ON THE HYDRODYNAMIC STABILITY OF TWO VISCOUS INCOMPRESSIBLE FLUIDS IN PARALLEL UNIFORM SHEARING MOTION

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ABSTRACT

A new problem in hydrodynamic stability is investigated. Given two contiguous plane sheets of viscous incompressible fluids, bounded on one side by a solid wall and unbounded on the other, the problem is to study the hydrodynamic stability when the fluids are in longitudinal, laminar, uniform shearing motion. The mathematical analysis, based on small disturbance theory, leads to a characteristic value problem in a system of two linear ordinary differential equations. The case for which forces due to gravity and surface tension are negligible compared with inertia and viscous forces is studied in detail, and the results are presented graphically for all fluid combinations of possible interest.

SUMMARY

The present investigation was started by the desire of obtaining some understanding of the stability of a thin liquid film when it is dragged over a flat solid surface by a high speed gas stream.

The problem arose originally in connection with film cooling applications, where an instability was found to occur at sufficiently high liquid flow rates. The instability resulted in a loss of liquid from the film at a much higher rate than the evaporation loss under laminar flow conditions.

In film cooling applications the gas stream will usually be turbulent, and all experiments so far have been concerned with such flows. In these experiments, small wave like disturbances were observed on the liquid-gas interface even at very low liquid flow rates, and it appeared that these might be the non-amplified response of the stable laminar flow to turbulent fluctuations in the gas. At higher liquid flow rates, a large scale instability of the interface appeared and it was this phenomenon that resulted in the increased liquid loss into the gas stream.

It was not clear that the large scale instability was related to the small waves; however, the first desirable step in a theoretical analysis of the problem, appeared to be a study of the laminar stability of the shear flow of two layers of fluid of different densities and viscosities. The calculated laminar sublayer thickness in the gas stream was of the same order of magnitude as the thickness of the liquid film in the experiments. Hence, it seemed reasonable to take as the idealized model for the analysis, a laminar shear flow in a liquid film bounded by a wall on

one side, and on the other by a semi-infinite gas stream also in uniform shearing motion.

It seemed possible that the observed large scale instability might result from the turbulence in the gas stream, and hence would not appear in the idealized model. The analysis given here shows that this is almost certainly the case. However, the stability analysis of the proposed model is of interest in itself because of the following fact: uniform shearing laminar motion of a single fluid bounded on one side by a wall is known to be stable with respect to any small disturbance (plane Couette motion); the question then arises of how a discontinuity in viscosity or density would affect the stability of uniform shearing motion.

The basic shear motion is perturbed by a small arbitrary disturbance and its stability is the subject of the analysis. The aim is to determine the value of the parameters for which neutral stability will result, i.e., for which the disturbance, as a function of time, will not grow or decay. The small disturbance theory has been applied in the past to several hydrodynamic stability problems of flow of a single fluid. The following are among the treated problems: a) plane Couette flow[8,13], b) initial stages of plane Couette flow when the moving wall starts from rest [21], c) plane Poiseuille flow[22,23], and d) Blasius boundary layer[22].

The dimensionless parameters in the problem are: wave number (α) expressed in terms of liquid film thickness, liquid Reynolds number (R), wave velocity (c) in terms of interface velocity, liquid to gas viscosity ratio (m), and gas to liquid density ratio (ν). It is to be noted that the liquid Reynolds number is simply proportional to the liquid flow rate, the reason being that the velocity profiles are linear functions of the distance away from the solid boundary.

The mathematical analysis leads to a characteristic value problem in two linear ordinary fourth order differential equations. The fact that the equations are linear is a very important one since superposition of solutions is permissible. This implies that any arbitrary disturbance can be decomposed into its Fourier components, and then the problem has to be solved only for the case of one general sinusoidal oscillation. The solution depends on the evaluation of certain contour integrals (cf. Appendix B). The key to the solution rests on the fact that the asymptotic evaluation of these integrals gives the possibility of expressing the complete characteristic value determinant as an asymptotic expansion in powers of $(\alpha R)^{-1/2}$. Keeping the first two leading terms of the series yields the solution in a conceptually simple way.

Since the physical case of interest was the one where gravity and surface tension forces were small, detailed calculations were carried out for the case where these forces vanished.

The conditions for neutral stability are usually represented by a curve of α vs. R with the dimensionless wave velocity c appearing as a parameter. In this analysis, m and r appear as additional parameters. The results obtained show that given a combination of two fluids, the shape (a loop) of the neutral stability line in the αR -plane, is similar to the one for the laminar boundary layer. The curve has a minimum Reynolds number below which all disturbances are damped, and two branches, one having $\alpha = 0$ as an asymptote, and the other some finite value of α. The quantities of interest are the values of α and c at the minimum

Reynolds number. A summary of the effects* of the physical properties of the fluids on these quantities follows:

(a) As the values of viscosity ratio (m) and the density ratio (r) tend to unity, the flow becomes completely stable.

(b) For a given density ratio (r), an increase in the value of the viscosity ratio (m) always increases the stability, and in regions away from the neutral stability curve, decreases the amplification or damping.

(c) For very small values of density ratio (r) the flow becomes completely stable.

(d) For a given value of viscosity ratio (m), there is always a density ratio (r) for which the flow is least stable.

(e) As the density ratio (r) tends to zero, the disturbance wave length approaches a value very close to ten film thicknesses, and the wave velocity becomes approximately a tenth of the liquid-gas interface velocity.

A check on the correctness of the analysis for some limiting cases is made by comparing some of the results obtained with the known universal stability of plane Couette flow between parallel walls of arbitrary spacing. Item (a), as well as (b) when $m \to \infty$, are equivalent to Couette flow, and the results show that, in effect, the flow has been completely stabilized.

Less detailed calculations are carried out for the case where

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* The influence of any parameter is said to be stabilizing when the result of this influence is to displace the neutral stability curve to the right in the αR -plane.

gravity* and surface tension forces are small but not zero. Intuitively it seems that they should both have the same effect on the neutral stability curve in the αR -plane. Consideration of a vertical displacement of a small region of the horizontal liquid-gas interface leads to the conclusion that surface tension and gravity forces act in the same direction. The results show this expectation to be correct, and the effect of gravity or surface tension forces is to destabilize the flow.

It can now be said that the model chosen for the analysis, although the simplest possible, has yielded a number of new and interesting results.

The most important conclusion to be drawn from this investigation is that a discontinuity of viscosity or density has a destabilizing effect on uniform shearing motion.

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* Gravity is assumed to act normally to the flow and in the direction of the solid boundary.

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NOMENCLATURE

a	quantity obtained by comparing Eqs. (B.21) through (B.26) with (B.27)
A	amplitude of disturbed liquid-gas interface
c	wave velocity (complex or real)
D	$= \frac{d}{dt}$
$\frac{D}{Dt}$	substantial derivative
F	$= \frac{\overline{U}_2^2}{g\delta}$ Froude number
g	acceleration due to gravity
$\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$	functions defined by Eqs. (C.8), (C.9) and (C.10) respectively
$\mathcal{I}m$	imaginary part of
$R\ell$	real part of
i	$\sqrt{-1}$
I_1, I_2	integrals defined by Eq. (B.30)
I_1^*, I_2^*, I_p	integrals defined on p.
k	$(12)^{1/4}\, e^{-i\frac{\pi}{6}}$
k^*	$(12)^{1/4}\, e^{i\frac{\pi}{6}}$
$k_1, \ldots k_4, K_1, \ldots K_4$	constants of integration
ℓ	variable defined in Eq. (B.57)
ℓ_r	radius of curvature of liquid-gas interface surface
$L_1^{*\prime}$	deformed L_1 path of integration
L_1, L_2	paths of integration in the complex plane (cf. Fig. 2)
L_1', L_2'	paths L_1, L_2 after rotation (cf. Fig. 15)
$m = \mu_\ell / \mu_g$	liquid-gas viscosity ratio
p	pressure

P	function defined by Eqs. (D.1) and (D.3)
Q	quantity defined by Eq. (B.60)
$r = \frac{\rho_g}{\rho_l}$	gas-liquid density ratio
$R = \frac{\bar{U}_2 \delta \rho}{\mu}$	Reynolds number
Res	residue of
t	when used in $e^{i\alpha(x-ct)}$ it denotes time, otherwise it is a variable of integration, real or complex, depending on whether it has or has not a subscript
T'	variable defined by Eq. (A.8)
$\mathcal{T}$	$\arg \tau$
u, v	disturbance velocities in x, y-direction, respectively; with a bar they denote a dimensional quantity
$\bar{U}_2$	liquid-gas interface dimensional velocity; reference velocity
	function defined by Eq. (B.55)
W	$\frac{\rho_l \delta \bar{U}_2^2}{\sigma}$ Weber number
x, y	Cartesian coordinates; with a bar they denote a dimensional quantity
z	complex variable defined by Eq. (1.8)
$\alpha = \frac{2\pi}{\lambda}$	wave number
γ	function defined by Eq. (B.86)
δ	dimensional thickness of liquid layer; reference length
ϵ	quantity defined by Eq. (8.15)
$\mathcal{S}_{1,2}$	functions defined by Eq. (1.15)
η	$\mathcal{I}m\, \tau$
θ	$\arg z$
λ	disturbance wave length
$\Lambda_k, T_k, V_k, a_k, b_k, c_k$	functions defined by Eq. (5.15)
μ	dynamic viscosity

ν	kinematic viscosity
ξ	$R\ell\,\tau$
ρ	density
σ	surface tension coefficient
σ_{yy}	normal stress component in y-direction
τ_1, τ_2	cols of integral (B.29), $\frac{i}{\sqrt{3}}$ and $-\frac{i}{\sqrt{3}}$ respectively
τ_s	Reynolds shearing stress
φ, Φ	dimensionless amplitude function of the disturbance stream function in the liquid and gas respectively
ψ, Ψ	dimensionless disturbance stream function in the liquid and gas respectively

<u>Subscripts</u>

crit.	denotes quantity evaluated at the critical Reynolds number
g	gas, upper fluid
i	interface, or imaginary; meaning is clear in each case
ℓ	liquid, lower fluid
$\sim$	denotes a quantity evaluated on the neutral stability line
P	pole
r	real, except when used with ℓ
T	due to surface tension
1,2,3	when used with y or with $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \Phi_1, \Phi_2, \Phi_3, \Phi_4$ and their derivatives indicate quantity evaluated at the wall, interface or at infinity, respectively

<u>Superscripts</u>

*	indicates quantities evaluated after deformation of the contour of integration
'	indicates derivative with respect to the independent variable

I. INTRODUCTION

The interest in the present problem stems from the desire of obtaining some theoretical understanding of the phenomenon that occurs when a liquid film flows over a flat surface, dragged along by a high speed gas. For certain liquid flow rates, the film surface becomes wavy and particles of liquid are detached from the main body, entrained by the gas, and carried downstream. This situation arises, among the numerous engineering applications, in connection with film cooling [1]* of a solid boundary.

A classical problem releated to the one mentioned above, is the one of generation of ocean waves by wind. This problem has been recently investigated theoretically by Lock[2], who is the first to have included in the analysis all the physical properties of the air and water. His calculations are incomplete and the results obtained rather cumbersome.

The problem of the stability of stratified motion of different fluids has been studied by Taylor[3] and Goldstein[4], who didn't include viscosity in their analyses. Taylor investigated continuous and discontinuous density and velocity distributions. Goldstein treated similar problems, his investigation being a generalization to heterogeneous stratified fluids of Rayleigh's[5] work on the homogeneous case.

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* Numerical superscripts refer to the references given at the end of this paper.

The problem of liquid film stability has been investigated experimentally by Kinney, Abramson and Sloop(6) and by Knuth(1), who were concerned with liquid film cooling applications, where the gas stream was always turbulent. York, Stubbs and Teck(7) studied the mechanism of disintegration of liquid sheets experimentally, and they proposed an inviscid model based on an extension of Lamb's work.

The philosophy of the present approach to the problem is to choose as simple a model of the physical situation as possible, so that the problem becomes tractable, but still keep enough of the physical features so that it will contribute to the understanding of the phenomenon.

The aim of this investigation is to solve the hydrodynamic stability problem when two viscous incompressible fluids, in two-dimensional, laminar, uniform shearing motion (Fig. 1) are perturbed by a small arbitrary disturbance. One of the fluids, from now on called liquid, is bounded in the direction normal to the flow by a solid wall and by the second fluid, called gas, of semi-infinite extent in the direction normal to the flow.

By "solving the problem" is meant to find the relationships to be satisfied by the parameters of the problem so that neutral stability will exist. This implies that a neutral stability hypersurface, whose coordinates are the physical variables involved, could be constructed that would separate regions of stability and instability. In other words, if the physical properties of the fluids are given, it is desired to determine the minimum critical Reynolds number* at which instability begins.

It is realized from the above remarks that the chosen model is quite an idealization of the physical situation. The most serious criticism is that the gas motion, in the actual case, is turbulent in most cases of interest, and the velocity profile is not a linear function of the distance away from the wall. Some justification for the present approach lies in the fact that at least in the laminar sublayer** of the turbulent flow, the gas is laminar with a near linear velocity profile. There is still one other reason for the approach used (besides the obvious one of being unable to analytically handle the turbulence), which is, that it has been definitely shown by Zondek and Thomas(8) that plane Couette flow of a semi-infinite fluid in the direction normal to the flow, is always stable. Now, this situation is a special case of the aforementioned model, and it is interesting from a theoretical point of view, to know what the answer to the proposed problem is, in other words, how does a discontinuity in density or viscosity affect the stability of pure shear motion?

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* Defined, for given flow physical properties, as the smallest value of R, for which incipient disturbances will become amplified, i.e., the smallest value of R of the neutral stability curve in the αR-plane.

** The ratio of gas laminar sublayer thickness to liquid film thickness in existing experiments is of the order of unity.

II. THE BOUNDARY VALUE PROBLEM AND ITS SOLUTION

1. The Orr-Sommerfeld Differential Equation and its General Solution for Plane Couette Flow

The task is now to mathematically formulate the problem of stability of two-dimensional laminar motion. This has been done in the past in numerous references (cf. Ref. 9). For the sake of completeness a brief description of the derivation of the disturbance equation will be given here.

Let all coordinates and velocities be made dimensionless by the use of a reference length δ, and a reference velocity $\overline{U}_2$. Consider a basic flow in the x-direction with a velocity profile $U(y)$. The Navier-Stokes and continuity equations in the xy-plane can be perturbed by assuming velocities in the x and y-directions of the form

$$U(y) + u(x,y,t)$$
$$v(x,y,t)$$

and a pressure

$$P + p(x,y,t) \qquad (1.1)$$

where the small symbols indicate small quantities, and t is the dimensionless time (i.e. $\overline{U}_2 \times$ time$/\delta$). Introducing Eq. (1.1) into the Navier-Stokes equations leads to two linear partial differential equations from which the pressure can be eliminated by cross differentiation and subtraction. The result is a linear partial differential equation containing u and v as the dependent variables, x, y and t being the independent variables. The equation of continuity guarantees the existence of a stream function $\psi(x,y,t)$, such that

$$u = \frac{\partial \psi}{\partial y}$$
$$v = -\frac{\partial \psi}{\partial x}$$

which, when used in the last differential equation obtained, reduces it to a linear partial differential equation in terms of only one dependent variable: $\psi(x,y,t)$. The fact that this equation is linear, is very important, since this means that superposition of solutions is allowable. This implies that if any arbitrary disturbance is decomposed into its Fourier components, it is then sufficient to solve the problem for one general sinusoidal oscillation. After the solution is obtained, it will be necessary to look at all possible frequencies and see how they affect the behavior of the solution.

In order to separate the variables in the partial differential equation for the stream function ψ , let

$$\psi(x,y,t) = \varphi(y)\, e^{i\alpha(x-ct)} \tag{1.2}$$

where $\varphi(y)$ is the complex amplitude, α is the wave number, assumed positive without any loss of generality, and c is the complex wave velocity

$$c = c_r + i\, c_i \tag{1.3}$$

where c_r is the wave velocity and c_i allows for amplification of disturbances if $c_i > 0$, damping of disturbances if $c_i < 0$, and neutral disturbances if $c_i = 0$. The partial differential equation for Ψ then reduces to the Orr-Sommerfeld equation

$$(U-c)(\varphi''-\alpha^2\varphi)-U''\varphi=-\frac{i}{\alpha R}(\varphi^{iv}-2\alpha^2\varphi''+\alpha^4\varphi) \qquad (1.4)$$

where the primes indicate derivatives with respect to y, and R is the Reynolds number: $\frac{\rho\delta U_2}{\mu}$, (ρ is the density of the fluid and μ the viscosity).

If the velocity profile of the undisturbed flow is a linear function of y, as in the case of interest, $U''=0$, and Eq. (1.4) becomes

$$(U-c)(\varphi''-\alpha^2\varphi)=-\frac{i}{\alpha R}(\varphi^{iv}-2\alpha^2\varphi+\alpha^4\varphi) \qquad (1.5)$$

which is a linear fourth order total differential equation in the complex y-plane. As first pointed out by Lin(9), Eq. (1.5) has four linearly independent solutions, analytic functions of y and entire functions of the parameters c, α and αR. We will now solve Eq. (1.5). Let*

$$\varphi''(y)-\alpha^2\varphi(y)=S(y) \qquad (1.6)$$

which when used in Eq. (1.5) yields

$$S''-\left[i\alpha R(U-c)+\alpha^2\right]S=0 \qquad (1.7)$$

Now, letting

$$z=\left\{\frac{\alpha^2}{(\alpha R)^{2/3}}+i(\alpha R)^{1/3}\left[U(y)-c\right]\right\}\left[U'(y)\right]^{-2/3} \qquad (1.8)$$

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* This transformation was first pointed out by Orr(24) and later independently by Sommerfeld(10).

and

$$\mathcal{S}(y) = h(z) \tag{1.9}$$

Eq. (1.7) could be written as

$$h''(z) + z\,h(z) = 0 \tag{1.10}$$

which is the so called Stokes' equation. The solution can be obtained in terms of contour integrals by using Laplace's method (cf. Ref. 11, pp. 582-585). The result is

$$h(z) = 2k_3\,h_1(z) + 2k_4\,h_2(z) \tag{1.11}$$

where the factor 2 is introduced for convenience in later calculations, k_3 and k_4 are arbitrary constants and

$$h_1(z) = \frac{k}{i\pi}\int_{L_1} e^{zt+\frac{t^3}{3}}\,dt \tag{1.12}$$

$$h_2(z) = -\frac{k^*}{i\pi}\int_{L_2} e^{zt+\frac{t^3}{3}}\,dt \tag{1.13}$$

$$\left.\begin{aligned} k &= (12)^{1/6}\,e^{-i\frac{\pi}{6}} \\ k^* &= (12)^{1/6}\,e^{i\frac{\pi}{6}} \end{aligned}\right\} \tag{1.14}$$

L_1 and L_2 being the contours of integration shown in Fig. 2, h_1 and h_2 are entire functions of z, and t is a complex variable of integration.

$h_1(z)$ and $h_2(z)$ could also be written in terms of Hankel functions of order one-third[12] as

$$h_1(z) = \left(\tfrac{2}{3} z^{3/2}\right)^{1/3} H^{(1)}_{1/3}\left(\tfrac{2}{3} z^{3/2}\right) \tag{1.15}$$

$$h_2(z) = \left(\tfrac{2}{3} z^{3/2}\right) H^{(2)}_{1/3}\left(\tfrac{2}{3} z^{3/2}\right) \tag{1.16}$$

It will be recalled[14] that the Hankel functions are of oscillating nature, $H^{(1)}_{1/3}$, the function of the first kind being damped exponentially as $\left|\tfrac{2}{3} z^{3/2}\right|$ becomes large, while $H^{(2)}_{1/3}$ increases exponentially under the same conditions.

From Eqs. (1.9) and (1.11)

$$S(y) = 2k_3 S_1(y) + 2k_4 S_2(y)$$

where

$$\left.\begin{aligned} S_1(y) &= h_1(z) \\ S_2(y) &= h_2(z) \end{aligned}\right\} \tag{1.17}$$

which when inserted in Eq. (1.6) gives

$$\varphi''(y) - \alpha^2 \varphi(y) = 2k_3 S_1(y) + 2k_4 S_2(y) \tag{1.18}$$

The solution of the homogeneous part is

$$\varphi_c(y) = k_1 e^{-\alpha y} + k_2 e^{\alpha y} \tag{1.19}$$

k_1 and k_2 being arbitrary constants. Eq. (1.18) can also be written as

$$(D^2 - \alpha^2)\varphi = f(y) \tag{1.20}$$

where

$$D \equiv \frac{d}{dy}$$

and

$$f(y) = 2k_3 S_1(y) + 2k_4 S_2(y)$$

Solving for φ gives

$$\varphi(y) = \frac{1}{2\alpha}\left(\frac{1}{D-\alpha} - \frac{1}{D+\alpha}\right) f(y) \tag{1.21}$$

The interpretation of the operator Eq. (1.21) gives the particular integral

$$\varphi_p(y) = \frac{k_3}{\alpha}\left\{e^{\alpha y}\int^y e^{-\alpha y} S_1(y)\,dy - e^{-\alpha y}\int^y e^{\alpha y} S_1(y)\,dy\right\}$$

$$+ \frac{k_4}{\alpha}\left\{e^{\alpha y}\int^y e^{-\alpha y} S_2(y)\,dy - e^{-\alpha y}\int^y e^{\alpha y} S_2(y)\,dy\right\}$$

(1.22)

where the constant of integration is arbitrary, since inclusion of this constant will give rise to two terms that can be merged with the complementary function, Eq. (1.19).

The complete solution of Eq. (1.5) is then

$$\varphi(y) = k_1\varphi_1(y) + k_2\varphi_2(y) + k_3\varphi_3(y) + k_4\varphi_4(y) \tag{1.23}$$

where

$$\varphi_1(y) = e^{-\alpha y}$$

$$\varphi_2(y) = e^{\alpha y}$$

$$\varphi_3(y) = \frac{1}{\alpha}\left\{\int^y e^{\alpha(y-t)} S_1(t)\,dt - \int^y e^{-\alpha(y-t)} S_1(t)\,dt\right\}$$

$$\varphi_4(y) = \frac{1}{\alpha}\left\{\int^y e^{\alpha(y-t)} S_2(t)\,dt - \int^y e^{-\alpha(y-t)} S_2(t)\,dt\right\} \tag{1.24}$$

and t is a variable of integration. S_1 and S_2 are given by Eq. (1.17).

2. Differential Equations for the Present Problem

For the case of flow of two stratified fluids two equations of the type (1.5) will have to be used, one for each fluid, with the proper matching conditions at the interface. Let δ and $\overline{U}_2$ used to render Eq. (1.5) dimensionless be, respectively, the height of the liquid sheet and its surface velocity; let small case and capital Greek letters denote conditions in the liquid and gas respectively, and let the subscripts l and g denote quantities evaluated in the liquid and gas. We then have

Liquid | **Gas**

Disturbance Stream Function*

$$\psi = \varphi(y)e^{i\alpha(x-ct)} \qquad\qquad \Psi = \Phi(y)e^{i\alpha(x-ct)} \tag{2.1}$$

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* The reason for using the same α and c in the liquid and gas stream functions, is that (since they are constants) the consideration of a disturbance occurring at the interface leads to the conclusion that the wave number and the wave velocity must be the same, whether considered in the liquid or in the gas.

Liquid **Gas**

Disturbance Velocities

$$u_l = \frac{\partial \psi}{\partial y} = \varphi'(y)\, e^{i\alpha(x-ct)} \qquad u_g = \frac{\partial \Psi}{\partial y} = \Phi'(y)\, e^{i\alpha(x-ct)}$$

$$v_l = -\frac{\partial \psi}{\partial x} = -i\alpha\, \varphi(y)\, e^{i\alpha(x-ct)} \qquad v_g = -\frac{\partial \Psi}{\partial x} = -i\alpha\, \Phi(y)\, e^{i\alpha(x-ct)} \tag{2.2}$$

Orr-Sommerfeld Equations

$$(U_l - c)(\varphi'' - \alpha^2\varphi) = -\frac{i}{\alpha R_l}(\varphi^{iv} - 2\alpha^2\varphi'' + \alpha^4\varphi) \qquad (0 \le y \le 1) \tag{2.3}$$

$$(U_g - c)(\Phi'' - \alpha^2\Phi) = -\frac{i}{\alpha R_g}(\Phi^{iv} - 2\alpha^2\Phi'' + \alpha^4\Phi) \qquad (1 \le y \le \infty) \tag{2.4}$$

where

$$R_l = \frac{\rho_l\, \delta\, \overline{U}_2}{\mu_l} \tag{2.5}$$

$$R_g = \frac{\rho_g\, \delta\, \overline{U}_2}{\mu_g} \tag{2.6}$$

$$U_l = y \tag{2.7}$$

and equating the shear stresses at the interface of the basic flow gives

$$U_g = 1 + \frac{\mu_l}{\mu_g}(y-1) \tag{2.8}$$

The solutions of Eqs. (2.3) and (2.4) can now be written, from Eq. (1.21), as

$$\varphi(y) = k_1\varphi_1(y) + k_2\varphi_2(y) + k_3\varphi_3(y) + k_4\varphi_4(y) \qquad (0 \leq y \leq 1) \tag{2.9}$$

$$\Phi(y) = K_1\Phi_1(y) + K_2\Phi_2(y) + K_3\Phi_3(y) + K_4\Phi_4(y) \qquad (1 \leq y \leq \infty) \tag{2.10}$$

where the φ 's are given by Eq. (1.24), and the Φ 's can be obtained also from the same equation with the precaution of using the appropriate value of z in the gas, given by Eq. (1.8).

3. Boundary Conditions

The boundary conditions are identical whether written in terms of dimensional or dimensionless velocities and coordinates. It will therefore be assumed, that dimensionless quantities are being used.

Let subscripts 1, 2 and 3 used with the coordinate y denote, respectively, the wall, interface and infinity, i.e. $y_1 = 0$, $y_2 = 1$ and $y_3 = \infty$.

At the wall, $(y = y_1 = 0)$, both components of the disturbance velocities must vanish

$$v_\ell(y_1) = 0 \tag{3.1}$$

$$u_\ell(y_1) = 0 \tag{3.2}$$

At the interface, $(y = y_2 = 1)$, the following conditions must hold

a) both fluids move together with no vacuum layer between them

$$v_\ell(y_2) - v_g(y_2) = 0 \tag{3.3}$$

b) no slip between the fluids in the direction of flow

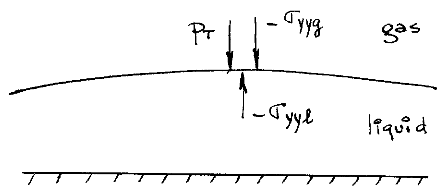
$$u_\ell(y_2) - u_g(y_2) = 0 \tag{3.4}$$

c) the shear stress must be continuous

$$\mu_\ell\left(\frac{\partial v_\ell}{\partial x} + \frac{\partial u_\ell}{\partial y}\right) - \mu_g\left(\frac{\partial v_g}{\partial x} + \frac{\partial u_g}{\partial y}\right) = 0 \tag{3.5}$$

d) because of surface tension effects, the normal stress jumps, i. e.

$$p_T - \sigma_{yyg} = -\sigma_{yy\ell}$$



where p_T (defined in Eq. (A.3), Appendix A) is the effective pressure caused by surface tension forces. The previous equation can be re-written as

$$p_g - 2\mu_g \frac{\partial v_g}{\partial y} - p_\ell + 2\mu_\ell \frac{\partial v_\ell}{\partial y} + p_T = 0 \tag{3.6}$$

<u>**At infinity**</u>, $(y = y_3 = \infty)$, the disturbances must vanish

$$v_g(y_3) = 0 \tag{3.7}$$

$$u_g(y_3) = 0 \tag{3.8}$$

Eqs. (3.1) through (3.8) are the eight boundary conditions necessary for solving the system of two fourth order differential equations given by Eqs. (2.9) and (2.10).

4. The Secular Equation

Eqs. (1.16), (1.17) and (1.24), show that when $y \to \infty$, $\Phi_2(y)$ and $\Phi_4(y) \to \infty$. In order to satisfy the boundary conditions given in Eqs. (3.7) and (3.8) it is necessary that in Eq. (2.10)

$$K_2 = K_4 = 0 \tag{4.1}$$

which when used with Eq. (2.2) allows the boundary conditions, Eqs. (3.1) to (3.8), to be rewritten as

$$k_1 \varphi_{11} + k_2 \varphi_{21} + k_3 \varphi_{31} + k_4 \varphi_{41} = 0 \tag{4.2}$$

$$k_1 \varphi'_{11} + k_2 \varphi'_{21} + k_3 \varphi'_{31} + k_4 \varphi'_{41} = 0 \tag{4.3}$$

$$k_1 \varphi_{12} + k_2 \varphi_{22} + k_3 \varphi_{32} + k_4 \varphi_{42} - K_1 \Phi_{12} - K_3 \Phi_{32} = 0 \tag{4.4}$$

$$k_1 \varphi'_{12} + k_2 \varphi'_{22} + k_3 \varphi'_{32} + k_4 \varphi'_{42} - K_1 \Phi'_{12} - K_3 \Phi'_{32} = 0 \tag{4.5}$$

$$\left.\begin{aligned} & k_1(\alpha^2 \varphi_{12} + \varphi''_{12}) + k_2(\alpha^2 \varphi_{22} + \varphi''_{22}) + k_3(\alpha^2 \varphi_{32} + \varphi''_{32}) + k_4(\alpha^2 \varphi_{42} + \varphi''_{42}) \\ & \qquad - \frac{K_1}{m}(\alpha^2 \Phi_{12} + \Phi''_{12}) - \frac{K_3}{m}(\alpha^2 \Phi_{32} + \Phi''_{32}) = 0 \end{aligned}\right\} \tag{4.6}$$

$$\begin{aligned} & k_1\left\{\left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right] R \varphi_{12} - \left[(1-c)R - i3\alpha\right]\varphi'_{12} - \frac{i}{\alpha}\varphi'''_{12}\right\} + k_2\left\{\left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right] R \varphi_{22} - \left[(1-c)R - i3\alpha\right]\varphi'_{22} - \frac{i}{\alpha}\varphi'''_{22}\right\} \\ & + k_3\left\{\left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right] R \varphi_{32} - \left[(1-c)R - i3\alpha\right]\varphi'_{32} - \frac{i}{\alpha}\varphi'''_{32}\right\} + k_4\left\{\left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right] R \varphi_{42} - \left[(1-c)R - i3\alpha\right]\varphi'_{42} - \frac{i}{\alpha}\varphi'''_{42}\right\} \\ & - \frac{K_1}{m}\left\{\left[m + \frac{1}{F(1-c)}\right] R r m \Phi_{12} - \left[(1-c)R r m - i3\alpha\right]\Phi'_{12} - \frac{i}{\alpha}\Phi'''_{12}\right\} - \frac{K_3}{m}\left\{\left[m + \frac{1}{F(1-c)}\right] R r m \Phi_{32} - \left[(1-c)R r m - i3\alpha\right]\Phi'_{32} - \frac{i}{\alpha}\Phi'''_{32}\right\} \\ & = 0 \end{aligned} \tag{4.7}$$

where the notation used is: $\varphi_1(y_1) \equiv \varphi_{11}$; $\varphi_2(y_3) \equiv \varphi_{23}$; etc, and

$$\left.\begin{aligned} m &= \frac{\mu_l}{\mu_g} \\ r &= \frac{\rho_g}{\rho_l} \\ R &\equiv R_l \end{aligned}\right\} \qquad (4.8)$$

R_g does not enter in the above relationships because it is clear, that from Eqs. (2.5) and (2.6)

$$R_g = R_l r m$$

For the details of the derivation of Eq. (4.7) reference should be made to Appendix A.

