THESIS

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TRANSIENT OSCILLATIONS IN ELASTIC SYSTEMS

CHAPTER I

GENERAL THEOREMS ON TRANSIENT OSCILLATIONS

TRANSIENT VIBRATIONS OF ELASTIC SYSTEMS

GENERAL PROPERTIES

We call elastic system any system in which the potential energy is a definite positive quadratic form of the coordinates. Such a system is generally realized if the forces are conservative and if we consider displacements in the neighborhood of a position of stable equilibrium. The theorem of reciprocity is verified and the relations between forces and displacements are linear.

Consider a continuous system. A volume force (XYZ) applied at the point (x'y'z') causes at the point (x y z) a displacement ($\xi\ \eta\ \zeta$).

$$\xi = \alpha_{11} X + \alpha_{12} Y + \alpha_{13} Z$$

$$\eta = \alpha_{21} X + \alpha_{22} Y + \alpha_{23} Z$$

$$\zeta = \alpha_{31} X + \alpha_{32} Y + \alpha_{33} Z$$

If the applied forces are harmonic and of same frequency and phase, the displacement has the same property. Taking in account the inertia force we get $X = X' \sin\omega t + \omega^2 \rho\, \xi \sin\omega t$ which gives three integral equations of the type,

$$\xi = \omega^2 \iiint \rho(\alpha_{11}\xi + \alpha_{12}\eta + \alpha_{13}\zeta)\, dx'dy'dz' + \iiint(\alpha_{11}X' + \alpha_{12}Y' + \alpha_{13}Z')\ldots$$

In order to simplify the notations we shall consider only the case where there is one displacement coordinate u

$$u(x) = \omega^2 \int_a^b \rho(t)\alpha(x\,t)\, u(t)\, dt + \int_a^b \alpha(x\,t) f(t)\, dt$$

From the reciprocity theorem $\alpha(x\,t) = \alpha(t\,x)$

The equation corresponding to the free oscillations is

$$u(x) = \omega^2 \int_a^b \rho(t)\,\alpha(x\,t)\,u(t)\,dt$$

The kernel is symmetrisable by putting $y(x) = u(x)\sqrt{\rho(x)}$

and the equation becomes,

$$y(x) = \omega^2 \int_a^b \alpha(x\,t)\sqrt{\rho(x)\rho(t)}\;y(t)\,dt$$

There exists an infinite number of positive characteristic values of ω^2 for which this equation has a solution. From Hilbert-Schmidt's theorem the solution of the non homogeneous equation

$$(1) \qquad y(x) = \omega^2 \int_a^b \alpha(x\,t)\sqrt{\rho(x)\rho(t)}\;y(t)\,dt + \int_a^b \alpha(x\,t)\,f(t)\,dt$$

may be expanded in a series absolutely and uniformly convergent of characteristic functions,

$$y = \sum A_i y_i$$

Substituting in equation (1)

$$\sum A_i y_i = \sum \frac{\omega^2}{\omega_i^2} A_i y_i + \int_a^b \alpha(x\,t)\,f(t)\,dt$$

multiplying then both sides by the orthogonal function y_i and integrating we get the required solution for the harmonic oscillation,

$$A_i = \frac{C_i}{\omega_i^2 - \omega^2} \qquad\qquad C_i = \frac{\int_a^b u_i(t) f(t)\,dt}{\int_a^b \rho\, u_i^2(t)\,dt}$$

$$u(x) = \sum \frac{C_i}{\omega_i^2 - \omega^2}\, u_i(x)$$

Hence an harmonic impulse $Bf(x)e^{i\omega t}$ causes an harmonic vibration.

$$v(x\,t) = B e^{i\omega t} \sum \frac{C_i}{\omega_i^2 - \omega^2} u_i(x)$$

In order to find the motion corresponding to a sudden applied distributed load f(x) we consider as before the integral

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\omega} d\omega = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

which leads to the solution

$$z(x\,t) = \sum \frac{C_i u_i(x)}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\omega(\omega_i^2 - \omega^2)} d\omega$$

$$z(x\,t) = - \sum \frac{C_i}{\omega_i^2} u_i(x) [\cos \omega_i t - 1]$$

It is interesting to note the order of magnitude of the coefficients C; they are generally of the order of $\frac{1}{\omega_i^2}$ because the characteristic functions u are asymptotic to the terms of a trigonometric Fourier series. Putting $C_i = \frac{B_i}{\omega_i^2}$ where B has an upper limit

$$z(x\,t) = - \sum \frac{B_i}{\omega_i^4} u_i(x) [\cos \omega_i t - 1]$$

The characteristic numbers w_i are of the order of the terms of the series of positive integers and therefore the convergence will generally be very fast.

If the loading of the system varies with time and is of the form $f(x)\,\psi(t)$ the motion may be expressed by

$$w(x\,t) = \int_0^t z[x, t-\tau] \frac{d\psi(\tau)}{d\tau} d\tau$$

$$w(x\,t) = \int_0^t \frac{\partial}{\partial \tau}[z(x, t-\tau)]\, \psi(\tau)\, d\tau$$

which by putting $\frac{B_i}{\omega_i^2} u_i(x) = K_i(x)$ takes the form

$$w(xt) = \sum K_i(x) \cos\omega_i t \int_0^t \sin\omega_i\tau\, \psi(\tau)\, d\tau - \sum K_i(x) \sin\omega_i t \int_0^t \cos\omega_i\tau\, \psi(\tau)\, d\tau$$

The Fourier integral gives

$$\psi(t) = \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^{+\infty} \psi(\tau) \cos\omega(\tau - t)\, d\tau$$

$$= \int_0^\infty d\omega \cos\omega t \cdot \frac{1}{\pi}\int_{-\infty}^{+\infty} \psi(\tau)\cos\omega\tau\, d\tau + \int_0^\infty d\omega \sin\omega t \cdot \frac{1}{\pi}\int_{-\infty}^{+\infty} \psi(\tau) \sin\omega\tau\, d\tau$$

$$= \int_0^\infty f_1(\omega)\cos\omega t\, d\omega + \int_0^\infty f_2(\omega) \sin\omega t\, d\omega$$

