SOME EMBEDDING THEOREMS FOR LATTICES

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ABSTRACT

In this thesis we give a general definition of a geometry on a set S and consider the lattices of the subspaces of these geometries.

First, we show that all such geometries on a fixed set S form a lattice and we investigate its properties.

Secondly, we show that the lattice of all geometries on a fixed set S is isomorphic to the lattice of subspaces of some geometry and we characterize all such geometries.

Finally, we show that every finite lattice can be embedded in the lattice of all geometries on some finite set S. This reduces the unsolved problem of embedding every finite lattice into a finite partition lattice to the problem of embedding every finite lattice of geometries into a finite partition lattice.

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SOME EMBEDDING THEOREMS FOR LATTICES.

1. Lattice Theory Foundations.

In the following paragraphswill be compiled the essentials of lattice theory which will be used in this manuscript.

A partially ordered set is a set P and a binary relation $\leq$ on the elements of P satisfying the postulates

P1: $a \leq a$ for all a in P,

P2: $a \leq b$ and $b \leq c$ imply that $a \leq c$.

For a and b in P, $a = b$ if and only if $a \leq b$ and $b \leq a$. $a < b$ shall mean $a \leq b$ but not $a = b$. An element c in P is said to be the union of a subset X of P, if $x \leq c$ for each x in X and $x \leq d$ for each x in X implies $c \leq d$. An element c in P is said to be the intersection of a subset X of P, if $c \leq x$ for each x in X and if $d \leq x$ for each x in X implies $d \leq c$.

A lattice is a partially ordered set L in which each pair of elements has a union and intersection in L. The union of a and b is designated by $a \cup b$ and the intersection $a \cap b$. The unions and intersections satisfy the following identities:

L1: $a \cap a = a \cup a = a$,

L2: $a \cap b = b \cap a$ and $a \cup b = b \cup a$,

L3: $a \cap (b \cap c) = (a \cap b) \cap c$ and $a \cup (b \cup c) = (a \cup b) \cup c$,

L4: $a \cap (a \cup b) = a$ and $a \cup (a \cap b) = a$.

A lattice is said to be complete if every non-empty set has a union and intersection. An element I is said to be the unit element of the lattice L if it is the union of all the elements of L. Dually, an element 0 is said to be the null element of the lattice L if it is the intersection of all elements of L. A lattice with a unit and null element is said to be complemented if for each a in L there exists an a' such that $a \cap a' = 0$ and $a \cup a' = I$. In a lattice an element a is said to cover an element b $(a \succ b)$ if $a \neq b$, and $a \geq p > b$ implies $p = a$.

A single valued mapping θ from a lattice L onto a lattice L' is a homomorphism, if for any a and b in L the two relations

$$\theta(a \cup b) = \theta(a) \cup \theta(b) \quad \text{and}$$
$$\theta(a \cap b) = \theta(a) \cap \theta(b)$$

are valid. If the mapping is one-to-one the homomorphism is referred to as an isomorphism. A subset L' of a lattice L is said to be a sublattice of L, if with any two elements it contains their union and intersection. A lattice L' is said to be embedded in a lattice L if L contains a sublattice isomorphic to L'.

Let S and R be any sets. Then their set union and set intersection will be denoted by $S \vee R$ and $S \wedge R$, respectively. $R \subseteq S$ will mean that R is a subset of S and $a \in S$ will mean that A is an element of S. If S is a finite set then $n[S]$ will be the number of elements in S.

2. Definition and Properties of Geometries.

Definition 1: A geometry G on a set S is a collection of subsets of S, such that any two distinct elements of S are contained in one and only one subset and every subset contains at least two distinct elements.

The elements of the set S are called points and the subsets of Definition 1 are called lines of the geometry G. The line defined by the two distinct points p and q is denoted by (p,q). A line (p,q) is said to be trivial if it contains only two points, non-trivial otherwise.

Definition 2: Let G be a geometry on a set S. Then a subset T of S is called a geometric object of the geometry if with any two distinct points of T the line defined by these points is contained in T.

Theorem 1: All geometric objects of a geometry form a complete point lattice under set inclusion.

Proof: The set of all geometric objects of G on S is a partially ordered set under set inclusion with the unit element I = S. We shall show that this partially ordered set is closed under arbitrary intersections. Let R be a subset of the set of all geometric objects of G. We have to show that $\wedge R$ is a geometric object. If $\wedge R$ is void or consists of only one point it is by Definition 2 a geometric object. If $\wedge R$ contains two distinct points p and q, then the line (p,q) is contained in every geometric object T in R and therefore (p,q) is contained in $\wedge R$. Thus by Definition 2, $\wedge R$ is a geometric object. Therefore the set of all geometric objects of G is a complete lattice, since every partially ordered set with a unit element and closed under arbitrary intersections is a complete lattice. Furthermore, it is a point lattice since by Definition 2 the points of G are geometric objects.

We shall call a lattice L a geometry if L is isomorphic to a lattice of all geometric objects of some geometry.

Theorem 2: A complete point lattice L with the set of points S is a geometry if and only if the two following conditions hold.

i) If p and q are distinct points in S then $p \cup q \succ p$ and q,

ii) If Q is a subset of S such that, $p \cup q \geq s$, $s \in S$, implies s is contained in Q whenever p and q are in Q, then $Q = \{p \in S | p \leq \cup Q\}$.

Proof: Let G be a geometry on S and L the lattice of this geometry. We shall show that the condition i) holds in L. If p and q are distinct points of G then $p \cup q = (p,q)$, since by Definition 2 every geometric object containing p and q has to contain the line (p,q). Let T be a geometric object such that $p \cup q \geq T > p,q$; then p and q are contained in T and therefore the line (p,q) is contained in T, thus $p \cup q \leq T \leq p \cup q$ and we have that $p \cup q = T$, which implies that $p \cup q \succ p$ and q. Let us now show that the condition ii) holds in L. If Q is a subset of S for which the condition ii) holds, then Q is a geometric object of G, since with any two distinct points in Q the line defined by these points is contained in Q. But then Q is an element of the complete point lattice L and therefore $Q = \{p \in S | p \leq \cup Q\}$.

Conversely, if L is a complete point lattice with the set of points S in which the conditions i) and ii) hold, then we can define a geometry on S so that L is isomorphic to the lattice of this geometry. To do this let the line which contains the two distinct points p and q be given by $(p,q) = \{ r \in S | r \leq p \cup q \}$. Let us now show that any two distinct points on the line define the line. Let r be a point on the line defined by p and q, $r \neq p$. Then $p \cup q \geq p \cup r > p$ and condition i) implies that $p \cup q = p \cup r$, so that $(p,q) = (p,r)$. Similarly we can replace p in (p,r) by any point s on this line if $s \neq r$. Thus any two distinct lines can have at most one point in common and since every two distinct points of S are contained in at least one line we see that these subsets (p,q) form a geometry on S. All the points of L contained in an element of L form a geometric object; therefore, to any two distinct lattice elements there correspond two distinct geometric objects. Condition ii) asserts that to any two distinct geometric objects there correspond two distinct lattice elements. Thus L is a geometry.

<u>Theorem 3</u>: The collection of all geometries on a set S forms a complete point lattice under its natural ordering.

<u>Proof</u>: Let G and H be two geometries on S, then $G \geq H$ if and only if for every line l of H there exists a line L of G such that l is contained in L. This relation satisfies P1 and P2 and thus is a partially ordered set. To show that it is a complete lattice under this ordering we first note that it has a unit element $I = \{S\}$ and we shall show that it is closed under arbitrary intersections. To do

this let $\{G_\alpha | \alpha \varepsilon A\}$ be any collection of geometries on the set S. For any two distinct points p and q of S there exists exactly one line $(p,q)_\alpha$ of G_α which contains p and q. Let $(p,q) = \bigwedge\{(p,q)_\alpha | \alpha \varepsilon A\}$. For any two such pairs p, q and r, s either $(p,q)_\alpha = (r,s)_\alpha$, all $\alpha \varepsilon A$, and therefore, $(p,q) = (r,s)$, or there exists an $\alpha \varepsilon A$ such that $(p,q)_\alpha \neq (r,s)_\alpha$. In the second case $(p,q)_\alpha$ and $(r,s)_\alpha$ are distinct lines of G_α and therefore they have at most one point in common which implies that (p,q) and (r,s) have at most one point in common. Thus any two distinct points of S are contained in one and only one set (p,q) and therefore all such distinct sets (p,q) containing two or more points form a geometry G on S. Clearly $G \leq G_\alpha$, $\alpha \varepsilon A$. If $H \leq G_\alpha$, $\alpha \varepsilon A$, then $H \leq G$ which shows that G is the intersection of the set $\{G_\alpha | \alpha \varepsilon A\}$. But then this partially ordered set has a unit element and is closed under arbitrary intersections and therefore is a complete lattice. Finally, let us show that it is a point lattice. We observe that a point P in this lattice is a geometry which has only one non-trivial line and this line consists of three points. If G is a geometry, we have to show that $G = \cup\{P | P \leq G\}$. First, G is larger then every point P contained in it. Secondly, we have to show that if $P \leq F$, all $P \leq G$, then $G \leq F$. To do this let us consider $H \leq G$; then there exists a line l of H such that l is properly contained in a non-trivial line L of G. Therefore we can pick three elements of S which are in L, but are not contained simultaneously in the same line of H. Thus there exists a point P such that $P \leq G$ and $P \nleq H$.