Thus, the important dimensionless parameters of the problem arise automatically and are: α, R, c, m, r, F and W. It is seen that R_g has disappeared from the problem as a parameter. This seems to be in line with experimental results (cf. Ref. 1, pp. 362-363), which show that the inception point of unstable disturbances is independent of the gas-stream Reynolds number.

If the preceding set of Eqs. (4.2) to (4.7) is to have a non-trivial solution for the k's and K's the following relation, the so-called secular equation, must hold:

$$
\begin{vmatrix}
\varphi_{11} & \varphi_{21} & \varphi_{31} & \varphi_{41} & 0 & 0 \\
\varphi_{11}' & \varphi_{21}' & \varphi_{31}' & \varphi_{41}' & 0 & 0 \\
\varphi_{12} & \varphi_{22} & \varphi_{32} & \varphi_{42} & -\Phi_{12} & -\Phi_{32} \\
\varphi_{12}' & \varphi_{22}' & \varphi_{32}' & \varphi_{42}' & -\Phi_{12}' & -\Phi_{32}' \\
\alpha^2\varphi_{12} + \varphi_{12}'' & \alpha^2\varphi_{22} + \varphi_{22}'' & \alpha^2\varphi_{32} + \varphi_{32}'' & \alpha^2\varphi_{42} + \varphi_{42}'' & -\frac{1}{m}\left(\alpha^2\Phi_{12} + \Phi_{12}''\right) & -\frac{1}{m}\left(\alpha^2\Phi_{32} + \Phi_{32}''\right) \\
\begin{array}{l}\Big\{\left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right]R\varphi_{12} \\ -\left[(1-c)R - i3\alpha\right]\varphi_{12}' - \frac{i}{\alpha}\varphi_{12}'''\Big\}\end{array} &
\begin{array}{l}\Big\{\left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right]R\varphi_{22} \\ -\left[(1-c)R - i3\alpha\right]\varphi_{22}' - \frac{i}{\alpha}\varphi_{22}'''\Big\}\end{array} &
\begin{array}{l}\Big\{\left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right]R\varphi_{32} \\ -\left[(1-c)R - i3\alpha\right]\varphi_{32}' - \frac{i}{\alpha}\varphi_{32}'''\Big\}\end{array} &
\begin{array}{l}\Big\{\left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right]R\varphi_{42} \\ -\left[(1-c)R - i3\alpha\right]\varphi_{42}' - \frac{i}{\alpha}\varphi_{42}'''\Big\}\end{array} &
\begin{array}{l}-\frac{1}{m}\Big\{\left[m + \frac{1}{F(1-c)}\right]R r m \Phi_{12} \\ -\left[(1-c)R r m - i3\alpha\right]\Phi_{12}' - \frac{i}{\alpha}\Phi_{12}'''\Big\}\end{array} &
\begin{array}{l}-\frac{1}{m}\Big\{\left[m + \frac{1}{F(1-c)}\right]R r m \Phi_{32} \\ -\left[(1-c)R r m - i3\alpha\right]\Phi_{32}' - \frac{i}{\alpha}\Phi_{32}'''\Big\}\end{array}
\end{vmatrix} = 0
$$

Eq. (4.9)

5. Solutions for the Boundary Value Problem

The purpose of our calculation from now on will be to find the function that the secular equation represents.

From Eqs. (2.7), (2.8) and (4.7) it is clear that

$$\left.\begin{array}{llll} \text{in liquid} & U = y\,; & U' = 1 & 0 \le y \le 1 \\ \text{in gas} & U = 1 + m(y-1)\,; & U' = m & 1 \le y \le \infty \end{array}\right\} \qquad (5.1)$$

The six solutions involved are, from Eqs. (1.8), (1.12), (1.13), (1.17), (1.24), (2.9), (2.10) and (4.1).

$$\varphi_1(y) = e^{-\alpha y} \qquad (5.2)$$

$$\varphi_2(y) = e^{\alpha y} \qquad (5.3)$$

$$\varphi_3(y) = \frac{1}{\alpha}\left\{\int_1^y e^{\alpha(y-t_l)} S_1(t_l)\,dt_l - \int_1^y e^{-\alpha(y-t_l)} S_1(t_l)\,dt_l\right\} \qquad (5.4)$$

$$\varphi_4(y) = \frac{1}{\alpha}\left\{\int_1^y e^{\alpha(y-t_l)} S_2(t_l)\,dt_l - \int_1^y e^{-\alpha(y-t_l)} S_2(t_l)\,dt_l\right\} \qquad (5.5)$$

$$\Phi_1(y) = e^{-\alpha y} \qquad (5.6)$$

$$\Phi_3(y) = \frac{1}{\alpha}\left\{\int_\infty^y e^{\alpha(y-t_g)} S_1(t_g)\,dt_g - \int_\infty^y e^{-\alpha(y-t_g)} S_1(t_g)\,dt_g\right\} \qquad (5.7)$$

where

$$S_1(t_l) = h_1(z_l) = \frac{k}{i\pi}\int_{L_1} e^{z_l t + \frac{t^3}{3}}\,dt \qquad (5.8)$$

$$S_2(t_\ell) = h_2(z_\ell) = -\frac{k^*}{i\pi}\int_{L_2} e^{z_\ell t + \frac{t^3}{3}}\,dt \tag{5.9}$$

$$S_1(t_g) = h_1(z_g) = \frac{k}{i\pi}\int_{L_1} e^{z_g t + \frac{t^3}{3}}\,dt \tag{5.10}$$

$$z_\ell = \left[\frac{\alpha^2}{(\alpha R)^{2/3}} + i\,(\alpha R)^{1/3}\,(t_\ell - c)\right] . \tag{5.11}$$

$$z_g = \left\{\frac{\alpha^2}{(\alpha R \nu m^2)^{2/3}} + i\,(\alpha R \nu m^2)^{1/3}\left[(t_g - 1) + \frac{1}{m}(1-c)\right]\right\} \tag{5.12}$$

and for clarity, instead of using y in the liquid and gas, the variables t_ℓ and t_g were introduced. In Eqs. (5.4) and (5.5) the lower limit of integration was taken as $y = 1$ because of convenience for later calculations.* Later on, many expressions will have to be evaluated at $y = y_2 = 1$. With the choice made the calculation is simplified since several integrals are going to vanish there, i. e. φ_{32}, φ'_{32}, φ''_{32}, φ_{42}, φ'_{42}, φ''_{42} and φ'''_{42}. In Eq. (5.7) the lower limit was taken as ∞ because two of the boundary conditions required the solutions to vanish there: no other choice would have been satisfactory. The functions needed in Eq. (4.9), will now be written down

$$\begin{array}{lll}
\varphi_{11} = 1 & \varphi_{21} = 1 & \\
\varphi'_{11} = -\alpha & \varphi'_{21} = \alpha & \\
\varphi_{12} = e^{-\alpha} & \varphi_{22} = e^{\alpha} & \Phi_{12} = e^{-\alpha} \\
\varphi'_{12} = -\alpha\, e^{-\alpha} & \varphi'_{22} = \alpha e^{\alpha} & \Phi'_{12} = -\alpha e^{-\alpha}
\end{array} \tag{5.13}$$

---

* For a different alternative refer to discussion on p. 23 of this paper.

$$\varphi_{12}'' = \alpha^2 e^{-\alpha} \qquad \varphi_{22}'' = \alpha^2 e^{\alpha} \qquad \Phi_{12}'' = \alpha^2 e^{-\alpha}$$

$$\varphi_{12}''' = -\alpha^3 e^{-\alpha} \qquad \varphi_{22}''' = \alpha^3 e^{\alpha} \qquad \Phi_{12}''' = -\alpha^3 e^{-\alpha}$$

$$\varphi_{31} = \frac{1}{\alpha}\left\{\int_1^0 e^{-\alpha t_\ell} S_1(t_\ell)\,dt_\ell - \int_1^0 e^{\alpha t_\ell} S_1(t_\ell)\,dt_\ell\right\}$$

$$\varphi_{31}' = \int_1^0 e^{-\alpha t_\ell} S_1(t_\ell)\,dt_\ell + \int_1^0 e^{\alpha t_\ell} S_1(t_\ell)\,dt_\ell$$

$$\varphi_{32} = 0$$

$$\varphi_{32}' = 0$$

$$\varphi_{32}'' = 2S_1(t_\ell = 1) = 2h_1[z_\ell(1)]$$

$$\varphi_{32}''' = 2S_1'(t_\ell = 1) = 2h_1'[z_\ell(1)]\, i\,(\alpha R)^{1/3}$$

$$\varphi_{41} = \frac{1}{\alpha}\left\{\int_1^0 e^{-\alpha t_\ell} S_2(t_\ell)\,dt_\ell - \int_1^0 e^{\alpha t_\ell} S_2(t_\ell)\,dt_\ell\right\}$$

$$\varphi_{41}' = \int_1^0 e^{-\alpha t_\ell} S_2(t_\ell)\,dt_\ell + \int_1^0 e^{\alpha t_\ell} S_2(t_\ell)\,dt_\ell \qquad (5.13)\ (\text{contd.})$$

$$\varphi_{42} = 0$$

$$\varphi_{42}' = 0$$

$$\varphi_{42}'' = 2S_2(t_\ell = 1) = 2h_2[z_\ell(1)]$$

$$\varphi_{42}''' = 2S_2'(t_\ell = 1) = 2h_2'[z_\ell(1)]\, i\,(\alpha R)^{1/3}$$

$$\Phi_{32} = \frac{1}{\alpha}\left\{\int_\infty^1 e^{-\alpha(t_g - 1)} S_1(t_g)\,dt_g - \int_\infty^1 e^{\alpha(t_g-1)} S_1(t_g)\,dt_g\right\}$$

$$\Phi_{32}' = \int_\infty^1 e^{-\alpha(t_g-1)} S_1(t_g)\,dt_g + \int_\infty^1 e^{\alpha(t_g-1)} S_1(t_g)\,dt_g$$

$$\Phi_{32}'' = 2S_1(t_g = 1) + \alpha^2\Phi_{32} = 2h_1[z_g(1)] + \alpha^2\Phi_{32}$$

$$\Phi_{32}''' = 2S_1'(t_g = 1) + \alpha^2\Phi_{32}' = 2h_1'[z_g(1)]\, i\,(\alpha R \nu m^2)^{1/3}$$

where $z_{\ell,g}(1)$ means the value that $z_{\ell,g}$ takes on, when $t_{\ell,g}=1$, the double subscript denoting either liquid or gas.

Thus far, no restrictions of any kind have been made in the previous analysis.

In the rest of this paper the term "viscous solutions" will often be used. It refers to any of the functions or derivatives of φ_3, φ_4 or Φ_3, regardless of where they have to be evaluated.

The behavior of the viscous solutions, for $\alpha R \gg 1$, will be discussed next. It turns out, that this will be sufficient to solve the problem. One of the major tasks is the evaluation of the integrals in Eq. (5.13). The only previously existent discussion of a similar integral, is the one by Hopf[13] who assumed αR small: his discussion is not satisfactory, especially in view of the results obtained here, which indicate that $\alpha R \gg 1$.

Before analyzing the behavior of the solutions for $\alpha R \gg 1$, the determinant of Eq. (4.9) will be changed into a form more convenient for calculation. The procedure is

a) divide last row by R;

b) multiply 3rd, 4th and 6th columns by $\frac{(\alpha R)^{1/2}}{{}'\varphi_{32}''}$, $\frac{(\alpha R)^{1/2}}{{}'\varphi_{42}''}$ and $\frac{(\alpha R \nu m^2)^{1/2}}{{}'\Phi_{32}''}$ respectively, where the $'$ to the left of a symbol means the leading term in the expansion of the corresponding function (cf. footnote on p. 25);

c) from Eq. (5.13), insert all the zeros for the terms that vanish;

d) multiply the 5th column by -1 and rearrange the columns, so that column 5 becomes 3, 3 becomes 4, and 4 becomes 5.

The result is then

$$\begin{vmatrix} \Lambda_1 & V_1 & 0 & a_1 & b_1 & 0 \\ \Lambda_2 & V_2 & 0 & a_2 & b_2 & 0 \\ \Lambda_3 & V_3 & T_3 & 0 & 0 & c_3 \\ \Lambda_4 & V_4 & T_4 & 0 & 0 & c_4 \\ \Lambda_5 & V_5 & T_5 & a_5 & b_5 & c_5 \\ \Lambda_6 & V_6 & T_6 & a_6 & b_6 & c_6 \end{vmatrix} = 0 \qquad (5.14)$$

where the first three columns involve the inviscid terms, (except for the last row that has terms of $O(\frac{1}{R})$, and will be neglected in comparison with the terms kept in the calculation) and the last three columns involve the viscous solutions. The meaning of the symbols in Eq. (5.14) is given by the following relationships

$$\begin{aligned} \Lambda_1 &= \varphi_{11} \\ \Lambda_2 &= \varphi_{11}' \\ \Lambda_3 &= \varphi_{12} \\ \Lambda_4 &= \varphi_{12}' \\ \Lambda_5 &= \alpha^2 \varphi_{12} + \varphi_{12}'' \\ \Lambda_6 &= \left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right] \varphi_{12} - (1-c)\varphi_{12}' + i\left(3\alpha^2 \varphi_{12}' - \varphi_{12}'''\right)(\alpha R)^{-1} \end{aligned} \qquad (5.15)$$

$$\begin{aligned} V_1 &= \varphi_{21} \\ V_2 &= \varphi_{21}' \\ V_3 &= \varphi_{22} \\ V_4 &= \varphi_{22}' \\ V_5 &= \alpha^2 \varphi_{22} + \varphi_{22}'' \\ V_6 &= \left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right] \varphi_{22} - (1-c)\varphi_{22}' + i\left(3\alpha^2 \varphi_{22}' - \varphi_{22}'''\right)(\alpha R)^{-1} \end{aligned}$$

$$T_3 = \Phi_{12}$$

$$T_4 = \Phi'_{12}$$

$$T_5 = \frac{1}{m}\left(\alpha^2\Phi_{12} + \Phi''_{12}\right)$$

$$T_6 = r\left\{\left[m + \frac{1}{F(1-c)}\right]\Phi_{12} - (1-c)\,\Phi'_{12}\right\} + \frac{i}{m}\left(3\alpha^2\Phi'_{12} - \Phi'''_{12}\right)(\alpha R)^{-1}$$

$$a_1 = (\alpha R)^{1/2}\varphi_{31}/'\varphi''_{32}$$

$$a_2 = (\alpha R)^{1/2}\varphi'_{31}/'\varphi''_{32}$$

$$a_5 = (\alpha R)^{1/2}\varphi''_{32}/'\varphi''_{32}$$

$$a_6 = -\frac{i}{(\alpha R)^{1/2}}\varphi'''_{32}/'\varphi''_{32}$$

$$b_1 = (\alpha R)^{1/2}\varphi_{41}/'\varphi''_{42} \qquad \text{(5.15) (contd.)}$$

$$b_2 = (\alpha R)^{1/2}\varphi'_{41}/'\varphi''_{42}$$

$$b_5 = (\alpha R)^{1/2}\varphi''_{42}/'\varphi''_{42}$$

$$b_6 = -\frac{i}{(\alpha R)^{1/2}}\varphi'''_{42}/'\varphi''_{42}$$

$$c_3 = (\alpha R r m^2)^{1/2}\,\Phi_{32}/'\Phi''_{32}$$

$$c_4 = (\alpha R r m^2)^{1/2}\,\Phi'_{32}/'\Phi''_{32}$$

$$c_5 = \frac{1}{m}\left[\alpha^2(\alpha R r m^2)^{1/2}\Phi_{32}/'\Phi''_{32} + (\alpha R r m^2)^{1/2}\,\Phi''_{32}/'\Phi''_{32}\right]$$

$$c_6 = \frac{1}{m}\left\{\left[m + \frac{1}{F(1-c)}\right] m r\,(\alpha R r m^2)^{1/2}\Phi_{32}/'\Phi''_{32} - \left[(1-c)mr - i\frac{3\alpha^2}{\alpha R}\right](\alpha R r m^2)^{1/2}\Phi'_{32}/'\Phi''_{32}\right.$$
$$\left. - \frac{i}{\alpha R}(\alpha R r m^2)^{1/2}\,\Phi'''_{32}/'\Phi''_{32}\right\}$$

It is worth remarking that the lower limit for the integrals of Eqs. (5.4) and (5.5) could have been taken, as an alternative, to be zero instead of unity. This would have lead to a fourth order determinant instead of the sixth order of Eq. (5.14). Each element of the alternate determinant would have been not only more complicated than the one of Eq. (5.14), but the clear separation between viscid and inviscid solutions (of which use is made in Appendix C for the calculations of the neutral stability lines) wouldn't have been possible.

6. Behavior of the Viscous Solutions for $\alpha R \gg 1$.

In order to bring out the behavior of the neutral stability curve (i.e. for given m, ν, F and W) for large values of the parameter αR, it will be sufficient to keep in the asymptotic expansion of the pertinent functions, the terms of highest order in αR. As will be shown later, (p. 29) the case of interest for neutral stability (cf. Eq. (1.3)), is the one where $c = c_r < 1$. The detailed description of the method used for obtaining the integrals, the order of magnitude of the errors involved, and the calculation of φ_{31} as an example of the procedure used, are given in Appendix B. The other functions are obtained in a similar way, and are just quoted here.

$$\left.\begin{aligned} \varphi_{31} &= \frac{2(12)^{1/6} e^{-i\frac{5}{12}\pi}}{\pi^{1/2} (\alpha R)^{2/3}} \left\{ z_{l_0}^{-5/4} e^{i\frac{2}{3} z_{l_0}^{3/2}} + O\left[(\alpha R)^{1/3}\right] \right\} \\ \varphi_{31}' &= -\frac{2(12)^{1/6} e^{-i\frac{5}{12}\pi}}{\pi^{1/2} (\alpha R)^{1/3}} \left\{ z_{l_0}^{-3/4} e^{i\frac{2}{3} z_{l_0}^{3/2}} + O(1) \right\} \end{aligned}\right\} \quad (6.1)$$

$$\left.\begin{aligned}
\varphi''_{32} &= \frac{2(12)^{1/6}\, e^{-i\frac{5}{12}\pi}}{\pi^{1/2}}\, z_{\ell_1}^{-1/4}\, e^{i\frac{2}{3} z_{\ell_1}^{3/2}} \left[1 - i\,\frac{5}{48}\, z_{\ell_1}^{-3/2} + O\!\left(z_{\ell_1}^{-3}\right)\right] \\
\varphi'''_{32} &= -\,\frac{2\,(12)^{1/6}\, e^{-i\frac{5}{12}\pi}}{\pi^{1/2}}\, (\alpha R)^{1/3}\, e^{i\frac{2}{3} z_{\ell_1}^{3/2}} \left[z_{\ell_1}^{1/4} + i\,\frac{7}{48}\, z_{\ell_1}^{-5/4} + O\!\left(z_{\ell_1}^{-\frac{11}{4}}\right)\right]
\end{aligned}\right\} \quad \begin{array}{l}(6.1)\\ \text{(contd.)}\end{array}$$

$$\left.\begin{aligned}
\varphi_{41} &= \frac{(12)^{1/6}\, e^{i\frac{5}{12}\pi}}{\alpha\,\pi^{1/2}\,(\alpha R)^{1/3}} \left\{\left(e^{\alpha} - e^{-\alpha}\right) z_{\ell_1}^{-3/4}\, e^{-i\frac{2}{3} z_{\ell_1}^{3/2}} - \left(e^{\alpha} + e^{-\alpha}\right) \frac{\alpha}{(\alpha R)^{1/3}}\, z_{\ell_1}^{-5/4}\, e^{-i\frac{2}{3} z_{\ell_1}^{3/2}} + O(1)\right\} \\
\varphi'_{41} &= -\,\frac{(12)^{1/6}\, e^{i\frac{5}{12}\pi}}{\pi^{1/2}\,(\alpha R)^{1/3}} \left\{\left(e^{\alpha} + e^{-\alpha}\right) z_{\ell_1}^{-3/4}\, e^{-i\frac{2}{3} z_{\ell_1}^{3/2}} - \left(e^{\alpha} - e^{-\alpha}\right) \frac{\alpha}{(\alpha R)^{1/3}}\, z_{\ell_1}^{-5/4}\, e^{-i\frac{2}{3} z_{\ell_1}^{3/2}} + O(1)\right\} \\
\varphi''_{42} &= \frac{2\,(12)^{1/6}\, e^{i\frac{5}{12}\pi}}{\pi^{1/2}}\, z_{\ell_1}^{-1/4}\, e^{-i\frac{2}{3} z_{\ell_1}^{3/2}} \left[1 - i\,\frac{5}{48}\, z_{\ell_1}^{-3/2} + O\!\left(z_{\ell_1}^{-3}\right)\right] \\
\varphi'''_{42} &= \frac{2(12)^{1/6}\, e^{i\frac{5}{12}\pi}}{\pi^{1/2}}\, (\alpha R)^{1/3}\, e^{-i\frac{2}{3} z_{\ell_1}^{3/2}} \left[z_{\ell_1}^{1/4} - i\,\frac{17}{48}\, z_{\ell_1}^{-5/4} - \left(\frac{35}{192} + \frac{385}{4608}\right) z_{\ell_1}^{-\frac{11}{4}} + O\!\left(z_{\ell_1}^{-\frac{17}{4}}\right)\right]
\end{aligned}\right\} \quad (6.2)$$

$$\left.\begin{aligned}
\Phi_{32} &= \frac{2(12)^{1/6}\, e^{-i\frac{5}{12}\pi}}{\pi^{1/2}\,(\alpha R r m^2)^{2/3}} \left\{z_{g_1}^{-5/4}\, e^{i\frac{2}{3} z_{g_1}^{3/2}} - \frac{\alpha}{(\alpha R r m^2)^{1/3}}\, z_{g_1}^{-7/4}\, e^{i\frac{2}{3} z_{g_1}^{3/2}} + O\left[\frac{z_{g_1}^{-9/4}\, e^{i\frac{2}{3} z_{g_1}^{3/2}}}{(\alpha R)^{2/3}}\right]\right\} \\
\Phi'_{32} &= -\,\frac{2(12)^{1/6}\, e^{-i\frac{5}{12}\pi}}{\pi^{1/2}\,(\alpha R r m^2)^{1/3}}\, z_{g_1}^{-3/4}\, e^{i\frac{2}{3} z_{g_1}^{3/2}} \left[1 - i\,\frac{41}{48}\, z_{g_1}^{-3/2} + O\!\left(\frac{z_{g_1}^{-5/2}}{(\alpha R)^{2/3}}\right)\right] \\
\Phi''_{32} &= \frac{2\,(12)^{1/6}\, e^{-i\frac{5}{12}\pi}}{\pi^{1/2}}\, z_{g_1}^{-1/4}\, e^{i\frac{2}{3} z_{g_1}^{3/2}} \left\{1 - i\,\frac{5}{4}\, z_{g_1}^{-3/2} - \frac{385}{4608}\, z_{g_1}^{-3} + \frac{\alpha^2}{(\alpha R r m^2)^{2/3}}\, z_{g_1}^{-1} \right. \\
&\qquad \left. - \frac{\alpha^3}{\alpha R r m^2}\, z_{g_1}^{-3/2} + O\left[\frac{z_{g_1}^{-2}}{(\alpha R)^{4/3}}\right]\right\}
\end{aligned}\right\} \quad (6.3)$$

$$\Phi_{32}''' = -\frac{2(12)^{1/6} e^{-i\frac{5}{12}\pi}}{\pi^{1/2}} e^{i\frac{2}{3}z_{g_1}^{3/2}} \left\{ (\alpha R r m^2)^{1/3} \left[z_{g_1}^{1/4} + i\frac{7}{48} z_{g_1}^{-5/4} + O\left(z_{g_1}^{-\frac{17}{4}}\right) \right] \right.$$

$$\left. + \frac{\alpha^2}{(\alpha R r m^2)^{1/3}} z_{g_1}^{-3/4} \left[1 - i\frac{41}{48} z_{g_1}^{-3/2} + O\left(\frac{z_{g_1}^{-5/2}}{(\alpha R)^{2/3}} \right) \right] \right\} \qquad \begin{matrix}(6.3)\\ \text{(contd.)}\end{matrix}$$

where it can be assumed that, (cf. statement on p. 58)

$$\left. \begin{aligned} z_{\ell_0} &\sim -i(\alpha R)^{1/3} c \\ z_{\ell_1} &\sim i(\alpha R)^{1/3}(1-c) \\ z_{g_1} &\sim i(\alpha R r m^2)^{1/3} \frac{(1-c)}{m} \end{aligned} \right\} \qquad (6.4)$$

Multiplying Eqs. (6.1), (6.2) and (6.3) by* $\frac{(\alpha R)^{1/2}}{{}^{1}\varphi_{32}''}$, $\frac{(\alpha R)^{1/2}}{{}^{1}\varphi_{42}''}$ and $\frac{\alpha R r m^2}{{}^{1}\Phi_{32}''}$ respectively, leads to

$$\left. \begin{aligned} (\alpha R)^{1/2} \frac{\varphi_{31}}{{}^{1}\varphi_{32}''} &= (\alpha R)^{-1/6} \left\{ z_{\ell_0}^{-5/4} z_{\ell_1}^{1/4} e^{i\frac{2}{3}\left(z_{\ell_0}^{3/2} - z_{\ell_1}^{3/2}\right)} + O\left[(\alpha R)^{1/3} z_{\ell_1}^{1/4} e^{-i\frac{2}{3} z_{\ell_1}^{3/2}} \right] \right\} \\ (\alpha R)^{1/2} \frac{\varphi_{31}'}{{}^{1}\varphi_{32}''} &= -(\alpha R)^{1/6} \left\{ z_{\ell_0}^{-3/4} z_{\ell_1}^{1/4} e^{i\frac{2}{3}\left(z_{\ell_0}^{3/2} - z_{\ell_1}^{3/2}\right)} + O\left[z_{\ell_1}^{1/4} e^{-i\frac{2}{3} z_{\ell_1}^{3/2}} \right] \right\} \\ (\alpha R)^{1/2} \frac{\varphi_{32}''}{{}^{1}\varphi_{32}''} &= (\alpha R)^{1/2} \left[1 - i\frac{5}{48} z_{\ell_1}^{-3/2} + O\left(z_{\ell_1}^{-3}\right) \right] \\ (\alpha R)^{1/2} \frac{\varphi_{32}'''}{{}^{1}\varphi_{32}''} &= -(\alpha R)^{5/6} \left[z_{\ell_1}^{1/2} + i\frac{7}{48} z_{\ell_1}^{-1} + O\left(z_{\ell_1}^{-5/2}\right) \right] \end{aligned} \right\} \qquad (6.5)$$

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* Where ${}^{1}\varphi_{32}''$ is the leading term of φ_{32}'' , i.e. ${}^{1}\varphi_{32}'' = \frac{2(12)^{1/6} e^{-i\frac{5}{12}\pi}}{\pi^{1/2}} z_{\ell_0}^{-3/4} e^{i\frac{2}{3} z_{\ell_1}^{3/2}}$; similarly for other functions used.

$$\left.\begin{aligned}
(\alpha R)^{1/2}\frac{\varphi_{41}}{{}^{1}\varphi''_{42}} &= \frac{(\alpha R)^{1/6}}{\alpha}\left\{ z_{\ell_1}^{-1/2}\sinh\alpha - \frac{\alpha z_{\ell_1}^{-1}}{(\alpha R)^{1/3}}\cosh\alpha + O\left(z_{\ell_1}^{1/4} e^{i\frac{2}{3} z_{\ell_1}^{3/2}}\right)\right\} \\
(\alpha R)^{1/2}\frac{\varphi'_{41}}{{}^{1}\varphi''_{42}} &= -(\alpha R)^{1/6}\left\{ z_{\ell_1}^{-1/2}\cosh\alpha - \frac{\alpha z_{\ell_1}^{-1}}{(\alpha R)^{1/3}}\sinh\alpha + O\left(z_{\ell_1}^{1/4} e^{i\frac{2}{3} z_{\ell_1}^{3/2}}\right)\right\} \\
(\alpha R)^{1/2}\frac{\varphi''_{42}}{{}^{1}\varphi''_{42}} &= (\alpha R)^{1/2}\left[1 - i\frac{5}{48} z_{\ell_1}^{-3/2} + O\left(z_{\ell_1}^{-3}\right)\right] \\
(\alpha R)^{1/2}\frac{\varphi'''_{42}}{{}^{1}\varphi''_{42}} &= (\alpha R)^{5/6}\left[z_{\ell_1}^{1/2} - i\frac{17}{48} z_{\ell_1}^{-1} - \left(\frac{35}{192}+\frac{385}{4608}\right) z_{\ell_1}^{-5/2} + O\left(z_{\ell_1}^{-4}\right)\right]
\end{aligned}\right\} \qquad (6.6)$$

$$\left.\begin{aligned}
(\alpha R r m^2)^{1/2}\frac{\Phi_{32}}{{}^{1}\Phi''_{32}} &= (\alpha R r m^2)^{-1/6}\left\{ z_{g_1}^{-1} - \frac{\alpha}{(\alpha R r m^2)^{1/3}} z_{g_1}^{-3/2} + O\left[\frac{z_{g_1}^{-2}}{(\alpha R)^{2/3}}\right]\right\} \\
(\alpha R r m^2)^{1/2}\frac{\Phi'_{32}}{{}^{1}\Phi''_{32}} &= -(\alpha R r m^2)^{1/6} z_{g_1}^{-1/2}\left[1 - i\frac{41}{48} z_{g_1}^{-3/2} + O\left(z_{g_1}^{-5/2}\right)\right] \\
(\alpha R r m^2)^{1/2}\frac{\Phi''_{32}}{{}^{1}\Phi''_{32}} &= (\alpha R r m^2)^{1/2}\left\{1 - i\frac{5}{4} z_{g_1}^{-3/4} - \frac{385}{4608} z_{g_1}^{-3} + \frac{\alpha^2}{(\alpha R r m^2)^{2/3}} z_{g_1}^{-1}\right. \\
&\qquad\left. - \frac{\alpha^3}{(\alpha R r m^2)} z_{g_1}^{-3/2} + O\left[\frac{z_{g_1}^{-2}}{(\alpha R)^{4/3}}\right]\right\} \\
(\alpha R r m^2)^{1/2}\frac{\Phi'''_{32}}{{}^{1}\Phi''_{32}} &= -\left\{(\alpha R r m^2)^{5/6}\left[z_{g_1}^{1/2} + i\frac{7}{48} z_{g_1}^{-1} - \frac{385}{4608} z_{g_1}^{-5/2} + O\left(z_{g_1}^{-4}\right)\right]\right. \\
&\qquad\left. + \alpha^2(\alpha R r m^2)^{1/6} z_{g_1}^{-1/2}\left[1 - i\frac{41}{48} z_{g_1}^{-3/2} + O\left(\frac{z_{g_1}^{-5/2}}{(\alpha R)^{2/3}}\right)\right]\right\}
\end{aligned}\right\} \qquad (6.7)$$