The density of energy of the impulse spectrum is,

$$\delta(\omega) = f_1^2(\omega) + f_2^2(\omega)$$

and the energy transmitted to each free oscillation of the system is proportional to

$$\omega_i^2 \int_c^b K_i^2(x)\, dx \cdot \delta(\omega_i)$$

Hence we get the following theorem:

When an impulse acts on an elastic system each free oscillation of frequency $\frac{\omega_i}{2\pi}$ receives an energy proportional to the product of a factor characteristic of that free oscillation and of the spatial loading distribution, by the density of energy of the impulse spectrum at the frequency $\frac{\omega_i}{2\pi}$

CHAPTER II

CALCULATION OF THE STRESSES OCCURRING IN A BUILDING DURING EARTHQUAKE

In order to give a clear idea of the method we shall at first suppose that the building has a simple structure and keep for the end the generalization to more complicated cases. Let it be of rectangular shape. The height of the first floor is generally greater than that of the others. The most important deformation is an horizontal shear as shown in fig. 1. This would naturally not be true for very high buildings where the bending would have to be taken into account. Furthermore the shearing rigidity and the mass of each floor is supposed to be constant from the second floor to the top. Only the first floor will be of a different rigidity.

Let h be the height of the building without the first floor, and x the coordinate counted downwards from the top as origin. We may consider this part of the building as an elastic continuous beam whose only possible deformation is shear. If M is its total mass, $m = \frac{M}{h}$ will be the mass per unit length. The number of upper floors being n and $h_1 = \frac{h}{n}$ their height, if we call K the force that is necessary to displace two consecutive floors so that the relative glide would be equal to the unit length, the coefficient of corresponding shearing rigidity of the continuous beam will be $\mu = \frac{K}{h_1}$. The rigidity of the first floor will be characterized by a coefficient G defined in the same way as K. (fig. 2.)

The ground is supposed to move horizontally with a variable acceleration j(t). The equation of relative motion of the beam is,

$$\mu \frac{\partial^2 u}{\partial x^2} = m \frac{\partial^2 u}{\partial t^2} + m j(t)$$

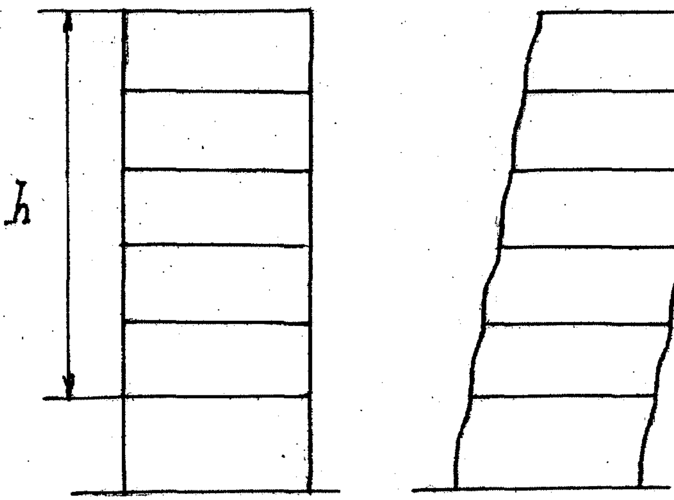


fig. 1.

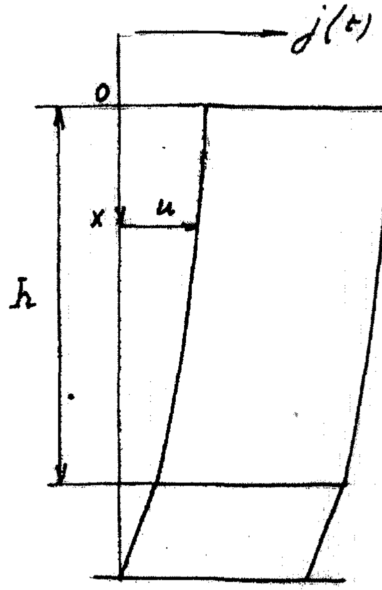


fig. 2.

We define the following notations:

$c = \sqrt{\frac{\mu}{m}}$ propagation speed of a shear wave,

$\frac{h}{c} = t_0$

$\frac{t}{t_0} = \tau$

$j(t) = j_0 \varphi(\tau)$

$\frac{x}{h} = \xi$

$\frac{u}{j_0 t_0^2} = y$

The equation of motion becomes

$$\frac{\partial^2 y}{\partial \xi^2} = \frac{\partial^2 y}{\partial \tau^2} + \varphi(\tau)$$

All the quantities figuring in this equation are dimensionless.

§ 1 FREE OSCILLATIONS

This motion is given by the homogeneous equation

$$\frac{\partial^2 y}{\partial \xi^2} = \frac{\partial^2 y}{\partial \tau^2}$$

which, by putting $y = z(\xi) e^{i\lambda\tau}$, becomes

$$\frac{d^2 z}{d\xi^2} + \lambda^2 z = 0$$

The general solution is

$$z = A \cos \lambda \xi + B \sin \lambda \xi$$

Consider the boundary conditions,

$$\frac{\partial u}{\partial x} = 0 \qquad x = 0$$

$$\mu \frac{\partial u}{\partial x} = -G u \qquad x = h$$

By putting $K = \frac{Gh}{\mu}$, ratio of the rigidity of the first floor to that of the others, and $\alpha = Rn$ these conditions take the form

