A METHOD OF PERFORMANCE ANALYSIS

FOR SAILING VESSELS

Thesis by

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TABLE OF SYMBOLS USED

SYMBOL	DEFINITION
Γ =	Total circulation or intensity of circulation.
Δ =	Displacement in 2240 lb. tons.
β =	Angle between apparent wind and boat course.
δ =	Draft loading, $\frac{H^2 L}{M}$
θ =	Angle of heel of vessel.
λ =	Leeway angle (angle of attack of fin).
μ =	Angle between sail C.P. line and normal to longitudinal hull axis.
ν_a, ν_w =	Kinematic viscosity of air or water, respectively.
ρ_a, ρ_w =	Mass density of air or water, respectively.
σ =	Sail area loading, $\frac{A L \rho_a}{M \rho_w}$.
ψ =	Angle between true wind and boat course.
$\mathfrak{M}$ =	Midship section submerged area.
ξ =	Angle between sail lift and plane perpendicular to rolling axis of vessel.
A =	Sail area, sq. ft.

TABLE OF SYMBOLS USED (continued)

SYMBOL	DEFINITION
$B =$	Maximum beam at load water line.
$C_D =$	Sail drag coefficient $\frac{D}{\frac{\rho_a}{2} U^2 A}$.
$C_{D_0} =$	Sail drag coefficient at zero lift.
$C_F =$	Sail driving force coefficient $\frac{F}{\frac{\rho_a}{2} U^2 A}$.
$C_f =$	Hull frictional drag coefficient $\frac{R_f}{\frac{\rho_w}{2} V^2 S}$.
$C_w =$	Hull wave drag coefficient $\frac{R_w}{\frac{\rho_w}{2} V^2 M/L}$.
$C_T =$	Total hull drag coefficient $\frac{R}{\frac{\rho_w}{2} V^2 M/L} \equiv C_f \frac{SL}{M} + C_w$
$D =$	Total sail drag
$F =$	Sail driving force (measured parallel to V).
$H =$	Maximum draft.
$H_m =$	Mean draft $= \frac{\text{Projected Lateral Area}}{\text{Waterline length (L)}}$.
$H_{EFF} =$	Effective draft for hull induced drag ($R_i = \frac{P^2}{\pi q_w H_{EFF}^2}$).
$L =$	Load waterline length.
$M =$	Total displacement volume.
$Q =$	Sail heeling force.

TABLE OF SYMBOLS USED (continued)

SYMBOL	DEFINITION
P =	Lateral water force.
R_f =	Frictional resistance of hull.
R_i =	Induced hull resistance (due to P).
R_w =	Wave making resistance of hull.
S =	Hull wetted surface.
T =	True wind speed ft./sec.
V =	True boat speed ft./sec. (along course, including leeway).
U =	Apparent wind speed ft./sec. (to an observer on boat).
X =	Distance from centroid of P to centroid of Q.
a =	Distance of foot of sail above W.L.
b =	Hoist of sail (normal to longitudinal axis).
ℓ =	Prismatic or longitudinal coefficient $\frac{M}{\otimes L}$.
q_a, q_w =	$\frac{\rho_a}{2} U^2$ or $\frac{\rho_w}{2} V^2$ the air or water dynamic pressure, respectively.

I. SUMMARY

A general method for calculating the performance of all types of wind driven vessels has been developed. The method is of such a nature that it may be used, without specific experimental assistance, as a qualitative guide in preliminary design. To this same procedure, data from a few simple tests of the model in question may be applied to obtain good quantitative performance information.

As a result, existing yacht model testing procedure {Reference (4)} can be considerably simplified while at the same time obtaining information of much broader applicability.

The use of the principles of this method, if not the letter, should tend to rationalize the arbitrary and extremely restrictive handicapping rules for ocean racing now in existence.

Further, the owner, in possession of a performance analysis of his vessel, can be guided in his choice of course and sail setting as these are fundamental parameters of the general performance equation. The optimum sail setting for any wind speed and boat direction is of a necessity obtained in the process of the solution of this equation.

The method provides several rather surprising general conclusions. It is indicated that angle of heel, contrary to Reference (4) is not a fundamental variable, and therefore stability only enters the picture in so far as the designer has a "gentlemen's agreement" not to provide more sail than will heel the vessel to a certain angle in a given wind. It is found that only draft is a monotonic parameter

in performance; all other form and sail variables exhibit optimums, some perhaps outside of the practical range.

Also, it is found from the experimental data of Reference (5) that yachtunderbodies, despite their extremely low aspect ratio, exhibit a polar of lateral force and induced resistance with a character in agreement with the simple Prandtl theory. The interference effect of the surface, however, is found to be at variance with the result obtained from a lifting line vortex arranged to satisfy the water boundary condition.

It is found that the usual triangular yacht sail does not have as bad a downwash distribution as one might first suspect on considering the plan form and twist. The presence of the water surface has the effect of providing a corrective upwash distribution.

It is found that simple static stability calculations with the water surface assumed undisturbed, at the boat, lead to surprisingly good agreement with the righting moments obtained experimentally under dynamic conditions.

II. INTRODUCTION

As the performance problem is one of steady state, force and moment equilibrium equations form the nucleus of the work. As is later shown, angle of heel has no appreciable direct influence on performance, therefore, the longitudinal force equilibrium carries the responsibility for the results.

In this equation appear the net (sail and rigging) driving force, the frictional and wave making resistances of the hull, together with the hull induced drag, (implicit in which is the heeling force). Neglected without justification is the windage of the hull, although, intuitively, for normal vessels without excessive freeboard this quantity seems small. The following sections discuss in some detail the properties of the forces involved and some of the equations involving wind and water velocities.

III. HULL RESISTANCE

By hull resistance is meant the water drag on the hull due to velocity and angle of heel but with no leeway, that is, no induced drag resulting from a lateral force.

For preliminary design purposes it is desirable to be able to predict the hull resistance as a function of velocity without recourse to experimental work. Taylor's Standard Series wave making resistance contours (Reference (1), Appendix B) are used for this purpose in power vessel design with considerable success, while in many phases of naval architecture the frictional resistance is calculated from von Kármán's logarithmic skin friction law,

$$\frac{0.242}{\sqrt{C_f}} = \log_{10}\left\{\frac{VL}{\nu_w} C_f\right\}$$

{References (2) and (3)}.

The question of angle of heel alone affecting hull resistance is rather well settled on pages 77 and 78 of Reference (1). Taylor, who was actually concerned with rolling resistances, showed that at a static angle of heel of 20° a normal merchant vessel model experienced little if any increase in resistance over that of 0° heel; in the worst case not over 3% increase was experienced. As sailing vessel hulls are, in the mass, considerably more axially symetric than merchant vessels, the above experiments were taken as justification for neglecting or averaging out the effects of angle of heel on hull resistance.

A major difference between sailing vessels and merchant ships lies in the extremely irregular profile of the former compared to the almost rectangular profile of the latter; therefore, while the $\frac{\Delta}{(L/100)^3}$ and ℓ parameters appear to be representative for sailing vessels as well as merchant ships, the $\frac{B}{H}$ ratio cannot have the same significance. Taylor's data is referred to the maximum draft of the ship, but for merchant vessels this value is nearly that of the mean draft. But to use the maximum draft as a parameter would be to ignore the difference between two sailing vessels, one with a broad shallow hull and a narrow keel, the other with a deep hull and broad keel. If these two boats had the same displacement, length, prismatic coefficient, maximum beam, and maximum draft (a perfectly possible situation), their wave making resistances would calculate equal using Taylor's method directly. That this result would be in error can be seen by observing that the former model would have most of its disturbance near the surface, while the latter, having an almost identical volumetric disturbance, would have this disturbance better distributed in a vertical direction so that the wave making effects would be smaller.

From the preceding discussion it appears that, for more general applicability of the Standard Series data, it is better to employ a mean draft. As beam-draft ratio corrections are moderate, perhaps a mean defined as the ratio of the projected lateral area to the waterline length will suffice until more rational methods are available. This mean is practically identical to the maximum draft when applied to merchant vessels.

The other major distinction between sailing vessels and merchant ships lies in the former's flaring bow sections above water. When the sailing boat is proceeding slowly this flare has no effect, but when speed-length ratios over about one are reached the wave shape above water is considerably changed. The effect is as if the prismatic coefficient were increased (although some prefer to think of it as an increase in length). The writer at present has no rational procedure to offer for evaluating the quantitative effects of flare or "overhang." Temporarily it is suggested that, if a vessel has pronounced overhang, the prismatic coefficient be assumed to be the optimum one for speed-length ratios over 1.0 (regardless of the actual prismatic coefficient), while if there is any doubt as to a vessel having good overhangs, the normal procedure using the actual prismatic coefficient be used in calculating the resistance. In any event, the differences are small particularly in the speed range wherein lies the main interest in performance.

Using the method outlined in the preceding paragraphs, the hull resistance for the yacht shown in Figs. 1 and 2 has been calculated and the results plotted in Fig. 3 along with the measured upright resistance as obtained at the Stevens Institute of Technology Experimental Towing Tank (Reference 5). Also shown are Stevens statistical resistances for other similar hulls (Fig. 4). It will be noted that in the medium speed range there is a discrepancy between the calculated results and the experimental data for hull No. 284 -- while the similar hulls do not exhibit this variance from the calculated values. It is probable then, that some peculiarity of form, not accounted for by this simple method has led to high resistances in this region.

For calculating frictional resistance it is Stevens Institute practice to use 0.7 of the Reynolds number based on waterline length in order to take into account the fact that the maximum downstream distance along the appendage is considerably less than that on the hull. This procedure was continued in the calculations for Fig. 3, but it is felt that a more general procedure would be to calculate separate wetted surfaces and Reynolds numbers for the hull and heel appendage and compute the frictional resistance in this manner. In this way the investigator is not dependent upon the assumption that the vessel in question is a "normal" cruiser for which the 0.7 Reynolds number friction coefficients give reasonable results.

It appears then that, for preliminary design purposes, a method is available for computing approximately the hull resistance. When the design is reasonably well settled in major proportions, it is a simple matter (if the facilities are available) to obtain very good values by the usual procedures for testing upright, small towing tank models.

IV. THE LATERAL PLANE

If one is satisfied that an angle of heel in itself produces no appreciable increase in resistance over the upright case, then the increase in resistance experienced by a real sailing vessel when heeled down by the wind must come from an induced drag effect. Of course, the aspect ratio of the usual keel-hull combination is extremely low, averaging from 0.2 to 0.4, but rather than becoming involved in the complicated but theoretically sound methods for calculating the properties of low aspect ratio airfoils, an attempt was made to fit the Prandtl equations to this case. Fig. 5 shows the result of calculating the usual polar for an elliptical plan form airfoil of the same aspect ratio in an infinite fluid. Also plotted on the same axes are the measured induced drags obtained by subtracting from the inclined resistance the upright resistance at the same speed. The third curve shown is that of lifting line airfoil perpendicular to and with one tip vortex absorbed in a solid boundary. The image effect necessary to satisfy the boundary condition has the result of producing an induced drag of roughly one-half that of the infinite fluid case.

In order to observe the effect of higher aspect ratio together with an absence of angle of heel, data from Reference (1), page 39, is plotted in Fig. 6 against the Prandtl theory. In this case an aspect ratio of 1.0 and a rectangular plan form is involved. A very similar curve is shown, and the comparison with the two Prandtl cases is surprisingly like that of the lower aspect ratio case. It is clear that in both cases the measured curve has the character of a parabola and could be fitted very closely by the usual aerodynamic methods. By this curve-fitting procedure an induced drag parameter, the effective draft (corresponding to the effective span), will be obtained.

However, for preliminary design purposes, it is felt that it is better to take the conservative view and calculate the induced resistance from:

$$R_i = \frac{P^2}{\pi \frac{\rho_w}{2} V^2 H^2}$$

The error can be fairly large in R_i itself, and still be only a minor fraction of the total resistance. As more experimental and theoretical work is done there will undoubtely be occasion to improve this rough assumption. Naturally, for the final performance calculation, there can be made available sufficient experimental information to obtain an accurate value of the particular effective draft. A suggested procedure would be to plot polars for the hull in question for the upright case and one or two fixed angles of heel. The parabola should then be fitted to a good average of the family. It is anticipated that the curves for the various heel angles will fall very close together, but if this were not the case a good average could be taken by fitting to the small angle of heel curve at the small lateral force end and to the large angle curve for large lateral forces.

V. HULL STABILITY

While angle of heel in itself does not appear to be of prime importance in the calculation of the performance of a given sailing vessel, the choice of the proper amount of sail area in the preliminary design stage depends almost entirely on some particular restriction as to angle of heel in some standard wind. And too, as will be shown later, it is only correct to say that angle of heel is not important if in normal winds the angle of heel of the vessel does not exceed about 30°.

To calculate the angle of heel of a given vessel for a given lateral force, **Q** , it is necessary to make several assumptions. First, it is assumed that the lift distribution on the sails is similar to the plan form. This choice is qualitatively justified in a following section.

Second, it is assumed that the lateral force distribution is similar to the projected lateral shape. This assumption, while very arbitrary, is considered good enough in as much as the distance between the centroid of **P** and the centroid of **Q** (called **X**) is the lever arm of the heeling moment; and any reasonable assumption as to **P**'s location will have little effect on **X**.

Third, containing these two assumptions two methods for calculating the dynamic stability of hull No. 284 were tried and compared with experiment. In these two methods it was assumed that the static stability of the hull was identical to the dynamic stability (stability in motion with a disturbed water surface). The two methods were the usual two used in merchant vessel work. One, the righting moment is taken equal to $W(GM)\sin\theta$ where (GM) is the initial metacentric height; and two, the righting moment is obtained by direct graphical integration for the righting arm for various angles of heel. The comparison with experiment is shown in Table I:

TABLE I

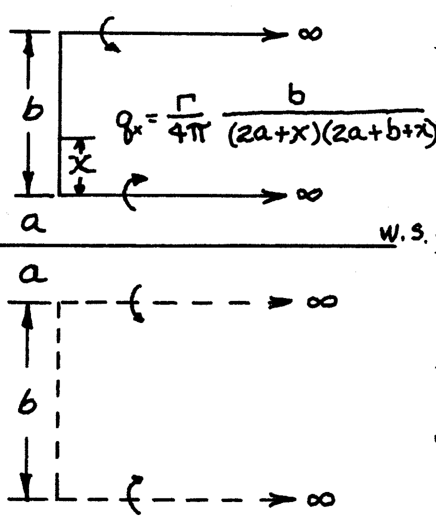
θ	$Q_{(EXP)}$	QX	$W(GM)\sin\theta$	$W(R.A.)$
10°	1250 lbs.	34,200 ft.lbs.	35,700 ft.lbs.	32,600 ft.lbs.
20°	2250 lbs.	64,000 ft.lbs.	69,000 ft.lbs.	60,000 ft.lbs.
30°	3170 lbs.	90,000 ft.lbs.	104,000 ft.lbs.	85,000 ft.lbs.

