THE ROLE OF COORDINATE SYSTEMS IN BOUNDARY LAYER THEORY

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ABSTRACT

The boundary layer approximation to a given flow problem is not invariant if different coordinate systems are used in the approximation process. However, a correlation theorem (Theorem 1) is given, which states that the boundary layer solution with respect to any given system can be found, by a simple substitution, from that with respect to any other system. On the basis of this theorem, the dependence of the solution on the choice of coordinates is investigated in detail. The skin friction is invariant, but the flow field is not invariant. At large distances from the wall, the flow field given by boundary layer theory depends almost entirely on the choice of coordinates, rather than on the physical problem.

This dependence may be used to obtain a complete matching between the boundary layer solution and the external flow, in the following sense: Theorem 2 states how a coordinate system can be found such that the boundary layer solution with respect to this system is valid as an approximation for the entire flow field. It contains the external flow and the flow due to displacement thickness.

The discussion is restricted to steady, two-dimensional, incompressible flow without separation. These restrictions, however, are not essential for many of the results.

I. INTRODUCTION

For a given problem, the approximate solution given by boundary layer theory depends on the system of coordinates used when the simplifying assumptions of boundary layer theory are applied to the Navier-Stokes equations. In general, different systems of coordinates lead to boundary layer equations which are not equivalent, that is, their solutions represent different flow fields. A well-known example of this is given by the boundary layer solutions for flow past a semi-infinite flat plate, when rectangular or parabolic coordinates are used in the approximation process.

The object of this thesis is to investigate in detail how different boundary layer solutions to the same flow problem are related to each other, and how a given solution is influenced by the coordinate system. It is found that the relation of boundary layer solutions to the external flow and to flow due to displacement thickness depends essentially on the choice of coordinate system. The discussion is restricted to incompressible, steady, two-dimensional flow without separation. However, many of the results hold much more generally and will be discussed in a later paper.

The main result of this paper is contained in Theorem 2. Normally one uses boundary layer theory in the following way in order to obtain a picture of the complete flow field of a viscous fluid (outside the wake): The flow field is divided into two separate regions, that is, a boundary layer region where the flow field is obtained from boundary layer equations, and an outer region where the Euler equations are used to obtain an external flow, corrected for the displacement effect of the boundary layer. There has been considerable discussion about where and how to patch the two parts of the flow field, and about how to proceed to higher order approximations. However, according to Theorem 2, a system of coordinates can be found such that the boundary layer solution with respect to this system gives an approximation which is valid in the whole flow field. Both the external flow and the flow due to displacement thickness are included analytically in this approximation and hence the problem of patching is automatically eliminated. A coordinate system with these properties will be referred to as optimal. In Section VI the problem is discussed to what order such an optimal boundary layer solution is valid. In general it gives a better approximation to the exact flow field than does the composite flow field described above. It also appears to form a reasonable starting point for finding higher order approximations, that is, approximations which are definitely outside the scope of boundary layer theory.

To find the optimal system of coordinates it is in general necessary first to compute the external flow and the flow due to displacement thickness. But once this system has been found one may use Theorem 1, which states that the boundary layer solution with respect to one coordinate system, say an optimal one, may be found by a simple substitution into the solution with respect to any other system, say a conventional one.

Concrete examples illustrating the application of the above general theorems are given in Section V.

In this paper the concepts of the boundary layer solution and the solutions for external flow and flow due to displacement thickness are defined with the aid of two special limiting processes. This viewpoint has previously been adopted by several authors, for example, H. Weyl (Ref. 1), K. O. Friedrichs (Ref. 2) and G. E. Latta (Ref. 3). This method is actually only a formal and more precise restatement of Prandtl's original method, and hence not widely used. However, it appears very natural for deriving the results of the present paper and the reasoning will actually be based on a systematic use of the two limiting processes.

There are many examples in the literature of the use of various systems of coordinates in connection with boundary layer theory, and in particular it has been noted that they give the same skin friction and agree approximately within the boundary layer region proper. To the author's knowledge, however, no systematic study of the relation between the solutions based on the various systems has been undertaken and, in particular, the two theorems referred to above appear to be new. The starting point for the present investigation was actually the following result due to M. D. Van Dyke (Ref. 4). Consider Oseen flow past a semi-infinite flat plate. If rectangular coordinates are used to make a boundary layer approximation, the resulting approximation resembles essentially the Blasius solution, and is valid only in the

the boundary layer proper. On the other hand, if parabolic coordinates are used, the boundary layer approximation satisfies exactly the full Oseen equations (cf. Example 2 in Section V). This result was further analyzed by P. A. Lagerstrom, who pointed out how in other special cases coordinates may be found which are optimal in the sense described above (Ref. 5). He also suggested to the author the general problem of finding optimal coordinates, and guided the present study. The ideas developed by G. E. Latta in Ref. (3) have also influenced the author. A closer comparison of Ref. (3) with the results and methods of the present paper would undoubtedly shed further light on the problems studied here. This comparison has, however, as yet not been carried out.

A preliminary announcement of the results of this paper was given in Ref. (6).

II. BOUNDARY LAYER APPROXIMATIONS AS LIMITS OF EXACT SOLUTIONS

The Navier-Stokes Equations. The equations for steady, viscous, incompressible flow are

$$(\vec{q}\cdot\nabla)\,\vec{q} = -\nabla p + \nu\nabla^2\vec{q} \tag{1a}$$

$$\nabla\cdot\vec{q} = 0 \tag{1b}$$

where ν is the kinematic viscosity and p is the pressure divided by density.

In two-dimensions, the kinematical form of (1) is obtained by taking the curl of (1a) and integrating (1b) by means of the stream function ψ, which gives

$$\vec{q}\cdot\nabla\omega = \nu\nabla^2\omega \tag{2a}$$

$$\omega = -\nabla^2\psi \tag{2b}$$

$$u = \frac{\partial\psi}{\partial y}\,, \qquad v = -\frac{\partial\psi}{\partial x} \tag{2c}$$

where ω is the vorticity. In arbitrary curvilinear coordinates, ξ^i, ($\xi^1 = \xi$, $\xi^2 = \eta$) the corresponding equations can be expressed as

$$u^j\frac{Du^i}{D\xi^j} = -|g|\,g^{ij}\frac{\partial p}{\partial\xi^j} + \nu\sqrt{|g|}\,g^{kl}\frac{D}{D\xi^k}\frac{Du^i}{D\xi^l} \tag{1a'}$$

$$\frac{\partial u^k}{\partial\xi^k} = 0 \tag{1b'*}$$

* $Du^i/D\xi^j$ here denotes the covariant derivative of a vector-density (cf. Ref. 7, p. 82).

and

$$\left(\frac{\partial\psi}{\partial\eta}\frac{\partial}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial}{\partial\eta}\right)\omega = \nu\frac{\partial}{\partial\xi^i}\left(\sqrt{|g|}\, g^{ij}\frac{\partial\omega}{\partial\xi^j}\right) \tag{2a'}$$

$$\omega = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial\xi^k}\left(\sqrt{|g|}\, g^{kl}\frac{\partial\psi}{\partial\xi^l}\right) \tag{2b'}$$

$$u = \frac{\partial\psi}{\partial\eta}\ , \qquad v = -\frac{\partial\psi}{\partial\xi} \tag{2c'}$$

Here ψ, ω and ν have been treated as absolute scalars. The velocity components u and v may be considered defined by (2c). In the language of tensor analysis (cf. Ref. 7) they are actually the contravariant vector density components of the velocity.* It should be noted that the square of the magnitude of the velocity vector is $u^i u^j g_{ij} / |g|$ so that the value of u^i is not equal to the value of the projection of the velocity vector on the ξ^i axis (cf. Section III below).

In this paper we shall be concerned with viscous flow past (or towards) a rigid body. The curve representing the surface of the body (called the wall) need not be closed, but may, for example, correspond to a semi-infinite flat plate or a wedge. It will be required, however, that no other boundaries are present.