If we now set $G \cap F = H$ then clearly $P \leq H$, all $P \leq G$, and therefore, H cannot be less than G. Thus $G \cap F = G$ and therefore $G \leq F$. This completes the proof.

We shall denote the lattice of all geometries on a set S by LG(S).

Theorem 4: The lattice of all geometries on a set S is a geometry.

Proof: By Theorem 3 the lattice of all geometries is a complete point lattice. Thus by Theorem 2 we just have to verify that the conditions i) and ii) of Theorem 2 hold in LG(S). To verify the condition i) let P and R be distinct points of LG(S). The points of LG(S) are geometries with only one non-trivial line and this line consists of three points, say, $P = \{(a,b,c)\}$, $R = \{(d,e,f)\}$. Let $n[(a,b,c) \wedge (d,e,f)] \leq 1$; then $P \cup R = \{(a,b,c),(d,e,f)\}$ and $P \cup R \succ P$ and R. If $n[(a,b,c) \wedge (d,e,f)] = 2$, say, $a = d$, $b = e$ and $c \neq f$, then $P \cup R = \{(a,b,c,f)\}$ and again $P \cup R \succ P$ and R. Thus property i) holds and therefore the sets of points of LG(S) which are contained in the unions of two distinct points form the set of lines for some geometry. To verify property ii) we have to show that if T is a geometric object in the above defined geometry then $P = \{(a,b,c)\} \leq \cup T$ implies that P is contained in T. By Theorem 3, $\cup T = \{K,L,M, \dots\}$ is a geometry on S and therefore a point $\{(a,b,c)\}$ is less than $\cup T$ if the elements a, b, c are contained in a line of $\cup T$, say, $a, b, c \in K$. Let us now consider all the points $\{(x,y,z)\}$ of LG(S) such that $x, y, z \in K$ and let us denote this set by N_K. N_K is a geometric object since it consists of all the points of LG(S) which are contained in an element of LG(S). This element is the geometry with the only non-trivial line K, let us denote it by $\{K\}$. Since LG(S) is a point lattice we have that $\cup N_K = \{K\}$. By Theorem 1 we know that the geometric objects are closed under intersections and therefore $T_K = T \wedge N_K$ is a geometric object. Furthermore we shall show that $\cup T_K = \{K\}$. We know that $\cup T = (\cup T_K) \cup (\cup T_L) \cup (\cup T_M) \ldots$, where T_L, T_M, ... are defined similarly to T_K. On the other hand $\cup T_K \leq \{K\}$, $\cup T_L \leq \{L\}$, ..., and thus

$$\{K, L, M, \ldots\} = \cup T \leq \{K\} \cup \{L\} \cup \{M\} \cdots = \{K, L, M, \ldots\}.$$

For the equality sign to hold we must have $\cup T_K = \{K\}$, $\cup T_L = \{L\}$, ... We shall now show that if $a, b, c \in K$ then $\{(a, b, c)\}$ is contained in T_K and therefore in T, which will complete the proof. To prove this we shall construct an auxiliary geometry G such that $G = \cup T_K$ and if $\{(a, b, c)\} \leq G$ then $\{(a, b, c)\}$ is contained in T_K. To construct G let us consider subsets U of S with the property that if the distinct elements $x, y, z \in U$, then $\{(x, y, z)\}$ is in T_K. Let us denote the collection of all such subsets by $\mathcal{F}$. Let us order the non-void set $\mathcal{F}$ under set inclusion. We shall now construct the lines $l(p,q)$ of G. Let $l(p,q)$ contain only p and q if there is no set U in $\mathcal{F}$ which contains p and q, otherwise let $l(p,q) = \bigvee \{U \in \mathcal{F} \mid p,q \in U\}$. To show that all the distinct sets $l(p,q)$ form a geometry on S we note that any two distinct points of S are contained in at least one of these lines. To show that any two distinct lines of G have at most one point in common we shall show that if $l(p,q)$ is non-trivial then it is a maximal element of $\mathcal{T}$. First, let us show that $l(p,q)$ is an element of $\mathcal{T}$. If the distinct elements x, y, z are contained in $l(p,q)$, then by definition of $l(p,q)$ each of the points x, y, z has to be contained in some element of $\mathcal{T}$ which also contains p and q. If the points p, q are contained in the set consisting of x, y, z, say, $p = x$, $q = y$, then $z, p, q \in U$, some element U of $\mathcal{T}$, and therefore, $\{(x,y,z)\} = \{(p,q,z)\}$ is contained in $\mathcal{T}$. If one of the points p or q is contained in the set consisting of x, y, z, say, $p = x$, then $\{(p, q, y)\}$ and $\{(p, q, z)\}$ are elements of T_K and since T_K is a geometric object we obtain that $\{(p, y, z)\} = \{(x, y, z)\}$ is contained in T_K. If p, q, x, y, z are distinct then $\{(x, p, q)\}$ and $\{(y, p, q)\}$ are contained in T_K and since T_K is a geometric object $\{(x,y,q)\}$ is contained in T_K. Similarly we obtain that $\{(x, z, q)\}$ is contained in T_K and therefore $\{(x, y, z)\}$ is contained in T_K, which shows that $l(p,q)$ is an element of $\mathcal{T}$. To see that $l(p,q)$ is a maximal element of $\mathcal{T}$ assume that $l(p,q) \subseteq U$, where U is an element of $\mathcal{T}$. Then p and q are contained in U and therefore by definition of $l(p,q)$, U is contained in $l(p,q)$ and thus $l(p,q) = U$. Let us now show that two distinct lines of G have at most one point in common. If the distinct points v, t are contained in $l(p,q)$ and $l(s,r)$ then since $l(p,q)$, $l(s,r)$ are elements of $\mathcal{T}$ we have that $l(p,q)$, $l(s,r) \subseteq l(v,t)$, but since they are maximal elements of $\mathcal{T}$ we obtain that $l(p,q) = l(s,r) = l(v,t)$. Thus the collection of distinct lines $l(p,q)$ formsa geometry G on S. Finally we see that if $\{(x,y,z)\} \leq G$

then x, y, z are contained in some U, $U \in \mathfrak{T}$, and therefore $\{(x, y, z)\} \in T_K$. Conversely if $\{(x, y, z)\} \in T_K$ then the set U consisting of x, y, z is an element of $\mathfrak{T}$ and therefore $\{(x, y, z)\} \leq G$. Thus $\cup T_K = G$ and if $\{(x, y, z)\} \leq \cup T_K$ then $\{(x, y, z)\}$ is contained in T_K, which completes the proof.

We shall now proceed to investigate the problem of complementation in the lattice of all geometries.

__Lemma 1:__ Let G and H be geometries on S and let H have at most one non-trivial line. Then $G \cup H$ has at most one line L which is not a line of G.

__Proof:__ If H has only trivial lines then $G \cup H = G$ and Lemma 1 holds. Let H have a non-trivial line and let L be the line of $G \cup H$ which contains it. Let F consist of L and the lines l of G which are not contained in L. We shall show that F is a geometry and $G \cup H = F$. Any two distinct points p and q in S are contained in a line l of G. Either l is a line of F or l is contained in L. Thus any two points are contained in some line of F. Note that no two lines of F have more than one point in common, since L is a line of $G \cup H$ and therefore if a line l of G has two or more points in common with L it is contained in L and clearly no two lines of G may have more than one point in common. Thus F is a geometry. Finally, we see that $G \leq F$, since every line l of G is either a line of F or contained in the line L of F. Also $H \leq F$ and therefore $G \cup H \leq F$. On the other hand $F \leq G \cup H$ so that $F = G \cup H$; which completes the proof.

