MOTION OF AN UNARTICULATED HELICOPTER BLADE WITH APPLICATION TO THE PROBLEM OF VIBRATION OF THE RIGID ROTOR HELICOPTER

Thesis by

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TABLE OF CONTENTS

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GENERAL INTRODUCTION

JUAN DE LA CIERVA'S WORK

The helicopter is a rotary wing aircraft which derives its lift and the force which propels it forward from one or more large rotors. The rotors are engine driven and are displaced with the axes of rotation in a more or less vertical position. This type of aircraft has achieved some prominence in recent years and little more will be said about its general characteristics.

It was preceded by the autogiro, a sister rotary wing ship which, in some respects, it resembles closely. The autogiro was the first of the rotary wing craft to be taken seriously by the aeronautical world. It was invented in 1924 by Juan de la Cierva, a highly talented Spanish aeronautical engineer. For some twenty years afterwards it was developed and manufactured by several small aircraft companies in this country and in England. At one time popular writers were captured by it much as by the helicopter today. However, serious question arose as to its usefulness as compared to the difficulties involved in its operation and the rotary wing manufacturers turned to the helicopter as soon as this craft showed some elements of practicality.

All of this is mentioned because the original work of Señor de la Cierva has been taken over with little change by present helicopter designers and because the problems encountered by de la Cierva form a convenient introduction to the subject treated in this paper. There is in particular a story told about the early work on the autogiro which may, or may not, be true but which is a good way to begin.

Cierva conceived of his first machine as having an airplane type fuselage, tail surfaces, landing gear and tractor propeller and one large horizontally displaced rotor as the lifting surface. The rotor had four blades rigidly attached to a central hub. The hub was free to turn on its vertical axis. In forward flight the rotor would be tilted slightly backward, would be turned by the wind and would generate a lifting force. A model built on this principle worked well but the full sized ship showed a regrettable tendency to roll over. Cierva found the reason for the behaviour of the large machine and developed what is to all intents and purposes the present day hinged bladed rotor (helicopter or autogiro) to solve the problem.

The difference in behaviour between the model and the full sized machine can be seen from a qualitative look at the rotor aerodynamics.

The airflow over a section of a rotor blade can be expected to be most unsteady. It will vary with time in both direction and magnitude. The blade turns about its hub at the same time as the hub (and ship) move forward through the air. This is shown in Figure (1). The most obvious effect of the superposition of the two airflows is to produce a greater relative velocity over the advancing blade (position a) than over the retreating blade (b). Consequently, it can be expected that the advancing side of the rotor will produce more lift than the retreating side and a rolling moment will result. There are other effects, a change of angle of attack of the blade section and a change of induced downwash at the section around the cycle, a stalled region of the rotor on the retreating side. The net result is an asymmetry of lift distribution which leads to a large rolling moment and a somewhat smaller pitching moment being transmitted to the fuselage by the rotor.

The large ship experienced these moments; the model seemed to avoid them. The answer lay in the large relative flexibility of the model rotor blades. The flexible blade responded to the moments of the lifting forces along it and deflected. Figure (2) shows the situation in that case. The blade is in static equilibrium under the influence of the aerodynamic, centrifugal, gravity and acceleration forces. The moments transmitted to the hub were now a great deal less than the moments produced by the aerodynamic forces. Relief was given mainly by the centrifugal forces which exerted opposite moments about the blade root. Thus all moments transmitted were much reduced and the model was able to fly.

Cierva went to the limit of the flexibility and on his large ship put a free horizontal hinge at the root of the blade. In this system no moment could be transmitted at the blade root. The blade itself experienced a flapping motion about the hinge which was mainly a harmonic motion at rotor frequency. Figure (3) shows such a blade in equilibrium under the influence of the various forces.

This was an excellent solution. The flapping motion was at most of the order of 10° and did not detract from the ships flying characteristics.

There was one more difficulty. Figure (3) shows that the projection of a point of the blade on a rotating radial line moves back and forth along the line as the blade flaps. This effective movement of a mass radially along a rotating axis gives rise to the so called Coriolis acceleration. This is shown replaced by its reversed effective force in Figure (4). The moment produced at the root about an axis perpendicular to the flapping pin by this force and the fluctuating drag forces made it necessary, from a structural point of view, to install a vertical pin at the root. This pin was put a bit outboard of the horizontal pin and the two formed more or less of a universal joint at the root. Figure (4) shows the equilibrium picture about this hinge. Again the centrifugal force is the restoring force. Motion about the vertical pin proved a bit more troublesome than motion about the horizontal pin. For reasons that will be stated later, it was felt necessary to experiment with and use various types of dampers to restrain motion about this pin.

This then was the form of rotor which Cierva developed for his autogiro, - a set of universally hinged blades variously damped about the vertical pin. The most generally used helicopter rotor designs are essentially the same. This type of helicopter rotor will now be more completely described, its characteristics discussed and the reasons given for the trend among many manufacturers to the unhinged or rigid rotor.

2. THE HINGED HELICOPTER ROTOR

A composite of the hinged rotors currently used might be described as follows:

Three light blades, constructed of steel tube spar, plywood ribs, fabric or plywood covering, each some 18-25 ft. in length are attached to a central hub. The attachment is made through a horizontal pin, a vertical pin and a bearing which allows the blade to turn about the axis of its spar (for controlled pitch change of the blade). The horizontal pin is inboard and its axis may be at some angle other than 90° to the spar axis. Figure (5) shows the angle to be 60°. The rotor is said to have a 30° δ_3 angle. Incorporating a δ_3 angle of this size in the rotor has two effects.

1 - The amplitude of flapping is reduced.

2 - The azimuth angle at which the maximum flapping occurs is shifted.

Both effects are described with respect to a $0° \delta_3$ rotor. The vertical pin is next outboard. An hydraulic damper connects the blade directly outboard of the vertical pin to the hub. The damper is installed primarily to prevent an unstable mechanical vibration from occurring during rev-up (ground resonance). The freedom of the blade about the vertical pin makes this condition possible. The pitch change bearing is outboard of the second pin and a control arm attaches somewhere beyond this.

The rotor has upper and lower stops on the flapping pin to prevent the blades from folding too far up or down during zero or slow rpm gust conditions or extraordinary maneuvers. Two stops are provided on the vertical pin. The blade may rest against the back stop during rev-up.

The machines of the Sikorsky, Kellett and P-V organizations use this rotor with minor modification.

Consider the ideal flapping rotor (i.e. omit for the moment questions of maintenance, ease of manufacture, etc.) The following advantages and disadvantages of the rotor have been found to exist: -

The principal advantages:

1. The problem of unbalanced moments is solved.
2. The blade stress situation is favorable because of the flapping hinge. The blades are light.
3. The rotor minimizes sudden effects of any sort (sharp gust, etc.), for the first effect of such conditions is a motion of the blade about the horizontal pin and not a transmission of moment.

There is one principle disadvantage. Both the inherent flexibility of the blades and the pin method of attachment to the hub leads to a rotor which is singularly unrestrained in its motion. Restraint is provided by various acceleration forces which under many circumstances may not exist or are very difficult to predict. In the tail rotor type of ship this means the design difficulty of mounting the rotor high above the ship to provide adequate ground (or tail) clearance. This is an annoying design consideration for the tail rotor ship but becomes an extremely serious problem in the case of ships with other rotor configurations. (Figure 6). In the coaxial rotor configuration, the large distance required between two flapping rotors to exclude collision has unfortunate effects on the design. The closely intermeshed rotor configuration (each rotor tilted outwards) using flapping rotors has not been able to avoid blade collision difficulties. In the side by side and tandem types, weight and space saving which might be brought about by the partial intermeshing of the rotors is made difficult to achieve because of the consequent increase in the possibility of blade collision. This, then, is the main disadvantage of the ideal flapping rotor.

Aside from these intrinsic qualities, the following practical difficulties exist: -

1. The hinge, damper and control systems are at best drag inducing, expensive and troublesome to adjust and maintain.
2. The rotor is temperamental. It must be

cared for constantly to achieve reasonably smooth operation. This characteristic is usually ascribed to inequality among dampers, or blades which are either improperly balanced or easily changed by weather condition. How much of this delicateness is due to components which are different in flapping and rigid rotors is not known. Rigid rotors have not yet been built and flown extensively.

3. THE RIGID ROTOR

A rigid rotor is one in which the blade attachment to the hub is much like that of a variable pitch airplane propeller. The only motion allowed the blade is the controlled angular motion about the pitch change axis. There is no other stipulation. Blade construction may be of the same type as the flapping rotor.

A listing of the properties of the rigid rotor is much a reversal of the properties given for the flapping rotor. Its most obvious disadvantages are: -

1. The unbalanced moment problem is not solved. In general some moment will always be transmitted to the rotor hub and use must be made of two oppositely turning rotors to prevent a resultant moment on the fuselage.

2. Construction of blade, hubs, etc. therefore tends to be heavier than that of the flapping rotor.
3. Sudden effects are transmitted.

The advantage: -

Strict restraint of the blades is maintained. The only appreciable motion possible is the bending of the rather stiff blades.

As has been mentioned, rigid rotors have not been extensively flown. Serious encouragement was given the helicopter in the United States only some 5 or 6 years ago. The first ships naturally enough followed autogiro practice and used the Cierva rotor. The industry is now in the stage where a number of small companies are experimenting with many different types of machines. Some of the designers, those concerned with the four rotor configurations of Figure (6), are considering or using the rigid rotor. The reason for the change is largely a matter of the freedom of the flapping rotor, although it is undoubtedly hoped that the other flapping rotor difficulties will be eliminated as well.

A few such ships have been built and flown. None are as yet successful. Little reliable information is available about any of them but the impression is unmistakably given that those which are far enough advanced to fly forward have encountered serious difficulty with rotor vibration. The vibrations are ill-defined. No strict investigation of them has been made known. This thesis takes the first step in the investigation of the most obvious cause of such vibration. A more precise statement of the work follows:

