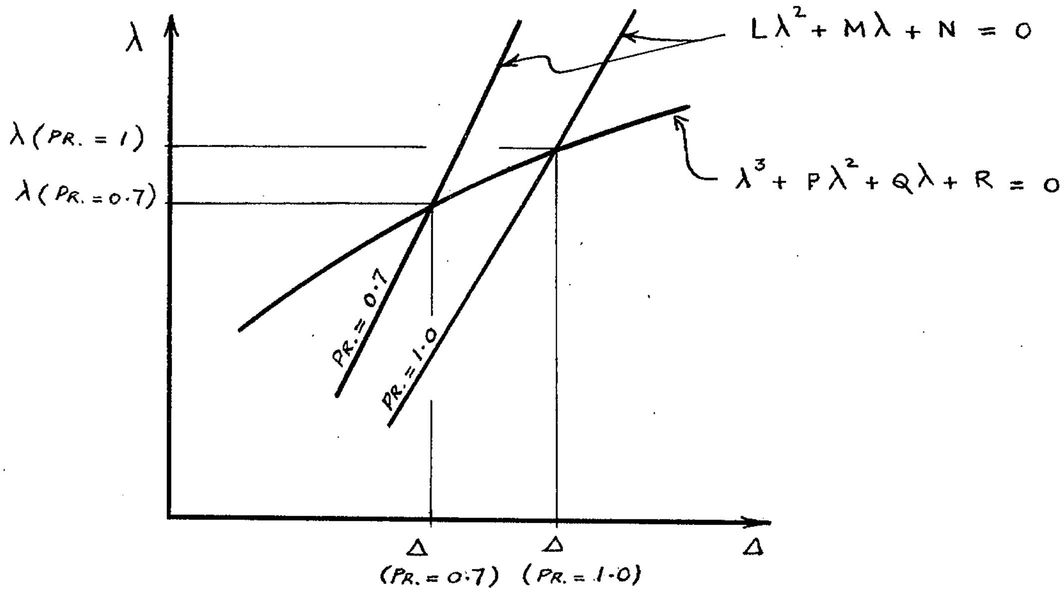

Fig. 3.

Once λ and Δ are determined, α , which is proportional to skin friction, is given by Eq. 4-11a and $\frac{1}{a}$, which is proportional to heat transfer, is given by Eq. 4-11b.

The boundary-layer profiles can now be easily determined. Since λ is known, the value of δ_V is calculated by using Eq. 4-9, and the velocity boundary-layer profile, $\frac{u}{u_e} = f'(\theta)$, as a function of θ , is given by Eq. 4-8a.

The profile of the temperature boundary-layer, as a function of the variable $\frac{\theta}{\delta_T}$, depends on the value of b alone, and is thus independent of the solution. However, since the value of δ_T depends on the value of Δ obtained as the solution, the temperature boundary-layer profile $\frac{h}{h_e} = \tilde{h}(\theta)$, as a function of θ also depends on the solution. Its shape may be obtained by determing the value of $\delta_T = \frac{\delta_v}{\Delta}$.

It should be noted, however, that the similarity variable θ is not formed from the physical coordinates x and y, but from x and η the Howarth-Stewartson coordinates for $M = 0$.

4.5. Illustrative Example.

As an example of the method, consider the flow of a compressible fluid of Prandtl Number 0.7 past a 45° wedge at $M = 0$.

For this case

$$u_e = A x^{1/3} \quad ; \quad l = 1/3 \quad ; \quad \lambda^* = 1/4$$

We shall assume that the wall temperature T_w is greater than T_e, the temperature in the external flow, more specifically that

$$\frac{T_w}{T_e} = \frac{h_w}{h_e} = \tilde{h}_w = 1 + b = 1.5$$

whence $b = 0.5$ and heat is transferred from the wall to the fluid.

For such a case Eqs. 4-12 become

$$\lambda^3 + 34.08\,\lambda^2 - \left(1{,}217 + \frac{453.6}{\Delta}\right)\lambda + 9{,}072 = 0 \quad (4\text{-}17a)$$

$$\left(168\,\Delta^3 - 180\,\Delta^2 + 81\Delta - 14\right)\lambda^2 + \left(2{,}016\,\Delta^3 - 324\,\Delta + 84\right)\lambda - 32{,}400\,\Delta^6 = 0 \quad (4\text{-}17b)$$

A graphical solution of Eqs. 4-17 yields

$$\lambda = 5.39$$

$$\Delta = 0.662$$

Thus

$$\delta_V = \sqrt{\frac{\lambda}{2\lambda^*(1+b)}} = 2.681 \quad ; \quad \alpha = \frac{\lambda+12}{6\delta_V} = 1.081$$

whence

$$\delta_T = \frac{\delta_V}{\Delta} = 4.050 \quad ; \quad \frac{1}{a} = \frac{2}{\delta_T} = 0.494$$

These values compare well with those obtained by Levy[5] by a trial and error method on the Differential Analyser. His values are

$$\alpha = 1.10$$
$$\left(\frac{1}{a}\right) = 0.49$$

4.6. Flat plate parallel to flow direction ($\lambda^* = 0$)

The flat plate is a simple case of the general problem for which exact solutions have been obtained. The general method outlined in the previous sections can be used as follows:

For the flat plate u_e = const., hence $\ell = 0$, and therefore $\lambda^* = 0$. Note that in this case Eq. 4-12a becomes independent of Eq. 4-12b. From Eq. 4-9 we see that since δ_V is finite, $\lambda = 0$. This, thus leaves δ_V indeterminate. We can, however, use Eq. 4-6a and this gives us

$$\alpha = \int_0^\infty ff''d\theta \doteq \int_0^{\delta_V} ff''d\theta \tag{4-18}$$

The integral of Eq. 4-18 can be evaluated by using Eq. 4-8a and this gives us

$$\alpha = \frac{5,328}{45,360}\,\delta_V \tag{4-19a}$$

Also, since $\lambda = 0$, Eq. 4-11a gives us

$$\alpha = \frac{2}{\delta_v} \tag{4-19b}$$

Whence, from Eqs. 4-19a and b

$$\alpha = \sqrt{\frac{2 \times 5,328}{45,360}} = 0.4846 \quad \text{(for all values of } \mathcal{b} \text{ and } P_R. \text{)}$$

This value of α is about 3 percent larger than the exact value of $\alpha = 0.4696$ obtained by Hartree[6].

The algebraic equatinn deduced from the integrated energy equation (i.e. Eq. 4-12b), can then be used to determine Δ . Direct substitution of $\lambda = 0$ and $\lambda^* = 0$ satisfies the equation trivially but does not determine Δ . So we proceed as follows. We divide Eq. 4-12b by $2\lambda^*(1+\mathcal{b})$ and then replace $\frac{\lambda}{2\lambda^*(1+\mathcal{b})}$ by δ_v^2. Then letting $\lambda \to 0$ gives us:

$$M\delta_v^2 - \frac{30,240}{P_R}\Delta^6 = 0 \tag{4-20}$$

But we know from Eq. 4-19b that

$$\delta_v^2 = \frac{4}{\alpha^2} = \frac{4}{0.2348} = 17.03 \tag{4-21}$$

Inserting the value of M from Eq. 4-13b and δ_v^2 from Eq. 4-21 in Eq. 4-20 gives us an equation in Δ :

$$\frac{1}{P_R}\Delta^6 - 1.353\,\Delta^3 + 0.1825\,\Delta - 0.0473 = 0 \tag{4-22}$$

The solution of this equation, for the appropriate value of P_R , gives us the required value of Δ . For $P_R = 1$, of course, $\Delta = 1.0$. From the values of Δ and δ_v one then finds δ_T , and $\left(\frac{1}{a}\right)$ is given by

Eq. 4-11b.

4.7. Incompressible flow ($b = 0$)

The so called incompressible flow is a special case of the above. In this case $b = 0$ and hence $\tilde{h}_w = 1 + b = 1$ and $h_w = h_e = h_\infty$. The energy equation, Eq. 4-3b, then has the trivial solution $\tilde{h} = 1$ and the momentum equation becomes the Falkner-Skan equation.

$$f''' + ff'' + 2\lambda^* \left[1 - (f')^2\right] = 0 \qquad (4\text{-}23)$$

If b is assumed to be small without necessarily being zero, i.e. when the flow is almost incompressible one may expand f and $\tilde{h}$ in powers of b :

$$f(\theta) = f_0(\theta) + b f_1(\theta) + b^2 f_2(\theta) + \cdots\cdots \qquad (4\text{-}24a)$$

$$\tilde{h}(\theta) = 1 + b \tilde{h}_1(\theta) + b^2 \tilde{h}_2(\theta) + \cdots\cdots \qquad (4\text{-}24b)$$

Inserting these expansions into Eqs. 4-3 one sees that f_0 still satisfies the Falkner-Skan equation for incompressible flow (Eq. 4-23), whereas $\tilde{h}_1$ satisfies the equation

$$\tilde{h}_1'' + Pr.\, f_0 \tilde{h}_1 = 0 \qquad (4\text{-}25)$$

with the boundary conditions

$$\tilde{h}_1(0) = 1 \qquad (4\text{-}26a)$$

$$\tilde{h}_1(\infty) = 0 \qquad (4\text{-}26b)$$

This is often referred to as the heat transfer equation in incompressible flow. The use of the term "incompressible" is, however, somewhat misleading here and Eq. 4-25 should be understood in the sense above.

Note that for almost incompressible flow the momentum equation (Eq. 4-23) is independent of the energy equation (Eq. 4-25) if one only considers the leading term in the expansion of f in Eq. 4-24a. Because of the fact that Eq. 4-23 is uncoupled from the energy equation its solution by momentum methods is especially easy. In this case Eq. 4-12a is an equation for λ only since b should be put equal to zero in the coefficients Q and R. The value of λ so obtained may be substituted into Eq. 4-12b which now becomes an equation for Δ. Solution of this equation then leads to an approximate solution of Eq. 4-25 and the heat-transfer coefficient is obtained to the first order in b.

The perturbation method just described is of course not restricted to those pressure gradients which lead to similarity solutions. However this idea will not be discussed further in this report.

4.8. Limiting case when $b = -1$.

The case when $b = -1$ corresponds to $\tilde{h}_w = \frac{T_w}{T_e} = 0$. This is true when either $T_w = 0$ or $T_e = \infty$ either of which are physically impossible. However, this is an interesting theoretical limit to consider.