The boundary conditions at the surface of the body are then

$$u = v = 0 \qquad \text{at the wall} \tag{3a}$$

ψ must then be a constant at the wall, which will be normalized to zero.

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*Weyl (Ref. 1) restricts himself to conformal coordinates in which case u and v are also covariant vector components of the velocity.

At infinity the customary boundary conditions are (in suitable Cartesian coordinates)

$$\vec{q} \to U\vec{\imath}, \quad p \to p_\infty \qquad \text{as} \quad x \to -\infty \quad \text{or} \quad |y| \to \infty \tag{3b}$$

However, for sufficiently divergent infinite bodies there is no potential solution of the equations for non-viscous flow which satisfies (3b). In this case we require that

$$\frac{\psi}{\psi_p} \to 1 \qquad \text{at infinity} \tag{3c}$$

where ψ_p represents one of the solutions for potential flow past the body. Evidently (3b) is a special case of (3c).

The term exact solution for flow past a body will henceforth be used to denote a solution of (1) or (2) with the boundary condition (3a) satisfied at the body and (3b) or (3c) at infinity. More precisely, the term will denote a class of solutions with ν as a parameter $(0 < \nu < \nu_0)$ where all solutions satisfy identical boundary conditions. Boundary layer theory is concerned with approximations to the exact flow field for small values of ν . It will be introduced below with the aid of two limit processes.

<u>First Limit Process</u>. Let $f(P, \nu)$ be a function of position and viscosity. Here P is used as an abbreviation for the coordinates (ξ, η) of a point. In the first limit process, denoted by $\lim_1$, P is held fixed while ν tends to zero: The resulting limit is denoted by subscript " e ".

$$f_e(P) = \lim_1 f(P, \nu) = \lim_{\nu \downarrow 0,\ P \text{ fixed}} f(P, \nu) \tag{4}$$

If this limit process is applied to the exact solution for flow past a solid, that is, to $\psi(P, \nu)$, $u^*(P,\nu)$ and $p(P,\nu)$, the resulting limiting functions represent a flow which will be referred to as the external flow.

In applying the first limit process to equations (1) and (2) at a point where the derivatives appearing in (1) or (2) actually tend to the corresponding derivatives of the limiting functions (which may be called a regular point), one may simply replace ψ by ψ_e, etc., and put $\nu = 0$ in the equations. The resulting equations are the Euler equations, that is, the equations of motion of a perfect fluid.

Applying the first limit process to the boundary conditions, however, it is seen that, since $u = v = 0$ at the wall, $u_e = v_e = 0$ at the wall, while $\psi_e / \psi_p \to 1$ at infinity. There is, in general, no solution of Euler's equations, satisfying the Euler equations at every point of the flow, which also satisfies these boundary conditions, so that all points of the flow cannot be regular in the above sense.

While many types of irregular behavior are conceivable, the type made prominent by experience consists of a discontinuity in the tangential velocity (a vortex-sheet discontinuity), which represents a line of slip in the perfect fluid. The behavior under the first limit process at such a discontinuity is not regular. As $\nu \to 0$ at a point on the line of discontinuity, the viscous terms in (1) or (2) do not tend to zero, and Euler's equations are not approached in the limit. Once such lines of slip in the perfect fluid are permitted, Euler's differential equations are not everywhere satisfied, and an infinite number of perfect-fluid flows becomes possible. The relevant one, that is, the limit which is actually approached, is decided by the action of viscosity, in a manner which, as yet, is not understood in many essential respects.

In ordinary problems, experience and physical arguments show that the line of slip occurs right at the wall, following the wall for some distance downstream from the point of impingement of the streamline coming from upstream infinity, but that it may eventually separate from the wall, defining a wake. It is to be expected that the limits $\psi_e(P)$, $u^i_e(P)$, $p_e(P)$, actually exist and satisfy Euler's equations, at least outside the region of the wake, which is essentially unexplored at the present time.* The present discussion will, however, be restricted to flows where no separation occurs. Then, all the streamlines come from upstream infinity, the flow is irrotational and satisfies Laplace's equation. The action of viscosity is restricted to the determination of circulation about the body, in accordance with the Kutta condition. ψ_e can then be found by solving the corresponding potential problem.

If the relevant solution of Euler's equations, u^i_e, is used as a starting point, the only essential effect of introducing ν small, but greater than zero, is to replace the discontinuity by a rapid, but continuous transition, which is nearly completed within a narrow region (boundary layer). Perturbation problems of this nature are referred to in recent literature as singular (see Ref. 3 and references given there).

In a regular perturbation problem, the successive perturbations are obtained by successive application of the first limit process, which is essentially equivalent to expanding the solution in powers of a small parameter. In a singular perturbation problem, however, $u^i \to u^i_e$ non-uniformly in any region containing boundary points. Hence, at

*This is briefly commented on further in Section VI.

$\nu > 0$; the limit μ^i_ε breaks down as an approximation, sufficiently close to the boundary. An approximation for the continuous transition across the boundary layer may be found by the application of a second limit process.

Second Limit Process. Let ε be a small parameter which in a sense measures the effective thickness of the transition region. In the present case ε will be taken proportional to $\sqrt{\nu}$. More generally ν may be a function of ε such that

$$\lim_{\varepsilon \to 0} \frac{\nu}{\varepsilon^2} = const. \neq 0 \tag{4}$$

A system of coordinates (ξ , η) will be chosen with the only restriction that η be zero at the wall. It is customary also to require (cf., for example, Ref. 8, p. 119) that the lines ξ = constant be orthogonal to the wall. This assumption will not be made here. The wider choice of coordinate systems thus provided will be essential for obtaining the principal result of this paper.

In the second limit process, the point P will not be fixed but will move toward the wall along a curve ξ = constant in such a way that the ratio

$$\bar{\eta} = \eta/\varepsilon \tag{5}$$

is fixed, in other words, keeping ξ and $\bar{\eta}$ constant. Let $f(P, \nu)$ as before be a physical quantity depending on position and viscosity. The limiting function approached in the second limit process will be denoted by $f_s(\xi, \bar{\eta})$ or $\lim_s f$ and defined by

$$f_s(\xi, \bar{\eta}) = \lim_s f = \lim_{\varepsilon \to 0;\ \xi, \bar{\eta}\ \text{fixed}} f(\xi, \bar{\eta}, \varepsilon) \tag{6}$$

Here ζ is used as an abbreviation for the coordinate system (ξ, η). Note that the first limit of $\mathfrak{f}$ depends on $\mathfrak{f}$ only; the second limit, on the other hand, depends also on the choice of coordinates. To emphasize this, ζ is used as a subscript in the notation introduced by (6).

Now consider an exact solution as described by ψ, u, v, ω, p. We define

$$\bar{\psi} = \frac{\psi}{\varepsilon}, \quad \bar{v} = \frac{v}{\varepsilon}, \quad \bar{\omega} = \varepsilon\,\omega \tag{7}$$

The boundary layer approximation to the quantities $\bar{\psi}$, u, $\bar{v}$, $\bar{\omega}$, p may then be defined as the limiting functions $\bar{\psi}_\zeta$, u_ζ, $\bar{v}_\zeta$, $\bar{\omega}_\zeta$, and p_ζ, that is

$$\bar{\psi}_\zeta = \lim_\zeta \bar{\psi}, \quad u_\zeta = \lim_\zeta u, \quad \text{etc.} \tag{8a}$$

The following notation will also be used

$$\psi_\zeta = \varepsilon\,\bar{\psi}_\zeta, \quad v_\zeta = \varepsilon\,\bar{v}_\zeta, \quad \omega_\zeta = \frac{\bar{\omega}_\zeta}{\varepsilon} \tag{8b}$$

The slight inconsistency in the notation should be kept in mind: ψ_ζ is not obtained by applying the second limit process to ψ, but by applying it to ψ/ε and afterwards multiplying it by ε. If now $\bar{\eta}$ is again replaced by η/ε, ψ_ζ, etc. are functions of ξ, η and ν. The flow field which has ψ_ζ as stream function is the flow field given by boundary layer theory. (The relation of ψ_ζ to u_ζ, ω_ζ, etc. will be discussed below, see (10b) and ff.) The definition of the boundary layer approximation thus introduced is formally different but actually equivalent to Prandtl's definition (cf. also Refs. (1), (2), and (3)). It is an approximation with respect to a coordinate system ζ. A different coordinate system would in general give rise to a different approximation as discussed in Section III.