Let

$$\left.\begin{aligned}
e^{+} &= \exp\left[\frac{2}{3}(\alpha R)^{1/2} e^{i\frac{5}{4}\pi}(1-c)^{3/2}\right] \\
e^{-} &= \exp\left[-\frac{2}{3}(\alpha R)^{1/2} e^{i\frac{5}{4}\pi}(1-c)^{3/2}\right] \\
E &= \exp\left\{i\frac{2}{3}(\alpha R)^{1/2}\left[e^{-i\frac{3}{4}\pi} c^{3/2} - e^{i\frac{3}{4}\pi}(1-c)^{3/2}\right]\right\}
\end{aligned}\right\} \qquad (6.8)$$

then, the terms of interest in Eq. (5.15) become, after use is made of Eqs. (6.5), (6.6) and (6.7)

$$\left.\begin{aligned}
a_1 &= a_{11}(\alpha R)^{-1/2}E + O\left[(\alpha R)^{1/4}e^-\right] \\
a_2 &= a_{21}E + O\left[(\alpha R)^{1/4}e^-\right] \\
a_5 &= a_{51}(\alpha R)^{1/2} + a_{52} + O(\alpha R)^{-1/2} \\
a_6 &= a_{61} + a_{62}(\alpha R)^{-1/2} + O(\alpha R)^{-1}
\end{aligned}\right\} \quad (6.9)$$

$$\left.\begin{aligned}
b_1 &= b_{11} + b_{12}(\alpha R)^{-1/2} + O\left[(\alpha R)^{1/4}e^+\right] \\
b_2 &= b_{21} + b_{22}(\alpha R)^{-1/2} + O\left[(\alpha R)^{1/4}e^+\right] \\
b_5 &= b_{51}(\alpha R)^{1/2} + b_{52} + O\left[(\alpha R)^{-1/2}\right] \\
b_6 &= b_{61} + b_{62}(\alpha R)^{-1/2} + b_{63}(\alpha R)^{-1} + O\left[(\alpha R)^{-3/2}\right]
\end{aligned}\right\} \quad (6.10)$$

$$\left.\begin{aligned}
c_3 &= c_{31}(\alpha R)^{-1/2} + c_{32}(\alpha R)^{-1} + O\left[(\alpha R)^{-3/2}\right] \\
c_4 &= c_{41} + c_{42}(\alpha R)^{-1/2} + O\left[(\alpha R)^{-1}\right] \\
c_5 &= c_{51}(\alpha R)^{1/2} + c_{52} + c_{53}(\alpha R)^{-1/2} + O\left[(\alpha R)^{-1}\right] \\
c_6 &= c_{61}(\alpha R)^{-1/2} + c_{62}(\alpha R)^{-1} + O\left[(\alpha R)^{-3/2}\right]
\end{aligned}\right\} \quad (6.11)$$

where

$$\left.\begin{aligned}
a_{11} &= e^{i\frac{3}{4}\pi}c^{-5/4}(1-c)^{1/4} & \qquad a_{52} &= -\frac{5}{48}e^{-i\frac{\pi}{4}}(1-c)^{-3/2} \\
a_{21} &= e^{-i\frac{\pi}{2}}c^{-3/4}(1-c)^{1/4} & \qquad a_{61} &= -e^{-i\frac{\pi}{4}}(1-c)^{1/2} \\
a_{51} &= 1 & \qquad a_{62} &= \frac{7}{48}e^{i\frac{\pi}{2}}(1-c)^{-1}
\end{aligned}\right\} \quad (6.12)$$

$$\left.\begin{aligned}
b_{11} &= e^{-i\frac{\pi}{4}} \frac{\sinh\alpha}{\alpha(1-c)^{1/2}} & b_{51} &= 1 \\
b_{12} &= e^{i\frac{\pi}{2}} \frac{\cosh\alpha}{1-c} & b_{52} &= -e^{-i\frac{\pi}{4}} \frac{5}{48} (1-c)^{-3/2} \\
b_{21} &= -e^{-i\frac{\pi}{4}} \frac{\cosh\alpha}{(1-c)^{1/2}} & b_{61} &= e^{-i\frac{\pi}{4}} (1-c)^{1/2} \\
b_{22} &= -e^{i\frac{\pi}{2}} \frac{\alpha\sinh\alpha}{1-c} & b_{62} &= \frac{17}{48} e^{i\frac{\pi}{2}} (1-c)^{-1} \\
 & & b_{63} &= -e^{i\frac{\pi}{4}} \left(\frac{35}{192} + \frac{385}{4608}\right)(1-c)^{-5/2}
\end{aligned}\right\} \tag{6.13}$$

$$\left.\begin{aligned}
c_{31} &= -e^{i\frac{\pi}{2}} (r m^2)^{-1/2} \frac{m}{1-c} & c_{51} &= r^{1/2} \\
c_{32} &= e^{i\frac{\pi}{4}} \alpha (r m^2)^{-1} \left(\frac{m}{1-c}\right)^{3/2} & c_{52} &= -\frac{5}{4} e^{-i\frac{\pi}{4}} \left(\frac{m}{1-c}\right)^{3/2} \\
c_{41} &= -e^{-i\frac{\pi}{4}} \left(\frac{m}{1-c}\right)^{1/2} & c_{53} &= -2 e^{i\frac{\pi}{2}} \alpha^2 (r m^2)^{-1/2} (1-c)^{-1} \\
c_{42} &= -\frac{41}{48} e^{i\frac{\pi}{2}} (r m^2)^{-1/2} \left(\frac{m}{1-c}\right)^2 & c_{61} &= -e^{i\frac{\pi}{2}} \frac{(r m^2)^{1/2}}{F(1-c)^2 m} \\
c_{62} &= e^{i\frac{\pi}{4}} \frac{1}{m} \left(\frac{m}{1-c}\right)^{1/2} \left\{ \left[m + \frac{1}{F(1-c)}\right] \frac{\alpha}{1-c} - 2\alpha^2 + \frac{385}{4608}\left(\frac{m}{1-c}\right)^2 \right\}
\end{aligned}\right\} \tag{6.14}$$

The secular equation (5.14) is expanded in Appendix C in terms of a sum of products of the viscid and inviscid terms. Eqs. (6.8) to (6.14) are then used for determining the important terms to be kept in the final result.

7. The Eigenvalue Problem for $\alpha R \to \infty$

It is shown in Appendix C, Eqs. (C.8) and (C.15), that for the case of $\alpha R \to \infty$ the secular equation reduces to

$$\mathfrak{F}(\alpha, c, r, m, F, W) = 0 \tag{7.1}$$

This equation turns out to be very simple and can be solved explicitly for c . The result is

$$c = 1 - \frac{\left(\frac{\alpha^2}{W}+1-rm\right)+\left\{\left(\frac{\alpha^2}{W}+1-rm\right)^2+4\alpha\left[g(\alpha,r,m)+r\right]\frac{1-r}{F}\right\}^{1/2}}{2\alpha\left[g(\alpha,r,m)+r\right]} \tag{7.2}$$

where

$$g(\alpha,r,m)=\left[\frac{e^{\alpha}}{\sinh\alpha}+\frac{(m/r)^{1/2}}{\tanh\alpha}-1\right]\frac{1}{1+\left(\frac{m}{r}\right)^{1/2}} \tag{7.3}$$

the radicand being always positive, i.e. c is always real, which means that for $R \to \infty$ the flow is neutrally stable. Also, since for cases of interest, $rm < 1$, c is always less than unity as $R \to \infty$. This is quite an important conclusion, since for finite Reynolds numbers, the computation is different, depending on whether c is less or more than unity. It will therefore be assumed in all later calculations for finite Reynolds number, that $c < 1$. The final calculations will bear out this assumption.

A different approach to the case of infinite Reynolds number, would be to neglect viscosity at the outset, i.e., in the Orr-Sommerfeld equations. The differential equations become then of second order and by relaxing the proper boundary conditions, the problem could be solved again. This has been done, (cf. Appendix C) and there is a discrepancy in the equation for c, that from a physical point of view, is a puzzling question that has remained unexplained. From a practical point of view, the discrepancy is immaterial since for fluids of interest, i. e., liquid-gas combination, $\frac{m}{r} \gg 1$, both results agree.

8. The Case of Finite Reynolds Number

The fundamental equations for the neutral stability curve in the αR- plane, for a given physical situation, (i.e., for fixed gas-liquid density ratio: r , liquid-gas viscosity ratio: m , Froude number: F , and Weber number: W) are*

$$c = c(\alpha) \tag{8.1}$$

$$(\alpha R)^{1/2} = -\frac{\mathcal{H}(\alpha, c)}{\mathcal{F}(\alpha, c)} \tag{8.2}$$

each point on the curve, having a particular wave velocity c .

The solution of Eq. (8.1) has to be obtained numerically** because of the impossibility of solving explicitly for c or α in terms, respectively, of α or c . Once the pair of values α and c is known, straightforward calculation leads to the value of $(\alpha R)^{1/2}$, it being possible to evaluate R immediately.

It is worth while to digress for a minute in order to point out the meaning of Reynolds number, $R \equiv R_\ell$, in the present problem. Since the velocity profile is a linear function of distance, for given liquid physical properties, R is proportional to the liquid flow rate, and therefore is a constant once the liquid flow rate is known.

Now that a method for determining the neutral stability curve has been described, it is necessary to find a way of deciding which region

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* Cf. Eqs. (C.13) and (C.14), Appendix C.

** All the numerical work was carried out on an IBM Card Programmed Electronic Calculator using an 8-digit floating decimal system.

on either side of it, is stable and unstable. The calculations (for the analysis refer to Appendix D) indicate, as is reasonable to expect, that for a disturbance of a given wave length, the flow is stable for Reynolds numbers smaller than a neutral Reynolds number R_n, and similarly, the flow is unstable for $R > R_n$.

The results obtained from the present analysis, of which Fig. 3 is a typical example, indicate that the shape of the neutral stability curves in the αR -plane is similar to the boundary layer case. The effect of varying the gas-liquid density ratio could be stabilizing or distabilizing, while increasing the liquid-gas viscosity ratio always stabilizes the flow, provided m is not very small compared with unity.

The general shape of the neutral stability curve is such that for $\alpha \rightarrow 0$, the equation for the curve is

$$\alpha R = \text{constant}, \tag{8.3}$$

this being the lower branch of the curve. Eq. (8.3) can be derived analytically from Eqs. (C.13) and (C.14), Appendix C. The upper branch seems to be nearly a horizontal tangent for $R \rightarrow \infty$; this was obtained numerically.

The influence of gravity* or surface tension forces on the above results should be alike, since they enter as a sum, in the secular relation Eq. (4.9). Fig. 4 shows, that for large Froude and Weber numbers, their effect is to destabilize the flow. Obviously, surface tension effects are negligible when the disturbance frequency is small,

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* The gravitational field was assumed to act downwards in the flow configuration given in Fig. 1.

but become very important at large frequencies. The effect on the neutral stability line is in this case to raise the upper branch so that its asymptote for $\alpha \gg 1$ is

$$\frac{\alpha}{R} = \text{constant} \tag{8.4}$$

which, can also be obtained analytically as Eq. (8.3).

All the effects discussed thus far, involved the stability of the flow, i.e. they indicate the direction (left or right) in which the neutral stability curve moves in the αR -plane, when the flow physical quantities are varied.

There are two more items that should be discussed before the important results of the large number of calculations made, are summarized, and they are

a) the behavior of the magnitude of the wave velocity.

b) the amplification or damping rate of disturbances in the neighborhood of the neutral stability line.

In answer to a), it will just be mentioned that all waves travel at speeds less than the velocity of the liquid-gas interface, a typical case being given in Fig. 5.

On the neutral stability curve, the imaginary part, c_i of the complex velocity c, vanishes. It is possible nevertheless, to compute (cf. Appendix D) the rate of change of c_i with respect to Reynolds number. The important result is that this is always a positive quantity, which as a function of α (Fig. 6), has a peak near the critical Reynolds number*. This means that the curves c_i= constant, in the αR- plane, would be packed close together when in the neighborhood of the critical value of α and R .

Since the important quantities are really the critical quantities, the summary of a large number of calculations will now be presented.

9. Values of Critical Quantities and Discussion of Results. Comparison with Experiments.

Since the physical case of interest is the one where gravity and surface tension forces are small, detailed calculations were carried out for the case of $F = W = \infty$. The critical values are presented in Figs. 7 through 10 from which, the following facts can be gathered:

(1) As m and $r \to 1$, the flow is completely stable.

(2) For a given gas-liquid density ratio (r), an increase in the liquid-gas viscosity ratio (m) always increases the stability, and decreases the amplification or damping in the region away from the neutral stability curve.

(3) For very small density ratio (r) the flow is completely stable.

(4) For a given viscosity ratio (m) there is always a density ratio (r) for which the flow is least stable.

(5) As $r \to 0$ the wave number (α) tends, approximately, to a value of 0.6, and the wave velocity (c) becomes, approximately, 0.1.

(6) For the special case of air and water at a temperature of 100° C and a pressure of one atmosphere ($m \cong 10$, $r \cong .001$), the value of the critical Reynolds number is 60,000.

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* cf. footnote on p. 3.

The significance of the results quoted in item 1 and 2 can be understood in terms of the known universal stability of plane Couette flow* between walls of arbitrary spacing. An explanation follows.

In the special case when both fluids have the same density and viscosity, (i.e., the case of a single fluid) the velocity profile becomes a single straight line, which coincides with the case of P.C.F. The result obtained (item 1) shows this flow to be universally stable, in agreement with the original result from P.C.F.; a check on the analysis is thus obtained.

When the result quoted in item 2 is interpreted for the limiting case of a very viscous liquid, the motion is again always stable. The very viscous liquid could just as well be considered as a solid and this case again reduces to Couette flow.

From item 2 and the above discussion there is a result that can be deduced for which calculations have not been made, i.e., for any arbitrary density ratio, the flow is completely stabilized when $m \to 0$. The reason for stabilization, is that this limiting case of the flow occurs when the gas becomes so viscous that it could be replaced by a solid wall, which again reduces to P.C.F. between two walls at a finite spacing. Therefore, in Fig. 7 there would exist a curve for some small value of m, that would be farthest to the left, and for smaller values of m the curves would again be displaced more and more to the right as the liquid-gas viscosity ratio is decreased.

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* From now on abbreviated as P.C.F.

Items 5 and 6 will now be compared with experimentally made observations. All experiments with liquid films have been carried out in horizontal round tubes where the liquid flows along the inner surface dragged by a high speed turbulent gas. It should be kept in mind that while the gas layer is of infinite thickness in the theoretical model, the ratio of laminar sublayer thickness in the gas to liquid film thickness in the experiments was of the order of unity.

The experimental neutral wave length (λ) observed by Knuth[1] for all liquid flow rates was about 10 film thicknesses. Considering that $\alpha = \frac{2\pi}{\lambda}$, Knuth's findings check with the critical value of item 5. Since the wave lengths in the neighborhood of the critical value are the most amplified, these might be the only ones visible in an experiment. Knuth might have observed these values, which, although seemingly neutral could have been slightly amplified.

Item 6 implies that for a liquid film .005 in. thick the critical liquid-gas interface velocity is 100 ft/sec, which is one order of magnitude larger than the values obtained from experiments in liquid films. This discrepancy would seem to indicate that the observed instability is not simply laminar instability of uniform shearing motion. The fact that the computed critical Reynolds number is too high as compared to experimental values could be due to the fact that the velocity profiles of both fluids in the analysis were assumed to be straight lines. If curved profiles were used it is conceivable that the critical Reynolds number could decrease. This possibility follows from a comparison of the change in the value of critical Reynolds number when going from plane Couette ($R_{crit} = \infty$) to plane Poiseuille flow ($R_{crit} = 11{,}560$, based on the maximum velocity and width of the channel). Curvature in the velocity profile of the liquid could exist in the case of laminar flow when there is a pressure gradient in the flow direction. There is one thing that has not been accounted for in the present or suggested analysis, and that is the effect of turbulence of the gas stream, which also possibly influences the stability of the flow.

Finite gravity and surface tension have a destabilizing effect (cf. Figs. 11 and 12), as expected from the discussion in Section 8. This influence of surface tension is reported in Ref. 6, p. 10 and is of the same order of magnitude (within a factor of at most 2) than the value found analytically.

Before concluding, it will be helpful to try to gain some physical insight into the stability problem by looking at the energy of the disturbed motion. This will be done in the next section.

III. THE REYNOLDS SHEARING STRESS

A different[18] way of looking at the stability problem, consists in following the time history of the disturbance energy, which, for damping or amplification, changes as a result of the action of the Reynolds shearing stress τ_s *. It will then be enlightning to know what its distribution is across the stream.

Foote and Lin[20] have shown that

$$\tau_s = -\rho\overline{uv} = -\rho\frac{\alpha}{4i}e^{2\alpha c_i t}(\varphi\bar{\varphi}' - \bar{\varphi}\varphi') \qquad (10.1a)$$

and

$$\frac{d\tau_s}{dy} = -\rho\frac{\alpha}{4i}e^{2\alpha c_i t}\frac{d}{dy}(\varphi\bar{\varphi}' - \bar{\varphi}\varphi') \qquad (10.1b)$$

where $\overline{uv}$ is the distance or time average of the product uv and $\bar{\varphi}$ and $\bar{\varphi}'$ indicate complex conjugates of φ and φ' .

At the interface, for the viscous case

$$\overline{u_l v_l} = \overline{u_g v_g} \qquad (10.2)$$

$$\left(\tau_{sl}/\tau_{sg}\right)_{interface} = \frac{\rho_l}{\rho_g} = \frac{1}{r} \qquad (10.3)$$

<u>The Inviscid Case.</u>

The amplitude functions φ and Φ for the inviscid case (cf.

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* The work done per unit volume per unit time by the basic flow[19], is $\tau_s\frac{dU}{dy}$, and converts energy from the basic flow into the disturbance when $\frac{dU}{dy} > 0$.

Eqs. (C.16), Appendix C) are real functions. Therefore Eq. (10.1a) shows that

$$(\varphi\bar{\varphi}' - \bar{\varphi}\varphi') \quad = \text{constant} = 0 \qquad (10.4)$$

from which, the Reynolds stresses are zero across the flow. This means that there is no mechanism for transferring energy between the basic flow and the disturbance, i.e., any disturbance will just subsist, without damping or amplification.

As can be seen, the present method of approach is extremely useful, since the important result of Section 7, regarding stability, has just been rederived in a few lines without any calculation.

The Viscid Case.

We will start the study of the viscous case by showing that the Reynolds stress is continuous across the layer where $U(y) = c_r$. The Orr-Sommerfeld Eq. (1.4) can be rewritten as (c is complex)

$$\varphi'' - \alpha^2\varphi = \frac{U''}{U-c}\varphi - \frac{i}{\alpha R(U-c)}\left(\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi\right) \qquad (10.5)$$

which on multiplying by $\bar{\varphi}$, subtracting its complex conjugate and regrouping yields

$$\frac{d}{dy}(\bar{\varphi}\varphi' - \bar{\varphi}'\varphi) = \frac{2c_i U''|\varphi|^2}{|U-c|^2} - \frac{i}{\alpha R}\left(\frac{\varphi^{IV}\bar{\varphi} - 2\alpha^2\varphi''\bar{\varphi} + \alpha^4\varphi\bar{\varphi}}{U-c} + \frac{\bar{\varphi}^{IV}\varphi - 2\alpha^2\bar{\varphi}''\varphi + \alpha^4\varphi\bar{\varphi}}{U-\bar{c}}\right) \qquad (10.6)$$

Eq. (10.6) can now be introduced into Eq. (10.1b) and since $U'' = 0$, we have

$$\frac{d\tau_s}{dy} = \frac{\rho}{4R} e^{2\alpha c_i t} \left(\frac{\varphi^{iv}\bar{\varphi} - 2\alpha^2 \varphi''\bar{\varphi} + \alpha^4 \varphi\bar{\varphi}}{U-c} + \frac{\bar{\varphi}^{iv}\varphi - 2\alpha^2 \bar{\varphi}''\varphi + \alpha^4 \varphi\bar{\varphi}}{U-\bar{c}} \right) \tag{10.7}$$

which can be integrated across the layer where $U=c$, and remembering that in our case $U=y$

$$\int d\tau_s = \tau_s(y=c+0) - \tau_s(y=c-0) = [\tau_s]$$

$$= \frac{\rho}{4R} e^{2\alpha c_i t} \lim_{\epsilon \to 0} \Big\{ \left[\varphi^{iv}\bar{\varphi} - 2\alpha^2\varphi''\bar{\varphi} + \alpha^4\varphi\bar{\varphi}\right]_{y=c} \left[\ln\epsilon - \ln(-\epsilon)\right]$$

$$+ \left[\bar{\varphi}^{iv}\varphi - 2\alpha^2\bar{\varphi}''\varphi + \alpha^4\varphi\bar{\varphi}\right]_{y=c} \left[\ln(c-\bar{c}+\epsilon) - \ln(c-\bar{c}-\epsilon)\right] \Big\} \tag{10.8}$$

Since

$$\ln\epsilon - \ln(-\epsilon) = \ln\epsilon - \ln\epsilon \pm i\pi(2n+1) \qquad n=0,1,2,\ldots$$

and letting

$$c_i = 0$$

Eq. (10.8) gives for the jump in Reynolds stress $[\tau_s]$ across the layer where $U=c$

$$[\tau_s] = \frac{\rho}{4R} \underbrace{\left[\varphi^{iv}\bar{\varphi} + \bar{\varphi}^{iv}\varphi - 2\alpha^2(\varphi''\bar{\varphi} + \bar{\varphi}''\varphi) + 2\alpha^2\varphi\bar{\varphi}\right]}_{\text{pure real}} \underbrace{\left[\pm i\pi(2n+1)\right]}_{\text{pure imaginary}} \qquad n=0,1,2,\ldots \tag{10.9}$$

where a particular n should be chosen for the branch of the logarithm being used. Since $[\tau_s]$ must be real, and $\frac{\rho}{4R} \neq 0$, we have at $y=c$

$$\varphi^{iv}\bar{\varphi} + \bar{\varphi}^{iv}\varphi - 2\alpha^2(\varphi''\bar{\varphi} + \bar{\varphi}''\varphi) + 2\alpha^4\varphi\bar{\varphi} = 0 \tag{10.10}$$

Therefore

$$[\tau_s] = 0 \qquad \text{at} \qquad y = c_r = c \tag{10.11}$$

Lin(18) has shown that the Reynolds stress in a viscous fluid grows positively and very rapidly with distance away from the wall in a very thin layer, and then stays about constant.

We now have enough information to build the complete picture of stress distribution for the viscous case. Starting at the wall, the Reynolds stress is zero and as we proceed outwards it grows at a rapid rate. It then levels off, and since viscosity is not very important Eq. (10.4), i.e.,

$$(\varphi\bar{\varphi}' - \bar{\varphi}\varphi') = \quad \text{constant}$$

will be almost satisfied (i.e. $\tau_s \sim$ constant). There is no jump across the layer where $U = c$. As we get to the liquid-gas interface the stress is discontinuous, the ratio of the values on both sides of the discontinuity being given by Eq. (10.3). The stress on the gas side can not be zero, since by Eq. (10.2) this would mean that the stress in the liquid is zero. Far away in the gas stream, the Reynolds stress must vanish. The diagram showing qualitatively this stress distribution, is given in Fig. 13.

Lin(18) found a prominent relationship in the theory of hydrodynamic stability in a very simple way. The principle used by him was to equate the stress in the fluid adjacent to the wall computed by two different methods: starting from the wall in one case, and from the main stream in the other. This method has not yet been made successful here, the reason being that the magnitude of the jump of stress across the interface is unknown, only the ratio is known. In order to compute the jump one would have first to compute the stress at the interface, on the gas side, by calculating the function Φ . This, would do away with the simplicity of Lin's method.

Therefore, looking at the Reynolds stress in the present case was not as fruitful as in Lin's case. It would nevertheless be interesting to find the quantitative distribution of stress for a self-excited disturbance and for a neutral disturbance. This would show whether most of the energy input into the disturbance (cf. footnote on p. 37) comes from the gas or the liquid. From the functions presented in this paper it would be possible to calculate the Reynolds stress for a neutral disturbance. The case of a self-excited disturbance is much more complicated and the necessary amount of numerical work, as envisioned at the present time, is prohibitive.

IV. CONCLUDING REMARKS

The results obtained here show that laminar instability is not responsible for the large scale disturbances observed in liquid film cooling experiments.

The study of the distribution of the Reynolds stress across the stream still remains a problem the solution of which will help in the physical understanding of the phenomenon of hydrodynamic stability.

It can now be said that the model chosen for the analysis, although the simplest possible, has yielded a number of new and interesting results.

The most important conclusion to be drawn from this investigation is that a discontinuity of viscosity or density has a destabilizing effect on uniform shearing motion.

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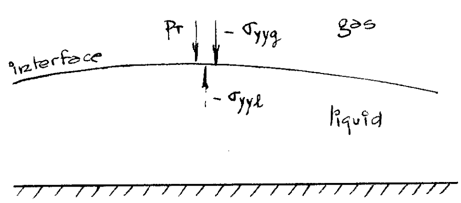
APPENDIX A

RELATIONSHIPS NEEDED IN THE BOUNDARY CONDITIONS

1. Boundary Condition Concerning the Normal Stress at the Interface

From the sketch below, the condition of equilibrium of normal forces at the interface is given by:

$$p_T - \sigma_{yyg} = -\sigma_{yyl} \tag{A.1}$$



where p_T is the equivalent pressure due to surface tension forces and the σ's indicate normal stresses.

Eq. (A.1) can be rewritten as

$$\bar{P}_g - 2\mu_g \frac{\partial \bar{v}_g}{\partial y} - \bar{P}_l + 2\mu_l \frac{\partial \bar{v}_l}{\partial y} + \bar{P}_T = 0$$

where the bars indicate dimensional quantities, the v's are the disturbance velocities in the y-direction and the μ's are the viscosities of the fluids of interest. Letting δ be the height, and $\bar{U}_2$ the surface velocity of the liquid, we can render the previous equation dimensionless by multiplying throughout by $\frac{\delta}{\bar{U}_2}$

$$\left[(P_l - P_T)\frac{\delta}{\bar{U}_2 \mu_l} - 2\frac{\partial v_l}{\partial y}\right] - \frac{1}{m}\left(P_g \frac{\delta}{\bar{U}_2 \mu_g} - 2\frac{\partial v_g}{\partial y}\right) = 0 \tag{A.2}$$

The various terms that enter in Eq. (A.2) will now be determined as a function of $\varphi(y)$, the amplitude of the disturbance stream function.

2. Determination of p_T

If σ is the surface tension coefficient (force per unit length) and l_r the radius of curvature of the interface surface, p_T can be written as

$$p_T = \frac{\sigma}{l_r} \tag{A.3}$$

where

$$\frac{1}{l_r} = - \frac{\dfrac{d^2\bar{y}}{d\bar{x}^2}}{\left[1 + \left(\dfrac{d\bar{y}}{d\bar{x}}\right)^2\right]^{3/2}} \tag{A.4}$$

the negative sign having been chosen because p_T is to be positive when $\frac{d^2\bar{y}}{d\bar{x}^2}$ is negative. The derivatives in Eq. (A.4) can be expressed in terms of velocities

$$\frac{d\bar{y}}{d\bar{x}} = \frac{d\bar{y}/d\bar{t}}{d\bar{x}/d\bar{t}} = \frac{\bar{v}}{\bar{U}_2 + \bar{u}}$$

$$\frac{d^2\bar{y}}{d\bar{x}^2} = \frac{1}{\bar{U}_2 + \bar{u}}\left(\frac{\partial\bar{v}}{\partial\bar{x}} + \frac{\partial\bar{v}}{\partial\bar{y}}\,\frac{d\bar{y}}{d\bar{x}}\right) - \frac{\bar{v}}{(\bar{U}_2 + \bar{u})^2}\left(\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{u}}{\partial\bar{y}}\,\frac{d\bar{y}}{d\bar{x}}\right) \tag{A.5}$$

and since $\bar{u}$ and $\bar{v} \ll \bar{U}_2$, the last equation can be rewritten as

$$\frac{d^2\bar{y}}{d\bar{x}^2} \cong \frac{1}{\bar{U}_2}\frac{\partial\bar{v}}{\partial\bar{x}} \tag{A.6}$$

Inserting Eqs. (A.5) and (A.6) into (A.4) and then into (A.3), p_T becomes, after non-dimensionalizing,

$$-p_T\frac{\delta}{\bar{U}_2\,\mu_l} = \frac{T'}{U_2}\frac{\partial v}{\partial x} \tag{A.7}$$

where

$$T' = \frac{\sigma}{\bar{U}_2 \mu_\ell} \tag{A.8}$$

the unbarred velocities and coordinates being dimensionless.

3. __Finding__ $\frac{\partial v_\ell}{\partial x}$, $\frac{\partial v_\ell}{\partial y}$ __and__ $\frac{\partial v_g}{\partial y}$

From Eq. (2.2)

$$\frac{\partial v_\ell}{\partial x} = \alpha^2 \varphi(y) e^{i\alpha(x-ct)}$$

$$\frac{\partial v_\ell}{\partial y} = -i\alpha \varphi'(y) e^{i\alpha(x-ct)} \tag{A.9}$$

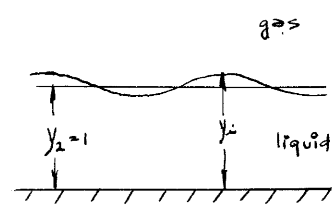
$$\frac{\partial v_g}{\partial y} = -i\alpha \Phi'(y) e^{i\alpha(x-ct)}$$

Before finding p_ℓ and p_g , we have to find the form of the deformed interface surface.