It is seen that either assumption is surprisingly well justified and that the dynamic stability can be calculated with good accuracy even by the simple equation method. This method then should satisfy the requirements of preliminary design while the more exact integration can be used as a check on the final design to avoid any possibility of a serious loss in stability due to peculiarities of form which might tend to invalidate the simple equation.

VI. SAIL FORCES

In the calculation of the sail forces, one of the primary questions is, the downwash distribution on a sail. The plan form of most modern yacht sails is triangular which in itself leads to a peculiar downwash distribution, but in addition the yacht sail has a twist in it of decreasing angle of attack toward the head. This twist will have the effect of aggravating the non-uniform downwash distribution due to plan form, and is a variable depending upon the wind velocity.

However, the presence of the water surface has the tendency to neutralize these unfavorable effects. The problem here is very similar to that of a longitudinally slotted wing, or that of a formation flight. In order to satisfy the boundary condition of a water surface, an image vortex system must be placed directly below the water surface as shown. As the result required is only qualitative, a simple horseshoe vortex system was chosen.



Applying the Biot-Savart law for semi-infinite vortices, $q = -\frac{\Gamma}{4\pi h}$ (the bound image vortex makes no contribution) the upwash distribution is found to be hyperbolic, . Notice that near $x = 0$ the plan form downwash is too large, but this is just where the correction upwash has its large values. To obtain an idea of the overall improvement of the induced drag from such an upwash a mean was taken over the span;

$$\omega' = \frac{\Gamma b}{4\pi} \int_0^b \frac{dx}{(2a+x)(2a+b+x)}$$

which yields,

$$\omega' = \frac{\Gamma}{4\pi} \ln\left(1 + \frac{b^2}{4a^2+4ab}\right)$$

If now, contrary to the previous assumption, the sail is taken to have uniform downwash -- the downwash angle will be from the Prandtl theory:

$$\frac{\omega}{u} = \frac{L}{\pi q_a b}.$$

The average upwash angle correction will be:

$$\frac{\omega'}{u} = \frac{-L}{\rho u^2 b^2 4\pi} \left\{ \ln\left(1+\frac{b^2}{4a^2+4ab}\right) \right\}$$

Therefore, the net average downwash is approximately given by:

$$\left(\frac{\omega}{u}\right)_{NET} \cong \frac{L}{\pi q_a b^2} \left\{ 1 - \frac{1}{8} \ln\left(1 + \frac{b^2}{4a^2+4ab}\right) \right\}$$

If in this approximate expression a is set equal to $0.15\,b$ (an average value for medium-sized sailing vessels), the correction part of this expression amounts to 11%. It can be seen from the above that a considerable loss in efficiency due to the unfavorable plan form and twist is offset by the presence of the water surface.

As it seemed very difficult to calculate the actual downwash distribution of a yacht sail, the next step was to see by comparison with experiment if the favorable correction noted above was good enough to make the Prandtl expression for the wing with uniform downwash a sufficiently good representation. As the available experimental results were considered a function of angle of heel, it was next necessary to investigate this parameter as to its effect on the sail forces.

One way in which angle of heel could cause a change in the sail forces is by an effective loss in aspect ratio. It is considered good aeronautical procedure to calculate the effect of sweepback as high as thirty degrees by projecting the line of aerodynamic centers into a plane perpendicular to the wind at infinity and considering this as the effective span (see for instance Reference 6.). By considering angle of heel as a sort of variable sweepback, an expression can be obtained for the loss in effective span or aspect ratio.

In Fig. 9 the line of aerodynamic centers is assumed to be a straight line, of length b, inclined at an angle μ to the perpendicular to the longitudinal axis of the vessel. The apparent wind is taken along the negative x direction with the vessel's course, V, in the xy plane in a positive direction at an angle β to the xy plane. The vessel is heeled an angle θ. It is seen that the component of the lifting line in the u direction is

$$b \cos\mu \sin\theta \sin\beta - b \sin\mu \cos\beta$$

and the required component which is in the bu plane perpendicular to u is

$$b\sqrt{1 - (\cos\mu \sin\theta \sin\beta - \sin\mu \cos\beta)^2}$$

Therefore, if we define $AR_0 = \frac{b^2}{A}$, then the effective aspect ratio can be calculated:

$$AR_{EFF} = AR_0 \left\{1 - (\cos\mu \sin\theta \sin\beta - \sin\mu \cos\beta)^2\right\}$$

For practical calculations, until more is known of the position of the aerodynamic center line for sail combinations, it is probably best to take μ as zero. This leads to the simpler expression:

$$A\!R_{EFF} = A\!R_0 \left\{ 1 - \sin^2\theta \sin^2\beta \right\}$$

However, it is noteworthy that if the sail lifting line position is calculable that there is an ideal μ for every β and θ. This is obtained by setting the correction to $A\!R_0 = 0$ which gives:

$$\tan\mu = \sin\theta \tan\beta$$

For instance, if $\theta = 15^\circ$ and $\beta = 30^\circ$, then $(\mu)_{IDEAL} = 8.5^\circ$ forward rake.

In the second, simpler expression for $A\!R_{EFF}$, it can be seen readily that this correction is always very small. Only at small β are large θ encountered in practice, with θ rarely exceeding 30°. If θ is taken equal to 30° and β likewise equal to 30° the $A\!R_{EFF} = 0.938\, A\!R_0$. For average conditions this correction is much less. It seems therefore justified to neglect the aspect ratio correction for angle of heel. The error involved will be less than that indicated above, for the forces of interest include components of both the sail induced drag and the lift, and the aspect ratio appears only in the induced drag.

The forces of interest coming from the sails are not really the lift and drag but the driving force, F, and the heeling force, Q. Defining the drag as the component of the resultant sail force parallel to the apparent wind, and defining the lift as the component perpendicular to the plane of the apparent wind and normal to the vessel's rolling axis, expressions are obtained for

F and Q. For the upright case:

$$F = L \sin\beta - D\cos\beta$$
$$Q = L\cos\beta + D\sin\beta$$

but for the inclined case the lift components will be functions of β and θ both. Here is another situation where the sail forces are functions of the angle of heel.

Considering Fig. 10, it is seen that the vessel's course is chosen, in this instance, along the x axis; π is at an angle β to the course in the xy plane; the lift is shown as defined previously. If the lift is to be projected along and perpendicular to the vessel's longitudinal axis the angle ξ must be known. By projecting the lift into the xy plane and calling this component L' it is seen from the geometry that:

$$L\cos\xi\cos\theta = L'\cos\beta$$
$$L\sin\xi = L'\sin\beta$$

Solving these two expressions for L' and ξ, it is found that:

$$\tan\xi = \tan\beta\cos\theta$$

However, the functions of ξ in which interest lies are the sine and cosine since for the inclined case:

$$F = L\sin\xi - D\cos\beta$$
$$Q = L\cos\xi + D\sin\beta$$

Simple trigonometric identities produce the following:

$$\sin\xi = \sin\beta\left\{\frac{\cos\theta}{\sqrt{\cos^2\beta + \sin^2\beta\cos^2\theta}}\right\}$$
$$\cos\xi = \cos\beta\left\{\frac{1}{\sqrt{\cos^2\beta + \sin^2\beta\cos^2\theta}}\right\}$$

So again expressions are obtained which are in the form of corrections to the upright case. For $\beta = 30^\circ$, $\theta = 30^\circ$:

$$\sin \xi = 0.895 \sin \beta$$
$$\cos \xi = 1.03 \cos \beta$$

while for $\beta = 30^\circ$, $\theta = 20^\circ$:

$$\sin \xi = 0.953 \sin \beta$$
$$\cos \xi = 1.01 \cos \beta$$

It is seen, then, that the correction to $\cos \beta$ for the heeling force is always very small but with such a sign as to augment the small correction for angle of heel due to the loss in AR_o in the drag components. On the other hand, the correction to $\sin \beta$ is somewhat larger. It too is in the same direction as the drag correction.

However, offsetting this effect is the fact that as the vessel heels down the image vortex system comes closer to the true vortex system, and the favorable upwash is thereby increased.

By now it can be seen that a great deal of responsibility is being placed on the water surface and without a rigorous analysis such assumptions are hard to justify. The only alternative would be a careful and complete experimental program designed to settle this question. Unfortunately, neither is available. The only comprehensive experiments on sails published are the "Gimcrack" tests which form the foundation for the model testing procedure at the Stevens Institute of Technology (Reference 4). In these tests a competent helmsman sailed the boat, "Gimcrack", close hauled and measurements of θ , β , and V were made. From tank tests of the "Gimcrack" hull the sail driving and heeling force coefficients could be calculated. These are shown in Fig. 8. Also shown are the C_L and C_D curves corresponding. The C_L and C_D curves are plotted in two ways: one, no angle of heel correction either to $\delta(\beta)$ or AR_o is applied; second, all angle of heel corrections are applied except for any change in the image vortex effects. Then from the four curves of C_L and C_D , two C_L vs C_D polars were plotted. It is seen that at the low C_L end of the corrected curves considerable irregularity appears. This corresponds to the larger angles of heel. It is felt then that since the measured curves (C_F and C_Q) were faired that the physical phenomenon associated with angle of heel is approximately as described; i.e., the angle of heel corrections should not be applied.

Furthermore, taking the faired C_L vs C_D polar, it is easy to fit a good parabola to the curve by usual aerodynamic methods, obtaining AR_o (which will be referred to hereafter as AR) and C_{D_o} . The values obtained for "Gimcrack" are:

$$C_{D_o} = 0.038$$

$$AR = 3.44$$

"Gimcrack" was a 20' W.L. boat with a sail area of approximately 450 square feet and a sail hoist of about 40 feet. Defining the aspect ratio, AR , as the sail hoist squared divided by the sail area the calculated AR = 3.54, indicating very good downwash distribution.

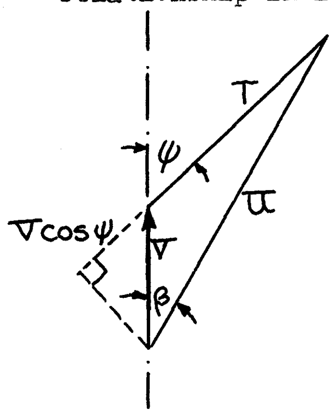
For these reasons, considering the limited amount of experimental information and despite the speculative nature of some of the physical reasoning involved, it is felt that good performance calculations may be made neglecting entirely the angle of heel. Only the major factor, mast height, is taken into account by this approach, and it is felt that unconventional sail plans could not be accurately appraised under these assumptions, but if the person making the performance analyses exercises some judgement the result should be within the accuracy required. It should be pointed out that on the completion of the performance analysis a check should be made to determine the angle of heel at the larger wind velocities and at the small ψ's. At angles of heel much over 30° the method must be considered inaccurate and performance curves should be so noted.

A noteworthy fact is the low value of C_{D_0} obtained for "Gimcrack". As sails are invariably operated at comparatively large C_L and much of the time near $C_{L_{MAX}}$, it is seen that the value of C_{D_0} has little significance. In other words, a C_{D_0} of 0.038 could be chosen for any sail plan without introducing a serious error.

It, therefore, can be said that eliminating the windage of the spars, rigging and even the hull of "Gimcrack" would not have changed her performance appreciably. (The windage of the hull must have been taken into account by the nature of the tests performed and the method of arriving at the force coefficients.)

VII. THE WINDWARD TRIANGLE

The sailing vessel, because it moves in the already moving air, experiences a wind which is the vector sum of the boat velocity and the wind velocity relative to fixed space. This geometrical relationship in itself explains several phenomena.



The diagram shown indicates the vector addition. By the nature of **triangles** the ratios of any two of the velocities can be expressed in terms of the angles β and ψ only. These relationships are of importance in performance calculations.

From the law of sines:

$$\frac{U}{T} = \frac{\sin\psi}{\sin\beta}$$

By projecting U and T on the course direction it is found that:

$$T\cos\psi + V = U\cos\beta$$

Dividing this equation by T and substituting $\frac{U}{T} = \frac{\sin\psi}{\sin\beta}$, a second important relationship is obtained:

$$\frac{V}{T} = \frac{\sin\psi}{\tan\beta} - \cos\psi$$

A third equation may be derived from the relationship:

$$\frac{U}{V} = \frac{\sin\psi}{\sin\beta}\,\frac{T}{V}$$

which becomes of importance when the reciprocal of the second result IS SUBSTITUTED:

$$\frac{U}{V} = \left\{\frac{1}{\cos\beta - \left(\frac{\sin\beta}{\tan\psi}\right)}\right\}$$

It is seen that if β and ψ are specified it is simple to obtain the relative velocities between the vessel and the wind.

Much of the time a sailing boat is able to sail in the direction desired, that is at arbitrary ψ, but on many occasions it is desired to go directly into the wind. In this case the vessel must bear off or tack, and the speed made good in a windward direction is only $V \cos \psi$.

For a given β, and T there is a ψ which will produce a maximum $V \cos \psi$. Said in another way, for a given angle to the apparent wind, the boat speed should not exceed some fraction of the apparent wind if the best speed made good to windward is to be obtained. It is easier to solve the problem as first stated, and then apply the previously obtained ratios to obtain the result as stated secondly:

$$V \cos \psi = \frac{T \sin \psi \cos \psi}{\tan \beta} - T \cos^2 \psi$$

Differentiating the expression and setting the derivative equal to zero gives the equation for ψ_{OPT}:

$$\cot 2\psi = \tan \beta$$

or finally:

$$\psi_{OPT} = \frac{\pi}{4} + \frac{\beta}{2}$$

This is clearly a maximum as when $\psi = 0$, $V = 0$; therefore, $V \cos \psi = 0$; and when $\psi = \frac{\pi}{2}$, $V \cos \psi = 0$. In between must lie a maximum as $V \cos \psi$ must have positive values.