The functions $\bar{\psi}_\xi$, u_ξ , $\bar{v}_\xi$, $\bar{\omega}_\xi$, and p_ξ depend on the two variables ξ and η/ε only, and their variation with the latter variable represents the transition across the boundary layer. Note that the boundary layer approximation is formally defined wherever the ξ -coordinates are defined, hence a flow field is formally given even outside the boundary layer proper. It is this complete flow field that will be discussed in the present paper.

ψ_ξ , u_ξ , etc. satisfy the Prandtl boundary-layer equations which actually may be obtained by applying the second limit process to Eqs. (1') or (2'). As an example, (2') become

$$\left(\frac{\partial \bar{\psi}_\xi}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial \bar{\psi}_\xi}{\partial \xi} \frac{\partial}{\partial \bar{\eta}} \right) \bar{\omega}_\xi = \sqrt{|g|_w}\; g^{22}_w \frac{\partial^2}{\partial \bar{\eta}^2} \bar{\omega}_\xi \tag{9a}$$

$$\bar{\omega}_\xi = - g^{22}_w \frac{\partial^2 \bar{\psi}_\xi}{\partial \bar{\eta}^2} \tag{9b}$$

Here the subscript "$_w$" indicates that the quantity in question has been evaluated at the wall: $g^{22}_w(\xi) = g^{22}(\xi, 0)$, etc.

Note that the relation of $\bar{\psi}$ to u and $\bar{v}$ is unaltered by the second limit process

$$\frac{\partial \bar{\psi}_\xi}{\partial \bar{\eta}} = u_\xi \quad , \quad \frac{\partial \bar{\psi}_\xi}{\partial \xi} = - \bar{v}_\xi \tag{10a}$$

Similarly

$$\frac{\partial \psi_\xi}{\partial \eta} = u_\xi \quad , \quad \frac{\partial \psi_\xi}{\partial \xi} = - v_\xi \tag{10b}$$

As a consequence (1') and (2') remain equivalent after the second limit process has been carried out. The flow field described by ψ_ξ is the same as that described by u_ξ and v_ξ . On the other hand, the relation of ω to this flow field has been altered. The vorticity of the flow field described by ψ_ζ is $-\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial \xi^k}\left(\sqrt{|g|}\, g^{kl}\, \frac{\partial \psi_\zeta}{\partial \xi^l}\right)$. As is seen by (9b) this is in general different from ω_ζ .

The boundary layer approximation may also be defined as the solution of (9) with certain approximate boundary conditions which will be discussed later. For the present, however, it will be regarded as the result of applying the second limit process, in the manner described above, to the exact solution of the Navier-Stokes equations.

III. COMPARISON OF DIFFERENT BOUNDARY LAYER SOLUTIONS

The boundary layer approximations to an exact solution of the Navier-Stokes equations are not uniquely determined, but depend both on the choice of dependent variables used to represent the flow field, and on the independent variables, that is, the coordinates.

As remarked above, the ψ and the (u, v) representations are equivalent. In general, however, different representations lead to different approximations. For example, when (ξ, η) are orthogonal coordinates the flow field is often represented by means of velocity components (U, V) in the sense of vector geometry (cf. Ref. (8), p. 101 and especially Ref. (7), p. 114). The magnitudes of U and V are then equal to the projections of the velocity vector on the local coordinates axes. They are connected with the contravariant vector densities u, v through the relations

$$u = h_2(\xi, \eta)\, U \quad , \qquad v = h_1(\xi, \eta)\, V \tag{11a}$$

where

$$ds^2 = h_1^2\, d\xi^2 + h_2^2\, d\eta^2$$

is the square of the line element. Applying the second limit process one obtains

$$u_\xi = h_{2w}\, U_\xi \quad , \quad \bar{v}_\xi = h_{1w}\, \bar{V}_\xi \tag{11b}$$

where

$$h_{iw} = h_i(\xi, 0) \quad \text{and} \quad \bar{V} = \frac{V}{\varepsilon} \; .$$

Thus the original metrical relations (11a) no longer hold except at the wall. The flow field corresponding to u_ξ and $\bar{v}_\xi$ will then be different from that corresponding to U_ξ and $\bar{V}_\xi$. The example of the

(U, V) representation also shows that, in general, the boundary layer approximation does not satisfy the exact continuity equation.* However, the ψ and the (u, v) representations have the special property of preserving the exact continuity equation under the boundary layer approximation. In the following only the ψ and the equivalent (u, v) representations will be used. While a further investigation of the role of the dependent variables may be of interest, it will not be carried out in the present paper, which is mainly concerned with the role of the dependent variables.

The flow field will, in general, also depend on the choice of the function ε . In the following it will therefore be assumed that ε = constant $\cdot \sqrt{\nu}$. The choice of the constant will not affect the flow field.

Main Correlation Theorem. Consider now two boundary layer approximations to the same exact solution, one based on the system of coordinates $\zeta = (\xi, \eta)$ and the other one on another system $\chi = (\rho, \sigma)$. Here η and σ respectively are assumed to be zero at the wall. The two flow fields may be represented by the stream functions ψ_ζ and ψ_χ respectively. Their relation is then described by the following correlation theorem:

Theorem 1. If $\bar{\psi}_\zeta = f(\xi, \bar{\eta})$ is given, then the boundary layer solution with respect to χ can be found directly by the substitution formula

$$\bar{\psi}_\chi = f(\xi_\chi, \bar{\eta}_\chi) \tag{12a}$$

*Cf. Ref. (8), p. 119.

where

$$\xi_x = \lim_x \xi = \xi(\rho, 0) \tag{12b}$$

$$\bar{\eta}_x = \lim_x \bar{\eta} = \left(\frac{\partial \bar{\eta}}{\partial \bar{\sigma}}\right)_{\bar{\sigma}=0} \cdot \bar{\sigma} \tag{12c}$$

The evaluations of the limits (12b) and (12c) are easily shown to be correct by considering ξ and η as functions of σ for a fixed ρ, and using the fact that η and σ vanish at the wall. Equation (12a) can be obtained directly, if it is assumed that $\psi(\xi, \bar{\eta}, \varepsilon)$ is a continuous function of ξ, $\bar{\eta}$ and ε (in the space of those variables), at $\varepsilon = 0$.

The validity of the theorem can be checked also by verifying that $\bar{\Psi}_x$, as given by (12a), satisfies the appropriate boundary layer equations (cf. (9)). The theorem states that the result of applying $\lim_x$ to $\bar{\Psi}$ is identical with that of first applying $\lim_\xi$ and then $\lim_x$.

The dependence of the boundary layer solution on the choice of coordinates can now be studied with the aid of (12). Since ψ is treated as an absolute scalar ψ_ξ and ψ_x represent the same flow field if and only if the systems ξ and x are connected by the relations

$$\xi(\rho, \sigma) = \xi(\rho, 0) \tag{13a}$$

$$\eta(\rho, \sigma) = \left(\frac{\partial \eta}{\partial \sigma}\right)_{\sigma=0} \cdot \sigma \tag{13b}$$

Equivalent relations are

$$\xi = \mathfrak{f}_1(\rho) \tag{13a'}$$

$$\eta = \mathfrak{f}_2(\rho)\,\sigma \tag{13b'}$$

where $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are arbitrary functions. According to (13a) only the shape of the curves ξ = constant, but not their spacing ("labelling") has an effect on the boundary layer approximation. Equation (13b) implies that a relabelling of the curves η = constant other than $\eta \rightarrow$ constant $\cdot\, \eta$, would alter the flow field given by the boundary layer approximation. On the other hand, certain changes of shape of the η-curves do not change the flow field.