<u>Theorem 5</u>: The lattice of all geometries on a set S is complemented.

<u>Proof</u>: Let G be a geometry on S. Let R be a subset of S such that R has at most two points in common with any line of G. Let us denote the collection of all such sets by $\mathfrak{h}$ and order $\mathfrak{h}$ under set inclusion. We shall show that if M is a maximal element of $\mathfrak{h}$ then the geometry H with the only non-trivial line M is a complement of G if $G \neq I$. First, let us show that there exists a maximal element in $\mathfrak{h}$. By the Maximal Principal every chain C of $\mathfrak{h}$ is contained in a maximal chain $\mathcal{C}$ of $\mathfrak{h}$. We assert that $\vee\mathcal{C}$ is an element of $\mathfrak{h}$. To see that, let the distinct points x, y, z be contained in $\vee\mathcal{C}$ and some line l of G. Then, since $\mathcal{C}$ is a chain, there exists some element R contained in $\mathcal{C}$ such that x, y, z are contained in R and l of G, contrary to the assumption that R is contained in $\mathfrak{h}$. Thus $\vee\mathcal{C}$ is an element of $\mathfrak{h}$. It clearly is a maximal element and we shall denote it by M. Let the geometry with the only non-trivial line M be denoted by H. Then $H \cap G = 0$. Let us show that $H \cup G = I$. By Lemma 1 the union $G \cup H$ can have at most one non-trivial line, let us denote this non-trivial line by L. Then $M \subseteq L$ and the geometry $G \cup H$ consists of L and all the lines l of G which are not contained in L. We shall show that $L = S$. Let us assume that there exists an element x contained in S which is not contained in L. If $M \vee \{x\}$ is an element of $\mathfrak{h}$ then since M is a maximal element of $\mathfrak{h}$ we have that x is contained in M and therefore in L, contrary to the assumption. Thus we may assume that $M \vee \{x\}$ is not an element of $\mathfrak{h}$ and therefore there exist distinct

points x, y, z contained in $M \vee \{x\}$ and some line l of G. But then y, z are contained in l and M, which implies that $l \subseteq L$ and therefore $x \in L$, contrary to the assumption. Thus $L = S$ and $G \cup H = \{S\} = I$, which shows that H is a complement of G.

From the proof of Theorem 5 we can see that we proved the somewhat stronger result:

<u>Corollary 1</u>: Let G be a geometry which is distinct from I. Then there exists a geometry H on S which is a complement of G, has only one non-trivial line, and this line contains any two prescribed points of S.

<u>Corollary 2</u>: If a geometry G has a unique complement then G is either the zero or the unit element of $LG(S)$.

<u>Proof</u>: Let $G \neq O, I$. Then G has a line which contains at least three distinct points, say, a, b, c. By Corollary 1 there exist two complements of G with only one non-trivial line and this line contains the points a, b and a, c, respectively. Clearly these two complements are distinct, which proves Corollary 2.

We shall now consider the homomorphisms of $LG(S)$.

<u>Lemma 2</u>: If Θ is a non-trivial homomorphism on a complete point lattice L, then there exists a point p of L such that $\Theta(p) = \Theta(o)$.

<u>Proof</u>: Let Θ be a non-trivial homomorphism on L. Then there exist two elements a, b contained in L, $a > b$, such that

$\theta(a) = \theta(b)$. Since L is a complete point lattice there exists a point p such that $a \cap p = p$, $b \cap p = o$, but then

$$\theta(p) = \theta(p \cap a) = \theta(p) \cap \theta(a) = \theta(p) \cap \theta(b) = \theta(p \cap b) = \theta(o) .$$

<u>Theorem 6</u>: There are only trivial homomorphisms on LG(S).

<u>Proof</u>: Let us consider the geometries on S whose non-trivial lines all contain a fixed point d of S. $I = \{S\}$ is one of these geometries and all such geometries are closed under arbitrary intersections; thus they form a sublattice L of LG(S). Any two of the non-trivial lines in one of these geometries have only the point d in common, so that after the removal of the point d the non-trivial lines can be considered as the non-trivial blocks of a partition on the set S - d. This yields a one-to-one order preserving mapping of L onto the set of all partitions on S - d. Thus L is isomorphic to a partition lattice. Let θ be a homomorphism on LG(S) which identifies at least two distinct elements. Then by Lemma 2 there exists a point $P = \{(a, b, c)\}$ of LG(S), such that $\theta(P) = \theta(O)$. If we set $a = d$ then P is an element of L and therefore θ induces a homomorphism on L which identifies two distinct elements. But Ore [1] proved that there are only trivial homomorphisms on a partition lattice, thus θ must identify all the elements in L. Since L and LG(S) have the same unit and zero elements θ identifies all elements of LG(S). Therefore θ is a trivial-homomorphism which was to be shown.

3. Characterization of LG(S).

In the previous part we investigated the lattice properties of LG(S) and we showed that it is isomorphic to the lattice of some geometry. We shall now characterize these geometries whose lattices are isomorphic to the lattice of all geometries on some set S.

We shall introduce some concepts which are essential for the following theory.

Definition 3: Two distinct points p and q of a geometry G are said to be *related* if the line defined by p and q is non-trivial.

Definition 4: The points p and q of a geometry G are said to be *close* if p is equal to q, p is related to q or there exists a point t of G such that p is related to t and t is related to q.

Definition 5: Let the line l of G consist of the four distinct points p_1, p_2, p_3, p_4 and let π denote the set consisting of the three collinear points p_1, p_2, and p_3. Then a point q of G is said to be *close* to π if one of the two following conditions holds:

i) q is equal to p_1, p_2 or p_3,

ii) q is close to p_1, p_2, p_3; q is distinct from p_4 and not related to p_4.

For the further discussion π_1 will denote a triplet of distinct collinear points of G.

Definition 6: π_1 is said to be close to π_2 if every point in π_1 is close to π_2.

Theorem 7: Let L be the lattice of a geometry G on W and let W consist of four or more points. Then L is isomorphic to the lattice of all geometries on some set S if and only if G satisfies the following five axioms:

Axiom 1: The non-trivial lines of G consist of four points and every point is contained in at least one non-trivial line.

Axiom 2: If a point p is close to π_1 and π_1 is close to π_2 then p is close to π_2.

Axiom 3: If π_1 is close to π_2 then π_2 is close to π_1.

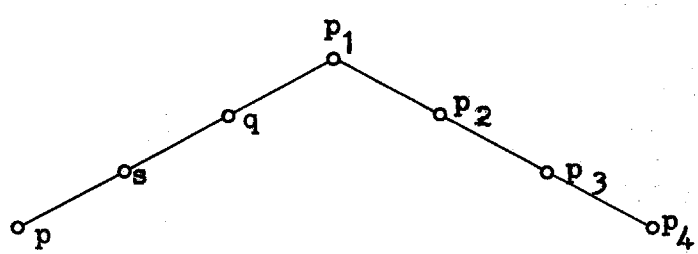
Axiom 4: If π_1, π_2, π_3 are distinct then there exists a point p such that p is close to π_1, π_2 and π_3.

Axiom 5: Let l be a non-trivial line and let p be a point which is not on this line but is close to every point on this line, then p is related to exactly two points of l.