$$\frac{dz}{d\xi} = 0 \qquad \xi = 0 \qquad (1)$$

$$\frac{dz}{d\xi} + \alpha z \qquad \xi = 1 \qquad (2)$$

From equation (1) $B = 0$ and $z = A \cos \lambda \xi$

From condition (2) $\lambda \, tg \, \lambda = \alpha$

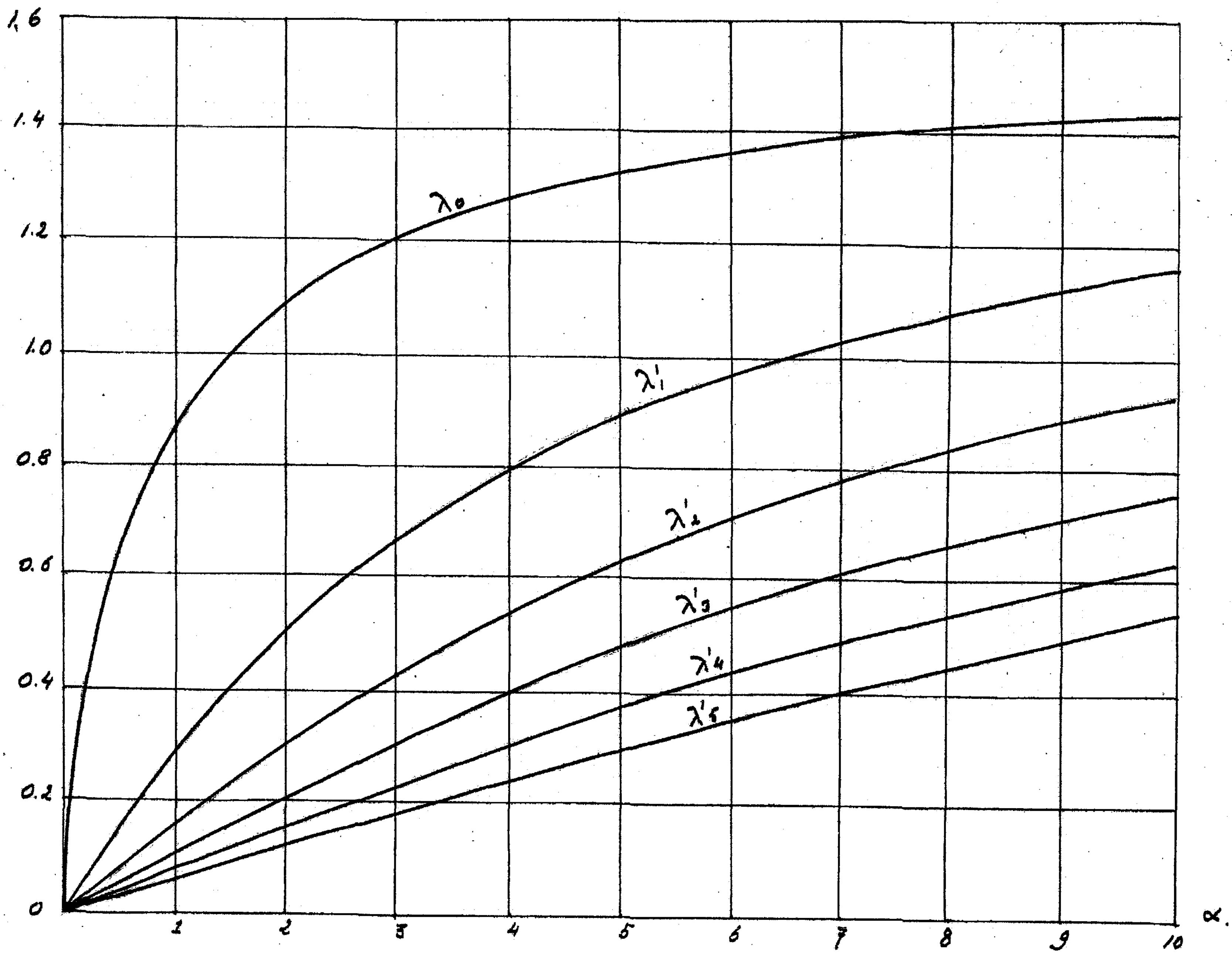
The roots λ_k of this equation correspond to the free oscillation frequencies of the building. We choose certain values of α corresponding to certain simple values of R and n as follows:

n \ R	1/9	1/6	1/3	1
15	1.66	2.50	5	
10	1.11	1.66	3.33	10
5	0.556	0.834	1.66	

The values of λ_k as a function of α are given in the following table:

α	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
0	0	3.14	6.28	9.42	12.56	15.70
0.556	0.68	3.31	6.37	9.48	12.60	15.73
0.834	0.80	3.38	6.41	9.51	12.62	15.75
1.11	0.89	3.45	6.45	9.54	12.65	15.77
1.66	1.03	3.58	6.53	9.59	12.69	15.80
2.50	1.15	3.73	6.65	9.67	12.76	15.85
3.33	1.23	3.86	6.74	9.75	12.82	15.91
5.00	1.32	4.04	6.91	9.90	12.93	16.0
10.0	1.44	4.30	7.22	10.13	13.20	16.24
∞	1.57	4.72	7.85	11.0	14.12	17.30

In fig. 3 are plotted the values of λ'_k where $\lambda_k = k\pi + \lambda'_k$



1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
1
2
3
4
5
6
7
8
9
10
α.
λ_0
λ'_1
λ'_2
λ'_3
λ'_4
λ'_5

fig 3.

The period T_k corresponding to λ_k is,

$$T_k = \frac{2\pi t_o}{\lambda_k}$$

It is interesting to compare the fundamental period T_0 to that T_0' which would occur if the building would be perfectly rigid from the second floor to the top, the only elasticity being due to the first floor. We get

$$M \frac{d^2u}{dt^2} + Gu = 0 , \qquad T_o' = 2\pi\sqrt{\frac{M}{G}} = 2\pi \frac{t_o}{\sqrt{\alpha}}$$

The ratio of frequencies $\frac{f_o}{f_o'} = \frac{T'}{T_o'} = \frac{\lambda_o}{\alpha}$ is a function of α

α	f_o/f_o'
0	1
0.556	0.910
0.834	0.875
1.11	0.844
1.66	0.800
2.50	0.725
3.33	0.674
5	0.590
10	0.455
∞	0

§ 2. FORCED HARMONIC OSCILLATION

In this case $\varphi(\tau) = e^{i\lambda\tau}$, and the equation of motion becomes

$$\frac{\partial^2 y}{\partial \xi^2} = \frac{\partial^2 y}{\partial t^2} + e^{i\lambda\tau} ,$$

Which, by putting $y = z(\xi) e^{i\lambda\tau}$, takes the form

$$\frac{d^2 z}{d\xi^2} + \lambda^2 z = 1$$

The solution of this equation can be expressed as a sum of the orthogonal functions $z_k = \cos\lambda_k \xi$ satisfying the corresponding homogeneous equation and the given boundary conditions

$$z = \sum_0^\infty A_K z_K$$

Carrying this expression into the differential equation

$$\sum_0^\infty \left[A_i \frac{d^2 z_i}{d\xi^2} + \lambda^2 A_i z_i\right] = 1$$