As in the case of $\lambda^* = 0$, we see here from Eq. 4-9, that λ must be zero in order that δ_v be finite. So δ_v is again indeterminate from Eq. 4-9. We can again use a similar limiting procedure as we did for the energy equation when $\lambda^* = 0$. We divide both Eqs. 4-12a and 4-12b by $2\lambda^*(1+b)$ and then replace $\dfrac{\lambda}{2\lambda^*(1+b)}$ by

δ_v^2 . If we now let $\lambda \to 0$, and remember that is finite, we end up with two equations which can be reduced to the following form:

$$\frac{4}{\delta_v^2} = 0.235\,(1+2\lambda^*) + 1.2\,(1-\frac{1}{\Delta})\,\lambda^* \tag{4-26a}$$

$$\frac{4}{\delta_v^2} = \frac{P_R.\,(2{,}016\,\Delta^3 - 324\,\Delta + 84)}{7{,}560} \tag{4-26b}$$

By equating the right hand sides of the above equations one obtains a comparatively simple equation for Δ alone, which is easily solved for any given λ^* and $P_R.$. Once Δ is found δ_v is directly given by either equation. Since λ is zero the equation for α (Eq. 4-11a) again reduces to the simple form of Eq. 4-19 and, as before, $(\frac{1}{a})$ is found from Eq. 4-11b.

4.9. Summary of Results.

Based on the above method the values of α and $(\frac{1}{a})$, which essentially represent skin friction and heat transfer, have been computed for λ^* = 0, 1/4, and 1/2; b = +1.0, +0.5, 0, -0.5 and -1.0; $P_R.$ = 1.0 and 0.7. The values are tabulated in Tables IA and IB. Figs. 4, 5, 6, and 7 give a plot of these parameters.

The values of Δ , the ratio of the boundary-layer thicknesses are also tabulated in the same Tables, and plotted on Figs. 8 and 9. The values obtained by Levy[5] do not cover as wide a range as those obtained in this report. In the range that is common, however, these values agree to within 3 percent of those obtained by him.

One point needs special mention and that is the variation of α with $P_R.$ for $b > 0$ and $b < 0$. In this report it is found that if

$b > 0$, α decreases as P_R. increases for any given λ^* (other then $\lambda^* = 0$, in which case λ^* is independent of both P_R. and b). This is in agreement with the results of Levy. However, for $b < 0$, α increases as P_R. increases. This reversal of trend is not shown by Levy's results, which show the same trend for $b < 0$ as for $b > 0$. However, the results obtained in the present report seem qualitatively plausible, at least for small b and P_R. near unity. Putting $P_R. = 1 + \varepsilon$ and expanding α around $\varepsilon = 0$ and $b = 0$ one obtains

$$\alpha = \alpha_{00} + b\alpha_{10} + \varepsilon\alpha_{01} + b^2\alpha_{20} + b\varepsilon\alpha_{11} + \varepsilon^2\alpha_{02} + \cdots\cdots\cdots$$

The fact that P_R. does not influence the value of α for $b = 0$ means that α_{01}, α_{02} etc. are zero. If we now consider a fixed small value of b at $P_R. = 1 + \varepsilon$ and $P_R. = 1$ it is seen that the difference of the value of α in the two cases is $\varepsilon b \alpha_{11}$. Since α_{11} is a constant this correction term changes sign with b.

Figures 5 and 8 show the variation of $\left(\frac{1}{\alpha}\right)$. We see here that the rate of heat-transfer is greater, the greater the P_R. No.

Lastly, the variation of Δ is shown in Figs. 6 and 9. We see that the ratio Δ increases as P_R. increases, because the thickness δ_T decreases with increase in P_R. . Δ is smaller for $b > 0$ when heat is transferred from the wall to the fluid, than it is for $b < 0$ when heat is transferred from the fluid to the wall.

V. ARBITRARY PRESSURE GRADIENT (FOR $M \doteq 0$)

5.1. Transformation of the Differential Equations.

In the case of arbitrary pressure gradient, when the problem cannot be reduced to a 'similarity' case, $\lambda' = \frac{d\lambda}{dx}$ and $\Delta' = \frac{d\Delta}{dx}$ are no longer zero. The complete equations (Eq. 3-18) have then to be used. These complete equations are

$$\ell \lambda' + m = 0 \qquad (5\text{-}1a)$$

$$p\lambda' + q\Delta' + r = 0 \qquad (5\text{-}1b)$$

where the functions ℓ , m , p , q and r are defined in Eqs. 3-19.

Following a method somewhat similar to that which Holstein and Bohlen[7] used for the case of incompressible fluids, we can rewrite the above differential equations in a simpler form. Apart from making the equations a bit easier to handle, the main simplification lies in the fact that they no longer contain the second derivative of the known velocity in the external flow.

The simplified form of the equations is:

$$\frac{d(\delta^{**})^2}{dx} = \frac{\nu_\infty}{u_e} F(\lambda, \Delta) \qquad (5\text{-}2a)$$

$$\frac{d(\vartheta^{**})^2}{dx} = \frac{\nu_\infty}{u_e} G(\lambda, \Delta) \qquad (5\text{-}2b)$$

where the functions $F(\lambda,\Delta)$ and $G(\lambda,\Delta)$ are defined below

$$F(\lambda,\Delta) = \left[\frac{2f_2}{1+b}\right]\left[\frac{(\lambda+12)(1+b)}{6} - \lambda f_1 - \frac{3b}{10}\frac{\lambda}{\Delta} - 2\lambda f_2\right] \tag{5-3a}$$

$$G(\lambda,\Delta) = \left[\frac{2b^2}{1+b}\cdot\frac{g(\lambda,\Delta)}{15,120\,\Delta^6}\right]\left[g_1\lambda^2 + g_2\lambda - \frac{30,240(1+b)\Delta^6}{Pr.}\right] \tag{5-3b}$$

and the values of $(\delta^{**})^2$ and $(\vartheta^{**})^2$ are obtained from Eqs. 3-15b, d and Eq. 3-4 as

$$(\delta^{**})^2 = \delta_v^2 f_2^2(\lambda) = \frac{1}{1+b}\cdot\frac{\nu_\infty}{u_e'}\cdot\lambda f_2^2(\lambda) \tag{5-4a}$$

$$(\vartheta^{**})^2 = b^2\delta_T^2 g^2(\lambda,\Delta) = \frac{b^2}{1+b^2}\cdot\frac{\nu_\infty}{u_e'}\cdot\frac{\lambda}{\Delta^2}\cdot g^2(\lambda,\Delta) \tag{5-4b}$$

5.2. Correspondence with the Holstein-Bohlen Equations when $b = 0$.

For the case of incompressible fluids $\tilde{h}_w = 1$ and $b = 0$. Eqs. 5-2a and 5-2b now become uncoupled and the velocity equation can be solved independent of the temperature equation. By putting $b = 0$ in Eq. 5-2a we obtain

$$\frac{d(\delta^{**})^2}{dx} = \frac{\nu_\infty}{u_e}\cdot 2 f_2(\lambda)\cdot\left[\frac{\lambda+12}{6} - \frac{(36-\lambda)\lambda}{120} - 2\lambda f_2(\lambda)\right] \tag{5-5}$$

where

$$f_2(\lambda) = -\frac{1}{45,360}\left[5\lambda^2 + 48\lambda - 5,328\right] = \frac{1}{63}\left[\frac{37}{5} - \frac{1}{15}\lambda - \frac{1}{144}\lambda^2\right] \tag{5-6}$$

Eq. 5-5 is the same as that deduced by Holstein and Bohlen for incompressible fluids.

5.3. Range of variation of λ and Δ

An examination of Eq. 3-3 shows that $\lambda = -12$ makes $\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0} = 0$ and thus gives us the shape of the separation profile. A value of λ less than - 12 is not admissible. Further, for $\lambda > 12$ there results values of $\frac{u}{u_e} > 1$ in the boundary layer, which is physically impossible. Thus the form parameter λ is limited to the values

$$-12 \leqslant \lambda \leqslant +12 \qquad (5-7)$$

Since both δ_V and δ_T are positive, the only restriction on will be that

$$\Delta \geqslant 0 \qquad (5-8)$$

5.4. Stagnation Values.

The integration of the equations, in either the form given in Eqs. 5-1 or in Eqs. 5-2, has to start at the stagnation point which is taken to be at $x = 0$. We, therefore, require some initial values at the stagnation point in order to start the integration. A typical set of initial values for Eqs. 5-1 would be the values of λ and Δ at $x = 0$. These are not given. However, we impose the condition that $\frac{d\lambda}{dx}$ and $\frac{d\Delta}{dx}$ be finite at $x = 0$. As will be seen later this requirement determines uniquely λ and Δ at $x = 0$, at least within the range of physically admissible values of λ and Δ (cf. Eqs. 5-7 and 5-8). It will be seen later that if the external velocity field represents flow past a blunt-nosed body, the values of λ and Δ at $x = 0$ agree with the values found in Section 4.9 for flow towards a flat plate normal to the stream (i.e. $\lambda^* = \frac{1}{2}$). In a similar way one finds the initial values of $(\delta^{**})^2$ and $(\vartheta^{**})^2$ for Eqs. 5-2.

We shall first use Eqs. 5-1. At the stagnation point (denoted by the subscript 'o' and assumed to be blunt) where $u_e = 0$, we see from Eqs. 3-19 that

$$l_o = p_o = q_o = 0$$

Thus, the requirement that the slopes λ_o' and Δ_o' be not equal to infinity, gives us the following conditions that must be satisfied

$$m_o = 0 \qquad (5\text{-}9a)$$

$$r_o = 0 \qquad (5\text{-}9b)$$

If we substitute the values of m_o and r_o from Eqs. 3-19 in the above equations, we get two equations in λ and Δ, of the form

$$\frac{(\lambda+12)(1+b)}{6} - \lambda f_1 - \frac{3b}{10}\frac{\lambda}{\Delta} - 2\lambda f_2 = 0 \qquad (5\text{-}10a)$$

$$g_1 \lambda^2 + g_2 \lambda - \frac{30,240(1+b)\Delta^6}{P_R} = 0 \qquad (5\text{-}10b)$$

These two equations, solved simultaneously, yield $\lambda = \lambda_o$ and $\Delta = \Delta_o$ the values of λ and Δ at $x = 0$.

It should be noted that Eqs. 5-9 for determining λ_o and Δ_o are very similar to Eqs. 4-15 used for determining λ and Δ in the Falkner-Skan similarity case. The difference lies in the fact that in using Eqs. 4-15 $\frac{u_e'' u_e}{(u_e')^2}$ was put equal to $\frac{l-1}{l}$ (where l is the exponent in $u_e \sim x^l$), whereas in using Eqs. 5-9 we considered the case of a blunt nosed body for which $\frac{u_e'' u_e}{(u_e')^2} = 0$ at $x = 0$. Thus solutions of Eqs. 5-9 are the same as solutions of Eqs. 4-15 when $l = 1$ (i.e. $\lambda^* = \frac{1}{2}$).

The above then tells us that the values of λ and Δ at the stagnation point of flow past a blunt nosed body are the same as the values of λ and Δ obtained for flow towards a flat plate normal to the free stream. Of course this was to be expected, for in both cases $u_e \sim x$. The values of λ_o and Δ_o , so obtained, are completely general and independent of the particular pressure gradient of the problem. For $P_R = 0{\cdot}7$, the values of λ_o and Δ_o so obtained, are tabulated in Table III.