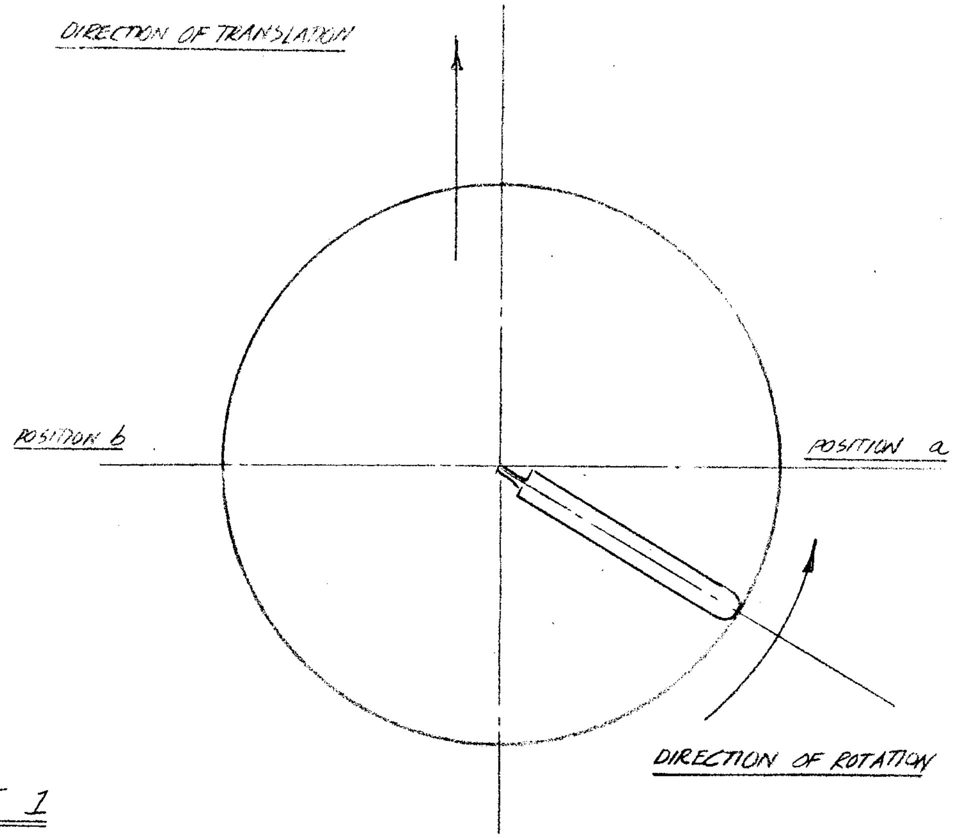

FIGURE 1

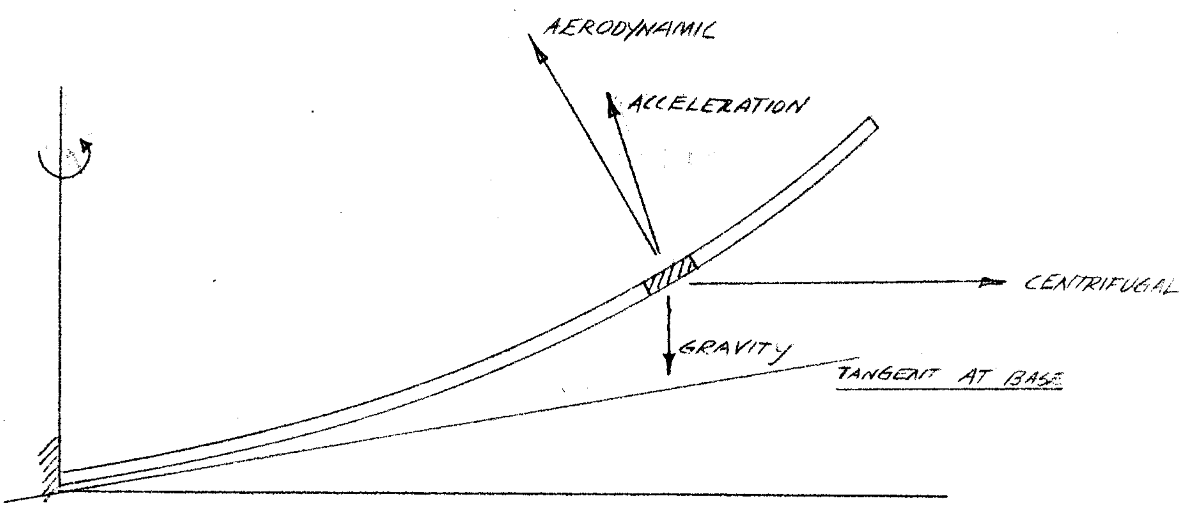

FIGURE 2

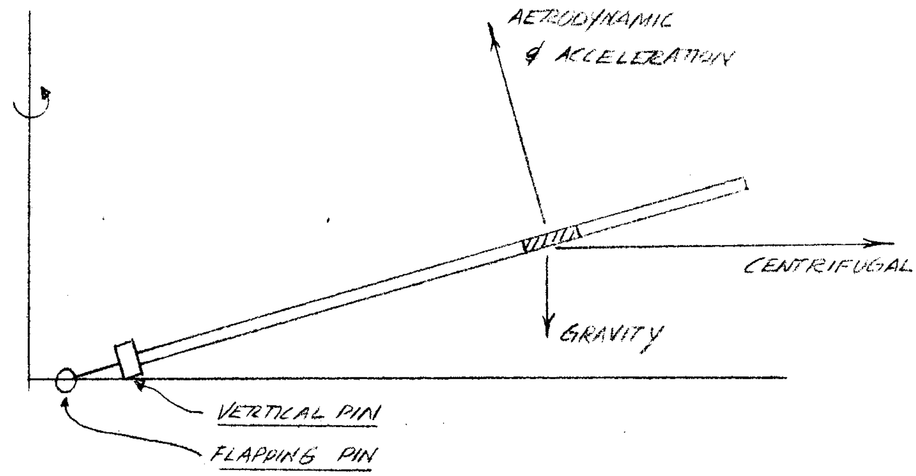

FIGURE 3

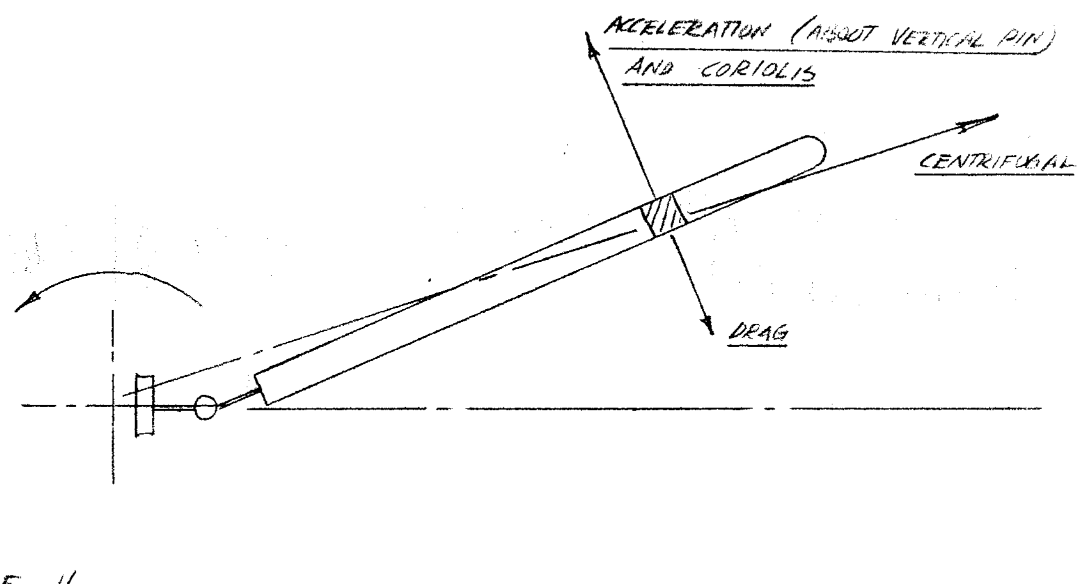

FIGURE 4

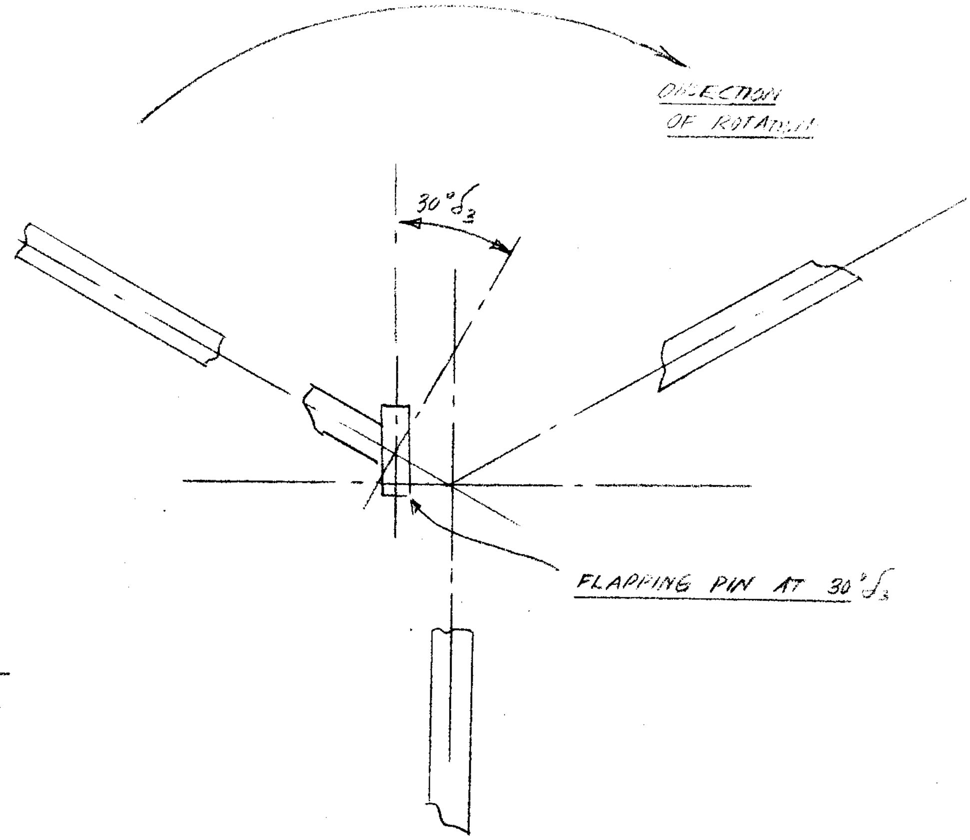

FIGURE 5

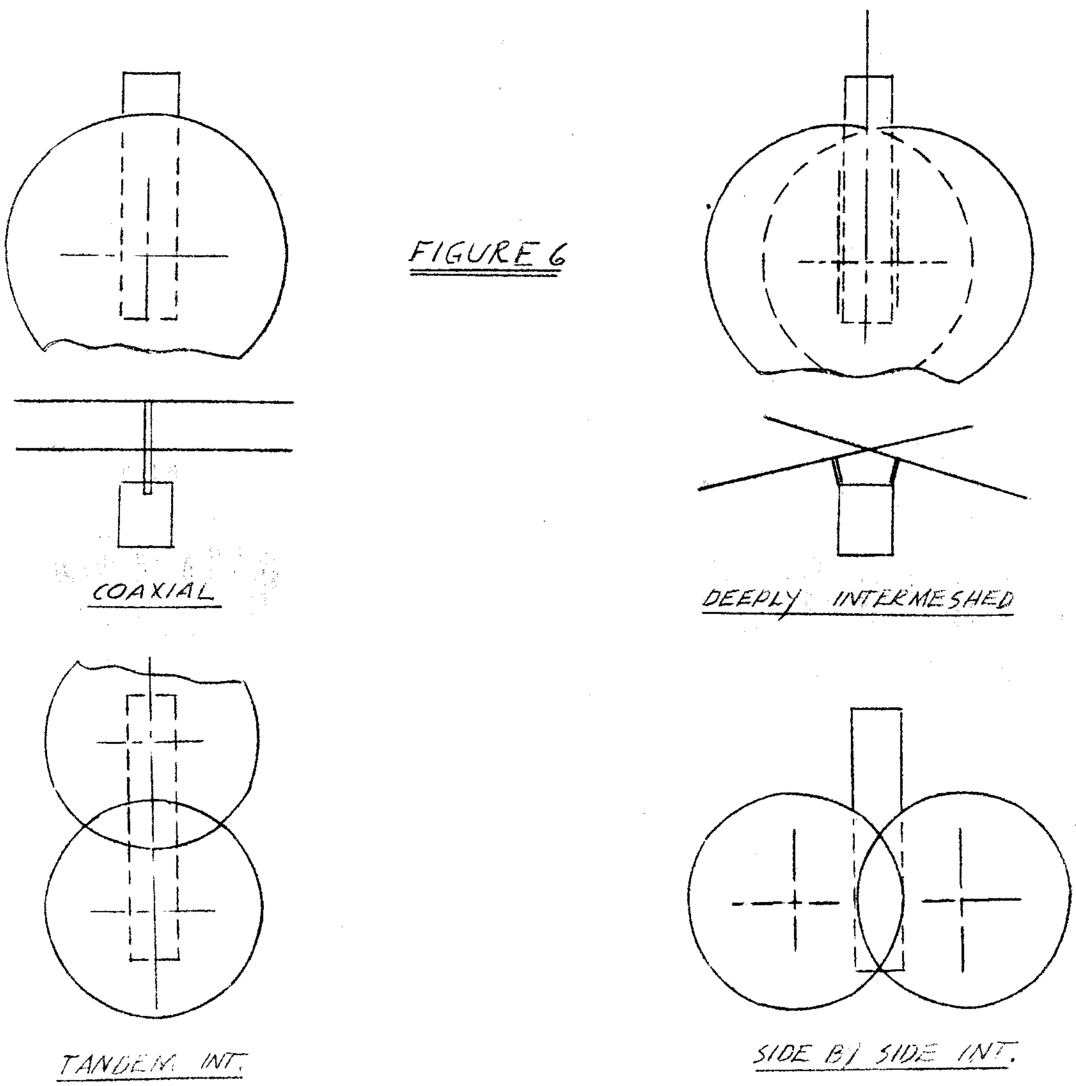

FIGURE 6

SUMMARY

The present investigation concerns itself almost completely with the derivation and solution of the equations of motion of the helicopter blade. The fluctuating aerodynamic forces which cause flapping in the hinged rotor bend the blades of the rigid rotor in a vertical plane. The effect of the bending is to cause fluctuating moments on the rotor hub. This pure bending oscillation is the most obvious cause of vibration and is treated here. Coupled motions are not considered and are effectively eliminated by the assumption that the mass center, aerodynamic center and elastic axis are at the point on each cross section of the blade and that the elastic axis is a straight line.

Although this work was undertaken to investigate rotor vibration, it has broader significance. It is, in effect, the ess ential step which makes possible the rational analysis of all rigid rotor dynamics and aerodynamics. It is directly comparable in importance to the solution of the problem of the blade flapping motion in hinged rotor work. For example, once the bending of the blade in forward flight is known such information as

1. Fatigue stresses in the blades
2. Control lag angle required
3. Maximum deflection of the blades, etc.

is immediately available.

The Thesis is Divided into the Following 4 Sections

I. The derivation of the bending forces which exist in forward flight along a blade. The differential force is expressed as a 3 harmonic Fourier series in terms of the azimuth position of the blade. The coefficient of each term is a power series in blade radius. The aerodynamic theory is based on work by Glauert (Ref. 1).

II. The calculation of the natural modes and frequencies of the blade.

a) The method of Myklestad (ref. 2) is summarized. By considering the blade made up of a finite number of discreet masses, a tabular method is developed which allows the calculation of the natural modes and frequencies of the blade in its centrifugal field.

b) A sample blade is considered and the 3 lowest modes and frequencies are found.

III. The derivation and solutions of the equations of motion. LaGrange's equation is applied to the system using as generalized coordinates arbitrary functions of time which multiply the two lowest modes. The generalized force is derived in terms of the aerodynamic force. There result two simultaneous second order, linear differential equations with variable coefficients. The solution of the equations is a pair of infinite Fourier series in terms of the azimuth angle. The series are approximated by their constant and first harmonic terms. The final solution for the blade motion is given by a set of 6 simultaneous algebraic equations in terms of the blade and flight parameters. This application of the LaGrangian was suggested by work of M. A. Biot. (ref. 3)

IV A Numerical Example

a) The calculation of the motion of the sample blade is carried out and plotted at a condition of forward flight at 100 mph.

b) The moments on the hub of a coaxial ship using 2, 2 bladed rotors (blades of calculation IVa) are plotted.

The immediate practical application of this theory has been the object through this investigation and a sufficient number of variables are included to describe blades of current design.

SECTION I

THE AERODYNAMIC FORCES ON A ROTATING BLADE

NOMENCLATURE

R radius of the blade.

r coordinate distance along the blade span.

y coordinate distance perpendicular to blade span, defining position of deflected blade with respect to its unbent position.

c chord length at any blade section.

c_0 chord length at the root of any tapered portion of the blade.

K_c coefficient describing the rate of decrease of chord with span for any tapered portion of the blade. For a blade linearly tapered from root to tip. $c = c_0 - K_c \frac{r}{R}$

The plane to the rotor disc is the plane perpendicular to rotor shaft passed through the rotor hub.

θ_0 geometrical blade setting of the blade at its root measured from the line of no lift of the airfoil section.

K_θ coefficient describing the rate of change of blade angle with span in a linearly twisted blade.

Thus $\theta = \theta_0 - K_\theta \frac{r}{R}$ indicates linear washout from root to tip.

θ geometrical blade angle at any station along the span, in general a function of r and time.

ψ azimuth position of the blade measured in the direction of rotor rotation. The blade in its downwind position is at zero angle of ψ.

u velocity of flow induced through the rotor disc, assumed constant over the disc.

V_r component of the resultant velocity of the air relative to the blade in a plane perpendicular to the blade span.

V_{ry} component of V_r perpendicular to the blade span and lying in the plane formed by the blade span and the rotor axis of rotation.