The flow fields given by ψ_ζ and ψ_x will now be compared in the case when ζ and x are arbitrary systems, not necessarily related by (13).

<u>Conditions at the Wall.</u> In general it should be noted that, since $\eta = \bar{\eta} = 0$ at the wall, the first and second limit processes are identical there. Since the first limit is independent of ζ, the same must then be true for the second limit at the wall. It is then a direct consequence of the boundary condition (3a) that at the wall

$$u_\zeta(\xi, 0) = v_\zeta(\xi, 0) = 0 \tag{14}$$

independent of the system of coordinates.

Furthermore, we have the important relation that the skin friction obtained from the boundary layer approximation is independent of the choice of coordinates. This can be seen directly, as follows. Let τ denote the exact skin friction (the subscript "w" being omitted from the usual notation for convenience). In analogy with (8) define $\bar{\tau} = \frac{\tau}{\varepsilon}$, $\bar{\tau}_\zeta = \lim_\zeta \bar{\tau}$ and $\tau_\zeta = \varepsilon\, \bar{\tau}_\zeta$. Then

$$\tau_\zeta = \varepsilon\, \bar{\tau}_\zeta = \varepsilon \lim_{\varepsilon \to 0} \left[\varepsilon^{-1}\, \tau(\zeta, \varepsilon) \right] \tag{15}$$

Since the limit on the right does not depend on the nature of coordinates, $\tau_\zeta = \tau_x$ at any point on the wall.

Whenever the no-slip condition (3a) holds, the exact skin friction is simply μ times the vorticity at the wall

$$\tau = \mu\omega_w = -\mu g^{22}_w \left(\frac{\partial^2\psi}{\partial\eta^2}\right)_w . \tag{16a}$$

Passing to the limit

$$\tau_\xi = \frac{\mu}{\varepsilon}\bar{\omega}_{\xi w} = -\mu g^{22}_w \left(\frac{\partial^2\psi_\xi}{\partial\eta^2}\right)_w \tag{16b}$$

Thus, in computing τ_ξ from the velocity field given by the boundary layer approximation, one may use the exact relation between stress and rate of strain. However, if the no-slip condition does not hold, τ_ξ would denote the skin friction computed by means of an appropriate approximate relation, derived from the exact relation by the passage to the limit. If the exact stress relation is used, the skin friction may not be invariant when slipping occurs.

<u>Differences at a Distance from the Boundary</u>. At the boundary the choice of coordinates is irrelevant in the sense discussed above. Within the boundary layer proper, that is, within a distance of order ε from the wall, the flow associated with ψ_ξ and ψ_x should agree up to and excluding terms of order ε. This follows from Theorem 1.

However, at a distance from the boundary, ψ_ξ and ψ_x may differ radically. According to (12), the flow (ψ_x) may be regarded as an image of the flow (ψ_ξ), obtained by the following mapping: let a point P go into an image point Q in such a way that, if the ξ - and η -affices of P are

$$\xi(P) = a, \quad \eta(P) = b, \tag{12e}$$

the point Q is determined by the relations

$$\xi_x(Q) = a, \quad \varepsilon\bar{\eta}_x(Q) = b \tag{12f}$$

Then, the stream function has the same numerical values at Q and at P , that is

$$\psi_\chi(Q) = \psi_\zeta(P) . \tag{12g}$$

Thus, a streamline $\psi_\chi = c$ will be the direct image of the streamline $\psi_\zeta = c$, but, in general, the coordinate curves will not be direct images under this mapping. However, if χ and ζ are connected at the wall by the special relations

$$\rho_w = \xi_w , \quad \left(\frac{\partial \eta}{\partial \sigma}\right)_w = 1 , \tag{12h}$$

ξ_χ becomes equal to ρ and $\varepsilon\bar{\eta}_\chi$ to σ , so that the χ -affices of Q become

$$\rho(Q) = a = \xi(P) , \quad \sigma(Q) = b = \eta(P) . \tag{12j}$$

Thus, if (12h) holds, the coordinate curves as well as the streamline curves map as direct images. This means that a given streamline, $\psi = c$, passes through the same coordinate affices in both flows.

Now, keeping conditions (12h) imposed at the wall (with respect to some fixed ζ -system), one may deform the χ -coordinates in an otherwise general manner. In particular, in any region external to the wall, of the type $\eta \geq \eta_0 > 0$, any desired continuous deformation of the χ- coordinates can be performed without violating (12h). Since the direct-image relations then hold, it is clear that, within a region of this type, the family of streamlines may be deformed into any desired continuous image of itself by a suitable choice of χ .

Thus, at a distance from the boundary the flow field obtained from the boundary layer approximation depends almost exclusively on the choice of coordinates rather than on the physical problem. For this reason it is customary to disregard the boundary layer approximation entirely outside the boundary layer. The boundary layer solution is assumed valid as a certain approximation up to the edge of the boundary layer where it is patched on to a potential solution. However, the viewpoint taken in this paper is different. One may make positive use of the dependence of ψ_ζ on the choice of ζ by searching for special systems of coordinates such that ψ_ζ approaches the exact solution ψ as closely as possible. Then ψ_ζ would be a valid approximation even outside the boundary layer region proper. The problem of such "optimal" systems of coordinates will be discussed in the following section.

IV. COMPARISON WITH EXACT SOLUTION. CHOICE OF OPTIMAL COORDINATES

In the preceding section it was shown how the boundary layer solution, as described, for example, by ψ_ζ , varies radically with the choice of ζ . The question was raised whether one could turn this fact to good advantage by choosing ζ in such a way that ψ_ζ is in some sense very close to ψ , even outside the boundary layer region proper. The conditions that such a ψ_ζ would have to satisfy, need to be specified more precisely. It is certainly natural to require that u_ζ and v_ζ satisfy the same boundary conditions as u and v at infinity. A stronger requirement would be that near infinity ψ_ζ agrees with ψ up to a certain order in ε . One may actually require that ψ and ψ_ζ agree up to and including first order terms in ε everywhere outside the boundary. An equivalent requirement is that ψ_ζ contain, in a sense to be defined, the external flow and the flow due to displacement thickness. (The latter flow is actually the first order perturbation of ψ_e outside the boundary.)

Below it will be shown that systems of coordinates ζ may be found such that ψ_ζ satisfies this requirement. Such systems will be called optimal. The Discussion in Section VI will briefly touch upon the question whether other requirements on ψ_ζ would be preferable.

<u>Behavior of ψ_ζ for $\bar{\eta}$ Large. First Order Approximation to ψ for $\eta \neq 0$</u>. A fundamental assumption of boundary layer theory is that, for a flow property f

$$\lim_{\bar{\eta}\to\infty} (\lim_\zeta f) = \lim_{\eta\to 0} (\lim_1 f) \tag{17}$$

provided the limits exist. Applying (17) to u yields

$$u_\zeta \to u_{ew} \qquad \text{as } \bar{\eta} \to \infty \tag{18}$$

Applying (17) to v or $\bar{v}$, or to ψ or $\bar{\psi}$, however, leads in general to trivial results. To find the behavior of ψ_ζ and v_ζ for $\bar{\eta}$ large, higher order external perturbations have to be considered.

Previously, the external flow had been defined as

$$\psi_e = \lim_1 \psi \;, \quad u_e = \lim_1 u \;, \quad \text{etc.} \tag{19a}$$

We shall now consider this definition to be valid only for $\eta \neq 0$ and extend it by continuity to the wall:

$$\psi_{ew} = \lim_{\eta \to 0} \psi_e(\xi, \eta) \,, \quad u_{ew} = \lim_{\eta \to 0} u_e(\xi, \eta), \text{ etc.} \tag{19b}$$

Note that if (19a) is applied even for $\eta = 0$, ψ_e would have been continuous there whereas u_e in general would be discontinuous.