Taking into account the identity,

$$\frac{d^2 z_i}{d\xi^2} = -\lambda_i^2 z_i$$

If we multiply both sides of that equation by z_K and integrate from 0 to 1 with respect to ξ , we get

$$\sum_0^\infty A_i\left[\lambda^2 - \lambda_i^2\right]\int_0^1 z_i z_K \, d\xi = \int_0^1 z_K \, d\xi$$

The condition of orthogonality $\int_0^1 z_K z_i \, d\xi = 0$ for $K \neq i$ gives

$$A_K = \frac{\sigma_K}{\lambda^2 - \lambda_K^2} \quad , \qquad \sigma_K = \frac{\int_0^1 z_K \, d\xi}{\int_0^1 z_K^2 \, d\xi}$$

The required solution is

$$z = \sum_0^\infty \frac{\sigma_K}{\lambda^2 - \lambda_K^2} z_K \ .$$

The value of σ_K may be given more explicitly

$$\sigma_K = 2\alpha \frac{\beta_K}{\lambda_K^2} \quad , \qquad \beta_K = \frac{\cos\lambda_K}{1 + \alpha \frac{\cos^2\lambda_K}{\lambda_K^2}}$$

$$z = 2\alpha \sum_0^\infty \frac{\beta_K}{\lambda_K^2} \frac{z_K}{\lambda^2 - \lambda_K^2}$$

In this series the values of the coefficients β_K tend to unity when K increases indefinitely and the convergence is absolute and uniform.

§3. EFFECT OF A SUDDEN CONSTANT ACCELERATION

With the purpose of finding a more general solution we shall use the preceding results for building a solution which corresponds to

$$\varphi(\tau) = 0 \qquad \tau < 0$$

$$\varphi(\tau) = 1 \qquad \tau > 0$$

Consider the integral in the complex plane,

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\lambda\tau}}{\lambda}\,d\lambda,$$

taken along the real axis (fig. 4). For $\tau<0$ the real part of $i\lambda\tau$ is negative on the half circle of infinite radius ADC. Hence we may add this path to the contour of integration without any effect on the value of the integral. For $\tau<0$ it can then be written

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\lambda\tau}}{\lambda}\,d\lambda=\frac{1}{2\pi i}\oint_{ACDA}\frac{e^{i\lambda\tau}}{\lambda}\,d\lambda=0$$

For the same reason, when $\tau>0$ we may add the path ABC and

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\lambda\tau}}{\lambda}\,d\lambda=\frac{1}{2\pi i}\oint_{ACBA}\frac{e^{i\lambda\tau}}{\lambda}\,d\lambda=1$$

The function $\varphi(\tau)$ defined by

$$\varphi(\tau)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\lambda\tau}}{\lambda}\,d\lambda$$

is discontinuous at the origin where it jumps from 0 to 1 and is equal to a constant anywhere else. (fig. 5). The solution corresponding to each element of this integral is

$$d\zeta=2\alpha\sum_{0}^{\infty}\frac{\beta_K}{\lambda_K^2(\lambda^2-\lambda_K^2)}\,z_K\,\frac{e^{i\lambda\tau}}{\lambda}\,d\lambda$$

Integrating term by term, we get the required solution in the form of contour integrals

$$\zeta=2\alpha\sum_{0}^{\infty}\frac{\beta_K z_K}{\lambda_K^2}\cdot\frac{1}{2\pi i}\oint_{ACBA}\frac{e^{i\lambda\tau}}{\lambda(\lambda^2-\lambda_K^2)}\,d\lambda,$$

and by taking the residues

$$\frac{1}{2\pi i}\oint_{ACBA}\frac{e^{i\lambda\tau}}{\lambda(\lambda^2-\lambda_K^2)}\,d\lambda=\frac{1}{2\pi i\,\lambda_K^2}\oint\left[\frac{1}{2}\left(\frac{e^{i\lambda\tau}}{\lambda+\lambda_K}+\frac{e^{i\lambda\tau}}{\lambda-\lambda_K}\right)-\frac{e^{i\lambda\tau}}{\lambda}\right]d\lambda$$

$$=\frac{1}{\lambda_K^2}\left[\cos\lambda_K\tau-1\right],$$

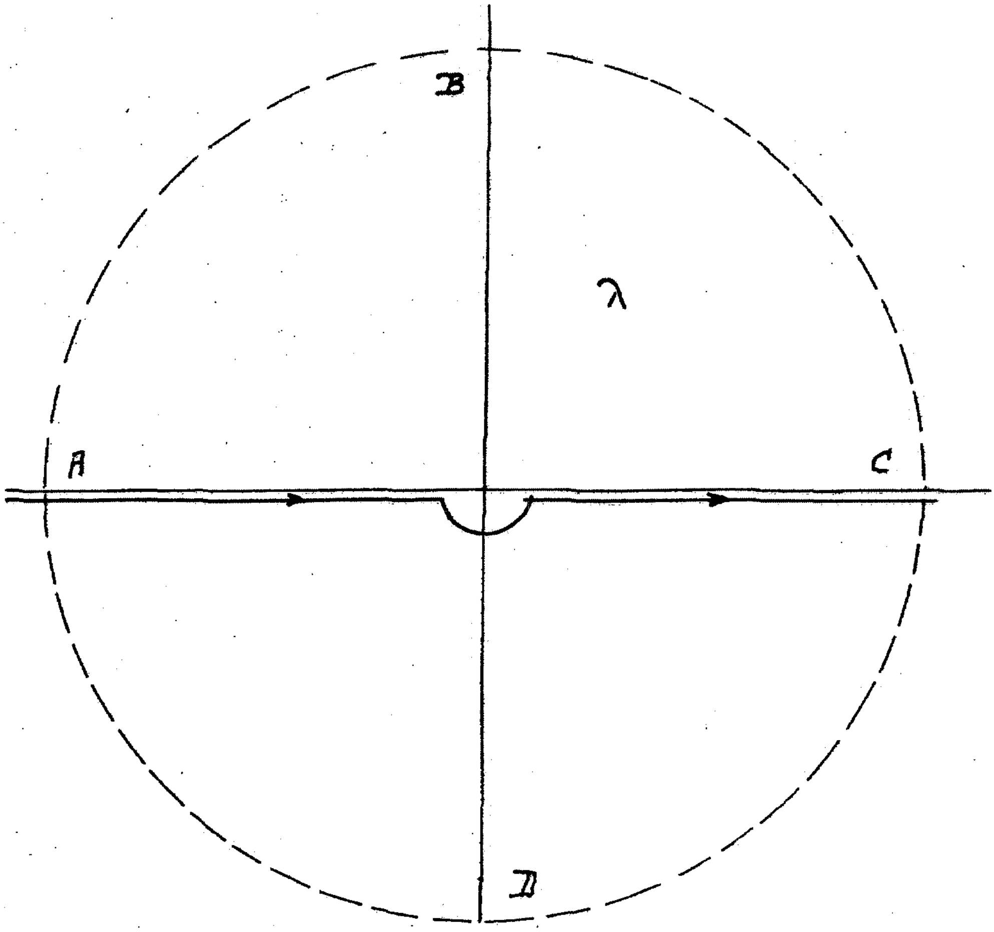


fig 4.