4. __Shape of the Deformed Interface Surface__

At the interface, $y_2 \cong 1$ and $U_2 \cong 1$. It is also clear that

$$v_2 = \left(\frac{Dy}{Dt}\right)_{y=y_2}$$



or upon using Eq. (2.2)

$$-i\alpha \varphi(y_2) e^{i\alpha(x-ct)} = \frac{\partial y}{\partial t} + U_2 \frac{\partial y}{\partial x} \tag{A.10}$$

Assuming that the surface ordinate is given by

$$y_i = 1 + A e^{i\alpha(x-ct)} \tag{A.11}$$

Eqs. (A.11) and (A.10) lead to:

$$A = -\frac{\varphi(y_2)}{U_2 - c}$$

and

$$y_i = 1 - \frac{\varphi(y_2)}{\bar{U}_2 - c} e^{i\alpha(x-ct)} \tag{A.12}$$

5. <u>Determination of</u> p_l <u>and</u> p_g

The Navier-Stokes equation for the perturbed flow in the y-direction, including body forces, is:

$$\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{U}\frac{\partial \bar{v}}{\partial \bar{x}} = -g - \frac{1}{\rho}\frac{\partial(P+p)}{\partial \bar{y}} + \nu\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right) \tag{A.13}$$

where g is the acceleration due to gravity and has a negative sign because it acts in the negative direction of the y-axis. Eq. (A.13) can be made dimensionless by multiplying it by $\frac{\delta}{\bar{U}_2^2}$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{1}{F} - \frac{1}{\rho \bar{U}_2^2}\frac{\partial(P+p)}{\partial y} + \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{A.14}$$

where

$$F = \frac{\bar{U}_2^2}{g\,\delta}\,, \qquad R = \frac{\bar{U}_2\,\delta}{\nu}$$

From Eq. (2.2)

$$\begin{aligned}
\frac{\partial v}{\partial t} &= -\alpha^2 c\,\varphi(y)\,e^{i\alpha(x-ct)} \\
\frac{\partial^2 v}{\partial x^2} &= i\alpha^3\,\varphi(y)\,e^{i\alpha(x-ct)} \\
\frac{\partial v}{\partial x} &= \alpha^2\,\varphi(y)\,e^{i\alpha(x-ct)} \\
\frac{\partial^2 v}{\partial y^2} &= -i\alpha\,\varphi''(y)\,e^{i\alpha(x-ct)}
\end{aligned} \tag{A.15}$$

Eq. (A.15) can now be used to rewrite Eq. (A.14) in terms of $\varphi(y)$

$$\frac{1}{\rho\bar{U}_2^2}\frac{\partial(P+p)}{\partial y} = -\frac{1}{F} - \left\{\alpha^2\left[U - c - i\frac{\alpha}{R}\right]\varphi + i\frac{\alpha}{R}\varphi''\right\}e^{i\alpha(x-ct)} \tag{A.16}$$

From the Orr-Sommerfeld equation

$$(U-c)(\varphi''-\alpha^2\varphi)-U''\varphi=-\frac{i}{\alpha R}(\varphi^{iv}-2\alpha^2\varphi''+\alpha^4\varphi)$$

one obtains

$$-\left\{\alpha^2\left[U-c-i\frac{\alpha}{R}\right]\varphi+i\frac{\alpha}{R}\varphi''\right\}=-\frac{i}{\alpha R}\varphi^{iv}+\left[i\frac{\alpha}{R}-(U-c)\right]\varphi''+U''\varphi \tag{A.17}$$

which when used in Eq. (A.16) yields

$$\frac{1}{\rho\bar{U}_2^2}\frac{\partial(P+p)}{\partial y}=\left\{\left[i\frac{\alpha}{R}-(U-c)\right]\varphi''+U''\varphi-\frac{i}{\alpha R}\varphi^{iv}\right\}e^{i\alpha(x-ct)}-\frac{1}{F} \tag{A.18}$$

and since

$$-(U-c)\varphi''+U''\varphi=\left[-(U-c)\varphi''-U'\varphi'\right]+U'\varphi'+U''\varphi=-\frac{d}{dy}\left[(U-c)\varphi'\right]-\frac{d}{dy}(\varphi U')$$

Eq. (A.18) becomes

$$\frac{1}{\rho\bar{U}_2^2}\frac{\partial(P+p)}{\partial y}=\left\{i\frac{\alpha}{R}\varphi''-\frac{d}{dy}\left[(U-c)\varphi'\right]+\frac{d}{dy}(\varphi U')-\frac{i}{\alpha R}\varphi^{iv}\right\}e^{i\alpha(x-ct)}-\frac{1}{F}$$

which after integrating with respect to y can be written as

$$\frac{1}{\rho\bar{U}_2^2}(P+p)=\left\{U'\varphi+\left[i\frac{\alpha}{R}-(U-c)\right]\varphi'-\frac{i}{\alpha R}\varphi'''\right\}e^{i\alpha(x-ct)}-\frac{y}{F}+f(x,t) \tag{A.19}$$

For steady motion, the basic flow satisfies the equation

$$\frac{P}{\rho\bar{U}_2^2}=-\frac{y_2}{F}+G(x) \tag{A.20}$$

since

$$f(x,t)\equiv G(x)$$

because there can be no dependence on time.

Using Eqs. (A.12) and (A.20), and considering that the conditions of present interest are at the interface, Eq. (A.19) becomes

$$\rho_\ell \frac{\delta}{\overline{U}_2 \mu_\ell} = \left\{\left[U_{2\ell}' + \frac{1}{F(U_2-c)}\right]\varphi + \left[i\frac{\alpha}{R_\ell} - (U_2-c)\right]\varphi' - \frac{i}{\alpha R_\ell}\varphi'''\right\} R_\ell e^{i\alpha(x-ct)} \tag{A.21}$$

where the subscript ℓ indicates liquid.

Similarly, for the gas

$$p_g \frac{\delta}{\overline{U}_2 \mu_g} = \left\{\left[U_{2g}' + \frac{1}{F(U_2-c)}\right]\Phi + \left[i\frac{\alpha}{R_g} - (U_2-c)\right]\Phi' - \frac{i}{\alpha R_g}\Phi'''\right\} R_g e^{i\alpha(x-ct)} \tag{A.22}$$

6. <u>The Equation Satisfied by the Normal Stress</u>

The insertion of Eqs. (A.7), (A.9), (A.21) and (A.22) into (A.2) leads to

$$\left\{\left[\frac{\alpha^2}{U_2}\frac{T'}{R_\ell} + U_{2\ell}' + \frac{1}{F(U_2-c)}\right]\varphi + \left[i\frac{3\alpha}{R_\ell} - (U_2-c)\right]\varphi' - \frac{i}{\alpha R_\ell}\varphi'''\right\} R_\ell$$
$$- \left\{\left[U_{2g}' + \frac{1}{F(U_2-c)}\right]\Phi + \left[i\frac{3\alpha}{R_g} - (U_2-c)\right]\Phi' - \frac{i}{\alpha R_g}\Phi'''\right\}\frac{R_g}{m} = 0 \tag{A.23}$$

where

$$m = \frac{\mu_\ell}{\mu_g}$$

Since

$$\frac{T'}{R_\ell} = \frac{\sigma}{\rho_\ell \delta \overline{U}_2^2} \equiv \frac{1}{W}$$

$$R_g = \frac{\mu_\ell}{\mu_g}\frac{\rho_g}{\rho_\ell} R_\ell = mr R_\ell$$

$$U_2 = U_{2\ell}' = 1$$

$$U_{2g}' = m$$

and letting

$$R_\ell \equiv R$$

Eq. (A.23) becomes

$$\left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right] R\varphi - \left[(1-c)R - i3\alpha\right]\varphi' - \frac{i}{\alpha}\varphi'''$$

$$- \frac{1}{m}\left\{\left[m + \frac{1}{F(1-c)}\right] R r m \Phi - \left[(1-c)R r m - i3\alpha\right]\Phi' - \frac{i}{\alpha}\Phi'''\right\} = 0 \qquad \text{(A.24)}$$

APPENDIX B

ASYMPTOTIC FORM OF THE VISCOUS SOLUTIONS ($\alpha R \gg 1$)

1. Asymptotic Form of $h_1(z)$, $h_2(z)$, $h_1'(z)$, $h_2'(z)$ and Related Functions

From Ref.12, p. XVII and XVIII

$$h_1(z) \sim 12^{1/6}\,\pi^{-1/2}\,e^{-i\frac{5}{12}\pi}\,z^{-1/4}\,e^{i\frac{2}{3}z^{3/2}}\left[1 - i\frac{5}{48}z^{-3/2} - \frac{385}{4608}z^{-3} + O\left(z^{-9/2}\right)\right] \tag{B.1}$$

$$-\tfrac{2}{3}\pi < \arg z < \tfrac{4}{3}\pi$$

$$h_2(z) \sim 12^{1/6}\,\pi^{-1/2}\,e^{i\frac{5}{12}\pi}\,z^{-1/4}\,e^{-i\frac{2}{3}z^{3/2}}\left[1 - i\frac{5}{48}z^{-3/2} - \frac{385}{4608}z^{-3} + O\left(z^{-9/2}\right)\right] \tag{B.2}$$

$$-\tfrac{4}{3}\pi < \arg z < \tfrac{2}{3}\pi$$

Since the values of z of interest will always lie in the right half plane, or for large αR nearly on the imaginary axis, the region of validity of the above expansions is satisfactory. h_1 and h_2 are analytic functions, therefore their asymptotic representations can be formally differentiated to obtain

$$h_1'(z) \sim \frac{\beta}{2}e^{-i\frac{5}{12}\pi}\left[i z^{1/4} - \frac{7}{48}z^{-5/4} + O\left(z^{-\frac{11}{4}}\right)\right]e^{i\frac{2}{3}z^{3/2}} \tag{B.3}$$

$$h_2'(z) \sim -\frac{\beta}{2}e^{i\frac{5}{12}\pi}\left[i z^{1/4} + \frac{17}{48}z^{-5/4} + O\left(z^{-11/4}\right)\right]e^{-i\frac{2}{3}z^{3/2}} \tag{B.4}$$

where

$$\beta = 2 \times 12^{1/6}\,\pi^{-1/2} \tag{B.5}$$

From Eqs. (5.11), (5.12), (5.13) and (B.1) to (B.4) the following can be obtained

$$\varphi''_{32} \sim \beta\, e^{-i\frac{5}{12}\pi}\, z_{\ell_1}^{-1/4}\, e^{i\frac{2}{3}z_{\ell_1}^{3/2}} \left[1 + O\left(z_{\ell_1}^{-3/2}\right)\right] \tag{B.6}$$

$$\varphi'''_{32} \sim -\beta\, e^{-i\frac{5}{12}\pi} (\alpha R)^{1/3} \left(z_{\ell_1}^{1/4} + \frac{i}{4} z_{\ell_1}^{-5/4}\right) e^{i\frac{2}{3}z_{\ell_1}^{3/2}} \left[1 + O\left(z_{\ell_1}^{-3/2}\right)\right] \tag{B.7}$$

$$\varphi''_{42} \sim \beta\, e^{i\frac{5}{12}\pi}\, z_{\ell_1}^{-1/4}\, e^{-i\frac{2}{3}z_{\ell_1}^{3/2}} \left[1 - i\frac{5}{48} z_{\ell_1}^{-3/2} + O\left(z_{\ell_1}^{-3}\right)\right] \tag{B.8}$$

$$\varphi'''_{42} \sim \beta\, e^{i\frac{5}{12}\pi} (\alpha R)^{1/3} e^{-i\frac{2}{3}z_{\ell_1}^{3/2}} \left[z_{\ell_1}^{1/4} - i\frac{17}{48} z_{\ell_1}^{-5/4} - \left(\frac{35}{192} + \frac{385}{4608}\right) z_{\ell_1}^{-11/4} + O\left(z_{\ell_1}^{-17/4}\right)\right] \tag{B.9}$$

$$\Phi''_{32} \sim \beta\, e^{-i\frac{5}{12}\pi}\, e^{i\frac{2}{3}z_{g_1}^{3/2}} \left[z_{g_1}^{-1/4} - i\frac{5}{4} z_{g_1}^{-7/4} - \frac{385}{4608} z_{g_1}^{-13/4} + O\left(z_{g_1}^{-19/4}\right)\right] + \alpha^2 \Phi_{32} \tag{B.10}$$

$$\Phi'''_{32} \sim -\beta\, e^{-i\frac{5}{12}\pi} \left(\alpha R r m^2\right)^{1/3} e^{i\frac{2}{3}z_{g_1}^{3/2}} \left[z_{g_1}^{1/4} + i\frac{7}{48} z_{g_1}^{-5/4} + O\left(z_{g_1}^{-11/4}\right)\right] + \alpha^2 \Phi'_{32} \tag{B.11}$$

In the above expressions, the following notation has been used, (cf. Eq. (1.8)).

$$z_{\ell_0} \equiv z_\ell(0) = \frac{\alpha^2}{(\alpha R)^{2/3}} - i(\alpha R)^{1/3} c \tag{B.12}$$

$$z_{\ell_1} \equiv z_\ell(1) = \frac{\alpha^2}{(\alpha R)^{2/3}} + i(\alpha R)^{1/3}(1-c) \tag{B.13}$$

$$z_{g_1} \equiv z_g(1) = \frac{\alpha^2}{(\alpha R \nu m^2)^{2/3}} + i(\alpha R \nu m^2)^{1/3} \frac{1-c}{m} \tag{B.14}$$

$$z_{g\infty} \equiv z_g(\infty) = \frac{\alpha^2}{(\alpha R \nu m^2)^{2/3}} + i\infty \tag{B.15}$$

In the last equation $i\infty$ is preceded by a positive sign because as shown on p. 29, c real, and $c<1$ is the case of interest. The general plan of the calculation is to transform these integrals in such a way that asymptotic methods can be used in their evaluation.

2. Determination of Necessary Integrals. General Plan of Calculation

The six integrals that need evaluation can be written in condensed form as

$$\left.\begin{aligned} &\int_0^1 e^{\pm\alpha t_\ell}\, \mathcal{S}_{(1,2)}(t_\ell)\, dt_\ell \\ &\int_1^\infty e^{\pm\alpha(t_g-1)}\, \mathcal{S}_1(t_g)\, dt_g \end{aligned}\right\} \tag{B.16}$$

The integrals of Eq. (B.16) can be expressed in terms of the integral representation of the functions they involve as given by Eqs. (1.12) and (1.13). The independent variables t_ℓ and t_g, will be expressed in terms of z_ℓ and z_g by means of Eqs. (1.9), (5.11) and (5.12)

$$t_l = c + i\left[\frac{\alpha}{R} - \frac{z_l}{(\alpha R)^{1/3}}\right] \qquad dt_l = -\frac{i}{(\alpha R)^{1/3}}\, dz_l$$

$$t_q - 1 = -\frac{1-c}{m} + i\left[\frac{\alpha}{R r m^2} - \frac{z_q}{(\alpha R r m^2)^{1/3}}\right] \qquad dt_q = -\frac{i}{(\alpha R r m^2)^{1/3}}\, dz_q$$

so that

$$\int_0^1 e^{\pm\alpha t_l} S_1\, dt_l = \frac{k}{\pi(\alpha R)^{1/3}}\, e^{\pm\alpha(c + i\frac{\alpha}{R})} \int_{z_{l_1}}^{z_{l_0}} e^{\mp i \frac{\alpha}{(\alpha R)^{1/3}} z_l} \left(\int_{L_1} e^{z_l t + \frac{t^3}{3}}\, dt\right) dz_l \tag{B.17}$$

$$\int_0^1 e^{\pm\alpha t_l} S_2\, dt_l = -\frac{k^*}{\pi(\alpha R)^{1/3}}\, e^{\pm\alpha(c + i\frac{\alpha}{R})} \int_{z_{l_1}}^{z_{l_0}} e^{\mp i \frac{\alpha}{(\alpha R)^{1/3}} z_l} \left(\int_{L_2} e^{z_l t + \frac{t^3}{3}}\, dt\right) dz_l \tag{B.18}$$

$$\int_1^\infty e^{\pm\alpha(t_q - 1)} S_1(t_q)\, dt_q = \frac{k}{\pi(\alpha R r m^2)^{1/3}}\, e^{\pm\alpha\left[-\frac{1-c}{m} + i\frac{\alpha}{R r m^2}\right]} \times$$

$$\times \int_{z_{q\infty}}^{z_{q_1}} e^{\mp i \frac{\alpha}{(\alpha R r m^2)^{1/3}} z_q} \left(\int_{L_1} e^{z_q t + \frac{t^3}{3}}\, dt\right) dz_q \tag{B.19}$$

Since the integrals on the paths L_1 and L_2 are uniformly convergent, the order of integration can be reversed, so that, in general

$$\int_{z_1}^{z_2} e^{\mp i a z} \left(\int_{L_1,L_2} e^{zt + \frac{t^3}{3}}\, dt\right) dz = \int_{L_1,L_2} e^{\frac{t^3}{3}} \left(\int_{z}^{z_2} e^{z(t \mp i a)}\, dz\right) dt$$

$$= \int_{L_1,L_2} \frac{1}{t \mp i a}\left[e^{z_2(t \mp i a) + \frac{t^3}{3}} - e^{z_1(t \mp i a) + \frac{t^3}{3}}\right] dt \tag{B.20}$$

where z_1 and z_2 are, respectively, the lower and upper limit for the integration on z (z_l or z_q) and a is the appropriate coefficient obtained by comparing with Eqs. (B.17) through (B.19). Therefore, the six integrals in Eq. (B.16) become

$$\mathcal{L}_{\ell_1} = \int_0^1 e^{\alpha t_\ell} S_1(t_\ell)\, dt_\ell = \frac{k}{\pi(\alpha R)^{1/3}} e^{\alpha(c+i\frac{\alpha}{R})} \Bigg\{ e^{-i\frac{\alpha}{(\alpha R)^{1/3}} z_{\ell_0}} \int_{L_1} \frac{e^{z_{\ell_0} t + \frac{t^3}{3}}}{t - i\frac{\alpha}{(\alpha R)^{1/3}}}\, dt$$

$$- e^{-i\frac{\alpha}{(\alpha R)^{1/3}} z_{\ell_1}} \int_{L_1} \frac{e^{z_{\ell_1} t + \frac{t^3}{3}}}{t - i\frac{\alpha}{(\alpha R)^{1/3}}}\, dt \Bigg\} \tag{B.21}$$

$$\mathcal{L}_{\ell_2} = \int_0^1 e^{-\alpha t_\ell} S_1(t_\ell)\, dt_\ell = \frac{k}{\pi(\alpha R)^{1/3}} e^{-\alpha(c+i\frac{\alpha}{R})} \Bigg\{ e^{i\frac{\alpha}{(\alpha R)^{1/3}} z_{\ell_0}} \int_{L_1} \frac{e^{z_{\ell_0} t + \frac{t^3}{3}}}{t + i\frac{\alpha}{(\alpha R)^{1/3}}}\, dt$$

$$- e^{i\frac{\alpha}{(\alpha R)^{1/3}} z_{\ell_1}} \int_{L_1} \frac{e^{z_{\ell_1} t + \frac{t^3}{3}}}{t + i\frac{\alpha}{(\alpha R)^{1/3}}}\, dt \Bigg\} \tag{B.22}$$

$$\mathcal{M}_{\ell_1} = \int_0^1 e^{\alpha t_\ell} S_2(t_\ell)\, dt_\ell = -\frac{k^*}{\pi(\alpha R)^{1/3}} e^{\alpha(c+i\frac{\alpha}{R})} \Bigg\{ e^{-i\frac{\alpha}{(\alpha R)^{1/3}} z_{\ell_0}} \int_{L_2} \frac{e^{z_{\ell_0} t + \frac{t^3}{3}}}{t - i\frac{\alpha}{(\alpha R)^{1/3}}}\, dt$$

$$- e^{-i\frac{\alpha}{(\alpha R)^{1/3}} z_{\ell_1}} \int_{L_2} \frac{e^{z_{\ell} t + \frac{t^3}{3}}}{t - i\frac{\alpha}{(\alpha R)^{1/3}}}\, dt \Bigg\} \tag{B.23}$$

$$\mathcal{M}_{\ell_2} = \int_0^1 e^{-\alpha t_\ell} S_2(t_\ell)\, dt_\ell = -\frac{k^*}{\pi(\alpha R)^{1/3}} e^{-\alpha(c+i\frac{\alpha}{R})} \Bigg\{ e^{i\frac{\alpha}{(\alpha R)^{1/3}} z_{\ell_0}} \int_{L_2} \frac{e^{z_{\ell_0} t + \frac{t^3}{3}}}{t + i\frac{\alpha}{(\alpha R)^{1/3}}}\, dt$$

$$- e^{i\frac{\alpha}{(\alpha R)^{1/3}} z_{\ell_1}} \int_{L_2} \frac{e^{z_{\ell_1} t + \frac{t^3}{3}}}{t + i\frac{\alpha}{(\alpha R)^{1/3}}}\, dt \Bigg\} \tag{B.24}$$

$$\mathcal{N}_1 = \int_1^\infty e^{\alpha(t_g-1)} S_1(t_g)\, dt_g = \frac{k}{\pi(\alpha R r m^2)^{1/3}}\, e^{\alpha\left[-\frac{1-c}{m} + i\frac{\alpha}{R r m^2}\right]} \times$$

$$\times \left\{ e^{-i\frac{\alpha}{(\alpha R r m^2)^{1/3}} z_{g_1}} \int_{L_1} \frac{e^{z_{g_1} t + \frac{t^3}{3}}}{t - i\frac{\alpha}{(\alpha R r m^2)^{1/3}}}\, dt - e^{-i\frac{\alpha}{(\alpha R r m^2)^{1/3}} z_{g\infty}} \int_{L_1} \frac{e^{z_{g\infty} t + \frac{t^3}{3}}}{t - i\frac{\alpha}{(\alpha R r m^2)^{1/3}}}\, dt \right\} \tag{B.25}$$

$$\mathcal{N}_2 = \int_1^\infty e^{-\alpha(t_g-1)} S_1(t_g)\, dt_g = \frac{k}{\pi(\alpha R r m^2)^{1/3}}\, e^{-\alpha\left[-\frac{1-c}{m} + i\frac{\alpha}{R r m^2}\right]} \times$$

$$\times \left\{ e^{i\frac{\alpha}{(\alpha R r m^2)^{1/3}} z_{g_1}} \int_{L_1} \frac{e^{z_{g_1} t + \frac{t^3}{3}}}{t + i\frac{\alpha}{(\alpha R r m^2)^{1/3}}}\, dt - e^{i\frac{\alpha}{(\alpha R r m^2)^{1/3}} z_{g\infty}} \int_{L_1} \frac{e^{z_{g\infty} t + \frac{t^3}{3}}}{t + i\frac{\alpha}{(\alpha R r m^2)^{1/3}}}\, dt \right\} \tag{B.26}$$

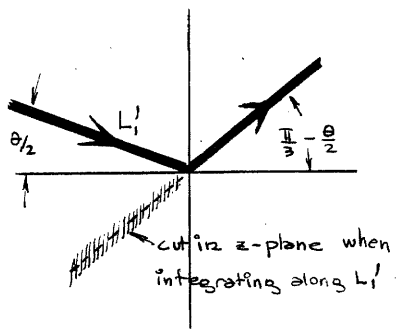
It can be seen from the preceding equations that the fundamental integral to be evaluated is

$$I = \int_L \frac{e^{zt + \frac{t^3}{3}}}{t + ia}\, dt \tag{B.27}$$

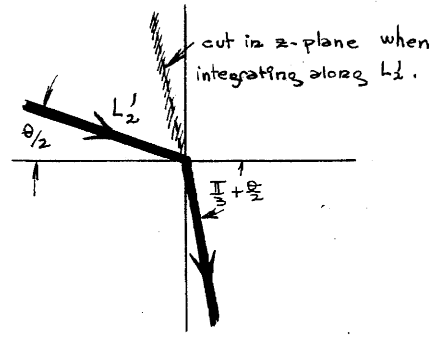
where L could be either L_1 or L_2 and a is real, positive or negative. Let

$$t = \tau\sqrt{3z} \qquad dt = \sqrt{3z}\, d\tau \tag{B.28}$$

where the cuts in the z-plane are taken as indicated in the following sketch, where the τ and z-plane are superposed



$$z = |z|\, e^{i\theta}$$



The use of Eqs. (B.27) and (B.28) leads to

$$I = \int_{L'} \frac{1}{\tau + \frac{ia}{\sqrt{3z}}}\, e^{\sqrt{3}|z|^{3/2}\left[(\tau+\tau^3)\, e^{i\frac{3}{2}\theta}\right]}\, d\tau \tag{B.29}$$

where the two necessary paths of integration are indicated in the preceding sketch.

It should be remarked that thus far Eqs. (B.21) to (B.26) and (B.29) are <u>completely general</u> and no restrictive assumption has been made on any of the parameters involved.

It will now be assumed that <u>$\alpha R \gg 1$</u>. This implies that $|z|^{3/2}$, according to Eq. (5.12), is a large parameter in Eq. (B.29).

3. Detailed Asymptotic Evaluation of Integral (B.29)

The integral (B.29) can be rewritten as

$$I_{1,2}(z) = \int_{L_1', L_2'} \varphi(\tau, z)\, e^{\sqrt{3}|z|^{3/2} f(\tau)}\, d\tau \tag{B.30}$$

where

$$\varphi(\tau, z) = \frac{1}{\tau + \frac{ia}{\sqrt{3z}}} \tag{B.31}$$

$$f(\tau) = (\tau + \tau^3)\, e^{i\frac{3}{2}\theta} \tag{B.32}$$

$$z = |z|\, e^{i\theta} \tag{B.33}$$

the paths of integration being indicated in figure on p. 58 .

It turns out that because of the form in which these two integrals enter into the calculation, only the leading term of the asymptotic expansion is needed when dealing with I_2 , while several terms are needed when using I_1 . Therefore, the simplest case, I_2 will be evaluated first by the saddle point method, and I_1 will be later evaluated by using the method of steepest descent. Both methods are completely described in Ref. 14. In the course of evaluating I_2 we will also determine the leading term of I_1 .

3a. Evaluation of the Leading Terms of I_1 and I_2 by the Saddle Point Method

The saddle points or cols are given by the zeros of $f'(\tau)$, i.e., the points

$$\tau_1 = \frac{i}{\sqrt{3}}$$
$$\tau_2 = -\frac{i}{\sqrt{3}} \tag{B.34}$$

Since $f''(\tau) \neq 0$, these are simple cols and it can, therefore, be expected that two steepest curves intersect at each col: one of steepest ascent and one of steepest descent. The paths of steepest descent are given by

$$\mathcal{I}m\ f(\tau) = \text{constant} \tag{B.35}$$

the constant being adjusted so that they go through the cols. Let

$$\tau = \xi + i\eta$$

which when inserted in Eq. (B.35) leads, after using Eqs. (B.32) and (B.34), to

$$\mathcal{I}m\left\{\left[(\xi^3+\xi-3\xi\eta^2)\cos\tfrac{3}{2}\theta-(\eta+3\xi^2\eta-\eta^3)\sin\tfrac{3}{2}\theta\right]+i\left[(\xi^3+\xi-3\xi\eta^2)\sin\tfrac{3}{2}\theta\right.\right.$$
$$\left.\left.+(\eta+3\xi^2\eta-\eta^3)\cos\tfrac{3}{2}\theta\right]\right\} = \mathcal{I}m\left\{\mp\frac{2}{3\sqrt{3}}\sin\tfrac{3}{2}\theta \pm i\frac{2}{3\sqrt{3}}\cos\tfrac{3}{2}\theta\right\}$$

where the top sign goes with the path through the col τ_1 , and the bottom one with the path through τ_2 . The <u>equation for the steepest paths</u> then is

$$\eta(1+3\xi^2-\eta^2)+\xi(1-3\eta^2+\xi^2)\tan\tfrac{3}{2}\theta = \pm\frac{2}{3\sqrt{3}} \tag{B.36}$$

where the positive or negative sign is for the path through τ_1 or τ_2 respectively.

The <u>equation for the level lines</u> through the cols is given by

$$\mathcal{R}\ell\ f(\tau) = \text{constant}$$

or

$$\xi(1-3\eta^2+\xi^2)\cot\tfrac{3}{2}\theta-\eta(1+3\xi^2-\eta^2) = \mp\frac{2}{3\sqrt{3}} \tag{B.37}$$

where the negative or positive sign is for the path through τ_1 or τ_2, respectively.

For $\alpha R \gg 1$, the arguments of the several values of z of interest are $\pm \frac{\pi}{2}$.

Consider the case $\theta = -\pi/2$. Eqs. (B.36) and (B.37) then become

$$\eta(1+3\xi^2-\eta^2)+\xi(1-3\eta^2+\xi^2) = \pm\frac{2}{3\sqrt{3}} \quad \text{steepest paths through cols}$$

$$\eta(1+3\xi^2-\eta^2)-\xi(1-3\eta^2+\xi^2) = \pm\frac{2}{3\sqrt{3}} \quad \text{level lines through cols} \tag{B.38}$$

where the positive or negative sign is for the path through τ_1 or τ_2, respectively.

In order to determine the location of the valleys and mountains, set $\tau = |\tau| e^{i\mathcal{E}}$ in Eqs. (B.30), (B.31) and (B.32). For the valleys, the real part of the exponential in Eq. (B.30) should be negative when $|\tau| \to \infty$ i.e.

$$\cos\left(3\mathcal{E} + \tfrac{3}{2}\theta\right) < 0$$

and since the case of interest is $\theta = -\frac{\pi}{2}$

$$2n\pi + \frac{\pi}{2} < 3\mathcal{E} - \frac{3}{4}\pi < \frac{3}{2}\pi + 2n\pi \qquad n = 0, \pm 1, \pm 2, \ldots$$

or

$$\frac{2}{3}n\pi + \frac{5}{12}\pi < \mathcal{E} < \frac{9}{12}\pi + \frac{2}{3}n\pi \tag{B.39}$$

Setting $n = 0, 1$ and 2 the sectors in the τ-plane containing the valleys are determined;

$$\left.\begin{array}{l} \frac{5}{12}\pi < \tau < \frac{3}{4}\pi \\ \frac{13}{12}\pi < \tau < \frac{17}{12}\pi \\ \frac{21}{12}\pi < \tau < \frac{25}{12}\pi \end{array}\right\} \quad \text{(B.40)}$$

The sectors excluded in Eq. (B.40) are the mountains. The result is shown in Fig. 14.