Due to the presence of the boundary layer, it is to be expected that the first external perturbation is of order ε, that is,

$$\psi = \psi_e(P) + \varepsilon\, \psi_e'(P) + \sigma(\varepsilon) \tag{20}$$

where $\lim_1 \frac{\sigma(\varepsilon)}{\varepsilon} = 0$. Thus the first order perturbation may be defined by a repeated application of the first limit process

$$\psi_e' = \lim_1 \frac{\psi - \psi_e}{\varepsilon} \qquad (\eta \neq 0) \tag{21a}$$

$$\psi_{ew}' = \psi_e'(\xi, 0) = \lim_{\eta \to 0} \psi_e'(\xi, \eta) \tag{21b}$$

Again it is necessary to extend the definition to $\eta = 0$ by continuity since $\lim_1 \frac{\psi_w - \psi_{ew}}{\varepsilon}$ is zero whereas ψ_{ew}', as will be seen by (23), is in general different from zero.

Now, for η small,

$$\psi_e = u_{ew}\,\eta + \left(\frac{\partial^2 \psi_e}{\partial \eta^2}\right)_w \frac{\eta^2}{2} + \ldots$$

Hence

$$\lim_\zeta \left(\frac{\psi_e}{\varepsilon}\right) = u_{ew}\,\bar{\eta} \quad \text{and} \quad \lim_\zeta \left(\frac{\psi - \psi_e}{\varepsilon}\right) = \bar{\psi}_\zeta - u_{ew}\,\bar{\eta}$$

Applying (17) to $\frac{\psi - \psi_e}{\varepsilon}$ one then finds that

$$\bar{\psi}_\zeta - u_{ew}\,\bar{\eta} \to \psi'_{ew} \qquad \text{as } \bar{\eta} \to \infty \tag{22}$$

or

$$\psi_\zeta \sim u_{ew}\,\eta + \varepsilon\,\psi'_{ew} \qquad \text{for } \bar{\eta} \text{ large} \tag{22'}$$

Since

$$\bar{\psi}_\zeta - u_{ew}\,\bar{\eta} = \int_0^{\bar{\eta}} (u_\zeta - u_{ew})\, d\bar{\eta}$$

it follows from (22) that

$$\psi'_{ew} = \int_0^\infty (u_\zeta - u_{ew})\, d\bar{\eta} \tag{23}$$

This relation shows how ψ'_{ew} may be evaluated from any given boundary layer approximation. On the other hand, since ψ'_e may be defined by (21), the integral in (23) must be independent of the choice of ζ . This may also be checked from (12). It may be seen that ψ'_e satisfies the Laplace equation and hence represents potential flow. ψ'_e may then be determined if one uses (23) as a boundary condition, together with the additional condition that $\psi'_e/\psi_e \to 0$ at infinity. It follows from the preceding discussion, in particular from (23), that ψ'_e represents the external perturbation flow due to displacement thickness. ψ_e represents a flow past the body so that the body surface is a streamline $\psi_e = 0$. ψ'_e , on the other hand, represents a flow that seems to emanate from a source distribution at the wall.

First Order Approximation to ψ_ζ for $\eta \neq 0$. The first limit process may be applied to the boundary layer solution ψ_ζ in the same manner as was done to ψ. The flow field represented by the limit $\psi_{\zeta e}$ will be called the external flow contained in the boundary layer solution. Since, for a fixed η, $\bar{\eta}$ tends to infinity as ε tends to zero, one may use equation (22) in evaluating the limit:

$$\psi_{\zeta e} = \lim_1 \psi_\zeta = u_{ew}\eta \tag{24a}$$

The associated velocity component $u_{\zeta e} = \dfrac{\partial \psi_{\zeta e}}{\partial \eta} = u_{ew}$ may be found directly as $\lim_1 (u_\zeta)$ for $\eta \neq 0$, and extended to the wall by continuity. Thus, for a fixed ξ, the curves of u_ζ versus η with ε as parameter will have the line $u = u_{ew}$ = constant as a limiting curve for ε tending to zero. The external flow field contained in the boundary layer approximation has thus a constant ξ-component of velocity. This is of course due to the fact that u_ζ depends on η and ε only in the combination η/ε so that the limiting value $u_{\zeta e}$ is $u_\zeta(\xi, \infty)$. On the other hand, the ξ-component of the exact solution may have a more complicated dependence on η and ε. Hence, its corresponding limiting curve $u_e(\xi, \eta)$ may vary with η and need coincide with $u_{\zeta e}$ only at the wall.

Similarly, one may define the external first perturbation contained in the boundary layer approximation and use (23) in its evaluation:

$$\psi'_{\zeta e} = \lim_1 \frac{\psi_\zeta - \psi_{\zeta e}}{\varepsilon} = \psi'_{ew} \qquad (\eta \neq 0) \tag{24b}$$

Its value for $\eta = 0$ is defined as the limiting value when $\eta \to 0$. Since $\psi'_{\zeta e}$ is independent of η it follows that $u'_{\zeta e}$ is zero and $v'_{\zeta e} = -\dfrac{\partial \psi'_{ew}}{\partial \xi} = v'_{ew}$. Hence $v'_{\zeta e}$ and v'_e, regarded as functions of η, must coincide for $\eta = 0$, but as η varies, the former must remain constant, whereas the latter may vary.

Optimal Coordinates. The boundary layer solution will be said to contain the external flow of the exact solution, whenever

$$\psi_{\zeta e} \equiv \psi_e \qquad \text{that is} \qquad \psi_e = \mu_{ew}\,\eta \tag{25a}$$

and to contain the flow due to displacement thickness, whenever

$$\psi'_{\zeta e} \equiv \psi'_e \qquad \text{that is} \qquad \psi'_e = \psi'_{ew} \tag{25b}$$

A coordinate system will be called optimal if both these equations are satisfied. Assuming that ψ_e and ψ'_e are given, let $\chi = (\rho, \sigma)$ be such an optimal system. It can be seen immediately that (25b) is satisfied if and only if ψ'_e is a function of ρ only. In other words, it is sufficient (and necessary) to choose the curves ρ = constant to coincide with streamlines of the flow due to displacement thickness. (25a) is satisfied if and only if $\mu_e = \frac{\partial \psi_e}{\partial \eta} = \mu_{ew}$, so that μ_e is a function of ρ only. This can be accomplished by taking $\sigma = \psi_e$. Most generally, one may choose $\sigma = f(\rho)\,\psi_e$ where $f(\rho)$ is an arbitrary function of ρ. Then $\mu_e = \frac{1}{f(\rho)} = \mu_{ew}$. Hence, the following theorem has been proved.

Theorem 2. The coordinate system $\zeta = (\xi, \eta)$ is a particular optimal system, in the sense that (25a) and (25b) are satisfied, if

$$\xi = \psi'_e \,, \qquad \eta = \psi_e \tag{26a}$$

Any other system $\chi = (\rho, \sigma)$ is optimal if and only if it is related to the above system by

$$\rho = f_1(\xi) \,, \qquad \sigma = f_2(\xi)\,\eta \tag{26b}$$

where f_1 and f_2 are arbitrary functions. The flow field given by the boundary layer approximation is the same for all optimal systems, but will be different if any other system is chosen.

The last statement follows directly from a comparison of (26b) and (13').

It will now be shown that the optimal boundary layer solution, as formally defined by the requirement (25) also has the other properties discussed at the beginning of this section. First it will be checked that it satisfies all the boundary conditions imposed on the exact solution. The conditions at the wall (3a) are satisfied according to (14). At infinity ψ approaches a potential solution which actually is ψ_ℓ, whereas ψ_ζ approaches $\psi_{\zeta\ell}$. If now $\psi_\ell \equiv \psi_{\zeta\ell}$ it follows that ψ_ζ satisfies the correct boundary condition. Note that according to the same boundary condition μ_ζ should approach $\mu_\ell(\xi, \infty)$. On the other hand, for any boundary layer solution, $\mu_\zeta(\xi, \infty) = \mu_\ell(\xi, 0)$. To satisfy the correct boundary condition at infinity for μ_ζ one should then make $\mu_\ell(\xi, 0) = \mu_\ell(\xi, \infty)$. An easy way of achieving this is to choose $\eta = \psi_\ell$ as in (26a) since in this case μ_ℓ = constant = 1.