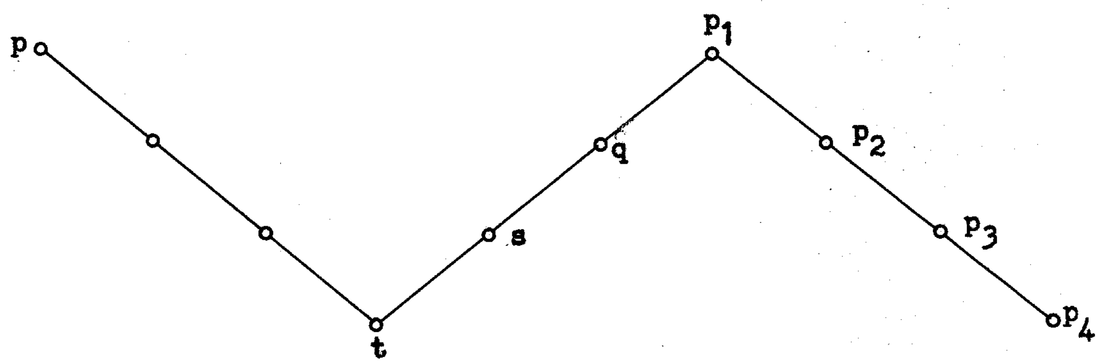
To show that L is isomorphic to the lattice of all geometries on some set S we have to show that there exists a one-to-one mapping of the set W onto the set of points of LG(S) and that this mapping preserves lines. To do this we shall introduce the concept of a star of G. Let π be a triplet of distinct collinear points of G. Then the set of all points which are close to π will be called the star of G defined by π. This set will be denoted by $\Delta(\pi)$. We shall show that the lattice of all geometries on the set of stars of G is isomorphic to L. We know that a point of LG(S) is a geometry with only one non-trivial line and this line consists of three points. Thus every point of LG(S) is characterized by the three elements of S which are contained in its non-trivial line. Therefore we first have to establish a one-to-one mapping of the set W onto the set of all triplets consisting of distinct stars of G. We shall do this by showing that every point of G is contained in exactly three distinct stars and that any three distinct stars have exactly one point of G in common. The proof consists of the following Lemmas.

Lemma 3: If the point p is close to π_1 then there exists π_2 such that p is contained in π_2 and π_2 is close to π_1.

Proof: Let π_1 consist of the distinct points p_1, p_2, p_3 and let the fourth collinear point of this line l be p_4. If p is contained in l then p must be equal to p_1, p_2 or p_3 since by Definition 5 the fourth collinear point p_4 is not close to π_1. Thus we may set $\pi_2 = \pi_1$ since then p is contained in π_2 and by Definition 5 and Definition 6 we see that π_1 is close to π_1. Let us now assume that p is not on the line l but is related to a point of π_1. Let this point be p_1 as indicated in the following figure:



Then p_4 is related to p_1 and p_1 is related to every point of the line defined by p and p_1. Thus by Definition 4, p_4 is close to every point on this line and therefore by Axiom 5, p_4 is related to exactly two of the four distinct points of this line. We know that p_4 is related to p_1 and let us denote the second point to which it is related by s. Since the line defined by p and p_1 consists of four points there must exist a point q on this line which is distinct from p and is not related to p_4. Let π_2 consist of p, p_1 and q. We see that p is contained in π_2 and we shall show that π_2 is close to π_1. By Definition 6 we have to show that every point of π_2 is close to π_1. p_1 is contained in π_1 and therefore by Definition 5 close to π_1. p and q are close to every point in π_1 and not related to p_4. Thus by Definition 5 they are close to π_1. It follows that π_2 is close to π_1. We may now assume that p is not related to any point on the line l. Since p is close to p_1 there exists a point t of G such that p is related to t and t is related to p_1 as shown in the following figure:



Using the result of the previous case we know that there exists a triplet π_3 on the line defined by p_1 and t such that t is contained in π_3 and π_3 is close to π_1. Let us denote the point of this line which is not contained in π_3 by s as we did in the previous case. We recall that s is related to p_4. Thus p cannot be related to s since otherwise p is related to s and s related to p_4 which implies that p is close to p_4 and therefore close to every point on l. From this we would conclude that p is related to exactly two points on the line l, contrary to assumption. But then p is close to π_3 and is related to t which is contained in π_3. Using again the result of the previous case there exists a triplet π_2 on the line defined by p and t such that p is contained in π_2 and π_2 is close to π_3. Now we have that p is contained in π_2, π_2 is close to π_3, and π_3 is close to π_1. Thus by Axiom 2 we conclude that π_2 is close to π_1. This completes the proof of Lemma 3.

<u>Lemma 4</u>: Any three distinct collinear points contained in a star define the star.

<u>Proof</u>: Let π_1 be contained in $\Delta(\pi_2)$. Then every point of π_1 is close to π_2 and therefore π_1 is close to π_2. By Axiom 3, π_2 is close to π_1 and therefore by Axiom 2 every point which is close to π_1 is close to π_2 and vice versa. From this it follows that $\Delta(\pi_1) = \Delta(\pi_2)$.

Lemma 5: Every point p of G is contained in at least three distinct stars.

Proof: By Axiom 1 a point p of G is contained in at least one non-trivial line l and this line consists of four points. There are exactly three distinct triplets π_1, π_2 and π_3 of l which contain p. The fourth collinear point of l which is not contained in π_1 is by Definition 5 not contained in $\Delta(\pi_1)$. But this point is contained in π_2 and π_3 and therefore it is contained in $\Delta(\pi_2)$ and $\Delta(\pi_3)$. Thus $\Delta(\pi_1)$ is distinct from $\Delta(\pi_2)$ and $\Delta(\pi_3)$. Similarly we show that $\Delta(\pi_2)$ is distinct from $\Delta(\pi_3)$. This shows that there are at least three distinct stars which contain p.

Lemma 6: There are exactly three distinct stars containing every point p of G.

Proof: Let p be contained in a non-trivial line l and let π_1, π_2 and π_3 be the distinct triplets of l which contain p. By the previous result we know that the stars $\Delta(\pi_1)$, $\Delta(\pi_2)$ and $\Delta(\pi_3)$ are distinct. Let p be contained in some star $\Delta(\pi)$; we shall show that $\Delta(\pi)$ is equal to $\Delta(\pi_1)$, $\Delta(\pi_2)$, or $\Delta(\pi_3)$. Note that if p is contained in $\Delta(\pi)$ then by Lemma 3 there exists a triplet π' such that p is contained in π' and π' is close to π. If π' is contained in the line l then it must be equal to π_1, π_2 or π_3 and therefore by Axiom 3 and Axiom 2 we conclude that $\Delta(\pi)$ is equal to $\Delta(\pi_1)$, $\Delta(\pi_2)$ or $\Delta(\pi_3)$. Thus we may assume that π' is contained in a non-trivial line l' and that l' is distinct from l. Let us denote the point of l' which is not contained in π' by q. Since p is contained in π' and therefore in l' we see that q is related to p and p is related to every element of the line l. Thus q is close to every point on the line l and therefore related to exactly two points of l. We know that q is related to p. Let the second point to which q is related be denoted by s. One of the triplets π_1, π_2 or π_3 does not contain the point s, say π_1. Then π_1 is close to π' since p is contained in π_1 and the two remaining points of π_1 are close to every point in π' and not related or equal to q. From the fact that π_1 is close to π' it follows by Axiom 3 and Axiom 2 that $\Delta(\pi_1) = \Delta(\pi')$; this proves Lemma 6.

By Axiom 4 any three distinct stars have at least one point in common. The next lemma will show that there cannot be more than one such point.

<u>Lemma 7</u>: Any three distinct stars have exactly one point in common.

<u>Proof</u>: Let p and q be distinct points of L. We shall show that the three distinct stars which contain p cannot all contain q. Let p be contained in the non-trivial line l and let π_1, π_2 and π_3 be the distinct triplets of l which contain p. These triplets define the three distinct stars which contain p. If q is also contained in the line l then q is not contained in one of these triplets. Let this triplet be π_1. Then q is not contained in the star $\Delta(\pi_1)$ since q is the point of l which is not contained in π_1. Thus we may assume that p and q are not related, which implies that q cannot be close to π_1, π_2 and π_3. Since if q would be

close to π_1, π_2 and π_3 then q would be close to every point on l and therefore q would be related to exactly two elements of l. But then q would be related to an element of l which is not contained in one of the triplets π_1, π_2 or π_3 and therefore q would not be close to one of these triplets, contrary to the assumption. Thus q is not contained in one of the three stars which contain p. This proves Lemma 7.

So far we have shown that there exists a one-to-one mapping of the set W onto the set of points of LG(S), where S is the set of stars of G. Let us denote this mapping by θ.

<u>Lemma 8</u>: The mapping θ preserves lines.

