TRANSIENT RESPONSE OF NON-LINEAR SPRING-MASS SYSTEMS

Thesis by

Erdem I. Ergin

In Partial Fulfillment of the Requirements
For the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1954

ACKNOWLEDGMENTS

The author wishes to thank Dr. D. E. Hudson for patient guidance and many helpful suggestions in the execution of the work and the final preparation of this thesis. He is also grateful to Dr. C. R. De Prima for taking an active interest in this work, and to his fellow student, T. K. Caughey, for sharing many stimulating discussions and arguments on the field of non-linear mechanics.

ABSTRACT

The purpose of this thesis is:

1) To investigate the applicability and to compare the accuracy of existing perturbation methods of non-linear mechanics for the solution of transient response problems, and

2) To describe a new analytical approximate method for the solution of certain types of non-linear problems involving pulse excitation. This new method combines the advantages of engineering accuracy with ease of applicability.

In the course of this study it is found that the solution of homogeneous non-linear equations can be obtained readily and with sufficient accuracy by the perturbation methods of Kryloff and Bogoliuboff or Lindstedt, even for large non-linearities. Greater accuracy can be attained by the use of the newly developed bi-linear approximation. The advantage of the bi-linear method becomes more pronounced when the step function or the single pulse response of the system is investigated. It is shown that the bi-linear method is the only convenient analytical approximate method available for the solution of general pulse excitation problems involving non-linear spring-mass systems.

TABLE OF CONTENTS

LIST OF NOMENCLATURE

x	Displacement
y	Transformed displacement
a	Amplitude of the fundamental mode
ω_o	Linear frequency
k	Linear spring constant
ε	Non-linear parameter
ω	Non-linear frequency
$\bar{k}$	Non-linear "equivalent" spring constant
$\bar{\lambda}$	Non-linear "equivalent" damping constant
Ω	Non-linear frequency obtained by Kryloff and Bogoliuboff Method
ϕ	Phase angle
ψ	$t + \phi$
$f(x)$	Non-linear portion of the spring force
$h(x)$	Linear plus non-linear spring force: $kx + f(x)$
k_1	First spring constant of the bi-linear approximation
k_2	Second spring constant of the bi-linear approximation
x_t	Transition amplitude
$g_1(x)$	$k_1 x$
$g_2(x)$	$k_2 x + x_t (k_1 - k_2)$
t_t	Transition time
x_o	or $x(0)$, initial amplitude
x_o	or $x(0)$, initial velocity
$F(t)$	Forcing function
τ	Pulse length

LIST OF NOMENCLATURE (Cont'd)

x_m	Approximate maximum deflection
$\bar{x}_m$	Exact maximum deflection
λ	Modulus of elliptic functions of first kind
c	Linear damping coefficient

I. INTRODUCTION

A. Engineering Significance of the Non-linear Transient Problem

Most linear equations dealt with in engineering are the results of idealization of non-linear systems. When the non-linearities are small such an idealization is justified by the fact that it enables one to describe the system with sufficient accuracy and relative ease. However, no matter how small the non-linearities are, their removal from the problem may cause significant changes in the characteristic behaviour of the solution. For example, ultra and subharmonic forced oscillations, and the so-called "jump phenomenon" are only found in non-linear systems. In the case of transient, or pulse excitation, such drastic changes in the character of the motion are ordinarily not involved. However, there are modifications in the displacement and frequency which can be of the order of fifteen-twenty percent even if the non-linear force is never in excess of ten percent of the linear force. Since engineering design in many fields is pushing closer and closer to the limits of critical stresses and optimum utilization such changes cannot be neglected.

It is also quite probable that with better understanding of non-linear problems it will be possible to make intentional use of them, instead of discarding them as parasitic effects. There is already a move in this direction, for example, in the design of shock mounts, which employ non-linear elements to improve the response characteristics of the system.

The primary purpose of the treatment of transient response in this thesis is to investigate this relatively neglected aspect of non-linear mechanics. To the author's knowledge there has never been a systematic search for a method of solution for problems involving pulse excitation, and there exists no approximate method suitable for at least a large class of problems involving transients.

Numerical and graphical methods are, of course, available which have an almost unlimited range of applicability. They are, however, long and tedious, and never exhibit answers in a general form. Even though in theory their errors can be made extremely small, in practice there is a maximum accuracy which cannot be exceeded. This is particularly true of the graphical methods where increasing accuracy necessitates the measurement of smaller angles and distances.

B. Applications of Existing Techniques

Unfortunately most of the classical approximate methods are not useful in the general treatment of the transient problem. With the exception of the Kryloff and Bogoliuboff method they are all developed for steady state oscillations only. The applicability of these methods to the homogeneous and step function response of non-linear equations justifies a careful analysis of their inherent limitations, which can be briefly summarized as follows:

1. Kryloff and Bogoliuboff Method: This is most useful of the classical approximate methods. It is, in principle, a perturbation procedure in which the elimination of secular terms gives the required frequency and amplitude corrections. Its application to pulse excitations is limited to the solution of the rectangular pulse, where the system exhibits oscillations of a single

frequency only; i.e., when the pulse length has no effect on the frequency of the oscillations.

2. Equivalent Linearization Method: When the non-linearities are sufficiently small that the amplitude and the frequency are slowly varying functions of time, the equivalent linearization method, or as it is sometimes called, the "first order Kryloff Method" gives good results. Its main difference from the classical Kryloff and Bogoliuboff procedure lies in the elimination of higher harmonic terms by an averaging process. It is, therefore, subject to the same limitations in application to transient oscillations.

3. Classical Perturbation: In the Kryloff procedure the frequency and the amplitude are assumed to be slowly varying functions of time. In the classical perturbation method they are taken as constants and the correction terms are again found by the elimination of secular terms. This method is also limited, in its application to transient problems, to the solution of rectangular pulses only. Its value lies in its simplicity, and as will be shown in the body of the thesis in its accuracy for a greater range of non-linearities.

C. Line Segment Approximation of Functions and the Bi-Linear Method

Perhaps the most important new technique to be presented in this thesis is a method for line-segment approximation of the non-linear functions. It is a well-known principle that any function can be approximated by a number of straight line segments "properly chosen". Each line can be determined by its slope and a point through which it passes; this point is usually picked to be the transition point where two segments meet. By "properly chosen" it is meant that the slope and the transition points are determined such that the mean square error between the function and the curve formed by the series of line segments is minimized within a range of variables. It is obvious that by increasing the number of line segments the approximation is improved, but also the complexity of the problem increased.

Extension of this idea to the solution of non-linear equations is evident. Any non-linearity, which is a function of the dependent variable only, can be described by a number of line segments which are determined to approximate the non-linear function as closely as possible. The problem, then, resolves itself to the solution of the same number of linear equations as there are line segments, with the proper matching of displacement and velocity at the transition points. Solution of this problem becomes more and more involved as the number of line segments is increased. The determination of optimum slopes and the transition points requires the solution of (2n-1) non-linear algebraic simultaneous equations if the number of line segments is (n).

There are certain simplifications which bring the method of linear approximations within the realm of practicability. Most functions encountered in practice can be satisfactorily approximated by two line segments only. In addition it is possible to choose the slope of the first line segment as the slope of the function at the origin without appreciable loss of accuracy. This simplifies the problem to the determination of the second slope and the transition point only by minimizing the mean square error. In the text of the thesis the above procedure will be referred to as the "improved bi-linear approximation", and the term "bi-linear approximation" itself will be used to indicate a simpler approach in which the second slope is chosen as the slope of the non-linear function at the point of the maximum linear deflection.

It will be shown that for any non-linearity which can be expressed as a power in the dependent variable, the second slope and the transition point are simple functions of the maximum amplitude for both the bi-linear and the improved bi-linear approximations. It will also be shown that for small non-linearities the solution of the problem is insensitive to the exact choice of the maximum deflection used to calculate the parameters of the bi-linear approximation, and hence that the choice of the linear maximum amplitude for this purpose gives satisfactory results. For very large non-linearities an iterative process can be developed to converge on a solution which has a small constant error in the maximum displacement.

Resolution of the non-linear differential equation into two linear ones makes it possible to treat problems of pulse excitation with relative ease. Even for problems to which other approximate methods are applicable, the accuracy of the solution can be increased by the bi-linear approach. However, one of the most significant applications of the line segment approximations will be in the field of electrical analog computing. It has always been difficult to form or synthesize exactly the non-linearity in an equation for the purposes of electrical analog computations. The analysis indicates that the approximation of the non-linearities by two or more line segments will give results which are well within the accuracy of an analog computer.

In principle, the steady state forced oscillations and the response of the systems with linear damping can be solved by the bi-linear method. However, in these cases the problem is somewhat complicated by the necessity of solving transcendental equations involving trigonometric as well as exponential functions. However, a few qualitative experiments carried out on an electrical analog computer showed that the line segment approximation of the non-linearities did not change the character of the solution of forced oscillations. Jump phenomenon as well as the ultra and sub harmonic oscillations can be obtained by such an approximation.

D. Outline of Thesis

To conclude this introduction, a brief outline of the succeeding chapters will be presented to clarify the relationship between the various parts of the thesis.

Chapter I treats the homogeneous non-linear equation describing a spring-mass system with a cubic non-linearity. Several previously mentioned methods are compared by means of numerical examples, and the application of the bi-linear method of this thesis is shown.

In Chapter II, extensions of several classical methods to the problem of step excitation are made and specific examples are worked out and compared. The new bi-linear method is then applied and a detailed study of the error involved in this method presented.

Chapter III shows the application of the bi-linear technique to the solution of single pulse response of several problems which could not be solved by the usual approximate methods. These solutions are compared with results of a numerical iteration method to establish the accuracy to be expected.

In Chapter IV a brief treatment of the influence of small linear damping on the transient problem is given.

II. TRANSIENT RESPONSE OF A SPRING-MASS SYSTEM WITH CUBIC NON-LINEARITY IN SPRING FORCE

Chapter I. Homogeneous Solution

In order to treat a pure impulse, or a single rectangular pulse forcing function, it is important to solve first the homogeneous non-linear equation. It will be shown that when the initial velocity is zero, there is an exact analytic solution in terms of elliptic cosines for a cubic non-linearity in spring force. Some of the so-called approximate methods based on the assumption of small non-linearity are shown to give good results even when the non-linear forces are several times the linear ones.

A. Kryloff and Bogoliuboff Method (Refs. 1, 2)

Let the differential equation be

$$\ddot{x} + \omega_o^2 x + \varepsilon x^3 = 0 \tag{1-1}$$

with the initial conditions that

$$\text{at } t = 0 \quad x(0) = x_o, \text{ and } \dot{x}(0) = \dot{x}_o$$

Following the usual Kryloff procedure (see Appendix A), the deflection x, amplitude (a), and the frequency (ψ) are expanded in terms of the non-linear parameter ε. (a) and (ψ) are assumed to be slowly varying functions of time such that

$$\dot{a} = A(a)\,\varepsilon, \quad \dot{\psi} = \omega_o + \varepsilon\,\overline{\psi}(a) \tag{1-2}$$

and hence the second derivatives of (a) and (ψ) are small compared to the first derivatives and can be neglected. Since

$$x = a \cos \Psi + \varepsilon x_1(a,\Psi) + 0(\varepsilon^2) \tag{1-3}$$

$$\dot{x} = -a\omega_o \sin\Psi + \varepsilon \left[A \cos\Psi - a\bar{\Psi} \sin\Psi + \omega_o \frac{dx_1}{d\Psi}\right]$$

and

$$\ddot{x} = -a\omega_o^2 \cos\Psi + \varepsilon \left[-2A\omega_o \sin\Psi - 2a\omega_o\bar{\Psi} \cos\Psi + \omega_o^2 \frac{d^2x_1}{d\Psi^2}\right] \tag{1-4}$$

Substituting (1-2) and (1-3) in Equation (1-1) and collecting terms one gets:

$$\frac{d^2x_1}{d\Psi^2} + x_1 = \left(-\frac{3a^3}{4\omega_o^2} + \frac{2a}{\omega_o}\bar{\Psi}\right) \cos\Psi + \frac{2A}{\omega_o} \sin\Psi - \frac{a^3}{4\omega_o^2} \cos 3\Psi \tag{1-5}$$

In order that x_1 have no secular terms

$$A = 0 \quad \text{and} \quad \bar{\Psi} = \frac{3a^2}{8\omega_o} \tag{1-6}$$

putting A and $\bar{\Psi}$ from Equation (1-6) into Equations (1-2) and integrating:

a = constant and

$$\Psi = \left(\omega_o + \frac{3}{8}\varepsilon\frac{a^2}{\omega_o}\right)t + \phi \tag{1-7}$$

where ϕ is a constant of integration to be determined from the initial conditions. It is obvious that if $x_o = 0$ $\phi = \frac{\pi}{2}$; and if $\dot{x}_o = 0$ $\phi = 0$. The solution of Equation (1-5) gives x_1 as

$$x_1 = \frac{a^3}{32\omega_o^2} \cos 3\Psi, \tag{1-8}$$

and the solution will then be:

$$x = a \cos\Psi + \frac{\varepsilon a^3}{32\omega_o^2} \cos 3\Psi \tag{1-9}$$

to the first approximation.

Initial Conditions:

1) Pure impulse - $x(0) = 0$, $\dot{x}(0) = \dot{x}_o$

In this case $\phi = \frac{\pi}{2}$ and the Equation (1-9) takes the form:

$$x = a \sin \Omega t + \frac{\varepsilon a^3}{32 \omega_o^2} \sin 3\Omega t \text{ where } \Omega = \omega_o + \frac{3}{8}\varepsilon \frac{a^2}{\omega_o} \tag{1-10}$$

since $\dot{x}(0) = \dot{x}_o$, an equation for the amplitude (a) is obtained as:

$$\frac{9\varepsilon^2}{256\omega_o^3} a^5 + \frac{15\varepsilon}{32\omega_o} a^3 + a\omega_o - \dot{x}_o = 0 \tag{1-11}$$

if the ε^2 term is again neglected:

$$a^3 + \frac{32\omega_o^2}{15\varepsilon} a - \frac{32\omega_o}{15\varepsilon} \dot{x}_o = 0 \tag{1-12}$$

It is seen that the Equation (1-12) has only one real positive root which gives the required amplitude.

2) Pure displacement - $x(0) = x_o$ $\dot{x}(0) = 0$

In this case $\phi = 0$, and the Equation (1-9) becomes:

$$x = a \cos \Omega t + \frac{\varepsilon a^3}{32 \omega_o^2} \cos 3\Omega t \text{ with } \Omega = \omega_o + \frac{3}{8}\varepsilon \frac{a^2}{\omega_o} \tag{1-13}$$

If the initial condition $x(0) = x_o$ is substituted in Equation (1-13) an amplitude equation is obtained as

$$a^3 + 32\frac{\omega_o^2}{\varepsilon} a - 32 \frac{\omega_o^2}{\varepsilon} x_o = 0$$

Again the cubic equation gives the amplitude uniquely, since it has only one positive real root.

3) Mixed Initial Conditions: $x(0) = x_o$, $\dot{x}(0) = \dot{x}_o$

If these conditions are applied to the Equation (1-9), they give rise to two non-linear transcendental equations which determine ϕ. These equations are:

$$a \cos\phi + \frac{\varepsilon a^3}{32\omega_o^2}\cos 3\phi = x_o \qquad (1\text{-}14)$$

$$a \sin\phi + \frac{3\varepsilon a^3}{32\omega_o^2}\sin 3\phi = -\frac{\dot{x}_o}{\Omega} \qquad (1\text{-}15)$$

It will be apparent from the exact solution of the differential equation that the higher harmonic amplitudes of the solution are less than 5 percent of the fundamental one even with very large non-linearities. If, on this basis, one neglects the terms of $\cos 3\Psi$ in the differential equation, the mixed initial conditions can be solved readily. Equations (1-14) and (1-15) will then be:

$$a \cos\phi = x_o \qquad (1\text{-}16)$$

$$a \sin\phi = -\frac{\dot{x}_o}{\Omega} \qquad (1\text{-}17)$$

which will give the amplitude equation as:

$$a^2 = x_o^2 + \frac{\dot{x}_o^2}{\Omega^2} \text{ or } a^2\Omega^2 - \Omega^2 x_o^2 - \dot{x}_o^2 = 0$$

$$(1\text{-}18)$$

but $\Omega^2 = \omega_o^2 + \varepsilon\frac{3a^2}{4}$

Hence:

$$a^4 + \left(\frac{4}{3}\frac{\omega_o^2}{\varepsilon} - x_o^2\right)a^2 - \frac{4}{3\varepsilon}\left(\dot{x}_o^2 + x_o^2\omega_o^2\right) = 0 \qquad (1\text{-}19)$$

This is a quadratic equation in (a^2), but it will always have one and only one real positive root. Then:

$$a^2 = -\frac{1}{2}\left(\frac{4}{3}\frac{\omega_o^2}{\varepsilon} - x_o^2\right) + \sqrt{\frac{1}{4}\left(\frac{4}{3}\frac{\omega_o^2}{\varepsilon} + x_o^2\right)^2 + \frac{4}{3\varepsilon}\dot{x}_o^2} \qquad (1\text{-}20)$$

The phase angle ϕ will be found to be:

$$\phi = \cos^{-1} \frac{x_o}{a} \qquad (1\text{-}21)$$

B. The Method of Equivalent Linearization (Ref. 2)

This method will give very much the same result as the above Kryloff and Bogoliuboff except that the solution will no longer contain the third harmonic. The frequency squared correction term as found from Equation (5-21) in Appendix A will be $\frac{3}{4}\varepsilon a^2$, and the solution can be expressed as

$$x = a\cos(\Omega t + \phi) \quad \text{with } \Omega^2 = \omega_o^2 + \frac{3}{4}\varepsilon a^2 \qquad (1.22)$$

Determination of initial conditions in this case is trivial and the amplitude Equation (1-20), and the phase Equation (1-21) will remain unchanged.

C. Classical Perturbation or Lindstedt Procedure (Refs. 3,4)

The well-known method of perturbation will be used here in the solution of the homogeneous equation. The results thus obtained give errors less than five percent in both frequency and amplitude even when the non-linear forces exceed the linear ones by several fold.

