# AN INVESTIGATION OF THE ROUND JET IN A MOVING AIR STREAM

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SUMMARY

A theoretical investigation is made of a circular jet issuing from an orifice into an air stream flowing parallel to the jet axis and in the same direction as the jet. The analysis is divided into two portions:

1. The region immediately downstream of the orifice in which exists the potential cone of uniform velocity.
2. The region downstream of the end of the potential cone in which the jet is fully developed.

The solution for these two portions are fitted together at the section at which the potential cone disappears giving a complete solution for the whole jet.

Numerical computations are made for several ratios of free stream velocity to jet velocity to determine the spread of the jet, velocity on the axis of the jet, and spread of the surface on which the velocity is one half the sum of the jet velocity and free stream velocity.

Squire and Trouncer's[1] results are plotted as a comparison with those obtained in this analysis.

# I. INTRODUCTION

The purpose of this analysis is to provide a comparison of the characteristics of a round turbulent jet discharging into a moving air stream with the known characteristics of a jet discharging into air at rest. Both jet and free stream are assumed to have the same density and the flow is assumed to be incompressible.

At the present time there exists no satisfactory theories for turbulent mixing. However, the integrated forms of the equations of motion, along with some dimensional reasoning, can be used to give some useful results.

The solution of the round jet in a general stream has been already carried out by Squire and Trouncer. In their solution the shear stress is based on Prandtl's momentum transfer theory[2] which is known to be in error. It therefore seems worth while to investigate this problem using the integral equations and also making use of experimental data for obtaining the shear stress in the mixing regions.

Use of the integral form of the equations was first proposed by von Karman[3]; one of the earliest applications was by Polhausen[4], in investigating laminar boundary layers in a pressure gradient. Sutton[5] originally used the integrated form of the mechanical energy equation in laminar boundary layer investigations. Liepmann and Laufer[6] have also applied this method to some simple two dimensional turbulent flows.

In the integrated form of the equations of motion the only assumption necessary is that for the velocity profile across the mixing region. This can be made quite accurately and for a closer comparison to the work of Squire and Trouncer the same velocity profile that they use will be assumed in this solution.

In transforming the momentum equation to the integrated form the term containing the shear drops out. It is therefore necessary to introduce the mechanical energy equation. This equation, while not independent of the momentum equation, is necessary in order to introduce a term containing the shear stress.

The following notation will be used.

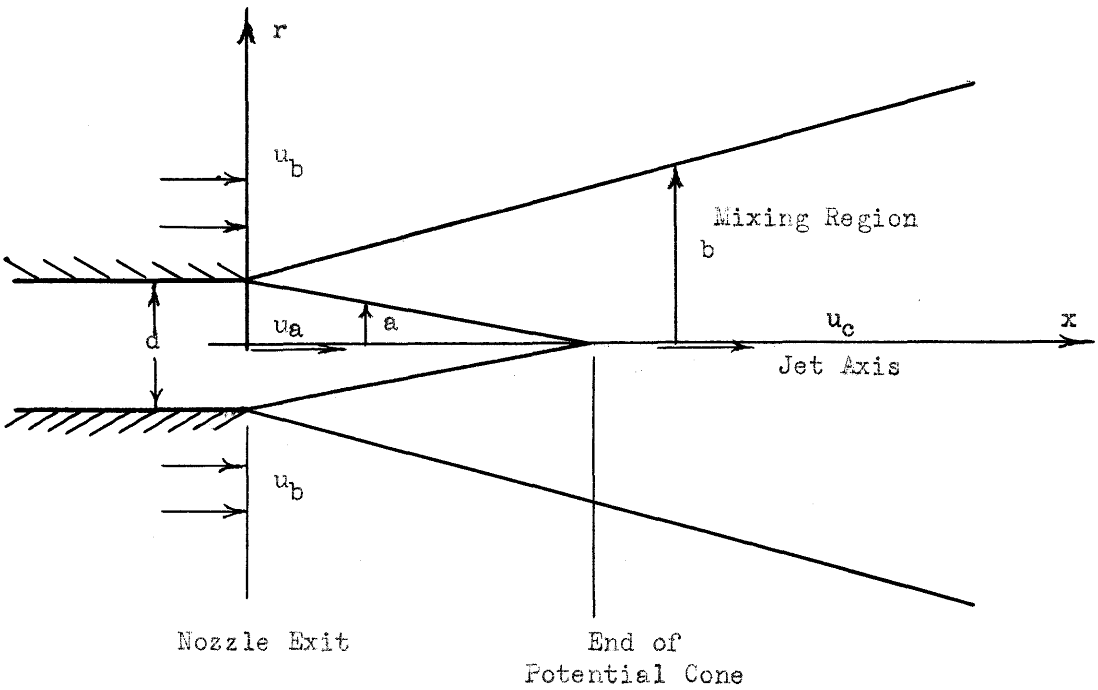


Fig. 1

| | |
|---|---|
| $x$ | Distance from nozzle exit along jet axis |
| $r$ | Distance from jet axis |
| $x_c$ | Distance from nozzle exit to end of the potential cone |
| $a$ | Radius of the potential cone at any point |
| $b$ | Radius of the outer jet boundary at any point |
| $b$ | Diameter of nozzle exit |
| $u$ | Velocity parallel to jet axis at any point |
| $u_a$ | Jet velocity at nozzle exit |
| $u_b$ | Free stream velocity |
| $u_c$ | Velocity on the jet axis at $x = x_c$ |
| $v_b$ | Velocity of the stream normal to the jet axis |
| $\rho$ | Stream density |
| $\tau$ | Shear stress |

# III. ANALYSIS

## A. Region between the exit and the end of the potential cone.

The three fundamental equations are

Continuity

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r} = 0$$

Momentum

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau)$$

Mechanical energy

$$\rho u^2 \frac{\partial u}{\partial x} + \rho u v \frac{\partial u}{\partial r} = \frac{u}{r}\frac{\partial}{\partial r}(r\tau)$$

These three equations are integrated between the boundaries of the mixing region and are then

Continuity

$$\frac{d}{dx}\int_a^b u r\,dr + v_b b + u_a a \frac{da}{dx} - u_b b \frac{db}{dx} = 0$$

Momentum

$$\frac{d}{dx}\int_a^b u^2 r\,dr + u_b v_b b + u_a^2 a \frac{da}{dx} - u_b^2 b \frac{db}{dx} = 0$$

Mechanical energy

$$\frac{1}{2}\frac{d}{dx}\int_a^b u^3 r\,dr + \frac{1}{2}u_b^2 v_b b + \frac{1}{2}u_a^3 a \frac{da}{dx} - \frac{1}{2}u_b^3 b \frac{db}{dx} = -\frac{1}{\rho}\int_a^b \tau r \frac{\partial u}{\partial r}\,dr$$

A function is assumed for the velocity profile across the jet such that

$$u = u_b + \frac{u_a - u_b}{2}\left[1 - \cos \pi \eta\right]$$

or

$$u = u_b + \frac{u_a - u_b}{2} f(\eta)$$

where
$$\eta = \frac{b-r}{b-a}$$

$$r = b - (b-a)\eta$$
$$dr = -(b-a)d\eta$$

Substituting into the continuity equation

$$\frac{d}{dx}\int_1^0 \left[u_b + \frac{u_a-u_b}{2} f(\eta)\right]\left[b-(b-a)d\eta\right]\left[-(b-a)d\eta\right]$$
$$+v_b b + u_a a \frac{da}{dx} - u_b b \frac{db}{dx} = 0$$