So far only the relation $\psi_{\zeta\ell} = \psi_\ell$ has been used. If in addition $\psi'_{\zeta\ell} = \psi'_\ell$ then ψ and ψ_ζ agree to first order at infinity and actually in any region which excludes the boundary. This follows directly from the fact that for $\eta \neq 0$,

$$\psi = \psi_\ell + \varepsilon\psi' + o(\varepsilon) \tag{20}$$

and also

$$\psi_\zeta = \psi_{\zeta\ell} + \varepsilon\psi'_{\zeta\ell} + o(\varepsilon)$$

A further evaluation of the result will be attempted in Section VI.

Construction of Optimal Approximation. In general, the optimal system cannot be determined a priori. However, one may proceed as follows. First the external flow ψ_e is found by solving a potential flow problem. Then a boundary layer solution ψ_ζ with respect to any convenient system of coordinates is found by solving boundary layer equations (9), or equivalent equations, using (14) and (18) as boundary conditions. By evaluating the integral in (23) one finds ψ'_{ew} and then the flow due to displacement thickness by solving Laplace's equation for ψ'_e . ψ_e and ψ'_e being known, an optimal system χ is directly given by (26). Finally, to find ψ_χ it is not necessary to solve the corresponding equation. ψ_χ may be obtained from ψ_ζ by a direct substitution in accordance with (12).

V. EXAMPLES

Example 1. Flow Towards a Flat Plate

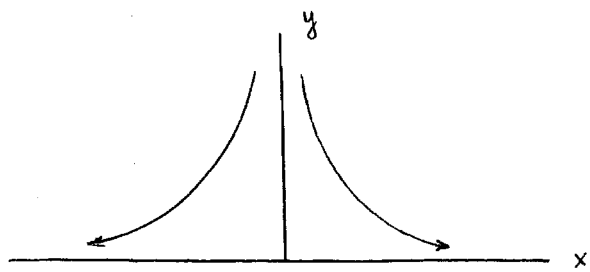

Fig. 1

The stream function

$$\psi = \sqrt{\nu U}\cdot x\, f(\bar{y}) \tag{27a}$$

where $\bar{y} = \sqrt{\frac{U}{\nu}}$, and

$$f''' + f f'' - (f')^2 + 1 = 0 \tag{27b}$$

$$f(0) = f'(0) = 0\ ; \quad f'(\infty) = 1\ , \quad f(\infty) \sim \bar{y} - c \tag{27c}$$

$$c = \int_0^\infty \left[1 - f'(\bar{y})\right] d\bar{y} \tag{27d}$$

represents an exact solution of the Navier-Stokes equations for $y \geq 0$, satisfying the conditions $\psi = \frac{\partial\psi}{\partial y} = 0$ at $y = 0$, $\psi \sim Uxy - cx\sqrt{\nu U}$ as $y \to \infty$ (cf. previous comments and Eq. (3c)). Thus ψ represents a flow normal to an infinite wall, as in Fig. 1 (see Refs. (1) and (2)).

Holding U, x and y fixed, and letting $\nu \to 0$, we obtain the external flow. Thus,

$$\psi_e = \lim_{1} \left(\sqrt{\nu U}\cdot x\, f\left(\sqrt{\tfrac{U}{\nu}}\, y\right)\right) = -cUx \tag{28}$$

Choosing the boundary layer thickness parameter as $\varepsilon = \sqrt{\frac{\nu}{U}}$, and regarding U as constant under the limit process, the boundary layer approximation with respect to the x-y coordinates is found by applying the $\lim_2$ process. Thus,

$$\bar{\Psi}_2 = \lim_2 \frac{\Psi}{\varepsilon} = \lim_2 Ux\, f(\bar{y}) = Ux\, f(\bar{y}) \tag{29a}$$

or,

$$\Psi_2 = \varepsilon \bar{\Psi}_2 = \sqrt{\nu U}\; x\, f(\bar{y}) \tag{29b}$$

Finally, the flow due to displacement thickness is found as:

$$\Psi_\ell' = \lim_1 \left(\frac{\sqrt{\nu U}\; x\, f(\bar{y}) - Uxy}{\varepsilon} \right) = -cUx \tag{30}$$

Thus, the optimal ξ -coordinates are $\xi = f_1(\Psi_\ell')$, or, since the choice of f_1 is inconsequential, $\xi = x$. The optimal η -coordinates are $\eta = f_2(\xi)\cdot\Psi_\ell$. Since the choice of f_2 does not affect the result, we can take $f_2(\xi) = 1/U\xi$, that is, $\eta = y$. Thus, in the present case, the x-y coordinates lead to the optimal boundary layer approximation, which, as may be seen by comparing Ψ_2 with Ψ , happens to satisfy the full Navier-Stokes equation.

Example 2. Oseen Flow Past a Semi-Infinite Flat Plate

In this example the Oseen equations will be regarded as a model for the Navier-Stokes equations rather than as an approximation. In other words, the Oseen equations will be taken as the fundamental "exact" equations and we shall be concerned with approximations to these equations. The results of the present paper apply, of course, equally well to the Oseen equations as to the Navier-Stokes equations.

The Oseen equation for ψ is:

$$\left(\nu \nabla^2 - U \frac{\partial}{\partial x}\right) \nabla^2 \psi = 0 \tag{31a}$$

The boundary conditions for a flat plate are

$$\psi = \frac{\partial \psi}{\partial y} = 0 \quad \text{at } y = 0, \quad x > 0 \tag{31b}$$

$$\frac{\partial \psi}{\partial y} \to U, \quad \frac{\partial \psi}{\partial x} \to 0 \quad \text{as } |y| \to \infty, \text{ or as } x \to -\infty \tag{31c}$$

The stream function ψ_e is obtained from the equation resulting by the analogous passage to the limit, that is,

$$\frac{\partial}{\partial x} \nabla^2 \psi_e = 0 \tag{32a}$$

and the boundary conditions:

$$\psi_e = 0 \text{ at } y = 0, \ x > 0; \quad \frac{\partial \psi_e}{\partial y} \to U, \ \frac{\partial \psi_e}{\partial x} \to 0 \text{ as } |y| \to \infty, \ x \to -\infty \tag{32b}$$

(cf. Section 2), corresponding to the assumption of no separation. Furthermore, in view of symmetry of the viscous problem about the x -axis, it is expected that the circulation is zero. Thus,

$$\psi_e = U y \tag{32c}$$

Next, choosing $\varepsilon = \sqrt{\frac{\nu}{U}}$, (31a) is written in the form:

$$\left[\left(\frac{\partial^2}{\partial \bar{y}^2} + \varepsilon^2 \frac{\partial^2}{\partial x^2}\right) - \frac{\partial}{\partial x}\right] \left(\frac{\partial^2}{\partial \bar{y}^2} + \varepsilon^2 \frac{\partial^2}{\partial x^2}\right) \bar{\Psi} = 0 \tag{33a}$$

Holding x, $\bar{y}$ (and nominally, U) constant, and letting $\varepsilon \to 0$, $\bar{\Psi} \to \bar{\Psi}_z$, that is, to the boundary layer approximation with respect to rectangular coordinates, we obtain:

$$\left(\frac{\partial^2}{\partial \bar{y}^2} - \frac{\partial}{\partial x}\right) \frac{\partial^2 \bar{\Psi}_z}{\partial \bar{y}^2} = 0 \tag{33b}$$

$$\bar{\Psi}_z = \frac{\partial \bar{\Psi}_z}{\partial \bar{y}} = 0 \quad \text{at } \bar{y} = 0, \ x > 0 \tag{33c}$$

Furthermore, at infinity, the boundary conditions are:

$$\bar{\Psi}_z \sim U\bar{y} + \Psi'_{\ell w} \tag{33d}$$

where $\Psi'_{\ell w}$ is unknown.