<u>Proof</u>: Let l be a non-trivial line of G. Then l contains four distinct triplets $\pi_1, \pi_2, \pi_3, \pi_4$ and these triplets define the four distinct stars $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, respectively. Every point of the line l is contained in three of these triplets and therefore in three of these stars. Under the above established mapping θ the line l is mapped into the line l' of LG(S) which consists of the four points $\{(\Delta_1, \Delta_2, \Delta_3)\}$, $\{(\Delta_1, \Delta_2, \Delta_4)\}$, $\{(\Delta_1, \Delta_3, \Delta_4)\}$ and $\{(\Delta_2, \Delta_3, \Delta_4)\}$. Which shows that every point on the line l is mapped into a point of the corresponding line l' of LG(S). Conversely, let l' be a non-trivial line of LG(S) and let this line consist of the four points $\{(\Delta_1, \Delta_2, \Delta_3)\}$, $\{(\Delta_1, \Delta_2, \Delta_4)\}$, $\{(\Delta_1, \Delta_3, \Delta_4)\}$ and $\{(\Delta_2, \Delta_3, \Delta_4)\}$. Let $\{(\Delta_1, \Delta_2, \Delta_3)\}$ and $\{(\Delta_1, \Delta_2, \Delta_4)\}$ be mapped into the points p and q, respectively. Let p be contained in

a non-trivial line l of G. We know that the triplets π_1, π_2, π_3 of l which contain p define the stars $\Delta_1, \Delta_2, \Delta_3$, respectively. q is contained in Δ_1 and Δ_2 and therefore q is close to every point on l. Thus q is related to exactly two points of l. q has to be related to p since otherwise one of the triplets π_1 or π_2 would not contain a point of l to which q is related and therefore q would not be close to π_1 or π_2, contrary to assumption. Thus p and q are related. Without loss of generality we may assume that p and q are contained in l and that $\pi_1, \pi_2, \pi_3, \pi_4$ are the distinct triplets of l. π_1, π_2, π_3 and π_4 define the stars $\Delta_1, \Delta_2, \Delta_3$ and Δ_4, respectively. Clearly every three distinct triplets of the set $\pi_1, \pi_2, \pi_3, \pi_4$ have a point in common and this point is contained in l. This shows that every point of the line l' is mapped into a point on the corresponding line l. Thus the mapping preserves lines and we conclude that L is isomorphic to $LG(S)$.

We now show that the five axioms of Theorem 7 hold in the geometry of $LG(S)$, whenever S consists of four or more elements. Let the elements of S be denoted by $a, b, c, \dots$. Let $\{(a, b, c)\}$ and $\{(d, e, f)\}$ be points of $LG(S)$. Then these points are related if and only if their non-trivial lines consisting of a, b, c and d, e, f, respectively, have two elements in common. If these non-trivial lines have one element in common, say $a = d$, then $\{(a, b, c)\}$ is related to $\{(a, b, e)\}$ and $\{(a, b, e)\}$ is related to $\{(a, e, f)\}$. From this follows that two points of $LG(S)$ are close if their non-trivial lines have one or more points in common. Let us recall that every non-trivial line of $LG(S)$ consists of four points and they are of the form $\{(a, b, c)\}$, $\{(a, b, d)\}$, $\{(a, c, d)\}$ and $\{(b, c, d)\}$. Thus Axiom 1 holds if the set S contains four or more elements. We note that any three of these four collinear points are such that their non-trivial lines have an element in common. For example the three first points which we shall denote by π have a as the element which is common to their non-trivial lines. Thus every point of the form $\{(a, y, z)\}$ is close to every point in π and is related to $\{(b, c, d)\}$ if and only if it is contained in π. Therefore every point of the form $\{(a, y, z)\}$ is close to π. Conversely, if $\{(x,y,z)\}$ is close to π, then its non-trivial line must have at least one element in common with the non-trivial line of every point in π. Thus its non-trivial line must contain a or if it does not contain a it must contain any two of the three elements b, c, d. For the second case $\{(x, y, z)\}$ will be related or equal to $\{(b, c, d)\}$ and therefore not close to π. Thus it must be of the form $\{(a, y, z)\}$.

It follows that π_1 is close to π_2 if and only if the non-trivial lines of the points in π_1 and π_2 have the same element in common. From this we see that Axiom 2 and Axiom 3 hold. Let $\Delta(\pi_1)$, $\Delta(\pi_2)$ and $\Delta(\pi_3)$ be distinct stars. Then the common elements a_1, a_2 and a_3 of π_1, π_2 and π_3, respectively, are distinct and the point $\{(a_1, a_2, a_3)\}$ is contained in these three stars. This shows that Axiom 4 holds. To verify Axiom 5 we note that every point $\{(x, y, z)\}$ which is close to the four collinear points $\{(a, b, c)\}$, $\{(a, b, d)\}$, $\{(a, c, d)\}$, $\{(b, c, d)\}$ and not equal to any one of them must be such that its non-trivial line contains exactly two of the four elements a, b, c, d. But then the point $\{(x, y, z)\}$ is related to exactly two of the four collinear points. This finally completes the proof of Theorem 7.

4. Embedding Theorems.

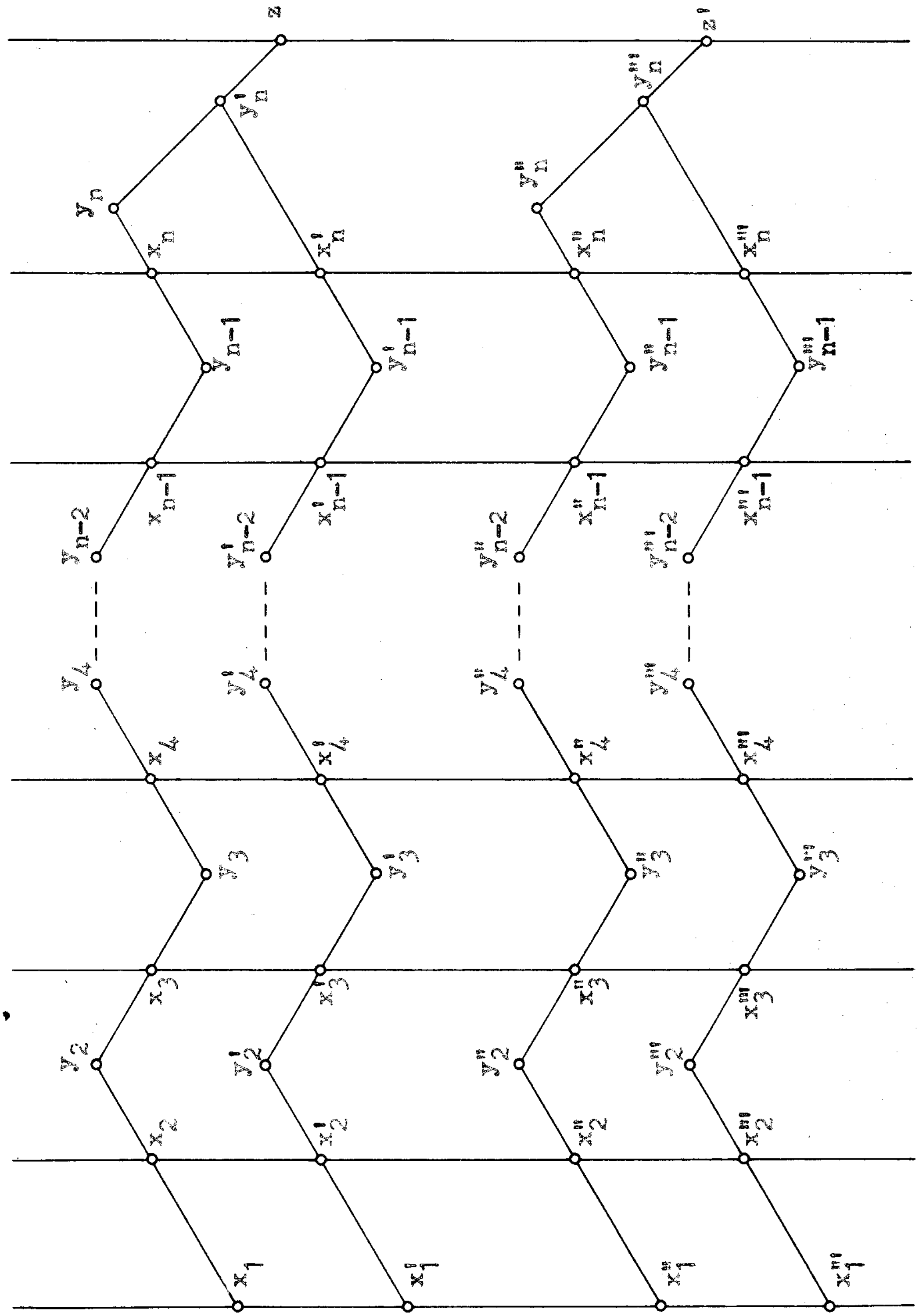
Theorem 8: Every finite point lattice can be embedded in the lattice of some finite geometry.