Now the helmsman is only able to observe $\mathcal{U}$, β , and V ; therefore, it is desirable to express the result in terms of these quantities:

$$\frac{V}{\mathcal{U}} = \cos\beta - \frac{\sin\beta}{\tan\psi}$$

and applying $\psi_{OPT} = \frac{\pi}{4} + \frac{\beta}{2}$ the preceeding equation becomes:

$$\left\{\frac{V}{\mathcal{U}}\right\}_{V\cos\psi\ MAX} = \cos\beta - \frac{\sin\beta}{\tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right)}$$

which can be simplified by trigonometric identities to:

$$\left(\frac{V}{\mathcal{U}}\right)_{OPT} = \frac{\cos\beta}{1 - \sin\beta}$$

This interesting sidelight of the performance problem should be of value to a helmsman, giving him a sort of ideal to shoot for at large $\mathcal{U}$ and a value to stay down to at small $\mathcal{U}$.

VIII. DOWNWIND PERFORMANCE

Downwind performance is arbitrarily defined for this paper as being the directions of sailing relative to the true wind which require that the sail be operated at an angle of attack greater than that for $C_{L_{MAX}}$. In this region of sailing ($\beta \geqq$ about 100^{o}), it is customary to set innumerable types of light balloon jibs and spinnakers, therefore any attempt to predict performance in this region would have to involve the type and area of light sails set. As airfoils or scoops operated beyond the stall are not subject to theoretical analysis even if their geometry is known, the task appears hopeless.

However, very good comparative downwind characteristics can be obtained by considering the relative upright resistance versus velocity curves of the vessels in question. If any question of relative sail area arises, some arbitrary figure of merit such as the ratio of the resistance to the mast height squared or the reaching sail area should be adopted. Very little interest settles in downwind performance except for special applications as this is somewhat of a "brute force" situation at best.

IX. REACHING PERFORMANCE

By reaching performance is meant all directions of sailing in which the boat course is the one desired and on which the sails can be operated at $C_{L_{MAX}}$ or less without structural interference. Thus, β's from 10° or 20° to 100° are included. The only distinction from windward performance is that in that case it is desired to travel closer to the wind than the course chosen.

For the reaching case, all important forces and parameters have been investigated. It is only necessary to write the longitudinal force equilibrium equation and consider its solution. In words this equation states that the sail driving force is equal to the hull resistance including induced hull drag due to lateral sail forces. Literally:

$$F = R_f + R_w + R_i$$

or rearranging and expressing this in coefficient form (see "definition of symbols used"):

$$C_F \frac{\rho_a}{2} \mathcal{V}^2 A - \frac{C_Q^2 \left[\frac{\rho_a}{2} \mathcal{V}^2 A\right]^2}{\pi \frac{\rho_w}{2} V^2 H_{EFF}^2} = \frac{\rho_w}{2} V^2 \left[C_f S + C_w \frac{M}{L}\right]$$

Dividing both sides by $\frac{\rho_w}{2} V^2$, the equation becomes:

$$C_F \left[\frac{\rho_a}{\rho_w} A \frac{\mathcal{V}^2}{V^2}\right] - \frac{C_Q^2}{\pi H_{EFF}^2} \left[\frac{\rho_a}{\rho_w} A\right]^2 \left[\frac{\mathcal{V}^2}{V^2}\right]^2 = \frac{M}{L} \left[C_f \frac{SL}{M} + C_w\right]$$

Defining $C_f \frac{SL}{M} + C_w$ as $C_T \equiv \dfrac{R_T}{\frac{\rho_w}{2} V^2 \frac{M}{L}}$ a further simplification is obtained:

$$C_F \left[\frac{\rho_a}{\rho_w} \frac{AL}{M}\right] \frac{\mathcal{V}^2}{V^2} - \frac{C_Q^2}{\pi \frac{H_{EFF}^2 L}{M}} \left[\frac{\rho_a}{\rho_w} \frac{AL}{M}\right]^2 \left[\frac{\mathcal{V}^2}{V^2}\right]^2 = C_T$$

Now, calling the sail area loading,

$$\sigma = \frac{\rho_a \; AL}{\rho_w \; M}$$

and the draft loading,

$$\delta = \frac{H_{EFF}^2 L}{M}$$

the equation reduces to:

$$C_F \sigma \left(\frac{\mathcal{U}}{V}\right)^2 - \frac{C_Q^2 \sigma^{-2}}{\pi \delta}\left[\left(\frac{\mathcal{U}}{V}\right)^2\right]^2 = C_T$$

It was shown from the geometry of the apparent wind that:

$$\left(\frac{\mathcal{U}}{V}\right) = \left\{\frac{1}{\cos\beta - \left(\frac{\sin\beta}{\tan\psi}\right)}\right\}$$

so that the dimensionless performance equation finally becomes:

$$C_T = \frac{C_F \sigma}{\left\{\cos\beta - \frac{\sin\beta}{\tan\psi}\right\}^2} - \frac{C_Q^2 \sigma^{-2}}{\pi \delta \left\{\cos\beta - \frac{\sin\beta}{\tan\psi}\right\}^4}$$

In this equation is contained all the major independent variables involved in reaching performance. The helmsman has at his choice the course in space, ψ , and the sail setting {contained in C_F and C_Q since $C_F = C_F(C_Q, \beta)$. Implicit in C_T is the boat velocity, while β expresses the relation between V and T for an arbitrary ψ .

For performance purposes it is only necessary to assume that the vessel is being sailed by the ideal helmsman who sets the sails at the angle of attack which will produce a maximum V for a given ψ and T . All other sail settings are not of interest as the best sail setting costs nothing. Since C_T vs. V (see Fig. 11) is practically always a single valued function in the speed range of interest, maximizing C_T in the dimensionless performance equation will maximize the boat velocity for a given ψ and β .

Calling C_Q the independent variable, the performance equation can be differentiated:

$$\frac{\partial C_T}{\partial C_Q} = \frac{\partial C_F}{\partial C_Q}\left\{\frac{\sigma}{\left(\cos\beta - \frac{\sin\beta}{\tan\psi}\right)^2}\right\} - \frac{2 C_Q \sigma^2}{\pi\delta\left(\cos\beta - \frac{\sin\beta}{\tan\psi}\right)^4}$$

and by setting $\frac{\partial C_T}{\partial C_Q} = 0$ to obtain a maximum, it is found that:

$$\frac{\partial C_F}{\partial C_Q} = \frac{2 C_Q \sigma}{\pi\delta\left(\cos\beta - \frac{\sin\beta}{\tan\psi}\right)^2}$$

This is an equation for $(C_Q)_{OPT}$ in terms of the independent variables β and ψ .

$\frac{\partial C_F}{\partial C_Q}$ can be expressed as a function of C_Q and β alone by recalling that:

$$C_F = C_L \sin\beta - \left(C_{D_0} + \frac{C_L^2}{\pi AR}\right)\cos\beta$$

$$C_Q = C_L \cos\beta + \left(C_{D_0} + \frac{C_L^2}{\pi AR}\right)\sin\beta$$

C_L can be eliminated from the above pair of equations to obtain:

$$C_F = \frac{\pi AR\cos\beta}{2}\left\{\sqrt{1 + \frac{4\tan^2\beta}{\pi AR}\left(\frac{C_Q}{\sin\beta} - C_{D_0}\right)} - 1\right\} - \left[C_{D_0} + \frac{\pi AR}{4\tan^2\beta}\left\{\sqrt{1 + \frac{4\tan^2\beta}{\pi AR}\left(\frac{C_Q}{\sin\beta} - C_{D_0}\right)} - 1\right\}^2\right]\cos\beta$$

From this,

$$\frac{\partial C_F}{\partial C_Q} = \frac{1 - \cos^2\beta\sqrt{1 + \frac{4\tan^2\beta}{\pi AR}\left(\frac{C_Q}{\sin\beta} - C_{D_0}\right)}}{\sin\beta\cos\beta\sqrt{1 + \frac{4\tan^2\beta}{\pi AR}\left(\frac{C_Q}{\sin\beta} - C_{D_0}\right)}}$$

Theoretically this expression could be set equal to:

$$\frac{2 C_Q \sigma}{\pi \delta \left(\cos\beta - \frac{\sin\beta}{\tan\psi}\right)^2}$$

and the equation solved for C_Q , but the result would be a high degree algebraic equation which for practical purposes would require graphical or successive approximation methods for its solution. It is much more convenient to solve graphically the equation:

$$\frac{\partial C_F}{\partial C_Q} = \frac{2 C_Q \sigma}{\pi \delta \left(\cos\beta - \frac{\sin\beta}{\tan\psi}\right)^2}$$

This has been done in Fig. 12 for hull No. 284 using a C_{D_0} of 0.038 and a AR = 3.44. The choice of "Gimcrack's" sail parameters instead of the true ones for hull No. 284 was made in order to test this performance method against the Stevens experimental results. (At the Hoboken tank all vessels are assumed to have a sail plan similar to that of "Gimcrack" varying only in area and center of pressure location.)

The curves of $\frac{\partial C_F}{\partial C_Q}$ vs. C_Q were first plotted for lines of constant β . Then as:

$$\frac{2 C_Q \sigma}{\pi \delta \left(\cos\beta - \frac{\sin\beta}{\tan\psi}\right)^2} \quad vs \quad C_Q$$

are straight lines through the origin it was only necessary to plot one point for each ψ and β , obtaining with a straight edge the intersection with a given $\frac{\partial C_F}{\partial C_Q}$ curve.

It is necessary to carry out this entire plot for every vessel analysed, but for preliminary design purposes, once the AR is settled the $\frac{\partial C_F}{\partial C_Q}$ curves are fixed and much of the work is simplified.

Sail setting having been eliminated as a variable, C_T can now be solved for in the dimensionless performance equation. A value of C_T will be obtained for every pair of values of ψ and β chosen. A tabular form makes this calculation simple. The work is facilitated by making a plot of C_F vs. C_Q for lines of constant β. This curve for the "Gimcrack" sail proportions is shown in Fig. 13. It is more convenient to calculate this curve parametrically than to use the explicit expression given previously.

The real values of C_T cover only a small range of values for a given vessel so that an intermediate plot of C_T vs. β for lines of constant ψ is convenient. This plot is shown in Fig. 14.

Because C_δ decreases with velocity monotonically and C_ω increases almost always monotonically, there will be a minimum value of C_T. Physically this means that the right hand side of the dimensionless performance equation must be above a certain value or the vessel will not sail. Lower velocities than correspond to this value of C_T are possible but correspond to an unstable equilibrium. A small disturbance will cause the vessel to increase speed to the min. value or above. For purposes of curve fairing the final performance curves are plotted through the origin as this situation could occur hypothetically with a vessel having a decreasing wetted surface with decreasing velocity.

Having C_T as a function of β and ψ, the curve of C_T vs. V can be entered and the corresponding velocity obtained. The true wind speed can be obtained from the relationship:

$$\frac{V}{T} = \frac{\sin\psi}{\tan\beta} - \cos\psi$$

so that the final reaching performance curves V vs. T for an arbitrary direction in space, ψ , can be plotted. This plot for hull No. 284 is shown in Fig. 15.

It should be pointed out that these curves are all over and above the information obtained by the Stevens Tank testing procedure where only windward performance is considered. Also it should be said that while only the direct performance plots are discussed here, the allied problems of angle of heel, best sail setting and others can be obtained readily as the method is entirely analytic.

X. WINDWARD PERFORMANCE

Having the reaching performance curves, obtaining any windward performance curve is a matter of simple cross plotting. In practice, one might wish to sail at any small angle to the true wind and should then make use of the procedure indicated in the following paragraphs. Often a navigator or helmsman in an important ocean race comes up against the problem of what course to sail to make the best speed in a direction closer to the wind than the vessel can sail.

For present purposes, only the performance in the direction of the true wind will be considered. All other similar directions can be computed in the same way. It is desired to make $\mathcal{V} \cos\psi$ a maximum for a given $\mathcal{T}$ and cross plots from the reaching performance curves for this arrangement of the variables can be made. Those for hull No. 284 are shown in Fig. 16. The expected maximums appear and are plotted on the windward performance curve, Fig. 17, $(\mathcal{V} \cos\psi)_{MAX}$ vs. $\mathcal{T}$.

It is here that the comparison with the Stevens experimental results is obtained. Shown also on Fig. 17 are the measured values for $(\mathcal{V} \cos\psi)_{MAX}$ vs. $\mathcal{T}$. It should be remembered that these points, while obtained experimentally, contain in them the assumption that the sails were set, at a given angle of heel, the same as "Gimcrack's" helmsman set his sails. As has been shown, the sail setting for best speed is a function of the underwater geometry (δ) and the sail area loading (σ) so that even if one allows the common

AR and C_{D_0} implied by the "Gimcrack" sail coefficients it should not be taken to indicate some error in the calculations if the experimental and calculated curves do not coincide. Actually, the character of the experimental curve is that to be expected of a smaller boat. At the high speeds a greater proportional wind velocity is measured to be required than is calculated, a result to be expected if hull No. 284 were constrained by a small boat's ideal sail settings. The discrepancy in sail settings is shown.

However, the agreement is rather good, and this should be expected if the calculations are to be correct since the comparison of Stevens tests with a few full-scale vessel performances showed reasonable good values for all sizes of vessels.

XI. DESIGN PROBLEMS

From the position of the preliminary designer the dimensionless performance equation offers much of interest. By inspection of the equation it is seen that only H_{EFF} can be increased indefinitely to improve the performance.

In C_T are found the two components $C_f \frac{SL}{M}$ and C_w where C_f is a function only of Reynolds number, and C_w a function primarily of $\frac{\Delta}{(L/100)^3}$, ℓ, and $\frac{B}{H_M}$. S is mainly sensitive to $\frac{\Delta}{(L/100)^3}$ and $\frac{B}{H_M}$. For a given speed and underwater volume, OR Δ, it is then possible to calculate an ideal L to minimize C_T. This can best be done graphically as S must be obtained from lines drawings. However, if a family of similar lines varying only in length or displacement is being investigated, Taylors approximate equation $S = C\sqrt{\Delta L}$ (Reference 1) can be used to reduce this problem to an analytic one.