Let the linear frequency squared (ω_o^2) be expanded in terms of the unknown frequency ω^2 and the correction term of the order of ε. The solution (x) will also be expanded in a similar fashion. Hence:

$$\omega_o^2 = \omega^2 + \varepsilon\omega_1^2 \quad \text{and} \qquad (1\text{-}23)$$

$$x = x^o + \varepsilon x_1, \qquad (1\text{-}24)$$

Here x^o corresponds to the zeroth order solution that satisfies the initial conditions. Then:

$$\ddot{x} = \ddot{x}^o + \varepsilon \ddot{x}_1 \qquad (1\text{-}25)$$

putting Equations (1-23) and (1-25) in the differential Equation (1-1)

$$(\ddot{x}^o + \varepsilon \ddot{x}_1) + (\omega^2 + \varepsilon \omega_1^2)(x^o + \varepsilon x_1) + \varepsilon (x^o + \varepsilon x_1)^3 = 0$$

Collecting terms of the like powers of ε and equating them to zero, one gets:

$$\ddot{x}^o + \omega^2 x^o = 0 \qquad (1\text{-}26)$$

$$\ddot{x}_1 + \omega^2 x_1 = -\omega_1^2 x^o - (x^o)^3 \qquad (1\text{-}27)$$

Initial conditions will be defined as:

$$x^o(0) = x_o, \quad \dot{x}^o(0) = \dot{x}_o, \quad x_1(0) = 0, \quad \dot{x}_1(0) = 0 \qquad (1\text{-}28)$$

The solution of Equation (1-27) satisfying the initial conditions (1-28) will be:

$$x^o = \frac{x_o}{\cos \phi} \cos (\omega t + \phi);$$

for the sake of simplicity the initial velocity will be taken as zero such that $\phi = 0$. Hence:

$$x^o = x_o \cos \omega t \qquad (1\text{-}29)$$

Substituting the value of x^o into the Equation (1-27), expanding and collecting terms one gets:

$$\ddot{x}_1 + \omega^2 x_1 = -\left(\omega_1^2 x_o + \frac{3}{4} x_o^3\right) \cos \omega t - \frac{x_o^3}{4} \cos 3\omega t \qquad (1\text{-}30)$$

In order to eliminate the secular terms, the coefficient of $\cos\omega t$ must be equal to zero:

$$\omega_1^2 x_o + \frac{3}{4} x_o^3 = 0 \quad \text{or} \quad \omega_1^2 = -\frac{3}{4} x_o^2$$

Hence from Equation (1-23):

$$\omega^2 = \omega_o^2 + \frac{3}{4}\varepsilon x_o^2 \tag{1-31}$$

Equation (1-30) will then become:

$$\ddot{x}_1 + \omega^2 x_1 = -\frac{x_o^3}{4}\cos 3\omega t \tag{1-32}$$

or

$$x_1 = b\cos\omega t + \frac{x_o^3}{32\omega^2}\cos 3\omega t \tag{1-33}$$

in order to satisfy the initial condition

$$x_1(0) = 0 \qquad b = -\frac{x_o^3}{32\omega^2}$$

hence

$$x_1 = -\frac{x_o^3}{32\omega^2}\left(\cos\omega t - \cos 3\omega t\right) \tag{1-34}$$

The complete solution will then be:

$$x = \left(x_o - \frac{\varepsilon x_o^3}{32\omega^2}\right)\cos\omega t + \varepsilon\frac{x_o^3}{32\omega^2}\cos 3\omega t \tag{1-35}$$

D. Exact Solution of the Homogeneous Non-Linear Equation with Cubic Non-linearity in Spring Force (Ref. 3)*

The differential equation will again be taken to be

----

*N. W. McLachlan obtains this same exact solution and compares it with the solutions obtained by assuming x to be a finite Fourier Series of only the fundamental and the third harmonic terms. This approach gives only slightly better results, and is considerably more tedious to solve.

$$\ddot{x} + \omega_o^2 x + \varepsilon x^3 = 0 \tag{1-36}$$

with the initial conditions

$$x(0) = x_o, \text{ and } \dot{x}(0) = 0 \tag{1-37}$$

Let, now, $v = \dot{x}$ such that $\ddot{x} = vv'$ where $v' = \frac{dv}{dx}$. Putting these values in Equation (1-36) and separating the variables:

$$vdv = -(\omega_o^2 x + \varepsilon x^3)dx$$

integrating between the limits of x_o to x:

$$\int_o^v vdv = -\int_{x_o}^x (\omega_o^2 x + \varepsilon x^3)dx$$

$$v^2 = -2 \left[\frac{\omega_o^2}{2} x^2 + \frac{\varepsilon}{4} x^4 \right]_{x_o}^{x} = \omega_o^2(x_o^2 - x^2) + \frac{\varepsilon}{2}(x_o^4 - x^4)$$

or since the velocity has to be negative

$$v = \frac{dx}{dt} = -\omega_o(x_o^2 - x^2)^{1/2} \left\{ 1 + \frac{\varepsilon}{2\omega_o^2}(x_o^2 + x^2) \right\}^{1/2} \tag{1-38}$$

solving for t one gets:

$$t = -\frac{1}{\omega_o} \int_{x_o}^{x} \frac{dz}{(x_o^2 - z^2)^{1/2} \left\{ 1 + \frac{\varepsilon}{2\omega_o^2}(x_o^2 + z^2) \right\}^{1/2}} . \tag{1-39}$$

(In Equation (1-39) the dummy variable x has been replaced by z.)

Let $z = x_o \cos \Psi$ so that $dz = -x_o \sin \phi \, d\phi$; Equation (1-39) becomes

$$t = +\frac{1}{\omega_o}\int_0^{\cos^{-1}\frac{x}{x_o}} \frac{d\phi}{\left\{1 + \frac{\varepsilon x_o^2}{2\omega_o^2}(1+\cos^2\phi)\right\}^{1/2}} \text{ , or}$$

$$t = \frac{1}{(\omega_o^2+\varepsilon x_o^2)^{1/2}}\int_0^{\cos^{-1}\frac{x}{x_o}} \frac{d\phi}{(1-\lambda^2\sin^2\phi)^{1/2}} \text{ where } \qquad (1-40)$$

$$\lambda^2 = \frac{\varepsilon x_o^2}{2(\omega_o^2+\varepsilon x_o^2)}$$

The integral in Equation (1-40) is an elliptic integral of the first kind with amplitude $\phi = \cos^{-1}\frac{x}{x_o}$, and the modulus $\lambda = \frac{1}{\sqrt{2(1+\frac{\omega_o^2}{\varepsilon x_o^2})}}$

Thus in standard notation t will be given by:

$$t = \frac{1}{(\omega_o^2+\varepsilon x_o^2)^{1/2}} F(\phi,\lambda) \qquad (1-41)$$

where

$$F(\phi,\lambda) = \int_0^{\phi} \frac{d\phi}{(1-\lambda^2 \sin^2\phi)^{1/2}} \qquad (1-42)$$

Since $x = x_o \cos\phi$, from the definition of elliptic functions, x is found to be:

$$x = x_o \text{ cn } u \qquad (1-43)$$

where cu is the elliptic cosine and $u = F(\phi,\lambda)$.

The complete period is found by letting $\phi = 2\pi$. Thus:

$$T = \frac{4}{(\omega_o^2+\varepsilon x_o^2)^{1/2}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-\lambda^2\sin^2\phi)^{1/2}} = \frac{4}{(\omega_o^2+\varepsilon x_o^2)^{1/2}} F(\tfrac{\pi}{2},\lambda) \qquad (1-44)$$

(because of the symmetry the integration is performed over a quarter of a cycle and the result multiplied by four).

$F(\frac{\pi}{2}, \lambda)$ is a complete elliptic integral of the first kind with a modulus λ, and is readily obtainable from tables.

The solution of Equation (1-36) for mixed initial conditions resolves itself to the determination of the maximum amplitude of the oscillations. Once this value is obtained, one can treat the problem simply by taking the maximum deflection to correspond to the initial displacement with zero initial velocity. The period can then be determined exactly from Equation (1-44) in which (x_o) is replaced by the maximum deflection (x_m).

If the initial displacement is again x_o, with an initial velocity $\dot{x}_o$, the maximum amplitude is found by integrating the Equation (1-36) between the limits of x_o and x_m and noting that for $x = x_m$ $v = 0$. Thus:

$$\int_{v_o}^{o} v dv + \int_{x_o}^{x_m} \omega_o^2 x + \varepsilon x^3 \ dx = 0$$

performing the integration and solving for x_m:

$$x_m^2 = -\frac{\omega_o^2}{\varepsilon} + \sqrt{\left(\frac{\omega_o^2}{\varepsilon}\right)^2 + \frac{1}{\varepsilon} \ 2v_o^2 + 2\omega_o^2 x_o^2 + \varepsilon x_o^4} \qquad (1\text{-}45)$$

E. Effect of Large Non-linearities

The effect of large non-linearities on frequency and harmonic content can be examined as follows:

Let

$$A_1 = x_o - \frac{\varepsilon x_o^3}{32\omega^2} \text{ and } A_3 = \frac{\varepsilon x_o^3}{32\omega^2} \qquad (1\text{-}46)$$

such that the Equation (1-35) can be written as

$$x = A_1 \cos\omega t + A_3 \cos 3\omega t \qquad (1\text{-}47)$$

Then

$$\frac{A_3}{A_1} = \frac{1}{\frac{32\omega^2}{\varepsilon x_o^2} - 1} \qquad (1\text{-}48)$$

putting the value of ω^2 from Equation (1-31) into Equation (1-48)

$$\frac{A_3}{A_1} = \frac{1}{\frac{32\omega_o^2}{\varepsilon x_o^2} + 23} \qquad (1\text{-}49)$$

It is seen that as ε becomes very large for a given initial displacement x_o, $\frac{A_3}{A_1} \longrightarrow \frac{1}{23}$. This means that the maximum amplitude of the third harmonic in the solution will be only about 4.5 percent of the fundamental one.

If one expands the function (cn u) in a Fourier series for $\lambda^2 = \frac{1}{2}$ (which is the limiting value for ε very large), the third harmonic amplitude is found to be about five percent of the fundamental. This is the main reason why a simple approximate approach to the problem as outlined above gives such good results.

Only a first correction to the frequency also gives satisfactory results even for large non-linearities. It can be shown that the higher order correction terms to the frequency will be of the form $0(\varepsilon A_1 A_3)+0(\varepsilon^2 A_3^2) + \ldots$. Since A_3 is shown to be very small even for large ε these terms are small compared to the first correction term which is of the order of (εA_1^2). Hence the series (1-23) for ω^2 converges rapidly enough to allow a good approximation by taking only one correction term.

F. Comparison of Methods by a Numerical Example

Since it was shown that even for large non-linearities the results should be reasonably accurate, an extreme example will be treated here.

Let $\omega_o^2 = 1$, $x_o = 1$, $\varepsilon = 10$ such that the non-linear spring force is ten times the linear one at maximum deflection.

Then from (1-44) the exact period is found to be:

$$T = 2.19 \text{ (since modulus } \lambda = .674, \text{ and } F(\tfrac{\pi}{2}, \lambda) = 1.82 \qquad (1\text{-}50)$$

From Equation (1-31) the approximate frequency is

$$\omega = \sqrt{\omega_o^2 + \frac{3}{4}\varepsilon x_o^2} = \sqrt{8.5} = 2.92 \qquad (1\text{-}51)$$

or the period:

$$T = \frac{2\pi}{2.92} = 2.15 \qquad (1\text{-}52)$$

which is only about two percent less than the exact period.

Equivalent linearization will also give the same result as (1-52). It will, of course, not include the third harmonic, and hence its wave form will be a pure cosine instead of an elliptic cosine.

The accuracy of the Kryloff and Bogoliuboff solution depends on the smallness of the non-linearity. If $\frac{3}{4}\frac{\varepsilon x_o^2}{\omega_o^2} << 1$, (Equation (1-31) can be written as:

$$\omega = \omega_o(1 + \frac{3}{4}\frac{\varepsilon x_o^2}{\omega_o^2})^{1/2} \cong \omega_o(1 + \frac{3}{8}\frac{\varepsilon x_o^2}{\omega_o^2}) \qquad (1\text{-}53)$$

which is the frequency found from the Kryloff and Bogoliuboff method. It is seen that for large non-linearities such a binomial expression of the square root term is no longer accurate.

G. Bi-linear Approximation

The line segment approximation outlined in the Introduction will now be used to find the solution of the homogeneous non-linear equation. The cubic non-linear spring force will be replaced by two straight line segments whose slopes match those of the non-linear function at the points of origin and the maximum deflection.

1) General Method

Let the slopes be defined as follows:

$$\begin{aligned} & k = k_1 \qquad 0 < |x| < |x_t| \\ \text{and} \quad & k = k_2 \qquad |x| > |x_t| \end{aligned} \tag{1-54}$$

where k_1 and k_2 are the two slopes and x_t the point at which the slope is increased from k_1 to k_2.

The spring forces $g_1(x)$ and $g_2(x)$ will then be:

$$\left.\begin{aligned} g_1(x) &= k_1 x \qquad 0 < |x| < |x_t| \quad \text{and} \\ g_2(x) &= k_2 x + x_t(k_1 - k_2) \qquad |x| > |x_t| \end{aligned}\right\} \tag{1-55}$$

These satisfy the additional condition that for $x = x_t$

$$g_1(x_t) = g_2(x_t) \tag{1-56}$$

The differential equations of motion can be written as:

$$\ddot{x}_1 + k_1 x_1 = 0 \qquad |x| < x_t \tag{1-57}$$

$$\ddot{x}_2 + k_2 x_2 + x_t(k_1 - k_2) = 0 \qquad |x| > x_t \tag{1-58}$$

Since the initial displacement $x_o > x_t$, the conditions to be satisfied are:

at t = 0 $x_2(0) = x_o$, $\dot{x}_2(0) = 0$ and

at $t = t_t$ $x_2(t_t) = x_1(t_t) = p$, $\dot{x}_2(t_t) = \dot{x}_1(t_t) = r$

where t_t is the transition time (time required for x_2 to change from x_o to x_t).

Solving (1-58) and putting in the initial conditions:

$$x_2 = -\frac{x_t(k_1-k_2)}{k_2} + \left\{x_o + \frac{x_t(k_1-k_2)}{k_2}\right\} \cos \omega_2 t \quad \text{where } \omega_2 = \sqrt{k_2} \qquad (1\text{-}59)$$

t_t will be found by letting $x_2 = x_t$ and solving for t, or

$$t_t = \frac{1}{\omega_2} \cos^{-1}\left\{\frac{k_1 x_t}{k_2(x_o - x_t) + k_1 x_t}\right\} = \frac{1}{\omega_2} \cos^{-1}\left\{\frac{1}{\frac{k_2}{k_1}\left(\frac{x_o}{x_t} - 1\right) + 1}\right\} \qquad (1\text{-}60)$$

Hence:

$$r = \dot{x}_2(t_t) = -\frac{k_2(x_o - x_t) + k_1 x_t}{\omega_2} \sin \omega_2 t_t \qquad (1\text{-}61)$$

The solution of Equation (-157) will be:

$$x_1 = A \cos(\omega_1 t + \phi) \qquad (1\text{-}62)$$

with the conditions that at $t = t_t$ $x_1(t_t) = x_2(t_t) = p$, and $\dot{x}_1(t_t) = \dot{x}_2(t_t) = r$

Then:

$$x_t = p = A \cos(\omega_1 t_t + \phi)$$

$$(-163)$$

$$r = -A\omega_1 \sin(\omega_1 t_t + \phi)$$

Equations (-163) determine the amplitude (A) and the phase angle (ϕ). The quarter period can be found from the condition that at $t = \frac{T}{4}$ $x_1 = 0$. Hence

$$\omega_1 \frac{T}{4} + \phi = \frac{\pi}{2} \quad \text{or} \quad T = \frac{4}{\omega_1}\left(\frac{\pi}{2} - \phi\right) \tag{1-64}$$

The only parameter not determined explicitly so far is the transition amplitude x_t. There are three logical bases for its selection:

It can be picked so that:

a) the work done per cycle by the bi-linear and the non-linear spring forces are the same,

b) the mean square error between the bi-linear and the non-linear spring forces are minimized, or

c) the spring forces are equal at the maximum displacement point.

Since it is the non-linear spring force that is being approximated and not the energy, the second of the first two choices is more reasonable. The third alternative is somewhat more arbitrary than the rest, but it is simpler to apply and gives results which are very nearly as good as those obtained by using the second alternative. In all the examples worked out the accuracy of the solution showed a distinct decline where the first alternative was used instead of the second one.

One can also improve the accuracy by taking the second slope k_2 at somewhat lower than the maximum amplitude, say at about 90 to 95 percent of x_m, but this seems more artificial since the proper choice will depend on the particular problem to be solved. A better method for the improvement of the accuracy will be discussed in Chapter 2.

In this example alternative (2) will be used to determine x_t

and the results will be compared in frequency and the amplitude with the exact solution already obtained.

Let (E^2) be the mean square error between the bi-linear and the non-linear spring forces. Then

$$E^2 = \frac{1}{x_m} \int_o^{x_t} \left[g_1(x)-h(x)\right]^2 dx + \int_{x_t}^{x_m} \left[g_2(x, x_o)-h(x)\right]^2 dx \qquad (1\text{-}65)$$

where $g_1(x)$ and $g_2(x)$ are defined by Equations (1-55) and $h(x)$ is the non-linear spring force. x_t and x_m have already been defined as the transition and the maximum amplitudes.

The condition that E have an extremum is:

$$\frac{\partial(E^2)}{\partial x_t} = 0$$

Since $g_1(x)$ and $h(x)$ are not functions of x_t

$$\frac{\partial(E^2)}{\partial x_o} = \frac{1}{x_m} \left\{ \left[g_1(x_t)-h(x_t)\right] + 2 \int_{x_t}^{x_m} \left[g_2(x, x_t)-h(x)\right] \frac{\partial g_2(x, x_t)}{\partial x_t} dx - g_2(x_t)+h(x_t) \right\} = 0$$

Since $g_1(x_t) = g_2(x_t)$ by definition, and $\dfrac{\partial g_2(x, x_t)}{\partial x_t}$ is a constant with respect to x, the minimum for E will be given by the value of x_t which is the solution of the equation

$$\int_{x_t}^{x_m} \left[g_2(x, x_t)-h(x) \right] dx = 0 \qquad (1\text{-}66)$$

for $\quad g_2 = k_2x+x_t(k_1-k_2) \quad$ and

$\quad\quad h = k_1x + \varepsilon x^3$

Equation (1-66) gives:

$$\frac{\varepsilon}{4}x_t + \frac{\varepsilon}{4}x_m x_t^2 + \frac{1}{2}(k_2-k_1) + \frac{\varepsilon}{4}x_m^2 \quad x_t - x_m \quad \frac{1}{2}(k_2-k_1) - \frac{\varepsilon}{4}x_m^2 \quad = 0$$

But from definition $k_2 = k_1 + 3\mathcal{E} x_m^2$ or $k_2 - k_1 = 3\mathcal{E} x_m^2$.