If now the values of λ_o and Δ_o are substituted into Eqs. 5-1 one sees that λ_o' and Δ_o' are indeterminate ("Prime" denotes derivative with respect to x). These derivatives can, however, be easily determined by first differentiating Eqs. 5-1 with respect to x and then putting $x = 0$. Using the fact that l_o , p_o and q_o are equal to zero one finds

$$l_o' \lambda_o' + m_o' = 0 \tag{5-11a}$$

$$p_o' \lambda_o' + q_o' \Delta_o' + r_o' = 0 \tag{5-11b}$$

Again, substituting the values of l_o' , m_o' , p_o' , q_o' and r_o' which can be obtained from Eqs. 3-19, in the above, we obtain two simultaneous equations for determining λ_o' and Δ_o' .

This process of differentiating Eqs. 5-1 with respect to x and then using the fact that l_o , p_o and q_o are zero, can be repeated as many times as desired and the equations for determining the derivatives of λ and Δ to any order, at the stagnation point, can be derived. Thus the equations for evaluating the second order derivatives

λ_o'' and Δ_o'' are

$$2\, l_o'\lambda_o'' + l_o''\lambda_o' + m_o'' = 0 \tag{5-12a}$$

$$2\, p_o'\lambda_o'' + p_o''\lambda_o' + 2 q_o'\Delta_o'' + q_o''\Delta_o' + r_o'' = 0 \tag{5-12b}$$

and so on.

Using Eqs. 5-2 instead of Eqs. 5-1 yields the same value for λ, Δ and their derivatives at $x = 0$. From Eqs. 5-2 we see that since $u_e = 0$, unless the values of $F(\lambda,\Delta)$ and $G(\lambda,\Delta)$ are also zero, the slopes $\left[(\delta^{**})^2\right]_0'$ and $\left[(\vartheta^{**})^2\right]_0'$ will be infinity. This then gives us the following conditions to be satisfied at the stagnation point:

$$F(\lambda, \Delta) = 0 \tag{5-13a}$$

$$G(\lambda, \Delta) = 0 \tag{5-13b}$$

Both the functions $F(\lambda,\Delta)$ and $G(\lambda,\Delta)$ as given in Eqs. 5-3 have two factors, and it turns out that the only physically significant values of λ_o and Δ_o, satisfying the restrictions of Eqs. 5-7 and 5-8, are obtained when the second of the two factors is zero in both cases. This gives two equations for determining λ_o and Δ_o which are exactly the same as Eqs. 5-10 obtained earlier.

Knowing the values of λ_o and Δ_o, the values of δ_o^{**} and ϑ_o^{**} can be obtained from Eqs. 5-4.

Since both $F(\lambda,\Delta)$ and $G(\lambda,\Delta)$ are now zero at the stagnation point, the initial slope of the integral curve for $(\delta^{**})^2$ and $(\vartheta^{**})^2$ at that point has the indeterminate value $\frac{0}{0}$. By using L'Hospital's rule, however, we get from Eqs. 5-2

$$\left[\frac{d(\delta^{**})^2}{dx}\right]_o = \frac{\nu_\infty}{(u_e')_o}\left[\frac{dF}{dx}\right]_o \qquad (5\text{-}14a)$$

$$\left[\frac{d(\vartheta^{**})^2}{dx}\right]_o = \frac{\nu_\infty}{(u_e')_o}\left[\frac{dG}{dx}\right]_o \qquad (5\text{-}14b)$$

Substituting the values $(\delta_o^{**})^2$, $(\vartheta_o^{**})^2$, $F(\lambda,\Delta)$, and $G(\lambda,\Delta)$ in Eqs. 5-14 we obtain two simultaneous equations for λ_o' and Δ_o' which are the same as those deduced earlier from Eqs. 5-11.

Differentiating Eqs. 5-3 with respect to x and using the values of λ_o, Δ_o, λ_o' and Δ_o' gives us the magnitudes of $\left[dF/dx\right]_o$ and $\left[dG/dx\right]_o$. These can now be substituted in Eqs. 5-14, to give us the slopes $\left[(\delta^{**})^2\right]_o'$ and $\left[(\vartheta^{**})^2\right]_o'$.

5.5. Check on stagnation values for $\ell = o$.

For incompressible flows, when $\ell = o$, the equation for determining λ_o is no longer coupled to the equation for Δ_o. This equation is obtained by putting $\ell = 0$ in Eq. 5-10a and is given by

$$\lambda^3 + 47.4\,\lambda^2 - 1{,}670.4\,\lambda + 9{,}072 = 0$$

Note that this equation is independent of the Prandtl Number. The solutions to this equation are λ = -72.255, 7.052 and 17.805. Of these, the only value that is physically admissible, is

$$\lambda = 7.052 \qquad (5\text{-}15)$$

Similarly, the value of λ_o' is obtained from Eq. 5-11a by putting $\ell = o$. This gives

$$- 0.4437 \; \lambda_0' = - \left(\frac{u_e''}{u_e'} \right)_0 0.7369$$

which yields

$$\lambda_0' = 1.6608 \left(\frac{u_e''}{u_e'} \right)_0 \qquad (5\text{-}16)$$

By differentiating Eq. 5-3a the value of $\left(dF/dx \right)_0$ is obtained to be

$$\left[\frac{dF}{dx} \right]_0 = - 0.0652 \left(\frac{u_e''}{u_e'} \right)_0$$

and hence, from Eq. 5-14a

$$\frac{d(\delta^{**})^2}{dx} = -0.0652 \; \frac{\nu_\infty (u_e'')_0}{(u_e')_0^2} \qquad (5\text{-}17)$$

These results given in Eqs. 5-15 and 5-17 agree with those given by Holstein and Bohlen[7].

5.6. Example to illustrate the method of calculating stagnation values.

To illustrate the method outlined above, we shall compute the magnitude of these parameters for the case when $b = 1.0$ and $P_R = 0.7$.

For this case Eqs. 5-10 become

$$\lambda_0^3 + 47.4 \; \lambda_0^2 - \left(914.4 + \frac{1,360.8}{\Delta_0} \right) \lambda_0 + 18,144 = 0$$

$$g_1(\Delta_0) \; \lambda_0^2 + g_2(\Delta_0) \, \lambda - 86,400 \; \Delta_0^6 = 0$$

and a graphical solution of these yields

$$\lambda_0 = 4.595 \qquad (5\text{-}18a)$$

$$\Delta_0 = 0.416 \qquad (5\text{-}18b)$$

as the only physically admissible values. These values check with the values for λ and Δ obtained in Section 4.9 for $\lambda^* = 1/2$ (and $b = 1.0$, $P_R = 0.7$).

Eqs. 3-16 now give the magnitude of the various functions and their derivatives (with respect to their arguments) as

$f_1(\lambda_0)$	$= 0.26171$	;	$f_2(\lambda_0)$	$= 0.11027$
$[f_1'(\lambda)]_0$	$= -0.00833$	;	$[f_2'(\lambda)]_0$	$= -0.002071$
$[f_1''(\lambda)]_0$	$= 0$	;	$[f_2''(\lambda)]_0$	$= -0.0002205$
$g_1(\Delta_0)$	$= 0.6350$	;	$g_2(\Delta_0)$	$= 94.134$
$[g_1'(\Delta)]_0$	$= 18.442$	;	$[g_2'(\Delta)]_0$	$= 721.13$
$[g_1''(\Delta)]_0$	$= 59.026$	;	$[g_2''(\Delta)]_0$	$= 5,028.3$

$$g(\lambda_0, \Delta_0) = -0.2149$$

Using these in Eqs. 5-11 yields

$$-1.7178\,\lambda_0' + 15.953\,\Delta_0' = -0.50666\left(\frac{u_e''}{u_e'}\right)_0$$

$$302.83\,\lambda_0' - 8,786.6\,\Delta_0' = 445.92\left(\frac{u_e''}{u_e'}\right)_0$$

which solved simultaneously, give

$$\lambda_0' = -0.2594\left(\frac{u_e''}{u_e'}\right)_0 \qquad (5\text{-}19a)$$

$$\Delta_0' = -0.05969\left(\frac{u_e''}{u_e'}\right)_0 \qquad (5\text{-}19b)$$

Similarly the equations for determining the second derivatives are obtained from Eqs. 5-12 as

$$+0.90451\,\lambda_o'' - 7.9766\,\Delta_o'' = -0.92056\left(\frac{u_e''}{u_e'}\right)_o^2 + 0.506666\left(\frac{u_e'''}{u_e'}\right)_o$$

$$-202.86\,\lambda_o'' + 6053.8\,\Delta_o'' = +95.985\left(\frac{u_e''}{u_e'}\right)_o^2 - 445.92\left(\frac{u_e'''}{u_e'}\right)_o$$

and when solved simultaneously give

$$\lambda_o'' = -1.2462\left(\frac{u_e''}{u_e'}\right)_o^2 - 0.1260\left(\frac{u_e'''}{u_e'}\right)_o \qquad (5\text{-}20a)$$

$$\Delta_o'' = -0.02590\left(\frac{u_e''}{u_e'}\right)_o^2 - 0.07791\left(\frac{u_e'''}{u_e'}\right)_o \qquad (5\text{-}20b)$$

The values of $(\delta_o^{**})^2$, $(\vartheta_o^{**})^2$, $\left[(\delta^{**})^2\right]_o'$, and $\left[(\vartheta^{**})^2\right]_o'$ can also be obtained, if desired, from Eqs. 5-4 and Eqs. 5-14 as

$$(\delta_o^{**})^2 = 0.02793\,\frac{\nu_\infty}{(u_e')_o}$$

$$(\vartheta_o^{**})^2 = 0.61412\,\frac{\nu_\infty}{(u_e')_o}$$

and

$$\left[(\delta^{**})^2\right]_o' = -0.02924\,\frac{\nu_\infty (u_e'')_o}{(u_e')_o^2}$$

$$\left[(\vartheta^{**})^2\right]_o' = -0.37791\,\frac{\nu_\infty (u_e'')_o}{(u_e')_o^2}$$

5.7. Point of velocity maxima.

At that point along the wall where the external velocity $u_e(x)$ reaches a maximum value (at $x = x_m$, say, which will be known from the given distribution of $u_e(x)$), we have

$$(u_e')_m = 0 \qquad (5\text{-}21)$$

where the subscript " m " refers to conditions at this point of velocity maxima.