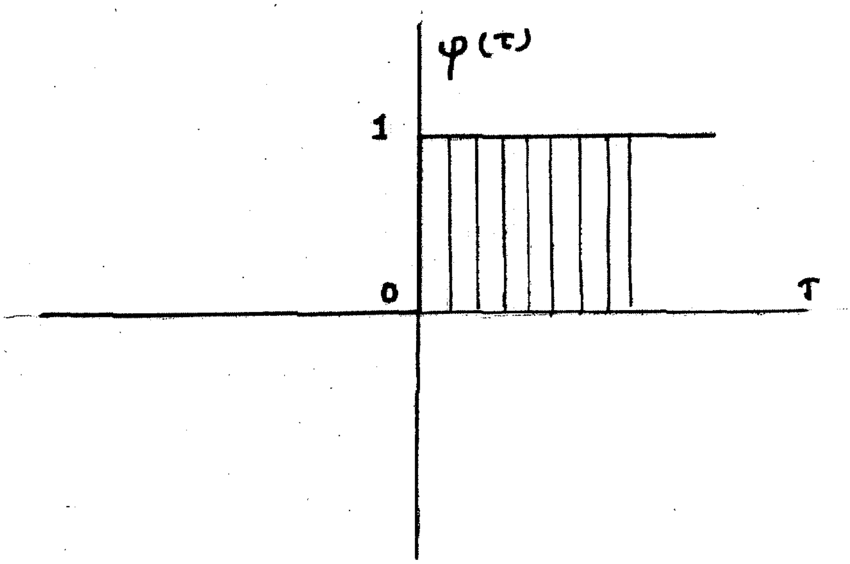


fig. 5.

we obtain finally $\quad \zeta = 2\alpha \sum_0^\infty \frac{\beta_k z_k}{\lambda_k^4}\left[\cos\lambda_k\tau - 1\right]$

By putting $B_k = \dfrac{2\alpha\beta_k}{\lambda_k^4}$,

$$\zeta = \sum_0^\infty B_k \cos\lambda_k\xi\left[\cos\lambda_k\tau - 1\right].$$

Let us study the convergence of this series and compute the order of magnitude of the rest. Each of its terms, the first one excepted, satisfies the inequality,

$$\left|\frac{\beta_k z_k(\cos\lambda_k\tau - 1)}{\lambda_k^4}\right| < \frac{2}{\lambda_k^4} < \frac{2}{(\pi k)^4}$$

The following inequality gives an upper limit for the rest,

$$|R_p| < \sum_p^\infty\left|\frac{\beta_k z_k(\cos\lambda_k\tau - 1)}{\lambda_k^4}\right| < \frac{2}{\pi^4}\sum_p^\infty \frac{1}{k^4}$$

The value of $\sum_1^\infty \frac{1}{k^4}$ is given by the Bernoulli number B_2,

$$\sum_1^\infty \frac{1}{k^4} = \frac{\pi^4}{90} = 1{,}080$$

On the other hand,

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} = 1.074,$$

so that $\sum_4^\infty \frac{1}{k^4} = 0{,}006$, and finally $|R_4| < \frac{2}{\pi^4}\times 0.006$

Hence if we take only the four initial terms of the series

$$\zeta = \sum_0^3 B_k \cos\lambda_k\xi\left[\cos\lambda_k\tau - 1\right],$$

the error due to the terms neglected will be smaller than

$$\frac{4\alpha}{\pi^4}\times 0.006 = \alpha \times 0.00025.$$

We also see that this series is absolutely and uniformly convergent.

Displacement

The dimensionless function ζ corresponds to a sudden acceleration j_0. The real displacement is

$$u = j_0 t_0^2 \zeta$$

The values of B_k are only functions of α, and are given in the following table:

α	B_0	B_1	B_2	B_3
0.	$\infty\left(\frac{1}{\alpha}\right)$	0	0	0
0.556	2.336	-0.00890	0.000692	-0.000138
0.834	1.734	-0.0116	0.000970	-0.000208
1.11	1.425	-0.0137	0.00122	-0.000272
1.66	1.091	-0.0167	0.00177	-0.000386
2.50	0.902	-0.0191	0.00234	-0.000544
3.33	0.802	-0.0203	0.00274	-0.000682
5	0.710	-0.0210	0.00334	-0.000912
10	0.612	-0.0212	0.00404	-0.00125
∞	0.518	-0.0192	0.00416	-0.00150

The maximum displacement takes place at the top, and its value is given very accurately by

$$u_0 = 2 j_0 t_0^2 B_0$$

It is practically twice the statical deformation that would occur under the same acceleration.