The sail area parameter σ likewise exhibits an optimum for a given sail setting, δ, β, and ψ. This is easily obtained by differentiating C_T with respect to σ. The equation in σ_{OPT} can, by inspection, be seen to give a maximum. This equation is:

$$\sigma_{OPT} = \frac{C_F \pi \delta \left(\cos\beta - \frac{\sin\beta}{\tan\psi}\right)^2}{2 C_Q^2}$$

While σ_{OPT}, is a function of all the independent variables, the designer can emphasize one particular type of performance at will or choose for a good average. To test that σ_{OPT} was not out of a practical range, it was calculated for hull No. 284 with

$\beta = 30^\circ$, $\psi = 45^\circ$, $C_F = 0.5$ and $C_Q = 1.5$. The sail area obtained was 1300 square feet, by coincidence very close to the actual design sail area.

If the designer restricts himself to a given heeling moment, the product of the $\sqrt{AR}$ and $\sigma^{3/2}$ is fixed, and for this constant value again the righthand side of the performance equation can be maximized for given conditions. This problem is more complicated than the preceding ones but should certainly be carried out for a few vessels to give an idea of the proper sail geometry.

The fact that these optimums occur as functions of the independent variables tends to explain why vessels are spoken of as having a point of sailing or a kind of weather particularly to their liking. Given a boat with a good clean hull and enough effective draft, there must almost always be some situation where she is well proportioned throughout.

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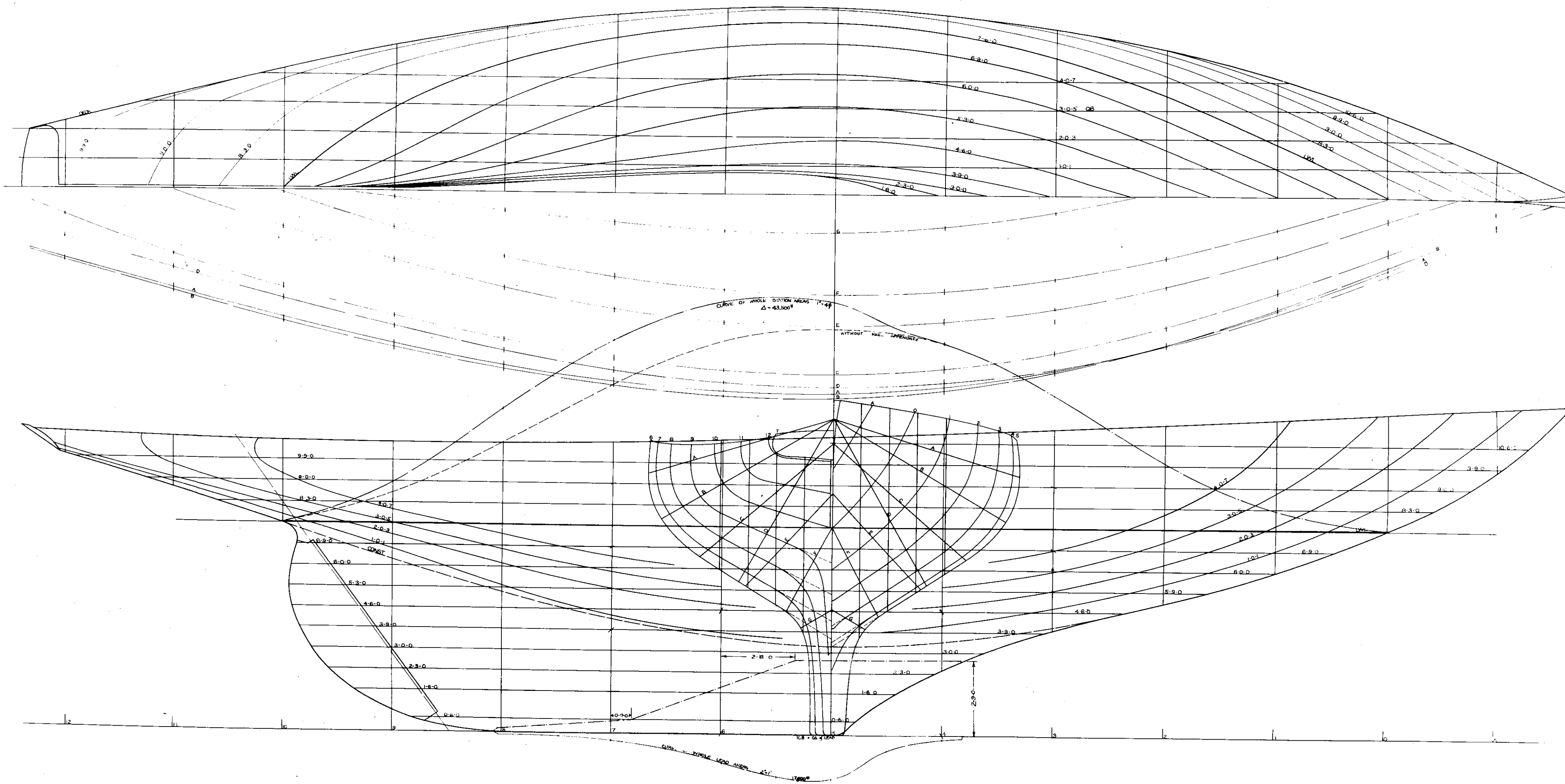
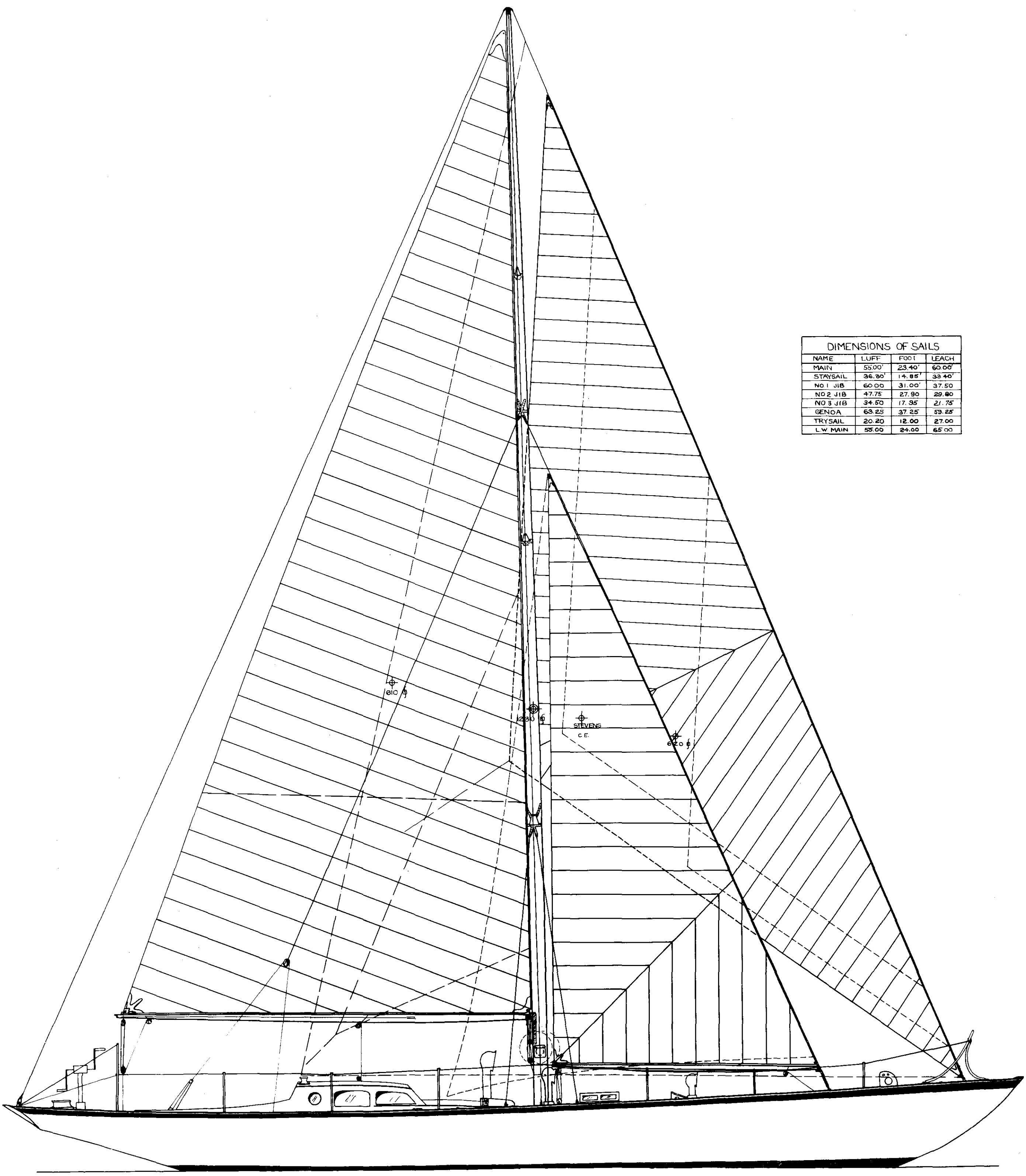


FIG 1

HULL 284 LINES DRAWING



DIMENSIONS OF SAILS			
NAME	LUFF	FOOT	LEACH
MAIN	55.00'	23.40'	60.00'
STAYSAIL	36.30'	14.85'	33.40'
NO. 1 JIB	60.00	31.00'	37.50
NO 2 JIB	47.75	27.90	29.80
NO 3 JIB	34.50	17.95	21.75
GENOA	63.25	37.25	59.25
TRYSAIL	20.20	12.00	27.00
L.W. MAIN	55.00	24.00	65.00