Hence, the boundary layer approximation with respect to rectangular coordinates is:

$$\bar{\Psi}_z = 2U\sqrt{x}\left\{\frac{\bar{y}}{2\sqrt{x}}\ \mathrm{erf}\ \frac{\bar{y}}{2\sqrt{x}} + \frac{1}{\sqrt{\pi}}\left(e^{-\frac{\bar{y}^2}{4x}} - 1\right)\right\} \tag{33e}$$

This gives

$$\Psi'_{\ell w} = -\frac{2U\sqrt{x}}{\sqrt{\pi}} \tag{34a}$$

Putting $\Psi = \Psi_\ell + \varepsilon\Psi'$ in Eq. (31a) and letting $\varepsilon \to 0$, the perturbation equation is:

$$\frac{\partial}{\partial x}\nabla^2\Psi'_\ell = 0 \tag{34b}$$

or, since Ψ represents an irrotational flow, as $x \to -\infty$,

$$\nabla^2\Psi'_\ell = 0 \tag{34c}$$

The solution, corresponding to a distribution of fundamental sources at the wall, which satisfies (34c) and (34a) is:

$$\Psi'_\ell = -2U\sqrt{\frac{r + x}{2\pi}} \tag{34d}$$

Since Ψ'_ℓ is constant along confocal parabolas, the optimal ξ-co-ordinates may conveniently be taken as:

$$\xi = \mathrm{Re}\ \zeta\ ,\quad \zeta = \sqrt{x + iy}\ ,\quad 0 \leq \arg(x+iy) < 2\pi \tag{35a}$$

The optimal η-coordinates are of the form $\eta = f(\xi)\,y$, or, choosing $f(\xi) = \frac{1}{2\xi}$, we may take

$$\eta = \mathrm{Im}\ \zeta \tag{35b}$$

The classical approximation is now transformed to optimal coordinates, using the transformation (12). The quantities x_ξ, $\bar{y}_\xi$, are, from (35):

$$x_\xi = \xi^2 \tag{36a}$$

$$\bar{y}_\xi = 2\xi\bar{\eta} \tag{36b}$$

Hence, the optimal approximation is:

$$\bar{\psi}_\xi = 2U\xi \left\{ \bar{\eta} \operatorname{erf} \bar{\eta} + \frac{1}{\sqrt{\pi}} \left(e^{-\bar{\eta}^2} - 1 \right) \right\} \tag{36c}$$

or,

$$\psi_\xi = 2\sqrt{U\nu} \cdot \xi \left\{ \bar{\eta} \operatorname{erf} \bar{\eta} + \frac{1}{\sqrt{\pi}} \left(e^{-\bar{\eta}^2} - 1 \right) \right\} \tag{36d}$$

While it can be seen that the classical approximation has a non-vanishing v -component of velocity at infinity and does not satisfy all the boundary conditions for ψ, and is further characterized by all the "anomalies" of the Blasius solution, it can be verified that the optimal boundary layer approximation satisfies exactly the Oseen equation and all the boundary conditions (31).*

Example 3. <u>Flow Against a Wedge</u>

For this class of flows, the exact solution of the Navier-Stokes equation is known only when the wedge angle is 180° (cf. Example 1). ψ_e is therefore found by solving

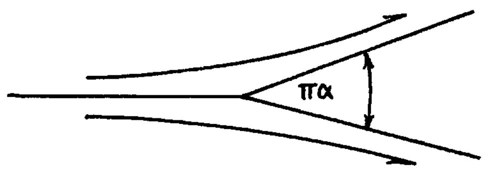

Fig. 2

*The solution of Oseen's equations is due to Lewis and Carrier (Ref. 9) and the phenomenon that the boundary layer approximation in parabolic coordinates is exact, has first been noticed by Van Dyke (Ref. 4).

the corresponding potential problem which gives

$$\psi_e = \mathrm{Im}\left(U \frac{2-\alpha}{2} e^{-i\frac{\pi\alpha}{2-\alpha}} z_1^{\frac{2}{2-\alpha}} \right) \tag{37}$$

where $z_1 = x_1 + iy_1$ is the complex variable in the physical plane, U is the speed of the external flow at $z_1 = -1$, and $\pi\alpha$ is the apex angle (cf. Fig. 2).

The boundary layer solution for the wedge has been investigated, among other authors, by H. Weyl (Ref. 1), who uses "streamline potential" coordinates as a basis for the approximation, that is, the coordinates:

$$x = \frac{\varphi_e}{U} \quad , \quad y = \frac{\psi_e}{U} \tag{38}$$

where φ_e is the harmonic conjugate of ψ_e, normalized to be zero at the apex. We then have $\psi_e = Uy$, or, $u_e \equiv U = u_{ew}$. The boundary layer equations based on the streamline potential coordinates admit a solution of the "similarity" type, namely, of the form:

$$\bar{\Psi}_z = U\sqrt{x}\, f_\alpha(\lambda) \quad , \quad \lambda = \frac{\bar{y}}{\sqrt{x}} \tag{39a}$$

where,

$$f_\alpha(0) = f_\alpha'(0) = 0 \; , \quad f_\alpha'(\infty) = 1 \tag{39b}$$

Furthermore, as $\lambda \to \infty$, $f_\alpha(\lambda) \sim \lambda - c_\alpha$ where $c_\alpha = \int_0^\infty [1 - f'(\lambda)]d\lambda$. Thus, $\psi_{ew}' = Uc_\alpha\sqrt{x}$, and, therefore, solving Laplace's equation, we obtain:

$$\psi_e' = Uc_\alpha\xi \; , \quad \text{where } \xi = \mathrm{Re}\sqrt{x+iy} \tag{40}$$

It is thus seen that the streamline potential coordinates are not optimal. They have the limited advantage that ψ_e is contained in the boundary layer approximation, but ψ'_ℓ is not contained, and all the singularities of the Blasius approximation are present.

The optimal ξ-coordinates may be taken as

$$\xi = \mathcal{R}\ell\, \zeta \tag{41a}$$

and the optimal η-coordinates may be taken as

$$\eta = \mathcal{J}m\, \zeta \tag{41b}$$

where $\zeta = \sqrt{z}$. This gives a conformal set of coordinates, which may be called streamline parabolic. The optimal coordinates in Examples 1 and 2 are special cases of this system.

We have $x_\xi = \xi^2$, $\bar{y}_\zeta = 2\xi\bar{\eta}$, and so the optimal boundary layer approximation is:

$$\bar{\psi}_\zeta = U\xi\, f_\alpha(2\bar{\eta}) \tag{42}$$

where f_α is the same function as that in Eq. (39).

Now, as $\bar{\eta} \to \infty$, $\psi_\zeta \sim 2U\xi\eta - \varepsilon U c_\alpha \xi = \psi_\ell + \varepsilon\psi'_\ell$, and further, the solution extends over the entire plane and is free from anomalies. Unfortunately, however, the Navier-Stokes equation is not satisfied by the optimal approximation, except if $\alpha = 1$, corresponding to Example 1.

A special case is $\alpha = 0$, which represents flow past a semi-infinite flat plate, in other words, the non-linear version of Example 2. In this case parabolic coordinates are optimal. They have been used for this case in Refs. (10), (11), and (12) where, however, the inclusion of flow due to displacement thickness is not commented upon.

It should be pointed out that, in general, the optimal coordinates cannot be taken as conformal. This is evident, for example, if the streamlines ψ_e' = constant meet the wall at angles other than normal. An example of this is flow past a finite flat plate.

VI. DISCUSSION

The significance of Theorem 2 will now be discussed.