Stresses

The total stress due to the shearing deformation is

$$F = \mu \frac{\partial u}{\partial x} = j_0 M \frac{\partial \zeta}{\partial \xi}$$

It is the total inertia force $j_0 M$ multiplied by a dimensionless function

$$\frac{\partial \zeta}{\partial \xi} = -\tau\alpha \sum_0^\infty \frac{\beta_k}{\lambda_k^3} \sin \lambda_k \xi \left[\cos \lambda_k \tau - 1\right]$$

which, by putting $B'_k = -\frac{2\alpha\beta_k}{\lambda_k^3} = -\lambda_k B_k$ takes the form,

$$\frac{\partial \zeta}{\partial \xi} = \sum_0^\infty B'_k \sin \lambda_k \xi \left[\cos \lambda_k \tau - 1\right].$$

The values of the B'_k as functions of α are given in the following table

α	B'_0	B'_1	B'_2	B'_3
0		0	0	0
0.556	-1.588	0.0294	-0.00440	0.00131
0.834	-1.388	0.0394	-0.00622	0.00198
1.11	-1.268	0.0473	-0.00788	0.00260
1.66	-1.123	0.0598	-0.0115	0.00370
2.50	-1.036	0.0712	-0.0155	0.00526
3.33	-0.987	0.0774	-0.0185	0.00665
5	-0.937	0.0850	-0.0235	0.00912
10	-0.880	0.0912	-0.0291	0.0127
	-0.813	0.900	-0.0325	0.0165

As previously, we can easily compute the rest of this series. We encounter the expression $\sum_1^\infty \frac{1}{k^3} = 1.20205$ (Stieltjes, Act. Math. vol. 10, p. 299). But $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} = 1.162$. Hence

$$\sum_0^\infty \frac{1}{k^3} = 0.04$$

The error made by taking only the four first terms is smaller than

$$\frac{4\alpha}{\pi^3} \times 0.04 = 0,0055\,\alpha$$

The maximum possible value of $\frac{\partial \zeta}{\partial \xi}$ is

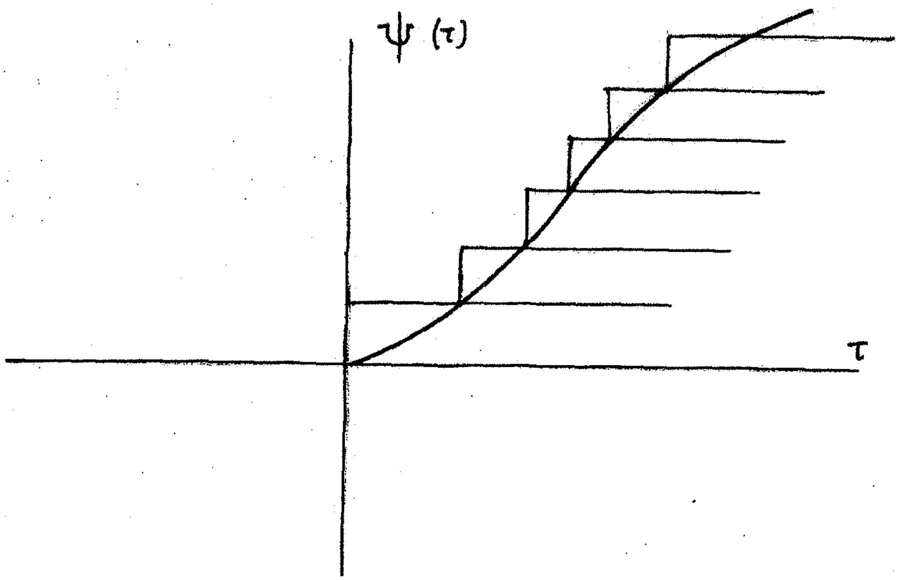
$$\left|\frac{\partial \zeta}{\partial \xi}\right|_{max} = 2\sum_{k=0}^{\infty} \left| B'_k \sin \lambda_k \right| = 2S$$

The following table gives the values of these terms:

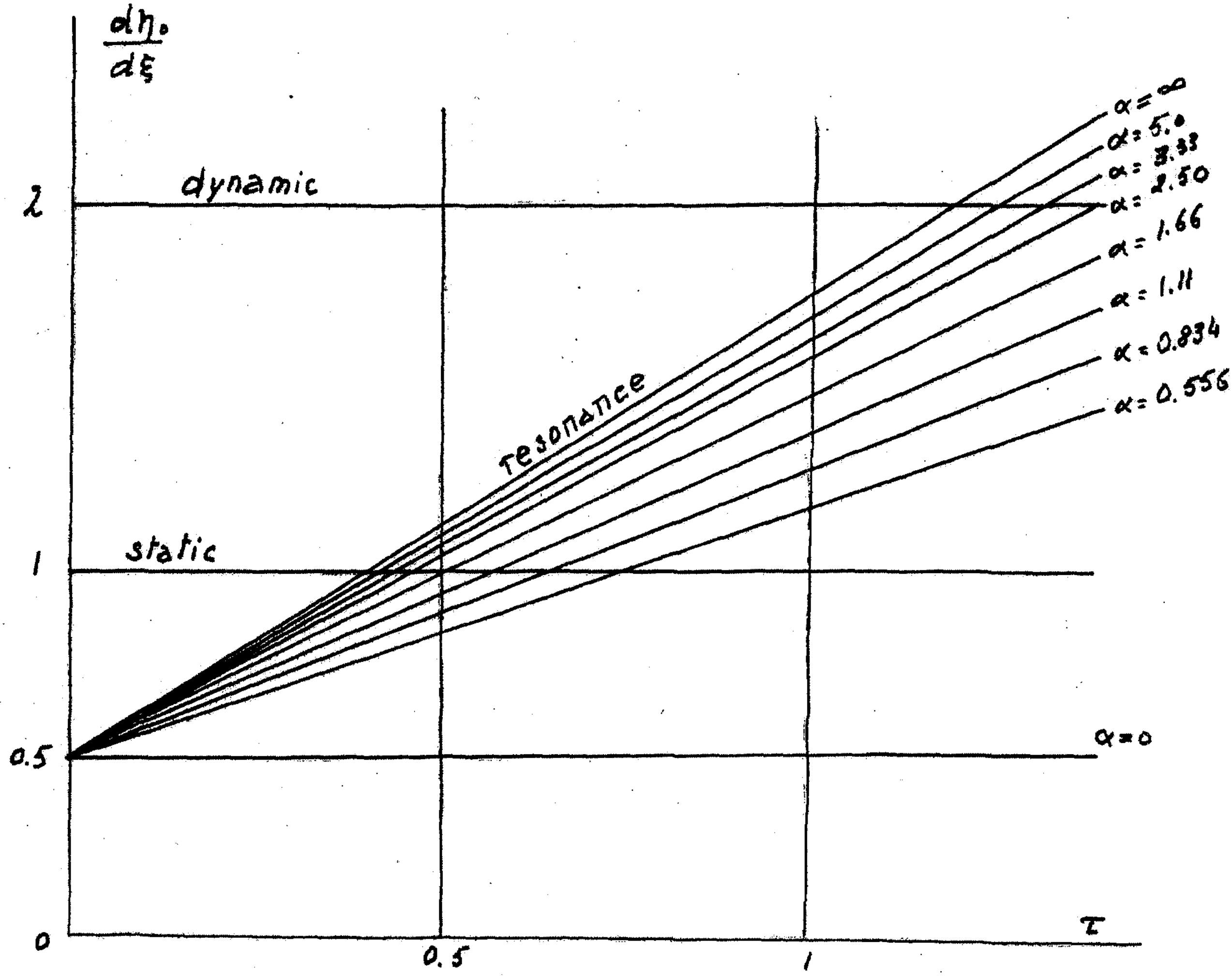
α	$\lvert B_0' \sin\lambda_0 \rvert$	$\lvert B_1' \sin\lambda_1 \rvert$	$\lvert B_2' \sin\lambda_2 \rvert$	$\lvert B_3' \sin\lambda_3 \rvert$	S
0	1	0	0	0	1.00
0.556	0.999	0.00500	0.000421	0.0000777	1.004
0.834	0.994	0.00950	0.000809	0.000179	1.003
1.11	0.986	0.0144	0.00132	0.000311	1.00
1.66	0.965	0.0254	0.00245	0.000640	0.99
2.50	0.946	0.0398	0.00562	0.00132	0.98
3.33	0.931	0.0511	0.00825	0.00218	0.98
5	0.908	0.0666	0.0138	0.00420	0.97
10	0.870	0.0830	0.0234	0.00875	0.97
	0.813	0.0900	0.0325	0.0165	0.95