HULL 284 SAIL PLAN

FIG 2

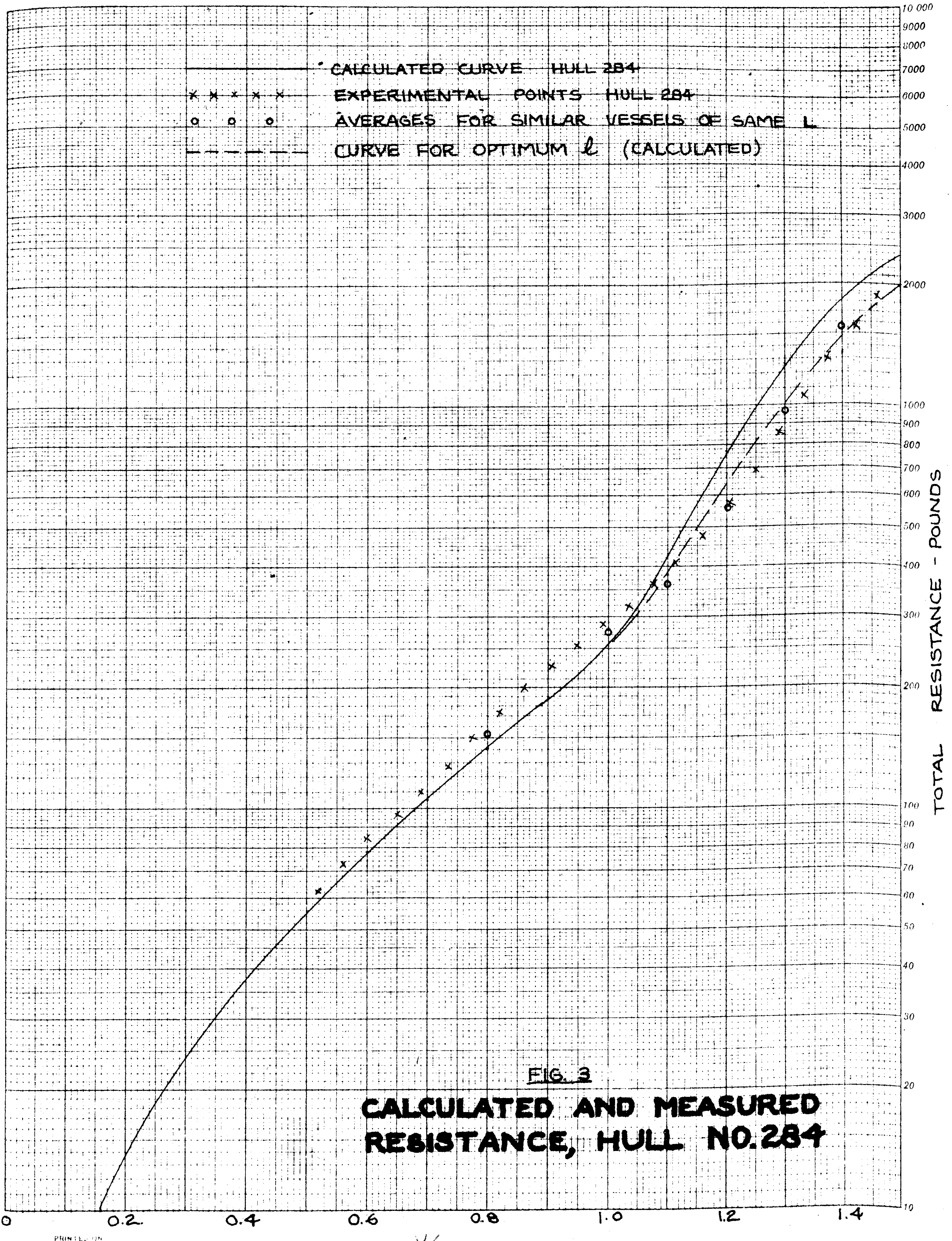


FIG. 3

CALCULATED AND MEASURED RESISTANCE, HULL NO. 284

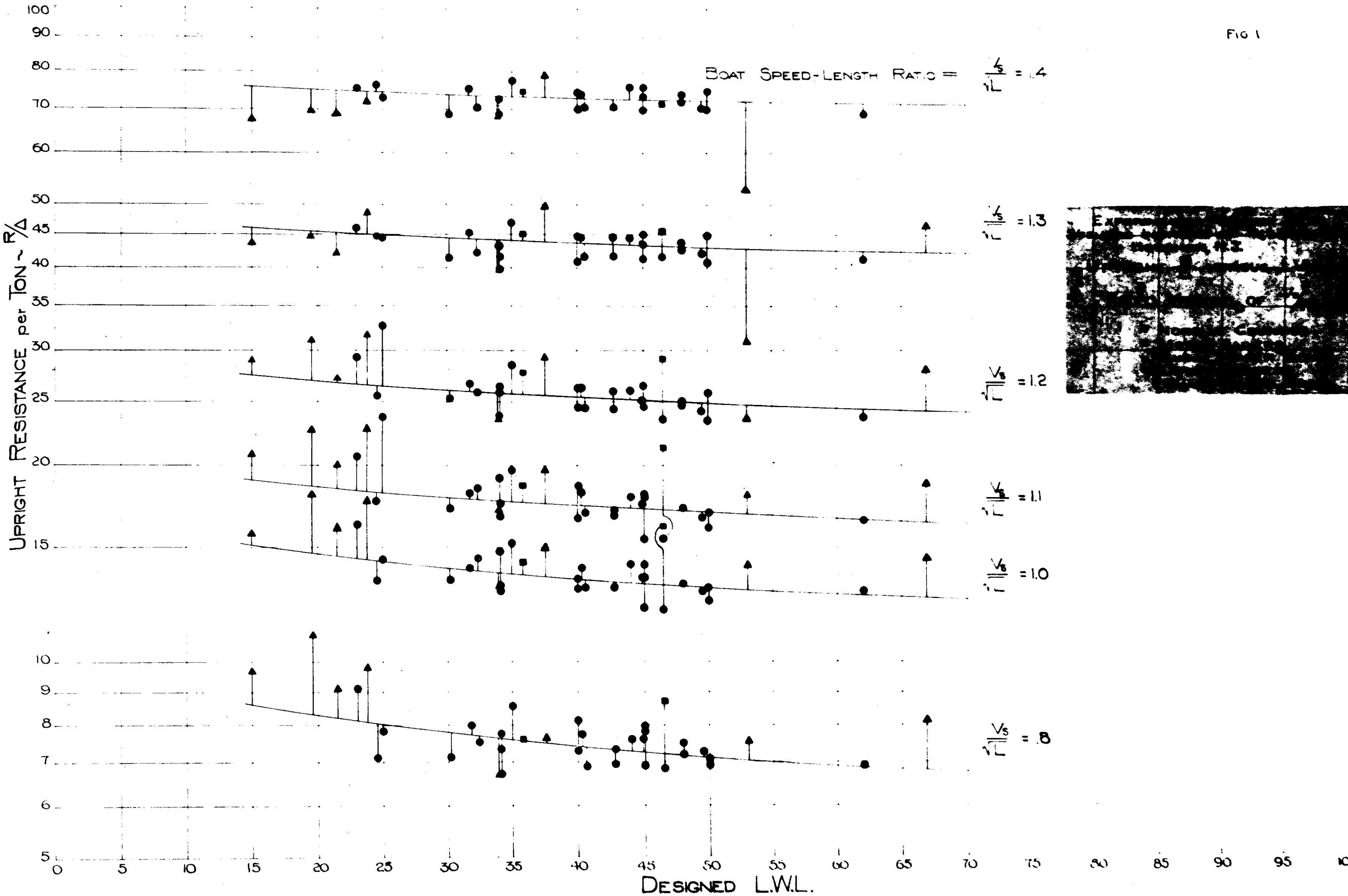
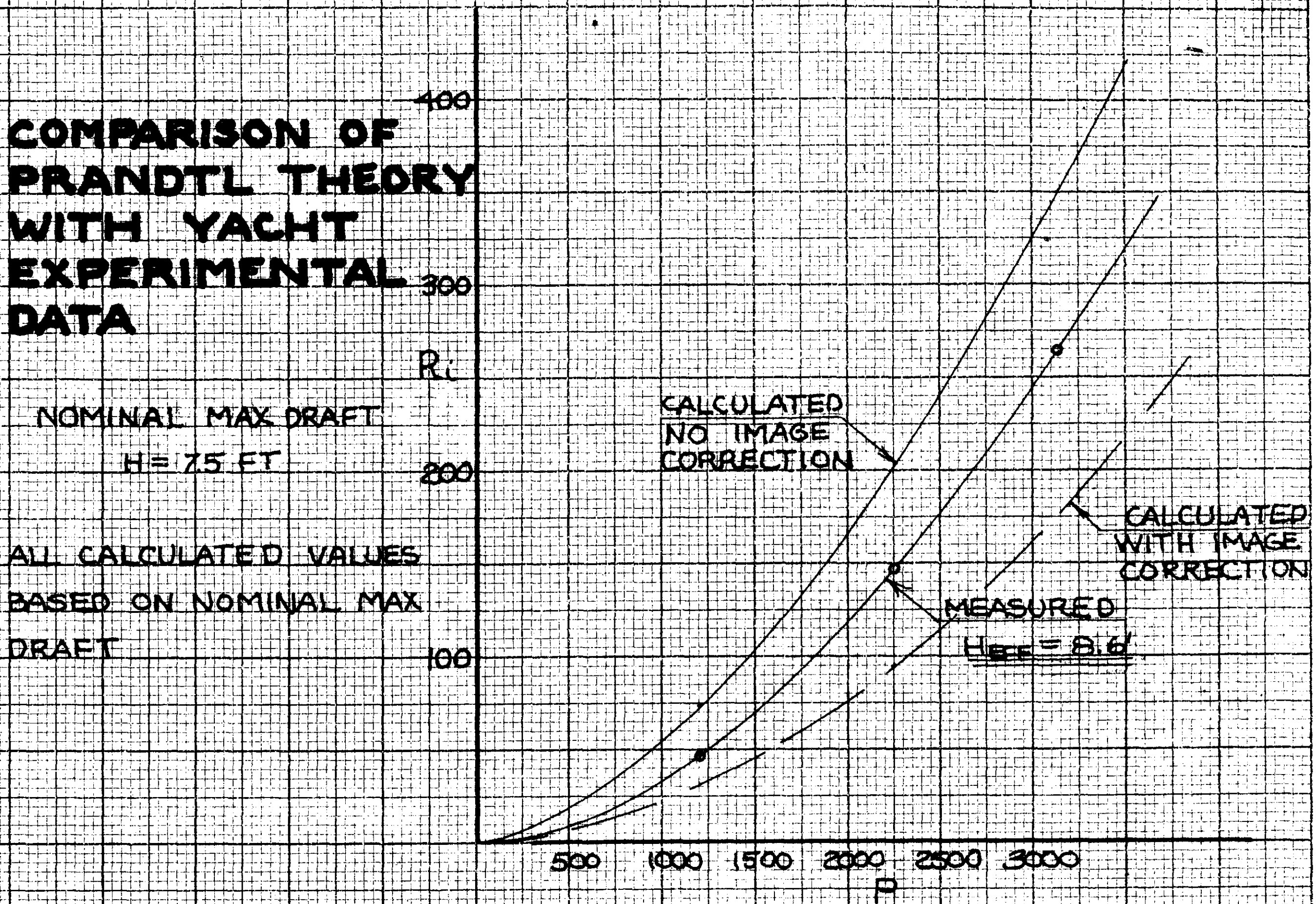
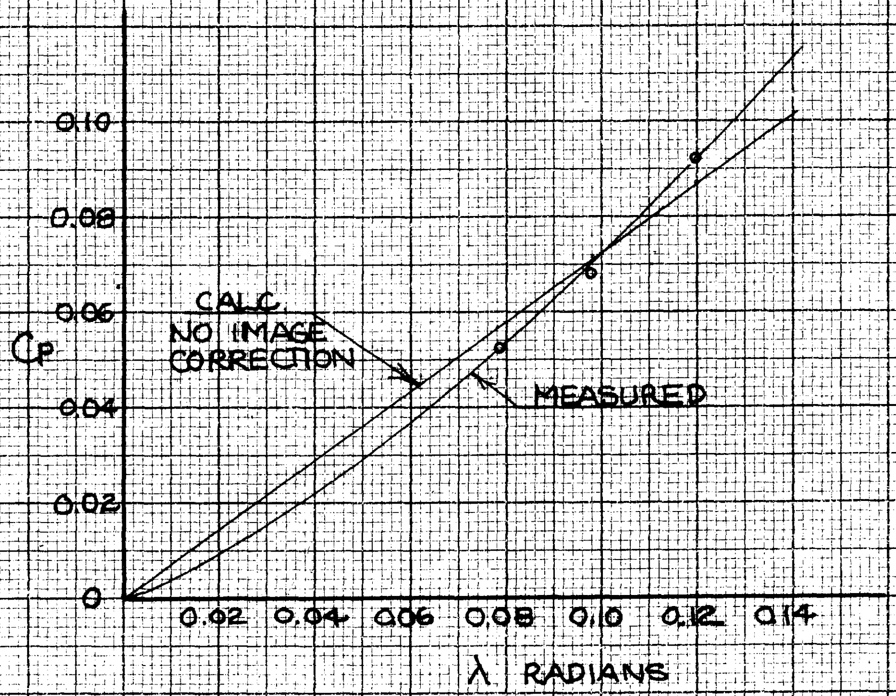


FIG 4

STEVENS STATISTICAL UPRIGHT RESISTANCES



COMPARISON OF
PRANDTL THEORY
WITH YACHT
EXPERIMENTAL
DATA
NOMINAL MAX DRAFT
H = 7.5 FT
ALL CALCULATED VALUES
BASED ON NOMINAL MAX
DRAFT
Ri
400
300
200
100
500
1000
1500
2000
2500
3000
P
CALCULATED
NO IMAGE
CORRECTION
CALCULATED
WITH IMAGE
CORRECTION
MEASURED
H_EFF = 8.6'



0.10
0.08
0.06
0.04
0.02
0
CP
0.02
0.04
0.06
0.08
0.10
0.12
0.14
λ RADIANS
CALC.
NO IMAGE
CORRECTION
MEASURED

FIG 5

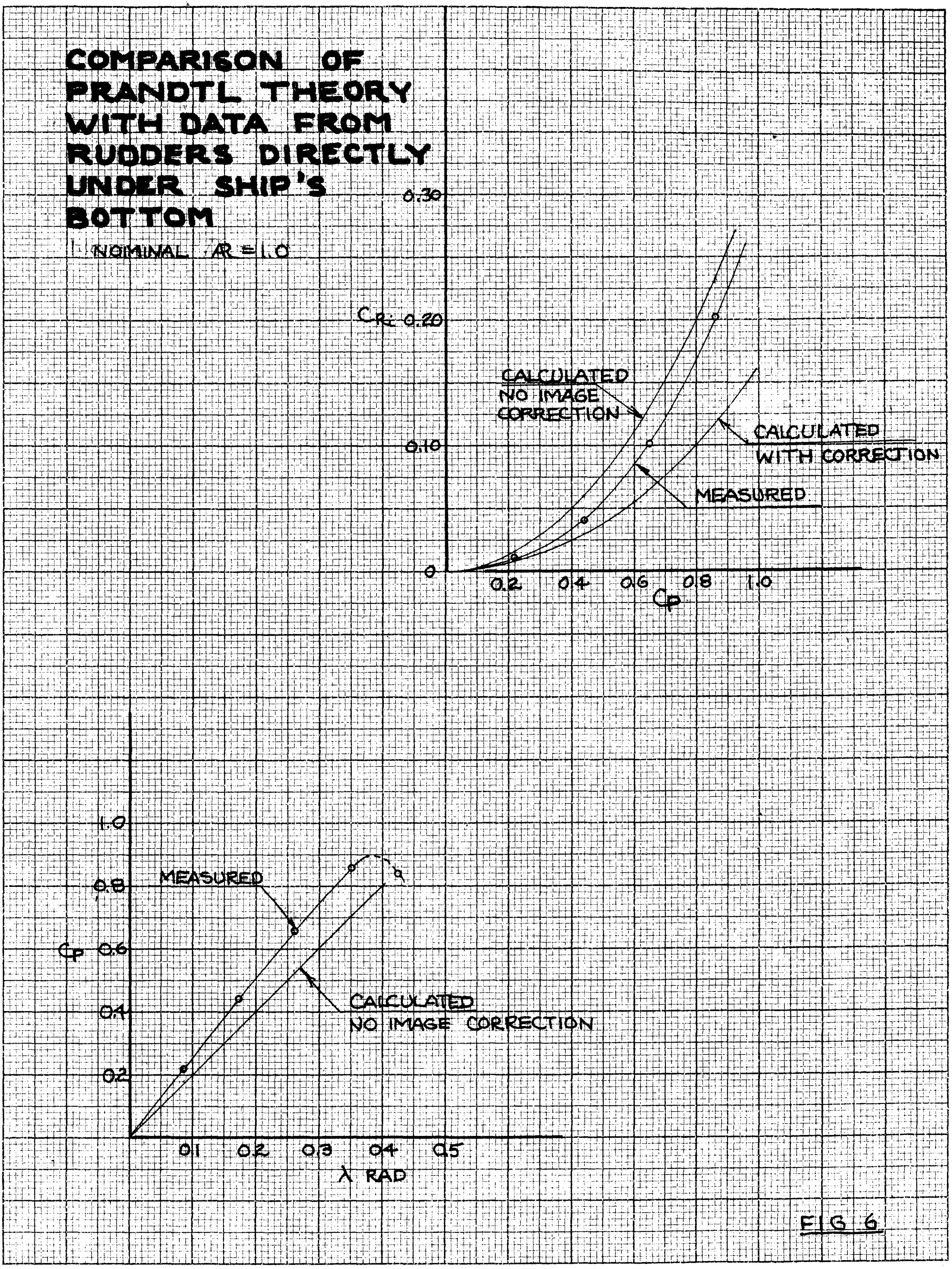

COMPARISON OF PRANDTL THEORY WITH DATA FROM RUDDERS DIRECTLY UNDER SHIP'S BOTTOM
NOMINAL AR = 1.0
C_{Ri}
0.30
0.20
0.10
0
0.2
0.4
0.6
0.8
1.0
C_P
CALCULATED NO IMAGE CORRECTION
CALCULATED WITH CORRECTION
MEASURED
1.0
0.8
0.6
0.4
0.2
C_P
MEASURED
CALCULATED NO IMAGE CORRECTION
0.1
0.2
0.3
0.4
0.5
λ RAD