From Eqs. 3-4 we see that the value of $\lambda = \lambda_m$ at this point is given by

$$\lambda_m = \frac{1+b}{\nu_\infty} \cdot (\delta_v^2)_m \cdot (u_e')_m = 0 \tag{5-22}$$

Although both λ_m and $(u_e')_m$ are zero at $x = x_m$, the ratio $\left(\frac{\lambda}{u_e'}\right)$ tends to a finite limit as $x \to x_m$. This limit can be obtained from either Eq. 5-la or 5-lb and is done as follows: Multiplying, say, Eq. 5-la by u_e' we have

$$(u_e' \ell)\lambda' + (u_e' m) = 0 \tag{5-23}$$

As $x \to x_m$, $\lambda \to 0$ and $u_e' \to 0$, and hence from Eqs. 3-19 we see that

$$(u_e' \ell) \to \left[\frac{u_e f_2(\lambda)}{2}\right]_m$$

$$(u_e' m) \to -\left[\frac{u_e f_2(\lambda)}{2}\right]_m \left(\frac{\lambda}{u_e'}\right)_m (u_e'')_m$$

Using these in Eq. 5-23 we have

$$\left(\frac{\lambda'}{\lambda}\right)_m = \left(\frac{u_e''}{u_e'}\right)_m \tag{5-24}$$

Eq. 5-24 when integrated gives

$$\left(\frac{\lambda}{u_e'}\right)_m = \text{const.} = C \quad \text{(say)} \tag{5-25}$$

and using the relation in Eq. 5-24 above gives us the value of λ' at $x = x_m$ as

$$\lambda'_m = C\,(u''_e)_m \tag{5-26}$$

The constant C is, as yet, arbitrary. Its actual magnitude, in a particular case, will depend on the initial values at $x = 0$. All that can be definitely said about C is that it must be a positive constant. For, from Eqs. 5-22 and 5-25 above, we see that

$$C = \lim_{x \to x_m} \left(\frac{\lambda}{u'_e}\right) = \left(\frac{\lambda}{u'_e}\right)_m = \frac{1+b}{\nu_\infty}\left(\delta_v^2\right)_m$$

So for $b \geq -1$, $C \geq 0$. Furthermore, since at the velocity maxima point $(u''_e)_m < 0$, we can conclude that at this point

$$\lambda'_m < 0 \tag{5-27}$$

Performing a similar limit procedure, as $x \to x_m$, on Eq. 5-1b leads to the same result as quoted in Eq. 5-26. In fact, taking the n^{th} derivative of either of the equations of Eqs. 5-1, multiplying the result by $(u'_e)^n$, and then taking the limit as $x \to x_m$, yields, each time, the same result as given in Eq. 5-26. Thus, these equations give us the value of only the first derivative of $\lambda(x)$ at $x = x_m$ (i.e. the value of λ'_m) up to a multiplicative constant C. They do not yield any information regarding the higher order derivatives of $\lambda(x)$ at this point. Further, the equations give us no a priori knowledge regarding $\Delta(x)$ or any of its derivatives at $x = x_m$.

3.12. Methods of solution.

The equations developed, Eqs. 5-1 for $\lambda(x)$ and $\Delta(x)$ or Eqs. 5-2 for $\delta^{**}(x)$ and $\vartheta^{**}(x)$, together with the initial values of the parameters involved and their first derivatives with respect to x , can be solved in various ways. Only two methods will be discussed below.

(A). Numerical integration

(b). Approximate methods

Of course, various schemes of iteration can be used to improve the results of the approximate solutions to any degree of accuracy desired.

(A). Numerical integration.

Apart from it being tedious, numerical integration in this case has the added disadvantage that the starting point of the integration (the stagnation point at $x = o$) is a singular point. To avoid this difficulty, it is best to use a Taylor's expansion for $\lambda(x)$ and $\Delta(x)$ about the point $x = o$ up to some small distance out from the stagnation point, say $x = x_1$.

In this range $o \leq x \leq x_1$,

$$\lambda(x) = \lambda(o) + \lambda'(o)\,x + \frac{\lambda''(o)}{2}\,x^2 + \ldots\ldots\ldots \quad (5\text{-}28a)$$

$$\Delta(x) = \Delta(o) + \Delta'(o)\,x + \frac{\Delta''(o)}{2}\,x^2 + \ldots\ldots\ldots \quad (5\text{-}28b)$$

where the values of

$$\lambda(o) = \lambda_o \quad ; \quad \lambda'(o) = \lambda'_o \quad ; \ldots\ldots\ldots\ldots \text{etc.}$$

$$\Delta(o) = \Delta_o \quad ; \quad \Delta'(o) = \Delta'_o \quad ; \ldots\ldots\ldots\ldots \text{etc.}$$

can be calculated as shown in section 5.4. From the point $x = x_1$ onwards, simultaneous numerical integration of Eqs. 5-1 can be continued without any difficulty.

(B). Approximate Methods.

(i). Approximate solution I - A polynomial solution.

We have two reference points: the stagnation point at $x = o$, and the point of velocity maxima at $x = x_m$. A Taylor's expansion of $\lambda(x)$ and $\Delta(x)$ is known and the value of $\lambda(x)$ at $x = x_m$ is also known. We can, therefore, choose a polynomial of any desired degree, which satisfies the above conditions. Thus, if we choose to satisfy the first two derivatives only, at $x = o$, and the value of $\lambda(x)$ at the points $x = o$ and $x = x_m$, we can use a cubic of the form

$$\lambda(x) = \lambda_o + \lambda_o' x + \frac{\lambda_o''}{2} x^2 - \frac{1}{x_m^3}\left(\lambda_o + \lambda_o' x_m + \frac{\lambda_o''}{2} x_m^2\right) x^3 \qquad (5\text{-}29)$$

Differentiating Eq. 5-29 gives $\lambda'(x)$ as

$$\lambda'(x) = \lambda_o' + \lambda_o'' x + \frac{3}{x_m^3}\left(\lambda_o + \lambda_o' x_m + \frac{\lambda_o''}{2} x_m^2\right) x^2 \qquad (5\text{-}30)$$

As a first approximation for $\Delta(x)$ we may assume that $\Delta(x)=$ constant $= \Delta_o$. This may be improved on as follows. Eq. 5-lb is equivalent to the following integral

$$\Delta(x) = \Delta_o + \int_o^x f(\Delta, x)\, dx \qquad (5\text{-}31a)$$

where

$$f(\Delta, x) = \frac{r + p\lambda'}{q} \qquad (5\text{-}31b)$$

In evaluating $f(\Delta, x)$ we may assume $\lambda(x)$ and $\lambda'(x)$ to be given by the polynomials of Eqs. 5-29 and 5-30 and $\Delta(x)$ to be Δ_o. The integral in Eq. 5-31a can then be evaluated. In principle this iteration method of solving Eq. 5-31a for $\Delta(x)$ may be continued.

(ii). Approximate Solution II - An integral solution.

If the value for $\Delta(x)$ is assumed to be a constant, then if certain linearizing approximations be made, the equation for $\lambda(x)$ can be integrated in a manner similar to that which Holstein and Bohlen[7] used for incompressible fluids.

The equation, in the form given in Eqs. 5-2a, is

$$\frac{d(\delta^{**})^2}{dx} = \frac{\nu_\infty}{u_e} F(\lambda, \Delta)$$

If we use a new parameter K defined by

$$K = \frac{(\delta^{**})^2}{\nu_\infty} u_e' (1+b) = \lambda f_2^2(\lambda) \tag{5-32}$$

and assume $\Delta(x) = \Delta_0$, then the function $F(\lambda, \Delta)$ can be closely approximated by a linear function of K. Thus for $F(\lambda,\Delta)$ we can write

$$F(\lambda, \Delta) = A + BK \tag{5-33}$$

where A is a constant and B is dependent on the values of b and Δ_0 alone. The range of variation of x has to be split in two and a different value of B used in each range, in order that the approximations be good. The values of these constants are given in Table II.

TABLE II.

VALUES OF "A" AND "B"

Range	Stagnation point to point of velocity maxima	Point of velocity maxima to separation point.
x	$0 \leqslant x \leqslant x_m$	$x_m \leqslant x \leqslant x_s$
A	0·47	0·47
B	$\frac{1}{1+b}\left[2.52b - \frac{5.75b}{\Delta_0} - 6.1\right]$	$\frac{1}{1+b}\left[3b - \frac{5.26b}{\Delta_0} - 8.0\right]$

It should be noted that for incompressible fluids where $b = 0$ the approximations reduce to:

$$0 \leqslant x \leqslant x_m : \qquad F(\lambda, \Delta) = 0.47 - 6.1\,K$$

$$x_m \leqslant x \leqslant x_s : \qquad F(\lambda, \Delta) = 0.47 - 8.0\,K$$

and these compare well with the approximation suggested by Walz[8] for incompressible fluids, for the whole range variation of x , as

$$F(\lambda) = F(K) = 0.47 - 6\,K$$

If this approximation is used then Eq. 5-2a becomes

$$\frac{d(\delta^{**})^2}{dx} = \frac{\nu_\infty}{u_e}\left[A + BK\right] \tag{5-34}$$

Using the expression for K from Eq. 5-32 makes the above equation

$$\frac{d(\delta^{**})^2}{dx} - \frac{u_e'}{u_e} B(1+b)(\delta^{**})^2 = \frac{\nu_\infty}{u_e'} A \qquad (5\text{-}35)$$

Eq. 5-34 can be integrated by using an integrating factor. This gives the solution as

$$u_e^{-B(1+b)} (\delta^{**})^2 \Big]_{x_1}^{x_2} = A\nu_\infty \int_{x_1}^{x_2} u_e^{-B(1+b)-1} dx \qquad (5\text{-}36)$$

Noting that for all reasonable values of b, (say, $b < 5$), the factor $B(1+b)$ is always negative, we have, for the two ranges under consideration

<u>$0 \leq x \leq x_m$</u>

$$(\delta^{**})^2 = \frac{A\nu_\infty}{u_e^{-B(1+b)}} \int_0^x u_e^{-B(1+b)-1} dx \qquad (5\text{-}37a)$$

<u>$x_m \leq x \leq x_s$</u>

$$(\delta^{**})^2 = A\nu_\infty \left[\frac{1}{u_e^{-B(1+b)}} \int_0^{x_m} u_e^{-B(1+b)-1} dx + \frac{1}{u_e^{-B(1+b)}} \int_{x_m}^{x} u_e^{-B(1+b)-1} dx \right] \qquad (5\text{-}37b)$$

where, in Eq. 5-37b the value of B is different for the two expressions within the brackets in accordance with Table II.

Once the distribution of $(\delta^{**})^2$ is known, that of $\lambda(x)$ can be found graphically by using Eq. 5-32, i.e. the relation

$$(\delta^{**})^2 = \frac{\nu_\infty}{u_e'} \frac{\lambda f_2^2(\lambda)}{1+b}$$

The distribution of $\lambda(x)$ and $\Delta(x)$ having been obtained by using any of the methods outlined above, that of $\alpha(x)$ and $\frac{1}{a(x)}$ can be deduced by using Eqs. 3-11 and 3-13.