Expanding

$$\frac{d}{dx}\left[u_b(ab-b^2)\int_1^0 d\eta + u_b(a-b)^2\int_1^0 \eta\, d\eta + \frac{u_a-u_b}{2}(ab-b^2)\int_1^0 f(\eta)d\eta\right.$$
$$\left. + \frac{u_a-u_b}{2}(a-b)^2\int_1^0 \eta f(\eta)d\eta\right] + v_b b + u_a a\frac{da}{dx} - u_b b\frac{db}{dx} = 0$$

Integrating

$$\frac{d}{dx}\left[-u_b(ab-b^2) - \frac{u_b}{2}(a-b)^2 + \frac{u_a-u_b}{2}(ab-b^2)I_1\right.$$
$$\left. + \frac{u_a-u_b}{2}(a-b)^2 I_2\right] + v_b b + u_a a\frac{da}{dx} - u_b b\frac{db}{dx} = 0$$

where
$$I_1 = \int_1^0 f(\eta)d\eta$$
$$I_2 = \int_1^0 \eta f(\eta)d\eta$$

Differentiating and collecting terms

$$\left[Aa + Bb\right]\frac{da}{dx} + \left[Ba + Cb\right]\frac{db}{dx} + v_b b = 0 \qquad (1)$$

where

$$A = (u_a - u_b)(I_2 + 1)$$

$$B = (u_a - u_b)\left(\frac{I_1}{2} - I_2\right)$$

$$C = (u_a - u_b)(I_2 - I_1)$$

Substituting into the momentum equation

$$\frac{d}{dx}\int_1^0 \left[u_b + \frac{u_a - u_b}{2} f(\eta)\right]^2 \left[(ab - b^2) + (a-b)^2\eta\right] d\eta$$

$$+ u_b v_b b + u_a^2 a\frac{da}{dx} - u_b^2 b\frac{db}{dx} = 0$$

Expanding and integrating

$$\frac{d}{dx}\Big[-u_b^2(ab - b^2) - \frac{u_b^2}{2}(a-b) + u_b(u_a - u_b)(ab - b^2)I_1$$

$$+ u_b(u_a - u_b)(a-b)^2 I_2 + \frac{(u_a - u_b)^2}{4}(ab - b^2)I_3$$

$$+ \frac{(u_a - u_b)^2}{4}(a-b)^2 I_4\Big] + u_b v_b b + u_a^2 a\frac{da}{dx} - u_b^2 b\frac{db}{dx} = 0$$

where

$$I_3 = \int_1^0 f^2(\eta)\,d\eta$$

$$I_4 = \int_1^0 \eta f^2(\eta)\,d\eta$$

Differentiating and collecting terms

$$\left[Da + Eb\right]\frac{da}{dx} + \left[Ea + Fb\right]\frac{db}{dx} + u_b v_b b = 0 \qquad (2)$$

where

$$D = u_a^2 - u_b^2 + \tfrac{1}{2} I_4 (u_a - u_b)^2 + 2 I_2 u_b (u_a - u_b)$$

$$E = (\tfrac{1}{4} I_3 - \tfrac{1}{2} I_4)(u_a - u_b)^2 + (I_1 - 2I_2) u_b (u_a - u_b)$$

$$F = (\tfrac{1}{2} I_4 - \tfrac{1}{2} I_3)(u_a - u_b)^2 + (2I_2 - 2I_1) u_b (u_a - u_b)$$

The shearing stress now appears in the mechanical energy equation and an assumption will be made for the shear profile. The shear stress is a variable and could be determined from the velocity profile . However, a shear profile derived from an already assumed velocity profile would have a greater degree of error due to small errors in the assumed velocity profile. For simplicity an average shear profile will be assumed for the whole region.

From experiments with a jet discharging into still air shear profiles have been found as shown in Fig. 2. At the jet exit the shear profile is symmetrical. Past the end of the potential cone the maximum shear occurs at about $\eta$=.667 to .75 and the shear profile has a steep slope on the high velocity side. An average shear profile for the potential cone region is shown in Fig. 2 and is given by

$$\tau = \rho \overline{u'v'}$$

which is assumed to be of the form

$$\tau = \rho (u_a - u_b)^2 g(\eta)$$

where

$$g(\eta) = \frac{\overline{u'v'}}{(u_a - u_b)^2}$$

In particular it will be assumed here that the function $g(\eta)$ can be represented by

$$g(\eta) = k\,\eta^3 \cos^2 \pi (\eta - \tfrac{1}{2})$$

Substituting into the mechanical energy equation

$$\frac{1}{2}\frac{d}{dx}\int_1^0 \left[u_b + \frac{u_a - u_b}{2} f(\eta)\right]^3 \left[(ab - b^2) + (a-b)^2\eta\right] d\eta$$

$$+ \frac{1}{2} u_b^2 v_b b + \frac{1}{2} u_a^3 a \frac{da}{dx} - \frac{1}{2} u_b^3 b \frac{db}{dx}$$

$$= -\frac{1}{\rho}\int_1^0 \rho (u_a - u_b)^2 g(\eta)\left[b - (b-a)\eta\right]\left[-(b-a)d(\eta)\right]\left[\frac{u_a - u_b}{2} f'(\eta)\right]$$

Expanding and integrating

$$\frac{d}{dx}\Big\{ \Big[-u_b^3 + \frac{3}{2} u_b^2 (u_a - u_b) I_1 + \frac{3}{4} u_b (u_a - u_b)^2 I_3$$

$$+ \frac{1}{8}(u_a - u_b)^3 I_5\Big]\left[ab - b^2\right] + \Big[-\frac{1}{2} u_b^3 + \frac{3}{2} u_b^2 (u_a - u_b) I_2$$

$$+ \frac{3}{4} u_b (u_a - u_b)^2 I_4 - \frac{1}{8}(u_a - u_b)^3 I_6\Big]\left[a - b\right]^2 \Big\}$$

$$+ u_b^2 v_b b + u_a^3 a \frac{da}{dx} - u_b^3 b \frac{db}{dx}$$

$$= \pi (u_a - u_b)^3 (I_7 - I_8) b + \pi (u_a - u_b)^3 I_8 a$$

where

$$I_5 = \int_1^0 f^3(\eta)\,d\eta$$
$$I_6 = \int_1^0 \eta f^3(\eta)\,d\eta$$
$$I_7 = \int_1^0 f'(\eta)\,g(\eta)\,d\eta$$
$$I_8 = \int_1^0 \eta f'(\eta)\,g(\eta)\,d\eta$$

Differentiating and collecting terms

$$\left[Ga+Hb\right]\frac{da}{dx} + \left[Ha+Jb\right]\frac{db}{dx} + u_b^2 v_b\, b = Ka + Lb \qquad (3)$$

where

$$G = u_a^3 - u_b^3 + 3u_b^2(u_a-u_b)I_2 + \tfrac{3}{2}u_b(u_a-u_b)^2 I_4 + \tfrac{1}{4}(u_a-u_b)^3 I_6$$
$$H = u_b^2(u_a-u_b)\left(\tfrac{3}{2}I_1 - 3I_2\right) + u_b(u_a-u_b)^2\left(\tfrac{3}{4}I_3 - \tfrac{3}{2}I_4\right) + (u_a-u_b)^3\left(\tfrac{1}{8}I_5 - \tfrac{1}{4}I_6\right)$$
$$J = u_b^2(u_a-u_b)(3I_2 - 3I_1) + u_b(u_a-u_b)^2\left(\tfrac{3}{2}I_4 - \tfrac{3}{2}I_3\right) + (u_a-u_b)^3\left(\tfrac{1}{4}I_6 - \tfrac{1}{4}I_5\right)$$
$$K = \pi(u_a-u_b)^3 I_8$$
$$L = \pi(u_a-u_b)^3(I_7 - I_8)$$