For the case $\theta = \frac{\pi}{2}$ Eqs. (B.36) and (B.37) reduce to

$$\left.\begin{array}{ll} \eta(1+3\xi^2-\eta^2) - \xi(1-3\eta^2+\xi^2) = \pm\frac{2}{3\sqrt{3}} & \text{steepest paths} \\ \eta(1+3\xi^2-\eta^2) + \xi(1-3\eta^2+\xi^2) = \pm\frac{2}{3\sqrt{3}} & \text{level lines} \end{array}\right. \quad \text{(B.41)}$$

where the positive or negative sign, is for the path through τ_1 or τ_2, respectively. Comparing these equations with the ones given by Eq. (B.38), it can be seen that the paths are now just a reflection on the $\xi=0$ -axis of the ones of Eq. (B.38). Similarly, the valleys are now given by

$$\left.\begin{array}{l} -\frac{\pi}{12} < \tau < \frac{\pi}{4} \\ \frac{7}{12}\pi < \tau < \frac{11}{12}\pi \\ \frac{15}{12}\pi < \tau < \frac{19}{12}\pi \end{array}\right\} \quad \text{(B.42)}$$

In order to apply the saddle point method, it is necessary to deform the original contour of integration (cf. Fig. 15a b) to the one that goes through the steepest descent paths (Fig. 14). This can be done without changing the value of the integral, provided no singularity of the integrand is crossed in the deformation process. In the present case, the integrand (cf. Eq. (B.29) has a pole at

$$\tau_p = -\frac{ia}{\sqrt{3z}} \tag{B.43}$$

and since $|z| \gg 1$, the pole is very near the origin. If this pole is crossed in the deformation process, it will give a contribution to the integral of $\pm 2\pi i \times$ Residue at $\tau = -\frac{ia}{\sqrt{3z}}$. The choice of sign depends on the direction of the path around the pole, while the location of the poles depends on the values a and z as given by Eqs. (B.21) to (B.26). It can be seen from Eqs. (B.12) through (B.15) that the poles location will be different, according to whether $c>1$ or $c<1$ (i.e. assuming c is real). Since, as mentioned on p. 29 the case of interest is $c = c_r < 1$, the poles will be located with this criterion in mind. In the following calculations, we take, for definiteness

$$\arg i = +\frac{\pi}{2}$$

and because of the cuts taken in the z-plane (cf. p. 58)

$$i^{1/2} = e^{i\frac{\pi}{4}}$$

$$(-i)^{1/2} = e^{-i\frac{\pi}{4}}$$

Case of z_{l_0} ; $\theta = -\frac{\pi}{2}$

$$\left.\begin{aligned} \tau_p &= i\frac{\alpha}{(\alpha R)^{1/3}\sqrt{3}\,(\alpha R)^{1/6}c^{1/2}(-i)^{1/2}} = \frac{\alpha}{\sqrt{6(\alpha R)c}}(-1+i) && (a<0) \\ \tau_p &= -\frac{\alpha}{\sqrt{6(\alpha R)c}}(-1+i) && (a>0) \end{aligned}\right\} \tag{B.44}$$

The location of the poles and paths of integration is shown in Fig. 15a, the important fact is the sector in which they are located. Similarly for

z_{q_1} ; $\theta = \frac{\pi}{2}$

$$\left.\begin{aligned} \tau_p &= \frac{\alpha}{\sqrt{6\alpha R \tau m (1-c)}} (1+i) && (a<0) \\ \tau_p &= -\frac{\alpha}{\sqrt{6\alpha R \tau m (1-c)}} (1+i) && (a>0) \end{aligned}\right\} \tag{B.45}$$

and

z_{q_∞} ; $\theta = \frac{\pi}{2}$

$$\left.\begin{aligned} \tau_p &= \frac{1+i}{\infty} && (a<0) \\ \tau_p &= -\frac{1+i}{\infty} && (a>0) \end{aligned}\right\} \tag{B.46}$$

<u>Evaluation of the Integral (B.29)</u>

After the contour of integration has been deformed, the steepest paths are made to go through the cols, (Fig. 14) the contributions to the integral I_1 and I_2 , Eq. (B.29) will then consist, in general, of three parts

a) contribution of integrand when in the neighborhood of col $\tau_1 : I_1^*$

b) contribution of integrand when in the neighborhood of col $\tau_2 : I_2^*$

c) contribution of pole, if a pole is crossed in the process of deforming the contour: I_p

The contribution of a col $\tau_{1,2}$ to the integral of interest is given by (cf. Ref. 2, pp. 50-53)

$$\int \phi(\tau) e^{x f(\tau)} d\tau = e^{x f(\tau_{1,2})} \phi(\tau_{1,2}) \left[- \frac{2\pi}{x f''(\tau_{1,2})} \right]^{1/2}$$

where, in the present case

$$\phi(\tau) = \frac{1}{\tau + \frac{ia}{\sqrt{3z}}} \quad ; \quad f(\tau) = (\tau + \tau^3) e^{i\frac{3}{2}\theta} \quad ; \quad x = \sqrt{3}\, |z|^{3/2}$$

$$\tau_1 = \frac{i}{\sqrt{3}} \quad ; \qquad \tau_2 = -\frac{i}{\sqrt{3}} \quad ; \qquad f''(\tau) = 6\tau e^{i\frac{3}{2}\theta}$$

The use of the above equation, together with Eq. (B.33), leads to

$$\boxed{I_1^*(z) = \frac{\pi^{1/2}}{1+az^{-1/2}} z^{-3/4} e^{i\left(\frac{2}{3}z^{3/2}-\frac{\pi}{4}\right)}} \tag{B.47}$$

$$\boxed{I_2^*(z) = \frac{\pi^{1/2}}{1-az^{-1/2}} z^{-3/4} e^{-i\left(\frac{2}{3}z^{3/2}-\frac{\pi}{4}\right)}} \tag{B.48}$$

The contribution of a pole to the integral, is given by

$$I_p = \pm 2\pi i \,\mathrm{Res}\left[\frac{1}{\tau + \frac{ia}{\sqrt{3z}}} e^{\sqrt{3}\, z^{3/2}(\tau+\tau^3)}\right]_{\tau = -\frac{ia}{\sqrt{3z}}}$$

where the positive sign is used if the direction of integration around a pole is counterclockwise, and the negative sign, if the pole is encircled clockwise. Now

$$\mathrm{Res}\left[\frac{1}{\tau+\frac{ia}{\sqrt{3z}}} e^{\sqrt{3}\, z^{3/2}(\tau+\tau^3)}\right]_{\tau=-\frac{ia}{\sqrt{3z}}} = \left[e^{\sqrt{3}\, z^{3/2}(\tau+\tau^3)}\right]_{\tau=-\frac{ia}{\sqrt{3z}}}$$

and

$$\tau+\tau^3 = -\frac{ia}{\sqrt{3z}} + \frac{ia^3}{3z\sqrt{3z}} = \frac{i}{3\sqrt{3}\, z^{3/2}}\left(a^3 - 3az\right)$$

from which

$$\boxed{I_p(z) = 2\pi i\, e^{i\left(\frac{a^3}{3} - az\right)}} \qquad (B.49)$$

3b. Asymptotic Expansion of I_1^* by the Method of Steepest Descents

The integral under consideration is

$$I_1^*(z) = \int_{L_1^{1*}} \varphi(\tau, z)\, e^{\sqrt{3}\,|z|^{3/2} f(\tau)}\, d\tau \qquad (B.50)$$

where L_1^{1*} is the part of the deformed path that goes through the col $\tau_1 = \frac{i}{\sqrt{3}}$, and as before

$$\varphi(\tau, z) = \frac{1}{\tau + \frac{ia}{\sqrt{3z}}} \qquad (B.51)$$

$$f(\tau) = (\tau + \tau^3)\, e^{i\frac{3}{2}\theta} \qquad (B.52)$$

$$z = |z|\, e^{i\theta} \qquad (B.53)$$

$f(\tau)$ could be expanded as a power series around $\tau = \tau_1$, i.e.

$$f(\tau) = f(\tau_1) + f''(\tau_1)(\tau - \tau_1)^2 + f'''(\tau_1)(\tau - \tau_1)^3$$

since $f'(\tau_1) = 0$, and all derivatives of $f(\tau)$ higher than the third vanish. Using the values of the derivatives leads to

$$f(\tau) = f(\tau_1) - w \qquad (B.54)$$

where

$$w = (\tau - \tau_1)^2\, e^{i\frac{3}{2}\theta} \left[-3\tau_1 - (\tau - \tau_1)\right] \qquad (B.55)$$

On a path of steepest descent, (or ascent)

$$\operatorname{Im} f(\tau) = \text{constant}$$

Taking the imaginary part of Eq. (B.54), and using the preceding relation gives

$$\operatorname{Im} f(\tau) = \operatorname{Im} f(\tau_1) - \operatorname{Im} w$$

from which $\operatorname{Im} w = 0$, i.e. w is real. Along a path of steepest descent from a col, w is positive, while along a path of steepest ascent w is negative. From now on, w will then be considered positive and real.

Using Eqs. (B.54) and (B.50), this last one could be rewritten as the sum of two integrals like

$$I(z) = e^{\sqrt{3}\,|z|^{3/2} f(\tau_1)} \int_0^{\infty} e^{-\sqrt{3}\,|z|^{3/2} w}\, \varphi(\tau, z)\, \frac{d\tau}{dw}\, dw \qquad \text{(B.56)}$$

The purpose of the following analysis is to find the correct expression for $\varphi(\tau,z)\frac{d\tau}{dw}$ along the two paths down from the col.

<u>Finding τ as a Function of w.</u>

Since w, as given by Eq. (B.55), is regular in a neighborhood of τ_1, its positive square root is therefore single valued in that neighborhood. Then, from Eq. (B.55)

$$\sqrt{w} = l = (\tau-\tau_1)\, e^{i\frac{3}{4}\theta} \left[-3\tau_1 - (\tau-\tau_1)\right]^{1/2} \qquad \text{(B.57)}$$

$$-\sqrt{w} = -l = (\tau-\tau_1)\, e^{i\frac{3}{4}\theta} \left[-3\tau_1 - (\tau-\tau_1)\right]^{1/2} \qquad \text{(B.58)}$$

From the theory of inverse functions (cf. Ref. 15, pp. 121-123), Eqs. (B.57) and (B.58) have each a solution for τ which vanishes at $\sqrt{w}=0$ and is regular near $\sqrt{w}=0$. Instead of proceeding this way for finding τ, we will consider Eq. (B.54) where

$$f(\tau)-f(\tau_1)=-w$$

and using Eq. (B.52)

$$\left(\tau+\tau^3-\frac{2}{3\sqrt{3}}i\right)e^{i\frac{3}{2}\theta}=-w \tag{B.59}$$

Let

$$\tau=\frac{i}{\sqrt{3}}-iQ$$

then

$$\tau^3=-i\left(\frac{1}{3\sqrt{3}}-Q-\sqrt{3}\,Q^2-Q^3\right) \tag{B.60}$$

which when used in Eq. (B.59) leads to

$$-i\left(\sqrt{3}\,Q^2-Q^3\right)e^{i\frac{3}{2}\theta}=-w$$

or

$$Q\left(\sqrt{3}-Q\right)^{1/2}=e^{-i\left(\frac{3}{4}\theta+\frac{\pi}{4}\right)}(\pm l) \tag{B.61}$$

where $l=+\sqrt{w}$ and $(\sqrt{3}-Q)^{1/2}$ means the branch that reduces to $3^{1/4}$ when $Q=0$. Using now the theory of inverse functions, (cf. Ref.14, pp.34, Eq. 8-32 and 8-33)

$$Q=\sqrt{3}\sum_{n=1}^{\infty}\frac{\Gamma\left(\frac{3}{2}n-1\right)}{n!\,\Gamma\left(\frac{1}{2}\right)}\left[\frac{e^{-i\left(\frac{3}{4}\theta+\frac{\pi}{4}\right)}}{3^{3/4}}\right]^n(\pm l)^n \tag{B.62}$$

Denoting by τ^+ the parametric equation of the path of our integral when using the positive square root of w, and by τ^- when using the negative square root, the previous equation, when inserted into Eq. (B.60) yields

$$\tau^{\pm} = \frac{i}{\sqrt{3}} - i\sqrt{3}\sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{3}{2}n-1\right)}{n!\,\Gamma\left(\frac{n}{2}\right)}\left[\frac{e^{-i\left(\frac{3}{4}\theta+\frac{\pi}{4}\right)}}{3^{3/4}}\right]^{n}(\pm\ell)^{n} \tag{B.63}$$

Since ℓ is real and positive, $\left(\frac{d\tau^{+}}{d\ell}\right)_{\ell=0}$ will indicate the direction in the τ-plane of the paths of integration in Eqs. (B.50) or (B.56) at the col $\tau=\tau_1$

$$\left(\frac{d\tau^{+}}{d\ell}\right)_{\ell=0} = -\frac{e^{-i\left(\frac{3}{4}\theta-\frac{\pi}{4}\right)}}{3^{1/4}} \tag{B.64}$$

$$\left(\frac{d\tau^{-}}{d\ell}\right)_{\ell=0} = \frac{e^{-i\left(\frac{3}{4}\theta-\frac{\pi}{4}\right)}}{3^{1/4}} \tag{B.65}$$

In order to determine the correct direction of integration in the two integrals of the type given by Eq. (B.56), two special cases, for which the paths of integration are known, will be considered, i.e. $\theta=-\frac{\pi}{2}$ and $\theta=\frac{\pi}{2}$. This gives

for $\theta=-\frac{\pi}{2}$

$$\left.\begin{aligned} \left(\frac{d\tau^{+}}{d\ell}\right)_{\ell=0} &= -\frac{e^{i\frac{5}{8}\pi}}{3^{1/4}} \\ \left(\frac{d\tau^{-}}{d\ell}\right)_{\ell=0} &= \frac{e^{i\frac{5}{8}\pi}}{3^{1/4}} \end{aligned}\right\} \tag{B.66a}$$

and for $\theta=\frac{\pi}{2}$

$$\left.\begin{aligned} \left(\frac{d\tau^{+}}{d\ell}\right)_{\ell=0} &= -\frac{e^{-i\frac{\pi}{8}}}{3^{1/4}} \\ \left(\frac{d\tau^{-}}{d\ell}\right)_{\ell=0} &= \frac{e^{-i\frac{\pi}{8}}}{3^{1/4}} \end{aligned}\right\} \tag{B.66b}$$

By comparing the results of Eq. (B.66a) with the paths shown in Fig. 14, it is seen that $\frac{d\tau^-}{d\ell}$ goes in the direction of the desired path of integration while $\frac{d\tau^+}{d\ell}$ goes in the opposite sense. Eq. (B.50) can now be rewritten in complete form as

$$I_1^*(z) = e^{\sqrt{3}\,|z|^{3/2} f(\tau_1)} \int_0^\infty e^{-\sqrt{3}\,|z|^{3/2} w} g(w)\, dw \tag{B.67}$$

where

$$g(w) = \left[\varphi(\tau^-,z)\frac{d\tau^-}{dw} - \varphi(\tau^+,z)\frac{d\tau^+}{dw}\right] = \left[\varphi(\tau^-,z)\frac{d\tau^-}{d\ell^-}\frac{d\ell^-}{dw} - \varphi(\tau^+,z)\frac{d\tau^+}{d\ell^+}\frac{d\ell^+}{dw}\right] \tag{B.68}$$

From Eq. (B.63) it is clear that $\frac{d\tau}{dw}$ can be expanded in a power series in $\sqrt{w}$. Now $\varphi(\tau,z)$ is analytic in a neighborhood of $\tau=\tau_1$, and therefore can be expanded in a power series in $\sqrt{w}$. Thus, it is desired to determine the coefficients A_m of

$$\varphi(\tau^+,z)\cdot\frac{d\tau^+}{d\ell} = \sum_{m=0}^{\infty} A_m \ell^m \tag{B.69}$$

when

$$u = \ell^2 = f(\tau_1) - f(\tau) \tag{B.70}$$

By Cauchy's theorem of residues

$$A_m = \frac{1}{2\pi i}\int_C \varphi(\tau^+,z)\,\frac{d\tau^+}{d\ell}\,\frac{d\ell}{\ell^{m+1}} \tag{B.71}$$

where the contour C encloses the point $\ell=0$. Eq. (B.71) can be rewritten, after use is made of Eq. (B.70), as

$$A_n = \frac{1}{2\pi i}\int_{\Gamma'} \varphi(\tau^+, z)\left[f(\tau_1) - f(\tau)\right]^{-\frac{n+1}{2}} d\tau \tag{B.72}$$

where Γ' encloses the point $\tau = \tau_1$. The function $\varphi(\tau^+, z)\left[f(\tau_1) - f(\tau)\right]^{-\frac{n+1}{2}}$ will next be determined. From Eqs. (B.54) and (B.55)

$$f(\tau_1) - f(\tau) = (\tau - \tau_1)^2 e^{i\frac{3}{2}\theta}\left[-3\tau_1 - (\tau - \tau_1)\right]$$

from which

$$\left[f(\tau_1) - f(\tau)\right]^{-\frac{n+1}{2}} = (\tau - \tau_1)^{-(n+1)} \sum_{k=0}^{\infty} d_k(n)\,(\tau - \tau_1)^k \tag{B.73}$$

where the d_k's are given by $\left(s = \frac{n+1}{2}\right)$

$$\left.\begin{aligned}
d_0(n) &= (3\tau_1)^{-s}(-1)^{-s} e^{-i\frac{3}{2}s\theta} \\
d_1(n) &= (3\tau_1)^{-s-1}\, s\,(-1)^{-s+1} e^{-i\frac{3}{2}s\theta} \\
d_2(n) &= (3\tau_1)^{-s-2}\, s(s+1)\frac{1}{2!}(-1)^{-s} e^{-i\frac{2}{3}s\theta} \\
d_3(n) &= (3\tau_1)^{-s-3}\, s(s+1)(s+2)\frac{1}{3!}(-1)^{-s+1} e^{-i\frac{2}{3}s\theta} \\
&\;\vdots \\
d_k(n) &= (3\tau_1)^{-s-k}\, s(s+1)(s+2)\ldots(s+k-1)\frac{1}{k!}(-1)^{-s+1,\,-s} e^{-i\frac{2}{3}s\theta}
\end{aligned}\right\} \tag{B.74}$$

In the last equation the exponent $-s+1$ is used when k is odd and $-s$ when k is even.

Since $\varphi(\tau^+, z)$ is regular in a neighborhood of $\tau = \tau_1$, it can be expanded as

$$\varphi(\tau^+, z) = \sum_{j=0}^{\infty} \varphi_j(\tau_1, z)\,(\tau^+ - \tau_1)^j$$

where

$$\varphi_j = \left(\tau_1 + \frac{ia}{\sqrt{3z}}\right)^{-(j+1)} (-1)^j \tag{B.75}$$

From Eqs. (B.73) through (B.75)

$$\varphi(\tau^+,z)\left[f(\tau_1)-f(\tau)\right]^{-\frac{n+1}{2}} = (\tau-\tau_1)^{-(n+1)}\sum_{h=0}^{\infty} A_h(n)\,(\tau-\tau_1)^h \tag{B.76}$$

where

$$\left.\begin{aligned}
A_0(n) &= d_0(n)\,\varphi_0\\
A_1(n) &= d_0(n)\,\varphi_1 + d_1(n)\,\varphi_0\\
A_2(n) &= d_0(n)\varphi_2 + d_1(n)\varphi_1 + d_2(n)\varphi_0\\
&\vdots
\end{aligned}\right\} \tag{B.77}$$

The use of Eq. (B.76) in conjunction with Eq. (B.72) gives

$$A_n = \frac{1}{2\pi i}\int_{C'} (\tau-\tau_1)^{-(n+1)}\sum_{h=0}^{\infty} A_h(n)\,(\tau-\tau_1)^h\,d\tau$$

from which it follows that

$$A_n = A_n(n) \tag{B.78}$$

which when used with Eq. (B.77) gives

$$\begin{aligned}
A_0 &= d_0(0)\,\varphi_0\\
A_1 &= d_0(1)\,\varphi_1 + d_1(1)\,\varphi_0\\
A_2 &= d_0(2)\,\varphi_2 + d_1(2)\,\varphi_1 + d_2(2)\,\varphi_0\\
&\vdots
\end{aligned} \tag{B.79}$$

Eq. (B.79) gives the coefficients to be used in Eq. (B.69), which, together with

$$\frac{d\ell^+}{dw} = \frac{1}{2\ell} \qquad \text{and} \qquad \frac{d\ell^-}{dw} = -\frac{1}{2\ell}$$

gives

$$\varphi(\tau^+, z)\frac{d\tau^+}{dw} = \frac{1}{2}\sum_{n=0}^{\infty} A_n l^{n-1}$$

Similarly

$$\varphi(\tau^-, z)\frac{d\tau^-}{dw} = \frac{1}{2}\sum_{n=0}^{\infty} A_n (-l)^{n-1}$$

so that $g(w)$ of Eq. (B.68) can be expressed as

$$\begin{aligned} g(w) &= \frac{1}{2}\sum_{n=0}^{\infty} A_n\left[(-l)^{n-1} - l^{n-1}\right] \\ &= \frac{1}{2}\sum_{n=0}^{\infty} A_{2n}\left[(-l)^{2n-1} - l^{2n-1}\right] + \frac{1}{2}\sum_{n=0}^{\infty} A_{2n+1}\left[(-l)^{2n} - l^{2n}\right] \\ &= -\sum_{n=0}^{\infty} A_{2n} l^{2n-1} = -\sum_{n=0}^{\infty} A_{2n} w^{n-\frac{1}{2}} \end{aligned}$$

or

$$g(w) = -\sum_{n=0}^{\infty} A_{2n} w^{\left(\frac{2n+1}{2}-1\right)} \tag{B.80}$$

It will now be shown that the function $g(w)$ satisfies the conditions prescribed by Watson's Lemma (cf. Ref. 15, p. 218). Since $g(w)$ is an analytic function (except for a branch point at the origin) it obviously has a circle of convergence. From inspection of Eqs. (B.74), (B.75) and (B.79) it can be seen that after n becomes large enough, the constants A_{2n} will become arbitrarily small because of the $(2n)!$ in the denominator of the d_k's. Let the largest coefficient A_{2n} be called K. Then

$$|g(w)| < K e^{w}$$

since every term of $g(w)$ is less than every term in the power series for the exponential. The two conditions required by Watson's Lemma[15] are then satisfied, and the integral of Eq. (B.67) can be written as (cf. Ref. 3, p. 218)

$$I_1^*(z) \sim e^{\sqrt{3}|z|^{3/2} f(\tau_1)} \frac{A_0 \pi^{-1/2}}{(\sqrt{3}|z|^{3/2})^{1/2}} \sum_{n=0}^{\infty} \frac{A_{2n}}{A_0} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{1}{(\sqrt{3}|z|^{3/2})^n} \tag{B.81}$$

The first few constants A_{2n} , for $n = 0$, 1 and 2 will now be determined by the use of Eqs. (B.74), (B.75) and (B.79).

<u>Determination of the Constants A_{2n}</u>

$$A_0 = \frac{(3\tau_1)^{-1/2}(-1)^{-1/2} e^{-i\frac{3}{4}\theta}}{\tau_1 + \frac{ia}{\sqrt{3z}}} = \frac{1}{i^{1/2}\, 3^{1/4} \frac{i^2}{\sqrt{3}} (1+az^{-1/2}) e^{i\frac{3}{4}\theta}}$$

$$= \frac{3^{1/4} e^{-i\frac{\pi}{4}}}{(1+az^{-1/2})\, e^{i\frac{3}{4}\theta}} \tag{B.82}$$

Similarly

$$\frac{A_2}{A_0} = \frac{1}{i\sqrt{3}\, e^{i\frac{3}{2}\theta}} \left[\frac{5}{8} + \frac{3}{2(1+az^{-1/2})} + \frac{3}{(1+az^{-1/2})^2} \right] \tag{B.83}$$

$$\frac{A_4}{A_0} = -\frac{1}{3e^{i3\theta}} \left[\frac{385}{384} + \frac{35}{16}\frac{1}{1+az^{-1/2}} + \frac{35}{8}\frac{1}{(1+az^{-1/2})^2} + \frac{15}{2}\frac{1}{(1+az^{-1/2})^3} + \frac{9}{(1+az^{-1/2})^4} \right] \tag{B.84}$$

Using Eqs. (B.52), (B.82), (B.83) and (B.84), Eq. (B.81) can be rewritten as

$$I_1^*(z) \sim \frac{\pi^{1/2}}{1+az^{-1/2}} z^{-3/4} e^{i(\frac{2}{3}z^{3/2} - \frac{\pi}{4})} \Bigg\{ 1 - \frac{i}{6z^{3/2}} \left[\frac{5}{8} + \frac{3}{2}\frac{1}{\gamma} + \frac{3}{\gamma^2} \right] \tag{B.85}$$

$$- \frac{1}{12}\frac{1}{z^3} \left(\frac{385}{384} + \frac{35}{16}\frac{1}{\gamma} + \frac{35}{8}\frac{1}{\gamma^2} + \frac{15}{2}\frac{1}{\gamma^3} + \frac{9}{\gamma^4} \right) + \cdots \Bigg\}$$

where

$$\gamma = 1 + a z^{-1/2} \tag{B.86}$$

Using the Binomial Theorem

$$\gamma^{-n} = \left(1 + a z^{-1/2}\right)^{-n} = \sum_{r=0}^{\infty} \frac{\Gamma(-n+1)\, a^r}{r!\,\Gamma(-n-r+1)} z^{-\frac{r}{2}} \qquad n>0 \tag{B.87}$$

which can be shown (cf. Ref. 4, p. 85, Eq. (14)) to be

$$\gamma^{-n} = \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(r+n)}{r!\,\Gamma(n)} z^{-\frac{r}{2}}$$

or

$$\gamma^{-n} = \sum_{r=0}^{\infty} (-1)^r \frac{(r+n-1)!\, a^r}{r!\,(n-1)!} z^{-\frac{r}{2}} \qquad n>0 \tag{B.88}$$

It will be assumed that the absolute value of z is large enough so that Eq. (B.88) is a convergent series. The various power of γ that enter in Eq. (B.85) can be evaluated from Eq. (B.88). $I_1^*(z)$ then becomes

$$I_1^*(z) \sim \frac{\pi^{1/2}}{1+a z^{-1/2}} z^{-3/4} e^{i\left(\frac{2}{3} z^{3/2} - \frac{\pi}{4}\right)} \left[1 - i \frac{41}{48} z^{-3/2} + i \frac{5}{4} a z^{-2} - i \frac{5}{4} a^2 z^{-5/2} + \left(\frac{9241}{4608} + i \frac{7}{4} a^3\right) z^{-3} + O\left(z^{-7/2}\right)\right] \tag{B.89}$$

which is the desired asymptotic expansion. It is seen that the leading term is identical with Eq. (B.47).

4. Calculation of the Viscous Solutions

4a. Detailed Calculation of φ_{31} .