Hence:

$$x_t^3 + x_m x_t^2 + 7x_m^2 x_t - 5x_m^3 = 0 \qquad (1\text{-}68)$$

To three place accuracy the solution of (1-68) is:

$$x_t = \frac{5}{8} x_m \qquad (1\text{-}69)$$

This means that the transition amplitude x_t is only a function of the maximum displacement x_m, and is independent of the non-linear parameter $\mathcal{E}$. It will be shown in Chapter 2 that such a simple relation exists for any power of non-linearity.

2) Numerical Example:

The example treated by the exact method will now be solved by the bi-linear method. Since $h(x) = x + 10x^3$ and $x_o = x_m = 1$, then $k_1 = 1$, $k_2 = 31$. The spring forces will be

$$g_1(x) = x, \qquad g_2(x) = 31x - 30x_t \qquad (1\text{-}70)$$

For a cubic non-linearity x_t was found to be $\frac{5}{8} x_m$.

Since $x_m = 1$, $x_t = .625$, then

from (1-60) $t_t = 0.272$

from (1-61) $r = \dot{x}_2(t_t) = -2.2$

from (1-63) $\phi = 1.021$ rad., $A = 2.29$

from (1-64) $T = 2.20$

It is seen that even for a non-linear force which is ten times as large as the linear one at the point of maximum deflection, such a simple approach to the problem gives a frequency which is less than one percent different from the exact value of 2.19.

A comparison of all of the solutions to the homogeneous equation obtained by different methods is shown in Figure 1.

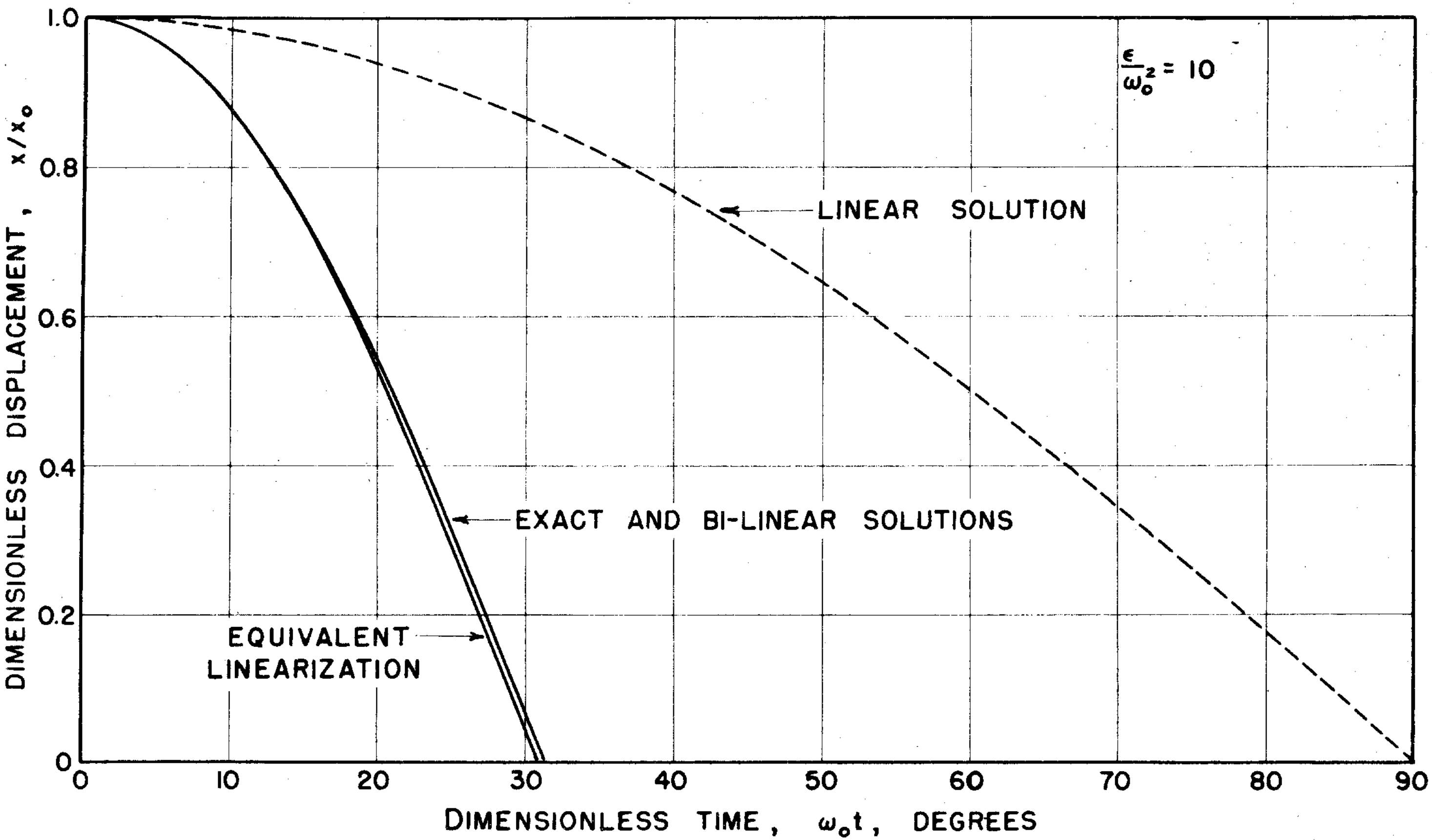

FIGURE 1- COMPARATIVE QUARTER PERIOD SOLUTION OF THE DIFFERENTIAL EQUATION $\ddot{x} + \omega_o^2 x + \epsilon x^3 = 0$ with $\dot{x}(0) = 0$, $x(0) = x_o$

Chapter 2.

Transient Response to a Step Function

Previously outlined methods will now be extended to the solution of the step function response of a non-linear spring-mass system. The solutions obtained will again be compared by means of several numerical examples.

A. Kryloff and Bogoliuboff Method

Let the differential equation of motion be:

$$m\ddot{x} + K \; x + bx^3 = F$$

or

$$\ddot{x} + \omega_o^2 x + \varepsilon x^3 = F_o \text{ where } \omega_o^2 = \frac{k}{m}, \varepsilon = \frac{b}{m}, \text{ and } F_o = \frac{F}{m} \qquad (2\text{-}1)$$

Furthermore let (y) be a new coordinate such that $y = x - \frac{F_o}{\omega_o^2}$, then Equation (2-1) becomes:

$$\ddot{y} + \omega_o^2 y = -\varepsilon(y + \frac{F_o}{\omega_o^2})^3 \qquad (2\text{-}2)$$

Following the classical Kryloff and Bogoliuboff procedure (see Appendix A), expand ($\ddot{y}$), the amplitude (a), and the unknown frequency (Ψ) in terms of the small parameter (ε). Hence:

$$y = -a \cos\Psi + \varepsilon y_1(a, \Psi) + 0(\varepsilon^2) \qquad (2\text{-}3)$$

$$\frac{d\Psi}{dr} = \omega_o + \varepsilon \tilde{\Psi}(a), \text{ and} \qquad (2\text{-}4)$$

$$\frac{da}{dt} = \varepsilon A + 0(\varepsilon^2) \qquad (2\text{-}5)$$

At this point some work can be saved by noticing that since the system is a conservative one, the amplitude (a) must remain constant; this means that $\frac{da}{dt} = 0$, so that Equation (2-5) can be dropped.

Differentiating y from Equation (2-3) with respect to time, and putting the value of $\dot{\Psi}$ in:

$$\frac{dy}{dt} = \dot{y} = a\dot{\Psi}\sin\Psi + \varepsilon\dot{\Psi}\frac{dy_1}{d\Psi} = a\omega_o \sin\Psi + \varepsilon(a\bar{\Psi}\sin\Psi + \omega_o\frac{dy_1}{d\Psi}) + 0(\varepsilon^2)$$

and differentiating it once again

$$\frac{d^2y}{dt^2} = \ddot{y} = a\omega_o^2 \cos\Psi + \varepsilon\ 2a\omega_o\bar{\Psi}\cos\bar{\Psi} + \omega_o^2\frac{d^2y_1}{d\Psi^2} \tag{2-6}$$

Substituting the values of y and $\ddot{y}$ in the differential Equation (2-2), and neglecting the terms of the order of ε^2 or higher one gets:

$$\omega_o^2 \left(\frac{d^2y_1}{d\Psi^2} + y_1\right) = a^3\cos^3\Psi - 3\left(\frac{F_o}{\omega_o^2}\right)a^2\cos^2\Psi + 3\left(\frac{F_o}{\omega_o^2}\right)a\cos\Psi - \left(\frac{F_o}{\omega_o^2}\right)^3 - 2a\omega_o\bar{\Psi}\cos\Psi \tag{2-7}$$

Expanding $\cos^2\Psi$ and $\cos^3\Psi$ and collecting terms, the final form of Equation (2-7) becomes:

$$\frac{d^2y_1}{d\Psi^2} + y_1 = \left[\frac{3a^3}{4\omega_o^2} + \frac{3a}{\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right)^2 - \frac{2a}{\omega_o}\bar{\Psi}\right]\cos\Psi - \frac{3a^2}{2\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right)\cos 2\Psi + \frac{a^3}{4\omega_o^2}\cos 3\Psi - \frac{1}{\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right)\left[\frac{3}{2}a^2 + \left(\frac{F_o}{\omega_o^2}\right)^2\right] \tag{2-8}$$

It can be seen from Equation (2-8) that in order to eliminate the possibility of secular terms in the solution, the coefficient of $\cos\Psi$ term must be equal to zero. This gives the frequency correction term as:

$$\bar{\Psi} = \frac{3}{8}\frac{a^2}{\omega_o} + \frac{3}{2}\frac{1}{\omega_o}\left(\frac{F_o}{\omega_o^2}\right)^2$$

Hence the frequency becomes:

$$\Psi = \left\{\omega_o + \frac{3}{8}\varepsilon\frac{a^2}{\omega_o} + \frac{3}{2}\frac{\varepsilon}{\omega_o}\left(\frac{F_o}{\omega_o^2}\right)^2\right\}t + \phi \tag{2-10}$$

Equation (2-8) will then take the form:

$$\frac{d^2y_1}{d\Psi^2} + y_1 = \frac{a^3}{4\omega_o^2}\cos 3\Psi - \frac{3a^2}{2\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right)\cos 2\Psi - \frac{1}{\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right) \times \left[\frac{3}{2}a^2 + \left(\frac{F_o}{\omega_o^2}\right)^2\right] \tag{2-11}$$

Solving the differential equation (2-11) one gets:

$$y_1 = -\frac{a^3}{32\omega_o^2}\cos 3\Psi + \frac{a^2}{2\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right)\cos 2\Psi - \frac{1}{\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right)\left[\frac{3}{2}a^2 + \left(\frac{F_o}{\omega_o^2}\right)^2\right] \tag{2-12}$$

since $x = y + \frac{F_o}{\omega_o^2} = -a\cos\Psi + \frac{F_o}{\omega_o^2} + \varepsilon y_1$ or

$$x = -a\cos\Psi + \frac{F_o}{\omega_o^2} + \varepsilon\ -\frac{a^3}{32\omega_o^2}\cos 3\Psi + \frac{a^2}{2\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right)\cos 2\Psi - \frac{1}{\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right) \times \left[\frac{3}{2}a^2 + \left(\frac{F_o}{\omega_o^2}\right)^2\right] \tag{2-13}$$

the initial conditions to be satisfied are:

$$x(0) = \dot{x}(0) = 0$$

It is seen that the condition $\dot{x}(0)=0$ is satisfied by taking the phase angle ϕ to be zero in Equation (2-10).

The condition that $x(0)=0$ gives the equation that determines (a). Hence:

$$0 = -a + \frac{F_o}{\omega_o^2} - \frac{\varepsilon}{32\omega_o^2}a^3 - \frac{\varepsilon a^2}{\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right) - \frac{\varepsilon}{\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right)^3$$

or

$$a^3 + 32\left(\frac{F}{\omega_o^2}\right)a^2 + \frac{32\omega_o^2}{\varepsilon}a + 32\frac{F_o}{\omega_o^2}\left[\left(\frac{F_o}{\omega_o^2}\right)^2 - \frac{\omega_o^2}{\varepsilon}\right] = 0 \tag{2-14}$$

This amplitude equation introduces several limitations on the non-linearity and the physical parameters of the system. It can be seen that in order to insure at least one positive real root

$$\frac{\omega_o^2}{\varepsilon} \geqslant \left(\frac{F_o}{\omega_o^2}\right)^2 .$$

This only means that the non-linear force should be small compared to the linear one. It also brings out the fact that the important factor that affects the accuracy of the solution is the size of the non-linear force. Since this is one of the original assumptions in the Kryloff and Bogoliuboff procedure, it is not really an added restriction on the method. An interesting fact is that this restriction does not appear when some of the other methods, such as equivalent linearization, are applied to the same problem.

One can put a further restriction on the amplitude equation (2-14) which at first glance seems to be artificial or unnecessary. Satisfying the above inequality insures only at least one real and positive root. Since from the physics of the problem, one must eliminate the possibilities of any negative amplitudes, the condition that one and only one real root should exist can be applied to the amplitude equation. This condition states that the discriminant of the cubic equation must be negative.

For a cubic equation of the form

$$z^3 + pz^2 + rz + s = 0$$

the discriminant is

$$\Delta = 18\,prs - 4p^3 s + p^2 r^2 - 4r^3 - 27s^2 \qquad (2\text{-}15)$$

For Equation (2-14), then, the discriminant becomes:

$$\Delta = 32^3 \left\{(\alpha - \beta)(18\,\tfrac{27}{32}\,\alpha\beta - 128\,\tfrac{27}{32}\,\alpha^2) + 32\alpha\beta^2 - 4\beta^3\right\} \qquad (2\text{-}16)$$

where $\alpha = \left(\frac{F_o}{\omega_o^2}\right)^2$ and $\beta = \frac{\omega_o^2}{\varepsilon}$

Writing the condition for the existence of one real root only:

$$32\beta^3 \left\{ \left(\frac{\alpha}{\beta} - 1\right) \quad 18\,\frac{27}{32}\left(\frac{\alpha}{\beta}\right) - 128\,\frac{27}{32}\left(\frac{\alpha}{\beta}\right)^2 \quad +32\left(\frac{\alpha}{\beta}\right) - 4 \right\} < 0 \tag{2-17}$$

Letting $\frac{\alpha}{\beta} = \gamma$, and noting that $\beta > 0$, Equation (2-17) can be re-written as:

$$(\gamma - 1)\left(18\,\frac{27}{32}\,\gamma - 128\,\frac{27}{32}\,\gamma^2\right) + 32\gamma - 4 < 0 \tag{2-18}$$

remembering that $0 < \gamma < 1$ for positive real root, the inequality (2-18) gives the result that γ must be between zero and 0.13; or

$$0 < \gamma \leqslant .13$$

putting this in terms of the initial parameters one gets:

$$0 < \left(\frac{F_o}{\omega_o^2}\right)^2 \left(\frac{\omega_o^2}{\varepsilon}\right) \leqslant .13 \tag{2-19}$$

This second condition limits the allowable non-linearity more than the initial one. It is an artificial restriction since it is difficult to find a physical justification for it. The question also arises as to whether one could allow three real roots, only one of which is positive, thus discarding the negative roots on physical grounds. Such an approach, of course, eliminates the limitation imposed upon by inequality (2-19). An interesting point, however, is that in the three examples treated in which the inequality (2-19) was not satisfied, the solutions were off as much as 100 percent or more. Although the complete mathematical explanation is not apparent this condition seems to guarantee results which are within about ten percent of the exact frequency and amplitude.

If, instead of stopping at the first approximation as it was done above, the treatment is carried out through second or higher approximations, it can reasonably be assumed that the restriction on the non-linearity will become less and less severe. However, the complexity of such an analysis makes it impractical to carry out the calculations. As was mentioned before, such a restriction does not occur when the problem is treated in somewhat different fashion, as will be shown below.

B. Method of Equivalent Linearization

The assumption of small non-linear forces made it possible to formulate an approximate solution of the non-linear equation by the use of Kryloff and Bogoliuboff's method. The accuracy of the approximation is limited only by the amount of work that can be carried out in a reasonable time. In general, however, it is found that a first approximation with an error of the order of ε^2 (non-linear parameter squared) gives satisfactory results.

In the solution of the homogeneous equation it was shown that the equivalent linearization method simplified the procedure of obtaining a first approximation without changing the results. In the case of a step function response, however, it gives answers which are different from those obtained by the Kryloff and Bogoliuboff method. For this reason, the method will be applied here in somewhat more detail than was done for the solution of the homogeneous problem. It will also be shown that a somewhat new approach to the same problem gives a better physical insight to the meaning of equivalent linearization.

Let the differential equation of motion be written in the form

$$\ddot{x} + \omega_o^2 x = -\varepsilon f(x) + F_o \tag{2-20}$$

where f(x) is some non-linear function of x. Since x is a periodic function of time, and since the solution of Equation (2-20) is assumed to be quasi-linear such that the resulting oscillations are "almost" sinusoidal, x can be taken in the form

$$x = a \cos\omega t \text{(the phase angle is neglected because of zero initial velocity)} \tag{2-21}$$

The problem is now to find a relationship between the amplitude (a) and the unknown frequency (ω) such that the differential equation (2-20) is satisfied as closely as possible by the solution (2-21). For this purpose, the criterion can be chosen as the mean square error over one cycle between the left and the right hand sides of Equation (2-20). This error is to be minimized.

If the mean square error is (E^2):

$$E^2 = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \left\{ \ddot{x} + \omega_o^2 x + \varepsilon f(x) - F_o \right\}^2 dt \tag{2-22}$$

Putting the value of (x) in from Equation (2-21):

$$E^2 = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \left\{ a(\omega_o^2 - \omega^2)\cos\omega t + \varepsilon f(x) - F_o \right\} dt \tag{2-23}$$

since (E^2) is to be minimized by the proper choice of (a) and (ω):

$$\frac{\partial (E)^2}{\partial a} = 0$$

Hence:

$$\frac{\omega}{2\pi} \int_0^{2\pi/\omega} 2 \left\{ a(\omega_o^2 - \omega^2)\cos\omega t + \varepsilon f(x) - F_o \right\} (\omega_o^2 - \omega^2)\cos\omega t \, dt = 0 \tag{2-24}$$

or letting $\omega t = \theta$

$$\int_0^{2\pi} \left\{a(\omega_o^2 - \omega^2)\cos\theta + \varepsilon f(x) - F_o\right\}\cos\theta\, d\theta = 0$$

Performing the integration

$$\frac{a}{2}(\omega_o^2 - \omega^2)(2\pi) + \varepsilon \int_0^{2\pi} f(x)\cos\theta\, d\theta = 0$$

solving for ω^2 :

$$\omega^2 = \omega_o^2 + \frac{\varepsilon}{\pi a}\int_0^{2\pi} f(x)\cos\theta d\theta \qquad (2\text{-}25)$$

It is seen that Equation (2-25) is exactly the same as Equation (5-21) derived in Appendix A.