Eliminating $v_b$ in equation (1) and equation (3)

$$\left[Ga+Hb\right]\frac{da}{dx} + \left[Ha+Jb\right]\frac{db}{dx} - u_b^2\left[Aa+Bb\right]\frac{da}{dx} - u_b^2\left[Ba+Cb\right]\frac{db}{dx} = Ka + Lb$$

$$\left[G-u_b^2A\right]\frac{da}{dx}+\left[H-u_b^2B\right]\left[b\frac{da}{dx}+a\frac{db}{dx}\right]+\left[J-u_b^2C\right]\frac{db}{dx}=Ka+Lb$$

$$\frac{d}{dx}\left[\alpha a^2+2\beta ab+\gamma b^2\right]=Ka+Lb$$

where

$$\alpha=\frac{G-u_b^2A}{2}\ ,\ \beta=\frac{H-u_b^2B}{2}\ ,\ \gamma=\frac{J-u_b^2C}{2}$$

Using the chain rule of differentiating, $\frac{d(\ )}{dx}=\frac{d(\ )}{db}\cdot\frac{db}{dx}$

the above equation becomes

$$\left[2\alpha a\frac{da}{db}+2\beta a+2\beta b\frac{da}{db}+2\gamma b\right]\frac{db}{dx}=Ka+Lb$$

$$\left[Ka+Lb\right]\frac{dx}{db}=(2\alpha a+2\beta b)\frac{da}{dx}+2\beta a+2\gamma b \qquad (4)$$

Eliminating $v_b$ in equations (1) and (2)

$$\left[Da+Eb\right]\frac{da}{dx}+\left[Ea+Fb\right]\frac{db}{dx}-u_b\left[Aa+Bb\right]\frac{da}{dx}$$
$$-u_b\left[Ba+Cb\right]\frac{db}{dx}=0$$

$$\left[D-u_bA\right]a\frac{da}{dx}+\left[E-u_bB\right]\left[a\frac{db}{dx}+b\frac{da}{dx}\right]+\left[F-u_bC\right]b\frac{db}{dx}=0$$

Integrating

$$\delta a^2+2\epsilon ab+\phi b^2+Z=0 \qquad (5)$$

where

$$\delta=\frac{D-u_bA}{2}\ ,\ \epsilon=\frac{E-u_bB}{2}\ ,\ \phi=\frac{F-u_bC}{2}$$

$Z$ = constant of integration

Differentiating with respect to b

$$2\delta a \frac{da}{db} + 2\epsilon a + 2\epsilon b \frac{da}{db} + 2\phi b = 0$$

$$\frac{da}{db} = -\frac{\epsilon a + \phi b}{\epsilon b + \delta a} \tag{6}$$

Substituting equation (6) into equation (4)

$$[Ka + Lb]\frac{dx}{db} = (2\alpha a + 2\beta b)\left(-\frac{\epsilon a + \phi b}{\epsilon b + \delta a}\right) + 2\beta a + 2\gamma b$$

$$\frac{1}{2}\frac{dx}{db} = \frac{(\delta\beta - \alpha\epsilon)a^2 + (\gamma\delta - \alpha\phi)ab + (\gamma\epsilon - \phi\beta)b^2}{\epsilon b + \delta a}$$

$$\frac{1}{2}\frac{dx}{db} = \lambda_1 + \frac{\lambda_2 b}{\delta a + \epsilon b} + \frac{\lambda_3 b}{Ka + Lb}$$

where

$$\lambda_1 = \frac{\delta\beta - \alpha\epsilon}{\delta K}$$

$$\lambda_2 = \frac{(\phi\delta - \epsilon^2)(\delta\beta - \alpha\epsilon)}{\delta(\epsilon K - \delta L)}$$

$$\lambda_3 = \frac{K\gamma(L\delta - K\epsilon) + K\phi(K\beta - L\alpha) + L^2(\alpha\epsilon - \delta\beta)}{K(L\delta - K\epsilon)}$$

From equation (5)

$$\delta a + \epsilon b = \sqrt{-Z\delta + b^2(\epsilon^2 - \phi\delta)}$$

$$\frac{1}{2}\frac{dx}{db} = \lambda_1 + \frac{\lambda_2 b}{\sqrt{-Z\delta + b^2(\epsilon^2 - \phi\delta)}} + \frac{\lambda_3 b}{Ka + Lb}$$

Integrating

$$\frac{x}{2} = \lambda_1 b - \frac{\lambda_2\sqrt{-Z\delta + b^2(\epsilon^2 - \phi\delta)}}{\phi\delta - \epsilon^2} + \lambda_3\int\frac{b\,db}{Ka + Lb}$$

This expression can be integrated and gives

$$\frac{x}{2} = \lambda_1 b - \lambda_2\frac{\sqrt{-Z\delta + b^2(\epsilon^2 - \phi\delta)}}{\phi\delta - \epsilon^2}$$

$$+\frac{\lambda_3 C_0}{2(C_2+1)}\left[2\left(b - \sqrt{C_1^2 - C_2 b^2}\right) + \frac{C_1}{\sqrt{C_2+1}}\log\frac{b\sqrt{C_2+1} - C_1}{b\sqrt{C_2+1} + C_1}\cdot\frac{\sqrt{(C_2+1)(C_1^2 - C_2 b^2)} + C_1}{\sqrt{(C_2+1)(C_1^2 - C_2 b^2)} - C_1}\right] + Y$$

where

$$C_0 = \frac{\delta}{L\delta - K\epsilon}$$

$$C_1^2 = -\frac{K^2 Z\delta}{(L\delta - K\epsilon)^2}$$

$$C_2 = \frac{K^2(\phi\delta - \epsilon^2)}{(L\delta - K\epsilon)^2}$$

$Y$ = constant of integration

$Y$ can be found by the boundary condition that $x = 0$ where $b = d/2$. Obtaining the parameters $x$ and $b$ in dimensionless form the result becomes

$$\frac{x}{d/2} = 2\lambda_1\left(\frac{b}{d/2}-1\right) - \frac{2\lambda_2\sqrt{-\frac{4z\delta}{d^2}-\left(\frac{b}{d/2}\right)^2(\phi\delta-\epsilon^2)}}{\phi\delta-\epsilon^2} + \frac{2\lambda_2\sqrt{-\frac{4z\delta}{d^2}-(\phi\delta-\epsilon^2)}}{\phi\delta-\epsilon^2}$$
$$+\frac{c_0\lambda_3}{c_2+1}\left[\frac{b}{d/2}-\sqrt{\frac{4c_1^2}{d^2}-c_2\left(\frac{b}{d/2}\right)^2}+\frac{c_1}{d\sqrt{c_2+1}}\log\frac{\frac{b}{d/2}\sqrt{c_2+1}-\frac{2c_1}{d}}{\frac{b}{d/2}\sqrt{c_2+1}+\frac{2c_1}{d}}\cdot\frac{\sqrt{(c_2+1)\left[\left(\frac{2c_1}{d}\right)^2-c_2\left(\frac{b}{d/2}\right)^2\right]}+2\frac{c_1}{d}}{\sqrt{(c_2+1)\left[\left(\frac{2c_1}{d}\right)^2-c_2\left(\frac{b}{d/2}\right)^2\right]}-\frac{2c_1}{d}}\right]$$
$$-\frac{\lambda_3 c_0}{c_2+1}\left[1-\sqrt{\frac{4c_1^2}{d^2}-c_2}+\frac{c_1}{d\sqrt{c_2+1}}\log\frac{\sqrt{c_2+1}-\frac{2c_1}{d}}{\sqrt{c_2+1}+\frac{2c_1}{d}}\cdot\frac{\sqrt{(c_2+1)\left[\left(\frac{2c_1}{d}\right)^2-c_2\right]}+\frac{2c_1}{d}}{\sqrt{(c_2+1)\left[\left(\frac{2c_1}{d}\right)^2-c_2\right]}-\frac{2c_1}{d}}\right] \qquad (7)$$

The constant of integration $z$ is obtained from the boundary condition that $a = b = d/2$ at $x = 0$ so

$$z = -\frac{d^2}{8}u_a(u_a-u_b)$$

The only remaining constant to be determined is $k$ which appears in the shear term. From experimental data obtained by Liepmann and Laufer[6] it is found that near the jet exit $g(\eta)_{MAX.} = 0.0072$. From experimental data obtained by Corrsin[7] it is found that in the fully developed region $g(\eta)_{MAX.} = 0.022$. Both of these results are for the case of a jet discharging into still air or $\frac{u_b}{u_a} = 0$.