From Eqs. (5.4), (B.21) and (B.22)(for φ_{31} , $y=0$)

$$\varphi_{31} = \frac{1}{\alpha}\left(\mathcal{L}_1 - \mathcal{L}_2\right)$$

$$= \frac{k}{\alpha\pi(\alpha R)^{1/3}} \Bigg[e^{\alpha(c+i\frac{\alpha}{R})} \Bigg\{ e^{-i\frac{\alpha}{(\alpha R)^{1/3}} z_{l_0}} \left[I_1^{*-}(z_{l_0}) + I_2^{*-}(z_{l_0})\right] - e^{-i\frac{\alpha}{(\alpha R)^{1/3}} z_{l_1}} \left[I_1^{*-}(z_{l_1}) + I_P^{-}(z_{l_1})\right] \Bigg\}$$

$$- e^{-\alpha(c+i\frac{\alpha}{R})} \Bigg\{ e^{i\frac{\alpha}{(\alpha R)^{1/3}} z_{l_0}} \left[I_1^{*+}(z_{l_0}) + I_2^{*+}(z_{l_0}) - I_P^{+}(z_{l_0})\right] - e^{i\frac{\alpha}{(\alpha R)^{1/3}} z_{l_1}} \left[I_1^{*+}(z_{l_1})\right] \Bigg\} \Bigg]$$

where a + or − superscript indicates that a is, respectively, positive or negative. In this case $a = \pm \frac{\alpha}{(\alpha R)^{1/3}}$. It turns out that in the case of φ_{31} it is sufficient to use the leading term of the asymptotic expansions, i.e. using Eqs. (B.47), (B.48) and (B.49) in the above equation, one obtains:

$$\varphi_{31} = \frac{k}{\alpha\pi(\alpha R)^{1/3}} \Bigg[e^{\alpha(c+i\frac{\alpha}{R})} \Bigg\{ e^{-i\frac{\alpha}{(\alpha R)^{1/3}} z_{l_0}} \Bigg[\frac{\pi^{1/2}}{1-\frac{\alpha}{(\alpha R)^{1/3}} z_{l_0}^{-1/2}} z_{l_0}^{-3/4} e^{i\left(\frac{2}{3} z_{l_0}^{3/2} - \frac{\pi}{4}\right)} + \frac{\pi^{1/2}}{1+\frac{\alpha}{(\alpha R)^{1/3}} z_{l_0}^{-1/2}} z_{l_0}^{-3/4} e^{-i\left(\frac{2}{3} z_{l_0}^{3/2} - \frac{\pi}{4}\right)} \Bigg]$$

$$- e^{-i\frac{\alpha}{(\alpha R)^{1/3}} z_{l_1}} \Bigg[\frac{\pi^{1/2}}{1-\frac{\alpha}{(\alpha R)^{1/3}} z_{l_1}^{-1/2}} z_{l_1}^{-3/4} e^{i\left(\frac{2}{3} z_{l_1}^{3/2} - \frac{\pi}{4}\right)} + 2\pi e^{-i\left(\frac{\alpha^2}{3R} - \frac{\alpha}{(\alpha R)^{1/3}} z_{l_1} - \frac{\pi}{2}\right)} \Bigg] \Bigg\}$$

$$- e^{-\alpha(c+i\frac{\alpha}{R})} \Bigg\{ e^{i\frac{\alpha}{(\alpha R)^{1/3}} z_{l_0}} \Bigg[\frac{\pi^{1/2}}{1+\frac{\alpha}{(\alpha R)^{1/3}} z_{l_0}^{-1/2}} z_{l_0}^{-3/4} e^{i\left(\frac{2}{3} z_{l_0}^{3/2} - \frac{\pi}{4}\right)} + \frac{\pi^{1/2}}{1-\frac{\alpha}{(\alpha R)^{1/3}} z_{l_0}^{-3/2}} z_{l_0}^{-3/4} e^{-i\left(\frac{2}{3} z_{l_0}^{3/2} - \frac{\pi}{4}\right)}$$

$$- 2\pi e^{i\left[\frac{\alpha^2}{\alpha R} - \frac{\alpha}{(\alpha R)^{1/3}} z_{l_0} + \frac{\pi}{2}\right]} \Bigg] - e^{i\frac{\alpha}{(\alpha R)^{1/3}} z_{l_1}} \frac{\pi^{1/2}}{1+\frac{\alpha}{(\alpha R)^{1/3}} z_{l_1}^{-1/2}} z_{l_1}^{-3/4} e^{i\left(\frac{2}{3} z_{l_1}^{3/2} - \frac{\pi}{4}\right)} \Bigg\} \Bigg]$$

which on regrouping becomes

$$
\begin{aligned}
\varphi_{31} = \frac{k}{\alpha \pi^{1/2} (\alpha R)^{1/3}} \Bigg\{ & z_{l_0}^{-3/4} e^{i\left(\frac{2}{3} z_{l_0}^{3/2} - \frac{\pi}{2}\right)} \left[\frac{\exp\left\{\alpha c - i\left[\frac{\alpha}{(\alpha R)^{1/3}} z_{l_0} - \frac{\alpha^2}{R}\right]\right\}}{1 - \frac{\alpha}{(\alpha R)^{1/3}} z_{l_0}^{-1/2}} - \frac{\exp\left\{-\alpha c + i\left[\frac{\alpha}{(\alpha R)^{1/3}} z_{l_0} - \frac{\alpha^2}{R}\right]\right\}}{1 + \frac{\alpha}{(\alpha R)^{1/3}} z_{l_0}^{-1/2}} \right] \\
& + z_{l_0}^{-3/4} e^{-i\left(\frac{2}{3} z_{l_0}^{3/2} - \frac{\pi}{4}\right)} \left[\frac{\exp\left\{\alpha c - i\left[\frac{\alpha}{(\alpha R)^{1/3}} z_{l_0} - \frac{\alpha^2}{R}\right]\right\}}{1 + \frac{\alpha}{(\alpha R)^{1/3}} z_{l_0}^{-1/2}} - \frac{\exp\left\{-\alpha c + i\left[\frac{\alpha}{(\alpha R)^{1/3}} z_{l_0} - \frac{\alpha^2}{R}\right]\right\}}{1 - \frac{\alpha}{(\alpha R)^{1/3}} z_{l_0}^{-1/2}} \right] \\
& + z_{l_1}^{-3/4} e^{i\left(\frac{2}{3} z_{l_1}^{3/2} - \frac{\pi}{4}\right)} \left[\frac{\exp\left\{-\alpha c + i\left[\frac{\alpha}{(\alpha R)^{1/3}} z_{l_1} - \frac{\alpha^2}{R}\right]\right\}}{1 + \frac{\alpha}{(\alpha R)^{1/3}} z_{l_1}^{-1/2}} - \frac{\exp\left\{\alpha c - i\left[\frac{\alpha}{(\alpha R)^{1/3}} z_{l_1} - \frac{\alpha^2}{R}\right]\right\}}{1 - \frac{\alpha}{(\alpha R)^{1/3}} z_{l_1}^{-1/2}} \right] \\
& + 2\pi^{1/2} \left[\exp\left\{ i\left(\frac{\alpha^2}{R} + \frac{\pi}{2} - \frac{\alpha^2}{R}\right) - \alpha c \right\} - \exp\left\{ -i\left(\frac{\alpha^2}{3R} - \frac{\pi}{2} - \frac{\alpha^2}{R}\right) + \alpha c \right\} \right] \Bigg\}
\end{aligned}
\qquad \text{(B.90)}
$$

Assuming that $\alpha R \gg 1$ so that

$$\left[1 + \frac{\alpha}{(\alpha R)^{1/3}} z^{-1/2} \right]^{-1} \sim 1 - \frac{\alpha}{(\alpha R)^{1/3}} z^{-1/2}$$

and using Eq. (B.12) for z_{l_0}

$$i \frac{\alpha}{(\alpha R)^{1/3}} z_{l_0} = i \frac{\alpha^2}{R} + \alpha c$$

so that

$$-\alpha c + i\left[\frac{\alpha}{(\alpha R)^{1/3}} z_{\ell_0} - \frac{\alpha^2}{R}\right] = 0$$

Similarly, using Eq. (B.13)

$$i\frac{\alpha}{(\alpha R)^{1/3}} z_{\ell_1} = i\frac{\alpha^2}{R} - \alpha(1-c)$$

and

$$-\alpha c + i\left[\frac{\alpha}{(\alpha R)^{1/3}} z_{\ell_1} - \frac{\alpha^2}{R}\right] = -\alpha c + i\frac{\alpha^2}{R} - \alpha(1-c) - i\frac{\alpha^2}{R} = -\alpha$$

The preceding relationships, when inserted in Eq. (B.90) lead to

$$\varphi_{31} = -\frac{2k\, e^{-i\frac{\pi}{4}}}{\alpha \pi^{1/2} (\alpha R)^{1/2}} \left\{ \frac{1}{2} z_{\ell_1}^{-3/4} e^{i\frac{2}{3} z_{\ell_1}^{3/2}} \left(e^{\alpha} - e^{-\alpha}\right) + z_{\ell_1}^{-5/4} e^{i\frac{2}{3} z_{\ell_1}^{3/2}} \frac{\alpha}{(\alpha R)^{1/3}} \left(e^{\alpha} - e^{-\alpha}\right) \right.$$

$$\left. + z_{\ell_0}^{-5/4} \frac{\alpha}{(\alpha R)^{1/3}} \left[e^{-i\left(\frac{2}{3} z_{\ell_0}^{3/2} - \frac{\pi}{2}\right)} - e^{i\left(\frac{2}{3} z_{\ell_0}^{3/2}\right)} \right] + \pi^{1/2} \left[e^{\alpha c + i\left(\frac{2}{3}\frac{\alpha^2}{R} + \frac{3}{4}\pi\right)} - e^{-\alpha c - i\left(\frac{2}{3}\frac{\alpha^2}{R} - \frac{3}{4}\pi\right)} \right] \right\}$$

and keeping the largest order term

$$\varphi_{31} = \frac{2(12)^{1/6} e^{-i\frac{5}{12}\pi}}{\pi^{1/2} (\alpha R)^{2/3}} \left[z_{\ell_0}^{-5/4} e^{i\frac{2}{3} z_{\ell_0}^{3/2}} + O(\alpha R)^{1/3} \right] \tag{B.91}$$

where the definition of k given by Eq. (1.14) has been used.

4b. Comments on the Other Necessary Viscous Solutions

No details will be given of the calculation of $\varphi_{31}', \varphi_{41}, \varphi_{41}', \Phi_{32}$ and Φ_{32}' , except that in the evaluation of $\varphi_{41}, \varphi_{41}', \Phi_{32}$ and Φ_{32}' , more than the leading terms in the expansion of the integrals should be used.

5. Accuracy of Calculations

The accuracy of our calculations, based upon the viscous functions given in Section 6, Part II, is restricted by the fact that αR was assumed large. This was interpreted as meaning that Eq. (6.4) is a good approximation to Eqs. (B.12), (B.13) and (B.14), and permitted the use of asymptotic methods. Terms of $O\left(\frac{1}{\alpha R}\right)$ were neglected compared to terms of $O\left(\frac{1}{\sqrt{\alpha R}}\right)$ in the secular equation . A check on the assumption mentioned was done for each calculation of a point on the neutral stability curve. The result of this checking showed that the assumption was incorrect for small values of τ , shown as dotted on Fig. 7 only. Nevertheless, the trends in that region are probably correct. This is inferred by comparing the dotted sections with the curve for $m=50$ which is valid over almost all of the region shown: they seem to form a reasonable family.

APPENDIX C

CALCULATION OF THE NEUTRAL STABILITY CURVES

1. Expansion of the Secular Determinant

Using Laplace's method of expansion[17], the determinant of Eq. (5.14) can be put in the form of a sum of products, each product consisting of two third order determinants, the first and second including, respectively, only inviscid or viscous functions, within the restriction mentioned in parenthesis after Eq. (5.14):

$$
\begin{aligned}
&\underset{(1)}{\begin{vmatrix} \Lambda_1 & V_1 & 0 \\ \Lambda_2 & V_2 & 0 \\ \Lambda_3 & V_3 & T_3 \end{vmatrix}} \begin{vmatrix} 0 & 0 & c_4 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{vmatrix}
- \underset{(2)}{\begin{vmatrix} \Lambda_1 & V_1 & 0 \\ \Lambda_2 & V_2 & 0 \\ \Lambda_4 & V_4 & T_4 \end{vmatrix}} \begin{vmatrix} 0 & 0 & c_3 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{vmatrix} \\
&+ \underset{(3)}{\begin{vmatrix} \Lambda_1 & V_1 & 0 \\ \Lambda_2 & V_2 & 0 \\ \Lambda_5 & V_5 & T_5 \end{vmatrix}} \begin{vmatrix} 0 & 0 & c_3 \\ 0 & 0 & c_4 \\ a_6 & b_6 & c_6 \end{vmatrix}
- \underset{(4)}{\begin{vmatrix} \Lambda_1 & V_1 & 0 \\ \Lambda_2 & V_2 & 0 \\ \Lambda_6 & V_6 & T_6 \end{vmatrix}} \begin{vmatrix} 0 & 0 & c_3 \\ 0 & 0 & c_4 \\ a_5 & b_5 & c_5 \end{vmatrix} \\
&+ \underset{(5)}{\begin{vmatrix} \Lambda_1 & V_1 & 0 \\ \Lambda_3 & V_3 & T_3 \\ \Lambda_4 & V_4 & T_4 \end{vmatrix}} \begin{vmatrix} a_2 & b_2 & 0 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{vmatrix}
- \underset{(6)}{\begin{vmatrix} \Lambda_1 & V_1 & 0 \\ \Lambda_3 & V_3 & T_3 \\ \Lambda_5 & V_5 & T_5 \end{vmatrix}} \begin{vmatrix} a_2 & b_2 & 0 \\ 0 & 0 & c_4 \\ a_6 & b_6 & c_6 \end{vmatrix} \\
&+ \underset{(7)}{\begin{vmatrix} \Lambda_1 & V_1 & 0 \\ \Lambda_3 & V_3 & T_3 \\ \Lambda_6 & V_6 & T_6 \end{vmatrix}} \begin{vmatrix} a_2 & b_2 & 0 \\ 0 & 0 & c_4 \\ a_5 & b_5 & c_5 \end{vmatrix}
+ \underset{(8)}{\begin{vmatrix} \Lambda_1 & V_1 & 0 \\ \Lambda_4 & V_4 & T_4 \\ \Lambda_5 & V_5 & T_5 \end{vmatrix}} \begin{vmatrix} a_2 & b_2 & 0 \\ 0 & 0 & c_3 \\ a_6 & b_6 & c_6 \end{vmatrix}
\end{aligned}
\qquad \text{(C.1)}
$$

continues on next page

$$
\begin{aligned}
& - \underset{(9)}{\begin{vmatrix} \Lambda_1 & V_1 & 0 \\ \Lambda_4 & V_4 & T_4 \\ \Lambda_6 & V_6 & T_6 \end{vmatrix}} \begin{vmatrix} a_2 & b_2 & 0 \\ 0 & 0 & c_3 \\ a_5 & b_5 & c_5 \end{vmatrix} + \underset{(10)}{\begin{vmatrix} \Lambda_1 & V_1 & 0 \\ \Lambda_5 & V_5 & T_5 \\ \Lambda_6 & V_6 & T_6 \end{vmatrix}} \begin{vmatrix} a_2 & b_2 & 0 \\ 0 & 0 & c_3 \\ 0 & 0 & c_4 \end{vmatrix} \\
& - \underset{(11)}{\begin{vmatrix} \Lambda_2 & V_2 & 0 \\ \Lambda_3 & V_3 & T_3 \\ \Lambda_4 & V_4 & T_4 \end{vmatrix}} \begin{vmatrix} a_1 & b_1 & 0 \\ a_5 & b_5 & c_5 \\ a_6 & b_6 & c_6 \end{vmatrix} + \underset{(12)}{\begin{vmatrix} \Lambda_2 & V_2 & 0 \\ \Lambda_3 & V_3 & T_3 \\ \Lambda_5 & V_5 & T_5 \end{vmatrix}} \begin{vmatrix} a_1 & b_1 & 0 \\ 0 & 0 & c_4 \\ a_6 & b_6 & c_6 \end{vmatrix} \\
& - \underset{(13)}{\begin{vmatrix} \Lambda_2 & V_2 & 0 \\ \Lambda_3 & V_3 & T_3 \\ \Lambda_6 & V_6 & T_6 \end{vmatrix}} \begin{vmatrix} a_1 & b_1 & 0 \\ 0 & 0 & c_4 \\ a_5 & b_5 & c_5 \end{vmatrix} - \underset{(14)}{\begin{vmatrix} \Lambda_2 & V_2 & 0 \\ \Lambda_4 & V_4 & T_4 \\ \Lambda_5 & V_5 & T_5 \end{vmatrix}} \begin{vmatrix} a_1 & b_1 & 0 \\ 0 & 0 & c_3 \\ a_6 & b_6 & c_6 \end{vmatrix} \\
& + \underset{(15)}{\begin{vmatrix} \Lambda_2 & V_2 & 0 \\ \Lambda_4 & V_4 & T_4 \\ \Lambda_6 & V_6 & T_6 \end{vmatrix}} \begin{vmatrix} a_1 & b_1 & 0 \\ 0 & 0 & c_3 \\ a_5 & b_5 & c_3 \end{vmatrix} - \underset{(16)}{\begin{vmatrix} \Lambda_2 & V_2 & 0 \\ \Lambda_5 & V_5 & T_5 \\ \Lambda_6 & V_6 & T_6 \end{vmatrix}} \begin{vmatrix} a_1 & b_1 & 0 \\ 0 & 0 & c_3 \\ 0 & 0 & c_4 \end{vmatrix} \\
& + \underset{(17)}{\begin{vmatrix} \Lambda_3 & V_3 & T_3 \\ \Lambda_4 & V_4 & T_4 \\ \Lambda_5 & V_5 & T_5 \end{vmatrix}} \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_6 & b_6 & c_6 \end{vmatrix} - \underset{(18)}{\begin{vmatrix} \Lambda_3 & V_3 & T_3 \\ \Lambda_4 & V_4 & T_4 \\ \Lambda_6 & V_6 & T_6 \end{vmatrix}} \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_5 & b_5 & c_5 \end{vmatrix} \\
& + \underset{(19)}{\begin{vmatrix} \Lambda_3 & V_3 & T_3 \\ \Lambda_5 & V_5 & T_5 \\ \Lambda_6 & V_6 & T_6 \end{vmatrix}} \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & c_4 \end{vmatrix} - \underset{(20)}{\begin{vmatrix} \Lambda_4 & V_4 & T_4 \\ \Lambda_5 & V_5 & T_5 \\ \Lambda_6 & V_6 & T_6 \end{vmatrix}} \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & c_3 \end{vmatrix} = 0
\end{aligned}
\tag{C.1 contd.}
$$

In what follows, the circled numbers stand for the inviscid part of each term of (C.1), including its algebraic sign, i.e. $\textcircled{4} \equiv \begin{vmatrix} \Lambda_1 & V_1 & 0 \\ \Lambda_2 & V_2 & 0 \\ \Lambda_6 & V_6 & T_6 \end{vmatrix}$. Eq. (C.1) can be rewritten as

$$\textcircled{1}\left[c_4(a_5b_6-a_6b_5)\right]+\textcircled{2}\left[c_3(a_5b_6-a_6b_5)\right]+\textcircled{5}\left[c_6(a_2b_5-a_5b_2)-c_5(a_2b_6-a_6b_2)\right]$$

$$+\textcircled{6}\left[c_4(a_6b_2-a_2b_6)\right]+\textcircled{7}\left[c_4(a_5b_2-a_2b_5)\right]+\textcircled{8}\left[c_3(a_6b_2-a_2b_6)\right]$$

$$+\textcircled{9}\left[c_3(a_5b_2-a_2b_5)\right]+\textcircled{11}\left[c_5(a_6b_1-a_1b_6)+c_6(a_1b_5-a_5b_1)\right]+\textcircled{12}\left[c_4(a_6b_1-a_1b_6)\right]$$

$$+\textcircled{13}\left[c_4(a_5b_1-a_1b_5)\right]+\textcircled{14}\left[c_3(a_6b_1-a_1b_6)\right]+\textcircled{15}\left[c_3(a_5b_1-a_1b_5)\right]$$

$$+\textcircled{17}\left[c_6(a_1b_2-a_2b_1)\right]+\textcircled{18}\left[c_5(a_1b_2-a_2b_1)\right]+\textcircled{19}\left[c_4(a_1b_2-a_2b_1)\right]$$

$$+\textcircled{20}\left[c_3(a_1b_2-a_2b_1)\right]=0 \qquad \text{(C.2)}$$

The use of Eqs. (5.15), (6.8) through (6.14), and (C.1) in conjunction with Eq. (C.2), indicates that this last equation can be rewritten, after division by $\sqrt{\alpha R}$ as

$$\left\{H_0(\alpha,c)+H_1(\alpha,c)\frac{1}{\sqrt{\alpha R}}+H_2(\alpha,c)\frac{1}{\alpha R}+O\left[(\alpha R)^{-3/2}\right]\right\}\exp\left\{i\frac{2}{3}(\alpha R)^{1/2}\left[c^{3/2}e^{-i\frac{3}{4}\pi}-(1-c)^{3/2}e^{i\frac{3}{4}\pi}\right]\right\}$$

$$+J_0(\alpha,c)+J_1(\alpha,c)\frac{1}{\sqrt{\alpha R}}+J_2(\alpha,c)\frac{1}{\alpha R}+O\left[(\alpha R)^{-3/2}\right]=0 \qquad \text{(C.3)}$$

where the H 's and J 's are complex functions of α and c only, provided the physical properties of the fluids, i.e. r, m, F and W are fixed. Since for neutral stability, c is real, and the case of interest is $\alpha R \gg 1$, Eq. (C.3) will be reduced, in order to solve the problem, to the bare essentials, and still keep the Reynolds number of dependence, i.e.

$$H_0(\alpha, c) + H_1(\alpha, c)(\alpha R)^{-1/2} = 0 \tag{C.4}$$

or

$$\Big[-\textcircled{5}\, a_{21} b_{61} c_{51} - \textcircled{7}\, a_{21} b_{51} c_{41} - \textcircled{18}\, a_{21} b_{11} c_{51}\Big] + \Big\{\textcircled{5}\, a_{21}(b_{51} c_{51} - b_{61} c_{52} - b_{62} c_{51})$$

$$- \textcircled{6}\, a_{21} b_{61} c_{41} - \textcircled{7}\, a_{21}(b_{51} c_{42} + b_{52} c_{41}) - \textcircled{9}\, a_{21} b_{51} c_{31} - \textcircled{11}\, a_{11} b_{61} c_{51}$$

$$- \textcircled{13}\, a_{11} b_{51} c_{41} + \textcircled{18}\big[a_{11} b_{21} c_{51} - a_{21}(b_{11} c_{52} + b_{12} c_{51})\big] - \textcircled{19}\, a_{21} b_{11} c_{41}\Big\}(\alpha R)^{-1/2} = 0 .$$

which* when multiplied by $-\dfrac{e^{-i\frac{\pi}{2}}(1-c)^{1/2}}{a_{21}\, r^{1/2}}$ and after using Eqs. (6.8) through (6.14) gives

$$-\frac{1+i}{\sqrt{2}}\left[\textcircled{5}(1-c) - \textcircled{7}\left(\frac{m}{r}\right)^{1/2} + \textcircled{18}\,\frac{\sinh\alpha}{\alpha}\right]$$

$$+\Bigg\{\textcircled{5}\,\frac{1}{(1-c)^{1/2}}\left[\frac{5}{4}\, m\left(\frac{m}{r}\right)^{1/2} + \frac{17}{48} + \frac{1}{F(1-c)}\right] + \textcircled{6}\left(\frac{m}{r}\right)^{1/2}(1-c)^{1/2}$$

$$-\textcircled{7}\left[\frac{1}{48}\left(\frac{m}{r}\right)^{1/2}\frac{1}{(1-c)^{3/2}}\right]\left[41\left(\frac{m}{r}\right)^{1/2} + 5\right] - \textcircled{9}\,\frac{1}{r}\,\frac{1}{(1-c)^{1/2}} + \textcircled{11}\, i\,\frac{(1-c)}{c^{1/2}} - \textcircled{13}\, i\left(\frac{m}{r}\right)^{1/2}\frac{1}{c^{1/2}}$$

$$+\textcircled{18}\left\{\frac{5}{4}\,\frac{\sinh\alpha}{\alpha}\left(\frac{m}{r}\right)^{1/2}\frac{m}{(1-c)^{3/2}} + \cosh\alpha\left[\frac{1}{(1-c)^{1/2}} + \frac{i}{c^{1/2}}\right]\right\}$$

$$+\textcircled{19}\,\frac{\sinh\alpha}{\alpha}\left(\frac{m}{r}\right)^{1/2}\frac{1}{(1-c)^{1/2}}\Bigg\}(\alpha R)^{-1/2} = 0 \tag{C.5}$$

---

* The meaning of the symbols is given by Eqs. (6.8) through (6.14).

Equating to zero the real and imaginary parts of Eq. (C.5), yields, respectively

$$\mathcal{F}(\alpha, c) + (\alpha R)^{-1/2}\, \mathcal{G}(\alpha, c) = 0 \tag{C.6}$$

$$\mathcal{F}(\alpha, c) + (\alpha R)^{-1/2}\, \mathcal{H}(\alpha, c) = 0 \tag{C.7}$$

where

$$\mathcal{F}(\alpha, c) = -\frac{1}{\sqrt{2}}\left[⑤(1-c) - ⑦\left(\frac{m}{r}\right)^{1/2} + ⑱\,\frac{\sinh\alpha}{\alpha}\right] \tag{C.8}$$

$$\begin{aligned}\mathcal{G}(\alpha, c) = {} & \frac{⑤}{(1-c)^{1/2}}\left[\frac{5}{4}\left(\frac{m}{r}\right)^{1/2} m + \frac{17}{48} + \frac{1}{F(1-c)}\right] + ⑥\left(\frac{m}{r}\right)^{1/2}(1-c)^{1/2} \\ & - ⑦\left(\frac{m}{r}\right)^{1/2}\frac{1}{48(1-c)^{3/2}}\left[5 + 41\left(\frac{m}{r}\right)^{1/2}\right] - ⑨\,\frac{1}{r(1-c)^{1/2}} \\ & + ⑱\,\frac{1}{(1-c)^{1/2}}\left[\frac{5}{4}\,\frac{m}{1-c}\left(\frac{m}{r}\right)^{1/2}\frac{\sinh\alpha}{\alpha} + \cosh\alpha\right] + ⑲\left(\frac{m}{r}\right)^{1/2}\frac{1}{(1-c)^{1/2}}\,\frac{\sinh\alpha}{\alpha}\end{aligned} \tag{C.9}$$

$$\mathcal{H}(\alpha, c) = \frac{1}{\sqrt{c}}\left[⑪(1-c) - ⑬\left(\frac{m}{r}\right)^{1/2} + ⑱\cosh\alpha\right] \tag{C.10}$$

and from the statement following Eq. (C.1)

$$\left.\begin{aligned} ⑤ &= -2\alpha \\ ⑥ &= 4\alpha^2 e^{-\alpha}\left(1 - \frac{1}{m}\right)\sinh\alpha \\ ⑦ &= e^{-\alpha}(\Lambda_6 - V_6) + 2T_6 \sinh\alpha \\ ⑨ &= -\alpha\left[(V_6 - \Lambda_6)e^{-\alpha} + 2T_6\cosh\alpha\right] \end{aligned}\right\} \tag{C.11}$$

$$
\left.
\begin{aligned}
&⑪ = -2\alpha^2 \\
&⑬ = -\alpha\left[(\Lambda_6+V_6)e^{-\alpha}-2T_6\cosh\alpha\right] \\
&⑱ = -2\alpha\left(T_6-\Lambda_6\right) \\
&⑲ = 2\alpha\left(1-\frac{1}{m}\right)\left(V_6 e^{-2\alpha}-\Lambda_6\right) \\
&\Lambda_6 = \left[\frac{\alpha^2}{W}+\frac{1}{F(1-c)}+1+(1-c)\alpha\right]e^{-\alpha} \\
&V_6 = \left[\frac{\alpha^2}{W}+\frac{1}{F(1-c)}+1-(1-c)\alpha\right]e^{\alpha} \\
&T_6 = \left[m+\frac{1}{F(1-c)}+(1-c)\alpha\right]r\, e^{-\alpha}
\end{aligned}
\right\}
\qquad \text{(C.11) (contd.)}
$$

Eqs. (C.6) and (C.7) are the fundamental equations for determining the neutral stability lines in the αR-plane. Conceptually, the procedure to be followed is extremely simple: subtract Eq. (C.7) from (C.6) and obtain, for αR finite and not zero,

$$\mathcal{G}(\alpha,c) - \mathcal{H}(\alpha,c) = 0 \qquad \text{(C.12)}$$

which can be solved numerically for c as

$$c = c(\alpha) \qquad \text{(C.13)}$$

The solution has to be obtained numerically because of the impossibility of solving explicitly for c or α in terms of α or c respectively. Once the value of c that goes with a value of α is known, $(\alpha R)^{1/2}$ becomes, from Eq. (C.7)

$$(\alpha R)^{1/2} = -\frac{\mathcal{H}(\alpha,c)}{\mathcal{F}(\alpha,c)} \qquad \text{(C.14)}$$

from which R can be obtained.

In the case of $R \to \infty$, Eqs. (C.6) and (C.7) reduce to

$$\mathfrak{F}(\alpha, c) = 0 \tag{C.15}$$

2. The Eigenvalue Problem for $\alpha R = \infty$.

It was just shown how the case of inviscid flow can be investigated by taking the limit of the secular determinant, Eq. (5.14) as $R \to \infty$. A different approach would be to neglect viscosity at the outset, i. e. in the Orr-Sommerfeld equations. The differential equations, for each fluid, then become, from Eqs. (2.3) and (2.4)

$$(U_\ell - c)(\varphi'' - \alpha^2 \varphi) = 0 \qquad 0 \leq y \leq 1 \tag{C.16}$$

$$(U_g - c)(\Phi'' - \alpha^2 \Phi) = 0 \qquad 1 \leq y \leq \infty \tag{C.17}$$

with the following boundary conditions

at the wall $(y=0)$: $v_\ell = 0$

at the interface $(y=1)$: $v_\ell = v_g$ and the pressures are discontinuous due to surface tension forces;

and at infinity $(y=\infty)$: $v_g = 0$.

These boundary conditions, in terms of φ and Φ , become, respectively

$$\varphi(0) = 0 \tag{C.18}$$

$$\varphi(1) = \Phi(1) \tag{C.19}$$

$$\left[\frac{\alpha^2}{W}+1+\frac{1}{F(1-c)}\right]\varphi(1)-(1-c)\varphi'(1)=r\left\{\left[m+\frac{1}{F(1-c)}\right]\Phi(1)-(1-c)\Phi'(1)\right\} \tag{C.20}$$

$$\Phi(\infty)=0 \tag{C.21}$$

The solutions of Eqs. (C.16) and (C.17) are

$$\varphi=k_1 e^{-\alpha y}+k_2 e^{\alpha y} \tag{C.22}$$

$$\Phi=K_1 e^{-\alpha y} \tag{C.23}$$

where Eq. (C.21) has been used.

The use of Eqs. (C.22) and (C.23) in (C.18) to (C.20) leads to

$$\left.\begin{aligned}
&k_1+k_2=0\\
&k_1 e^{-\alpha}+k_2 e^{\alpha}-K_1 e^{-\alpha}=0\\
&k_1\left[\frac{\alpha^2}{W}+1+\frac{1}{F(1-c)}+(1-c)\alpha\right]e^{-\alpha}+k_2\left[\frac{\alpha^2}{W}+1+\frac{1}{F(1-c)}-(1-c)\alpha\right]e^{\alpha}\\
&\qquad -K_1 r\left[m+\frac{1}{F(1-c)}+(1-c)\alpha\right]e^{-\alpha}=0
\end{aligned}\right\} \tag{C.24}$$

If Eq. (C.24) is to have a non-trivial solution, the following equation must be satisfied

$$\begin{vmatrix}
1 & -1 & 0\\
e^{-\alpha} & e^{\alpha} & -e^{-\alpha}\\
\left[\frac{\alpha^2}{W}+\frac{1}{F(1-c)}+1+(1-c)\alpha\right]e^{-\alpha} & \left[\frac{\alpha^2}{W}+\frac{1}{F(1-c)}+1-(1-c)\alpha\right]e^{\alpha} & -r\left[m+\frac{1}{F(1-c)}+(1-c)\alpha\right]e^{-\alpha}
\end{vmatrix}=0 \tag{C.25}$$

Multiplication of the last column by -1 and the use of Eqs. (5.13) and (5.15) show that Eq. (C.25) is equivalent to

$$\begin{vmatrix} \Lambda_1 & V_1 & 0 \\ \Lambda_3 & V_3 & T_3 \\ \Lambda_6 & V_6 & T_6 \end{vmatrix} = 0 \qquad (C.26)$$

the left hand member being identical with ⑦ * of Eq. (C.1), i.e., the equation for neutral stability is now

$$⑦ = 0 \qquad (C.27)$$

while in section 7, Eq. (7.1) (cf. Eq. (C.8)), it was

$$⑤(1-c) - ⑦\left(\frac{m}{r}\right)^{1/2} + ⑱\,\frac{\sinh\alpha}{\alpha} = 0 \qquad (C.28)$$

Before discussing this discrepancy, we will do a few detailed calculations using Eq. (C.27). More information will then be available for the discussion.

Eq. (C.27), or its equivalent, Eq. (C.25) can be put in the form

$$\left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right] - r\left[m + \frac{1}{F(1-c)} + (1-c)\alpha\right] - \frac{(1-c)\,\alpha}{\tanh\alpha} = 0 \qquad (C.29)$$

Let

$$1 - c = \epsilon \qquad (C.30)$$

and notice that when $F = \infty$, $\epsilon = 0$ is not a root of Eq. (C.29). The use

---

* For the exact meaning of the circled numbers cf. note following Eq. (C.1).

of Eq. (C.30), permits Eq. (C.29) to be written, after multiplying throughout by ϵ, as

$$\epsilon^2 \alpha \left(\frac{1}{\tanh \alpha} + r\right) - \epsilon \left(\frac{\alpha^2}{W} + 1 - rm\right) - \frac{1-r}{F} = 0$$

which is a quadratic equation in ϵ. Solving for ϵ yields two roots, one being extraneous, introduced when multiplying throughout by ϵ. The result is, in terms of c

$$c = 1 - \frac{\left(\frac{\alpha^2}{W} + 1 - rm\right) + \left\{\left(\frac{\alpha^2}{W} + 1 - rm\right)^2 + 4\alpha\left(\frac{1}{\tanh\alpha} + r\right)\frac{1-r}{F}\right\}^{1/2}}{2\alpha\left(\frac{1}{\tanh\alpha} + r\right)} \tag{C.31}$$

Here again, as in Eq. (7.2) c is always real, and for $rm < 1$, less than unity. A few special cases will now be treated and results will be re-written for convenience, together with similar results calculated from the equations of Section 7. They will be distinguished by using $R = \infty$ and $R \rightarrow \infty$ respectively.