We could have proved by direct considerations that the maximum value of the stress must be twice the static stress M/ρ. We actually find $2S/\rho M$ for this maximum, which is a very good approximation. The above calculation shows furthermore that this maximum stress is very nearly reached during the first fundamental oscillation.

fig 6.



ψ (τ)
τ



dη₀/dξ
2
1
0.5
0
dynamic
static
resonance
α = ∞
α = 5.0
α = 3.33
α = 2.50
α = 1.66
α = 1.11
α = 0.834
α = 0.556
α = 0
τ
0.5
1

fig 7

§4 OSCILLATIONS DUE TO ARBITRARY HORIZONTAL ACCELERATION

Let the horizontal acceleration of the ground be

$$\gamma = \gamma_0 \psi(\tau)$$

We will calculate the corresponding motion of the building by considering the curve $\psi(\tau)$ as composed of an infinite number of small jumps. (fig. 6). Each of these increments can be written $d\psi = \frac{d\psi}{d\tau} d\tau$. By using then the function previously mentioned, we get the elementary solution corresponding to the increment at the point θ,

$$\zeta(\tau-\theta) \frac{d\psi(\theta)}{d\theta} d\theta$$

The function giving the total motion is the integral

$$\eta(\xi\tau) = \int_0^\tau \zeta(\tau-\theta) \frac{d\psi(\theta)}{d\theta} d\theta$$

Integrating by parts and noting that $\psi(0) = 0$, we get the more convenient form

$$\eta(\xi\tau) = \int_0^\tau \psi(\theta) \frac{d}{d\theta} \zeta(\tau-\theta) d\theta$$

where $\frac{d}{d\theta}[\zeta(\tau-\theta)] = -\sum B_K \lambda_K \cos\lambda_K \xi \sin\lambda_K(\tau-\theta)$.

Hence, more explicitly,

(3) $$\eta(\xi\tau) = \gamma\alpha \sum_0^\infty \frac{\beta_K \cos\lambda_K\xi}{\lambda_K^3} \int_0^\tau \sin\lambda_K(\tau-\theta)\psi(\theta)d\theta.$$

Application of resonance

Consider the special case where

$$\begin{aligned} \psi(\theta) &= 0 & \theta &< 0 \\ \psi(\theta) &= \sin\lambda\theta & \theta &> 0 \end{aligned}$$

Put

$$\phi_K(\tau) = \int_0^\tau \sin\lambda_K(\tau-\theta)\sin\lambda\theta\, d\theta.$$

$$= \frac{1}{\lambda_K+\lambda}\left[\sin\left(\frac{\lambda+\lambda_K}{2}.\tau\right)\cos\left(\frac{\lambda-\lambda_K}{2}.\tau\right)\right] + \frac{1}{\lambda_K-\lambda}\left[\sin\left(\frac{\lambda-\lambda_K}{2}\tau\right)\cos\left(\frac{\lambda+\lambda_K}{2}\tau\right)\right],$$

$$\eta(\xi\tau) = -2\alpha \sum_{\kappa=0}^{\infty} \frac{\beta_\kappa}{\lambda_\kappa^3} \cos\lambda_\kappa \xi \, \phi_\kappa(\tau).$$

If $\lambda = \lambda_\kappa$, we have resonance, and the value of $\phi_\kappa(\tau)$ becomes

$$\phi_\kappa(\tau) = \frac{\tau}{2} \cos\lambda_\kappa\tau + \frac{1}{2\lambda_\kappa} \sin\lambda_\kappa\tau$$

After a certain time the value of $\eta(\xi\tau)$ is reduced to its principal term. The amplitude increases indefinitely.

$$\eta_\kappa(\xi\tau) = -\alpha \frac{\beta_\kappa\tau}{\lambda_\kappa^3} \cos\lambda_\kappa\xi \cos\lambda_\kappa\tau$$

Put

$$C_\kappa = \frac{\alpha\beta_\kappa}{\lambda_\kappa^3} = -\frac{1}{2} B_\kappa'$$

The maximum amplitude takes place at the top and its actual value is

$$u = \rho_0 t_0^2 C_\kappa \tau$$

The coefficients C_κ are given in the following table:

α	C_0	C_1	C_2	C_3 .
0	$\infty\left(\frac{1}{2\sqrt{\alpha}}\right)$	0	0	0
0.556	0.794	-0.0147	0.00220	-0.0139
0.834	0.694	-0.0197	0.00311	-0.0199
1.11	0.633	-0.0237	0.00394	-0.0254
1.66	0.562	-0.0299	0.00578	-0.0377
2.50	0.518	-0.0356	0.00778	-0.0517
3.33	0.493	-0.0387	0.00924	-0.0623
5	0.468	-0.0425	0.01175	-0.0812
10	0.440	-0.0456	0.0145	-0.00635
∞	0.406	-0.0452	0.0164	-0.00830

We have shown previously that the stress is

$$j_o M \frac{\partial \eta_K}{\partial \xi} = j_o M\, C_K' \tau \sin \lambda_K \xi \cos \lambda_K \tau \,,$$

where $C_K' = \dfrac{\alpha \beta_K}{\lambda_K^2} = -\dfrac{1}{2} B_K' \lambda_K$

The maximum stress occurs between the first and the second floor for the first harmonic and has the value $j_o M \tau C_o' \sin \lambda_o$.