Proof: Let L be a finite point lattice with the points $p_1, p_2, \dots, p_N$. We shall construct a geometry G on a finite set S such that L is isomorphic to a sublattice of the lattice L' of this geometry G. If N = 1 or 2 then L is a geometry. For $N \geq 3$ let S contain $2^N(6N^2 - 15N + 8)$ points and let $R, S_1, S_2, \dots, S_N$ be disjoint subsets of S. Let R contain $4 \cdot 2^N(N - 2)(N - 1)$ points and let S_i, i = 1, 2, ... , N, contain $2^N(2N - 3)$ points. At the end of this proof we shall show that these sets are sufficiently large to carry out the following construction of G. Let $S_1, S_2, \dots, S_N$ be non-trivial lines of G. Let every point p_i of L correspond to the line S_i of G. This establishes a one-to-one mapping θ of the set of points of L onto the set consisting of non-trivial lines $S_1, S_2, \dots, S_N$ of G. In general we shall add other non-trivial lines to G in order to preserve the unions and intersections under the mapping θ. We shall now describe the construction of the additional non-trivial lines. Assume that, after re-enumerating if necessary, the union of the points $p_1, p_2, \dots, p_n$ of L contains the point p_{n+1} and that no union of a smaller number of the points $p_1, p_2, \dots, p_n$

contains p_{n+1}. The additional non-trivial lines of G will be so constructed that the corresponding union of $S_1, S_2, \ldots, S_n$ in L' contains S_{n+1}. To do this let the distinct points x_i, x_i', x_i'', x_i''' be contained in S_i for $1 \leq i \leq n$, let y_i, y_i', y_i'', y_i''' be contained in R for $2 \leq i \leq n$ and let z, z' be contained in S_{n+1}. Let none of these points be contained in any other construction of additional non-trivial lines. Then we shall add to G the lines

$$(x_1,x_2,y_2),\ (y_2,x_3,y_3),\ (y_3,x_4,y_4),\ \ldots\ ,\ (y_{n-1},x_n,y_n),$$

the corresponding lines in the primed, double primed, and triple primed elements and finally the lines (y_n,y_n',z) and (y_n'',y_n''',z'). This construction is shown in Figure 1. We see that any two of these non-trivial lines have at most one point in common and since all the different constructions of additional non-trivial lines will be done on disjoint sets of points G will be a geometry. We also see that any geometric object T of G which contains the first n lines $S_1, S_2, \ldots, S_n$ of this construction will contain the points z, z' and therefore T will contain the last line S_{n+1} of this construction. Let us assume that we have adjoined the necessary non-trivial lines for all the unions as described above. We shall now characterize the union $\bigcup\{S_i | i \in A\}$ in L'. Let $\bigcup\{p_i \in L | i \in A\} = a$ be the corresponding union in L and let $\bar{A} = \{i | p_i \leq a\}$. With the index set $\bar{A}$ we associate a set $P_{\bar{A}}$ which is a subset of the elements of R which are used to construct additional non-trivial lines. To determine whether such a point y is contained in $P_{\bar{A}}$ let us re-enumerate the


z
z'
y_n'
y_n'''
y_n
y_n''
x_n
x_n'
x_n''
x_n'''
y_{n-1}
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lines and points as we did in Figure 1. Then $y = y_k$ and y_k is contained in $P_{\bar{A}}$ if and only if the first k lines $S_1, S_2, \ldots, S_k$ of this construction are contained in the set $\{S_i | i \varepsilon \bar{A}\}$. We assert that $\cup\{S_i | i \varepsilon A\} = \vee\{S_i | i \varepsilon \bar{A}\} \vee P_{\bar{A}}$. Let us denote the set $\vee\{S_i | i \varepsilon \bar{A}\} \vee P_{\bar{A}}$ by T. We first note that $S_i \subseteq T$, whenever $i \varepsilon A$. Secondly, $T \subseteq \cup\{S_i | i \varepsilon A\}$ since if $S_j \subseteq T$ then j is either contained in A and therefore S_j is contained in the union or j is not contained in A but is contained in $\bar{A}$. In this case $p_j \leq \cup\{p_i \varepsilon L | i \varepsilon A\}$ and therefore there exists a construction of additional lines of G which forces S_j to be contained in the union. If y_k is contained in $P_{\bar{A}}$ then by definition of $P_{\bar{A}}$ the first k lines of this construction are contained in the union, but any geometric object which contains these lines has to contain y_k. Thus we see that $T \subseteq \cup\{S_i | i \varepsilon A\}$. We shall show that T is a geometric object of G. To do this we shall show that with any two distinct points of T the line defined by these points is also contained in T. Let x and y be distinct points of T. If x and y do not belong to the same construction of additional non-trivial lines then the line defined by x and y is either trivial and therefore contained in T or it is equal to S_i, but then by the definition of T we see that S_i is contained in T. We may now assume that x and y are contained in the same construction of additional non-trivial lines of G. Let us re-enumerate the points and lines as we did when we described the construction of these lines (see Figure 1). If both points x and y are contained in S_i then clearly S_i is contained in T. The only other non-trivial lines contained in this construction are listed above where we described the construction. We see that the only pairs of distinct points which define non-trivial lines are,

1) x_1, x_2 3) x_i, y_i, $2 \leq i \leq n$,

2) x_1, y_2 4) x_{i+1}, y_i, $2 \leq i < n$,

the corresponding pairs of primed, double primed, and triple primed points and any two distinct points on the line (y_n, y_n', z) or (y_n'', y_n''', z').

For the first case the lines S_1 and S_2 of this construction are contained in T and therefore by definition of $P_{\bar{A}}$ the third collinear point y_2 of the line is contained in $P_{\bar{A}}$ and therefore in T.

In the second case y_2 is contained in T and therefore S_1 and S_2 are contained in T and thus x_2 is contained in T.

In the third case y_i is contained in T and therefore all the lines $S_1, S_2, \ldots, S_i$ of this construction are contained in T and thus y_{i-1} is contained in T.

In the fourth case y_i is contained in T and therefore the lines $S_1, S_2, \ldots, S_i$ of this construction are contained in T. We also have that x_{i+1} is contained in T which implies that S_{i+1} is contained in T. This together implies that y_{i+1} is contained in T.

The same proof holds for the primed, double primed, and triple primed points. Finally, we note that any distinct pair of points which is contained in the line (y_n, y_n', z) or (y_n'', y_n''', z') has to contain one of the points y_n, y_n', y_n'', y_n'''. But if any one of these points is contained in T then the lines $S_1, S_2, \ldots, S_n$ are contained in T and therefore the line S_{n+1} and the lines (y_n, y_n', z) and (y_n'', y_n''', z') are contained in T. This shows that T is a geometric object and therefore $T = \bigvee\{S_i | i \in \bar{A}\} \bigvee P_{\bar{A}}$. From this we see that $p_j \leq \bigcup\{p_i \in L | i \in A\}$ if and only if $S_j \leq \bigcup\{S_i | i \in A\}$, which shows that Θ defines a one-to-one order preserving mapping of L onto the collection of elements of L' which are unions of the elements of $\{S_1, S_2, \ldots, S_N\}$. We shall now show that these unions are closed under intersections and therefore form a sublattice of L' which is isomorphic to L. Let $\bigvee\{S_i | i \in \bar{A}\} \bigvee P_{\bar{A}}$ and $\bigvee\{S_i | i \in \bar{B}\} \bigvee P_{\bar{B}}$ be two unions and let $\bar{A} \wedge \bar{B} = \bar{C}$. We know that the sets $P_{\bar{A}}$ and $P_{\bar{B}}$ are disjoint from the sets $S_1, S_2, \ldots, S_N$. From the definition of the sets $P_{\bar{A}}$ and $P_{\bar{B}}$ it follows that $P_{\bar{A}} \wedge P_{\bar{B}} = P_{\bar{C}}$. Thus the intersection of these unions is given by $\bigvee\{S_i | i \in \bar{C}\} \bigvee P_{\bar{C}}$, which shows that they are closed under intersection.