V_{rx} component of V_r which is perpendicular to both V_{ry} and the blade span.

dT thrust component of resultant aerodynamic force on segment of blade dr. dT is parallel to V_{ry}

θ_1 AND θ_2 constants which are used to define the cyclic control imposed on the rotor. Arbitrary feathering control produced by a swashplate may be expressed as $\Delta\theta = \theta_1 \sin\psi + \theta_2 \cos\psi$

α absolute angle of attack of the blade element

ϕ the induced angle; angle between V_r & $V_{r\psi}$ in radians

ω speed of angular rotation of the rotor.

i Angle of attack of the rotor disc; the angle between the flight path and the rotor disc, positive when the disc is tilted forward.

The aerodynamic derivation is based on steady flow wing theory, i. e. aerodynamic forces are determined by instantaneous values of velocity and angle of attack at each blade element. The following facts are, however, incompatible with the assumption of steady flow.

1. The blade element oscillates vertically with an amplitude which is large as compared to the chord.
2. The blade element changes its pitch periodically.
3. The velocity at the element varies greatly.
4. The blade operates in a wake produced by itself, other blades of the rotor and other rotor (s).

The induced effects are obviously not as predicted by steady flow theory. Nevertheless, the Glauert steady flow theory (ref.1) has been widely applied in helicopter work and has given results which have been found reasonable. More specifically, the calculation of the flapping motion of hinged blades, a degenerate case of the present problem, has been carried out using this theory and has proved to be in satisfactory agreement with experiment.

A more exact theory has not yet been developed. Points 1 & 2 have been considered for the case of small oscillations by Von Kármán and Sears (Ref 4) and point 3 by Isaacs (Ref. 5). No rigorous evaluation of the error involved in the Glauert theory may be made on the basis of these partial refinements. The one calculation which can be made indicates a small error.*

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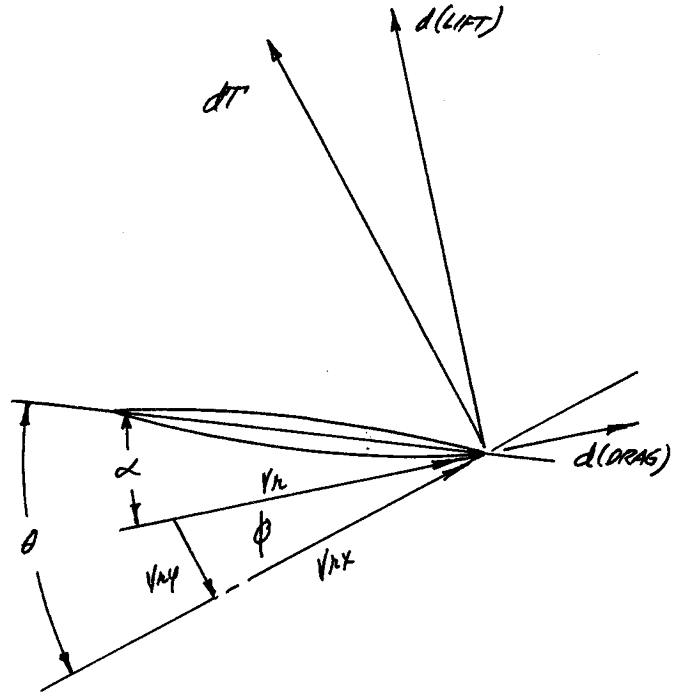
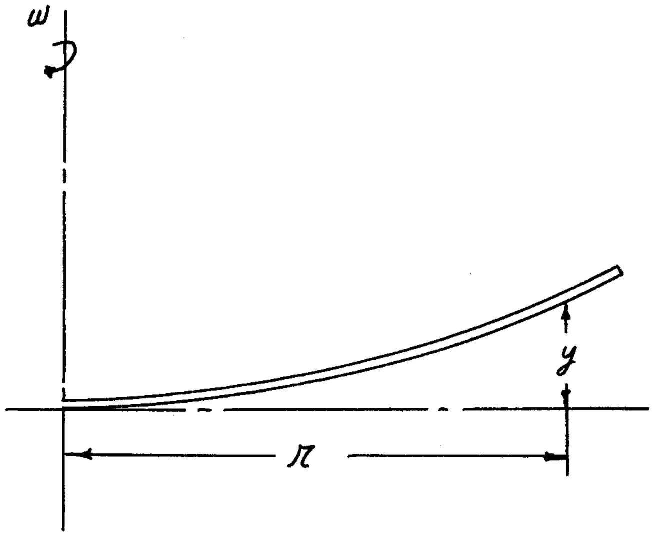
*Assuming small vertical oscillations, the effects of deviations 1 & 2 are found to be small. The reduced frequency of the system as defined in the Karman paper is of the order of .04.

THRUST LOADING OF THE BLADE





The following assumptions are made

1. The angle ϕ is small so that $\phi = \frac{V_{ry}}{V_{rx}}$ and $\cos\phi = 1$
2. $|V_r| = |V_{rx}|$
3. The blade sections do not stall

   i.e. $C_L = a_\infty \alpha$ (a_∞ = SLOPE OF LIFT CURVE FOR $AR = \infty$)
4. U is a constant over the disc.
5. The component of drag in the direction of thrust is negligible so that $d(\text{LIFT}) = dT$
6. $\frac{dy}{dr}$ is small so that $\sin\left(\tan^{-1}\frac{dy}{dr}\right) = \frac{dy}{dr}$

Then

$$dT = \frac{\rho}{2} C_L V_R^2 \, c\, dr$$

$$dT = \frac{\rho}{2} a_\infty \alpha V_{Rx}^2 \, c\, dr$$

$$\alpha = \theta - \frac{V_{Ry}}{V_{Rx}}$$

$$\theta = \theta_0 - K_\theta \frac{r}{R} + \theta_1 \sin\psi + \theta_2 \cos\psi$$

$$c = c_0 - K_c \frac{r}{R}$$

$$dT = \frac{\rho}{2} a_\infty \left(\theta_0 - K_\theta \frac{r}{R} + \theta_1 \sin\psi + \theta_2 \cos\psi - \frac{V_{Ry}}{V_{Rx}}\right)\left(c_0 - K_c \frac{r}{R}\right) V_{Rx}^2 \, dr$$

$$dT = \frac{\rho}{2} a_\infty \left(\theta_0 - K_\theta \frac{r}{R} + \theta_1 \sin\psi + \theta_2 \cos\psi\right)\left(c_0 - K_c \frac{r}{R}\right) V_{Rx}^2 \, dr$$

$$- \frac{\rho}{2} a_\infty \left(c_0 - K_c \frac{r}{R}\right)\left(V_{Ry} V_{Rx}\right) dr$$

$$V_{Rx} = \omega r + V \cos i \sin\psi$$

$$V_{Ry} = V \sin i + u + V \cos i \sin\cos\psi \frac{\partial y}{\partial r} + \frac{\partial y}{\partial t}$$

$$V_{Rx}^2 = \omega^2 r^2 + V^2 \cos^2 i \sin^2\psi + 2\omega r V \cos i \sin\psi$$

$$V_{rx} V_{ry} = \omega r (V\sin i + u) + \omega r V\cos i \cos\psi \frac{\partial y}{\partial r} + \omega r \frac{\partial y}{\partial t}$$

$$+ V\cos i \sin\psi (V \sin i + u) + V^2\cos^2 i \sin\psi\cos\psi \frac{\partial y}{\partial r}$$

$$+ V\cos i \sin\psi \frac{\partial y}{\partial t}$$

LET:

$$\mu = \frac{V\cos i}{\omega R} \quad ; \quad \nu = \frac{V\sin i + u}{\omega R}$$

THEN:

$$V_{rx}^2 = \omega^2 r^2 + \mu^2\omega^2R^2\sin^2\psi + 2\mu\omega^2 r R\sin\psi$$

$$V_{rx}V_{ry} = \nu\omega^2 r R + \omega^2 R r \mu\cos\psi \frac{\partial y}{\partial r} + \omega r \frac{\partial y}{\partial t} + \mu\nu\omega^2R^2\sin\psi$$

$$+ \mu^2\omega^2R^2\sin\psi\cos\psi\frac{\partial y}{\partial r} + \mu\omega R\sin\psi\frac{\partial y}{\partial t}$$

SUBSTITUTION IN THE EXPRESSION FOR dT LEADS TO:

$$dT = \frac{\rho}{2} a_0 \omega^2 dr \Big(r^2 C_0\theta_0 - \frac{C_0 K_\theta r^3}{R} + C_0\theta_1 \sin\psi\, r^2 + C_0\theta_2 \cos\psi\, r^2 - K_c\theta_0 \frac{r^3}{R}$$

$$+ K_c K_\theta \frac{r^4}{R^2} - K_c\theta_1 \sin\psi \frac{r^3}{R} - K_c\theta_2 \cos\psi \frac{r^3}{R} + C_0\theta_0 \mu^2 R^2 \sin^2\psi$$

$$- C_0 K_\theta \mu^2 R r \sin^2\psi + C_0\theta_1 \sin^3\psi\, \mu^2 R^2 + C_0\theta_2 \cos\psi\, \mu^2 R^2 \sin^2\psi$$

$$- K_c\theta_0 \mu^2 R r \sin^2\psi + K_c K_\theta r^2 \mu^2 \sin^2\psi - K_c\theta_1 \mu^2 R r \sin^3\psi$$

$$- K_c\theta_2 \cos\psi\, \mu^2 R r \sin^2\psi + 2C_0\theta_0 \mu r R \sin\psi - 2C_0 K_\theta r^2 \mu \sin\psi$$

$$+ 2C_0\theta_1 \sin^2\psi\, \mu r R + 2C_0\theta_2 \sin\psi\cos\psi\, \mu r R - K_c\theta_0 r^2 2\mu \sin\psi$$

$$+ K_c K_\theta \frac{r^3}{R} 2\mu \sin\psi - K_c r^2\theta_1 \sin^2\psi\, 2\mu - K_c\theta_2 \cos\psi \sin\psi\, r^2 2\mu$$

$$- C_0 \nu R r - C_0 r \frac{1}{\omega}\frac{dy}{dt} - C_0 r R\mu \cos\psi \frac{dy}{dr} - C_0 \mu \nu \sin\psi\, R^2$$

$$- C_0 \mu \sin\psi\, R \frac{1}{\omega}\frac{dy}{dt} - C_0 \mu^2 R^2 \sin\psi\cos\psi \frac{dy}{dr} + K_c \nu r^2$$

$$+ K_c \frac{r^2}{R}\frac{1}{\omega}\frac{dy}{dt} + K_c r^2 \mu \cos\psi \frac{dy}{dr} + K_c \mu \nu \sin\psi\, r R$$

$$+ K_c \mu \sin\psi\, r \frac{1}{\omega}\frac{dy}{dt} + K_c \mu^2 r R \sin\psi\cos\psi \frac{dy}{dr} \Big)$$

There are three parts to the expression for dT, one independent of blade shape, one dependent on blade shape and one dependent on the rate of change of the shape with time. Each part will be expressed as a fourier series in ψ.

It is first necessary to expand all trigonometric products, thus:

$$dT = \frac{\rho}{2} a_\infty W^2 dr \Big(r^2 C_0 \theta_0 - \frac{C_0 K_\theta r^3}{R} + C_0 \theta_1 \sin\psi\, r^2 + C_0 \theta_2 \cos\psi\, r^2 - K_c \theta_0 \frac{r^3}{R}$$

$$+ K_c K_\theta \frac{r^4}{R^2} - K_c \theta_1 \sin\psi \frac{r^3}{R} - K_c \theta_2 \cos\psi \frac{r^3}{R} + \frac{C_0 \theta_0 \mu^2 R^2}{2} - \frac{C_0 \theta_0 \mu^2 R^2}{2} \cos 2\psi$$

$$- \frac{C_0 K_\theta \mu^2 R r}{2} + \frac{C_0 K_\theta \mu^2 R r}{2} \cos 2\psi + \frac{3}{4} C_0 \theta_1 \mu^2 R^2 \sin\psi - \frac{1}{4} C_0 \theta_1 \mu^2 R^2 \sin 3\psi$$

$$+ \frac{1}{4} C_0 \theta_2 \mu^2 R^2 \cos\psi - \frac{1}{4} C_0 \theta_2 \mu^2 R^2 \cos 3\psi - \frac{1}{2} K_c \theta_0 \mu^2 R r + \frac{1}{2} K_c \theta_0 \mu^2 R r \cos 2\psi$$

$$+ \frac{K_c K_\theta r^2 \mu^2}{2} - \frac{K_c K_\theta r^2 \mu^2}{2} \cos 2\psi - \frac{3}{4} K_c \theta_1 \mu^2 R r \sin\psi$$

$$+ \frac{1}{4} K_c \theta_1 \mu^2 R r \sin 3\psi - \frac{1}{4} K_c \theta_2 \mu^2 R r \cos\psi + \frac{1}{4} K_c \theta_2 \mu^2 R r \cos 3\psi$$

$$+ C_0 \theta_0 2\mu r R \sin\psi - C_0 K_\theta r^2 \mu 2 \sin\psi + C_0 \theta_1 \mu r R - C_0 \theta_1 \mu r R \cos 2\psi$$

$$+ C_0 \theta_2 \mu r R \sin 2\psi - K_c \theta_0 r^2 2\mu \sin\psi + K_c K_\theta \frac{r^3}{R} 2\mu \sin\psi$$

$$- K_c r^2 \theta_1 \mu + K_c r^2 \theta_1 \mu \cos 2\psi - K_c \theta_2 r^2 \mu \sin 2\psi - C_0 \nu R r$$

$$- C_0 r R \mu \cos\psi \frac{dy}{dr} - C_0 \mu \nu \sin\psi R^2 - \frac{1}{2} C_0 \mu^2 R^2 \sin 2\psi \frac{dy}{dr} + K_c \nu r^2$$

$$+ K_c r^2 \mu \cos\psi \frac{dy}{dr} + K_c \mu \nu r R \sin\psi + \frac{K_c}{2} \mu^2 r R \sin 2\psi \frac{dy}{dr}$$

$$+ \frac{1}{W} \frac{dy}{dt} \left[- C_0 r - C_0 \mu \sin\psi R + \frac{K_c r^2}{R} + K_c \mu \sin\psi\, r \right] \Big)$$