5.9. Example to illustrate the methods of solution.

To demonstrate the methods of solution outlined in section 5.8, we consider the flow of a compressible fluid of $P_R. = 0.7$ at $M = 0$, over the heated surface of a body. The wall temperature is assumed to be twice the temperature in the external flows (i.e. $b = 1.0$) and the distribution of external velocity on the surface is approximated by a cubic of the form:

$$u_e(x) = x - \frac{x^3}{6} \tag{5-38}$$

where x is measured along the surface from the stagnation point. Of course, it is not necessary to have the distribution of $u_e(x)$ given as a explicit function of x. Its distribution, in general, will be known numerically, and the method can be used just as easily for these arbitrary distributions of $u_e(x)$. The above distribution is chosen only as a convenient illustrative example.

For this distribution

$$u_e(o) = 0 \quad ; \quad u_e'(o) = 1 \quad ; \quad u_e''(o) = 0 \quad ; \quad u_e'''(o) = -1$$

and using these in Eqs. 5-18 and 5-20 of the illustration example of section 5.6, gives us

$$\lambda_o = 4.595 \quad ; \quad \lambda_o' = 0 \quad ; \quad \lambda_o'' = 0.127$$

$$\Delta_o = 0.416 \quad ; \quad \Delta_o' = 0 \quad ; \quad \Delta_o'' = 0.078$$

The Taylor's expansions for these functions, about the point $x = 0$, then becomes

$$\lambda(x) = 4.595 + 0.127 \frac{x^2}{2} + \ldots\ldots\ldots\ldots \qquad (5\text{-}39a)$$

$$\Delta(x) = 0.416 + 0.078 \frac{x^2}{2} + \ldots\ldots\ldots\ldots \qquad (4\text{-}39b)$$

(A). Numerical Integration.

The above Taylor's expansions can be used up to a distance, say $x_1 = 0{\cdot}1$, where the values of $\lambda(x)$ and $\Delta(x)$ are

$$\lambda(0{\cdot}1) = 4.595$$

$$\Delta(0{\cdot}1) = 0.416$$

and the numerical integration started from this point. The results of this integration are given in Table IV and plotted on Figs. 10 and 11.

(B). Approximate Methods.

(i). Approximate polynomial solution.

Examination of the numerical solution obtained above reveals that the variation in $\Delta(x)$ is small when compared to the variation in $\lambda(x)$. As we go from stagnation point to the point of separation the value of $\Delta(x)$ varies approximately from 0.42 to 0.56 whereas that of $\lambda(x)$ varies from 4.6 to -12. So, as a first approximation $\Delta(x)$ may be assumed to be constant. This gives us

$$\Delta(x) = \Delta_0 = 0.416 \qquad (5\text{-}40)$$

The velocity $u_e(x)$ reaches a maxima at the point

$$x_m = 1.414$$

and this makes the polynomial solution for $\lambda(x)$ and $\lambda'(x)$

$$\lambda(x) = 4.595 + 0.064\, x^2 - 1.669\, x^3 \tag{5-41a}$$

$$\lambda'(x) = 0.128\, x - 5.007\, x^2 \tag{5-41b}$$

Eqs. 5-40, 5-41a and b may now be used in Eq. 5-31a to give a more accurate estimate of $\Delta(x)$. However, this is not done here.

The values of $\lambda(x)$, as calculated from Eq. 5-40 are given in Table IV and plotted on Fig. 10.

<u>(ii). Approximate integral solution.</u>

The solution for $\Delta(x)$ is, as stated before, assumed to be

$$\Delta(x) = 0.416$$

Since $b = 1.0$, the value of the factor $B(1+b)$, for the two ranges, is obtained from Table II as

$$0 \leqslant x \leqslant 1.414 \quad : \quad B(1+b) = -17.40$$

$$1.414 \leqslant x \leqslant x_s \quad : \quad B(1+b) = -17.64$$

The solution for $(\delta^{**})^2$ thus becomes, for $0 \leqslant x \leqslant 1.414$,

$$\frac{(\delta^{**})^2}{\nu_\infty} = \frac{0.47}{u_e^{17.40}} \int_0^x u_e^{16.40}\, dx \tag{5-42}$$

and for the range $1.414 \leqslant 0 \leqslant x_s$, Eq. 5-37b is used with the appropriate values of $B(1+b)$.

It should be pointed out that integration of Eq. 5-42 has to be performed rather carefully, specially in the neighborhood of the stagnation point, because of the large values of the exponents involved. For example, Simpson's rule will not give accurate answers, and more refined techniques will have to be used.

The curve for $\lambda(x)$ is obtained graphically from the $(\delta^{**})^2$ curve by using the relation

$$\frac{(\delta^{**})^2}{\nu_\infty} = \frac{\lambda f_2^2(\lambda)}{2 u_e'}$$

The values of $\lambda(x)$ thus obtained are given in Table IV and plotted on Fig. 10.

5.10. Results.

The variation of λ with x obtained for this special case (i.e. $u_e = x - \frac{x^3}{6}$ with $P_R = 0.7$ and $b = 1.0$) is shown in Fig. 10. The figure shows the results obtained by all the three methods.

The polynomial solution does not show good agreement with the numerical solution and the separation point is also moved too far down stream. It is felt, however, that for smaller values of the curve of λ versus x has, in general, a shape which can be better approximated by a polynomial of a low degree.

The integral solution, on the other hand, shows much better agreement with the numerical solution throughout the range of variation of x from $x = 0$ to $x = x_s$. The solution by this method is quick and easy to obtain. For any given P_R and b one obtains a value for Δ_0 by the methods of section 5.4. By substituting this value into the expression for $B(1+b)$ as given in Table II one obtains the exponents involved in the integral solution for the momentum thickness. This integral is then readily evaluated numerically and from it the curve of $\lambda = \lambda(x)$ obtained.

Fig. 11 shows the variation of $\Delta(x)$ with x as obtained by the numerical solution. It shows that the ratio of the velocity to the temperature boundary layer increases steadily from the stagnation to the separation point. However, this increase is not large, and as a first approximation $\Delta(x)$ may be assumed to be a constant.

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TABLE IA.

Values of α, $\left(\frac{1}{a}\right)$ and Δ for $M = 0$ and $u_e = Ax^{\ell}$

Prandtl Number = 1.0

Value of 'b'		+ 1.0	+ 0.5	0	- 0.5	- 1.0
$\lambda^* = 0$	α	0.485				
	$\left(\frac{1}{a}\right)$	0.485				
	Δ	1.000				
$\lambda^* = 1/4$	α	1.256	1.081	0.907	0.728	0.534
	$\left(\frac{1}{a}\right)$	0.509	0.494	0.478	0.459	0.437
	Δ	0.581	0.662	0.733	0.784	0.818
$\lambda^* = 1/2$	α	1.822	1.501	1.196	0.892	0.559
	$\left(\frac{1}{a}\right)$	0.550	0.523	0.500	0.475	0.442
	Δ	0.416	0.523	0.664	0.760	0.791

TABLE IB

Values of α, $\left(\frac{1}{a}\right)$ and Δ for $M = 0$ and $u_e = Ax^{\ell}$.

Prandtl Number = 0.7

Value of 'b'		+ 1.0	+ 0.5	0	- 0.5	- 1.0
$\lambda^* = 0$	α	0.485				
	$\left(\frac{1}{a}\right)$	0.426				
	Δ	0.878				
$\lambda^* = 1/4$	α	1.256	1.081	0.907	0.728	0.534
	$\left(\frac{1}{a}\right)$	0.509	0.494	0.478	0.459	0.437
	Δ	0.581	0.662	0.733	0.784	0.818
$\lambda^* = 1/2$	α	1.822	1.501	1.196	0.892	0.559
	$\left(\frac{1}{a}\right)$	0.550	0.523	0.500	0.475	0.442
	Δ	0.416	0.523	0.664	0.760	0.791

TABLE III.

Values of λ_0 and Δ_0

(Pr. = 0.7)

b	λ_0	Δ_0
+ 1.0	4.59	0.416
+ 0.8	5.12	0.455
+ 0.5	5.99	0.523
+ 0.2	6.82	0.609
0	7.052	0.664
- 0.2	6.76	0.713
- 0.5	5.13	0.760
- 0.8	2.35	0.782
- 1.0	0	0.791

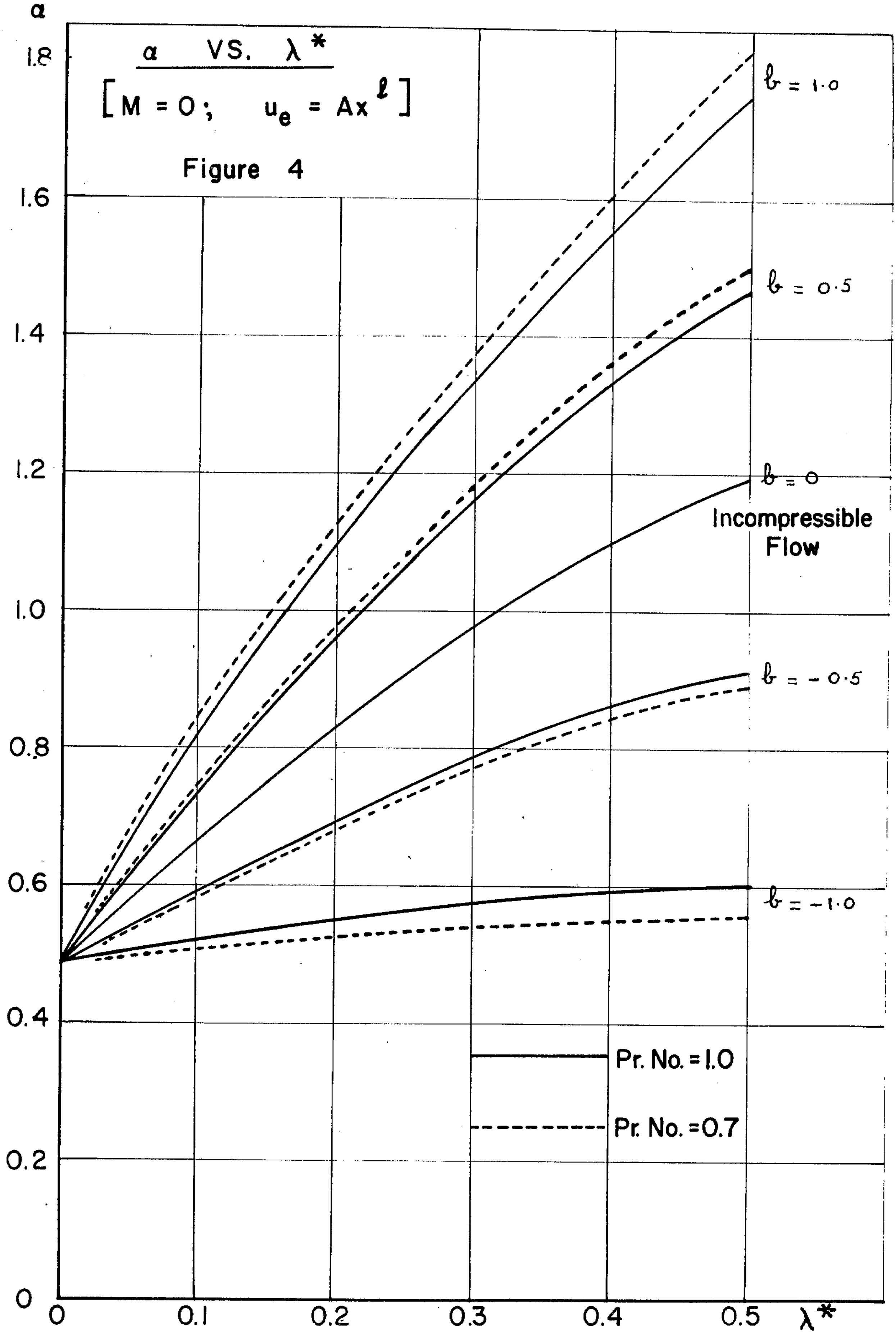
TABLE IV.