FIG 6

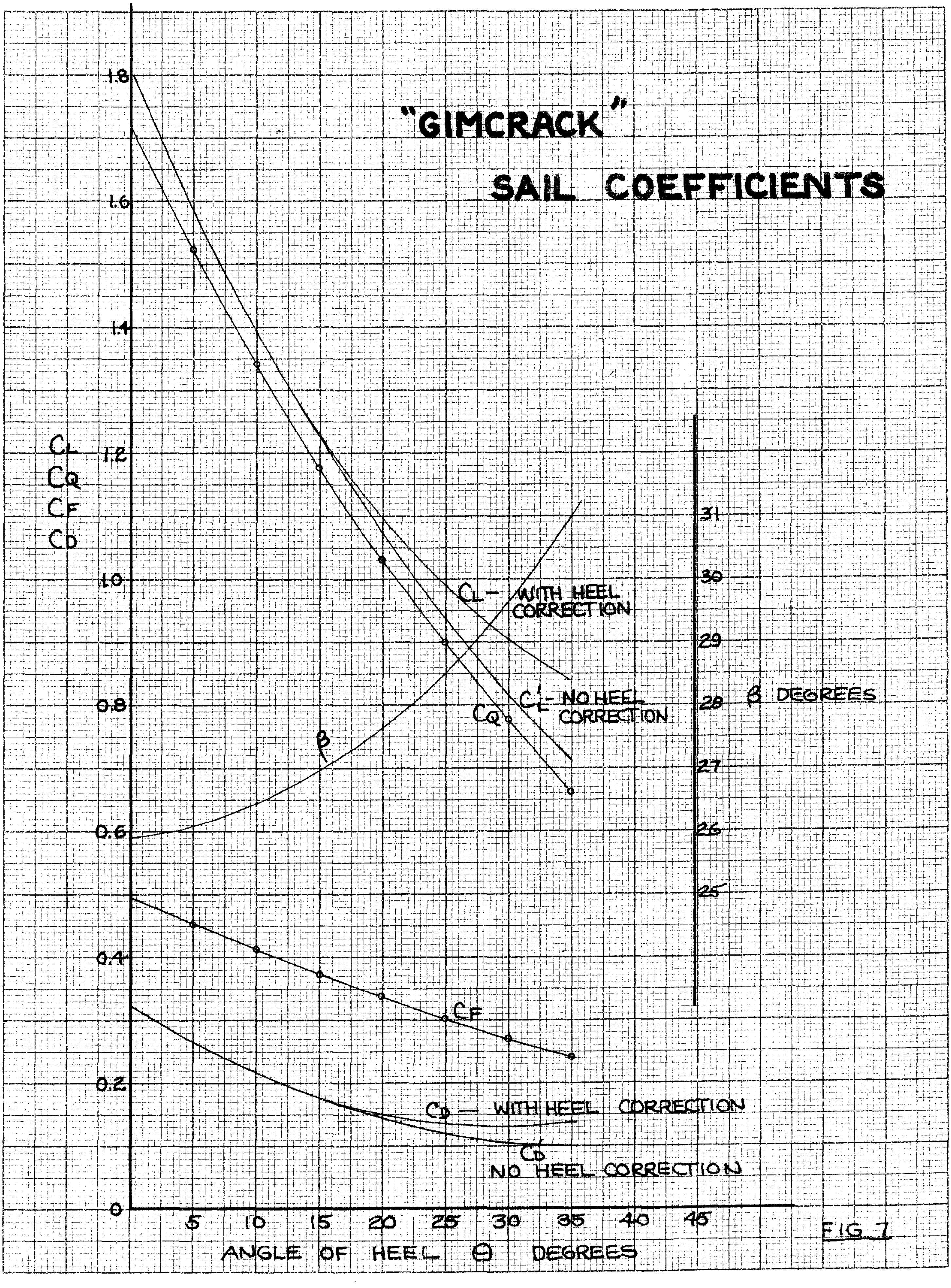

"GIMCRACK"
SAIL COEFFICIENTS
CL
CQ
CF
CD
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
CL - WITH HEEL CORRECTION
C'L - NO HEEL CORRECTION
CQ
β
31
30
29
28
27
26
25
β DEGREES
CF
CD — WITH HEEL CORRECTION
C'D
NO HEEL CORRECTION
5
10
15
20
25
30
35
40
45
ANGLE OF HEEL Θ DEGREES

FIG. 7

"GIMCRACK" POLAR

$C_{D_0} = 0.038 \quad AR_{EFF} = 3.44$

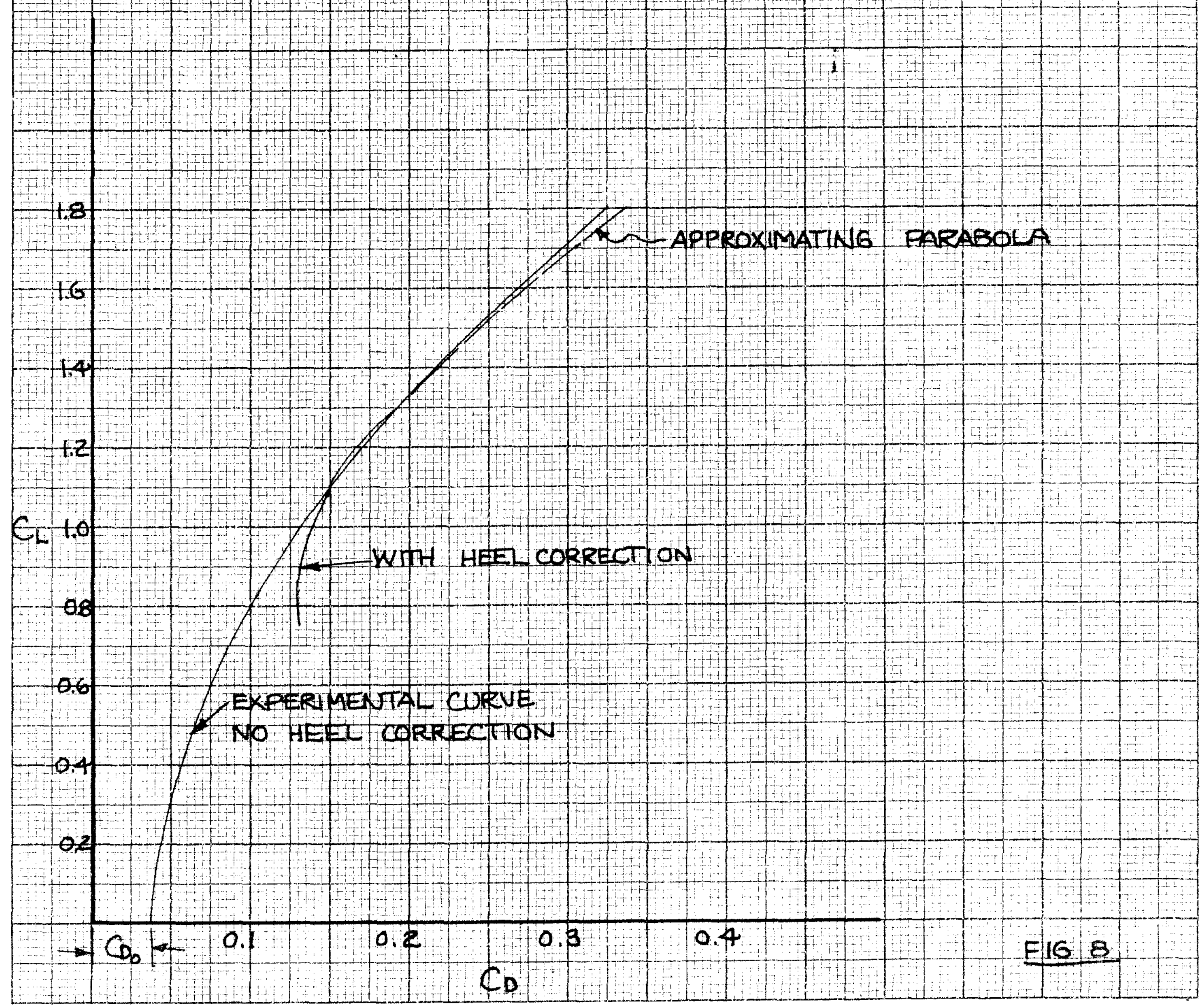


FIG 8

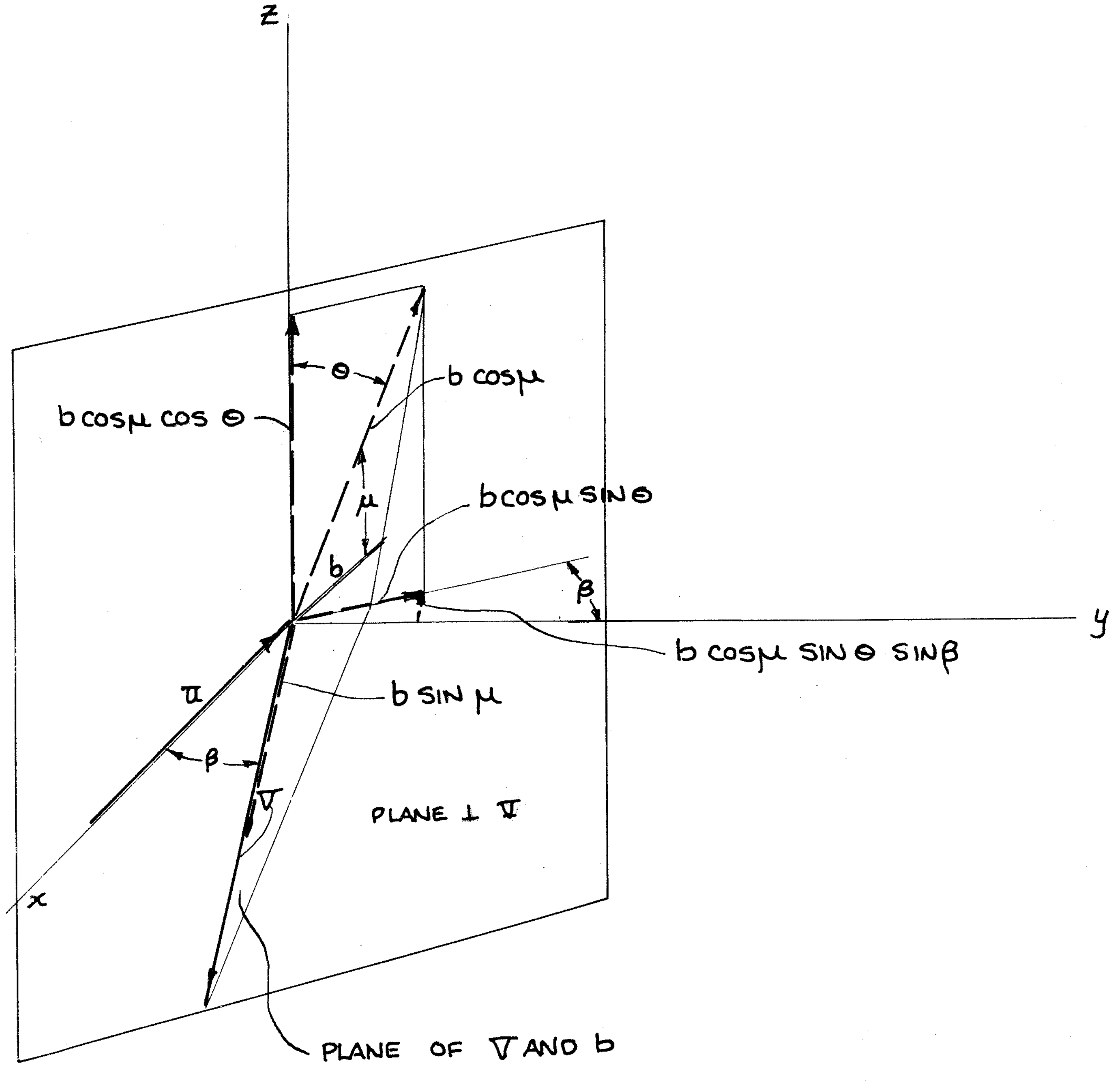


FIG 9

ASPECT RATIO CORRECTION DIAGRA

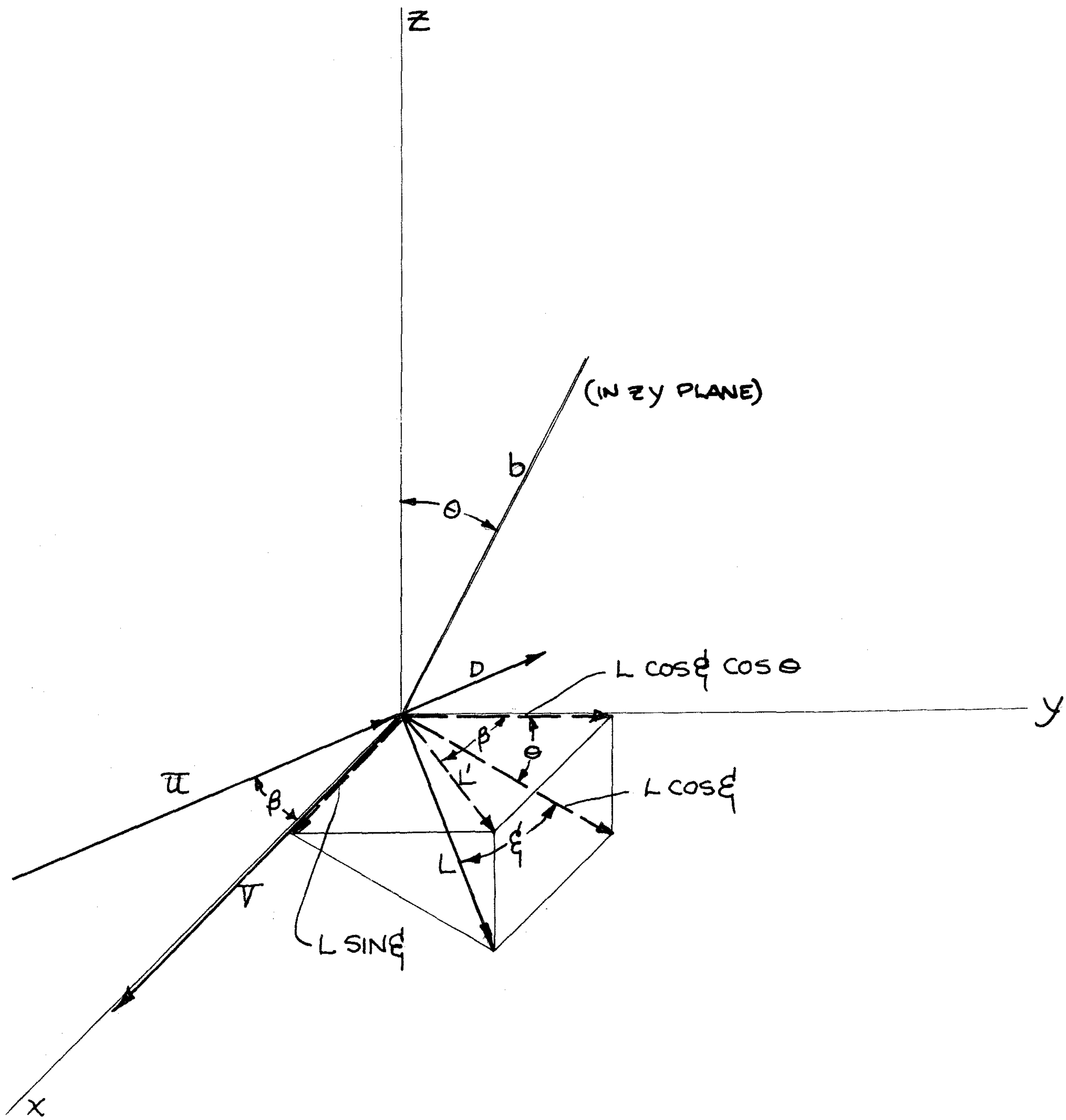


FIG 10

FORCE RESOLUTION DIAGRAM (SAIL)