In the above analysis nothing was said about f(x). The most accurate value of the frequency will be obtained if the exact value of (x) is substituted in f(x); however, if one is limited to first approximation only, f(x) must then be a function of zeroth approximation in (x), that is $(f(x) = f(a \cos\omega t)$.

There are no complications introduced if (f) is a function of $(\dot{x})$ as well as (x). The same analysis can also be carried out for forced oscillations with equal success.

The equivalent linearization method will now be used for the solution of the step function response of a system with cubic nonlinearity in the spring force.

Let the equation of motion be

$$m\ddot{x} + kx + bx^3 = F$$

or

$$\ddot{x} + \omega_o^2 x + \varepsilon x^3 = F_o \quad \text{where } \omega_o^2 = \frac{k}{m},\ \varepsilon = \frac{b}{m},\ \text{and } F_o = \frac{F}{m} \qquad (2\text{-}26)$$

Again making a transformation of the form

$$x = \frac{F_o}{\omega_o^2} + y \qquad (2\text{-}27)$$

(2-26) becomes

$$\ddot{y} + \omega_o^2 y = -\varepsilon (y + \frac{F_o}{\omega_o^2})^3 \qquad (2\text{-}28)$$

The zeroth approximation of this differential equation (that is to say for $\varepsilon = 0$) will be:

$$y^o = a \cos \Psi \quad \text{where } \Psi = \omega_o t + \phi(t) \qquad (2\text{-}29)$$

Then using Equation (5-23) the equivalent spring constant is found to be:

$$\bar{k} = k + \frac{\varepsilon}{\pi a} \int_o^{2\pi} (a \cos t + \frac{F_o}{\omega_o^2})^3 \cos\Psi d\Psi \qquad (2\text{-}30)$$

or, integrating (2-30)

$$\bar{k} = k + 3\varepsilon \left\{ \frac{a^2}{4} + (\frac{F_o}{\omega_o^2})^2 \right\} \qquad (2\text{-}31)$$

Since $\bar{\lambda} = 0$, the equivalent differential equation will be of the form:

$$\ddot{y} + \omega_o^2 + 3\varepsilon \left\{ \frac{a^2}{4} + (\frac{F_o}{\omega_o^2})^2 \right\} y=0 \qquad (2\text{-}32)$$

The solution of (2-32) is

$$y = a \cos \Psi \quad \text{where } \Psi = \left\{ \omega_o^2 + 3\varepsilon \ \frac{a^2}{4} + (\frac{F_o}{\omega_o^2})^2 \right\}^{1/2} t \qquad (2\text{-}33)$$

Assuming $\frac{3\varepsilon}{\omega_o^2} \left\{ \frac{a^2}{4} + (\frac{F_o}{\omega_o^2})^2 \right\} \ll 1$ (2-33) can be rewritten as:

$$\Psi = \omega_o \left\{ 1 + \frac{3}{2} \ \frac{\varepsilon}{\omega_o^2} \ \frac{a^2}{4} + (\frac{F_o}{\omega_o^2})^2 \right\} t \qquad (2\text{-}34)$$

It is seen that this is exactly the same frequency obtained by the Kryloff and Bogoliuboff method in Section A. (See Equation (2-10)).

It should also be noted that as ε gets larger the frequency computed by Equation (2-33) will give better results than that computed by (2-34), since the first two terms of the binomial expansion no longer give the required accuracy. It is reasonable to say that as the non-linearity gets large one should expand the square of the frequency in perturbation procedures and not the frequency as was done in the Kryloff and Bogoliuboff method.

The initial conditions for Equation (2-33) are

$$y(0) = -\frac{F_o}{\omega_o^2}\ , \text{ and } \dot{y}(0) = 0 \tag{2-35}$$

Since the phase angle is zero $\dot{y}(0) = 0$ is automatically satisfied.

The condition $y(0) = -\frac{F_o}{\omega_o^2}$ gives $a = -\frac{F}{\omega_o^2}$.

The solution of the differential equation (2-26) will then be

$$x = \frac{F_o}{\omega_o^2}(1 - \cos\Psi) \tag{2-35b}$$

This is the solution to the linear equation ($\varepsilon= 0$) with a correction in frequency. However, the Kryloff and Bogoliuboff method used before also indicated a correction for the so-called d-c level which is defined as the position of equilibrium about which the oscillations take place. It will now be shown that a different approach to the problem will give the required correction to the d-c level.

If one attacks the problem without first making the transformation, such that the differential equation to be solved is:

$$\ddot{x} + \omega_o^2 x + \varepsilon x^3 = F_o \tag{2-36}$$

the zeroth approximation to this non-linear equation will be:

$$x = A(1-\cos\omega t) \tag{2-37}$$

where again A and ω are slowly varying functions of time. Hence the equivalent frequency will be:

$$\omega^2 = \omega_o^2 + \frac{\varepsilon}{\pi A} \int_0^{2\pi} A^3(1-\cos\theta)^3 \cos\theta \, d\theta \tag{2-38}$$

or, integrating (2-38)

$$\omega^2 = \omega_o^2 + \frac{15}{4} \varepsilon A^2 \tag{2-39}$$

One can then write the equivalent linear differential equation as:

$$\ddot{x} + (\omega_o^2 + \frac{15}{4} \varepsilon A^2)x = F_o \tag{2-40}$$

The solution of (2-40) will give:

$$x = \frac{F_o}{\omega_o^2 + \frac{15}{4} \varepsilon A^2} (1-\cos\omega t) \tag{2-41}$$

where

$$\omega = \left\{\omega_o^2 + \frac{15}{4} \varepsilon A^2\right\}^{1/2} \tag{2-42}$$

or for cases when $\frac{15}{4} \varepsilon \frac{A^2}{2} << 1$ one gets:

$$\omega = \omega_o (1 + \frac{15}{8} \frac{\varepsilon A^{2^o}}{\omega_o^2}) \tag{2-43}$$

From the comparison of Equations (2-37) and (2-41) it is seen that

$$A = \frac{F_o}{\omega_o^2 + \frac{15}{4}\varepsilon A^2} \tag{2-44}$$

or

$$A^3 + \frac{4}{15}\frac{\omega_o^2}{\varepsilon} A - \frac{4}{15}\frac{F_o}{\varepsilon} = 0 \tag{2-45}$$

Equation (2-45) gives the required amplitude equation. It will be noticed that Equation (2-45) differs in character from Equation (2-14) in several respects. First and most important is that it always has one real positive root for the amplitude. There are no limitations on the size of the non-linearity from the physical considerations. Mathematically, of course, the results will not be accurate for large non-linearities.

Frequency correction terms corresponding to two methods will be the same if the amplitude (a) in Kryloff and Bogoliuboff's method is replaced by $\frac{F_o}{\omega_o^2}$.

A more complete comparison and analysis of the above and the following methods will be made after a sample problem is solved and the results are checked against the exact solution.

C. Classical Perturbation or Lindstedt Procedure

From the point of view of accuracy, it is obvious that Kryloff and Bogoliuboff's procedure will fail completely when the non-linearities become of the same order as the linear terms. Of course the main reason for this lies in the fact that the non-linear frequency was assumed to be in the neighborhood of the linear one and a power series expansion in terms of the non-linear parameter in this neighborhood was justified. For positive large non-linearities in the spring constant, however, the non-linear frequency may be several times the value of the linear term; in this case one would expect that an expansion of very large numbers of terms would be required for any reasonably accurate answer. The difficulty of such a procedure is self evident.

It will now be shown that in cases of large non-linear forces the so-called classical perturbation procedure gives much better answers because of the fact that it converges much more rapidly. The original Equation (2-1) will again be solved by this method.

The differential equation of motion was:

$$\ddot{x} + \omega_o^2 x = -\varepsilon x^3 + F_o \tag{2-46}$$

if the solution is to be taken of the form:

$$x = x_o + \varepsilon x_1 + 0(\varepsilon^2) \tag{2-47}$$

The initial conditions will be homogeneous, such that $x_o(0) = \dot{x}_o(0) = x_1(0) = \dot{x}_1(0) = 0$.

Let the linear frequency (ω_o^2) be expanded in the neighborhood of the unknown non-linear frequency (ω^2) in terms of the non-linear parameter (this is the opposite of the Kryloff and Bogoliuboff expansion in which ω^2 is expanded in the neighborhood of ω_o^2). If one neglects the terms of the order of ε^2:

$$\omega_o^2 = \omega^2 + \varepsilon \omega_1^2 \tag{2-48}$$

where ω_1^2 is a correction term to be determined from the elimination of secular terms.

Substituting Equations (2-47) and (2-48) in the differential equation (2-46) one gets:

$$(\ddot{x}_o+\varepsilon\ddot{x}_1)+(\omega^2+\varepsilon\omega_1^2)(x_o+\varepsilon x_1)=F_o-\varepsilon(x_o+\varepsilon x_1)^3$$

Collecting terms of like powers of ε, and neglecting the ones with an order of ε^2 or higher, two following equations are obtained:

$$\ddot{x}_o+\omega^2 x_o = F_o \tag{2-49}$$

and

$$\ddot{x}_1+\omega^2 x_1 = -x_o^3 - \omega_1^2 x_o \tag{2-50}$$

The solution of Equation (2-49) that satisfies the initial conditions will be:

$$x_o = \frac{F_o}{\omega^2}(1-\cos\omega t) \tag{2-51}$$

Putting this value of x_o into Equation (2-50):

$$\ddot{x}_1+\omega^2 x_1 = -\omega_1^2\left(\frac{F_o}{\omega^2}\right)(1-\cos\omega t)-\left(\frac{F_o}{\omega^2}\right)^3(1-\cos\omega t)^3 \tag{2-52}$$

If $(1-\cos\omega t)$ is cubed, and the values of $\cos^2\omega t$ and $\cos^3\omega t$ are substituted by their equivalents $\frac{1}{2}(\cos 2\omega t+1)$ and $\frac{1}{4}(\cos 3\omega t+3\cos\omega t)$ respectively, Equation (2-52) becomes:

$$\begin{aligned}\ddot{x}_1+\omega^2 x_1 = &-\omega_1^2\left(\frac{F_o}{\omega^2}\right)(1-\cos\omega t)\\ &-\left(\frac{F_o}{\omega^2}\right)^3\left\{1-3\cos\omega t+\frac{3}{2}(1+\cos 2\omega t)-\frac{1}{4}(\cos 3\omega t+3\cos\omega t)\right\}\end{aligned} \tag{2-53}$$

collecting terms and simplifying.

$$\begin{aligned}\ddot{x}_1+\omega^2 x_1 = &\left\{\omega_1^2\left(\frac{F_o}{\omega^2}\right)+\frac{15}{4}\left(\frac{F_o}{\omega^2}\right)^3\right\}\cos\omega t\\ &-\frac{3}{2}\left(\frac{F_o}{\omega^2}\right)^3\cos 2\omega t+\frac{1}{4}\left(\frac{F_o}{\omega^2}\right)^3\cos 3\omega t-\omega_1^2\left(\frac{F_o}{\omega^2}\right)-\left(\frac{F_o}{\omega^2}\right)^3\end{aligned} \tag{2-54}$$

In order that x_1 be free of any secular terms the coefficient of $\cos \omega t$ must be equal to zero. Hence:

$$\omega_1^2 \left(\frac{F_o}{\omega^2}\right) + \frac{15}{4}\left(\frac{F_o}{\omega^2}\right)^3 = 0$$

or

$$\omega_1^2 = -\frac{15}{4}\left(\frac{F_o}{\omega^2}\right)^2 \tag{2-56}$$

putting this value of ω_1^2 in the frequency equation (2-48) and solving for the unknown frequency ω^2:

$$\omega^2 = \omega_o^2 + \frac{15}{4}\varepsilon\left(\frac{F_o}{\omega^2}\right)^2 \tag{2-57}$$

or

$$(\omega^2)^3 - \omega_o^2(\omega^2)^2 - \frac{15}{4}\varepsilon F_o^2 = 0 \tag{2-58}$$

It is easy to see that for any positive value of the parameters F_o, ε, and ω_o, there will be only one positive root of the cubic Equation (2-58) which will give the unknown non-linear frequency (ω^2). This is where the classical procedure outlined above differs from the Kryloff and Bogoliuboff method. Here there is a cubic equation in frequency squared, while in the Kryloff procedure, the amplitude was determined from a cubic equation.

It is also important to note that the correction term in Equation (2-57) is of the form $\varepsilon\left(\frac{F_o}{\omega^2}\right)^2$, while in the Kryloff method it was $\varepsilon\left(\frac{F_o}{\omega_o^2}\right)^2$. Since with increased non-linearity the non-linear frequency (ω) also increases, the correction term in Equation (2-57) remains small compared to ω^2 even for quite large non-linearities; however the correction term $\varepsilon\left(\frac{F_o}{\omega_o^2}\right)^2$ from the Kryloff method will obviously be proportional to the non-linearity, and hence may be as large as the zeroth order term. This explains why one gets a better approximation to the frequency, and a wider range of applicability by using the classical approach.

If Equation (2-54) is solved for x_1, and the value of ω_1^2 is substituted from Equation (2-56):

$$x_1 = b \cos\omega t - \frac{1}{2}\left(\frac{F_o}{\omega^2}\right)^3 \cos 2\omega t + \frac{1}{32}\left(\frac{F_o}{\omega^2}\right)\cos 3\omega t + \frac{11}{4}\left(\frac{F_o}{\omega^2}\right)^3 \qquad (2\text{-}59)$$

Since the initial condition $x_1(0) = 0$ must be satisfied

$$b = -\frac{73}{32}\left(\frac{F_o}{\omega^2}\right)^3$$

Putting the above values in Equation (2-47), the solution for (x) is found to be:

$$x = \left(\frac{F_o}{\omega^2}\right)\left\{1 + \frac{11}{4}\varepsilon\left(\frac{F_o}{\omega^2}\right)^2\right\} - \left(\frac{F_o}{\omega^2}\right)\left\{1 + \frac{73}{32}\varepsilon\left(\frac{F_o}{\omega^2}\right)^2\right\}\cos\omega t + \varepsilon\left(\frac{F_o}{\omega^2}\right)^3\left\{-\frac{1}{2}\cos 2\omega t + \frac{1}{32}\cos 3\omega t\right\} \qquad (2\text{-}60)$$

Here again all of the correction terms are functions of $\varepsilon\left(\frac{F_o}{\omega^2}\right)$ and will tend to remain small even for large non-linearities.

In the following pages two examples will be solved by the above mentioned methods. Comparisons of maximum deflection and half period will be made with the exact solution obtained by numerical integration.

D. Numerical Examples

Before any actual calculations are made a short outline of the procedure used to obtain the exact solutions will be given.

As before the differential equation is:

$$\ddot{x} + \omega_o^2 x + \varepsilon x^3 = F_o \qquad (2\text{-}61)$$

with the initial conditions $x(0) = \dot{x}(0) = 0$.

Let $v = \dot{x}$ so that $\ddot{x} = v \frac{dv}{dx}$

Putting the value of $\ddot{x}$ in Equation (2-61) and integrating:

$$\int_0^v v dv = \int_0^x (F_o - \omega_o^2 x - \varepsilon x^3)\, dx \tag{2-62}$$

or

$$\frac{1}{2} v^2 = F_o x - \frac{\omega_o^2}{2} x^2 - \frac{\varepsilon}{4} x^4$$

or

$$v = \frac{dx}{dt} = \sqrt{2F_o x - \omega_o^2 x^2 - \frac{\varepsilon}{2} x^4} \tag{2-63}$$

Integrating once more

$$t = \int_0^x \frac{dx}{\sqrt{2F_o x - \omega_o^2 x^2 - \frac{\varepsilon}{2} x^4}} \tag{2-64}$$

The maximum deflection can directly be determined from the condition: at $x = x_m$, $v = 0$ or

$$g(x) = 2F_o x_m - \omega_o^2 x_m^2 - \frac{\varepsilon}{2} x_m^4 = 0 \tag{2-65}$$

The non-zero root of Equation (2-65) or the root of the cubic equation $\frac{\varepsilon}{2} x_m^3 + \omega_o^2 x_m - 2F = 0$ will give the required maximum deflection. The form of the cubic equation shows that there will be only one real positive root and no negative roots. This real root will be the maximum amplitude of the oscillations. The half period is then found by integrating (2-64) by numerical or graphical methods between the limits $x = 0$ and $x = x_m$. That is:

$$\frac{T}{2} = \int_0^{x_m} \frac{dx}{\sqrt{g(x)}} \tag{2-66}$$

Example 1. (See Figure 2 for graphical comparison).

The differential equation of motion is taken as

$$\ddot{x} + x + 0.1\,x^3 = 1$$

and the results of the solutions by different methods are shown in Figure 2.

a. Exact solution.

Equation (2-65) becomes:

$2x - x^2 - .05\,x^4 = 0$ which has the roots $x = 0$ and $x = 1.74$

Hence: Maximum deflection $\underline{x_m = 1.74}$

Half period $\underline{\frac{T}{2} = 2.85}$

The integration (2-66) was performed graphically with an accuracy of result of $\pm$ .5 percent.

From the above results the non-linear spring force can be calculated to be about 17 percent of the linear one.

b. Kryloff and Bogoliuboff solution:

From the values of the parameters chosen, it is seen that both inequalities (2-19) and $\frac{\omega_o^2}{\varepsilon} \geq \left(\frac{F_o}{\omega_o^2}\right)^2$ are satisfied. Equation (2-14) for the amplitude becomes:

$$a^3 + 32a^2 + 320a - 288 = 0 \qquad (2\text{-}67)$$

Solution of (2.67) gives $a = .83$.

From Equation (2-10)

$\psi = (1 + .026 + .15)t$ or half period

$$\frac{T}{2} = \frac{\pi}{1.176} = \underline{2.67}$$

The maximum displacement is found from Equation (2-13) by letting $\psi = \pi$ which gives: $\underline{x_m = 1.66}$

The deviation from the exact value is found to be 6.3 percent in frequency and 4.6 percent in maximum amplitude.

c. Equivalent linearization solution:

The amplitude (A) is found from Equation (2-45) which becomes:

$$A^3 + \frac{8}{3} A - \frac{8}{3} = 0 \quad \text{Solution of (2-68) is } A = .78 \qquad (2\text{-}68)$$

From Equation (2-39) the frequency is found to be

$\omega = \sqrt{1 + .228}$ or half period:

$$\frac{T}{2} = \frac{\pi}{1.06} = \underline{2.86}$$

Maximum deflection $x_m = 2A$, or $\underline{x_m = 1.56}$.