$\underline{F = W = \infty}$

$$c = 1 - \frac{1-rm}{\alpha} \; \frac{1}{q(\alpha, r, m) + r} \qquad \alpha R \rightarrow \infty \tag{C.32a}$$

$$c = 1 - \frac{1-rm}{\alpha} \; \frac{1}{\coth \alpha + r} \qquad \alpha R = \infty \tag{C.32b}$$

$F = W = \infty \; ; \; r \to 0$

$$c = 1 - \frac{1}{\alpha\, g(\alpha, r, m)}\left[1 - \left(m + \frac{1}{g(\alpha, r, m)}\right) r\right] \qquad \alpha R \to \infty \qquad (C.33a)$$

$$c = 1 - \frac{\tanh \alpha}{\alpha}\left[1 - \left(m + \tanh\alpha\right) r\right] \qquad \alpha R = \infty \qquad (C.33b)$$

$F = W = \infty \; ; \; m/r \gg 1 \; ; \; \alpha \to 0$

$$c = 0 \qquad \alpha R \to \infty \qquad (C.34a)$$

$$c = \frac{\alpha^2}{3} \qquad \alpha R = \infty \qquad (C.34b)$$

$F = W = \infty \; ; \; m/r \gg 1 \; ; \; \alpha \to \infty$

$$c = 1 - \frac{1}{\alpha} \qquad \alpha R \to \infty \qquad (C.35a)$$

$$c = 1 - \frac{1}{\alpha} \qquad \alpha R = \infty \qquad (C.35b)$$

Eqs. (C.34) and (C.35) indicate that the results for some special, extreme cases, are identical independent of the limiting process used. The agreement would be perfect, for any value of the parameters involved, if $g(\alpha, r, m)$ (cf. Eqs. (7.2), (7.3) and (C.31)) would satisfy the equation

$$g(\alpha, r, m) = \cot \alpha$$

which here holds only for $\frac{m}{r} \gg 1$

Mathematically, the fact that Eqs. (8.12) and (8.13) are not identical is not surprising, since there is no reason for them to be so, because the limiting processes used for obtaining each result was different. Let's recall them: in the case of Eq. (C.27) the viscous forces were considered negligible at the outset, and the differential equations changed from fourth to second order. This required the relaxing of four boundary conditions: slip was allowed, a) at the wall, b) at the interface, and c) at infinity; and d) at the interface, no condition is imposed on the shear (which vanishes here because the fluid is inviscid). In the case of Eq. (C.28), the Reynolds number was made to approach infinity, in the final neutral stability equation (cf. Eqs. (C.6) and (C.7)).

From a physical point of view, the disagreement is inacceptable; no answer has been found as yet.

From a practical standpoint, the discrepancy is immaterial, since for fluids of interest, liquid-gas combinations, $\frac{m}{r} >> 1$, both results agree.

APPENDIX D

DERIVATION OF EQUATION FOR $\frac{\partial c_i}{\partial R}$

The equation for neutral stability could be written in general as

$$P(\alpha, R, c, r, m, F, W) = 0 \tag{D.1}$$

where P is an analytic function of a set of real variables α, R, r, m, F and W, and a complex variable c where

$$c = c_r + i c_i \tag{D.2}$$

Since the variables r, m, F and W are physical constants that can be arbitrarily fixed, as well as α, the wave number, Eq. (D.1) is then equivalent to

$$\begin{aligned} P_r(R, c_r, c_i) &= 0 \\ P_i(R, c_r, c_i) &= 0 \end{aligned} \tag{D.3}$$

and taking their total differentials

$$\frac{\partial P_r}{\partial R} dR + \frac{\partial P_r}{\partial c_r} dc_r + \frac{\partial P_r}{\partial c_i} dc_i = 0 \tag{D.4}$$

$$\frac{\partial P_i}{\partial R} dR + \frac{\partial P_i}{\partial c_r} dc_r + \frac{\partial P_i}{\partial c_i} dc_i = 0 \tag{D.5}$$

From Eq. (D.4) and (D.5), $\frac{dc_i}{dR}$ and $\frac{dc_r}{dR}$ can be solved for

$$\frac{dc_i}{dR} = - \frac{\frac{\partial P_r}{\partial R} + \frac{\partial P_r}{\partial c_r}\frac{dc_r}{dR}}{\frac{\partial P_r}{\partial c_i}} \tag{D.6}$$

$$\frac{dc_r}{dR} = - \frac{\dfrac{\partial P_i}{\partial R} + \dfrac{\partial P_i}{\partial c_i}\dfrac{dc_i}{dR}}{\dfrac{\partial P_i}{\partial c_r}} \tag{D.7}$$

and using Eq. (D.7) in Eq. (D.6) gives

$$\frac{dc_i}{dR} = \frac{\dfrac{\partial P_r}{\partial R}\dfrac{\partial P_i}{\partial c_r} - \dfrac{\partial P_r}{\partial c_r}\dfrac{\partial P_i}{\partial R}}{\dfrac{\partial P_r}{\partial c_r}\dfrac{\partial P_i}{\partial c_i} - \dfrac{\partial P_i}{\partial c_r}\dfrac{\partial P_r}{\partial c_i}} \tag{D.8}$$

For an analytic function, the Cauchy-Riemann conditions are

$$\frac{\partial P_r}{\partial c_r} = \frac{\partial P_i}{\partial c_i}$$

$$\frac{\partial P_r}{\partial c_i} = -\frac{\partial P_i}{\partial c_r}$$

which, when used in Eq. (D.8) yield

$$\frac{\partial c_i}{\partial R} = \frac{\dfrac{\partial P_r}{\partial R}\dfrac{\partial P_i}{\partial c_r} - \dfrac{\partial P_r}{\partial c_r}\dfrac{\partial P_i}{\partial R}}{\left(\dfrac{\partial P_r}{\partial c_r}\right)^2 + \left(\dfrac{\partial P_i}{\partial c_r}\right)^2} \tag{D.9}$$

where the partial derivative on the left hand member has been used on account of the assumptions at the start of this section.

Eq. (D.9) was first given by Lock[2] who didn't point out the reason for its validity, the key being that we are dealing with an analytic function.

Eq. (D.9) is completely general and applies to any point on the αR -plane regardless whether on the neutral stability curve or not.

From Eqs. (C.6) and (C.7), Appendix C, it can be seen that P_r and P_i for the case of neutral stability are given by

$$\left.\begin{aligned} P_r &= \mathcal{F}(\alpha, c) + (\alpha R)^{-1/2}\, \mathcal{G}(\alpha, c) \\ P_i &= \mathcal{F}(\alpha, c) + (\alpha R)^{-1/2}\, \mathcal{H}(\alpha, c) \end{aligned}\right\} \qquad \text{(D.10)}$$

where $c = c_r$. Their derivatives with respect to R and c can be evaluated from Eqs. (C.8) to (C.11), Appendix C. The actual calculations are quite lengthy, as could easily be seen. When these results are used in Eq. (D.9), the region of stability in the αR -plane can be determined immediately.

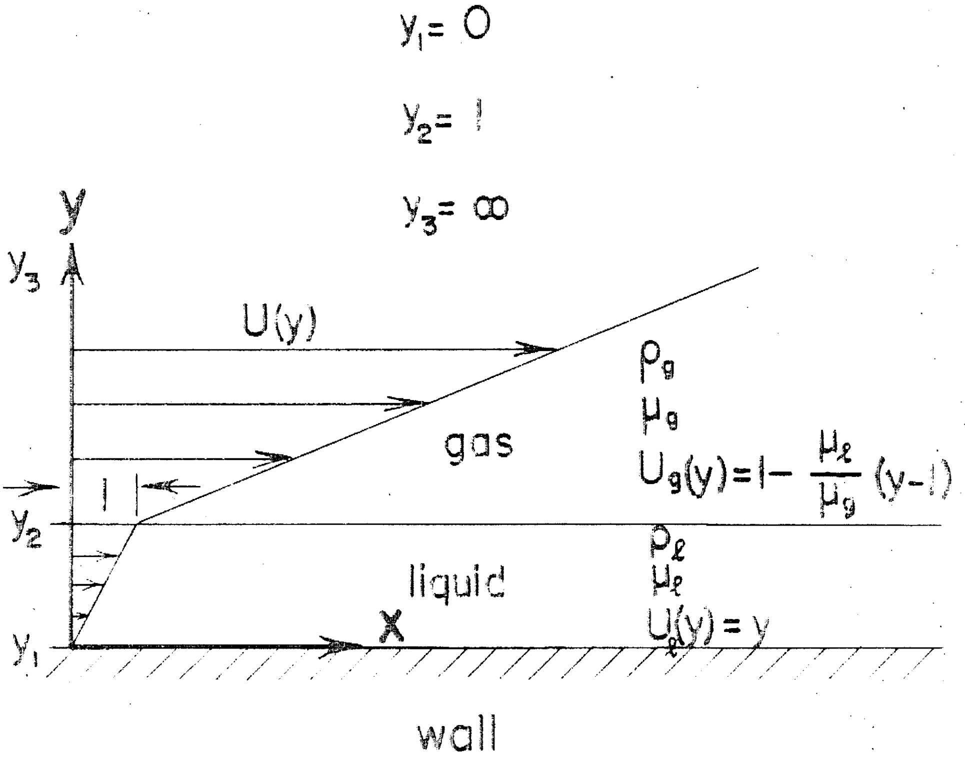


Fig I. Undisturbed Velocity Profile to be Investigated

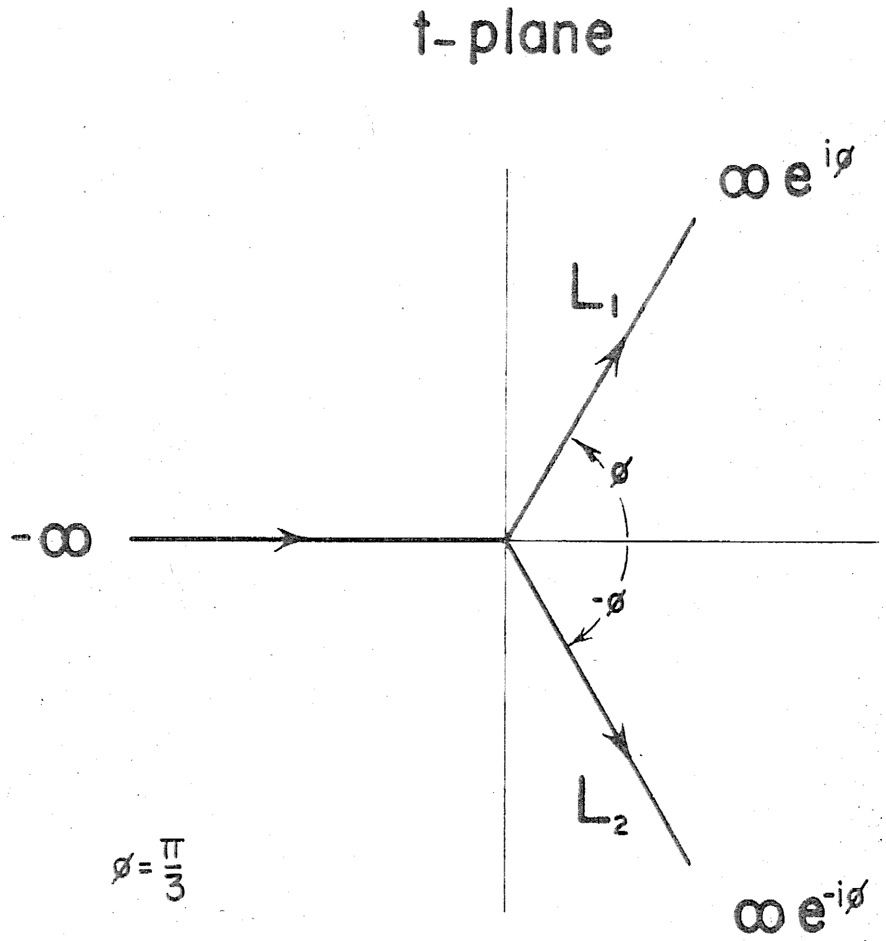


Fig. 2. Paths for Contour Integrals in the Solution of Stokes' Equation

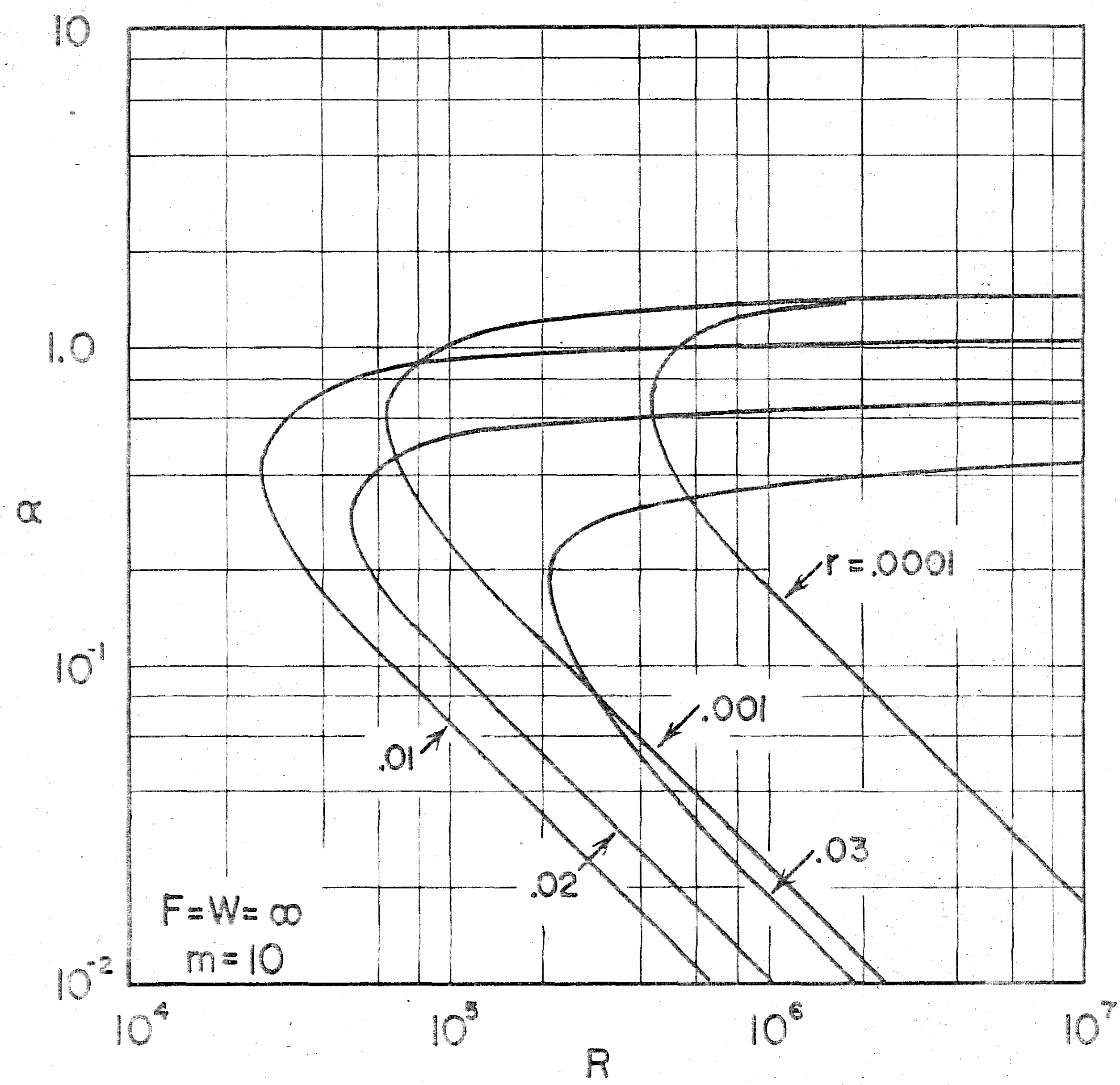


Fig. 3. Neutral Stability Curves: Wave Number vs. Liquid Reynolds Number. Gravity and Surface Tension Forces are Neglected. Liquid-Gas Viscosity Ratio = 10

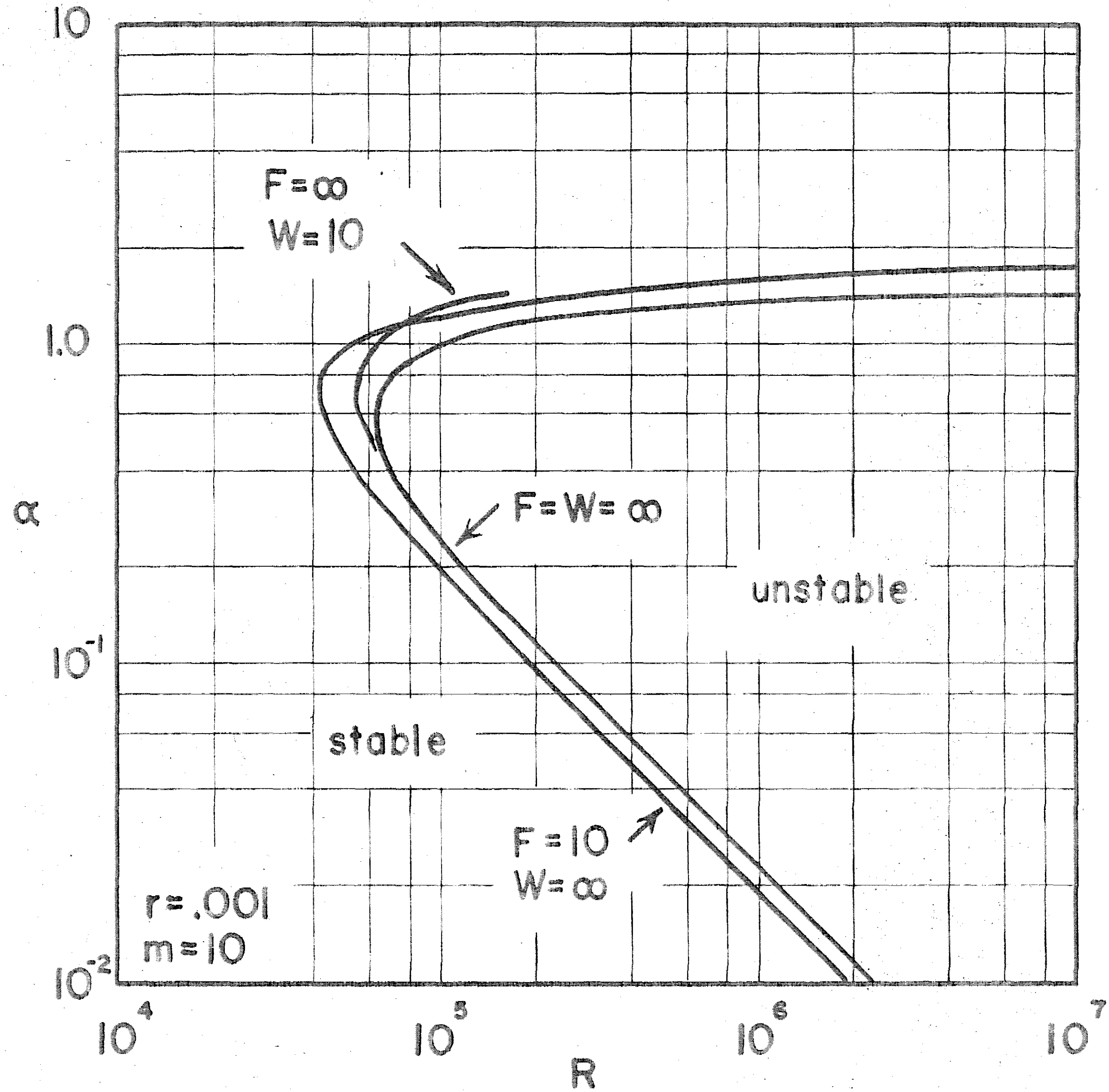


Fig.4. Effect of Gravity and Surface Tension Forces on Neutral Stability Curves

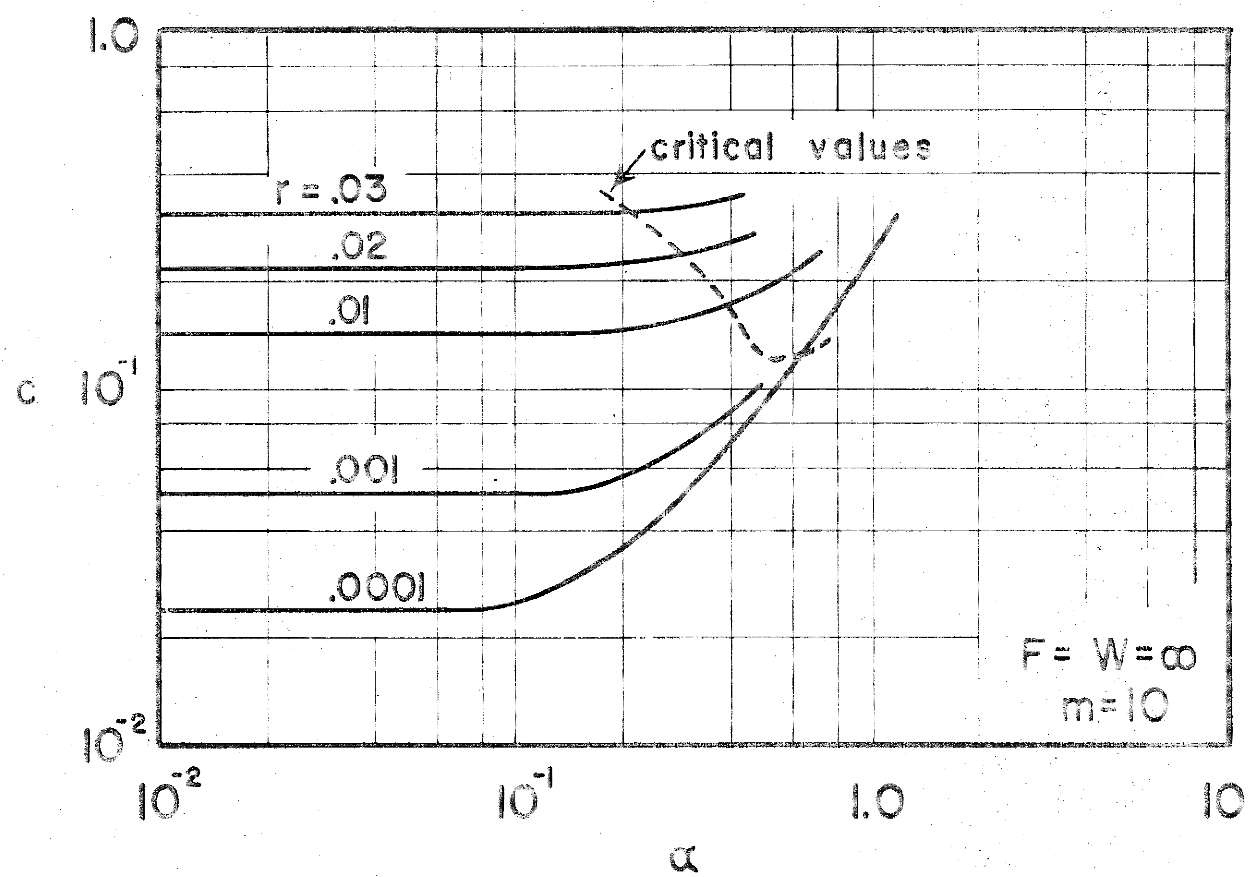


Fig. 5. Wave Velocity vs. Wave Number for Neutral Stability. Gravity and Surface Tension Forces are Neglected. Liquid-Gas Viscosity Ratio = 10

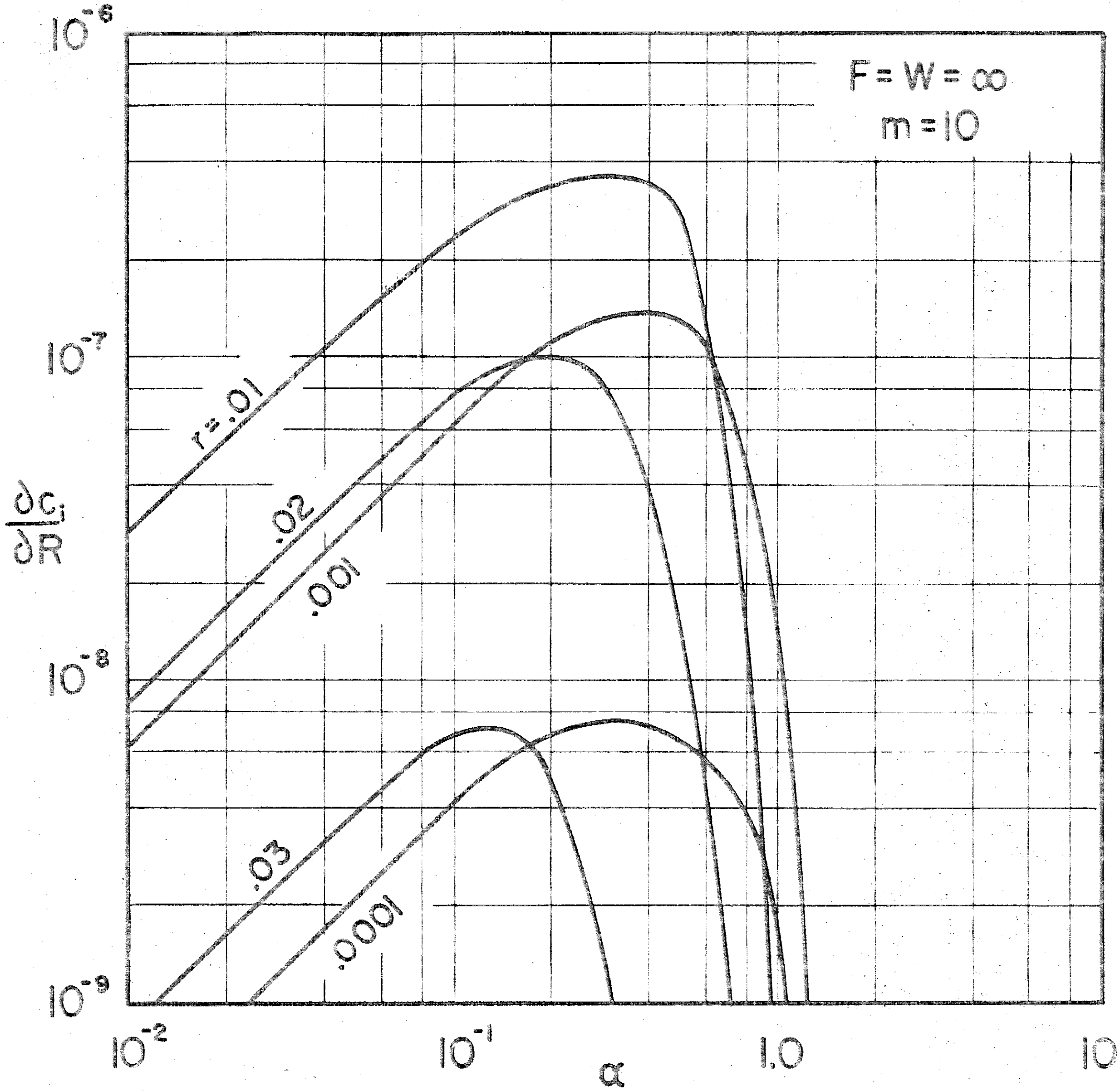


Fig. 6. Rate of Change of Amplification Factor with Reynolds Number as a Function of Wave Number and Gas-Liquid Density Ratio. Liquid-Gas Viscosity Ratio = 10