Differentiating our assumed function it is found that $g(\eta)_{MAX.}$ occurs at the point where $\eta = 0.692$ and $g(\eta)_{MAX.} = 0.224k$ The constant $k$ must be given some average value such that $g(\eta)_{MAX.}$ for the average shear profile will be between 0.0072 and 0.224.

From experimental data[8] it is known that the end of the potential cone is at a point about 9 radii downstream of the exit of the jet for the limiting case where $\frac{u_b}{u_a} = 0$. In order for $\frac{x}{d/2} = 9$ when $\frac{u_b}{u_a} = 0$ and $a = 0$

$$g(\eta)_{MAX.} = 0.01767$$

$$\text{and} \quad k = 0.07388$$

B. Region downstream of the end of the potential cone.

At the point downstream of the jet exit where $x = x_c$ the potential cone disappears and the variable, a, no longer appears. Downstream of this point the velocity on the jet axis, $u_c$, will become a variable decreasing with x. The assumption for the velocity profile in the region of the potential cone was

$$u = u_b + \frac{u_a - u_b}{2}\left[1 - \cos\pi\left(\frac{b-r}{b-a}\right)\right]$$

At $x = x_c$, $a = 0$ and $u_c = u_a$

$$u = u_b + \frac{u_c - u_b}{2}\left[1 - \cos\pi\left(\frac{b-r}{b}\right)\right]$$

This reduces to

$$u = u_b + \frac{u_c - u_b}{2}\left[1 + \cos\pi\frac{r}{b}\right]$$

or

$$u = u_b + \frac{u_c - u_b}{2} f(\mu)$$

where

$$f(\mu) = 1 + \cos\frac{\pi r}{b}$$

$$r = b\mu$$

$$dr = b\,d\mu$$

The continuity equation in this region then becomes

$$\frac{d}{dx}\left[u_b b^2\int_0^1 \mu\, d\mu + \frac{u_c-u_b}{2} b^2\int_0^1 f(\mu)\, d\mu\right] + v_b b - u_b b \frac{db}{dx} = 0$$

This can be reduced to

$$I_9 V \frac{db}{dx} + \frac{1}{2} I_9 b \frac{du_c}{dx} + v_b = 0 \qquad (8)$$

where

$$I_9 = \int_0^1 \mu f(\mu)\, d\mu$$

$$V = u_c - u_b$$

Substituting in the momentum equation

$$\frac{d}{dx}\left[u_b^2 b\int_0^1 \mu\, d\mu + u_b(u_c-u_b) b^2\int_0^1 \mu f(\mu)\, d\mu\right.$$

$$\left. + \frac{(u_c-u_b)^2}{4} b^2\int_0^1 \mu f^2(\mu) d\mu\right] + u_b v_b b - u_b^2 b \frac{db}{dx} = 0$$

Which reduces to

$$\left[4 I_9 u_b V + I_{10} V^2\right]\frac{db}{dx} + \left[2 I_9 u_b + I_{10} V\right] b \frac{du_c}{dx} + 2 u_b v_b = 0 \qquad (9)$$

where

$$I_{10} = \int_0^1 \mu f^2(\mu)\, d\mu$$

In the mechanical energy equation the shearing stress is given by

$$\tau = \rho\,(u_c - u_b)^2\, g(\mu)$$

The function $g(\mu)$ must be such that the shear profile will resemble that illustrated in Fig. 2. This will be

$$g(\mu) = l\,\frac{f(\mu)}{\mu}\int_0^{\mu} \mu\, f(\mu)\, d\mu$$

$$l = \text{constant}$$

Substituting into the mechanical energy equation

$$\frac{d}{dx}\Big[u_b^3 b^2 \int_0^1 \mu\, d\mu + \tfrac{3}{2} u_b^2 (u_c - u_b)\, b^2 \int_0^1 \mu\, f(\mu)\, d\mu$$

$$+ \tfrac{3}{4} u_b (u_c - u_b)^2 b^2 \int_0^1 \mu f^2(\mu)\, d\mu + \tfrac{1}{8}(u_c - u_b)^3 b^2 \int_0^1 \mu f^3(\mu)\, d\mu\Big]$$

$$+ u_b^2 v_b\, b - u_b^3 b\,\frac{db}{dx} = -b\,(u_c - u_b)^3 \int_0^1 \mu\, g(\mu)\, f'(\mu)\, d\mu$$

which reduces to

$$\Big[3 I_9 u_b^2 V + \tfrac{3}{2} I_{10} u_b V^2 + \tfrac{1}{4} I_{11} V^3\Big]\frac{db}{dx}$$

$$+\Big[\tfrac{3}{2} I_9 u_b^2 b + \tfrac{3}{2} I_{10} u_b b V + \tfrac{3}{8} I_{11} b V^2\Big]\frac{du_c}{dx} + u_b^2 v_b = -I_{12} V^3 \qquad (10)$$

where

$$I_{11} = \int_0^1 \mu f^3(\mu)\, d\mu$$

$$I_{12} = \int_0^1 \mu\, g(\mu)\, f'(\mu)\, d\mu$$

Eliminating $v_b$ in equation (8) and equation (9)

$$\left[4 I_9 u_b V + I_{10} V^2\right] \frac{db}{dx} + \left[2 I_9 u_b + I_{10} V\right] b \frac{dV}{dx}$$

$$+ 2 u_b \left[- I_9 V \frac{db}{dx} - \frac{1}{2} I_9 b \frac{dV}{dx}\right] = 0$$

$$\frac{db}{b} = - \frac{I_9 u_b + I_{10} V}{2 I_9 u_b + I_{10} V} \cdot \frac{dV}{V}$$

Integrating

$$\log b = -\frac{1}{2} \log\left[2 I_9 u_b + I_{10} V\right] - \frac{1}{2} \log V + \log C_1$$

$$b = \frac{C_1}{\sqrt{V(2 I_9 u_b + I_{10} V)}} \tag{11}$$

$$\frac{db}{dx} = - \frac{b}{V} \cdot \frac{I_9 u_b + I_{10} V}{2 I_9 u_b + I_{10} V} \cdot \frac{dV}{V}$$

$$\frac{db}{dx} = - \frac{C_1 (I_9 u_b + I_{10} V)}{V^{3/2} (2 I_9 u_b + I_{10} V)^{3/2}} \frac{dV}{dx}$$

Eliminating $v_b$ in equations (8) and (10) and substituting for $\frac{db}{dx}$ above we get a single equation which is

$$\frac{I_9 I_{10} u_b^2 + I_9 I_{11} u_b V + \frac{1}{4} I_{10} I_{11} V^2}{(2 I_9 u_b + I_{10} V)^{3/2}} \frac{dV}{dx} = - \frac{2 I_{12}}{C_1} V^{5/2}$$