The deviation from the exact solution in this case is only 3.8 percent in frequency but 10.3 percent in amplitude.

d. Classical perturbation solution:

The solution of the frequency equation gives:

$\omega^2 = 1.24$ or $\omega = 1.113$

Half period $\frac{T}{2} = \frac{\pi}{1.113} = \underline{2.82}$

The maximum deflection is calculated from Equation (2-60) as:

$\underline{x_m = 1.80}$

This method gives the most accurate results with a deviation of about 1 percent in frequency and 3.5 percent in amplitude.

In the example worked out above the second and third harmonic terms, that is to say terms in $\cos 2\omega t$ and $\cos 3\omega t$ are only about 4 percent of the fundamental one. That is why the assumption of a pure sine as a solution gives such good results. It was shown in Chapter I that even with larger non-linearities higher harmonics always remain small compared to the fundamental mode. In fact the maximum value the third harmonic can have is about 11 percent which corresponds to a triangular wave.

There are three important factors that come out from the above analysis:

1) In the case of hard spring the classical perturbation method will always give results which are in very good agreement with the exact solution for steady state oscillations. In fact, in the case of the homogeneous non-linear equation with cubic spring constant it is shown that as the non-linearity gets very large the approximate frequency with only one correction term is within three percent of the exact one. Convergence of the frequency of forced oscillations can also be shown to be of the same order of magnitude as in the homogeneous equation as long as the forcing frequency is in the neighborhood of the natural frequency of the system. Far enough away from resonance the effect of sub or ultra harmonic terms may come in which would tend to distort the wave form on which the convergence of the perturbation method depends.

2) Kryloff and Bogoliuboff's method is quite accurate for small enough non-linearities (an aribtrary upper limit of about 15 percent can be set for the non-linear force in order to keep the results within about ten percent of the exact value). Its most important advantage lies in the fact that one can get transient behavior from such an analysis, while the Lindstedt method gives only steady state solutions.

3) If the higher harmonic terms are neglected in the solution, and if only the first approximations are desired, the equivalent

linearization method gives the least mean square error in the differential equation.

Example 2.

In this case the non-linear parameter is taken as unity. The differential equation, then, becomes:

$$\ddot{x} + x + x^3 = 1$$

The results of the solution of this problem by various methods are shown in graphical form in Figure 3.

E. Bi-linear Approximation for the Step Function Response of a Non-linear Spring-mass System

The solution of a non-linear differential equation of the form (2-1) will now be approximated by the bi-linear approach. The procedure outlined for the homogeneous case will be followed here, and the results can be checked with the exact solution already obtained.

1) General Approach

The two linear spring forces will be:

$$g_1(x) = k_1 x, \quad g_2(x) = k_2 x + x_t(k_1 - k_2) \tag{2-69}$$

Here x_t is again used to indicate the transition amplitude (the value of x at which spring constant is switched from k_1 to k_2 or vice versa) and x_o and $\dot{x}_o$ are again defined to be the initial amplitude and the velocity respectively. x_t is found as the non-zero root of the equation:

$$\int_{x_t}^{x_m} \left[g_2(x) - h(x) \right] dx = 0 \tag{2-70}$$

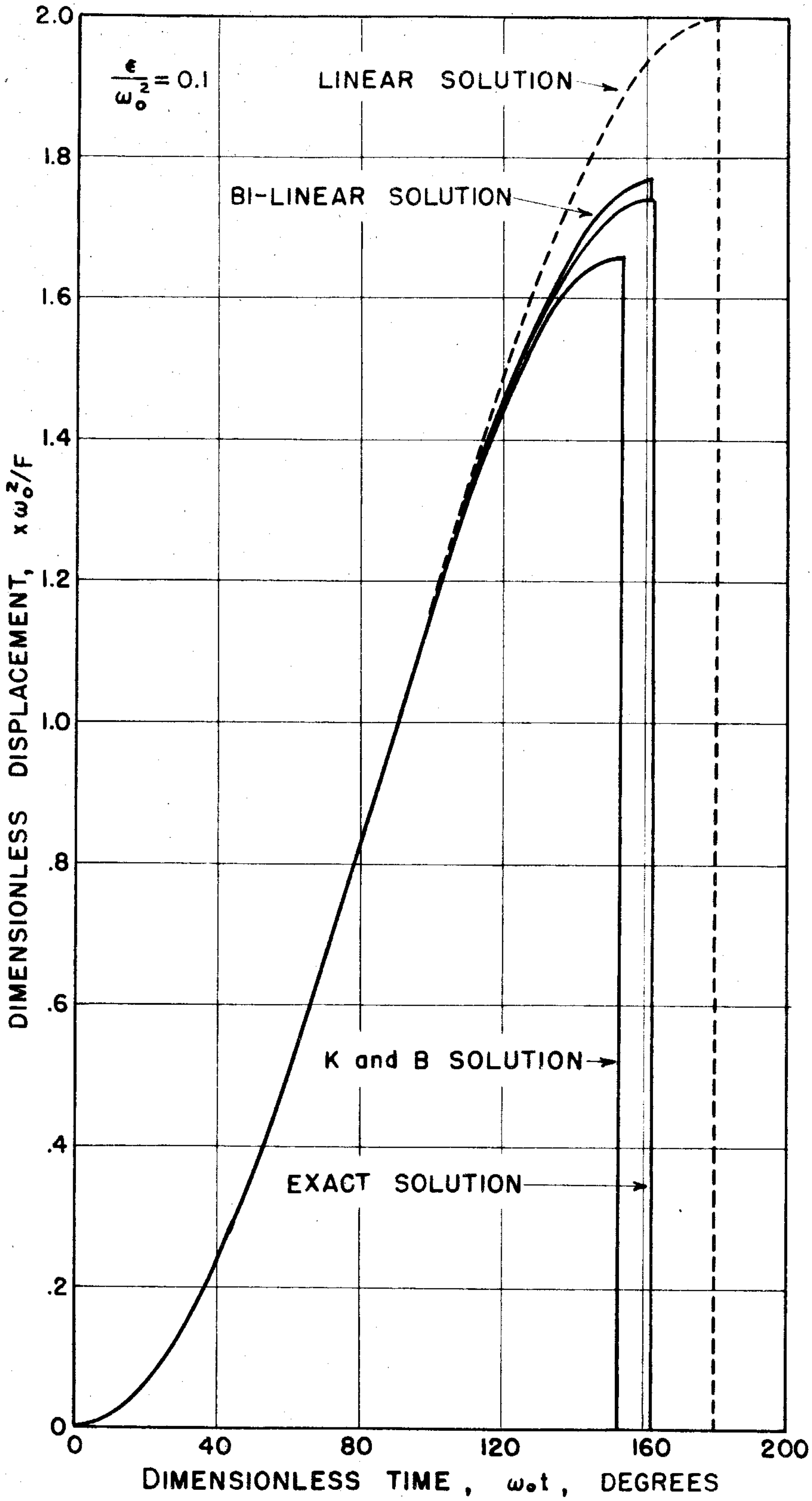

FIGURE 2 – COMPARATIVE HALF-PERIOD SOLUTIONS OF THE DIFFERENTIAL EQUATION $\ddot{x} + \omega_o^2 x + \epsilon x^3 = F$

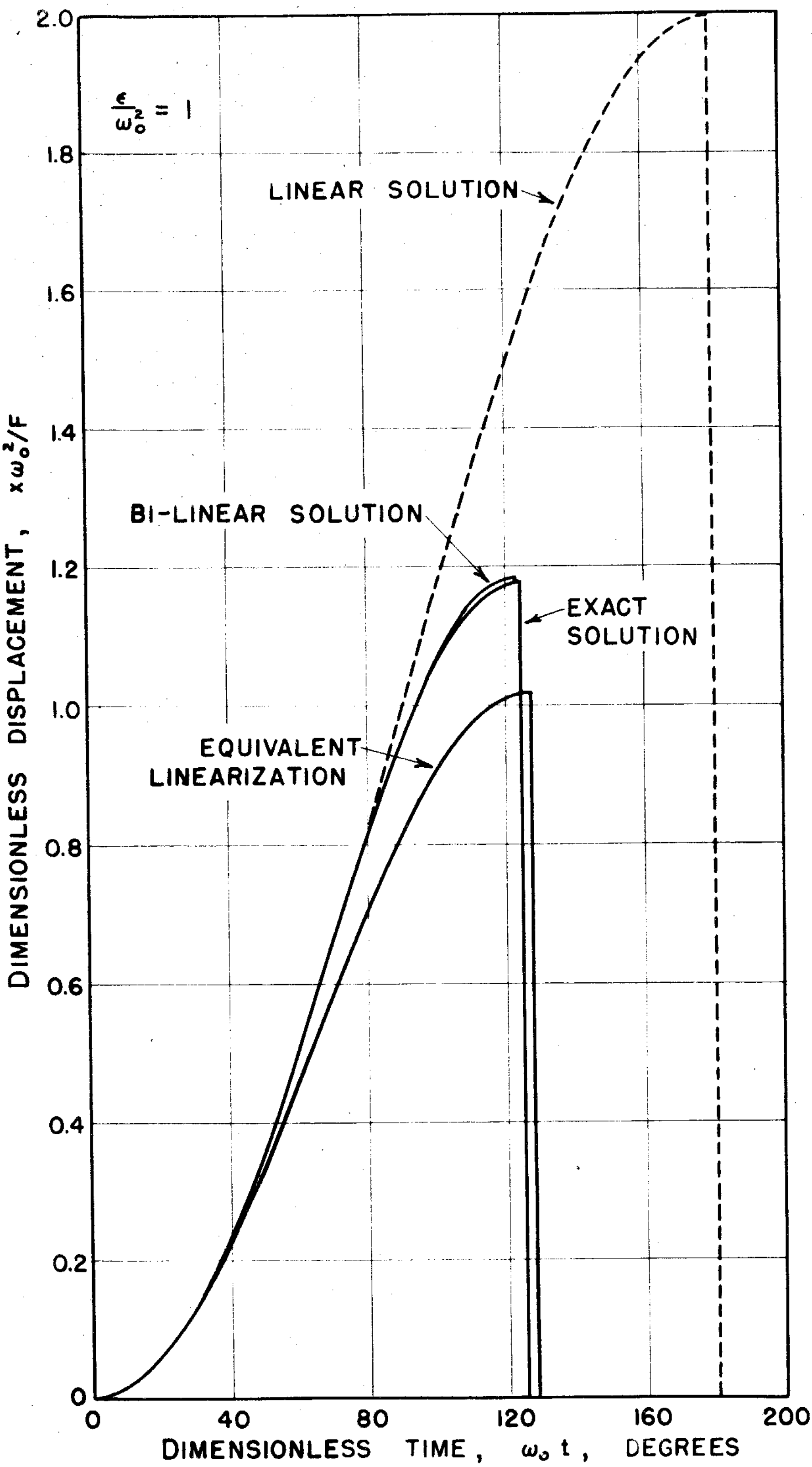

FIGURE 3 — COMPARATIVE HALF-PERIOD SOLUTIONS OF THE DIFFERENTIAL EQUATION $\ddot{x} + \omega_o^2 x + \epsilon x^3 = F$

where $h(x)$ is the non-linear spring force, and x_m the maximum displacement.

The initial conditions will be taken as:

$$x(0) = 0, \qquad \dot{x}(0) = 0 \tag{2-71}$$

The differential equations of motion are:

$$\ddot{x}_1 + k_1 x_1 = F_o \quad \text{and} \quad |x| < |x_t| \tag{2-72}$$

$$\ddot{x}_2 + k_2 x_2 + x_t (k_1 - k_2) = F_o \quad |x| > |x_t| \tag{2-73}$$

The initial conditions for x_1 will be:

$$\text{at } t = 0 \quad x_1(0) = \dot{x}_1(0) = 0 \qquad \text{and for } x_2\text{:} \tag{2-74}$$

at $t = t_t$ $\quad x_2(t_t) = x_1(t_t)$ and $\dot{x}_2(t_t) = \dot{x}_1(t_t)$

where t_t is the time required for x_1 to reach x_t.

The solution of Equation (2-72) satisfying the initial conditions is:

$$x_1 = \frac{F_o}{k_1}(1 - \cos\omega_1 t) \tag{2-75}$$

where $\omega_1 = \sqrt{k_1}$ since mass is taken as unity.

For $x_1 = x_t$

$$x_t = \frac{F_o}{k_1}(1 - \cos\omega_1 t_t)$$

or

$$t_t = \frac{1}{\omega_1} \cos^{-1}\left[1 - \frac{k_1 x_t}{F_o}\right] \tag{2-76}$$

Hence

$$\dot{x}_1(t_t) = \frac{F_o \omega_1}{k_1} \sin\omega_1 t_t = \sqrt{F_o x_t\left(2 - \frac{k_1 x_t}{F_o}\right)} \tag{2-77}$$

The solution of Equation (2-73) is:

$$x_2 = \frac{F_o - x_t(k_1 - k_2)}{k_2} + A\cos(\omega_2 t + \phi) \tag{2-78}$$

with the initial conditions that at $t = t_t$

$$x_2(t_t) = x_t \text{ and } \dot{x}_2(t_t) = \sqrt{F_o x_t \left(2 - \frac{k_1 x_t}{F_o}\right)}$$

Putting these conditions in:

$$x_t = \frac{F_o - x_t(k_1 - k_2)}{k_2} = -A\cos(\omega_2 t_t + \phi) \tag{2-79}$$

$$\sqrt{F_o x_t \left(2 - \frac{k_1 x_t}{F_o}\right)} = -A\omega_2 \sin(\omega_2 t_t + \phi) \tag{2-80}$$

Solving Equations (2-79) and (2-80) for A and ϕ:

$$A = \frac{F_o}{k_2}\sqrt{1 + (k_2 - k_1)\left(2\frac{x_t}{F_o} - k_1 \frac{x_t^2}{F_o^2}\right)} \tag{2-81}$$

and

$$\phi = \left(\tan^{-1}\sqrt{\frac{k_2 \frac{x_t}{F_o}\left(2 - k_1 \frac{x_t}{F_o}\right)}{1 - k_1 \frac{x_t}{F_o}}} - \omega_2 t_t\right) \tag{2-82}$$

2) Numerical example

The sample problem treated before will be now solved by the bi-linear method. The slopes k_1 and k_2 will be determined such that for $x = 0$ and $x = x_m$ they correspond to the slopes of the cubic non-linearity for which an exact value of half-period and maximum amplitude have already been obtained.

Since the spring force was taken to be of the form $(x+0.1\,x^3)$ and $x_m = 1.74$

$$k_1 = 1 \quad \text{and}$$
$$k_2 = 1.92 \tag{2-83}$$

For a cubic non-linearity x_t was calculated to be:

$$x_t = \frac{5}{8} x_m \qquad \text{(see Fig. 4)} \qquad (2\text{-}84)$$

for $k_1 = 1$, $k_2 = 1.92$, $x_m = 1.74$, and $x_t = 1.09$.

the bi-linear problem can be completely solved to give the period and the maximum amplitude. If the calculations are carried out one finds that the half period is 2.82, and that the maximum amplitude of oscillations is 1.77. As compared to the exact values of 2.85 for half period, and 1.74 for maximum amplitude they are in error by about one percent in period and two percent in amplitude. It should be mentioned here that as much as ten percent variation in x_m chosen to determine k_2 will change the results by only about one percent. This shows that the exact choice of transition point is not very critical in the results. A more general treatment of this question will be given later.

The complete solution of the bi-linear problem can be expressed as:

$$x_1 = 1 - \cos t \qquad \text{and for} \quad |x| < 1.09 \qquad (2\text{-}85)$$

$$x_2 = 1.05 + 0.72 \cos (1.38t + 137^{\circ}) \qquad |x| > 1.09 \qquad (2\text{-}86)$$

The Equations (2-85) and (2-86) are used in comparison of the wave forms in Figure 2.

3) Error in maximum displacement

It will first be shown that for any non-linearity as a power in x (not a polynomial in x) the transition point x_t is directly propor-

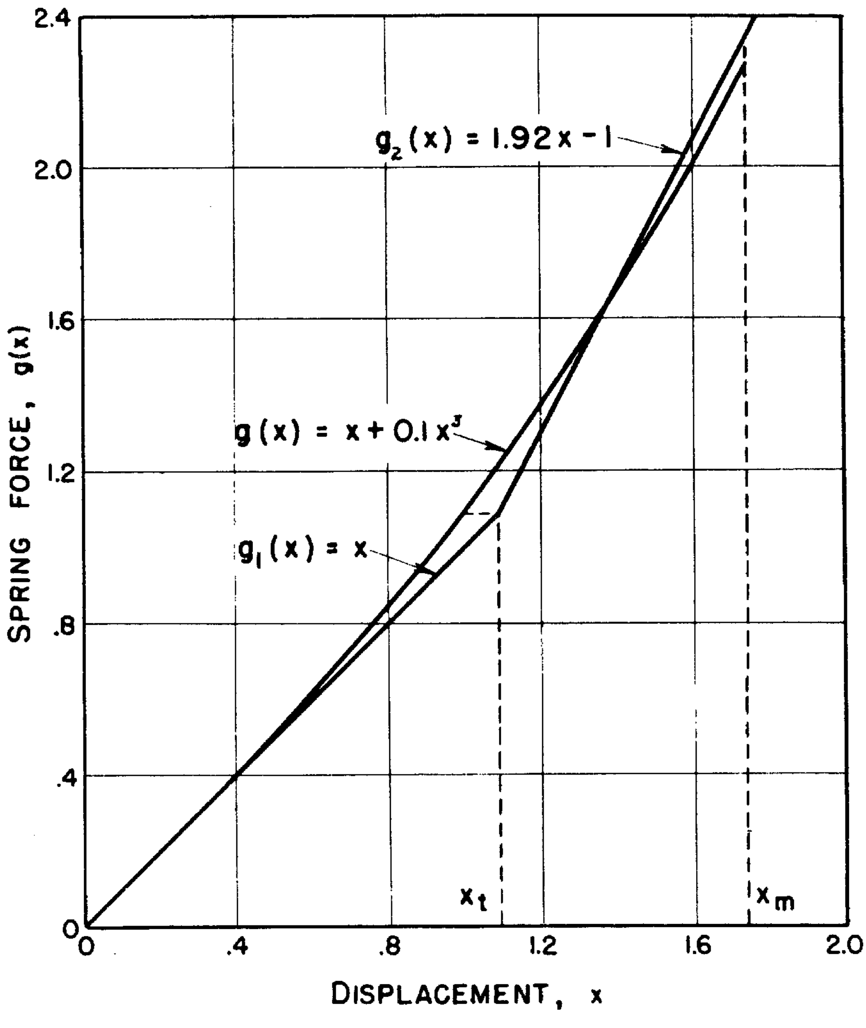

FIGURE 4 - NON-LINEAR CUBIC SPRING FORCE AND ITS BI-LINEAR APPROXIMATION

tional to the maximum displacement x_m with a proportionality constant (a).

x_t was determined from the equation:

$$\int_{x_t}^{x_m} \left\{ k_2 x + x_t(k_1 - k_2) - k_1 x - \varepsilon x^n \right\} dx = 0 \qquad (2\text{-}87)$$

where $k_1 x + \varepsilon x^n$ is the non-linear spring force. k_2 was defined as the slope of the non-linear spring force at the point of maximum deflection. Hence:

$$k_2 = k_1 + n \varepsilon x_m^{n-1} \qquad (2\text{-}88)$$

putting this value in Equation (2-87)

$$\int_{x_t}^{x_m} \left\{ n\varepsilon x_m^{n-1} x - n\varepsilon x_t x_m^{n-1} - \varepsilon x^n \right\} dx = 0 \text{ , or} \qquad (2\text{-}89)$$

$$\frac{n}{2} x_m^{n-1}(x_m^2 - x_t^2) - n x_t x_m^{n-1} (x_m - x_t) - \frac{1}{n+1}(x_m^{n+1} - x_t^{n+1}) = 0 \qquad (2\text{-}90)$$

Now let $x_t = a x_m$; Equation (2-90) becomes:

$$n(n+1)(1-a)^2 - 2(1-a^{n+1}) = 0 \qquad (2\text{-}91)$$

One root of the Equation (2-91) is a = 1. However this can be discarded since it is the value that will make the mean square error a maximum instead of a minimum.