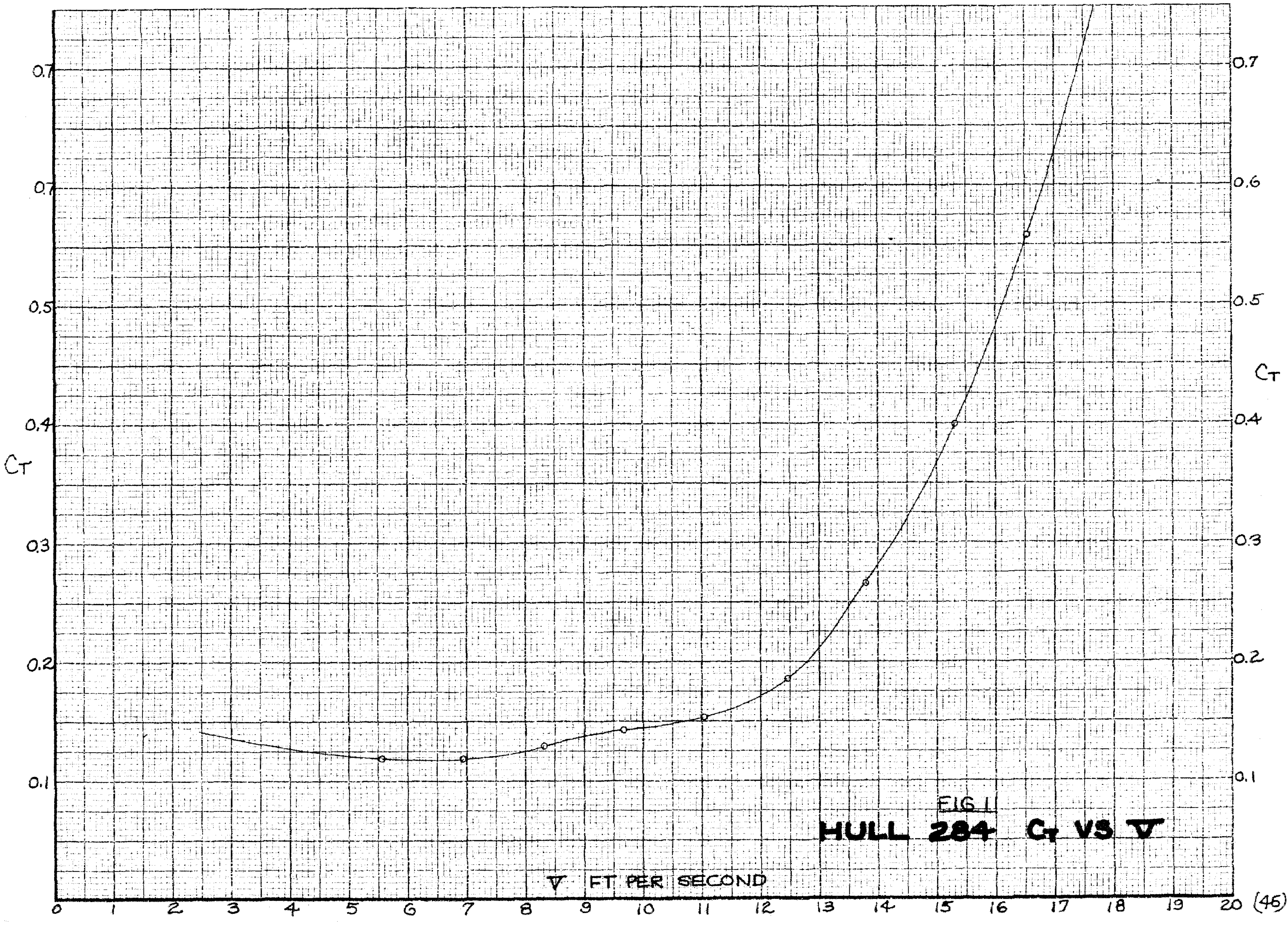


FIG 1
HULL 284 C_T VS V

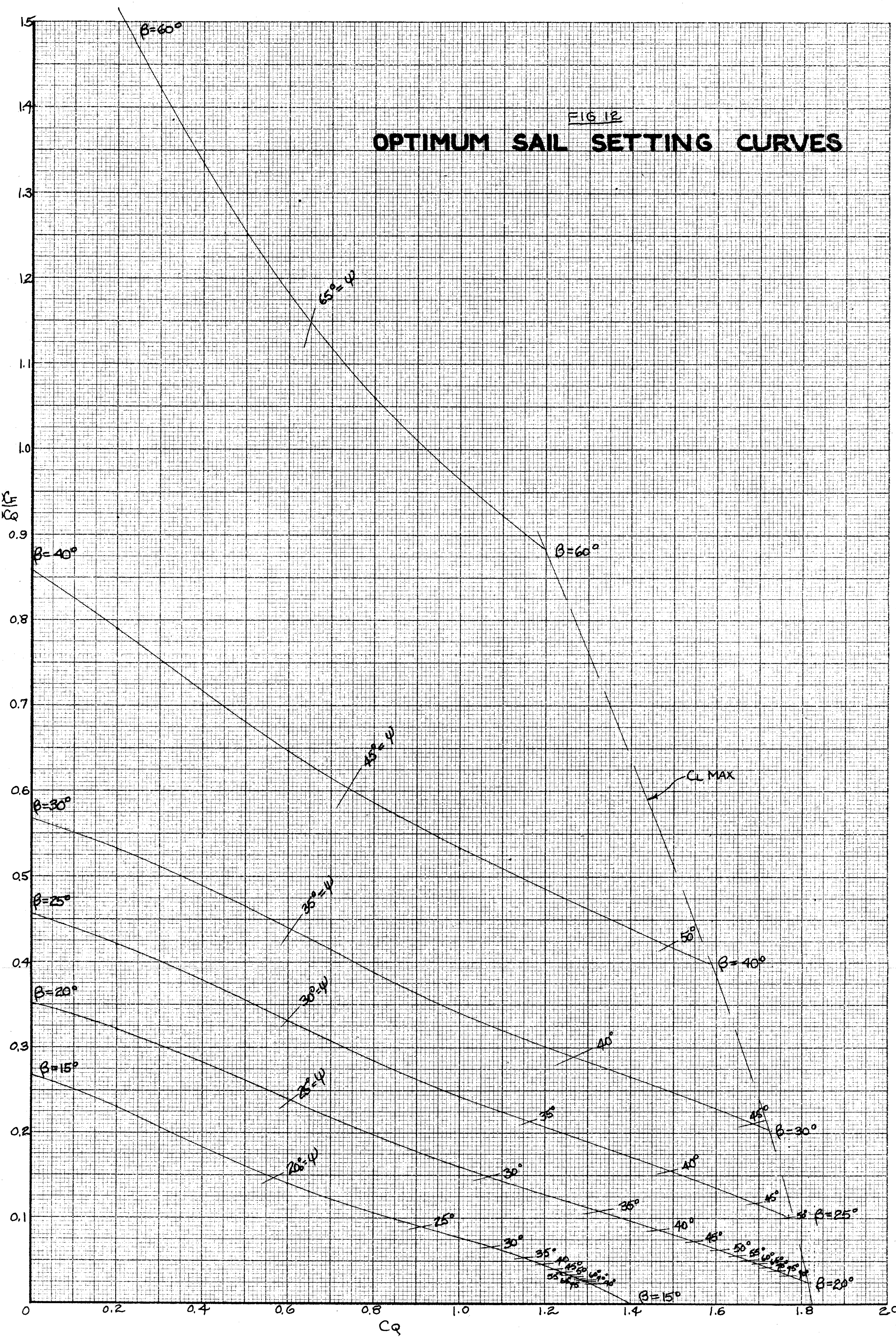


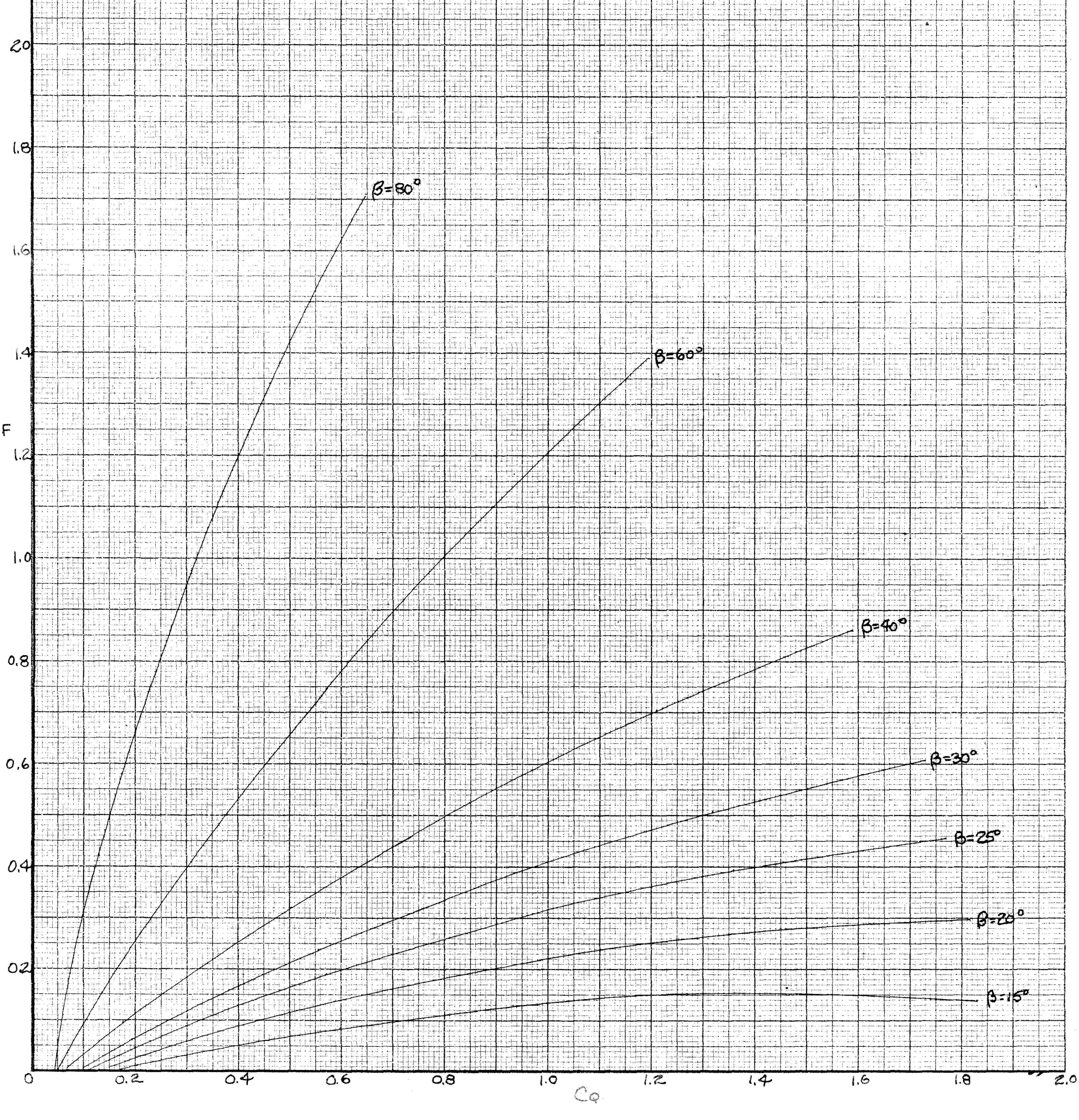
FIG 12

OPTIMUM SAIL SETTING CURVES

FIG 13

"GIMCRACK" C_F VERSUS C_Q

$C_{D_0} = 0.038 \quad AR = 3.44$



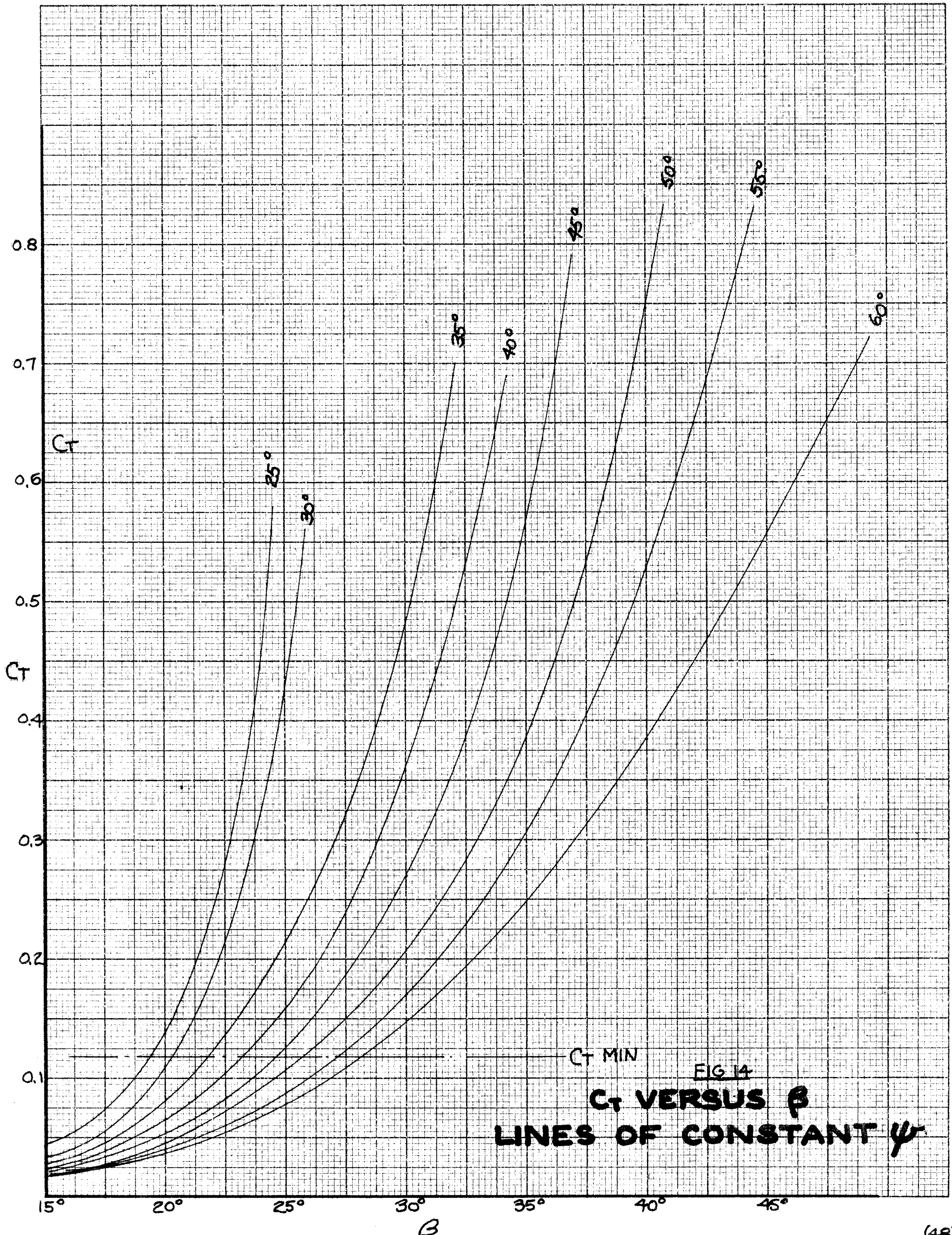