This may be integrated to give

$$- \frac{8 I_{12} I_9^3}{C_1 I_{11} I_{10}^{5/2}} u_b x = - \frac{I_9}{3 I_{11}} \left[ \frac{2 \frac{I_9}{I_{10}} u_b + V}{V} \right]^{3/2}$$

$$- 2 \left[ \left( \frac{I_9}{I_{10}} \right)^2 - \frac{I_9}{I_{11}} \right] \left[ \frac{2 \frac{I_9}{I_{10}} u_b + V}{V} \right]^{1/2}$$

$$+ \left[ \frac{I_9}{I_{11}} - \left( \frac{I_9}{I_{10}} \right)^2 \right] \left[ \frac{V}{2 \frac{I_9}{I_{10}} u_b + V} \right]^{1/2} + C_2 \qquad (12)$$

The constant of integration $C_2$ may be found by the boundary condition that at $x = x_c$, $V = u_a - u_b$.

$$C_2 = \frac{1}{3} \frac{I_9}{I_{11}} q_b^{3/2} + 2 \left[ \left( \frac{I_9}{I_{10}} \right)^2 - \frac{I_9}{I_{11}} \right] q_b^{1/2}$$

$$- \left[ \frac{I_9}{I_{11}} - \left( \frac{I_9}{I_{10}} \right)^2 \right] q_b^{-\frac{1}{2}} - \frac{8 I_{12} I_9^3}{C I_{11} I_{10}^{5/2}} u_b x_c$$

where

$$q_b = \frac{u_a - \left(1 - 2 \frac{I_9}{I_{10}}\right) u_b}{u_a - u_b}$$

The constant of integration $C_1$ may be evaluated from these same boundary conditions and from equation (5) when $a = 0$

Substituting these values of $C_1$ and $C_2$ the final solution is obtained.

$$\frac{x-x_c}{d/2} = \left\{ \frac{\left[\left(\frac{u_a}{u_b}\right)^2 - \frac{u_a}{u_b}\right]\left[2 I_9 \frac{u_b}{u_a} + I_{10}\left(1-\frac{u_b}{u_a}\right)\right]}{\left[I_9 - I_{10}\right]\frac{u_b}{u_a} + \frac{1}{2}\left[I_{11}-I_{12}\right]\left[1-\frac{u_b}{u_a}\right]} \right\}^{1/2}$$

$$\times \left\{ \left[\frac{I_{10}^{5/2}}{24 I_9^2 I_{12}}\right]\left[q_a^{3/2} - q_b^{3/2}\right] + \left[\frac{I_9 I_{10}^{1/2} I_{11} - I_{10}^{5/2}}{8 I_9^2 I_{12}}\right] \right.$$

$$\left. \times \left[2\left(q_a^{1/2} - q_b^{1/2}\right) + \left(q_a^{-1/2} - q_b^{-1/2}\right)\right] \right\} \qquad (13)$$

where

$$q_a = \frac{2 \frac{I_9}{I_{10}} u_b + V}{V}$$

Equation (11) becomes

$$\left(\frac{b}{d/2}\right)^2 = \frac{-\left[\left(\frac{u_a}{u_b}\right)^2 - \frac{u_a}{u_b}\right]\left[I_{10}\frac{u_a}{u_b} + (2 I_9 - I_{10})\right]}{\left[\frac{u_c}{u_b} - 1\right]\left[2 I_9 + I_{10}\left(\frac{u_c}{u_b} - 1\right)\right]\left[\left(\frac{I_{12}-I_{11}}{2}\right)\frac{u_a}{u_b} + I_{10} - I_9 - \frac{I_4}{2} + \frac{I_3}{2}\right]} \qquad (14)$$

$x_c$ may be found by solving for $b$ when $a = 0$ in equation (5). Using this value of $b$ find the corresponding value of $x$ in equation (7) which will be $x_c$.

Equation (13) gives $x$ as a function of $(u_c - u_b)$. Equation (14) gives $b$ as a function of $(u_c - u_b)$. It does not seem necessary to get an explicit expression for $x$ and $b$. By assuming values of $(u_c - u_b)$ corresponding values of $x$ and $b$ are found and $b$ is plotted against $x$ in Fig. 3.

Now the constant $l$ appearing in the shear stress expression must be evaluated. In the fully developed region downstream of the end of the potential cone $g(\mu)_{max} = 0.022$. From the assumed function of $g(\mu)$ it is found that $g(\mu)_{max} = 0.4382$ or $l = 0.05116$.

## III. DISCUSSION

The results plotted for the region containing the potential cone are the results obtained by Squire and Trouncer. In using the derived equation for x and b (equation 7) in the potential cone region the final results obtained did not appear physically reasonable, particularly at higher velocity ratios of free stream to jet. Due to the enormous amount of computation necessary to recheck these results Squire and Trouncer's results were used in this region.

The very close comparison of the two analyses in the fully developed region would seem to indicate that the integrated equation solution that has been used here should give fair agreement in the potential cone region.

The basic analysis of the potential cone region is known to be correct through equation (5). This equation gives a relation between the spread of the jet and the width of the potential cone. The remainder of the analysis within this same region or the computations may be in error. However, the method of solution is outlined for any future investigation of this problem.

## IV. REFERENCES


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APPENDIX I

Evaluation of dimensionless integrals appearing in the equations of motion.

The dimensionless integrals appearing in the equations of motion all may be expanded into integrals of forms found in standard integral tables. Following are the results of these integrations.

$$I_1 = \int_1^0 f(\eta)\,d\eta = \int_1^0 [1-\cos\pi\eta]\,d\eta = -1.00$$

$$I_2 = \int_1^0 \eta f(\eta)\,d\eta = \int_1^0 [1-\cos\pi\eta]\,\eta\,d\eta = -0.702$$

$$I_3 = \int_1^0 f^2(\eta)\,d\eta = \int_1^0 [1-\cos\pi\eta]^2\,d\eta = -1.500$$

$$I_4 = \int_1^0 \eta f^2(\eta)\,d\eta = \int_1^0 [1-\cos\pi\eta]^2\,\eta\,d\eta = -1.155$$

$$I_5 = \int_1^0 f^3(\eta)\,d\eta = \int_1^0 [1-\cos\pi\eta]^3\,d\eta = -2.500$$

$$I_6 = \int_1^0 \eta f^3(\eta)\,d\eta = \int_1^0 [1-\cos\pi\eta]^3\,\eta\,d\eta = -2.014$$

$$I_7 = \int_1^0 g(\eta) f'(\eta)\,d\eta = \int_1^0 [k\eta^3\cos^2\pi(\eta-\tfrac{1}{2})\sin\pi\eta]\,d\eta = -0.06875\times10^{-3}\,k$$

$$I_8 = \int_1^0 \eta\, g(\eta) f'(\eta)\,d\eta = \int_1^0 k\eta^4\cos^2\pi(\eta-\tfrac{1}{2})\sin\pi\eta\,d\eta = -0.04305\times10^{-3}\,k$$

$$I_9 = \int_0^1 \mu f(\mu) d\mu = \int_0^1 [1 + \cos \pi\mu] \mu d\mu = 0.298$$

$$I_{10} = \int_0^1 \mu f^2(\mu) d\mu = \int_0^1 [1 + \cos \pi\mu]^2 \mu d\mu = 0.346$$

$$I_{11} = \int_0^1 \mu f^3(\mu) d\mu = \int_0^1 [1 + \cos \pi\mu]^3 \mu d\mu = 0.485$$

$$I_{12} = \int_0^1 \mu g(\mu) f'(\mu) d\mu$$

$$= \int_0^1 \pi l \sin \pi\mu [1 + \cos \pi\mu] \left[\frac{\mu^2}{2} + \frac{1}{\pi^2}(\cos \pi\mu + \pi\mu \sin \pi\mu - 1)\right]$$

$$= -0.0398 l$$

APPENDIX II

Integrations necessary for solution of the equations.