To complete this proof we shall show that the sets S, R, $S_1, S_2, \ldots, S_N$ are sufficiently large to carry out the construction of G. If S_i is one of the first n lines of a construction consisting of n + 1 lines, then four points of the set S_i are contained in this construction. If S_i is the last line of this construction then two points of S_i are contained in this construction. There are at most 2^{N-1} subsets of the set $\{S_1, S_2, \ldots, S_N\}$ which contain the set S_i. Any one of these subsets can form the first n lines for at most N - 2 constructions. We also see that S_i can be the last line for at most 2^{N-1} constructions. Thus S_i has to contain at most $4 \cdot 2^{N-1}(N - 2) + 2 \cdot 2^{N-1} = 2^N(2N - 3)$ points. For every construction which contains n + 1 lines S_i we must have $4(n - 1)$ points contained in R. Since there are at most $2^N(N-2)$ constructions R has to contain at most $4 \cdot 2^N(N-2)(N-1)$ points. Thus S has to contain at most $2^N(6N^2-15N+8)$ points. This completes the proof of Theorem 8.

We know that every finite lattice can be embedded in a finite point lattice. Thus by Theorem 8 we obtain the more general result:

<u>Corollary 3</u>: Every finite lattice can be embedded in the lattice of a finite geometry.

<u>Lemma 9</u>: Let L be the lattice of a geometry G on a finite set S. Then L can be embedded in the lattice L' of the geometry G' on a finite set S' such that every point of G' is contained in at least one non-trivial line.

<u>Proof</u>: For every point p_i of G which is not contained in a non-trivial line of G we shall add two new points p_i', p_i'' to S and a new line consisting of p_i, p_i' and p_i'' to G. Let this be the geometry G' on S' and denote its lattice by L'. Every geometric object T of G is a geometric object of G' since we did not introduce any new non-trivial lines between the points of S. The set of all geometric objects of G is closed under intersections in L'. To see that it is also closed under unions let $T_1 \cup T_2 = T_3$ in L. Then $T_3 \subseteq T_1 \cup T_2$ in L' since we added new points and lines to G. But we know that $T_1, T_2 \subseteq T_3$ and T_3 is a geometric object of G', thus $T_1 \cup T_2 = T_3$ also in L'. This shows that L is a sublattice of L'.

<u>Theorem 9</u>: Let L be the lattice of a geometry G on a finite set S, then L can be embedded in the lattice of all geometries on some finite set S'.

<u>Proof</u>: Let G be a geometry with the points $p_1, p_2, \ldots, p_m$ and with the non-trivial lines $l_1, l_2, \ldots, l_n$. By Lemma 9 we may assume that every point of G is contained in at least one non-trivial line of G. We shall show that if the set S' contains at least $mn(6n + 1)$ points then L is a sublattice of LG(S'). To do this we shall map every point p_i of G onto a geometry G_i on S' and show that these geometries generate a sublattice of LG(S') which is isomorphic to L. Let us denote this mapping by θ. To construct the geometry G_i we shall first define certain subsets of S' which will be used in this construction. Let $R, S_o, S_1, S_2, \ldots, S_n$ be disjoint subsets of S' such that R and S_o each contain $2mn^2$ points and let S_i, $i = 1, 2, \ldots, n$, contain $m(2n + 1)$ points. With every point p_j of G we associate a set A_j which is contained in the set union of $S_1, S_2, \ldots, S_n$ and is such that A_j has exactly one point in common with the set S_i if the point p_j is on the line l_i of G and has no points in common with S_i otherwise. Let any two such sets A_i and A_j be disjoint if they are associated with distinct points p_i and p_j, respectively. Let l_k be a non-trivial line of G and let, after re-enumerating if necessary, $p_1, p_2, \ldots, p_t$ be the points of this line. Then we define

$$L_k = A_1 \vee A_2 \vee \cdots \vee A_t - S_k.$$

With every pair (a, L_k), $a \varepsilon L_k$, we associate the three subsets each consisting of three points: $\pi_1 = (x_o, x_k, z)$, $\pi_2 = (x_o', x_k', z')$ and $\pi_i = (z, z', a)$. Let the distinct points x_o, x_o' be contained in S_o, let x_k, x_k' be contained in S_k, and let z, z' be contained in R. Let none of these points except a be contained in A_i, $i = 1, 2, \ldots, m$. Let two such triplets of subsets which are associated with (a, L_k) and (b, L_j), respectively, have no points in common if $a \neq b$ and only the point a in common if $a = b$, $k \neq j$. We shall denote the set of all such subsets π_i by P. Z_k shall denote the set consisting of points z, z' of R which are contained in some set π_i associated with (a, L_k), $a \varepsilon L_k$.

Before we proceed with the proof let us show that the sets $S, R, S_o, S_1, S_2, \ldots, S_n$ contain a sufficient number of points to carry out the construction of the above defined subsets. Any one of the m sets $A_1, A_2, \ldots, A_m$ can have at most one point in common with the set S_k. Thus S_k has to contain at most m points for the construction of these sets. To construct the three subsets π_1, π_2, π_3 associated with the pair (a, L_k), the sets R, S_o and S_k each has to contain two distinct points. There are at most $m \cdot n$ points in the set $A_1 \vee A_2 \vee \cdots \vee A_m$ so that there are at most $m \cdot n$ pairs (a, L_k) for a fixed k. Thus S_k has to contain at most $2mn + m$ points and S_o and R have at most $2mn^2$ points each. Therefore we see that S has to contain at most $mn[6n + 1]$ points.

To construct the geometry G_1 on S' we let the set of non-trivial lines of G_1 consist of $S_o \vee A_1, S_1, S_2, \ldots, S_n$ and all the elements π_i of P. Any two of these non-trivial lines have at most one point in common which shows that it is a set of non-trivial lines of some geometry on S'. We recall that every point p_i of G is contained in at least one non-trivial line of G and therefore the set A_i, $i = 1, 2, \ldots, m$, is non-void. Since these sets are disjoint we see that the geometries $G_1, G_2, \ldots, G_m$ are non-comparable in $LG(S)$. Thus the mapping θ is a one-to-one mapping of the set of points of L onto the set consisting of $G_1, G_2, \ldots, G_m$. We also see that the intersection of any two such distinct geometries G_i and G_j is the geometry whose set of non-trivial lines consists of $S_o, S_1, S_2, \ldots, S_n$ and all the elements π_i of P.

If p_i and p_j are distinct points of G which are not contained in the same non-trivial line then $G_i \cup G_j = E$ and the set of non-trivial lines of E consists of $S_o \vee A_i \vee A_j, S_1, S_2, \ldots, S_n$ and all the elements π_i of P. First, we shall show that any two of these non-trivial lines have at most one point in common. To see this, let us denote the line of E which contains S_o by M and let us recall that the sets $R, S_o, S_1, S_2, \ldots, S_n$ are disjoint. The line M can have at most one point in common with the line S_k since otherwise A_i and A_j would have points in common with the set S_k which would imply that the points p_i and p_j are contained in the non-trivial line l_k of G, contrary to assumption. From the definition of the line π_k contained in P it follows that π_k can have at most one point in common with L and S_i, $i = 1, 2, \ldots, n$. Thus E is a geometry. We see that every line of G_i and G_j is contained in a line of E, thus $G_i, G_j \leq E$. Since $S_o \vee A_i$ and $S_o \vee A_j$ are lines of G_i and G_j, respectively, and they have more than one point in common

$S_o \vee A_i \vee A_j$ is contained in a line of $G_i \cup G_j$, but then every line of E is contained in a line of the union and therefore $E \leq G_i \cup G_j$. From this it follows that $E = G_i \cup G_j$.

If p_i and p_j are distinct points of G which are contained in the non-trivial line l_k then $G_i \cup G_j = E$ and the set of non-trivial lines of E consists of $S_o \vee S_k \vee L_k \vee Z_k$, S_1, S_2, ... , S_{k-1}, S_{k+1}, ... , S_n and all the elements of π_i of P which are not associated with a pair (a, L_k), $a \varepsilon L_k$.