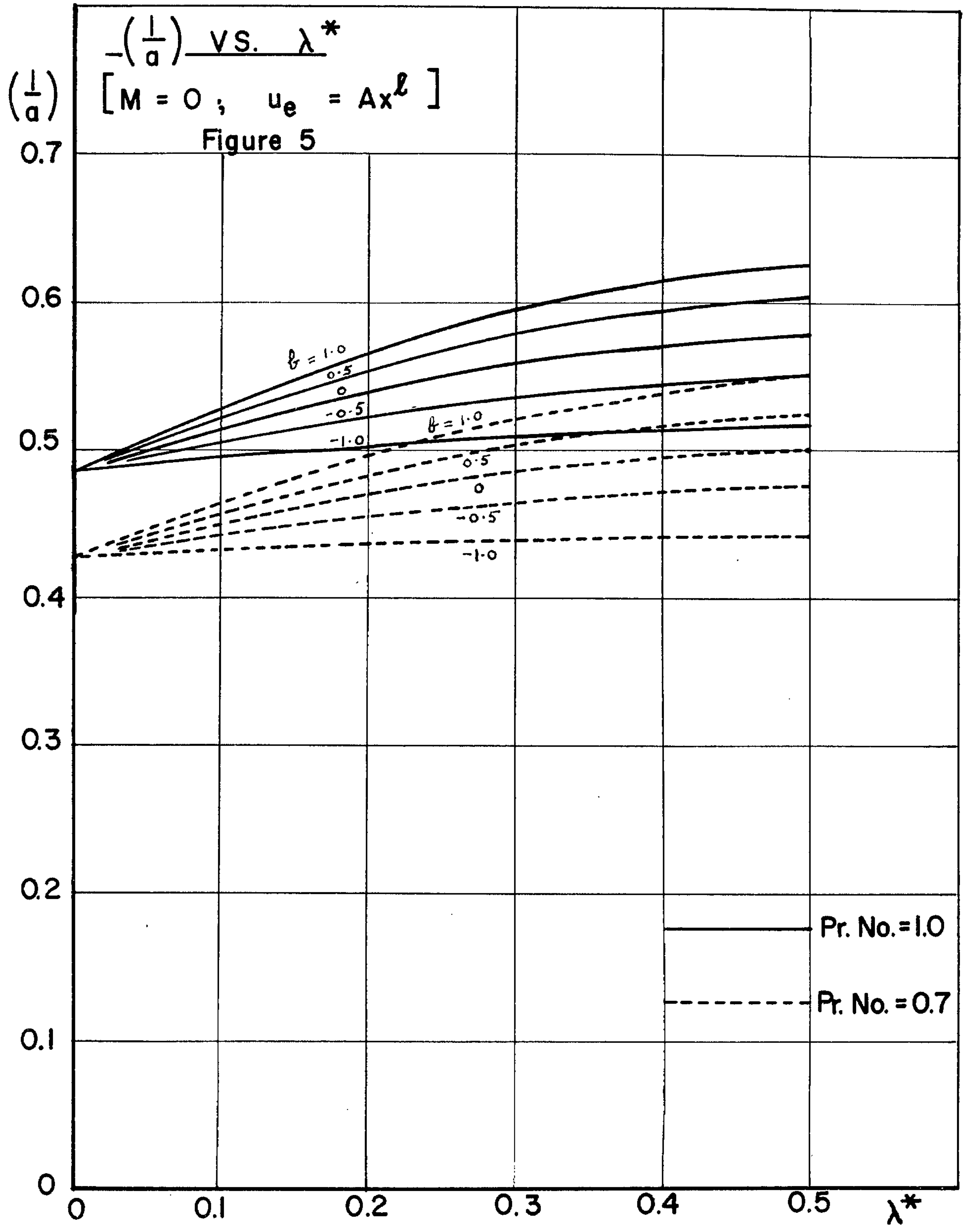
Values of $\lambda(x)$ and $\Delta(x)$

(PR. = 0.7 ; b = 1.0 ; $u_e = x - \frac{x^3}{6}$)

	Value of $\lambda(x)$			Value of $\Delta(x)$			Remarks
	NUM. INTG.	INTG. SOL.	POLY. SOL.	NUM. INTG.	INTG. SOL.	POLY. SOL.	
0	4.60	4.41	4.60	0.416	$\Delta(x)$ = 0.416	$\Delta(x)$ = 0.416	Stagnation point
0.1	4.60	4.41	4.59	0.416			
0.2	4.60	4.40	4.58	0.417			
0.3	4.60	4.39	4.56	0.419			
0.4	4.60	4.37	4.50	0.422			
0.5	4.59	4.33	4.40	0.426			
0.6	4.59	4.29	4.26	0.431			
0.7	4.57	4.22	4.05	0.437			
0.8	4.52	4.13	3.78	0.445			
0.9	4.44	4.01	3.43	0.455			
1.0	4.28	3.81	2.99	0.466			
1.1	3.98	3.50	2.45	0.480			
1.2	3.45	3.01	1.80	0.494*			
1.3	2.46	2.25	1.03	0.510*			
1.4	0.58	0.46	0.14	0.526*			
1.414	0	0	0	0.528*			Velocity maxima
1.45	-1.39	-1.36	-0.36	0.538*			
1.50	-3.54	-3.99	-0.90	0.543*			
1.55	-6.38	-8.45	-1.47	0.550			
1.570	--	-12.0	--	--			Sep. Pt. (Intg. Sol.)
1.60	-10.67	--	-2.08	0.560			
1.608	-12.0	--	--	0.561			Sep. Pt. (Num. Intg.)
1.7	--	--	-3.42	--			
1.8	--	--	-4.94	--			
1.9	--	--	-6.63	--			
2.0	--	--	-8.51	--			
2.1	--	--	-10.59	--			
2.163	--	--	-12.0	--			Sep. Pt. (Poly. Sol.)

* These values are taken from the faired curve for $\Delta(x)$


α VS. λ*
[M = 0; u_e = Ax^ℓ]
Figure 4
α
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
λ*
b = 1.0
b = 0.5
b = 0
Incompressible Flow
b = -0.5
b = -1.0
Pr. No. = 1.0
Pr. No. = 0.7


(1/a) vs. λ*
[M = 0 ; u_e = Ax^{ℓ}]
Figure 5
(1/a)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
λ*
ℓ = 1.0
0.5
0
-0.5
-1.0
ℓ = 1.0
0.5
0
-0.5
-1.0
Pr. No. = 1.0
Pr. No. = 0.7