For n = 3 (cubic non-linearity) equation (2-91) gives $a = \frac{5}{8}$ to three place accuracy.

So far nothing has been said about the choice of maximum deflection x_m from which the second slope k_2 is determined. It will be shown below that this choice is not very critical and an error relationship for the maximum deflection for any power of non-linearity will be derived.

Assume that the exact maximum deflection $\bar{x}_m$, as found from the solution of the bi-linear equation, differs by an amount δ from the assumed maximum deflection x_m such that $\frac{\delta}{\bar{x}_m} << 1$. Hence:

$$x_m = \bar{x}_m + \delta \quad \text{and } x_m^s = \bar{x}_m^s \left(1+s\frac{\delta}{\bar{x}_m}\right) \tag{2-92}$$

where s is any power.

The differential equations of motion were:

$$\ddot{x}_1 + k_1 x_1 = F \qquad |x| < |x_t| \tag{2-93}$$

$$\ddot{x}_2 + k_2 x_2 + x_t(k_1 - k_2) = F \qquad |x| > |x_t| \tag{2-94}$$

letting $x_1 = v_1$ and $x_2 = v_2$ and integrating Equation (2-93) between the limits of $0 < x_1 < x_t$ and Equation (2-94) between $x_t < x_2 < \bar{x}_m$ where $\bar{x}_m$ is the maximum deflection defined by the point where velocity is zero. Then:

$$\frac{1}{2} v_t^2 + \frac{k_1}{2} x_t^2 = F x_t \tag{2-95}$$

since initial displacement and velocity are zero. And

$$\frac{1}{2}(v_m^2 - v_t^2) + \frac{k_2}{2}(\bar{x}_m^2 - x_t^2) - x_t(k_2 - k_1)(\bar{x}_m - x_t) = F(\bar{x}_m - x_t) \tag{2-96}$$

But $k_2 = k_1 + n\varepsilon x_m^{n-1}$ and $x_t = ax_m$ putting these values in with the value of x_m from Equation (2-92) and simplifying:

$$\left\{(1-a)^2\left(1+n\frac{\delta}{\bar{x}_m}\right) - \frac{\delta}{\bar{x}_m}(1-a^2)\right\} \frac{n\varepsilon}{2}\bar{x}_m^n + \frac{k_1}{2}\bar{x}_m - F = 0 \tag{2-97}$$

This is an equation of nth order in the maximum deflection $\bar{x}_m$ and can be solved for any value of the parameters. However, without the necessity of solving Equation (2-97) several useful conclusions can be drawn from it.

a. Since $\frac{\delta}{\bar{x}_m}$ is of the same order as ε which in itself is a small quantity, the effect of δ on the determination of the amplitude is only of the order of ε^2. This means that the original assumption x_m can be made quite freely as long as it is in the neighborhood of the correct deflection. This suggests that one can choose, for example, the linear maximum deflection as the x_m in order to calculate the second slope without any appreciable loss of accuracy. In the example worked out previously x_m was assumed to be 1.74 which is the exact maximum deflection. If the problem is repeated with $x_m = 2$ which is the linear maximum deflection, one finds that the values of half period and maximum amplitude become 2.82 and 1.79 respectively as compared to 2.82 and 1.77 obtained previously.

b. Since the iterative procedure of determining x_m will converge on a value where $\delta = 0$ Equation (2-97) can be written as:

$$(1-a)^2 \frac{n\varepsilon}{2} \bar{x}_m^n + \frac{k_1}{2} \bar{x}_m - F = 0 \tag{2-98}$$

If the value of $(1-a)^2$ is substituted from Equation (2-91)

$$\frac{(1-a^{n+1})}{n+1} \varepsilon \bar{x}_m^n + \frac{k_1}{2} \bar{x}_m - F = 0 \tag{2-99}$$

This gives the best approximation one can obtain by using the bilinear method. However the exact maximum amplitude equation can be obtained from the original differential equation as:

$$\frac{\varepsilon}{n+1}\, \bar{x}_m^n + \frac{k_1}{2}\, \bar{x}_m - F = 0 \tag{2-100}$$

From the comparison of Equations (2-99) and (2-100) it is seen that the remainder is only

$$-\frac{\varepsilon a^{n+1}}{n+1}\, \bar{x}_m^n$$

Since $a < 1$ as n gets large the remainder will approach zero. This means that for a given maximum deflection the accuracy of the solution will increase with increasing powers of non-linearity.

In general, then, the bi-linear approach will converge on a value slightly larger than the exact one.

For cubic non-linearity the two amplitude equations (bi-linear and exact) are of the form:

$$.42\, \varepsilon\, \bar{x}_m^3 + k_1\, \bar{x}_m - 2F = 0 \qquad \text{bi-linear} \tag{2-101}$$

$$.50\, \varepsilon\, \bar{x}_m^3 + k_1\, \bar{x}_m = 2F = 0 \qquad \text{exact} \tag{2-102}$$

It is seen that even for very large non-linearities such that the linear term $k_1\bar{x}_m$ can be neglected compared to the non-linear term $\varepsilon\, \bar{x}_m^3$ the error in maximum amplitude is only about 5 percent. For any non-linearity the error will always be less than 5 percent.

In general for any power (n) and coefficient ε of the non-linearity the error in deflection will always be less than

$$Ex_m = 100\, \frac{a^{n+1}}{3} \text{ percent}$$

4) Improved bi-linear approximation

The accuracy of the results can be improved still further by the proper choice of the second slope k_2. In the following analysis k_2 as well as the transition amplitude x_t are assumed to be unknown. Again the mean square error in the spring forces are minimized by the proper determination of x_t and k_2 both. In other words, if E^2 is the mean square error

$$\frac{\partial(E^2)}{\partial x_t} = 0 \qquad \text{and} \qquad \frac{\partial(E^2)}{\partial k_2} = 0 \tag{2-103}$$

are satisfied simultaneously.

If from Equation (1-65) E^2 is obtained and the operations (2-103) are performed:

$$\int_{x_t}^{x_m} \left\{ k_2 x + x_t(k_1 - k_2) - k_1 x - \varepsilon x^n \right\} dx = 0 \tag{2-104}$$

$$\int_{x_t}^{x_m} \left\{ k_2 x + x_t(k_1 - k_2) - k_1 x - \varepsilon x^n \right\} (x - x_t)\, dx = 0 \tag{2-105}$$

Integration of (2-104) gives:

$$\frac{1}{2}(k_2 - k_1)\left\{(x_m^2 - x_t^2) - 2x_t(x_m - x_t)\right\} - \frac{\varepsilon}{n+1}(x_m^{n+1} - x_t^{n+1}) = 0 \tag{2-106}$$

from (2-105):

$$\frac{1}{3}(k_2 - k_1)\left\{(x_m^3 - x_t^3) - \frac{3}{2}x_t(x_m^2 - x_t^2)\right\} - \frac{\varepsilon}{n+2}(x_m^{n+2} - x_t^{n+2}) = 0 \tag{2-107}$$

Eliminating $(k_2 - k_1)$ between Equations (2-106) and (2-107) and again letting $x_t = ax_m$, an equation for the proportionality constant (a) is obtained:

$$a+2 = \frac{3(n+1)}{(n+2)} \frac{1-a^{n+2}}{1-a^{n+1}} \qquad 0 < a < 1 \tag{2-108}$$

Since $a < 1$, $\frac{1-a^{n+2}}{1-a^{n+}} \sim 1$ then

$$a = \frac{n-1}{n+2} \tag{2-109}$$

$(k2-k_1)$ can then be calculated from Equation (2-106) as:

$$k_2-k_1 = \frac{2\epsilon}{n+1} \frac{1-a^{n+1}}{(1-a)^2} x_m^{n-1} \tag{2-110}$$

The equation for the maximum deflection can again be calculated as outlined previously. This gives:

$$(1-a^{n+1}) \frac{\epsilon}{n+1} \bar{x}_m^n + \frac{k_1}{2} \bar{x}_m - F = 0 \tag{2-111}$$

The form of the equation is exactly the same as before. However the accuracy is improved because of the fact that (a) is smaller than it was before. For the cubic non-linearity $a = \frac{2}{5}$ so that (2-111) becomes

$$.487\, \epsilon\, \bar{x}_m^3 + k_1 \bar{x}_m - 2F = 0 \qquad \text{bi-linear} \tag{2-112}$$

$$.500\, \epsilon\, \bar{x}_m^3 + k_1 \bar{x}_m - 2F = 0 \qquad \text{exact} \tag{2-113}$$

The maximum error in amplitude cannot exceed one percent no matter how large the non-linearity is.

The discussion of convergence, and the effect of the error in the initial choice of the amplitude also applies to this case equally well, and hence will not be repeated here.

Chapter 3

Response of a Second Order Differential Equation with Cubic Non-linearity in Spring Force to a Single Pulse

The solution of step function response and the homogeneous non-linear equation makes it possible to treat the case of the rectangular pulse without any difficulty. It was shown in Chapter 2 that the maximum displacement due to a step function can be found exactly without the complete solution of the differential equation, and that the frequency can be approximated quite accurately even for large non-linearities. Hence the problem resolves itself to one of finding the step function solution for the duration of the rectangular pulse, and the homogeneous solution with initial conditions depending on the pulse length τ for $t > \tau$.

If the pulse length τ is very small compared to the period of free oscillations of the system, the problem can be solved as a case of pure impulse. In such instances the pulse shape will also be relatively unimportant, since the only function of the pulse is to impart an initial velocity to the system. However, when τ is of the order of, or larger than, the natural period of the system under study the pulse shape becomes very important.

The rectangular pulse is the easiest to treat from the point of view that it only involves solutions already obtained or readily obtainable. It is possible to obtain solutions for the half sine pulse by using the bi-linear approximation to the non-linear spring force. The calculation is somewhat more complex than that for the step function, since it involves the transient solution of forced oscillations.

Modulation of the forced amplitude by the homogeneous amplitude necessitates a cycle by cycle analysis of the problem. It is also necessary to assume a maximum amplitude x_m which may be used in evaluating k_2 and x_t. Since it was shown that the exact choice of x_m will not affect the results appreciably, as long as it is in the neighborhood of the correct answer, a most obvious approach is to take the linear maximum amplitude as x_m for determining x_t and k_2. The method will be illustrated below and several problems of varying order of non-linearity will be solved and checked against exact solutions obtained by iteration or graphical methods.

When the pulse length τ is large compared to the natural period T, from physical considerations one can assume that any transients excited had been damped out, and that there are only forced oscillations present. In this case, it is relatively simple to evaluate the amplitude and the velocity at the end of the pulse and to obtain the complete behavior of the system by any one of the methods mentioned in previous chapters.

For any pulse shape and duration, of course, one can theoretically always solve the problem exactly by the use of numerical or graphical integration methods. The solution by the bi-linear approximation is also theoretically possible, but becomes considerably more complicated than for the simpler cases treated below. It is likely that for any pulse shape, the bi-linear method would be shorter than numerical or graphical integration.

A. Rectangular Pulse

The treatment of the rectangular pulse is quite simple since the step function response and the homogeneous solutions are readily available. A particular example is solved and the results are shown in Figure 5. For this example the pulse height F is taken as unity, and the system has a differential equation of the form:

$$\ddot{x} + x + x^3 = 1 \qquad 0 < t < \tau$$

$$\text{and} \quad \ddot{x} + x + x^3 = 0 \qquad t > \tau \quad \text{and} \quad \tau = \pi \tag{3-1}$$

where τ is the pulse length. The initial velocity and displacement are again zero, and $x(\tau)$ and $\dot{x}(\tau)$ are the displacement and the velocity at the end of the pulse. The solution for $t > \tau$ is, then, simply a homogeneous one with initial conditions $x(\tau)$, $\dot{x}(\tau)$ and can be treated exactly as outlined in Chapter 1.

For comparison purposes the linear solution is drawn on the same graph. The effect of non-linearity on both the frequency and the amplitude becomes quite apparent.

B. Half Sine Pulse

Here the bi-linear method is expanded into a region where no analytic solutions, approximate or otherwise, have been obtained before. The only available tools for such problems as these have been the use of graphical integration or numerical iteration methods, both of which are very time-consuming.

The bi-linear method permits a quick and easy way of handling the class of problems which have solutions in linear mechanics. The purpose of the following examples is to show the applicability

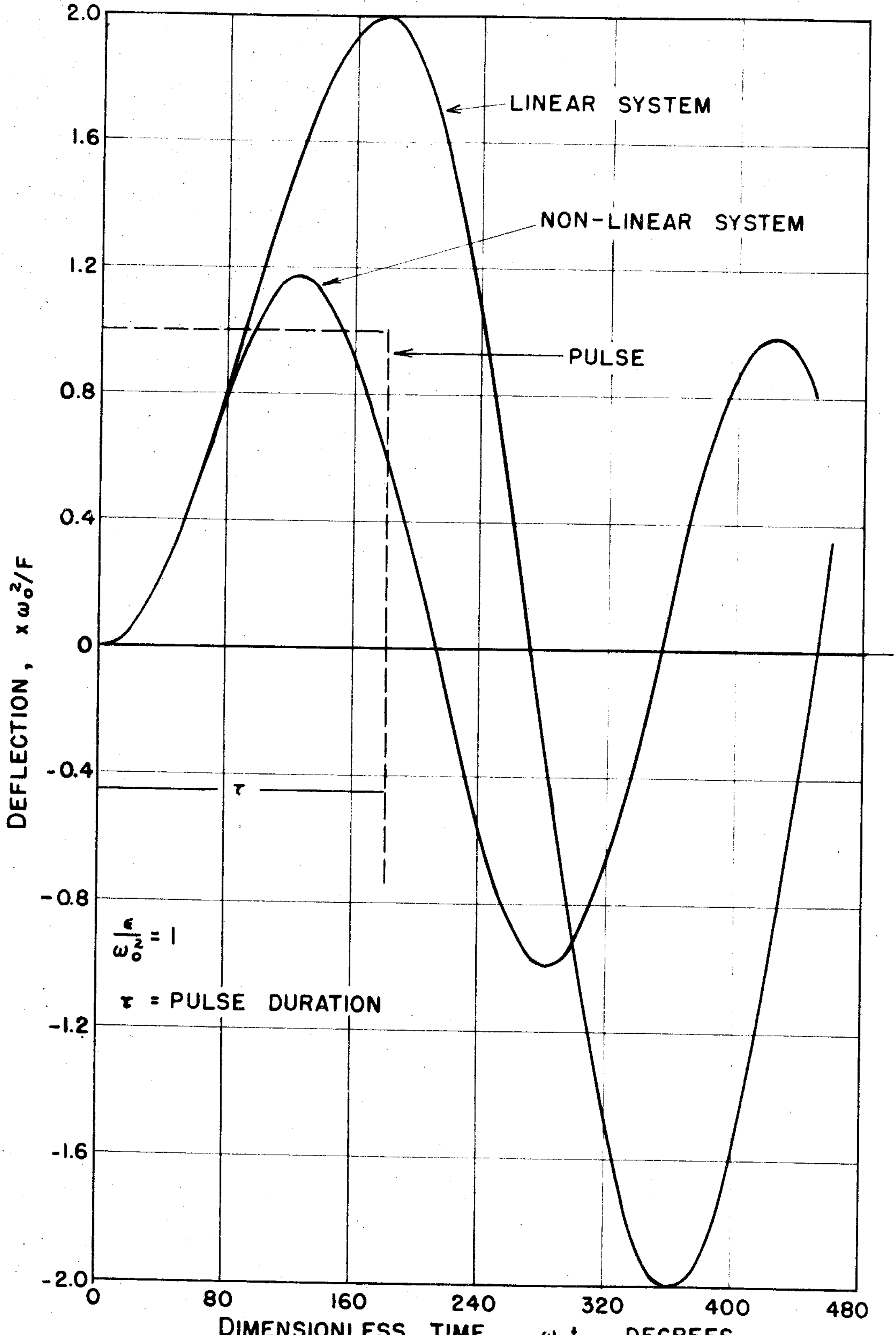

FIGURE 5 - EFFECT OF NON-LINEARITY ON THE RECTANGULAR PULSE RESPONSE OF THE DIFFERENTIAL EQUATION

$$\ddot{x} + \omega_o^2 x + \epsilon x^3 = \begin{cases} F & 0 < t < \frac{\pi}{\omega_o} \\ 0 & t > \frac{\pi}{\omega_o} \end{cases}$$

of the bi-linear approach to the analysis of a typical transient oscillation problem. Two cases of half sine pulse response, one with relatively small and one with large non-linearity in spring force, are treated.

To check the results obtained by the improved bi-linear approximation, exact solutions to the problems are obtained by numerical iteration of the differential equations.

Let the differential equation of motion be:

$$\ddot{x} + k_1 x + \varepsilon x^3 = \sin \omega t \qquad 0 < t < \tau \quad \text{where } \tau = \frac{\pi}{\omega} \tag{3-3}$$

$$\ddot{x} + k_1 x + \varepsilon x^3 = 0 \qquad t > \tau \tag{3-4}$$

For $t = \tau$ the solutions of the differential Equations (3-3) and (3-4) must correspond. The initial conditions are:

$$\text{at } t = 0 \qquad x(0) = \dot{x}(0) = 0 \tag{3-5}$$

Since an exact solution to Equation (3-4) exists, the main interest lies in finding an approximation to the differential Equation (3-3) such that the motion during the pulse can be obtained. The complete analysis requires the solution of a transcendental equation and hence cannot be done in a general form.