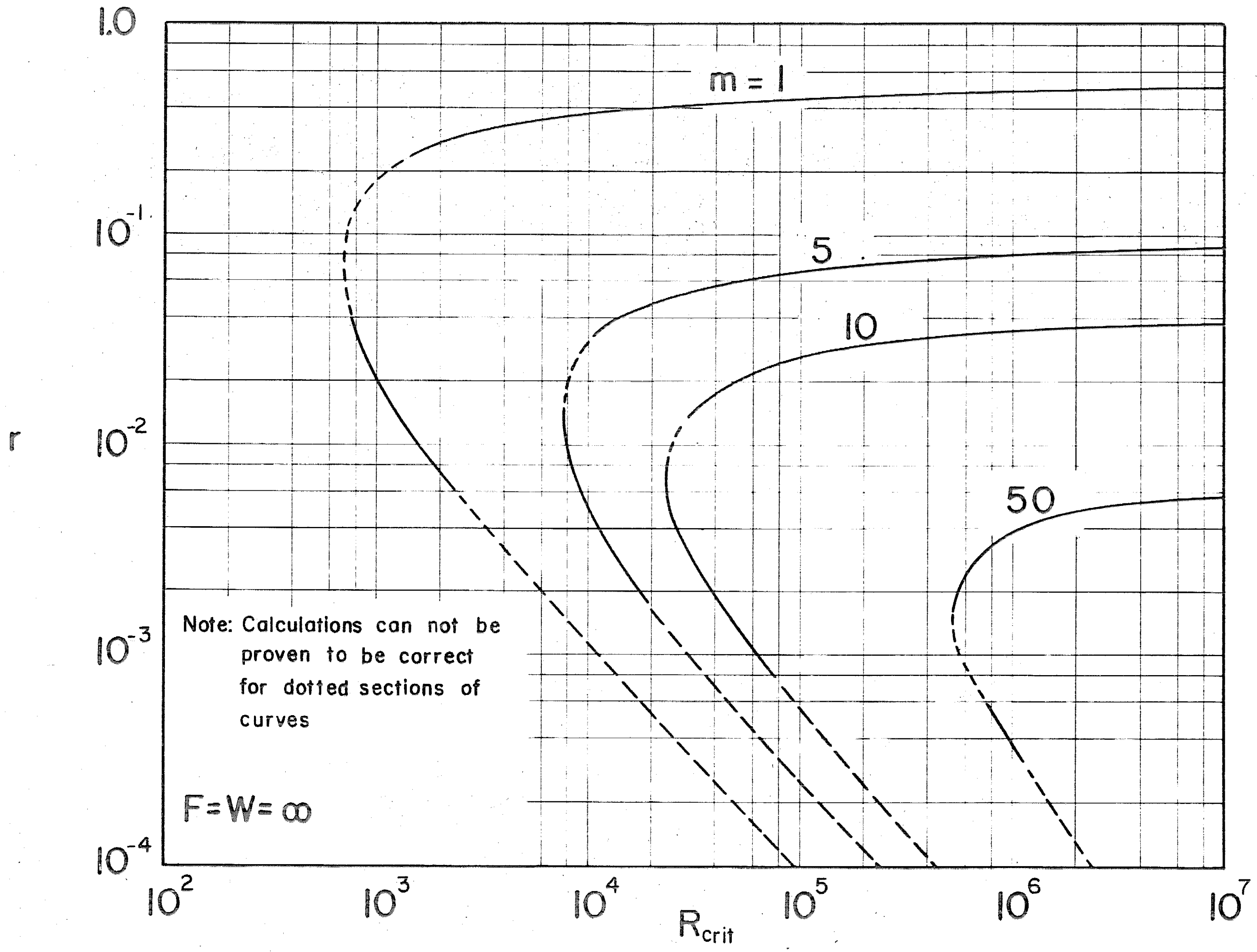


Fig. 7. Critical Reynolds Number

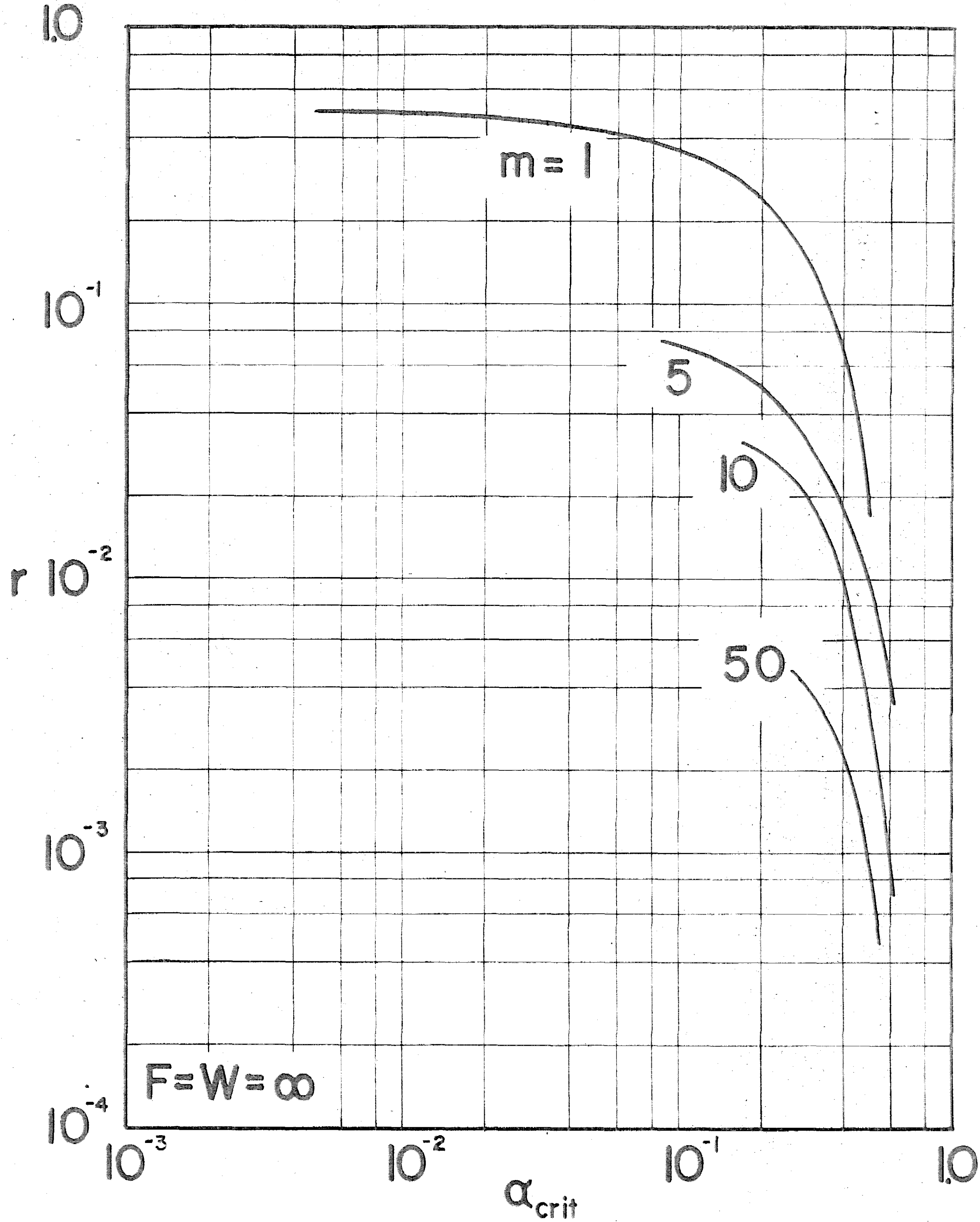


Fig. 8. Critical Wave Number

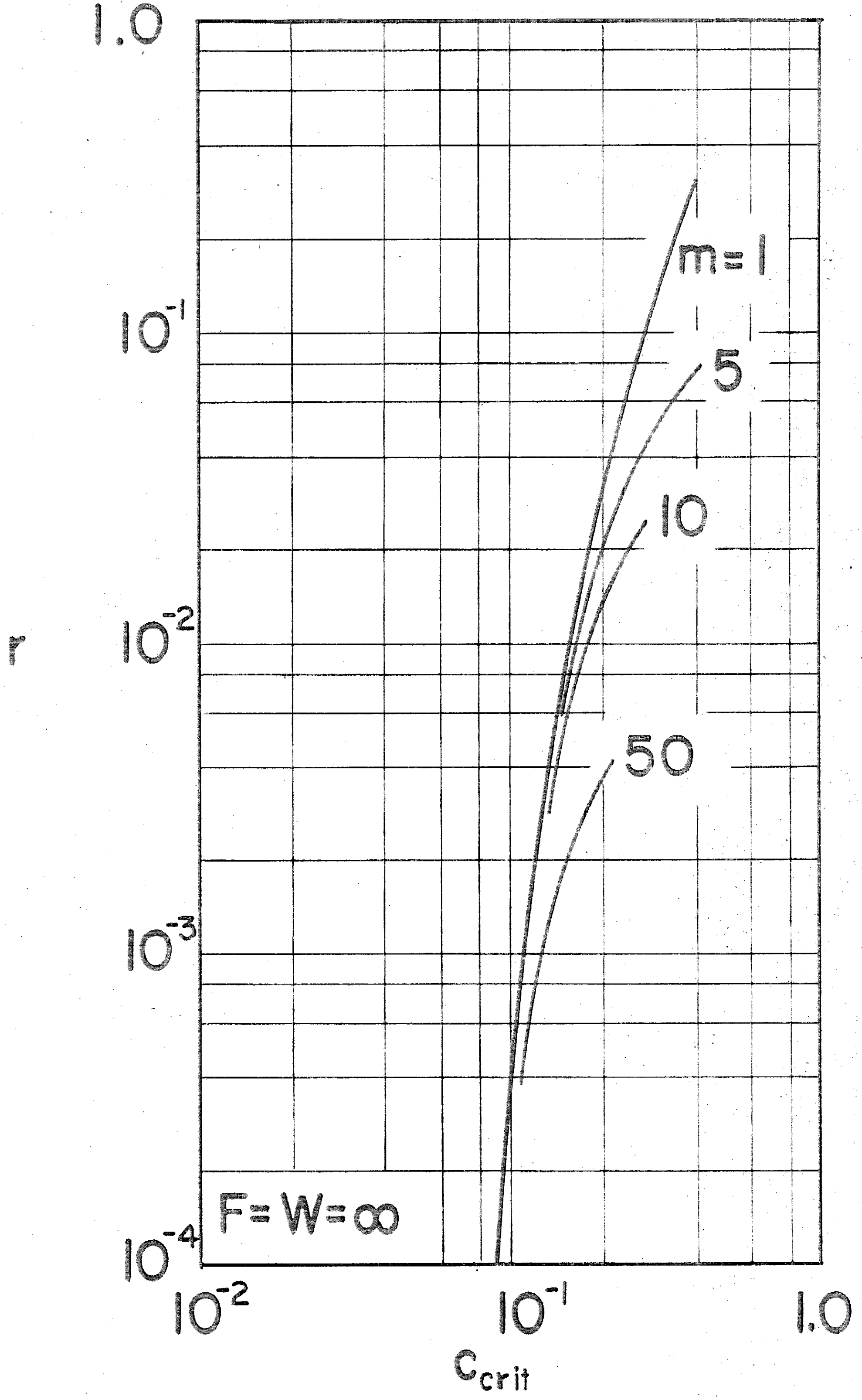


Fig. 9. Critical Wave Velocity

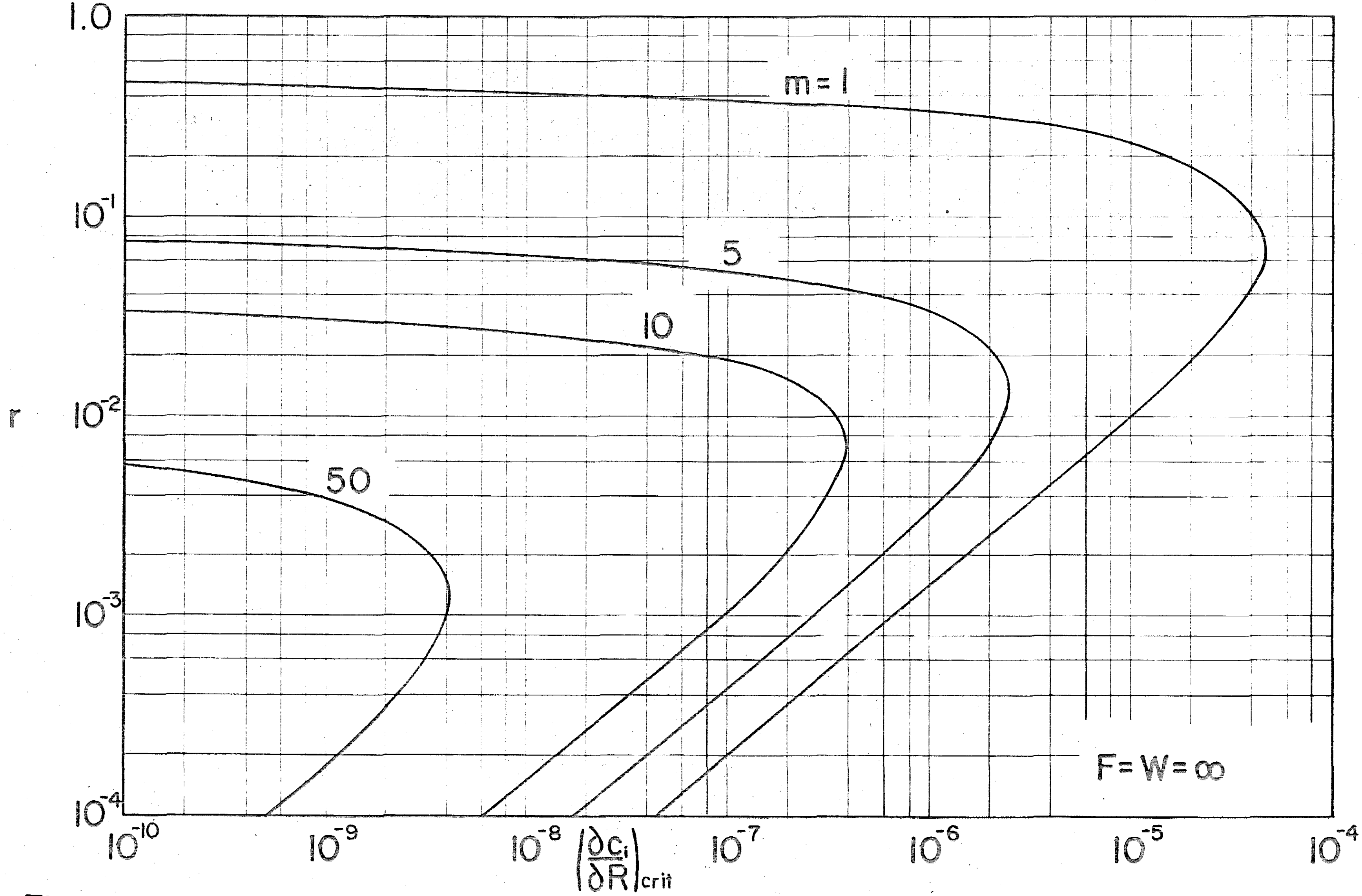


Fig. 10. Critical Values of the Rate of Change of Amplification Factor with Reynolds Number

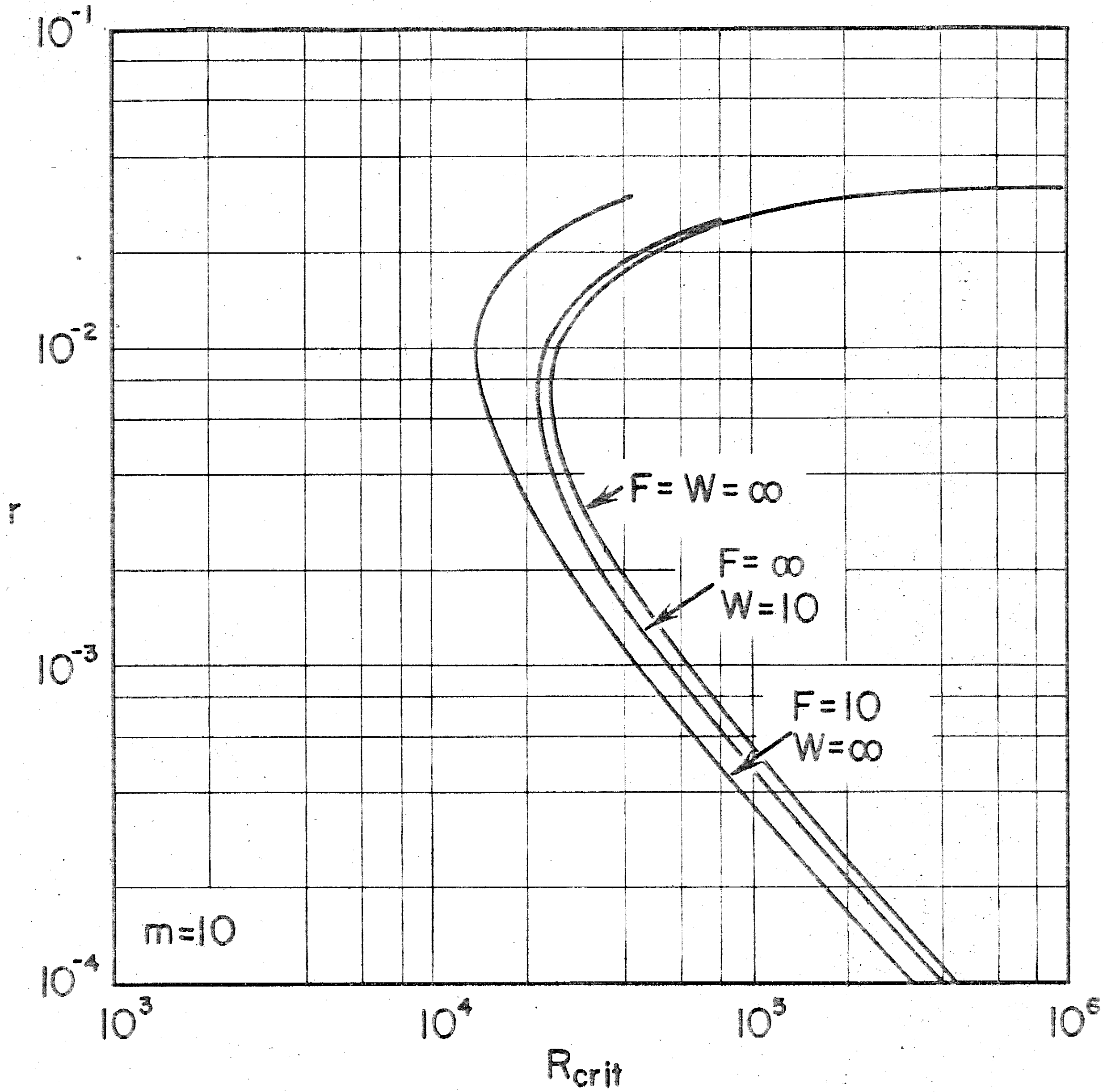


Fig. 11. Influence of Gravity and Surface Tension Forces on Critical Reynolds Number

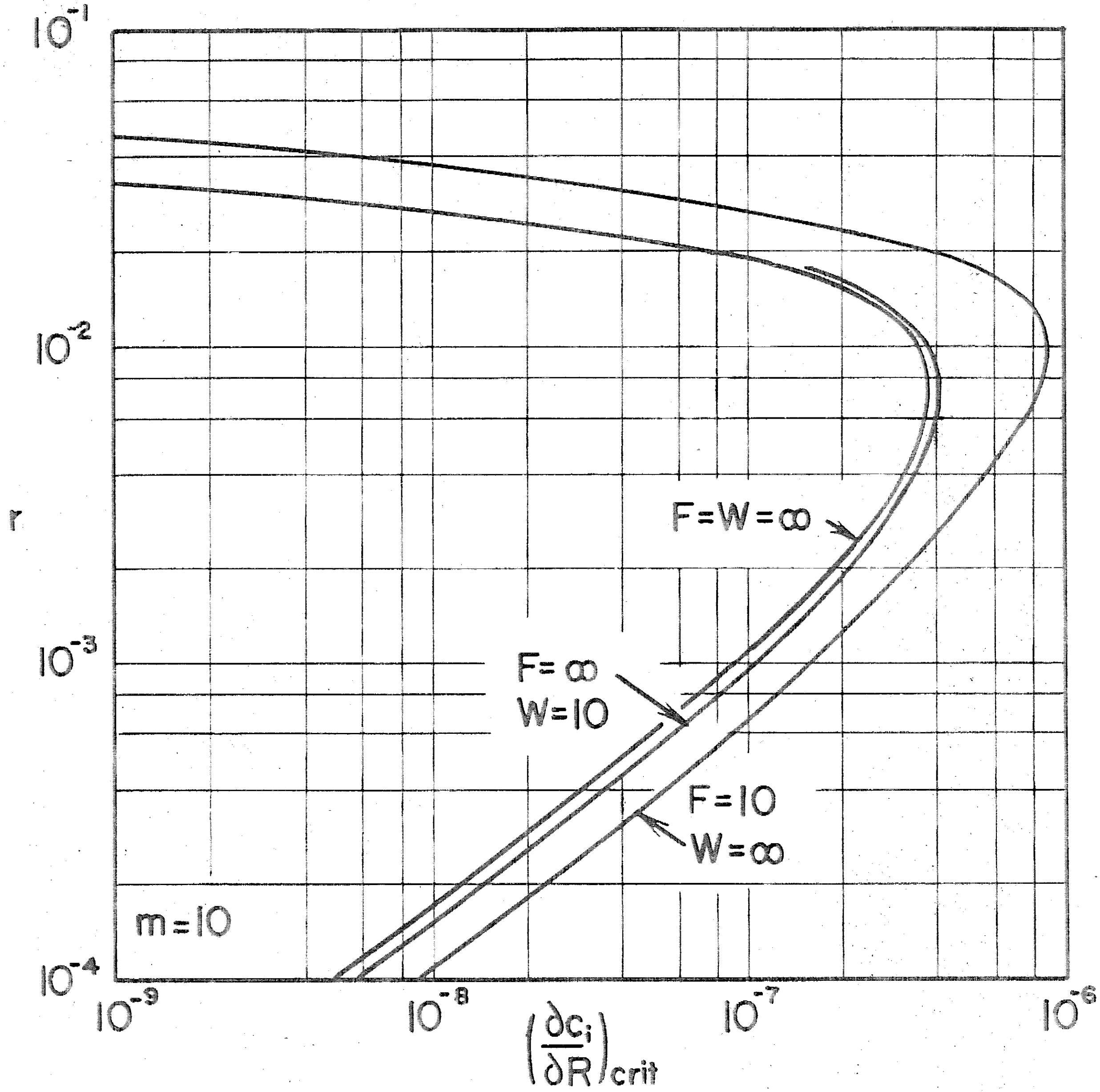


Fig. 12. Influence of Gravity and Surface Tension Forces on Critical of Rate of Change of Amplification Factor with Reynolds Number

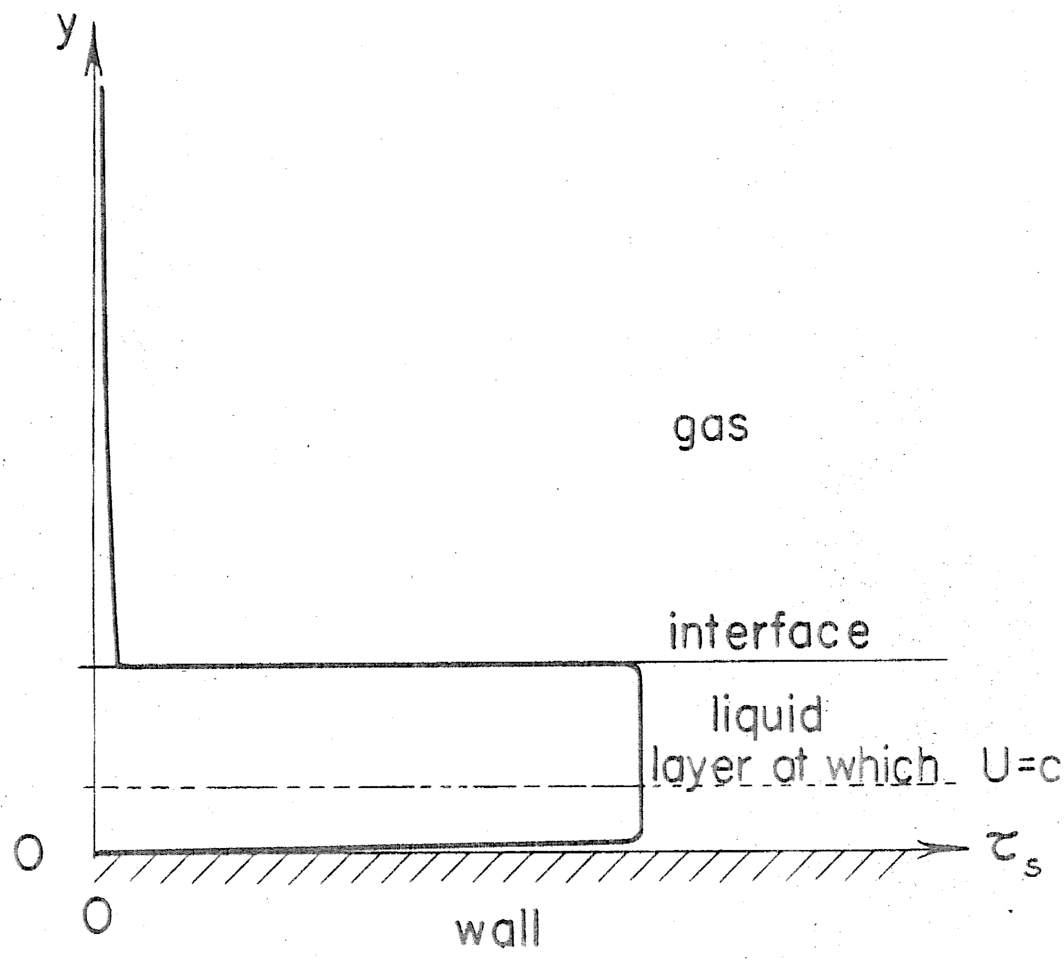


Fig. 13. Distribution of Reynolds Stress for a Neutral Oscillation

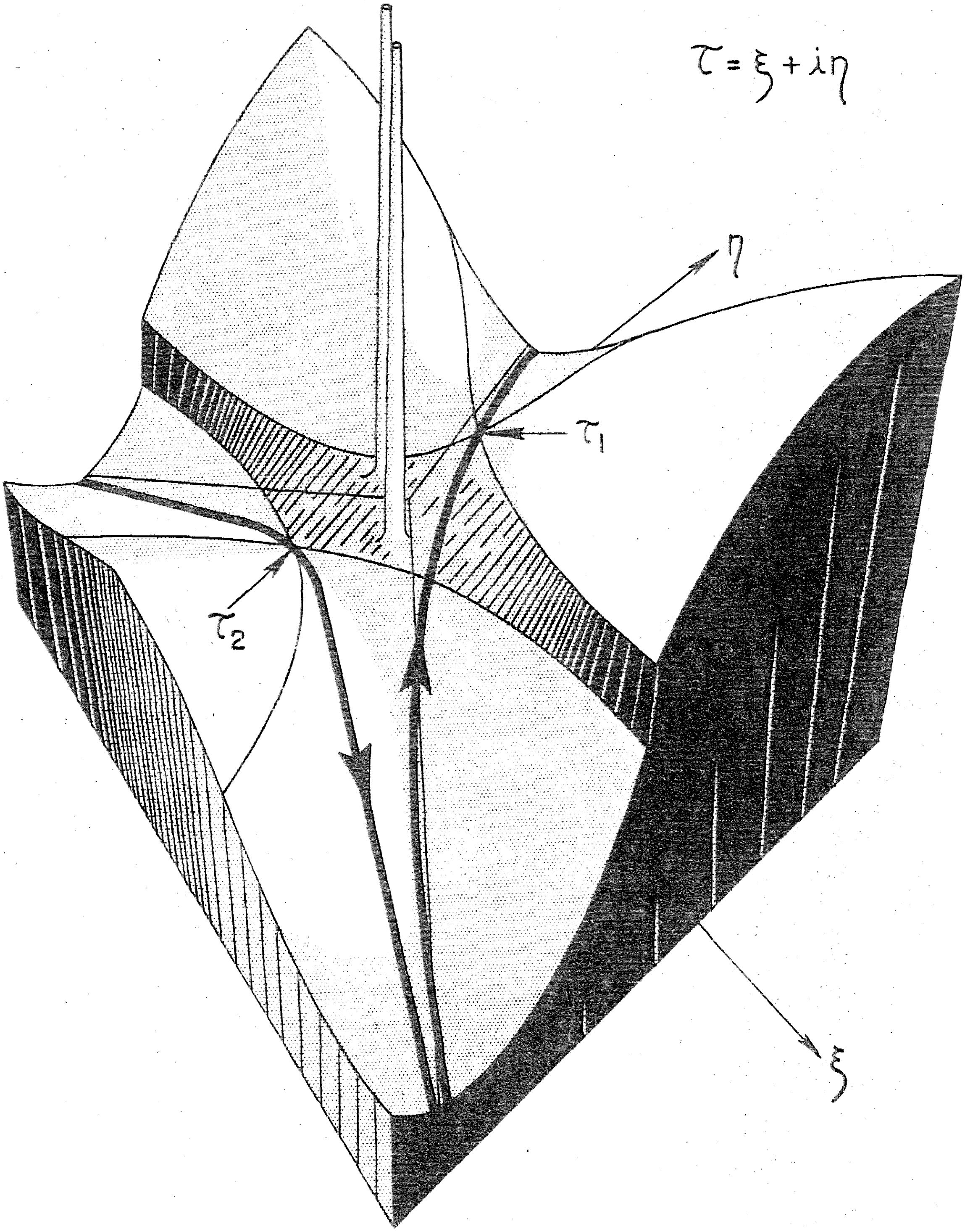


Fig. 14.- Perspective Drawing in the τ-Plane Showing the Absolute Value of the Integrand and the Deformed Path of Integration for the Integral (B.29) for the Special Case $\Theta = -\frac{\pi}{2}$.

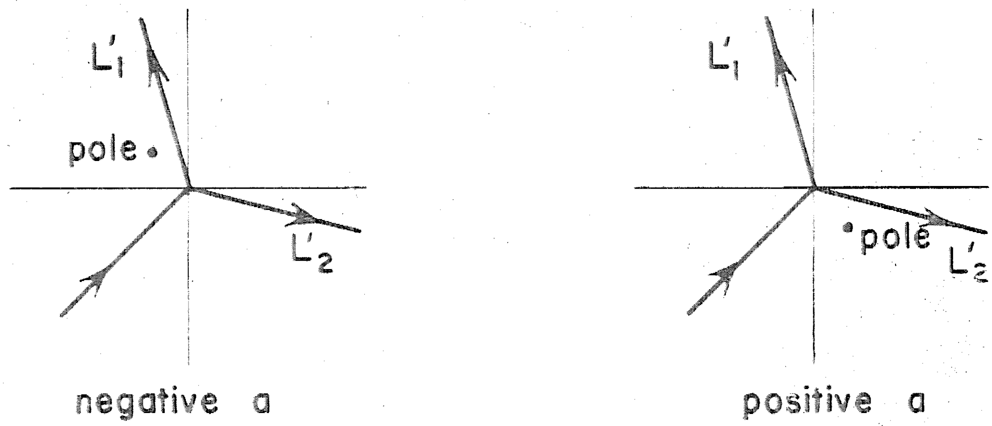


Fig. 15a. Location of Poles of Integrand of Eq. (B.29) for z_{1_0} $(c=c_r<1)$

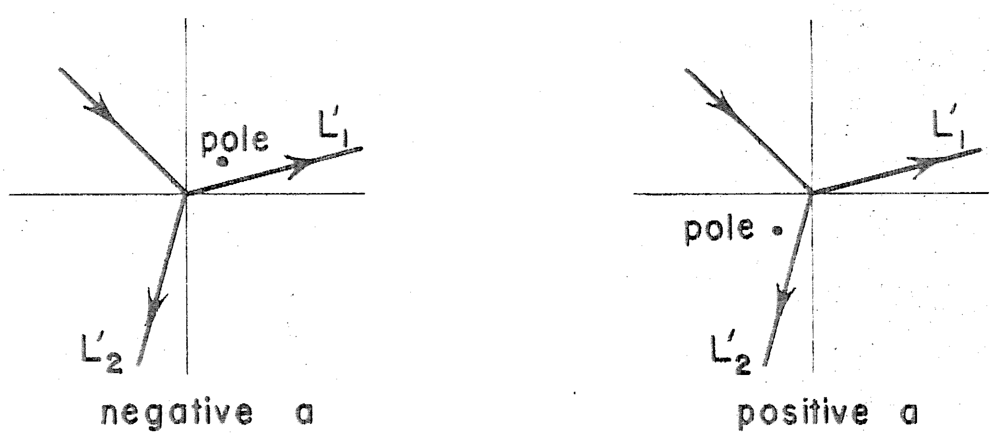


Fig. 15b. Location of Poles of Integrand of Eq. (B.29) for any z Other Than z_{1_0} $(c=c_r<1)$