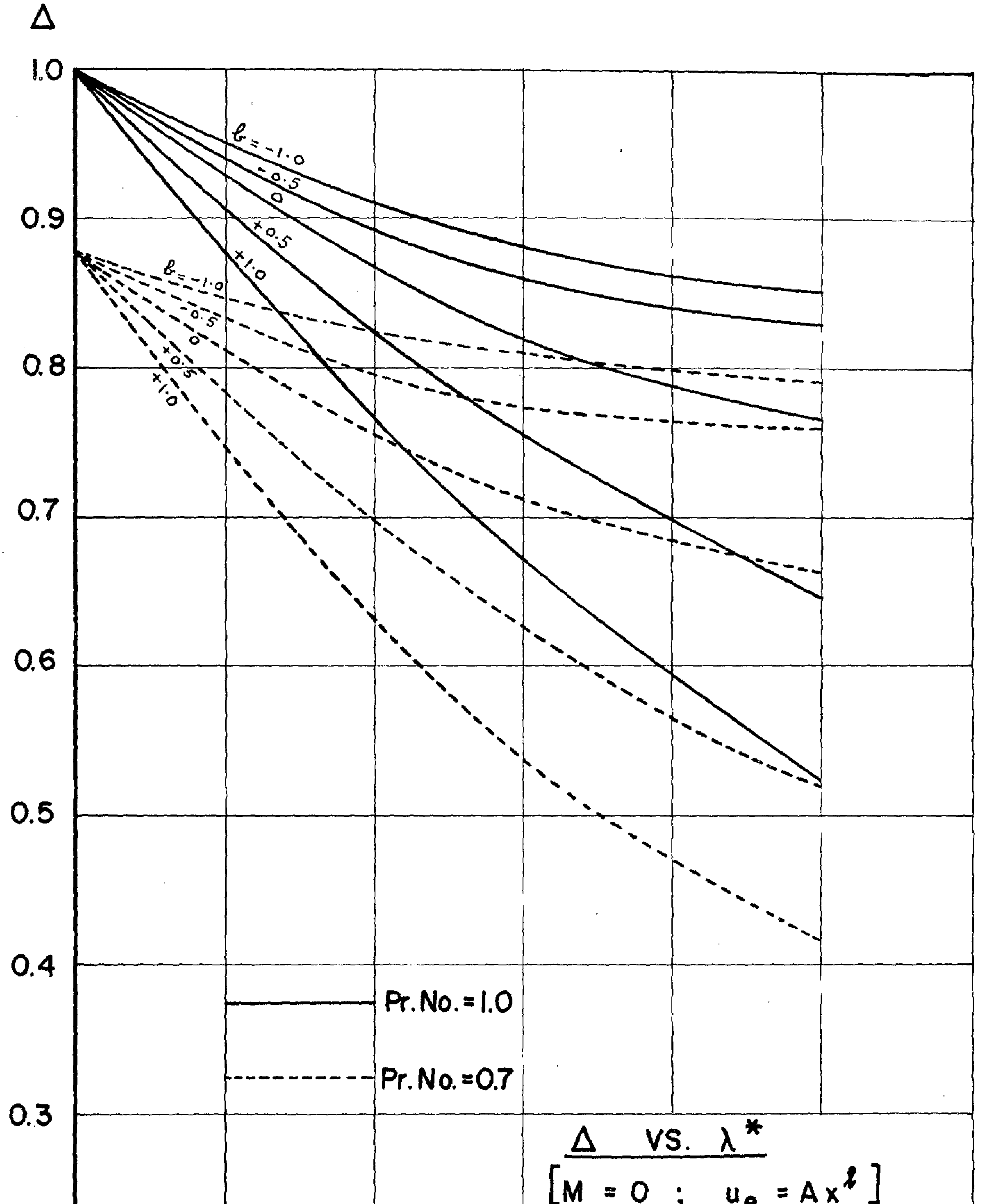
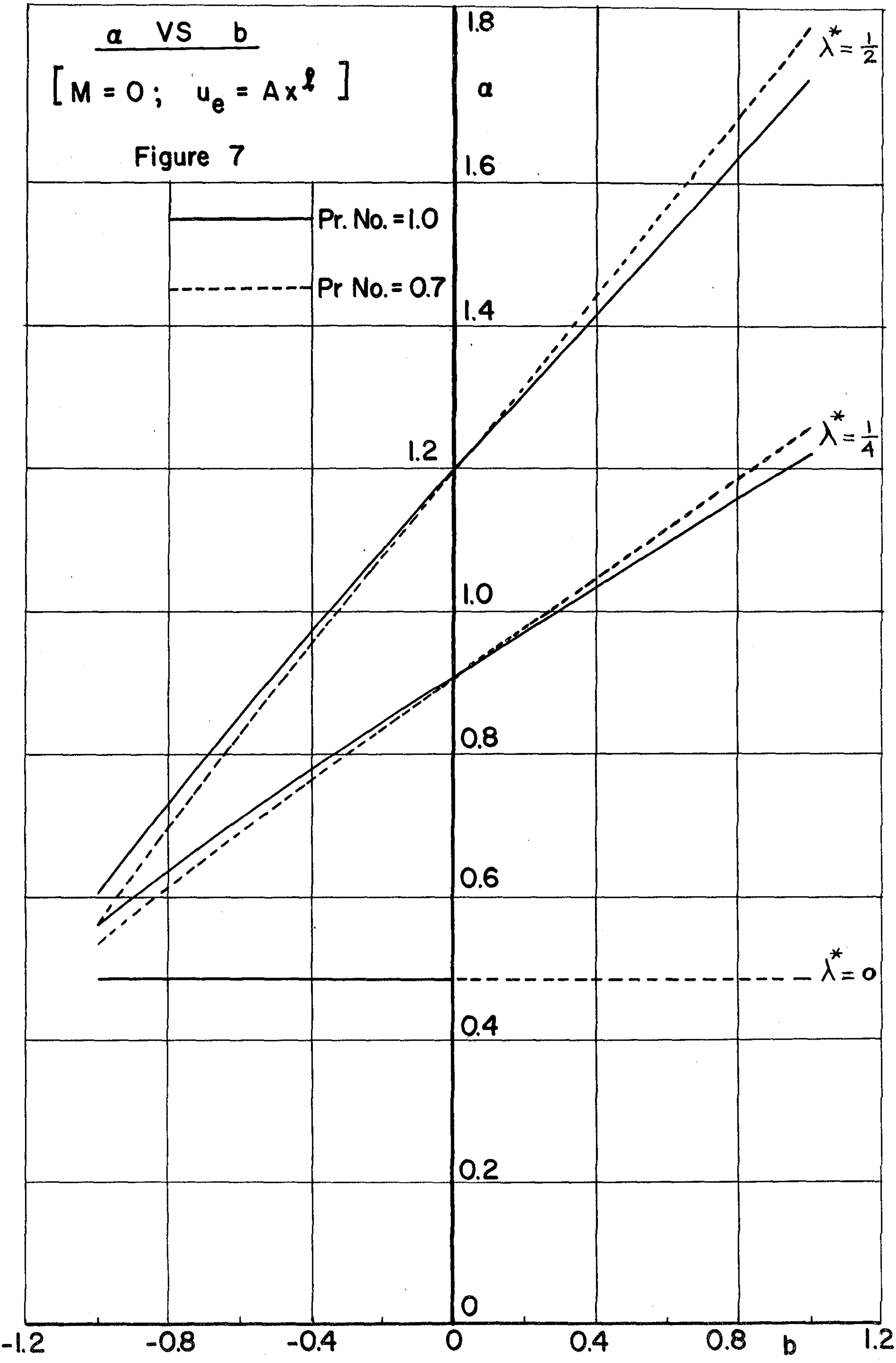
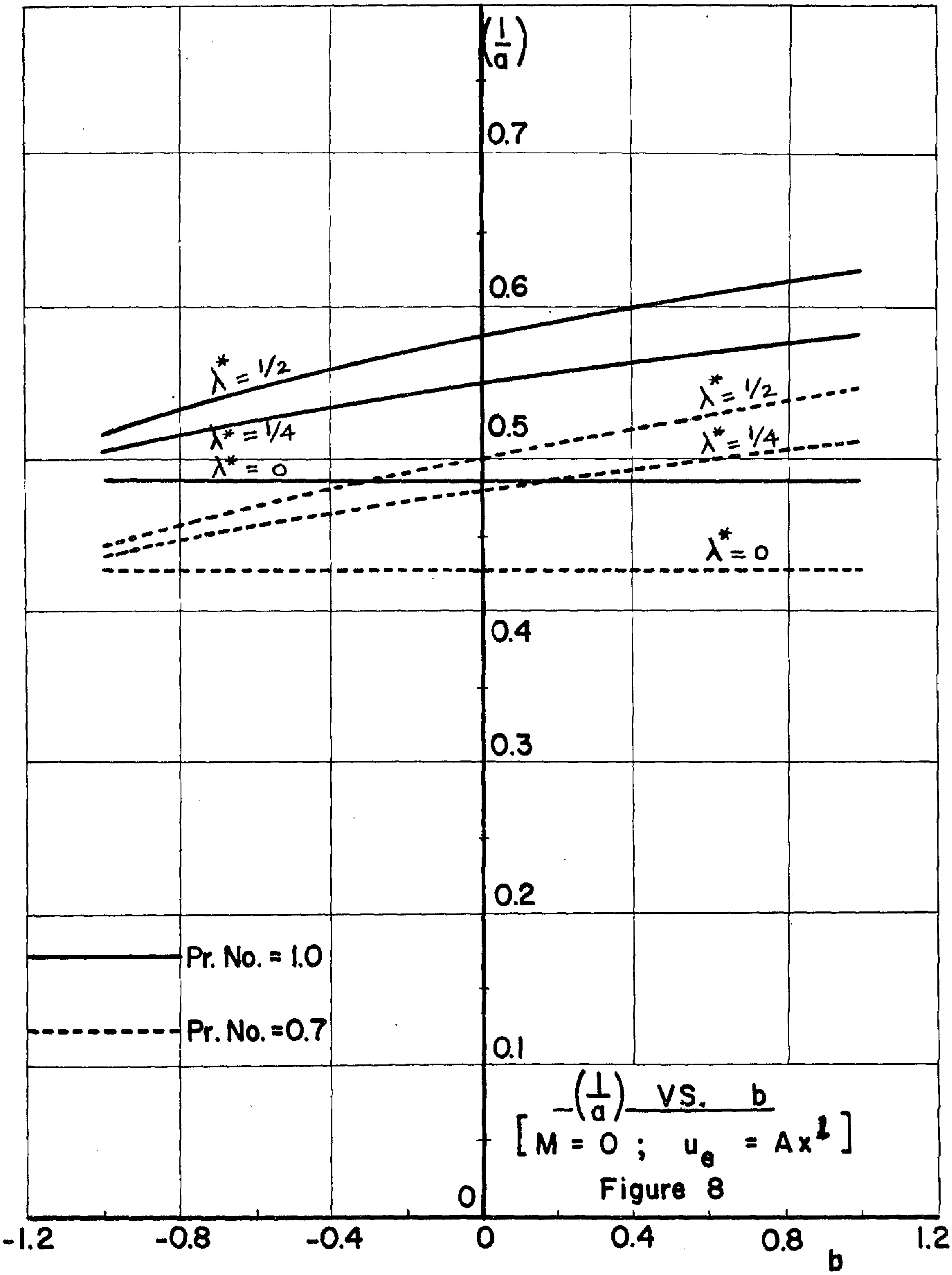

Δ
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
λ*
ℓ = -1.0
-0.5
0
+0.5
+1.0
ℓ = -1.0
-0.5
0
+0.5
+1.0
Pr. No. = 1.0
Pr. No. = 0.7
Δ VS. λ*
[M = 0 ; u_e = A x^ℓ]

Figure 6


a VS b
[M = 0; u_e = A x^ℓ]
Figure 7
Pr. No. = 1.0
Pr No. = 0.7
λ* = 1/2
λ* = 1/4
λ* = 0
a
b
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
-1.2
-0.8
-0.4
0
0.4
0.8
1.2