FIG 14

C_T VERSUS β
LINES OF CONSTANT ψ

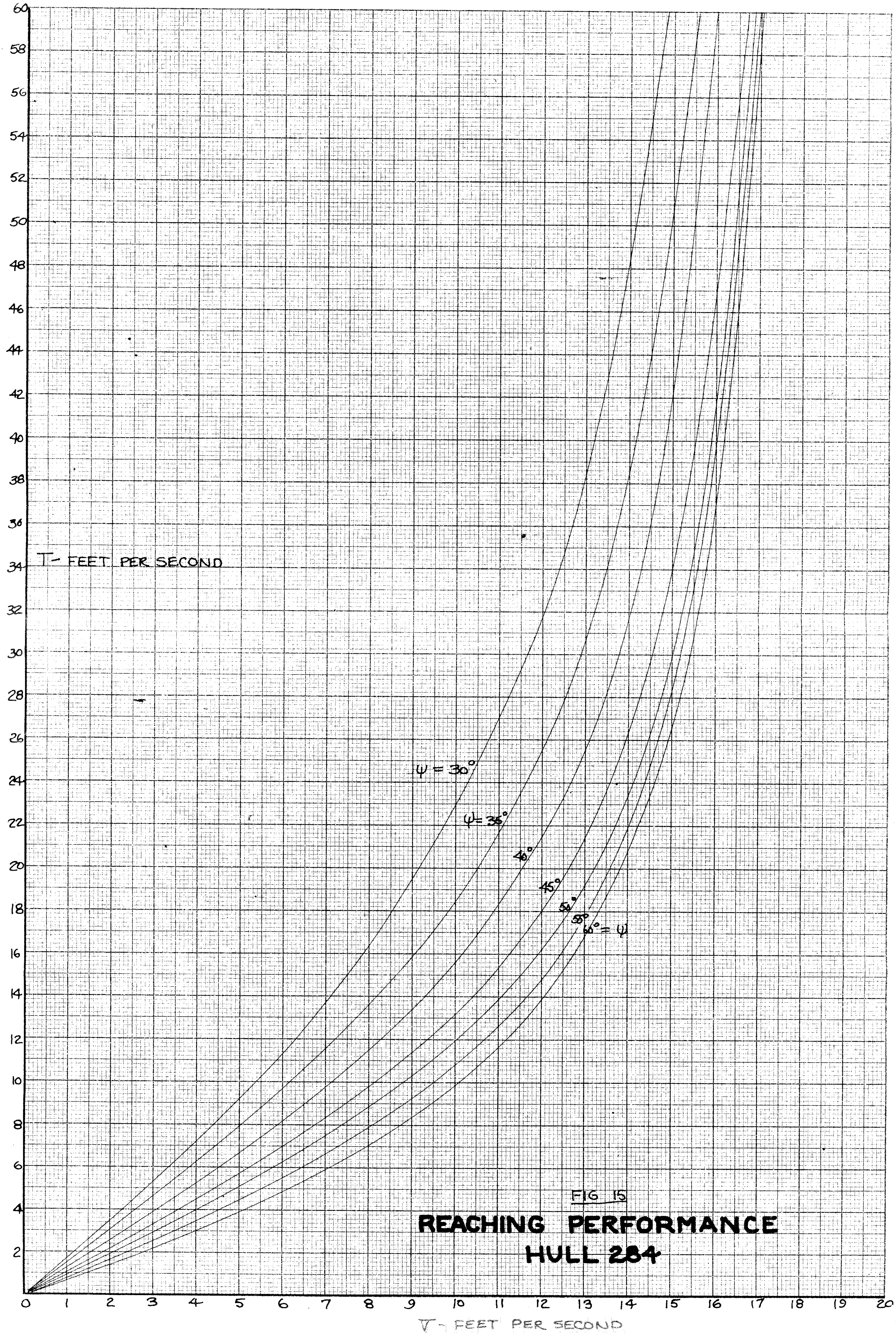


FIG 15

REACHING PERFORMANCE
HULL 284

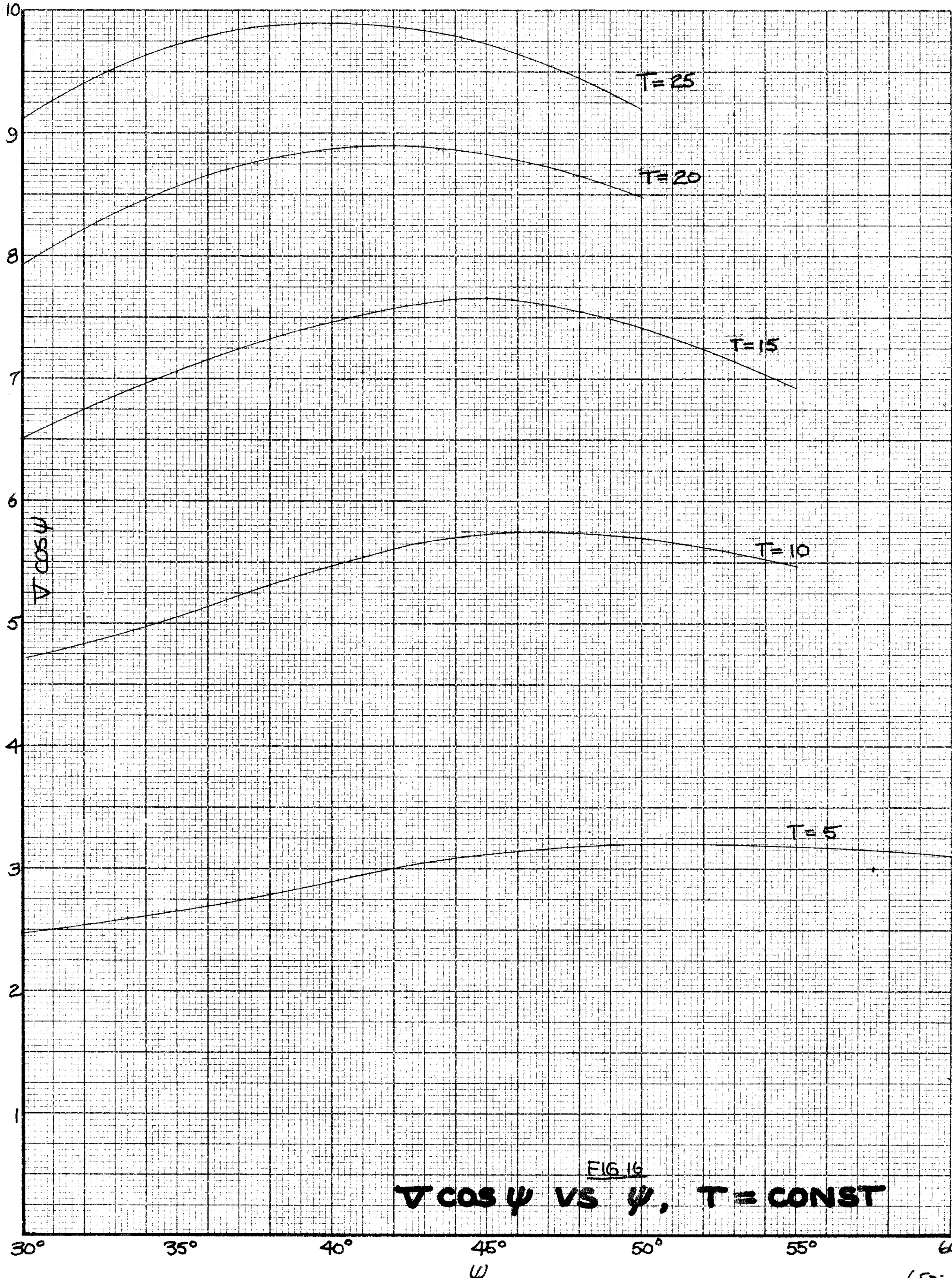


FIG 16

∇ COS ψ VS ψ, T = CONST

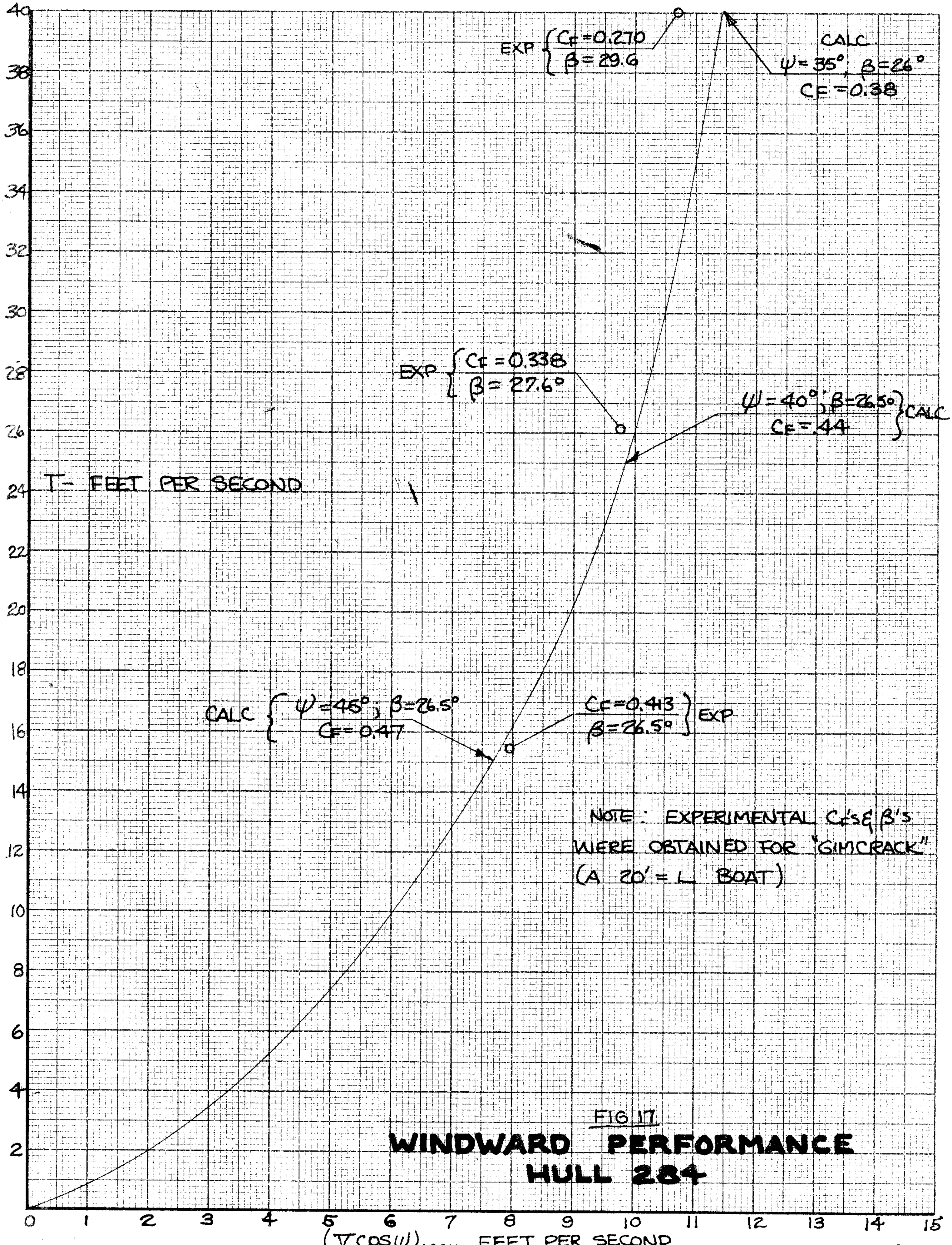


FIG 17

WINDWARD PERFORMANCE
HULL 284