Dealing first with the constant part and collecting harmonics

$$dT_{CONST.} = \frac{\rho}{2} a c \omega^2 dr \times$$

CONSTANTS

$$\left(C_0\theta_0 r^2 - \frac{C_0 K_\theta r^3}{R} - \frac{K_c}{R}\theta_0 r^3 + \frac{K_c K_\theta}{R^2} r^4 + \frac{C_0\theta_0\mu^2R^2}{2} - \frac{C_0K_\theta\mu^2Rr}{2} \right.$$

$$\left. -\frac{1}{2}K_c\theta_0\mu^2Rr + \frac{K_cK_\theta}{2}\mu^2r^2 + C_0\theta_1\mu Rr - K_c\theta_1\mu r^2 - C_0\nu Rr + K_c\nu r^2 \right)$$

Sin ψ TERMS

$$\left(C_0\theta_1 r^2 - \frac{K_c\theta_1 r^3}{R} + \frac{3}{4}C_0\theta_1\mu^2R^2 - \frac{3}{4}K_c\theta_1\mu^2Rr + 2C_0\mu\theta_0Rr \right.$$

$$\left. -2C_0K_\theta\mu r^2 - 2K_c\theta_0\mu r^2 + 2\frac{K_cK_\theta}{R}\mu r^3 - C_0\mu\nu R^2 + K_c\mu\nu Rr \right)$$

Cos ψ TERMS

$$\left(C_0\theta_2 r^2 - \frac{K_c}{R}\theta_2 r^3 + \frac{1}{4}C_0\theta_2\mu^2R^2 - \frac{1}{4}K_c\theta_2\mu^2Rr \right)$$

Sin 2ψ TERMS

$$\left(+C_0\theta_2\mu Rr - K_c\theta_2\mu r^2 \right)$$

Cos 2ψ TERMS

$$\left(-\frac{C_0\theta_0}{2}\mu^2R^2 + \frac{C_0K_\theta\mu^2Rr}{2} + \frac{1}{2}K_c\theta_0\mu^2Rr - \frac{K_cK_\theta}{2}\mu^2r^2 \right.$$

$$\left. -C_0\theta_1\mu rR + K_c\theta_1\mu r^2 \right)$$

SIN 3ψ TERMS

$$\left(-\frac{1}{4} C_0 \theta_1 \mu^2 R^2 + \frac{1}{4} K_C \theta_1 \mu^2 R r\right)$$

COS 3ψ TERMS

$$\left(-\frac{1}{4} C_0 \theta_2 \mu^2 R^2 + \frac{1}{4} K_C \theta_2 \mu^2 R r\right)$$

THE EXPRESSION FOR $dT_{CONST.}$ MAY THEN BE WRITTEN

$$dT_{CONST.} = \frac{\rho}{2} a_0 \omega^2 dr \left(A_0 + A_1 \sin\psi + A_2 \cos\psi + A_3 \sin 2\psi + A_4 \cos 2\psi + A_5 \sin 3\psi + A_6 \cos 3\psi\right)$$

WHERE EACH COEFFICIENT IS A POWER SERIES IN r.

THE COEFFICIENTS ARE TABULATED BELOW

$$A_0 = \left(\frac{C_0 \theta_0 \mu^2 R^2}{2}\right) + r\left(-\frac{\mu^2 R}{2}\left(C_0 K_\theta + \theta_0 K_C\right) + C_0 R\left(\theta_1 \mu - \nu\right)\right)$$

$$+ r^2\left(C_0 \theta_0 + \frac{K_C K_\theta \mu^2}{2} - K_C \theta_1 \mu + K_C \nu\right)$$

$$+ r^3\left(-\frac{C_0 K_\theta}{R} - \frac{K_C \theta_0}{R}\right) + r^4\left(\frac{K_C K_\theta}{R^2}\right)$$

$$A_1 = \left(\frac{3}{4}C_0\theta_1\mu^2R^2 - C_0\mu\nu R^2\right) + r\left(-\frac{3}{4}K_c\theta_1\mu^2R + 2C_0\theta_0\mu R + K_c\mu\nu R\right) + r^2\left(C_0\theta_1 - 2C_0K_\theta\mu - 2K_c\theta_0\mu\right) + r^3\left(-\frac{K_c\theta_1}{R} + \frac{2K_cK_\theta}{R}\mu\right)$$

$$A_2 = \left(\frac{1}{4}C_0\theta_2\mu^2R^2\right) + r\left(-\frac{1}{4}K_c\theta_2\mu^2R\right) + r^2\left(C_0\theta_2\right) + r^3\left(-\frac{K_c\theta_2}{R}\right)$$

$$A_3 = r\left(C_0\theta_2\mu R\right) + r^2\left(-K_c\theta_2\mu\right)$$

$$A_4 = \left(-\frac{C_0\theta_0\mu^2R^2}{2}\right) + r\left(\frac{C_0K_\theta\mu^2R}{2} + \frac{K_c\theta_0\mu^2R}{2} - C_0\theta_1\mu R\right) + r^2\left(-\frac{K_cK_\theta}{2}\mu^2 + K_c\theta_1\mu\right)$$

$$A_5 = -\frac{1}{4} C_0 \theta_1 \mu^2 R^2 + r\left(\frac{1}{4} K_c \theta_1 \mu^2 R\right)$$

$$A_6 = -\frac{1}{4} C_0 \theta_2 \mu^2 R^2 + r\left(\frac{1}{4} K_c \theta_2 \mu^2 R\right)$$

Considering the part of dT dependent on static blade shape

$$dT_2 = \frac{\rho}{2} a_\infty \omega^2 dr \left(B_2 \cos\psi + B_3 \sin 2\psi\right)\left(\frac{dy}{dr}\right)$$

WHERE

$$B_2 = r(-C_0 \mu R) + r^2 (K_c \mu)$$

$$B_3 = \left(-\frac{1}{2} C_0 \mu^2 R^2\right) + r\left(\frac{1}{2} K_c \mu^2 R\right)$$

Finally the part of dT dependent on $\frac{dy}{dt}$

$$dT_3 = \frac{\rho}{2} a_\infty \omega^2 dr \left(G_0 + G_1 \sin\psi\right)\left(\frac{1}{\omega}\frac{dy}{dt}\right)$$

WHERE

$$G_0 = r(-C_0) + r^2\left(\frac{K_c}{R}\right)$$

$$G_1 = (-C_0 \mu R) + r(K_c \mu)$$

Recapitulating

$$dT^{*} = \omega^2 dr \left(A_0 + A_1 \sin\psi + A_2 \cos\psi + A_3 \sin 2\psi + A_4 \cos 2\psi + A_5 \sin 3\psi + A_6 \cos 3\psi\right)$$

$$+ \left(\frac{d\psi}{dr}\right)\left[\omega^2 dr \left(B_2 \cos\psi + B_3 \sin 2\psi\right)\right]$$

$$+ \frac{1}{\omega}\frac{d\psi}{dt}\left[\omega^2 dr \left(C_0 + C_1 \sin\psi\right)\right]$$

---

*Let all constants absorb $\frac{\rho}{2} a_{\infty}$

SECTION IIA

THE NATURAL MODES AND FREQUENCIES OF THE ROTATING BLADE

The Myklestad method for the calculation of the natural modes and frequencies of a rotating blade is summarized here.

The theory of normal coordinates indicates that, at a natural frequency, the rotating cantilever beam considered will vibrate in such a manner that the following conditions are satisfied:

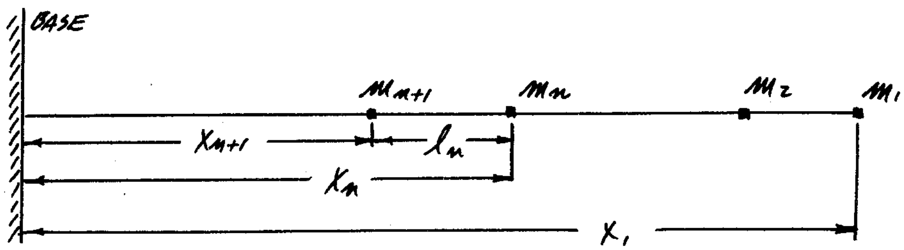
1. The transverse motion of all points on the beam will be harmonic and in phase, i.e. $y_i = \bar{y}_i \cos \omega t$ (SAY)
2. The motion will exist without application of external forcing functions. The external forces acting on a point mass will be only those of inertia and centrifugal force.
3. The motion will be consistent with the elastic characteristics of the beam.
4. The boundary conditions of the system will be satisfied.

   They are: a) deflection at the root

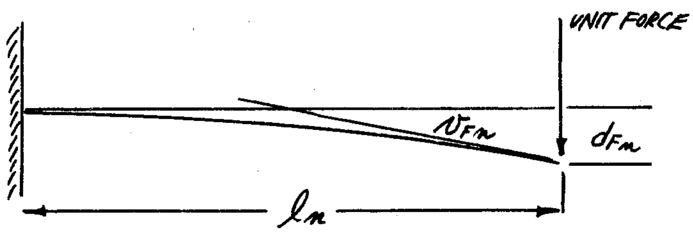
   b) slope at the root

By a proper mathematical statement of the four conditions it is possible to find both the frequencies at which this motion will exist and the mode of motion at each frequency.

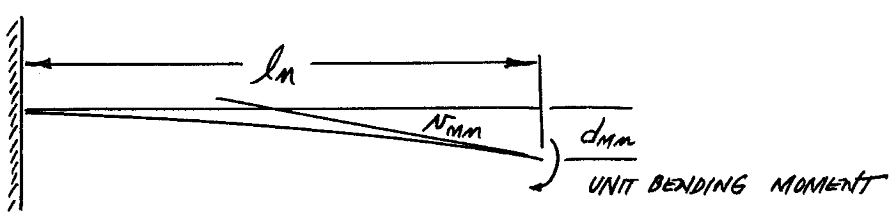
The system is assumed to consist of a finite number of discreet masses connected by a massless elastic beam.



The ELASTIC COEFFICIENTS at the n'th section are defined as follows:

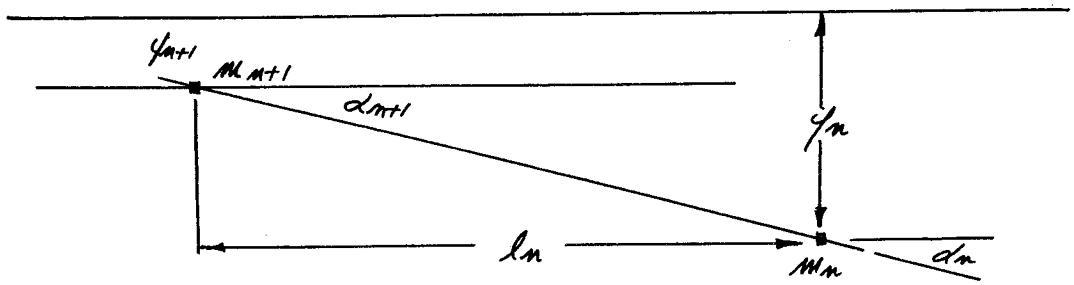


Slope ν_{Fn} and deflection d_{Fn} produced by unit force at the tip



Slope ν_{Mn} and deflection d_{Mn} produced by unit moment at the tip

The third of the necessary conditions is then expressed by the following relations:



$$\alpha_n = \alpha_{n+1} + S_n \nu_{Fn} + M_n \nu_{Mn} \qquad (1)$$

$$y_n = y_{n+1} + l_n \alpha_{n+1} + S_n d_{Fn} + M_n d_{Mn} \qquad (2)$$

Where S_N and M_n are the shear and bending moment at the n'th mass*

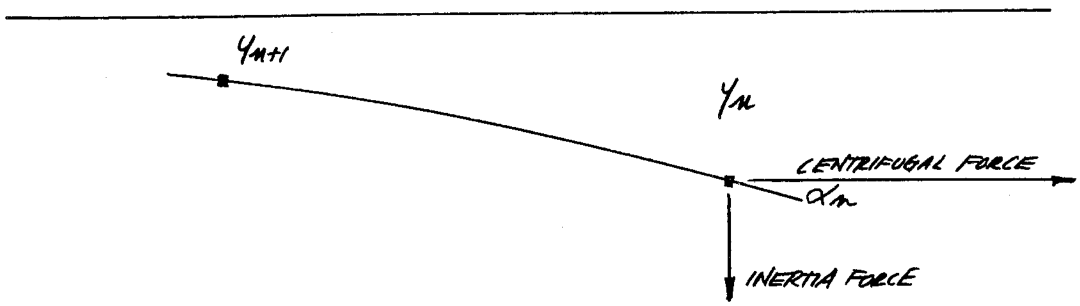
By the first condition the inertia loading on the i'th mass is $-m_i w^2 \bar{y}_i \cos wt$. S_n and M_n may now be expressed in terms of the inertia loading and the centrifugal force. The small angle and small deflection assumptions of beam theory are made. y_n and α_n are expressed harmonically as follows:

$$y_n = \bar{y}_n \cos wt$$

$$\alpha_n = \bar{\alpha}_n \cos wt$$

where in general $\alpha = \frac{dy}{dx}$

*Note the reversal of the sign of y in this section of the work.



$$S_n = \sum_{i=1}^{n} m_i \omega^2 \bar{y}_i \cos\omega t - \sum_{i=1}^{n} m_i x_i \Omega^2 \bar{\alpha}_n \cos\omega t \qquad (3)$$

$$M_n = \sum_{i=1}^{n-1} m_i \omega^2 \bar{y}_i \cos\omega t\,(x_i - x_n) - \sum_{i=1}^{n-1} m_i x_i \Omega^2 (\bar{y}_i - \bar{y}_n) \cos\omega t \qquad (4)$$

Where ω is a natural frequency

Ω is the speed of angular motion of the blade

Equations 1, 2, 3 and 4 now express all of the conditions necessary for motion in a natural mode except the boundary conditions.