A. Integral appearing in the equation for the region containing the potential cone.

$$I = \int \frac{b\,db}{Ka + Lb}$$

Substituting for a from equation (5)

$$I = \int \frac{b\,db}{\frac{-K\epsilon b + K\sqrt{4\epsilon^2 b^2 - 4\delta(\phi b^2 - z)}}{\delta} + Lb}$$

which reduces to

$$I = c_0 \int \frac{b\,db}{b + \sqrt{c_1^2 - c_2 b^2}}$$

where

$$C_0 = \frac{\delta}{L\delta - K\epsilon}$$

$$C_1^2 = -\frac{K^2 z \delta}{(L\delta - K\epsilon)^2}$$

$$C_2 = \frac{K^2(\phi\delta - \epsilon^2)}{(L\delta - K\epsilon)^2}$$

Rationalizing the denominator

$$I = c_0 \int \frac{\left(b^2 - b\sqrt{c_1^2 - c_2 b^2}\right) db}{b^2(c_2 + 1) - c_1^2}$$

$$I = c_0 \int \frac{b^2\,db}{b^2(c_2+1) - c_1^2} - c_0 \int \frac{\sqrt{c_1^2 - c_2 b^2}\; b\,db}{b^2(c_2+1) - c_1^2}$$

$$I' = \int \frac{b^2\,db}{b^2(c_2+1) - c_1^2}$$

$$I' = \int \frac{b^2\,db}{\left[b\sqrt{c_2+1} - c_1\right]\left[b\sqrt{c_2+1} + c_1\right]}$$

$$I' = \int \left[ \frac{1}{c_2+1} + \frac{\frac{c_1}{2(c_2+1)}}{b\sqrt{c_2+1} - c_1} - \frac{\frac{c_1}{2(c_2+1)}}{b\sqrt{c_2+1} + c_1} \right] db$$

$$I' = \int \left[ 2 + \frac{c_1}{b\sqrt{c_2+1} - c_1} - \frac{c_1}{b\sqrt{c_2+1} + c_1} \right] \frac{db}{2(c_2+1)}$$

$$I' = \frac{1}{2(c_2+1)} \left[ 2b + \frac{c_1}{\sqrt{c_2+1}} \log\left(b\sqrt{c_2+1} - c_1\right) - \frac{c_1}{\sqrt{c_2+1}} \log\left(b\sqrt{c_2+1} + c_1\right) \right]$$

$$I' = \frac{1}{2(c_2+1)} \left[ 2b + \frac{c_1}{\sqrt{c_2+1}} \log \frac{b\sqrt{c_2+1} - c_1}{b\sqrt{c_2+1} + c_1} \right]$$

$$I'' = \int \frac{\sqrt{c_1^2 - c_2 b^2}\; b\,db}{b^2(c_2+1) - c_1^2}$$

Let

$$c_1^2 - c_2 b^2 = c_2 R_0^2$$

$$-2c_2 b\,db = 2c_2 R_0\,dR_0$$

$$b\,db = -R_0\,dR_0$$

$$b^2 = \frac{c_1^2 - c_2 R_0^2}{c_2}$$

$$I'' = \int \frac{\sqrt{c_2}\; R_0 \,(-R_0\, dR_0)}{(c_2+1)\left(\frac{c_1^2}{c_2} - R_0^2\right) - c_1^2}$$

$$= \int \frac{\sqrt{c_2}\; R_0^2\, dR_0}{(c_2+1)R_0^2 - \frac{c_1^2}{c_2}}$$

$$= \frac{\sqrt{c_2}}{2(c_2+1)} \left[ 2R_0 + \frac{c_1 \log\left(R_0\sqrt{c_2+1} - \frac{c_1}{\sqrt{c_2}}\right)}{\sqrt{c_2(c_2+1)}} - \frac{c_1 \log\left(R_0\sqrt{c_2+1} + \frac{c_1}{\sqrt{c_2}}\right)}{\sqrt{c_2(c_2+1)}} \right]$$

$$= \frac{1}{2(c_2+1)} \left[ 2\sqrt{c_1^2 - c_2 b^2} + \frac{c_1}{\sqrt{c_2+1}} \log \frac{\sqrt{(c_2+1)(c_1^2 - c_2 b^2)} - c_1}{\sqrt{(c_2+1)(c_1^2 - c_2 b^2)} + c_1} \right]$$

B. Integral appearing in the equation for the region downstream of the end of the potential cone.

$$I = \int \frac{4\frac{I_9}{I_{11}}u_b^2 + 4\frac{I_9}{I_{10}}u_b V + V^2}{V^{5/2}\left(2\frac{I_9}{I_{10}}u_b + V\right)^{3/2}}\, dV$$