In the following example a numerical solution will be worked out by the bi-linear method to show the amount of work involved in such a calculation.

Let the differential equation be

$$\ddot{x} + x + x^3 = \begin{cases} \sin 2t & 0 < t < \frac{\pi}{2} \\ 0 & t > \frac{\pi}{2} \end{cases} \tag{3-6}$$

For cubic non-linearity the transition amplitude was calculated to be:

$$x_t = 0.4\, x_m \qquad \text{and the second slope} \tag{3-7}$$

$$k_2 = k_1 + 1.35\, x_m^2 = 1 + 1.35\, x_m^2 \tag{3-8}$$

(x_m) is calculated as the maximum amplitude of the linear solution of Equation (3-7). Hence:

$$\ddot{x} + x = \sin 2t$$

The solution with the required initial conditions will be:

$$x = \frac{1}{3}\,(2\ \sin t - \sin 2t) \tag{3-9}$$

The maximum value of x in Equation (3-9) within the duration of pulse is $x_m = 0.667$. Hence:

$$x_t = 0.267 \quad \text{and}\ k_2 = 1.60 \tag{3-10}$$

The two bi-linear equations to be solved are:

$$\ddot{x}_1 + x_1 = \sin 2t \qquad x_1(0) = \dot{x}_1(0) = 0 \tag{3-11}$$

$$\ddot{x}_2 + 1.6\, x_2 - 0.16 = \sin 2t \quad x_2(t_t)=x_1(t_t);\ \dot{x}_2(t_t)=\dot{x}_1(t_t) \tag{3-12}$$

where t_t is the transition time--time which it takes x_1 to reach the value x_t. The solution of Equation (3-11) is:

$$x_1 = \frac{1}{3}\,(2\sin t - \sin 2t) \tag{3-13}$$

from which the transition parameters can be determined as:

$$x_1(t_t)=x_t = 0.267,\ t_t = 58^{\circ}\ ,\quad \dot{x}_1\,(t_t) = 0.646 \tag{3-14}$$

The solution of Equation (3-12) which satisfies the conditions $x_2(t_t) = 0.267$ and $\dot{x}_2(t_t) = 0.646$ is:

$$x_2 = 0.1 + 0.582 \sin(\sqrt{1.6}\, t - 4.8^{\circ}) - \frac{1}{2.4} \sin 2t \qquad (3\text{-}15)$$

The displacement and the velocity at the end of the pulse can be calculated by letting $t = \frac{\pi}{2}$:

$$x_2\left(\frac{\pi}{2}\right) = 0.651 \qquad (3\text{-}16)$$

and

$$\dot{x}_2\left(\frac{\pi}{2}\right) = 0.592$$

The exact values obtained by numerical iteration to four place accuracy are:

$$x\left(\frac{\pi}{2}\right) = 0.6518 \quad \text{and} \quad \dot{x}\left(\frac{\pi}{2}\right) = 0.6001$$

The approximate results obtained are within about half a percent of exact amplitude and one and a half percent of exact velocity when the non-linear force is about 60 percent of the linear one at the maximum deflection point.

If more than one cycle of the system occurs during the pulse a cycle by cycle analysis is necessary to obtain the complete solution. In the following example the non-linearity is increased to investigate its effect on the accuracy of the solution.

Let the differential equation of motion be:

$$\ddot{x} + x + 5x^3 = \sin 2t \qquad (3\text{-}17)$$

Since the effect of non-linearity is to decrease the maximum amplitude, x_m will be assumed to be 0.60 instead of 0.667 which was the linear maximum displacement. This helps the results to converge faster and as it will be shown below makes it possible to get an accurate answer after only one step. Then

$$x_t = 0.24 \quad \text{and } k_2 = 3.5 \tag{3-18}$$

If one goes through the process of solving the two bi-linear equations:

$$\ddot{x}_1 + x_1 = \sin 2t \qquad |x| < x_t \tag{3-19}$$

$$\ddot{x}_2 + 3.5\, k_2 - 0.6 = \sin 2t \quad |x| > x_t \tag{3-20}$$

and matching the boundary conditions at $t = t_t$ (the transition point) the solutions are found to be:

$$x_1 = \frac{1}{3}(2\sin t - \sin 2t) \qquad 0 < t < t_t \tag{3-21}$$

$$x_2 = 1.98 \sin(\sqrt{3.5}\, t - 1^{\circ}) + 0.172 - 2\sin 2t \quad t_t < t < \frac{\pi}{2} \tag{3-22}$$

where $t_t = 55.5^{\circ}$. The conditions at the end of pulse are determined by letting $t = \frac{\pi}{2}$ and finding the displacement and the velocity from Equation (3-22). Hence:

$$x(\tfrac{\pi}{2}) = 0.581 \ , \qquad \dot{x}(\tfrac{\pi}{2}) = 0.38 \tag{3-23}$$

It is seen that the increased non-linearity has caused a reduction of about 13 percent in amplitude and 47 percent in velocity at the end of the pulse.

The accuracy of the results at any point can be readily checked without the necessity of obtaining an exact solution to the differential equation (3-17). Since Equations (3-21) and (3-22) satisfy the initial conditions, their value at any point can be substituted into the differential equation and an error in acceleration at that particular point calculated.

For the example solved above at $t = \frac{\pi}{2}$ the error in the differential equation (3-17) is about two percent; this means that the error in displacement and velocity must necessarily be less than this.

Figure 6 shows a comparison of the bi-linear and the exact solutions for the case treated above. The linear solution is also indicated on the graph for comparison purposes.

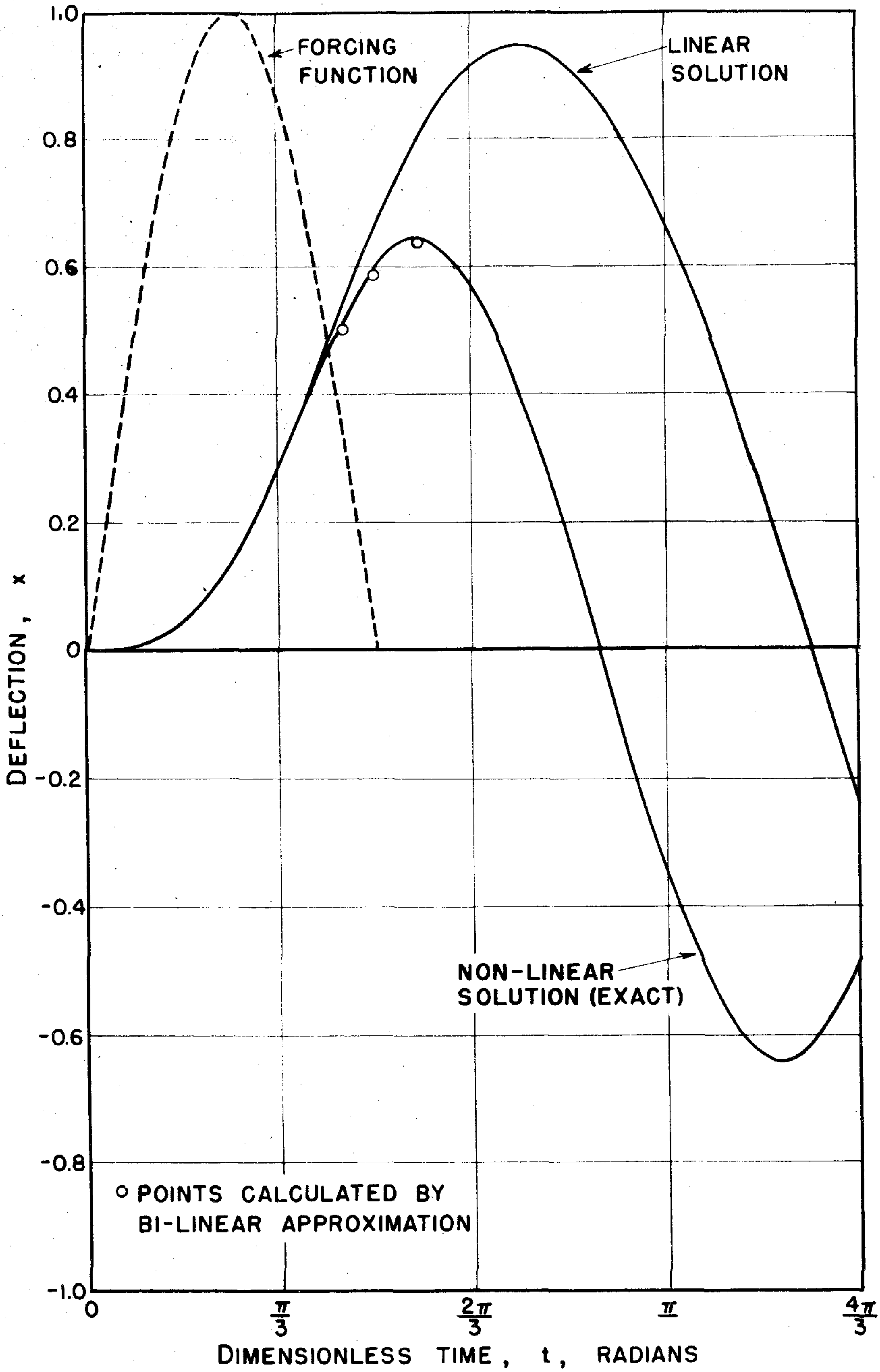

FIGURE 6- EFFECT OF NON-LINEARITY ON THE SOLUTION OF THE DIFFERENTIAL EQUATION

$$\ddot{x} + x + 5x^3 = \begin{cases} \sin 2t & 0 < t < \frac{\pi}{2} \\ 0 & t > 0 \end{cases}$$

Chapter 4

The Effect of Small Linear Damping on the Solution of the Non-linear Transient Problems

Since any physical system will inherently have a certain amount of damping, it is important to see the effect of such small damping on the transient solution of a non-linear differential equation.

For the sake of simplicity the step function response of a single degree of freedom non-linear system with small linear damping will be solved by the method of Kryloff and Bogoliuboff as outlined in Appendix A.

Let the differential equation of motion be:

$$\ddot{x} + c\dot{x} + \omega_o^2 x + \varepsilon x^3 = F_o \tag{4-1}$$

Again the transformation

$$x = \frac{F_o}{\omega_o^2} + y \tag{4-2}$$

will be made such that the Equation (4-1) can be written as:

$$\ddot{y} + c\dot{y} + \omega_o^2 y = -\varepsilon\left(y + \frac{F_o}{\omega_o^2}\right)^3 , \tag{4-3}$$

since the damping coefficient (c) is assumed to be small and of the order of magnitude as ε, (4-3) will be re-written as:

$$\ddot{y} + \omega_o^2 y = -\varepsilon\left\{\left(y + \frac{F_o}{\omega_o^2}\right)^3 + \frac{c}{\varepsilon}\dot{y}\right\} \quad \text{where } \frac{c}{\varepsilon} \sim 1 \tag{4-4}$$

This means that the zeroth order of approximation can be found as:

$$y_o = -a \cos \Psi$$

where Ψ and a are slowly varying functions of time.

Expanding y, $\frac{da}{dt}$, and $\frac{d\Psi}{dt}$ in terms of the small parameter ε and neglecting higher than first order terms in ε

$$y = -a \cos \Psi + \varepsilon y_1 + 0(\varepsilon^2) \qquad (4\text{-}5)$$

and

$$\frac{da}{dt} = \varepsilon A(a) \quad , \quad \frac{d\Psi}{dt} = \omega_o + \varepsilon \bar{\Psi}(a). \qquad (4\text{-}6)$$

Then, from Equation (4-5)

$$\dot{y} = a \omega_o \sin\Psi - \varepsilon \left\{A \cos\Psi - a \bar{\Psi} \sin \Psi - \omega_o \frac{\partial y_1}{\partial \Psi}\right\} + 0(\varepsilon^2) \qquad (4\text{-}7)$$

and

$$\ddot{y} = a \omega_o^2 \cos \Psi + \varepsilon \left\{2A \omega_o \sin \Psi + 2a \omega_o \bar{\Psi} \cos \Psi + \omega_o^2 \frac{\partial^2 y_1}{\partial \Psi^2}\right\} \qquad (4\text{-}8)$$

Putting $\ddot{y}$ and y back in Equation (4-4) and simplifying (terms higher than linear in (ε) are neglected),

$$\frac{d^2 y_1}{d\Psi^2} + y_1 = -\left(\frac{2A}{\omega_o} + \frac{c}{\varepsilon}\frac{a}{\omega_o}\right)\sin\Psi + \left\{\frac{3a^3}{4\omega_o^2} + \frac{3a}{\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right)^2 - \frac{2a}{\omega_o}\bar{\Psi}\right\}\cos\Psi \qquad (4\text{-}9)$$

$$- \frac{3a^2}{2\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right)\cos 2\Psi + \frac{a^3}{4\omega_o^2}\cos 3\Psi - \frac{1}{\omega_o^2}\left(\frac{F_o}{\omega_o^2}\right)\left\{\frac{3}{2}a^2 + \left(\frac{F_o}{\omega_o^2}\right)^2\right\}$$

In order to eliminate the secular terms:

$$2A + \frac{c}{\varepsilon} a = 0 \quad \text{or} \quad A = -\frac{c}{2\varepsilon} a \qquad (4\text{-}10)$$

and

$$\frac{3a^3}{4\omega_o} + \frac{3a}{\omega_o}\left(\frac{F_o}{\omega_o^2}\right)^2 - 2a\bar{\varphi} = 0 \quad \text{or} \quad \bar{\varphi} = \frac{3}{2\omega_o}\left\{\frac{1}{4}a^2 + \left(\frac{F_o}{\omega_o^2}\right)^2\right\} \tag{4-11}$$

Equations (4-6) can now be integrated to give (a), and φ in terms of t

$$a = a_o e^{-\frac{ct}{2}} \tag{4-12}$$

and

$$\varphi = \left\{\omega_o + \frac{3\varepsilon}{2\omega_o}\left(\frac{F_o}{\omega_o^2}\right)^2\right\} t - \frac{3\varepsilon}{8c\,\omega_o} a_o^2 e^{-ct} + \delta \tag{4-13}$$

where δ and a_o are the constants of integration and will be determined from initial conditions.

It is seen that the form of the solution is exactly the same as that of Chapter 2, and the amplitude equation determined from the initial conditions will also be the same with the same limitations on the non-linearity. The only difference lies in the fact that there is an exponential decay on the amplitude and hence on the frequency, and a phase shift due to the damping.

If one is interested in the response of the system during a very short interval of time after the application of the pulse the effect of small linear damping can justifiably be neglected.

The bi-linear approximation can, theoretically, be applied to problems involving linear damping. However, the work involved in matching the conditions at transition point is somewhat tedious due to the presence of the exponential terms.

III. CONCLUSIONS

Conclusions from each chapter will be summed up as follows:

A. Homogeneous Solution

1) The so-called classical approximate methods of non-linear mechanics (Lindstedt, Kryloff and Bogoliuboff, and Equivalent Linearization) are well suited for the solution of the homogeneous non-linear problem. Even though they are based on the assumption of small non-linearities, they give consistently good results for odd power non-linearities even when the non-linear force exceeds the linear one several times. In this respect the Equivalent Linearization and the Lindstedt methods give better approximation to the frequency than the Kryloff and Bogoliuboff method. This is due to the fact that in the latter method the frequency instead of its square is expanded in terms of the non-linear parameter. If, however, one is interested in higher accuracy than can be attained by the first approximation theory, the Kryloff and Bogoliuboff method proves to be easier than the Lindstedt method to apply even though the convergence is not as rapid.

2) The bi-linear approximation gives more accurate results than any of the above perturbation methods, and is easier to apply to most problems. The accuracy obtained by the bi-linear method is not significantly decreased by large non-linearities. The method is limited, however, to problems in which there are no mixed non-linearities, i.e., the non-linearity must be either in the displacement alone, or in the velocity alone.

B. Step Function Response

1) The effect of large non-linearities on the accuracy of the solutions obtained by the various classical approximate methods becomes more pronounced for the step function problem than for the homogeneous problem. For the Kryloff solution the limitation on the size of the non-linearity appears in a mathematical form in the amplitude equation. In the other methods larger non-linearities only result in lowered accuracy. For small non-linearities the Lindstedt method gives the closest approximation to both the frequency and the amplitude. The equivalent linearization is the least accurate one.

2) The bi-linear approximation gives better results than the Lindstedt or the Kryloff methods. The error introduced by the bi-linear approximation is small and essentially independent of the size of the non-linearity. It is shown that the initial choice of the maximum amplitude needed to determine the parameters of the bi-linear approximation is not critical. The results are not altered more than one or two percent by a variation of twenty percent in this initial choice. One can, then, take the maximum linear amplitude to start the calculation of the parameters. For very large non-linearities when the linear maximum amplitude may vary greatly from the non-linear one, it is possible to develop an iterative extension of the bi-linear method.

C. Single Pulse Response

1) The classical approximate methods are limited to the treatment of the rectangular pulse. The limitations outlined for

the step function response will, of course, apply to the single rectangular pulse response.

2) The bi-linear method proves most useful for the treatment of the single pulse forcing function. It can be used to find the response of the system to any pulse shape which can be treated by linear mechanics. Even though it involves a cycle by cycle analysis it is easier and faster than graphical or numerical methods. The limitation on the size of the non-linearity is not severe. However, if the problem is such that a large number of cycles occur during the pulse and if the non-linear force is big compared to the linear one, the cumulative error at the end of the pulse may be appreciable. In such cases, however, it is usually possible to treat the problem as a steady state forced oscillation, rather than as a transient problem, thus avoiding the cumulative error of cycle by cycle analysis.

D. Small Linear Damping

To the degree of accuracy implied by the first approximation theory described and developed above, the presence of small linear damping in the system affects the non-linear problem in the same way as it would a linear system. If one is interested in the first few cycles after the application of a pulse, the assumption of negligible linear damping is justified. The solution of the problems involving large linear damping is within the scope of the bi-linear approximation; however procedure in such a case becomes somewhat involved.

APPENDIX A.

Equivalent Linearization and Improved Kryloff Approximations

The method of Kryloff and Bogoliuboff or, as it is sometimes called, "the method of slowly varying parameters", is found in a few text books on non-linear mechanics (Refs. 2 and 3). It differs from classical perturbation methods not in principle but in detail. It is possible to investigate transient states by Kryloff procedure whereas the classical perturbation only gives steady state solutions. The equivalent linearization method is a simplification of the first order Kryloff solution and proves useful in most cases of forced steady state oscillations. For purposes of reference a brief outline of these two methods is given below.