<u>Substituting 3 & 4 in 1 and 2, rearranging, cancelling $\cos\omega t$</u> and dropping the bars above the amplitudes of deflection and slope, the following equations are obtained.

$$\alpha_{n+1} = \alpha_n \left(1 + \nu_{Fn} \sum_{i=1}^{n} m_i x_i \Omega^2\right) - \nu_{Fn} \sum_{i=1}^{n} m_i \omega^2 y_i - \nu_{Mn} \sum_{i=1}^{n-1} \left(m_i \omega^2 y_i (x_i - x_n)\right) - \nu_{Mn} \sum_{i=1}^{n-1} \left(m_i x_i \Omega^2 (y_i - y_n)\right) \qquad (5)$$

$$y_{n+1} = y_n - \alpha_n \left[l_n + (l_n N_{Fn} - d_{Fn}) \sum_{i=1}^{n} m_i x_i \Omega^2 \right]$$

$$+ (l_n N_{Fn} - d_{Fn}) \sum_{i=1}^{n} m_i \omega^2 y_i$$

$$+ (l_n N_{Mn} - d_{Mn}) \sum_{i=1}^{n-1} \left(m_i \omega^2 y_i (x_i - x_n) - m_i x_i \Omega^2 (y_i - y_n) \right) \qquad (6)$$

Let the amplitude of the slope at the tip of the blade be an arbitrary angle ϕ and the amplitude of vibration of the tip be 1. From equations 5 and 6 it can be seen that in general α_n & y_n are then expressable as:

$$\alpha_n = f_{\phi n} \phi - f_n \qquad (7)$$

$$y_n = -g_{\phi n} \phi + g_n \qquad (8)$$

$f_{\phi n}, f_n, g_{\phi n}, g_n$ are <u>AMPLITUDES COEFFICIENTS</u>. Their value at each n gives the shape of the curve of deflection.

<u>at the blade tip</u>

$\alpha_1 = \phi$ $\quad \therefore \quad$ $f_1 = 0$, $f_{\phi 1} = 1$

$y_1 = 0$ $\quad\quad\quad$ $g_1 = 1$, $g_{\phi 1} = 0$

<u>at the blade root $(n = b)$</u>

$\alpha_b = 0$ $\quad\quad\quad$ $f_{\phi b} - f_b = 0$

$y_b = 0$ $\quad \therefore \quad$ $\phi = \dfrac{g_b}{g_{\phi b}}$

Thus the boundary conditions are introduced.
Difference formulae may now be found for the amplitudes coefficients by substituting 7 & 8 in equations 5 & 6 and collecting coefficients of ϕ and constants. The formulae are:

$$f\phi_{n+1} = A_n f\phi_n + N_{Fn} G\phi_n + N_{Mn} G\phi_n' \quad (9)$$

$$f_{n+1} = A_n f_n + N_{Fn} G_n + N_{Mn} G_n' \quad (10)$$

$$g\phi_{n+1} = g\phi_n + A_n' f\phi_n + U_{Fn} G\phi_n + U_{Mn} G\phi_n' \quad (11)$$

$$g_{n+1} = g_n + A_n' f_n + U_{Fn} G_n + U_{Mn} G_n' \quad (12)$$

WHERE

$$U_{Fn} \equiv l_n N_{Fn} - d_{Fn}$$

$$U_{Mn} \equiv l_n N_{Mn} - d_{Mn}$$

$$\sum_{i=1}^{n} m_i x_i \Omega^2 \equiv a_n \qquad \text{AND} \qquad 1 + N_{Fn} a_n \equiv A_n$$

$$l_n + U_{Fn} a_n \equiv A_n'$$

$$G\phi_n \equiv \sum_{i=1}^{n} m_i w^2 g\phi_i$$

$$G_n \equiv \sum_{i=1}^{n} m_i w^2 g_i$$

$$(\Delta g)_{\phi n} \equiv g_{\phi n+1} - g_{\phi n}$$

$$(\Delta g)_n \equiv g_{n+1} - g_n$$

$$\sum_{i=1}^{n-1} m_i x_i \Omega^2 (g_n - g_i) = \sum_{i=1}^{n-1} (\Delta g)_i a_i$$ (AS FOLLOWS FROM PREVIOUS DEFINITIONS)

FINALLY

$$G_{\phi n}' \equiv \sum_{i=1}^{n-1} \left(l_i G_{\phi i} + a_i (\Delta g)_{\phi i} \right)$$

$$G_n' \equiv \sum_{i=1}^{n-1} \left(l_i G_i + a_i (\Delta g)_i \right)$$

By assuming a value for ω , it is possible to proceed from tip to root, finding all amplitudes coefficients, by means of equations 9 through 12 and the three boundary conditions $\alpha_1 = \phi$, $y_1 = 1$, $y_b = 0$

At the root a value is found for α_b. When α_b is zero a natural mode exists. A plot of α_b versus ω then yields as many of the natural frequencies and modes as are desired.

This method is now applied to the blade taken as the example in this thesis.

SECTION IIb

The step tapered hollow steel spar which is the main structural member of the blade is shown in Figure (7). So little stiffness in vertical bending is added by the covering that the steel spar is considered to be the only elastic member present. The weight of the covering, ribs, etc. is considered.

The essential characteristics of blade are given below

Station"	Spar Size"	Weight-Lbs
0-12	3.50 x .263	9.09
12-30	3.25 x .224	10.86
30-45	3.00 x .200	9.08
45-60	2.75 x .183	9.21
60-74	2.50 x .170	8.02
74-90	2.25 x .148	7.80
90-102	2.00 x .131	5.02
102-114	1.75 x .117	4.30
114-126	1.50 x .109	3.79
126-145	1.25 x .104	5.17
145-169	1.00 x .104	5.66
169-193	.875 x .102	4.90
193-210	.750 x .102	3.05
		85 . 95 Lbs

Spar size given as diameter x wall thickness

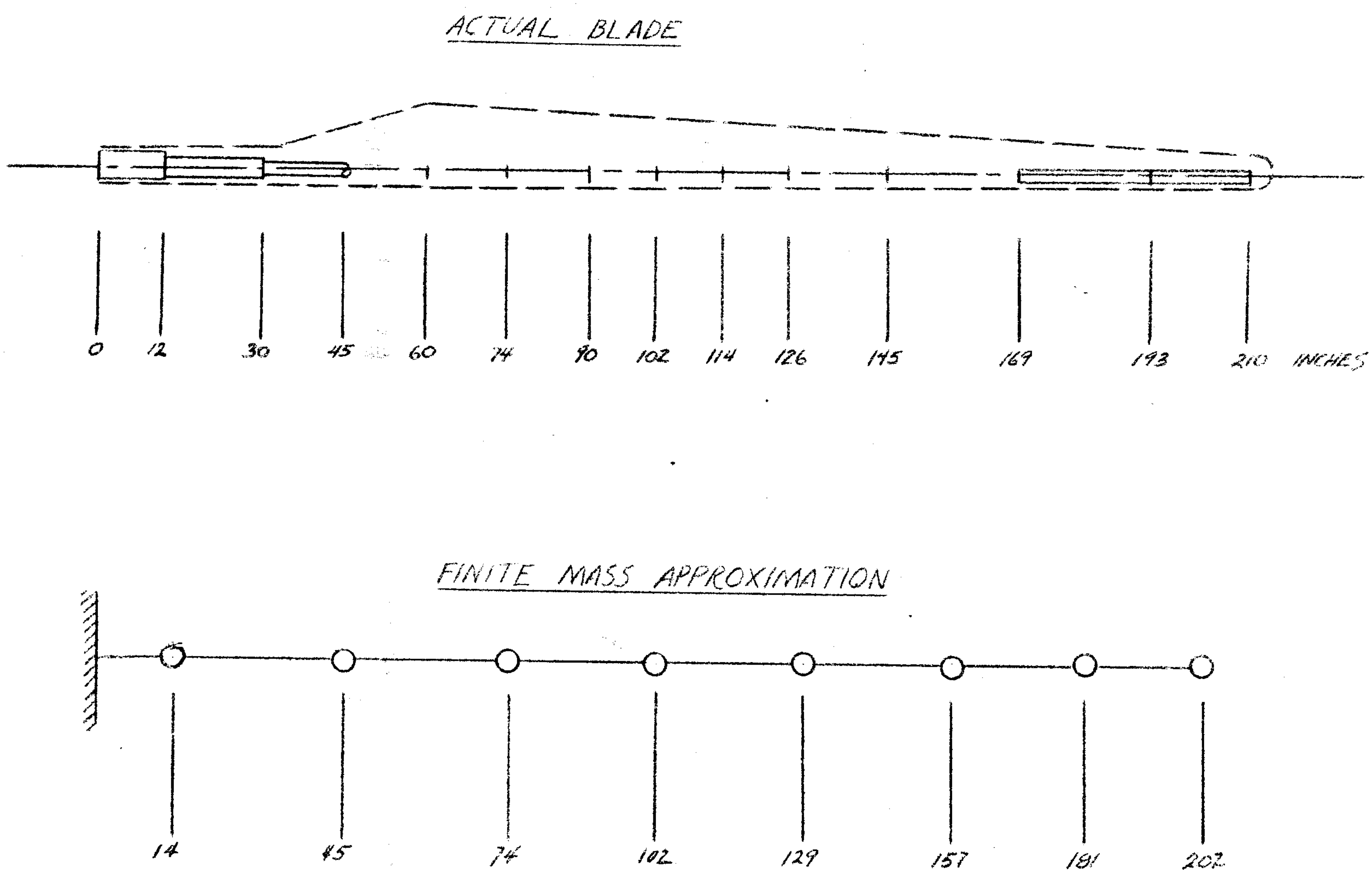

ACTUAL BLADE
0
12
30
45
60
74
90
102
114
126
145
169
193
210
INCHES
FINITE MASS APPROXIMATION
14
45
74
102
129
157
181
202

FIGURE 7

Moment of inertia of hollow circular cross section

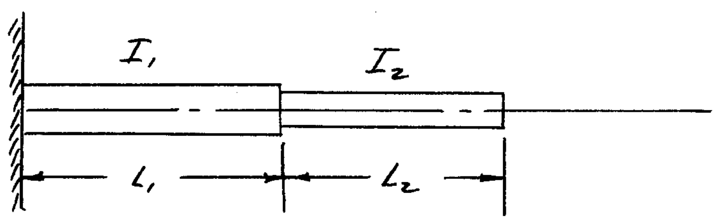
$$I = \frac{1}{4}\pi\left(R^4 - R_o^4\right)$$

Where R = OUTSIDE RADIUS

R_o = INSIDE RADIUS

Modulus of Elasticity $E = 29 \times 10^6$ #/SQ.IN.

Elastic coefficients



For a uniform section of length L

Unit Force products at end Deflection $= \frac{L^3}{3EI}$

Slope $= \frac{L^2}{2EI}$

Unit Moment produces at end Deflection $= \frac{L^2}{2EI}$

Slope $= \frac{L}{EI}$

By superposition of the effects at the end of each section

$$Ed_F = \frac{L_2^3}{3I_2} + L_2\left(\frac{L_1^2}{2I_1} + \frac{L_2 L_1}{I_1}\right) + \frac{L_1^3}{3I_1} + \frac{L_2 L_1^2}{2I_1}$$

$$Ed_F = \frac{L_2^3}{3I_2} + \frac{L_1^3}{3I_1} + \frac{L_1 L_2}{I_1}(L_1 + L_2)$$

$$E\theta_F = Ed_M = \frac{L_2^2}{2I_2} + \frac{L_1^2}{2I_1} + \frac{L_1 L_2}{I_1}$$

$$E\theta_M = \frac{L_1}{I_1} + \frac{L_2}{I_2}$$

All the material necessary for the investigation of the natural modes and frequencies has now been presented. The results of the calculation as carried out for the given blade are presented in the six charts and tables which follow. They are:

1 - A table of moments of inertia as calculated from the given blade data.

2 - A table of the constants $a, l, N_F, N_M, A, U_F, U_M, A'$ as required for the frequency work.

3 - A sample of the calculations by which a point on the frequency determination chart (α_b vs. ω) is found.

4 - The frequency chart with the three lowest natural frequencies indicated.

5 - A table of the natural modes as determined by the formula $y = g - g_b\phi$ where the various quantities are found from calculations such as shown in 3.

6 - A plot of the three lowest natural modes.

MOMENTS OF INERTIA

STATION LIMITS	INTERVAL	R	R_0	I - INCHES^4	$EI \times 10^{-6}$
0-12	12	1.750 00	1.487 00	3.52615	102.258
12-30	18	1.625 00	1.401 00	2.45068	71.0677
30-45	15	1.500 00	1.300 00	1.73290	50.2541
45-60	15	1.375 00	1.192 00	1.22177	35.4313
60-74	14	1.250 00	1.080 00	.848949	24.6195
74-90	16	1.125 00	.977 000	.542459	15.7313
90-102	12	1.000 00	.867 000	.337 510	9.78779
102-114	12	.875 000	.758 000	.221 107	5.83210
114-126	12	.750 000	.641 000	.115 911	3.36142
126-145	19	.625 000	.521 000	.0619741	1.79725
145-169	24	.500 000	.396 000	.0297135	.862 932
169-193	24	.437500	.335 500	.0158232	.458877
193-210	17	.375 000	.273 000	.0111 676	.323 960