First, it is obvious from (15) that if one is interested only in the first approximation to the skin friction, then the choice of coordinates is irrelevant. The purpose of choosing an optimal system is rather to obtain a good picture of the complete flow field. The problem studied here, is, vaguely speaking, that of obtaining as much information as possible within the limits of boundary layer theory. It is reasonable to require this information, before proceeding to approximations which are definitely beyond boundary layer theory, for example, in considering higher order approximations to the skin friction.

The flow field of the optimal solution will therefore be compared with a flow field obtained by standard methods, that is the composite flow field defined as follows: A certain definite edge, terminating the boundary-layer region, in introduced. Within the boundary-layer region, the flow field is taken from a non-optimal boundary-layer solution; outside the edge, a potential flow, consisting of the external flow and flow due to displacement thickness, is assumed. The viscous and non-viscous solutions are patched together along the edge.

First a general observation can be made. The unpatched solution, that is, the optimal boundary layer solution, is conceptually clearer. It is free from artificial singularities and discontinuities and therefore easier to handle analytically. The question of where and how to patch and the problems associated with this are avoided.

In order to make a more detailed comparison it is convenient to consider separately the potential part of the flow and the viscous shear waves. These terms are easily defined in the case of Oseen flow, since in that case the velocity field separates into a potential and a transversal wave in such a way that all pressure forces are associated with the former and the viscous shear only with the latter. A suitable definition for the non-linear case will be discussed later. For the time being we shall consider the vorticity as representing the viscous shear waves. Thus consider first the vorticity. In the patched solution it is assumed to be zero outside the boundary layer. In the optimal solution it vanishes continuously away from the boundary, but very rapidly (exponentially in Example 2), so that the actual numerical correction is small. However, in the neighborhood of the leading edge, the difference may be large. Consider, for example, the flat plate (Examples 2 and 3). If rectangular coordinates are used, the entire line $x = 0$ is singular. The vorticity tends to infinity as x decreases to zero for any value of y. Furthermore, no vorticity is obtained in the region upstream of the leading edge. These defects are remedied by the use of parabolic coordinates. The artificial singularity along the line $x = 0$ is removed, and a certain upstream vorticity is obtained. In the Oseen case, the optimal solution is, in fact, exact and thus holds in particular near the origin. In the non-linear case, the optimal solution is not correct near the origin, but, qualitatively, comes much closer to the anticipated actual behavior of the exact solution than the solution in rectangular coordinates. It follows from the elliptic nature of the Navier-Stokes equations that an upstream spreading of vorticity actually must occur. Similar remarks apply to the wedge (Example 3), in comparing the solutions in streamline-potential and streamline-parabolic coordinates. Inside the boundary layer region, no improvement in the values of the vorticity may be guaranteed by the use of optimal coordinates. This will be commented on later.

Next consider the potential part of the flow. Outside the boundary layer proper the patched flow field is, by construction, simply the potential part of the optimal solution. In considering the flow inside the boundary layer region it should be remembered that any boundary layer solution contains part of the external flow and flow due to displacement thickness. In the terminology of Section IV, the relevant comparison is thus between $\psi_\ell + \varepsilon \psi'_\ell$ and $\psi_{\zeta\ell} + \varepsilon \psi'_{\zeta\ell}$, the former being the correct value (to the order ε). In particular it is of interest to compare the corresponding first derivatives since these represent the velocity fields. $\psi_{\zeta\ell}$ and its first derivatives have the correct values at the wall but their normal derivatives of higher order are different. Hence, within the boundary layer region (of thickness $\sqrt{\nu}$) $\mu_{\zeta\ell}$ may show a mistake of order $\sqrt{\nu}$. Similarly, only the values of $\psi'_{\zeta\ell}$ are correct at the wall, while the normal derivatives are not generally correct. Thus, another error of order $\sqrt{\nu}$ generally appears in $\mu'_{\zeta\ell}$. These errors cause difficulty in connection with the patching at the edge of the boundary layer.

It should be remembered, however, that the criterion of optimality adopted here is still somewhat arbitrary. A more systematic approach would be to use boundary layer theory as a basis for finding the beginning of an expansion of the form

$$\psi \sim \psi_0(P,\nu) + \varepsilon\,\psi_1(P,\nu) + \ldots \tag{43}$$

where each ψ_n is of order unity uniformly in the complete flow field. (Certain singular points, like the leading edge, would require special consideration.) To the author's knowledge, the first systematic discussion of an asymptotic expansion of the type (43) for the case of singular perturbation problems was given by G. E. Latta in Ref. 3. Considering the optimal solution in this light, we see that it is valid uniformly up to and including terms of order ε in any region which does not include the boundary, since here only the external terms $\psi_\ell + \varepsilon\psi_\ell'$ are involved. On the other hand, if the region includes the boundary, the optimal solution is not necessarily accurate enough to include terms of order ε since terms of order ε are in general omitted from the boundary layer equations. Actually, the improvement of the boundary layer solution discussed in this paper was based mainly on a consideration of the potential part of the flow field, even if some incidential improvement of the viscous shear waves resulted. To investigate the matter further it then seems natural to treat the viscous shear waves and the potential part of the flow field separately.

These terms, used previously in a somewhat vague fashion, may be defined as follows. The potential part is the part obtained by a repeated application of the first limit process, namely $\psi_\ell + \varepsilon\psi_\ell' + \varepsilon^2\psi_\ell'' + \ldots$, where $\psi_\ell'' = \lim_1\left[\frac{1}{\varepsilon^2}(\psi - \psi_\ell - \varepsilon\,\psi_\ell')\right]$ etc. The viscous shear waves would be the remainder, namely the difference between ψ and its potential part.* Approximations to the potential part may be found by successively solving Laplace's equation with appropriate boundary conditions (which for the higher terms are related to the shear waves). Boundary layer theory may then be applied for the purpose of finding an approximation to the viscous shear waves. This procedure, in a sense, represents patching at the wall, rather than at some edge of the boundary layer. An equivalent procedure is actually used in Ref. 3. If this method is followed the choice of coordinates should be guided by a study of the shear waves alone. In general, different coordinates may then appear optimal. It should be noted, however, that for Example 2 and certain special cases of Example 3, Latta was led to the introduction of parabolic and streamline-parabolic coordinates from a consideration of a series expansion of the form (43). Thus in these special cases different methods yield the same results. A closer integration of the methods of Ref. 3 and the methods used in this paper would undoubtedly prove very fruitful.

Finally, a brief comment will be added about the case when separation occurs at the body. The use of boundary layer theory in this case is hampered by the fundamental difficulty that the relevant external flow ψ_e is different from the potential unseparated flow past the body. Then it is actually not correct to determine the point of

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*The series representing the potential part may be divergent. In this case, the procedure would only amount to a splitting of the asymptotic expansion (43), term by term, into potential and viscous parts. On the other hand, it may be possible to define the potential part exactly, by considering the behavior at ϑ fixed and r $(=\sqrt{x^2+y^2})$ large, that is, by subtracting from ψ harmonic terms of all finite orders in r^{-1} (r^{-1}) as $r \to \infty$.

separation from the pressure distribution corresponding to the latter flow. In principle, however, if ψ_e is known or assumed, boundary layer solution may still be regarded as a singular perturbation of ψ_e for $\nu > 0$. A semi-optimal solution, containing at least ψ_e, may then be found by choosing suitable η -coordinates (cf. (26)).

A similar problem arises in connection with ψ'_e when the flow separates at the body. The external disturbance is now influenced by the behavior of the separated region. It is not even clear, a priori, if the first external perturbation would be of order ε. However, if the disturbance is of this order, it may still be included, in principle, in the optimal boundary layer solution by a suitable choice of ξ -coordinates.

Furthermore, it is believed that a complicated singularity may occur at the point of separation, at which the v -velocity becomes infinite (Ref. 13). It may be possible that this infinity can be removed by an adequate choice of the ξ -coordinates, in analogy with the removal of the infinity along $x = 0$ in Example 2. This possibility, however, has not been investigated.

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