$\left(\frac{1}{a}\right)$ VS. b

$[M = 0 ; \quad u_e = Ax^{\ell}]$

Figure 8

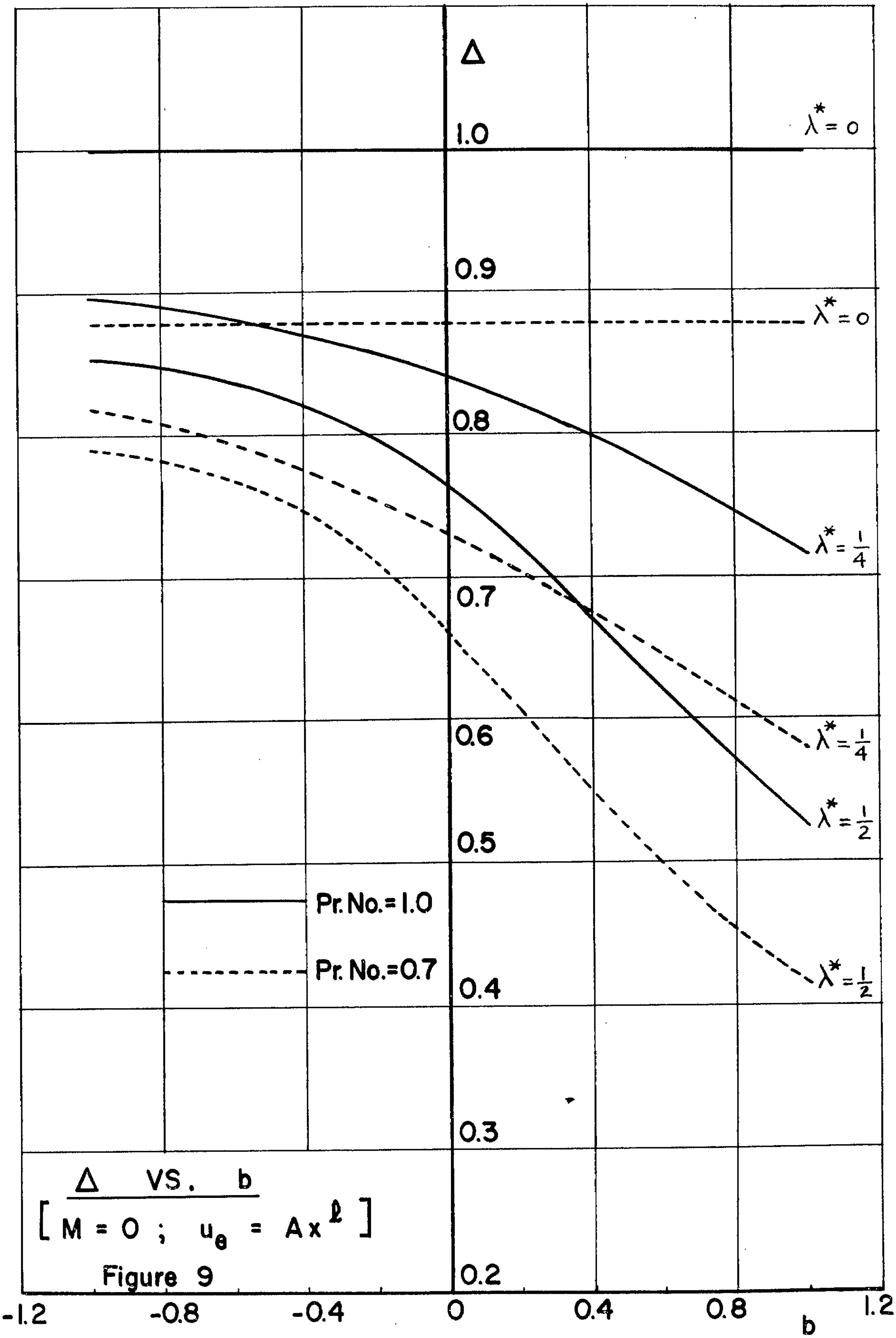

Δ
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
λ* = 0
λ* = 0
λ* = 1/4
λ* = 1/4
λ* = 1/2
λ* = 1/2
Pr. No. = 1.0
Pr. No. = 0.7
Δ VS. b
[M = 0 ; u_e = A x^ℓ]
-1.2
-0.8
-0.4
0
0.4
0.8
1.2
b

Figure 9

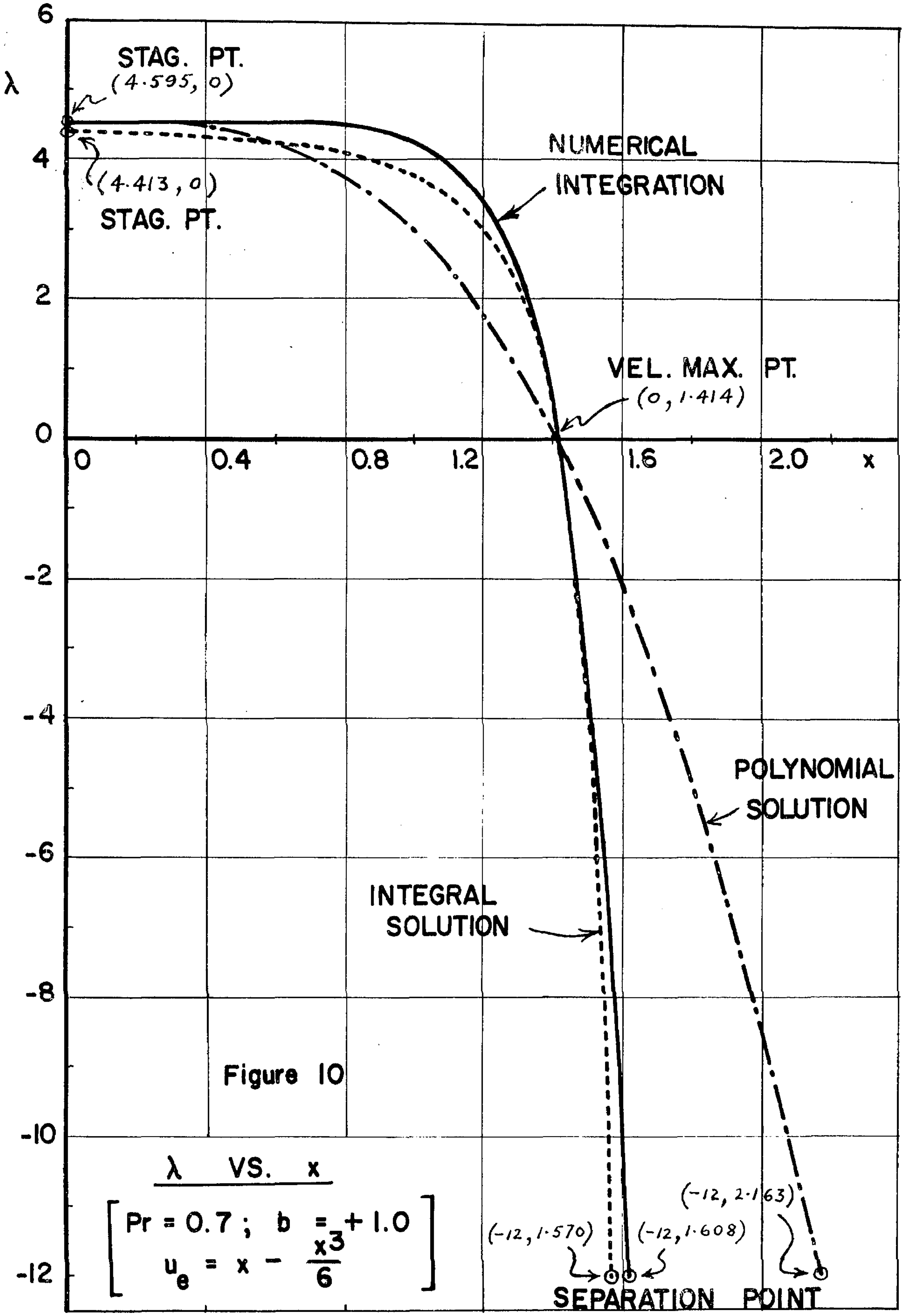

6
4
2
0
-2
-4
-6
-8
-10
-12
λ
0
0.4
0.8
1.2
1.6
2.0
x
STAG. PT.
(4·595, 0)
(4·413, 0)
STAG. PT.
NUMERICAL
INTEGRATION
VEL. MAX. PT.
(0, 1·414)
POLYNOMIAL
SOLUTION
INTEGRAL
SOLUTION
(-12, 2·163)
(-12, 1·570)
(-12, 1·608)
SEPARATION POINT
λ VS. x
Pr = 0.7 ; b = + 1.0
$u_e = x - \frac{x^3}{6}$

Figure 10

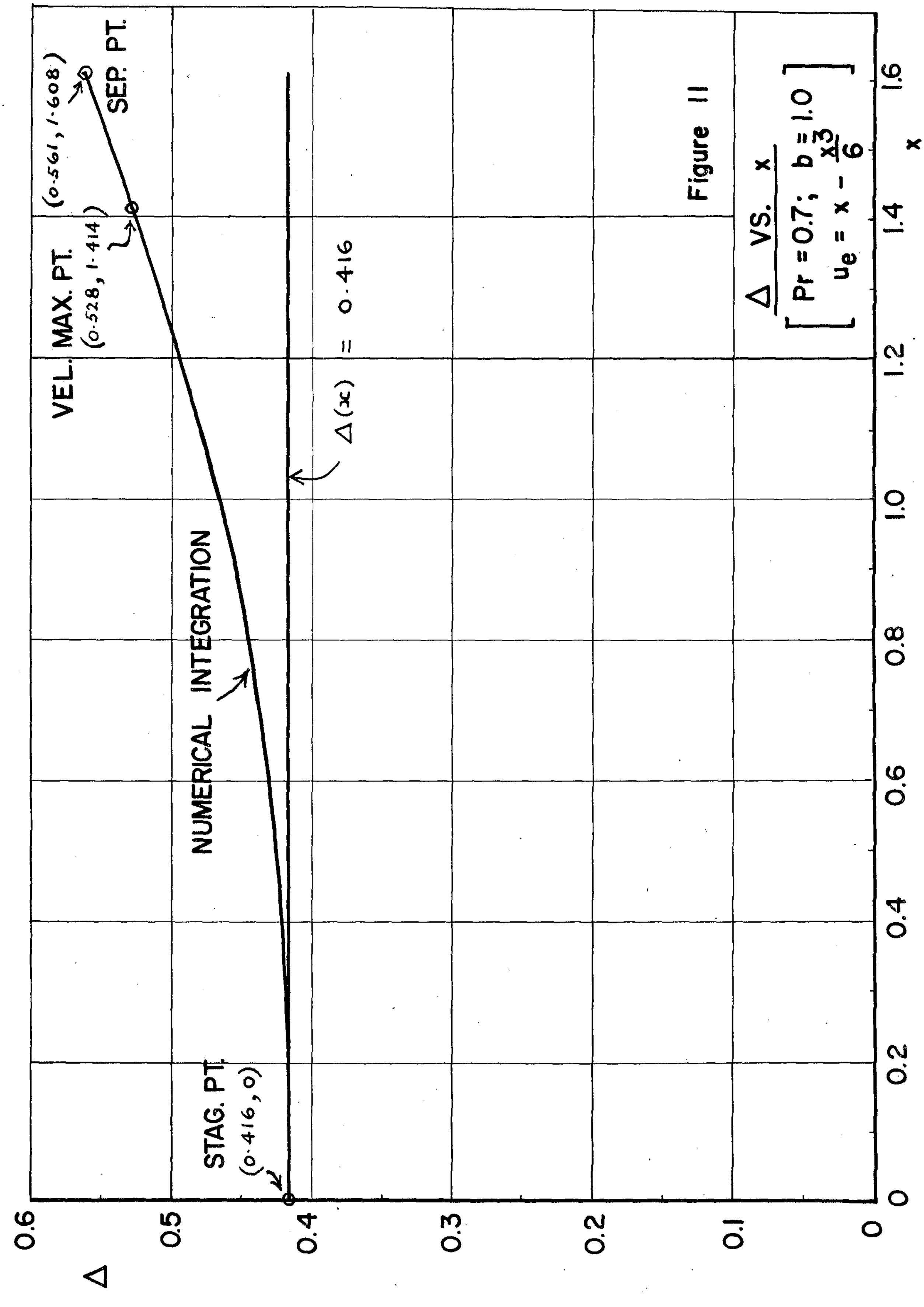

Δ
0.6
0.5
0.4
0.3
0.2
0.1
0
x
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
STAG. PT.
(0·416, 0)
NUMERICAL INTEGRATION
Δ(x) = 0·416
VEL. MAX. PT.
(0·528, 1·414)
SEP. PT.
(0·561, 1·608)
Δ VS. x
[Pr = 0.7; b = 1.0
u_e = x - x3/6]

Figure 11