CONSTANTS

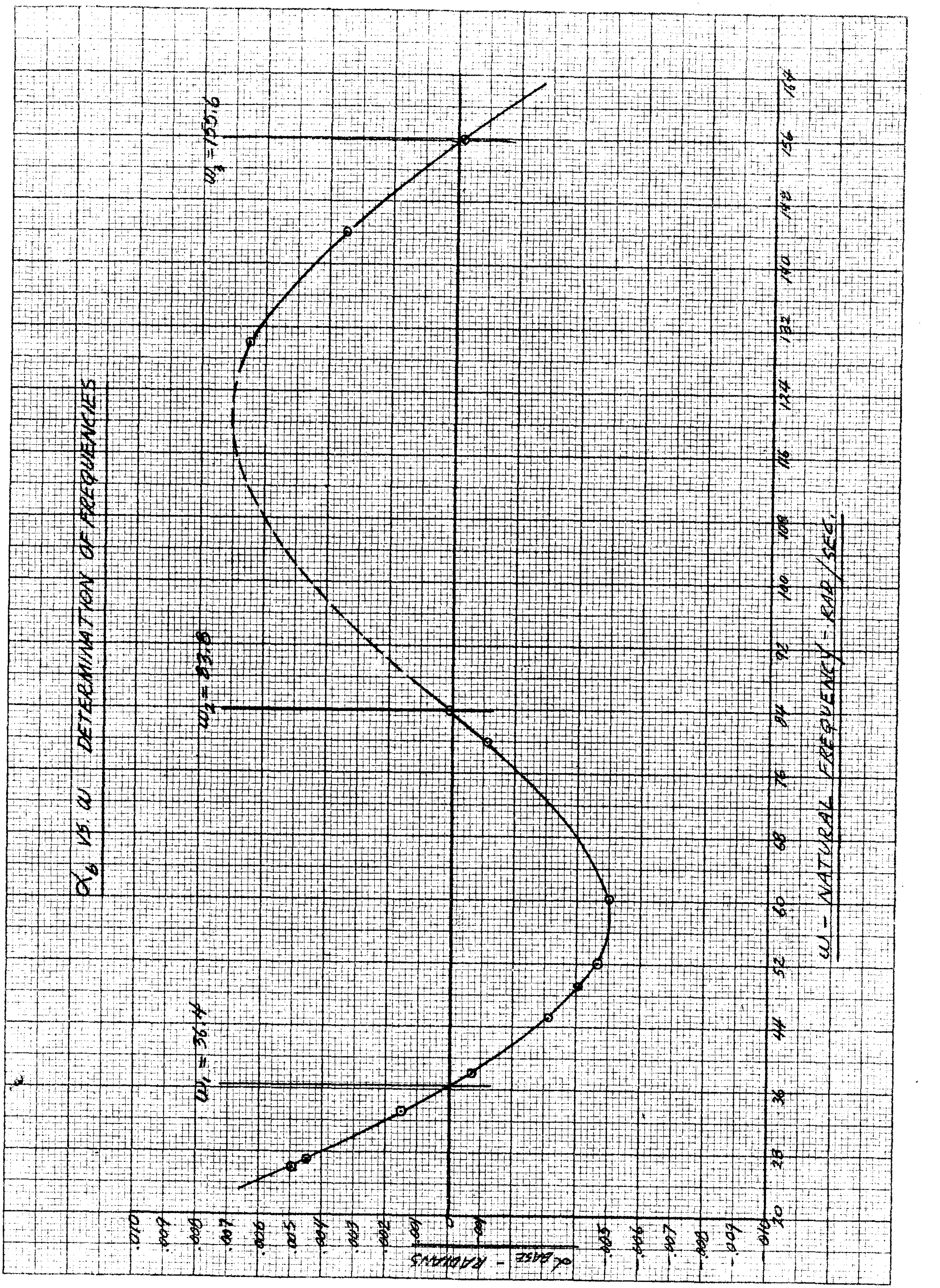
n	m	$a \times 10^{-6}$	ℓ	$\nu_F \times 10^6$	$\nu_M \times 10^6$	A	$u_F \times 10^6$	$u_M \times 10^6$	A'
1	.007 701 54	.001 078 97	21	454.801	47.7729	1.490 72	3592.50	590.4[illegible]	24.874[illegible]
2	.012 6993	.002 63[illegible] 19	24	382.063	35.881 1	[illegible]	3444.60	479.0[illegible]	33.[illegible]
3	.014 663 2	.004 1834[illegible]	28	261.438	[illegible]	2.035 01	2902.14	570.[illegible]	40.15[illegible]
4	.023 2124	.006 212 6[illegible]	27	76.6772	[illegible]	1.47657	813.514	[illegible]	21.9[illegible]
5	.024 145[illegible]	.007 87748	28	[illegible]7.6978	[illegible]	1.21819	288.159	[illegible]	[illegible]
6	.046 9844	.009 92768	29	15.0827	.9[illegible]	1.12988	108.61[illegible]	[illegible]	30.3[illegible]61
7	.047 383 4	.011 367 1	31	7.41665	.[illegible]614	1.08432	83.63[illegible]	9.217[illegible]	21.95[illegible]
8	.051 [illegible] 9	.011 858 2	14	.958 358	.136 [illegible]	1.011 [illegible]6	4.472 3[illegible]	.95[illegible]	14.[illegible]

$\omega = 36.4$ rad/sec $\omega^2 \times 1 = 1324.96$

n	$m\omega^2$	① $g_\phi = ① + ⑤$ for $n-1$	② $G_\phi \times 10^{-6} = \Sigma \frac{m\omega^2}{1}$ ①	③ $G_\phi' \times 10^{-6} = \Sigma(a② + l②)$ for $n-1$	④ $f_\phi = u_p ② + u_m ③ + A ④$ for $n-1$	⑤ $(\Delta g)_\phi = u_p ② + u_m ③ + A' ④$	n
1	10.4692	0	0	0	1	24.8740	1
2	16.8194	24.8740	.000418366	.0268383	1.49072	63.5922	2
3	19.4282	88.4662	.00213711	.204266	4.11271	248.355	3
4	26.7555	336.821	.0124962	1.30432	13.8323	582.164	4
5	31.9916	918.985	.0418956	5.25849	29.8839	1101.28	5
6	54.3027	2020.26	.151601	15.1069	49.3600	1757.34	6
7	62.7811	3777.60	.388753	36.9496	72.7404	2682.43	7
8	68.4791	6460.05	.831140	79.4981	101.105	1500.73	8
b	———	7960.76	———	———	113.934	———	b

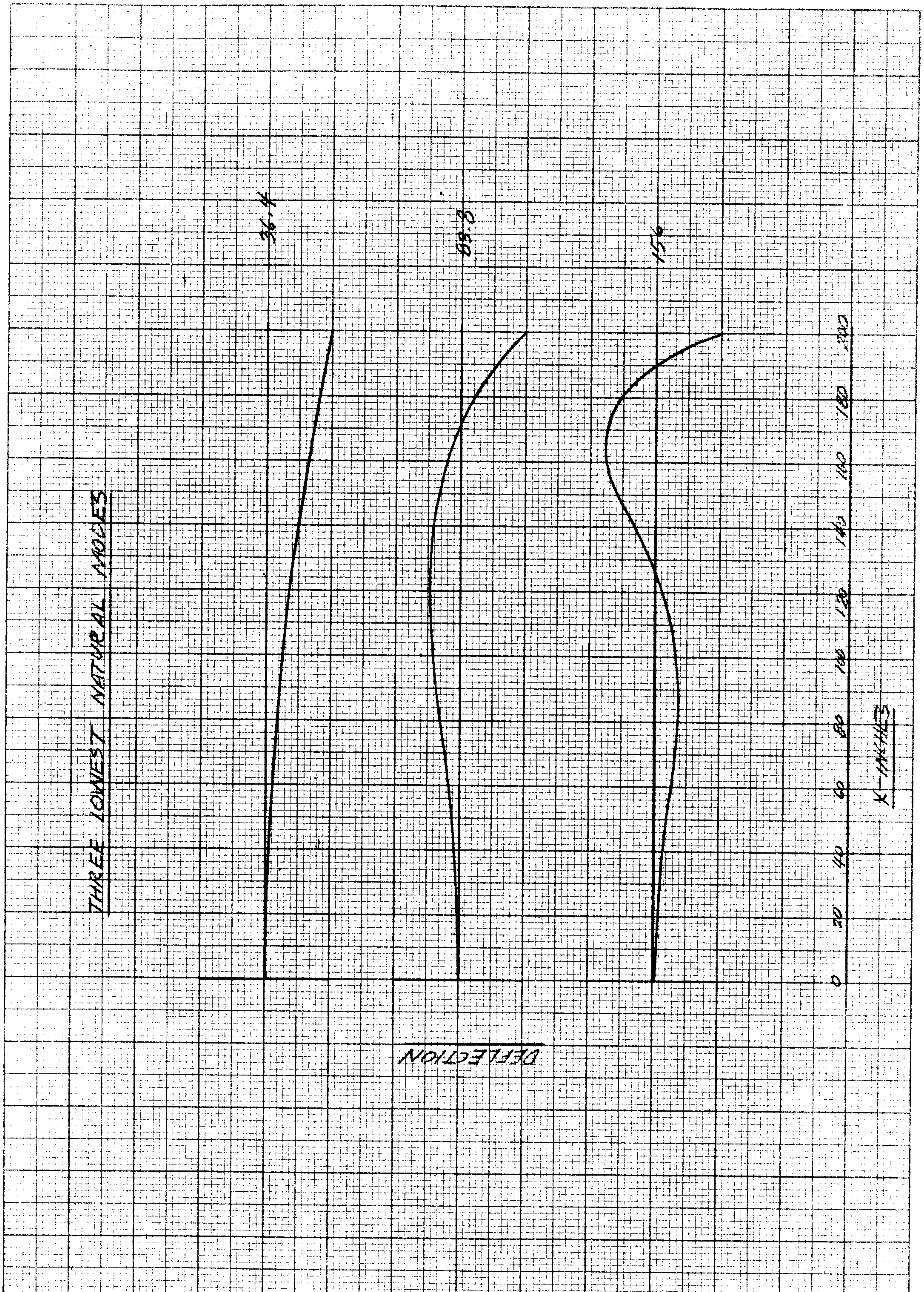
$\phi = \frac{g_b}{g_{\phi b}} = .00918547$ $\alpha_b = f_{\phi b}\,\phi - f_b = -.00004$

n	$m\omega^2$	① $g = ① + ⑤$ for $n-1$	② $G \times 10^{-3} = \Sigma \frac{m\omega^2}{1}$ ①	③ $G' \times 10^{-3} = \Sigma(a② + l②)$ for $n-1$	④ $f = u_p ② + u_m ③ + A ④$ for $n-1$ $\times(n②)$	⑤ $(\Delta g) = u_p ② + u_m ③ + A' ④$	n
1	10.4692	1	.0104692	0	0	.0375897	1
2	16.8194	1.03759	.0279208	.260411	4.76140	.378379	2
3	19.4282	1.41597	.0554305	1.92648	29.5611	2.07416	3
4	26.7555	3.49013	.162771	12.1660	120.347	5.18977	4
5	31.9916	8.07990	.440451	48.8024	269.460	7.99750	5
6	54.3027	18.6774	1.45468	139.891	449.947	16.0635	6
7	62.7811	34.7489	3.43575	341.550	666.190	24.6010	7
8	68.4791	59.3419	7.69943	733.949	928.169	13.7814	8
b	———	73.1233	———	———	1.04658	———	b


αB VS. ω DETERMINATION OF FREQUENCIES
ω1 = 36.4
ω2 = 83.8
ω3 = 155.6
ω - NATURAL FREQUENCY - RAD/SEC.
α BASE - RADIANS
.010
.009
.008
.007
.006
.005
.004
.003
.002
.001
0
.001
.005
.006
.007
.008
.009
.010
20
28
36
44
52
60
68
76
84
92
100
108
116
124
132
140
148
156
164

THREE LOWEST NATURAL MODES

n	y	y	y
FREQ. →	36.4	83.8	155.6
1	1	1	1
2	.809	+.270	−.515
3	.603	−.277	−.721
4	.396	−.473	−.024
5	.239	−.406	+.350
6	.120	−.247	+.355
7	.042	−.094	+.173
8	.004	−.007	+.018
b	0	0	0


THREE LOWEST NATURAL MODES
36.4
68.8
156
DEFLECTION
0
20
40
60
80
100
120
140
160
180
200
X - INCHES

SECTION III

MOTION OF THE BLADE

The motion of the blade under the applied aerodynamic forces will now be found. The system is considered continuous rather than made up of a finite number of discreet masses.

The motion, in general, will consist of some combina- of the natural modes. Only the effects of the lowest two natural modes are considered here but the work may be expanded with the addition of no new theory.*

The normal modes (normalized by taking unit deflection at the blade tip) may be described by power series

$$y = y_1(r) \qquad \text{AND} \qquad y = y_2(r)$$

The Complete Motion May Be Expressed As

$$y = q_1 y_1 + q_2 y_2$$

Where $\quad q_1 = q_1(t) \quad , q_2 = q_2(t)$

q_1 and q_2 are taken as the generalized coordinates in an application of the LaGrange equation. The generalized forces are found from the expressions for the aerodynamic force.

---

*The methods which follow were suggested by work of M. A. Biot. Proofs of the general statements made in this section may be found in Karman & Biot (ref.6)

LaGrange's Equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} = Q_i$$

Where

T = the kinetic energy of the system

U = the potential energy of the system

Q_i = the generalized force, as defined later

The Kinetic and Potential Energies

$$T = \frac{1}{2}\int_0^R \dot{y}^2\, dm$$

$$T = \frac{1}{2}\int_0^R (\dot{q}_1 y_1 + \dot{q}_2 y_2)^2\, dm \qquad \text{WHERE } \dot{q}_i = \frac{dq_i}{dt}$$

$$T = \frac{1}{2}\int_0^R \dot{q}_1^2 y_1^2\, dm + \frac{1}{2}\int_0^R \dot{q}_2^2 y_2^2\, dm + \int_0^R \dot{q}_1 \dot{q}_2 y_1 y_2\, dm$$

Because of the orthogonality of the principle modes

$$\int_0^R y_1 y_2\, dm = 0$$

$$\therefore T = \frac{1}{2}\dot{q}_1^2\int_0^R y_1^2\, dm + \frac{1}{2}\dot{q}_2^2\int_0^R y_2^2\, dm$$

The theory of normal coordinates also indicates that the potential energy is simply related to the kinetic energy through the natural frequencies and may be written as follows:

$$U = \frac{1}{2} q_1^2 \omega_1^2 \int_0^R y_1^2 \, dm + \frac{1}{2} q_2^2 \omega_2^2 \int_0^R y_2^2 \, dm$$

Where ω_1 AND ω_2 ARE THE LOWEST AND NEXT HIGHEST NATURAL FREQUENCIES OF THE BLADE

The Generalized Force

Q_i is defined as $\frac{\delta W}{\delta q_i}$

Where δW is the work done by the external forces on the system during a small change in the coordinate q_i indicated by δq_i.

In this case the only external load is that of the air of magnitude dT in length dr of the span.