1) Equivalent Linearization Method

Let the differential equation of motion be of the form

$$\ddot{x} + \omega_o^2 x + \varepsilon f(x, \dot{x}) = 0 \qquad (5\text{-}1)$$

where $f(x, \dot{x})$ is a non-linear function of x and $\dot{x}$. The zeroth order solution of this equation will be:

$$x = a \cos(\omega_o t + \phi) \qquad (5\text{-}2)$$

where (a) and ϕ are slowly varying functions of time such that the assumption of quasi-linearity for Equation (5-2) holds. Let $\Psi = \omega_o t + \phi$. Then:

$$\dot{x} = -a\,\omega_o \sin\Psi \qquad (5\text{-}3)$$

Differentiating (5-3) with respect to time, and keeping in mind that second derivatives of (a) and (ϕ) with respect to time are negligible:

$$\dot{x} = -\dot{a}\,\omega_o \sin\Psi - a\,\omega_o^2 \cos\Psi - a\,\omega_o\,\dot{\phi}\cos\Psi \tag{5-4}$$

From differentiating Equation (5-2) one gets:

$$\ddot{x} = \dot{a}\cos\Psi - a\omega_o \sin\Psi - a\dot{\phi}\sin\Psi \tag{5-5}$$

Equating (5-5) to (5-3) and simplifying:

$$-a\,\omega_o \sin\Psi = \dot{a}\cos\Psi - a\,\omega_o \sin\Psi - a\,\dot{\phi}\sin\Psi$$

or

$$\dot{\phi}\sin\Psi = \frac{\dot{a}}{a}\cos\Psi \quad \text{or} \quad \dot{\phi} = \frac{\dot{a}}{a}\,\mathrm{ctn}\,\Psi \tag{5-6}$$

Putting the value of $\dot{\phi}$ from Equation (5-6) into (5-4) and substituting the values of $\ddot{x}$ and x back in the differential Equation (5-1) one gets:

$$-\dot{a}\,\omega_o \sin\Psi - \dot{a}\,\omega_o \frac{\cos^2\Psi}{\sin\Psi} + \varepsilon\left[f\quad a\cos\Psi,\ -a\,\omega_o \sin\Psi\right] = 0$$

or simplifying

$$\dot{a} = \frac{\varepsilon}{\omega_o}\ f\left[a\cos\Psi,\ -a\,\omega_o \sin\Psi\right]\sin\Psi \tag{5-7}$$

and

$$\dot{\phi} = \frac{\varepsilon}{a\,\omega_o}\ f\left[a\cos\Psi,\ -a\,\omega_o \sin\Psi\right]\ \cos\Psi \tag{5-8}$$

It can be seen that (a) and (ϕ) are periodic functions with periods of $\frac{2\pi}{\omega_o}$. Since (f) is also a periodic function it can be expanded in a Fourier series such that:

$$f\left[a\cos\psi, -a\,\omega_o\sin\psi\right]\sin\psi = k_o(a) + \sum_{n=1}^{\infty} K_n(a)\cos n\psi + L_n(a)\sin n\psi \qquad (5\text{-}9)$$

and

$$f\left[a\cos\psi, -a\,\omega_o\sin\psi\right]\cos\psi = P_o(a) + \sum_{n=1} P_n(a)\cos n\psi + Q_n(a)\sin n\psi \qquad (5\text{-}10)$$

where

$$K_o = \frac{1}{2\pi}\int_o^{2\pi} f\left[a\cos\theta, -a\,\omega_o\sin\theta\right]\sin\theta\, d\theta \qquad (5\text{-}11)$$

and

$$P_o = \frac{1}{2\pi}\int_o^{2\pi} f\left[a\cos\theta, -a\,\omega_o\sin\theta\right]\cos\theta\, d\theta \qquad (5\text{-}12)$$

Then

$$\dot{a} = \frac{\varepsilon}{\omega_o}\left\{K_o(a) + \sum_{n=1}^{\infty} K_n(a)\cos n\psi + L_n(a)\sin n\psi\right\} \qquad (5\text{-}13)$$

and

$$\dot{\phi} = \frac{\varepsilon}{a\,\omega_o}\left\{P_o(a) + \sum_{n=1}^{\infty} P_n(a)\cos n\psi + Q_n(a)\sin n\psi\right\} \qquad (5\text{-}14)$$

Since (a) and (ϕ) are slowly varying functions of time, they can be assumed to remain constant over one period T; hence, finding the average value of the right-hand side of Equations (5-13) and (5-14) over one cycle:

$$\dot{a} = \frac{\varepsilon}{\omega_o}\int_t^{t+T}\left\{K_o(a) + \sum_{n=1}^{\infty} K_n(a)\cos n\theta + L_n(a)\sin n\theta\right\} d\theta \qquad (5\text{-}15)$$

and

$$\dot{\phi} = \frac{\varepsilon}{a\,\omega_o} \int_t^{t+T} \left\{ P_o(a) + \sum_{n=1}^{\infty} P_n(a)\cos n\theta + Q_n(a)\sin n\theta \right\} d\theta \qquad (5\text{-}16)$$

Since $\cos n\theta$ and $\sin n\theta$ are periodic functions of period $\frac{T}{n}$, their average over one cycle will be zero. Hence:

$$\dot{a} = \frac{\varepsilon}{\omega_o} K_o(a) \qquad (5\text{-}17)$$

and

$$\dot{\phi} = \frac{\varepsilon}{a\,\omega_o} P_o(a) \quad \text{or} \quad \dot{\Psi} = \omega_o + \frac{\varepsilon}{a\,\omega_o} P_o(a) \qquad (5\text{-}18)$$

Putting the values of $K_o(a)$ and $P_o(a)$ in from Equations (5-11) and (5-12), one gets:

$$\dot{a} = \frac{\varepsilon}{2\pi\,\omega_o} \int_o^{2\pi} f(x^o, \dot{x}^o) \sin\theta d\theta \qquad (5\text{-}19)$$

$$\dot{\Psi} = \omega_o + \frac{\varepsilon}{2\pi a\,\omega_o} \int_o^{2\pi} f(x^o, \dot{x}^o) \cos\theta d\theta \qquad (5\text{-}20)$$

where x^o and $\dot{x}^o$ are the solutions of the differential equation corresponding to the zeroth order of approximation ($\varepsilon = 0$), and are periodic functions with a period of $\frac{2\pi}{\omega_o}$.

If (5-20) is squared neglecting the terms of ε^2 or higher:

$$(\dot{\Psi})^2 = \Omega^2(a) = \omega_o^2 + \frac{\varepsilon}{\pi a} \int_o^{2\pi} f(x^o, x^o) \cos\theta d\theta \qquad (5\text{-}21)$$

Let:

$$\bar{\lambda} = -\frac{\varepsilon}{\pi a\,\omega_o} \int_o^{2\pi} f(x^o, \dot{x}^o) \sin\theta d\theta \qquad (5\text{-}22)$$

and

$$\bar{k} = k + \frac{\varepsilon}{\pi a} \int_o^{2\pi} f(x^o, \dot{x}^o) \cos\theta d\theta \qquad (5\text{-}23)$$

Then Equations (5-19) and (5-21) respectively become:

$$\dot{a} = -\frac{\bar{\lambda}}{2} a \quad \text{or} \quad a = a_o e^{-\frac{\bar{\lambda}}{2} t} \tag{5-24}$$

and

$$\Omega^2 (a) = \bar{k} \qquad \text{(note: mass is taken as unity)} \tag{5-25}$$

where $\bar{k}$ is some equivalent spring constant and $\bar{\lambda}$ an equivalent damping coefficient.

(5-24) and (5-25) correspond to the solution of a linear equation of the form

$$\ddot{x} + \bar{\lambda} \dot{x} + \bar{k} x = 0 \tag{5-26}$$

which can be called the equivalent linear equation.

Equivalence of the energy per cycle between the Equation (5-26) and the original non-linear Equation (5-1) can be shown quite easily and will not be given here (Ref. 2).

It is easy to see that for the equations of the form:

$$\ddot{x} + \omega_o^2 x + \epsilon f(x) = 0 \tag{5-27}$$

$\bar{\lambda}$ will be zero. Hence in such a case the amplitude (a) remains constant to the first approximation, but the frequency of oscillations is still a function of the amplitude.

2) Improved Kryloff and Bogoliuboff Method

In the equivalent linearization method the higher harmonics in the Fourier series expansion of the amplitude and the phase were dropped due to the averaging process employed. In reality, owing to the presence of these terms the slowly varying quantities (a) and (ϕ)

undergo oscillations of relatively high frequency. The improved first approximation theory takes into account these higher harmonic terms. Let the differential equation of motion be:

$$\ddot{x} + \omega_o^2 x + \varepsilon f(x, \dot{x}) = 0 \tag{5-28}$$

Again the displacement x will be expanded in terms of the small non-linear parameter such that

$$x = a \cos \psi + \varepsilon x_1(a, \psi) + 0(\varepsilon^2) \tag{5-29}$$

where (a) and ψ are assumed to be slowly varying functions of time such that

$$\frac{da}{dt} = \varepsilon A(a) \text{ and } \frac{d\psi}{dt} = \omega_o + \varepsilon \bar{\Psi}(a) + 0(\varepsilon^2) \tag{5-30}$$

Differentiating x from (5-29) and putting in the values of $\frac{da}{dt}$ and $\frac{d\psi}{dt}$ from Equations (5-30)

$$\dot{x} = -a\omega_o \sin\psi + \varepsilon\left[A \cos\psi - \bar{\Psi} a \sin\psi + \omega_o \frac{dx_1}{d\psi}\right] + 0(\varepsilon^2) \tag{5-31}$$

Differentiating (5-31) once more:

$$\ddot{x} = -a\,\omega_o^2 \cos\psi + \varepsilon\left[-2A\,\omega_o \sin\psi - 2\bar{\Psi} a\,\omega_o \cos\psi + \omega_o^2 \frac{d^2 x_1}{d\psi^2}\right] + 0(\varepsilon^2) \tag{5-32}$$

Putting the values of x and $\ddot{x}$ from Equations (5-29) and (5-32) into the differential Equation (5-28) and simplifying

$$\frac{d^2 x_1}{d\psi^2} + x_1 = \frac{2A}{\omega_o} \sin\psi + \frac{2\bar{\Psi} a}{\omega_o} \cos\psi - \frac{1}{\omega_o^2} f(a\cos\psi, -a\,\omega_o \sin\psi) \tag{5-33}$$

Since $f(a \cos\psi, a\,\omega_o \sin\psi)$ is a periodic function with a period of 2π, it can be expanded in a Fourier series such that:

$$f(a \cos\psi, -a\,\omega_o \sin\psi) = h_o + \sum_{u=1}^{\infty} h_u(a) \cos u\psi + g_u(a) \sin u\psi$$

It is seen that Equation (5-33) can now be written as

$$\frac{d^2x_1}{d\psi^2} + x_1 = \left[\frac{2A}{\omega_o} - \frac{g_1(a)}{\omega_o^2}\right] \sin\psi + \left[\frac{2\bar{\psi}a}{\omega_o} - \frac{h_1(a)}{\omega_o^2}\right] \cos\psi - \frac{h_o}{\omega_o^2} - \frac{1}{\omega_o^2} \sum_{n=2}^{\infty} h_n(a) \cos n\psi + g_n(a) \sin n\psi \tag{5-34}$$

In order to eliminate the secular terms, the coefficients of $\sin\psi$ and $\cos\psi$ terms in Equation (5-34) must be made equal to zero. This gives the required frequency, and amplitude correction terms as:

$$\bar{\psi} = \frac{h_1}{2a\,\omega_o} \quad \text{and } A = \frac{g_1}{2\,\omega_o} \tag{5-35}$$

Hence

$$\dot{a} = \frac{\varepsilon g_1}{2\,\omega_o} \quad \text{and} \quad \dot{\psi} = \omega_o + \frac{\varepsilon h_1}{2a\,\omega_o} \tag{5-36}$$

g_1 and h_1 are given as the first Fourier coefficients of the function (f) by the equations:

$$h_1 = \frac{1}{\pi} \int_0^{2\pi} f(a \cos\psi, -a\,\omega_o \sin\psi) \cos\psi \, d\psi$$

and

$$g_1 = \frac{1}{\pi} \int_0^{2\pi} f(a \cos\psi, -a\,\omega_o \sin\psi) \sin\psi \, d\psi$$

Equation (5-34) can now be solved to give x_1 as:

$$x_1 = -\frac{h_o}{\omega_o^2} + \frac{1}{\omega_o^2} \sum_{n=2}^{\infty} \frac{h_n(a) \cos n\psi + g_n(a) \sin n\psi}{n^2 - 1} \tag{5-37}$$

or the solution for x becomes:

$$x = a \cos \psi - \varepsilon \frac{h_o}{\omega_o^2} + \frac{\varepsilon}{\omega_o^2} \sum_{n=1}^{\infty} \frac{h_n \cos n\psi + g_n \sin n\psi}{n^2 - 1} \qquad (5\text{-}38)$$

It is seen that Equation (5-38) includes higher harmonic terms which were not present in the equivalent linearization solution. It is also evident that, however, for small non-linearities the higher harmonic terms will be small compared to the fundamental one.

APPENDIX B

Numerical and Graphical Integration Methods

There are a number of numerical and graphical methods available for the exact solution of non-linear differential equations. Each has certain characteristics which makes it more suited for the solution of a particular class of problems. In principle, however, they do not vary appreciably, and a representative method suffices to illustrate the principle.

1) Numerical Iteration (Refs. 5, 6)

The main advantage of numerical iteration methods lies in their ability to compensate automatically for the error that might occur during the process of solution. Theoretically there is no limitation to their accuracy if one is willing to repeat the procedure a sufficient number of times. For small non-linearities they converge quite readily to give accurate results within a reasonable time duration.

Let the differential equation of motion be:

$$\ddot{x} + \omega_o^2 x + \varepsilon f(x, \dot{x}, t) = 0 \qquad (6\text{-}1)$$

or this can be written as:

$$\ddot{x} = -\omega_o^2 x - \varepsilon f(x, \dot{x}, t) \qquad (6\text{-}2)$$

If one solves Equation (6-1) for $\varepsilon = 0$, the zeroth order approximation to the non-linear equation is obtained. Let this be x_o. The next approximation is found by putting the value of x_o in the right-hand side of Equation (6-2) and integrating it twice with respect to time. In general this integration has to be performed numerically since $f(x, \dot{x}, t)$ can be a complicated function. The procedure is repeated a number of times until the difference between successive approximations becomes small. Hence the nth approximation to Equation (6-1) is obtained by:

$$x_n = -\int_0^t \int_0^{\eta} \left\{ \omega_o^2 x_{n-1} + \varepsilon f(x_{n-1}, \dot{x}_{n-1}, \tau) \right\} d\tau \, d\eta \qquad (6\text{-}3)$$

It is possible to formulate certain refinements for the above procedure to make it more suitable for a certain class of problems. Since these refinements are generously represented in the literature, they will not be repeated here (Ref. 6).

2) Graphical Methods:

There are a great number of graphical methods available for the solution of non-linear differential equations. Most of these give solutions in the phase plane where the coordinates are the displacement and the velocity. Lienard's method is representative of one of the graphical approaches. This method is amply treated in any textbook on non-linear mechanics and will not be repeated here (Ref. 2).

Another graphical approach suggested by Jacobsen (see Ref. 7 in Bibliography) was used in this thesis for the solution of half-sine pulse response. A short outline of the method is given below:

Let the differential equation of motion be:

$$\ddot{x} + \omega_o^2 x + \varepsilon f(x, \dot{x}, t) = 0 \qquad (6\text{-}4)$$

and let

$$\sigma = \frac{\varepsilon}{\omega_o^2} f(x, \dot{x}, t) \tag{6-5}$$

such that Equation (6-4) can be written as:

$$x + \omega_o^2 (x + \sigma) = 0 \tag{6-6}$$

Now let $\dot{x} = v$ which gives $\ddot{x} = v\frac{dv}{dx}$, putting this value in (6-6) and simplifying:

$$\frac{dx}{dv} = - \frac{v}{x+\sigma} \tag{6-7}$$

Suppose that one is interested in the solution of Equation (6-7) for a very short time duration during which σ can be considered to remain constant. Equation (6-7) can now be integrated to give:

$$(x+\sigma)^2 + v^2 = c^2 = \text{constant} \tag{6-8}$$

It is seen that (6-8) is the equation of a circle with radius c and whose center lies on the point $x = -\sigma$, $v = 0$. Starting from any point (say initial conditions) all one has to do is to calculate σ from Equation (6-5), and as $x = -\sigma$ the center, swing a small circular arc to locate the next point on the phase plane. The phase trajectory will then be made up of a series of circular arcs whose center lies on the x-axis. It can be shown that the time is measured as the length of each arc and can thus be calculated.

The accuracy of the method depends on the smallness of each step so that during that step σ can be assumed. to remain constant. On the other hand the smaller the step the greater the error in measurement of angles and lines, and the larger the number of steps required to complete the trajectory. Since the errors in a graphical solution will be cumulative, there seems to be an optimum to the number of steps to be taken and hence a maximum to the accuracy that can be obtained. The errors induced in the solution of a problem will also depend largely on the person who performs the operation so that it is difficult to draw general conclusions as to the accuracy.

It was found that in the solution of half-sine pulse response the accuracy was not better than 2.5 percent with intervals as small as two degrees per arc.

The other graphical methods are also faced with similar difficulties and must be used with caution when a solution of high accuracy is required.

BIBLIOGRAPHY

1. Kryloff, N.: Bogoliuboff, N.: "Introduction to Non-linear Mechanics", Princeton University Press, 1947.

2. Minorsky, N.: "Introduction to Non-linear Mechanics", J. W. Edwards, Ann Arbor, Michigan, 1947.

3. McLachlan, N. W.: "Ordinary Non-linear Differential Equations in Engineering and Physical Sciences", Oxford Press, 1950.

4. Stoker, J. J.: "Non-linear Vibrations", Interscience Publishers, 1950.

5. Milne, W. E.: "Numerical Calculus", Princeton University Press, 1949.

6. Brock, J. E.: "An Iterative Numerical Method for Non-linear Vibrations", Journal of Applied Mechanics, March 1951, vol. 18, pp. 1-11.

7. Jacobsen, Lydik S.: "On a General Method of Solving Second-order Ordinary Differential Equations by Phase-Plane Displacements", Journal of Applied Mechanics, December 1952, vol. 19, pp. 543-553.

8. Evaldson, Rune L.; Ayre, Robert S.; Jacobsen, Lydik S.: "Response of an Elastically Non-linear System to Transient Disturbances", Journal of the Franklin Institute, December 1949, vol. 248, pp. 548-562.

9. Timoshenko, J.; Young, D. H.: "Advanced Dynamics", McGraw-Hill, 1948.

10. Timoshenko, S., "Vibration Problems in Engineering", D. Van Nostrand, 1937.