Let

$$t^2 = \frac{V}{2\frac{I_9}{I_{10}}u_b + V}$$

$$V = \frac{2\frac{I_9}{I_{10}}u_b t^2}{1-t^2}$$

$$dV = \frac{4\frac{I_9}{I_{10}}u_b t}{(1-t^2)^2}\, dt$$

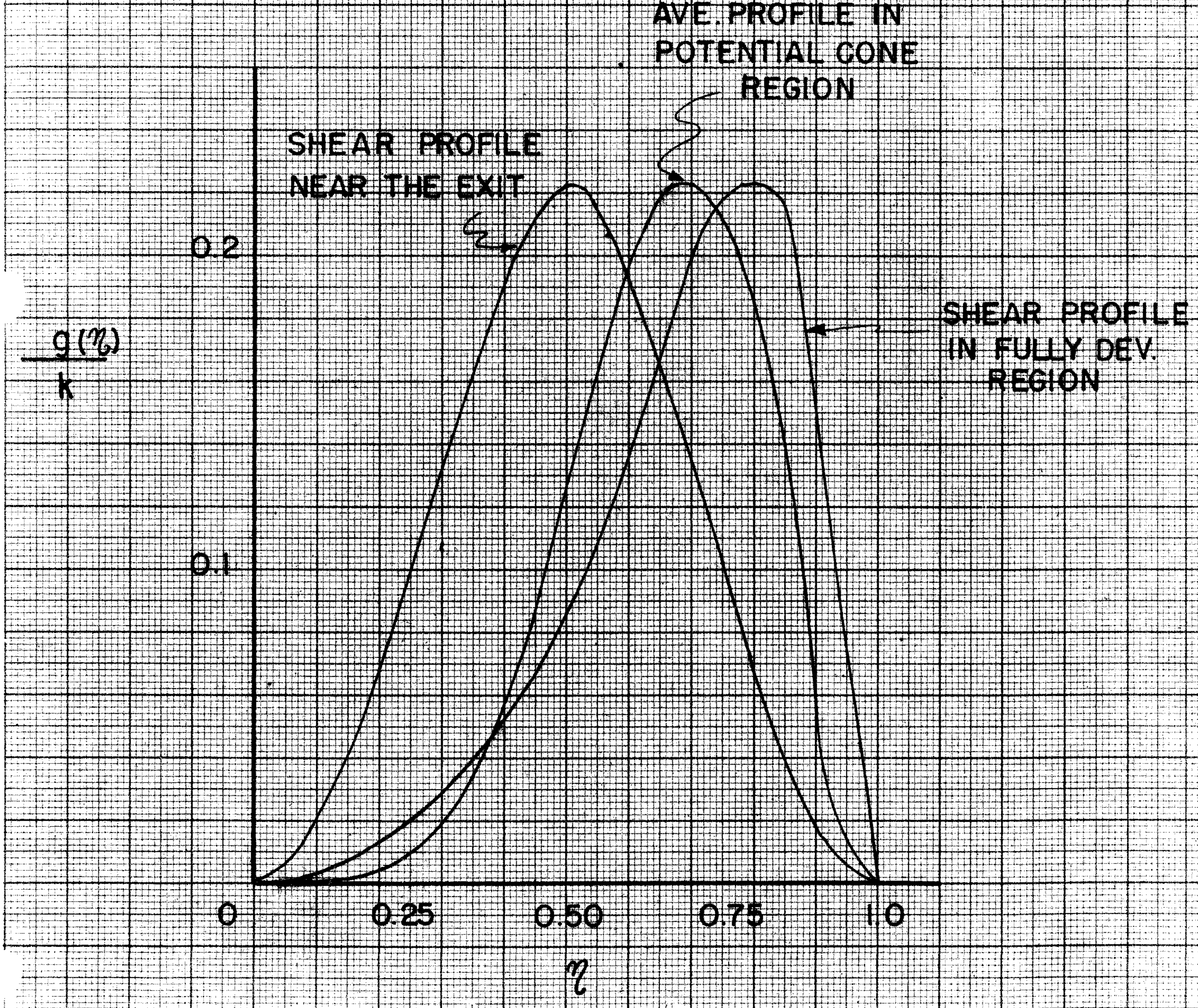
then

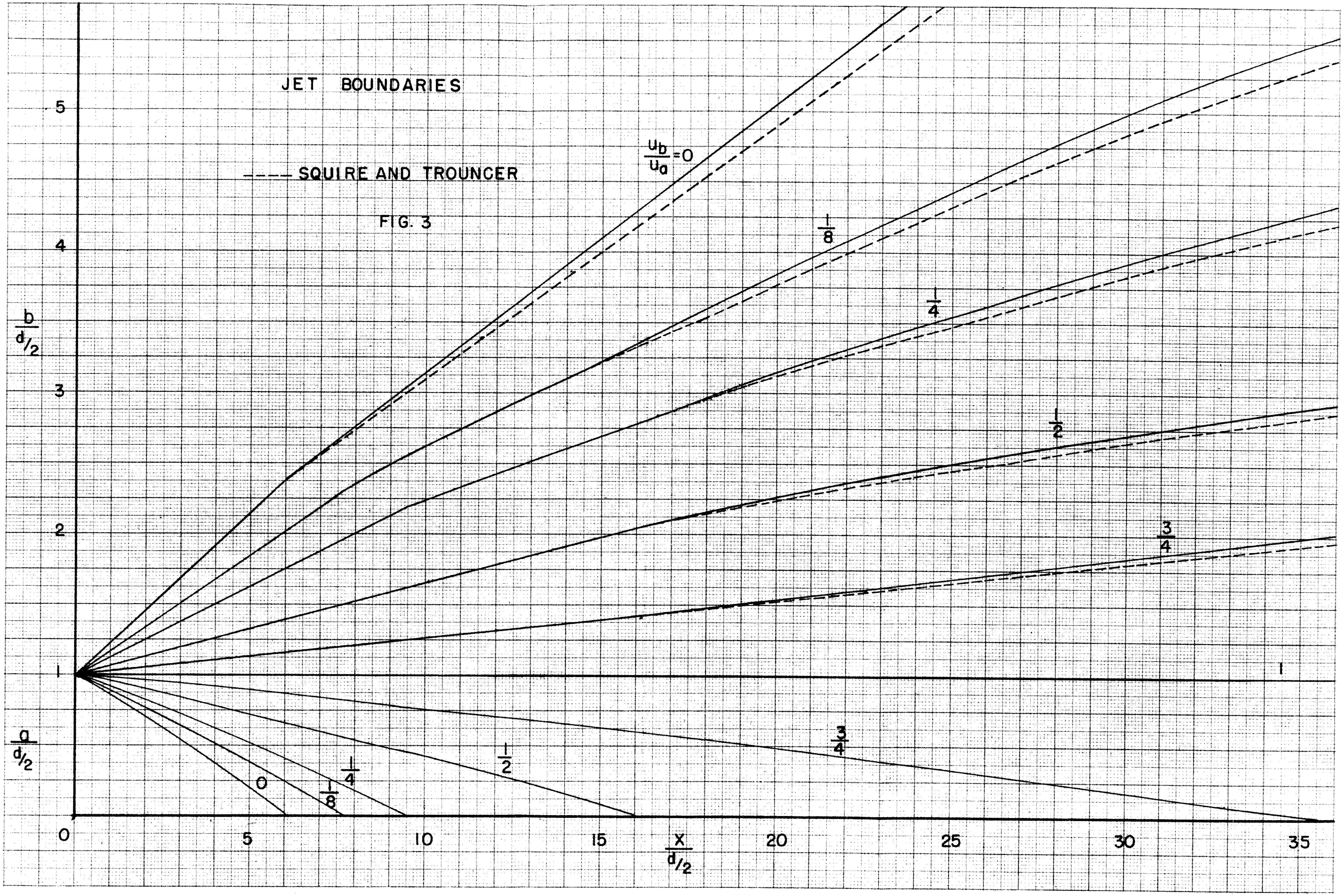
$$I = \frac{1}{\left(\frac{I_9}{I_{10}}\right)^3 u_b}\int \frac{\frac{I_9}{I_{10}} + \left[2\left(\frac{I_9}{I_{10}}\right)^2 - 2\frac{I_9}{I_{11}}\right]t^2 + \left[\frac{I_9}{I_{11}} - \left(\frac{I_9}{I_{10}}\right)^2\right]t^4}{t^4}\, dt$$