$\delta W = \int dT \, \delta y$ over the blade

Where δy occurs because of δq_i

$$\delta y = \delta q_i \, y_i$$

$$\delta W = \int dT \, y_i(r) \, \delta q_i = \delta q_i \int dT \, y_i(r)$$

$$Q_i = \int_0^R dT \, y_i(r)$$

REWRITING dT

$$dT = w^2 dr \left(A_0 + A_1 \sin\psi + A_2 \cos\psi + A_3 \sin 2\psi + A_4 \cos 2\psi + A_5 \sin 3\psi + A_6 \cos 3\psi\right)$$

$$+ w^2 dr \left(q_1 y_1' + q_2 y_2'\right)\left(B_2 \cos\psi + B_3 \sin 2\psi\right)$$

$$+ w^2 dr \left(\frac{1}{w} \dot{q}_1 y_1 + \frac{1}{w} \dot{q}_2 y_2\right)\left(C_0 + C_1 \sin\psi\right)$$

WHERE

$$y_i' = \frac{dy_i}{dr}$$

<u>THE EQUATION FOR THE COORDINATE q_1 MAY NOW BE WRITTEN</u>

$$\frac{\partial T}{\partial \dot{q}_1} = \dot{q}_1 \int_0^R y_1^2 \, dm$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \ddot{q}_1 \int_0^R y_1^2 \, dm$$

IF $\lambda_1 = \int_0^R y_1^2 \, dm$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \ddot{q}_1 \lambda_1$$

$$\frac{\partial U}{\partial q_1} = q_1 \omega_1^2 \lambda_1$$

$$\varphi_1 = \int_0^R dr\, w^2 y_1 \left(A_0 + A_1 \sin\psi + A_2 \cos\psi + A_3 \sin 2\psi + A_4 \cos 2\psi + A_5 \sin 3\psi + A_6 \cos 3\psi\right)$$

$$+ \int_0^R dr\, w^2 \left(g_1 y_1' + g_2 y_2'\right)\left(B_2 \cos\psi + B_3 \sin 2\psi\right) y_1$$

$$+ \int_0^R dr\, w^2 \left(\frac{1}{w} \dot{g}_1 y_1 + \frac{1}{w} \dot{g}_2 y_2\right)\left(C_0 + C_1 \sin\psi\right) y_1$$

<u>A SERIES OF CONSTANTS ARE NOW DEFINED</u>

$$\lambda_3 = \int_0^R y_1 A_0\, dr$$

$$\lambda_4 = \int_0^R y_1 A_1\, dr$$

$$\lambda_5 = \int_0^R y_1 A_2\, dr$$

$$\lambda_6 = \int_0^R y_1 A_3\, dr$$

$$\lambda_7 = \int_0^R y_1 A_4\, dr$$

$$\lambda_8 = \int_0^R y_1 A_5\, dr$$

$$\lambda_9 = \int_0^R y_1 A_6\, dr$$

$$\lambda_{10} = \int_0^R y_1 y_1' B_2\, dr$$

$$\lambda_{11} = \int_0^R y_1 y_1' B_3\, dr$$

$$\lambda_{12} = \int_0^R y_1 y_2' B_2\, dr$$

$$\lambda_{13} = \int_0^R y_1 y_2' B_3\, dr$$

$$\lambda_{14} = \int_0^R y_1^2 C_0\, dr$$

$$\lambda_{15} = \int_0^R y_1^2 C_1\, dr$$

$$\lambda_{16} = \int_0^R y_1 y_2 C_0\, dr$$

$$\lambda_{17} = \int_0^R y_1 y_2 C_1\, dr$$

THE EQUATION IS THEN

$$\ddot{q}_1 \lambda_1 + q_1 \omega_1^2 \lambda_1 = \omega^2 (\lambda_3 + \lambda_4 \sin\psi + \lambda_5 \cos\psi + \lambda_6 \sin 2\psi + \lambda_7 \cos 2\psi + \lambda_8 \sin 3\psi + \lambda_9 \cos 3\psi)$$
$$+ \omega^2 q_1 (\lambda_{10} \cos\psi + \lambda_{11} \sin 2\psi)$$
$$+ \omega^2 q_2 (\lambda_{12} \cos\psi + \lambda_{13} \sin 2\psi)$$
$$+ \omega \dot{q}_1 (\lambda_{14} + \lambda_{15} \sin\psi)$$
$$+ \omega \dot{q}_2 (\lambda_{16} + \lambda_{17} \sin\psi)$$

WRITING IN TERMS OF THE VARIABLE $\psi = \omega t$

$$\dot{q}_i = \frac{\omega \, dq_i}{d\psi} \quad \text{ETC.}$$

$$\lambda_1 \frac{d^2 q_1}{d\psi^2} + \frac{dq_1}{d\psi} (-\lambda_{14} - \lambda_{15} \sin\psi) + q_1 \left(\frac{\omega_1^2}{\omega^2} \lambda_1 - \lambda_{10} \cos\psi - \lambda_{11} \sin 2\psi \right)$$
$$+ \frac{dq_2}{d\psi} (-\lambda_{16} - \lambda_{17} \sin\psi) + q_2 (-\lambda_{12} \cos\psi - \lambda_{13} \sin 2\psi)$$
$$= \lambda_3 + \lambda_4 \sin\psi + \quad \cdots \cdots \quad + \lambda_9 \cos 3\psi$$

AND FOR q_2

$$\lambda_2 \frac{d^2 q_2}{d\psi^2} + \frac{dq_2}{d\psi} (-\lambda'_{16} - \lambda'_{17} \sin\psi) + q_2 \left(\frac{\omega_2^2}{\omega^2} \lambda_2 - \lambda'_{12} \cos\psi - \lambda'_{13} \sin 2\psi \right)$$
$$+ \frac{dq_1}{d\psi} (-\lambda'_{14} - \lambda'_{15} \sin\psi) + q_1 (-\lambda'_{10} \cos\psi - \lambda'_{11} \sin 2\psi)$$
$$= \lambda'_3 + \lambda'_4 \sin\psi + \quad \cdots \cdots \quad + \lambda'_9 \cos 3\psi$$

WHERE

$$\lambda_2 = \int_0^R y_2^2 \, dm$$

$\lambda'_n = \lambda_n$ WITH y_2 SUBSTITUTED FOR ONE y_1

SOLUTION OF THE EQUATIONS

ASSUME

$$q_1 = a_0 + a_1 \sin\psi + a_2 \cos\psi$$

$$q_2 = b_0 + b_1 \sin\psi + b_2 \cos\psi$$

THEN

$$\frac{dq_1}{d\psi} = a_1 \cos\psi - a_2 \sin\psi \qquad \frac{dq_2}{d\psi} = b_1 \cos\psi - b_2 \sin\psi$$

$$\frac{d^2 q_1}{d\psi^2} = -a_1 \sin\psi - a_2 \cos\psi \qquad \frac{d^2 q_2}{d\psi^2} = -b_1 \sin\psi - b_2 \cos\psi$$

SUBSTITUTING IN THE FIRST D.E.

$$-\lambda_1 a_1 \sin\psi - \lambda_1 a_2 \cos\psi + (a_1 \cos\psi - a_2 \sin\psi)(-\lambda_{14} - \lambda_{15} \sin\psi)$$

$$+ (a_0 + a_1 \sin\psi + a_2 \cos\psi)\left(\frac{\omega_1^2}{\omega^2}\lambda_1 - \lambda_{10} \cos\psi - \lambda_{11} \sin 2\psi\right)$$

$$+ (b_1 \cos\psi - b_2 \sin\psi)(-\lambda_{16} - \lambda_{17} \sin\psi) + (b_0 + b_1 \sin\psi + b_2 \cos\psi) \times$$

$$(-\lambda_{12} \cos\psi - \lambda_{13} \sin 2\psi) = \lambda_3 + \lambda_4 \sin\psi + \lambda_5 \cos\psi$$

$$+ \lambda_6 \sin 2\psi + \lambda_7 \cos 2\psi + \lambda_8 \sin 3\psi + \lambda_9 \cos 3\psi$$

COLLECTING COEFFICIENTS OF THE CONSTANT TERMS, SIN ψ TERMS AND COS ψ TERMS, THE FOLLOWING 3 EQUATIONS ARE OBTAINED

$$a_0 \frac{\omega_1^2}{\omega^2}\lambda_1 + \frac{a_2}{2}(\lambda_{15}-\lambda_{10}) + \frac{b_2}{2}(\lambda_{17}-\lambda_{12}) = \lambda_3$$

$$a_1\left(\left(\frac{\omega_1}{\omega}\right)^2\lambda_1-\lambda_1\right) + a_2\left(\lambda_{14}-\frac{\lambda_{11}}{2}\right) + b_2\left(\lambda_{16}-\frac{\lambda_{13}}{2}\right) = \lambda_4$$

$$a_0\lambda_{10} + b_0\lambda_{12} + a_1\left(\lambda_{14}+\frac{\lambda_{11}}{2}\right) + b_1\left(\lambda_{16}+\frac{\lambda_{13}}{2}\right) + a_2\left(-\left(\frac{\omega_1}{\omega}\right)^2\lambda_1+\lambda_1\right) = -\lambda_5$$

SUBSTITUTION IN THE SECOND D.E. YIELDS 3 ADDITIONAL EQUATIONS. THUS THE BLADE MOTION COEFFICIENTS ARE DETERMINED BY THE SIMULTANEOUS SOLUTION OF THE FOLLOWING 6 ALGEBRAIC EQUATIONS.

a_0	a_1	a_2	b_0	b_1	b_2	= CONST.
$\left(\frac{\omega_1}{\omega}\right)^2\lambda_1$	0	$\frac{\lambda_{15}-\lambda_{10}}{2}$	0	0	$\lambda_{17}-\lambda_{12}$	λ_3
0	0	$\frac{\lambda_{15}'-\lambda_{10}'}{2}$	$\left(\frac{\omega_2}{\omega}\right)^2\lambda_2$	0	$\lambda_{17}'-\lambda_{12}'$	λ_3'
0	$\lambda_1\left(\frac{\omega_1^2}{\omega^2}-1\right)$	$\lambda_{14}-\frac{\lambda_{11}}{2}$	0	0	$\lambda_{16}-\frac{\lambda_{13}}{2}$	λ_4
0	0	$\lambda_{14}'-\frac{\lambda_{11}'}{2}$	0	$\lambda_2\left(\frac{\omega_2^2}{\omega^2}-1\right)$	$\lambda_{16}'-\frac{\lambda_{13}'}{2}$	λ_4'
λ_{10}	$\lambda_{14}+\frac{\lambda_{11}}{2}$	$-\lambda_1\left(\frac{\omega_1^2}{\omega^2}-1\right)$	λ_{12}	$\lambda_{16}+\frac{\lambda_{13}}{2}$	0	$-\lambda_5$
λ_{10}'	$\lambda_{14}'+\frac{\lambda_{11}'}{2}$	0	λ_{12}'	$\lambda_{16}'+\frac{\lambda_{13}'}{2}$	$-\lambda_2\left(\frac{\omega_2^2}{\omega^2}-1\right)$	$-\lambda_5'$

RECAPITULATING, THE FOLLOWING QUANTITIES MUST BE EVALUATED IN A NUMERICAL PROBLEM

$y_1(r)$, $y_2(r)$; y_1', y_2' ; A_0, A_1, A_2 ; w_1, w_2 ;

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_{10} - \lambda_{17}$;

$\lambda_3', \lambda_4', \lambda_5', \lambda_{10}' - \lambda_{17}'$

SECTION IVa

NUMERICAL EXAMPLE

The motion of the sample blade in a given flight condition will now be found.

1 The Quantities $y_1(r), y_2(r), y_1'(r), y_2'(r)$

The following series are adequate for the representation of the two modes.

$$y_1 = .00354\, r^2 \qquad r \text{ in feet}$$

$$y_2 = -.009398 r^2 + .0004584 r^4$$

ϕ
$$y_1' = .00708\, r$$

$$y_2' = -.018796\, r + .0018336 r^3$$

The plot on the following page shows the degree of the approximation.

2 The Quantities A_0, A_1, A_2

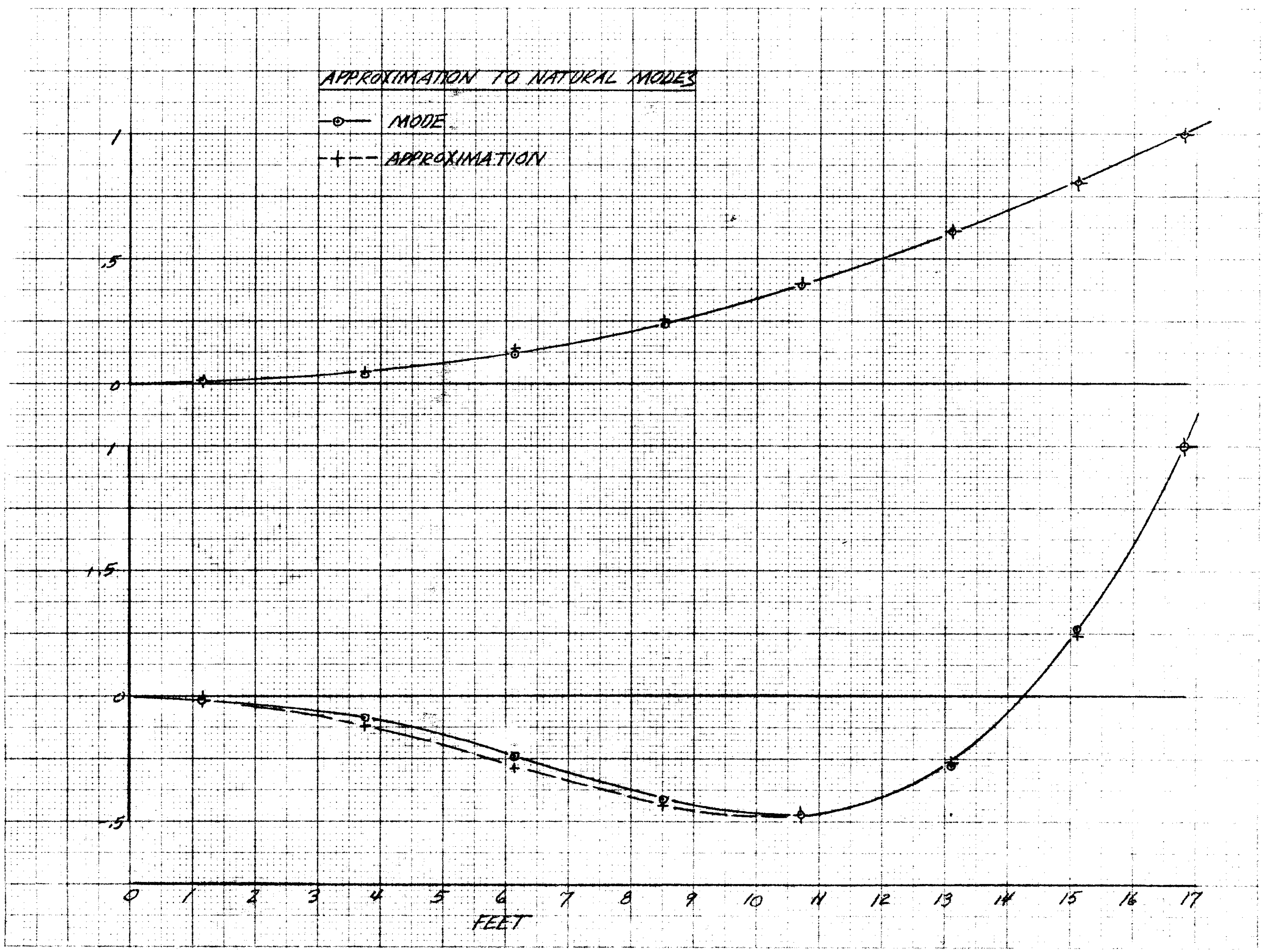
The following aerodynamic parameters are taken

$$\mu = .304 \qquad \theta_1 = \theta_2 = 0$$

$$\nu = .0853$$

$$\theta_0 = 13.0^\circ = .2269$$

This corresponds to flight at 100 mph of a ship using 2, 2 bladed rotors of the type being considered. The weight of the ship is approximately 3000 pounds.