Let M denote the line of E which contains the set S_o. If M and S_t, $t \neq k$, have a point in common then this point must come from one of the sets A_r contained in M. Thus the point p_r of G is contained in the non-trivial line l_t of G. By the definition of M the point p_r is also contained in the non-trivial line l_k. Therefore the sets M and S_t, $t \neq k$, can have at most one point in common since otherwise the two distinct non-trivial lines l_4 and l_t would have more than one point in common. The line π_i can have at most one point in common with the set S_j, $j = 1, 2, \ldots, n$, and if π_i is not related to (a, L_k) then it can have at most one point in common with the line M. Thus E is a geometry. We see that G_i, $G_j \leq E$ since every line of G_i and G_j is contained in a line of E. $S_o \vee A_i$ and $S_o \vee A_j$ are lines of G_i and G_j, respectively, and they have more than one point in common. Thus $S_o \vee S_i \vee A_j$ is contained in a line of the union $G_i \cup G_j$. We know that S_k has a point in common with A_i and a distinct point in common with A_j so that $S_o \vee S_k \vee A_i \vee A_j$ must be contained in a line of $G_i \cup G_j$.

But then by the definition of the sets π_i which are associated with (a, L_k), $a \varepsilon L_k$, we obtain that $S_o \vee S_k \vee L_k \vee Z_k$ is contained in a line of $G_i \cup G_j$. Thus $G_i, G_j \leq E$ and $E \leq G_i \cup G_j$ which implies that $G_i \cup G_j = E$.

From these results it follows that $p_k \leq p_i \cup p_j$ in L if and only if $G_k \leq G_i \cup G_j$ in $LG(S')$. Thus these unions define a set of lines on the set consisting of $G_1, G_2, \ldots, G_m$, and these lines are preserved under the mapping θ. To complete the proof we shall show that the geometries $G_1, G_2, \ldots, G_m$ generate a sublattice of $LG(S')$ and that this sublattice is the lattice of the above defined geometry on $\{G_1, G_2, \ldots, G_m\}$ and therefore isomorphic to L. Let $T = \{p_i \varepsilon L | i \varepsilon B\}$ be a geometric object of G and let $\{G_i | i \varepsilon B\}$ be the corresponding geometric object in the geometry on $\{G_1, G_2, \ldots, G_m\}$. Let, after re-enumerating if necessary, $l_1, l_2, \ldots, l_t$ be the non-trivial lines which are contained in T and let $p_1, p_2, \ldots, p_s$ be the points of T which are not contained in non-trivial lines in T. Then $\cup \{G_i | i \varepsilon B\} = H$ and the set of non-trivial lines of H consists of

$$S_o \vee S_1 \vee S_2 \vee \cdots \vee S_t \vee Z_1 \vee Z_2 \vee \cdots \vee Z_t \vee L_1 \vee L_2 \vee \cdots \vee L_t \vee$$

$$A_1 \vee A_2 \vee \cdots \vee A_s,$$

$S_{t+1}, S_{t+2}, \ldots, S_n$ and all the elements π_i of P which are not associated with (a, L_i), $i = 1, 2, \ldots, t$. The line of H which contains S_o is also given by

$$[S_o \vee S_1 \vee S_2 \vee \cdots \vee S_t \vee Z_1 \vee Z_2 \vee \cdots \vee Z_t] \vee [\vee \{Ai | i \varepsilon B\}].$$

Exactly as in the two previous cases we show that any two non-trivial lines can have at most one point in common and that $H = \cup\{G_i | i \varepsilon B\}$. From this follows that $G_j \leq H$ if and only if G_j is contained in the geometric object $\{G_i | i \varepsilon B\}$. Thus to complete the proof we need only to show that these unions are closed under intersections. Let $T_1 = \{G_i | i \varepsilon B\}$ and $T_2 = \{G_j | j \varepsilon C\}$ be geometric objects of the geometry on the set consisting of $G_1, G_2, \ldots, G_m$. Let $\{l_s | s \varepsilon \bar{B}\}$ and $\{l_s | s \varepsilon \bar{C}\}$ be the sets of non-trivial lines of T_1 and T_2, respectively. We wish to show that $(\cup T_1) \cap (\cup T_2) = \cup(T_1 \wedge T_2)$. Since every G_i which is contained in $T_1 \wedge T_2$ is contained in $(\cup T_1) \cap (\cup T_2)$ we see that $(\cup T_1) \cap (\cup T_2) \geq \cup(T_1 \wedge T_2)$. We shall now show that every line of the geometry $(\cup T_1) \cap (\cup T_2)$ is contained in a line of the geometry $\cup(T_1 \wedge T_2)$. This will imply that $(\cup T_1) \cap (\cup T_2) \leq \cup(T_1 \wedge T_2)$ and therefore $(\cup T_1) \cap (\cup T_2) = \cup(T_1 \wedge T_2)$. Let us denote the line of $\cup T_1$ which contained S_o by M_1. Then

$$M_1 = S_o \vee [\vee\{S_i | i \varepsilon \bar{B}\}] \vee [\vee\{Z_i | i \varepsilon \bar{B}\}] \vee [\vee\{A_i | i \varepsilon B\}] \ .$$

Let the corresponding line of $\cup T_2$ be M_2. Then

$$M_2 = S_o \vee [\vee\{S_i | i \varepsilon \bar{C}\}] \vee [\vee\{Z_i | i \varepsilon \bar{C}\}] \vee [\vee\{A_i | i \varepsilon C\}] \ .$$

We have to compute $M_1 \wedge M_2$. To do this let us recall that the sets Z_i, Z_j are disjoint whenever $i \neq j$, S_i, S_j are disjoint whenever $i \neq j$, S_i, Z_j are disjoint and so are A_i and Z_j. Thus after using the distributive law for set operations we obtain that

$$M_1 \wedge M_2 = S_0 \vee [\vee\{S_i | i \in \bar{B} \wedge \bar{C}\}] \vee [\vee\{Z_i | i \in \bar{B} \wedge \bar{C}\}]$$

$$\vee [\vee\{A_i | i \in A \wedge B\}] \vee [(\vee\{A_i | i \in B\}) \wedge (\vee\{S_j | j \in \bar{C}\})]$$

$$\vee [(\vee\{A_i | i \in C\}) \wedge (\vee\{S_j | j \in \bar{B}\})] .$$

Let us simplify this expression. Assume that a point x of the set S' is contained in $A_i \wedge S_j$, $i \in B$ and $j \in \bar{C}$ (or $i \in C$ and $j \in \bar{B}$). Note that i contained in B implies that the geometry G_i is contained in T_1, and if A_i and S_j have a point in common then the geometry G_i is contained in the non-trivial line l_j which is contained in T_2. Thus G_i is contained in $T_1 \wedge T_2$ and therefore x is contained in $\vee\{A_i | i \in B \wedge C\}$. From this it follows that

$$M_1 \wedge M_2 = S_0 \vee [\vee\{S_i | i \in \bar{B} \wedge \bar{C}\}] \vee [\vee\{Z_i | i \in \bar{B} \wedge \bar{C}\}]$$

$$\vee [\vee\{A_i | i \in B \wedge C\}] .$$

This is also the line of the geometry $\cup(T_1 \wedge T_2)$ which contains the set S_0; thus this line of the geometry $(\cup T_1) \cap (\cup T_2)$ is contained in a line of $\cup(T_1 \wedge T_2)$. Any one of the remaining non-trivial lines of the geometry $(\cup T_1) \cap (\cup T_2)$ is an intersection of some lines from the geometry $\cup T_1$ with some line from $\cup T_2$ and therefore this line must be contained in a set S_i or π_i. Clearly any one of these sets is contained in a line of the geometry $\cup(T_1 \wedge T_2)$. Thus we see that $(\cup T_1) \cap (\cup T_2) = \cup(T_1 \wedge T_2)$. This shows that the geometries $G_1, G_2, \ldots, G_m$ generate a sublattice of $LG(S')$ which is isomorphic to L as was to be shown.

From Theorem 8 and Theorem 9 we obtain the final result:

Theorem 10: Any finite lattice L can be embedded in the lattice of all geometries on some finite set S.

From P.M. Whitman's [2] result we know that every lattice can be embedded in a partition lattice on some infinite set. The corresponding problem of embedding every finite lattice in a finite partition lattice has not been solved yet. We see though that Theorem 10 reduces this problem to the problem of embedding every lattice of all geometries on a finite set into a finite partition lattice.

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