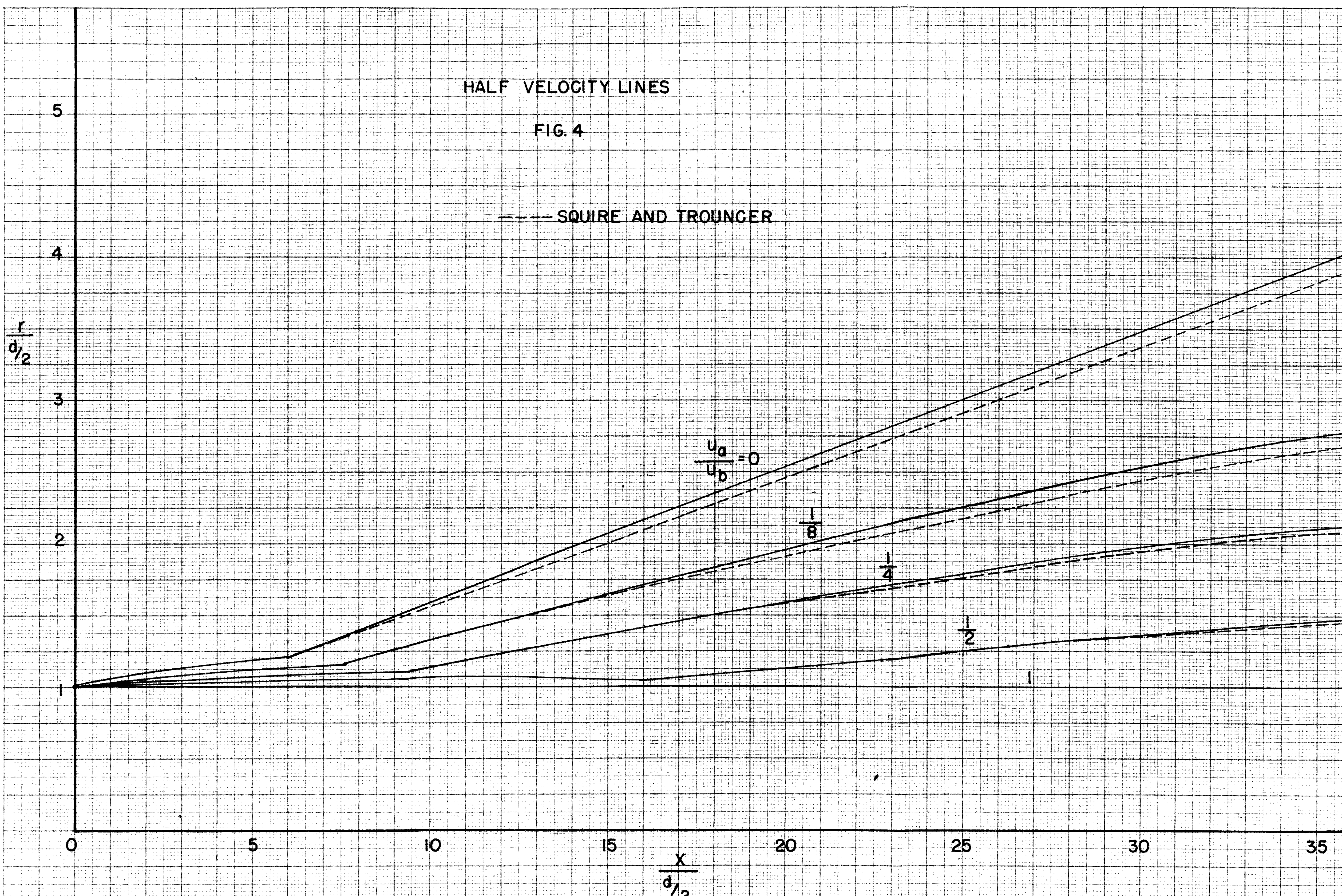
$$I = \frac{1}{\left(\frac{I_9}{I_{10}}\right)^3 u_b}\left[-\frac{1}{3}\cdot\frac{I_9}{I_{11}}\cdot\frac{1}{t^3} - \left[2\left(\frac{I_9}{I_{10}}\right)^2 - 2\frac{I_9}{I_{11}}\right]\frac{1}{t} + \left[\frac{I_9}{I_{11}} - \left(\frac{I_9}{I_{10}}\right)^2\right]t\right]$$

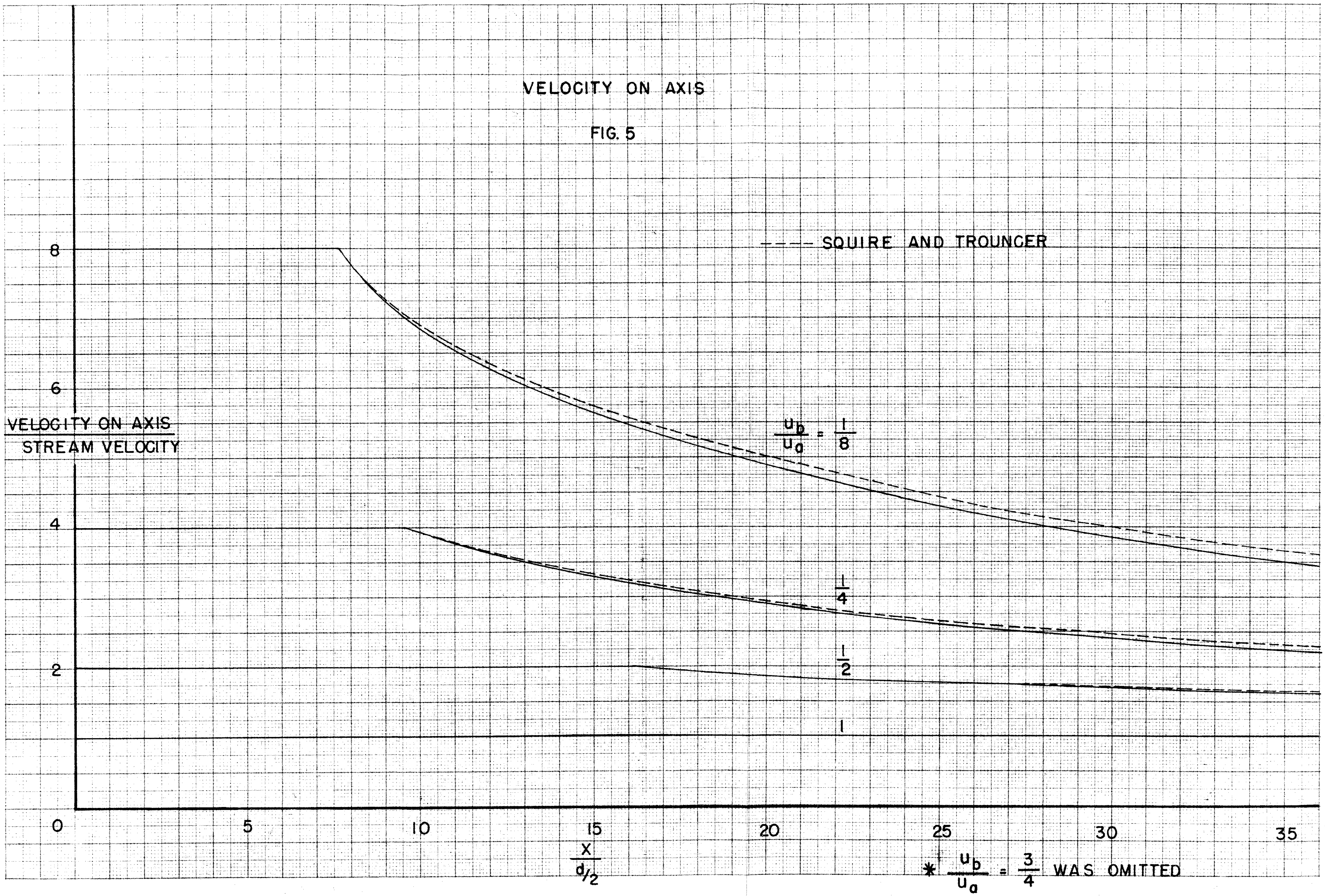
SHEAR PROFILES

FIG. 2




JET BOUNDARIES
---- SQUIRE AND TROUNCER
FIG. 3
$\frac{u_b}{u_a}=0$
$\frac{1}{8}$
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{3}{4}$
1
$\frac{b}{d/2}$
$\frac{a}{d/2}$
0
$\frac{1}{8}$
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{3}{4}$
$\frac{x}{d/2}$
0
5
10
15
20
25
30
35
1
2
3
4
5


HALF VELOCITY LINES
FIG. 4
SQUIRE AND TROUNCER
$\frac{u_a}{u_b} = 0$
$\frac{1}{8}$
$\frac{1}{4}$
$\frac{1}{2}$
1
$\frac{r}{d/2}$
$\frac{x}{d/2}$
0
5
10
15
20
25
30
35
1
2
3
4
5


VELOCITY ON AXIS
FIG. 5
---- SQUIRE AND TROUNCER
VELOCITY ON AXIS / STREAM VELOCITY
8
6
4
2
0
5
10
15
20
25
30
35
$\frac{u_b}{u_a} = \frac{1}{8}$
$\frac{1}{4}$
$\frac{1}{2}$
1
$\frac{X}{d/2}$
* $\frac{u_b}{u_a} = \frac{3}{4}$ WAS OMITTED