APPROXIMATION TO NATURAL MODES
MODE
APPROXIMATION
1
.5
0
1
.5
0
.5
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
FEET

THE PHYSICAL BLADE PARAMETERS ARE

$C_0 = 1.593$ (FT.)

$K_c = 1.039$

$K_\theta = 0$

$R = 17.5$ (FT.)

$W = 26$ RADIANS/SEC.

GENERAL CONSTANTS

$\rho = .002378$ SLUGS/CU.FT.

$a_0 = 5.3$

THEN

$$A_0 = .03223 - .01619\,r + .002837\,r^2 - .00008489\,r^3$$

$$A_1 = -.07972 + .02721\,r - .0009031\,r^2$$

$$A_2 = 0$$

$$B_2 = -.05341\,r + .001991\,r^2$$

$$B_3 = -.1421 + .005296\,r$$

$$C_0 = -.01004\,r + .0003741\,r^2$$

$$C_1 = -.05341 + .001990\,r$$

AND

$$dm = (.258 - .0120\,r)\,dr$$

FROM THESE ARE FOUND THE FOLLOWING VALUES OF λ

$\lambda_1 = +.3414$

$\lambda_2 = +.2914$

$\lambda_3 = +.7181$ $\qquad \lambda_3' = -.08594$

$\lambda_4 = +.7052$ $\qquad \lambda_4' = -.07300$

$\lambda_5 = 0$ $\qquad \lambda_5' = 0$

$\lambda_{10} = -.2005$ $\qquad \lambda_{10}' = -.00204$

$\lambda_{11} = -.03994$ $\qquad \lambda_{11}' = +.00866$

$\lambda_{12} = -.5364$ $\qquad \lambda_{12}' = -.3281$

$\lambda_{13} = -.08872$ $\qquad \lambda_{13}' = -.05793$

$\lambda_{14} = -.2657$ $\qquad \lambda_{14}' = -.04737$

$\lambda_{15} = -.1003$ $\qquad \lambda_{15}' = -.00099$

$\lambda_{16} = -.04737$ $\qquad \lambda_{16}' = -.1886$

$\lambda_{17} = -.00099$ $\qquad \lambda_{17}' = -.08088$

$\left(\frac{\omega_1}{\omega}\right)^2 = 1.9600$ $\qquad \left(\frac{\omega_2}{\omega}\right)^2 = 10.388$

THE EQUATIONS TAKE THE FORM

a_0	a_1	a_2	b_0	b_1	b_2	CONST.
.6691	0	+.0501	0	0	+.5354	+.7181
0	0	+.0005250	3.0271	0	+.2472	-.08594
0	+.3277	-.2457	0	0	-.003010	+.7052
0	0	-.05170	0	+2.736	-.1596	-.07300
-.2005	-.2857	-.3277	-.5364	-.09173	0	0
-.00204	-.04304	0	-.3281	-.2176	-2.736	0

WHICH YIELD THE FINAL SOLUTION OF

$a_0 = +1.20$ $\qquad b_0 = -.03$

$a_1 = +1.00$ $\qquad b_1 = -.06$

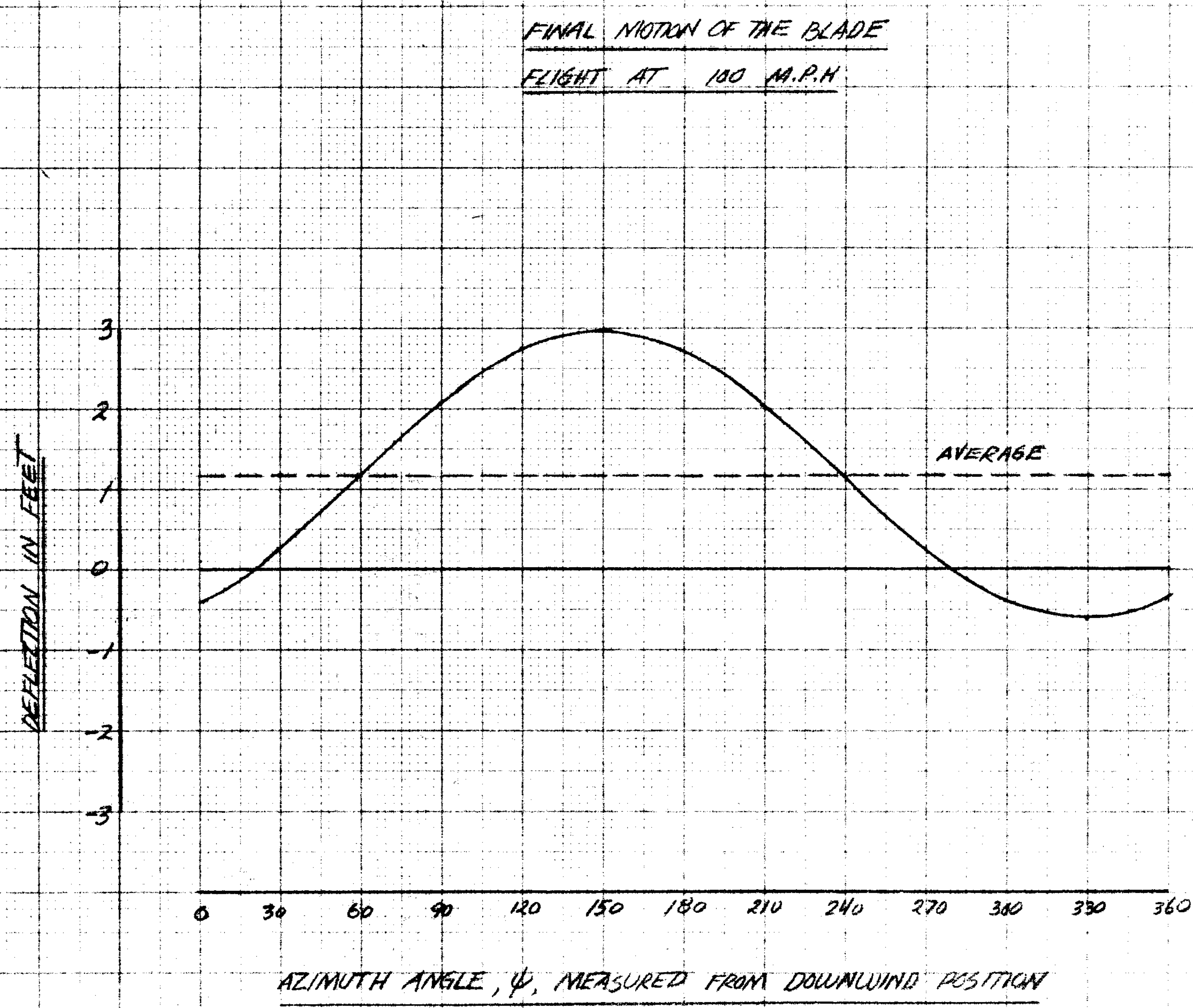
$a_2 = -1.54$ $\qquad b_2 = -.01$

THE MOTION THEN IS DESCRIBED BY

$$y = (1.20 + 1.00 \sin\psi - 1.54 \cos\psi)(.000354\, r^2)$$
$$+ (-.03 - .06 \sin\psi - .01 \cos\psi)(-.009398\, r^2 + .00004584\, r^4)$$

WHERE r IS IN FEET.

THE MOTION OF A POINT AT $r = 210$ INCHES $(\mu = 1)$ IS PLOTTED ON THE FOLLOWING PAGE.


FINAL MOTION OF THE BLADE
FLIGHT AT 100 M.P.H
AVERAGE
DEFLECTION IN FEET
3
2
1
0
-1
-2
-3
0
30
60
90
120
150
180
210
240
270
300
330
360
AZIMUTH ANGLE, ψ, MEASURED FROM DOWNWIND POSITION

SECTION IVb

MOMENTS ON THE HUB

The moment transmitted by the blade to the hub may now be found directly. From the derivation of Section IIa, the bending moment at the blade root is harmonic, in phase with the deflection and has the value

$$M_b = (-G_{\phi m'} + G_{m'}) \times 12$$

Where $m = b$

The moment is in inch pounds

This is for the condition of unit deflection at station $m=1$ ($y=1$ FT.)

The total moment is then

$$M_b = \left[g_1 (-G_{\phi b'}\phi + G_{b'})_1 + g_2(-G_{\phi b'}\phi + G_{b'})_2 \right] \times 12$$

For This Calculation

$$12\left(-(G_{\phi b'}\phi - G_{b'})_1\right) = 55{,}000 \text{ IN.LB.}$$

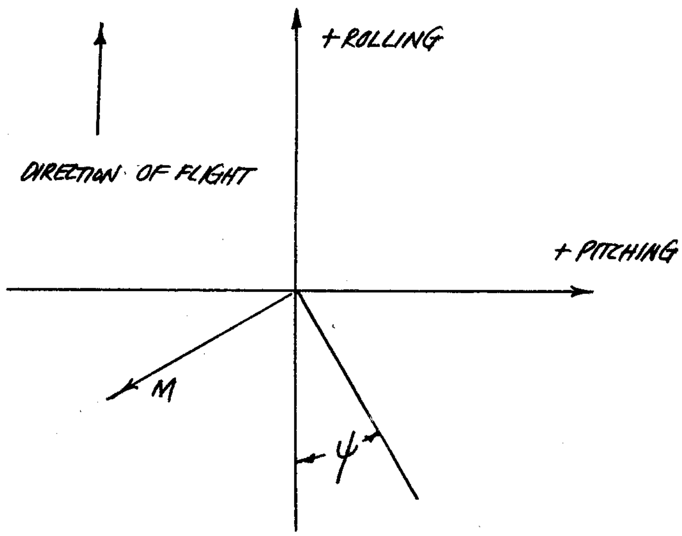
$$12\left(-(G_{\phi b'}\phi - G_{b'})_2\right) = -137{,}000 \text{ IN LB.}$$

The moments may be checked by the less reliable method of applying the equation $EI\frac{d^2y}{dx^2}$ to the approximation curves used in Section IVa. The corresponding values are 60,000 AND -160,000 IN LB. The greater discrepancy in the second case is to be expected from the poorer fit of the deflection power series to the deflection curve .

Finally

$$M = 61,890 + 46,780 \sin\psi - 83,330 \cos\psi$$

VIBRATORY LOADS ON THE HUB



The right hand rule is used for the representation of moments. The directions of positive rolling and pitching moments are indicated .

It has been found that M may be represented as

$$M = M_0 + M_1 \sin\psi + M_2 \cos\psi$$

$$M_P = -M \cos\psi$$

$$M_R = -M \sin\psi$$

$$M_P = -\cos\psi \left(M_0 + M_1 \sin\psi + M_2 \cos\psi\right)$$

$$= -M_0 \cos\psi - \frac{M_1}{2} \sin 2\psi - \frac{M_2}{2} - \frac{M_2}{2} \cos 2\psi$$

$$M_R = -\sin\psi \left(M_0 + M_1 \sin\psi + M_2 \cos\psi\right)$$

$$= -M_0 \sin\psi - \frac{M_1}{2} + \frac{M_1}{2} \cos 2\psi - \frac{M_2}{2} \sin 2\psi$$

The effect for a two bladed rotor is obtained by adding the loads produced by a blade at azimuth angle $\psi + 180°$ The moments are then:

$$M_P = -M_1 \sin 2\psi - M_2 - M_2 \cos 2\psi \quad ; \quad M_R = -M_2 \sin 2\psi - M_1 + M_1 \cos 2\psi$$

for a single 2 bladed rotor

The constant parts are balanced in steady flight. The other parts form a second harmonic forcing function for ship vibration.

For a coaxial helicopter using such rotors

$$M_P = \left(-M_1 \sin 2\psi - M_2 - M_2 \cos 2\psi\right) + \left(-M_1 \sin(-2\psi) - M_2 - M_2 \cos(-2\psi)\right)$$

$$M_P = -2M_2 - 2M_2 \cos 2\psi$$

$$M_R = \left(-M_2 \sin 2\psi - M_1 + M_1 \cos 2\psi\right) + \left(-M_2 \sin(-2\psi) - M_1 + M_1 \cos(-2\psi)\right)$$

$$M_R = -2M_1 + 2M_1 \cos 2\psi$$

This assumes that the rotors are geared so as to overlap at the azimuth positions $\psi = 0$ AND $90°$

<u>IN THE CASE WHERE THE BLADES OVERLAP AT $\psi = 45°$ AND $135°$</u>

$$M_P = -2M_2 - 2M_1 \sin 2\psi$$

$$M_R = -2M_1 - 2M_2 \sin 2\psi$$

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