The stresses due to higher harmonics are $j_o M \tau C_K'$

These stresses can be easily computed from the following table or fig. 8:

α	C_o'	$C_o' \sin \lambda_o$	C_1'	C_2'	C_3'
0	-0.500	0	0	0	0
0.556	-0.540	-0.340	0.0486	-0.0139	0.00622
0.834	-0.555	-0.397	0.0666	-0.0199	0.00942
1.11	-0.564	-0.438	0.0818	-0.0254	0.0123
1.66	-0.578	-0.496	0.107	-0.0377	0.0177
2.50	-0.596	-0.544	0.133	-0.0517	0.0254
3.33	-0.606	-0.572	0.149	-0.0633	0.0324
5.0	-0.619	-0.600	0.172	-0.0812	0.0451
10	-0.635	-0.629	0.196	-0.105	0.071
	-0.637	-0.637	0.212	-0.129	0.091

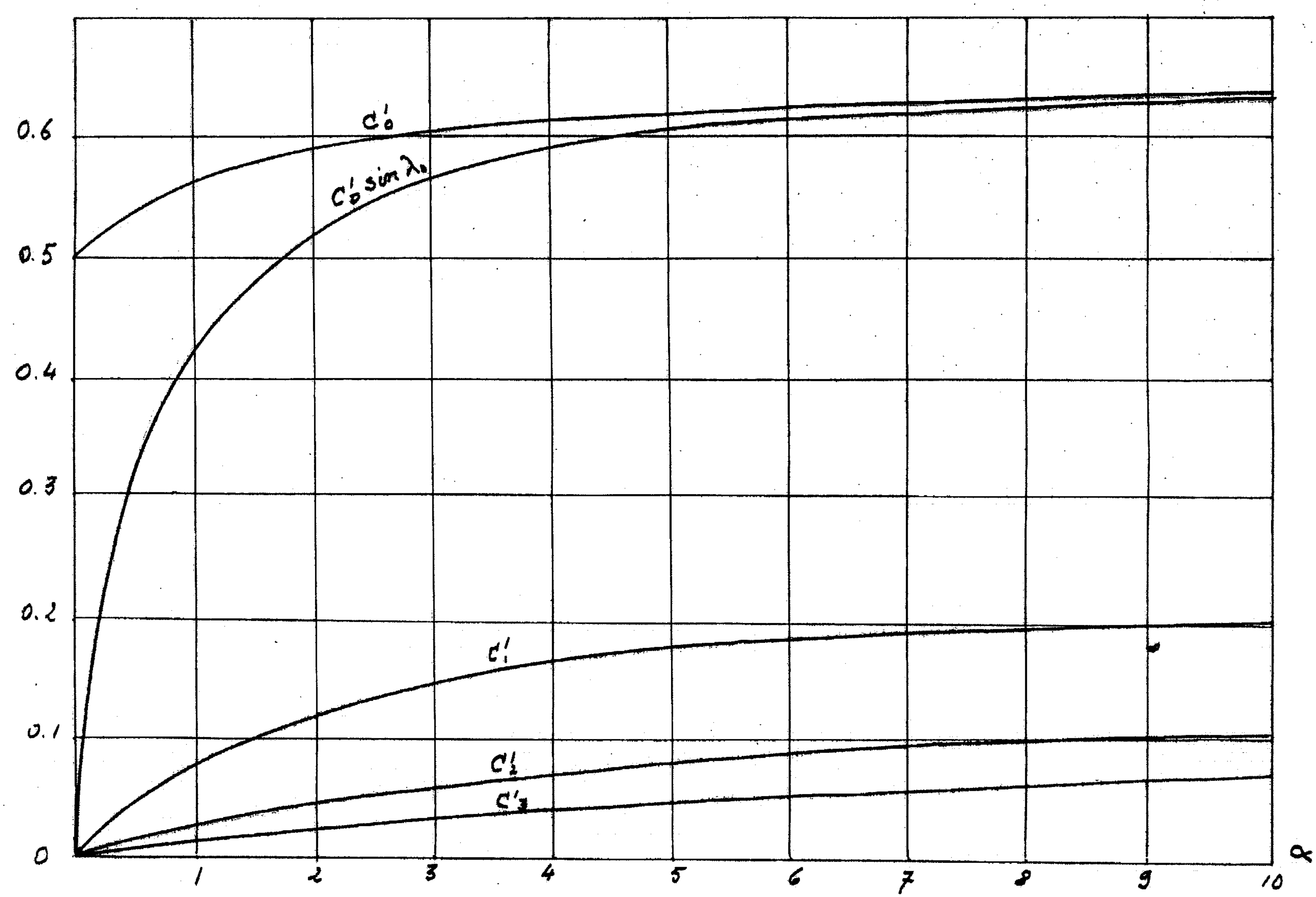


fig. 8.

Upper limit of resonance stresses

Consider the case of resonance with the fundamental harmonic

$$\frac{\partial\eta}{\partial\xi} = r\alpha\frac{\beta_0}{\lambda_0^2}\sin\lambda_0\xi\cdot\frac{\tau}{2}\cos\lambda_0\tau + \frac{1}{2}\cdot\frac{r\alpha\beta_0}{\lambda_0^3}\sin\lambda_0\xi\sin\lambda_0\tau + r\alpha\sum_{k=1}^{\infty}\frac{\beta_k}{\lambda_k^2}\sin\lambda_k\xi\,\phi_{k_0}(\tau)$$

The last term may be neglected, for

$$\left|\phi_{k_0}(\tau)\right| < \frac{1}{\lambda_k+\lambda_0} + \frac{1}{\lambda_k-\lambda_0} = \frac{2\lambda_k}{\lambda_k^2-\lambda_0^2} = \frac{2}{\lambda_k\left[1-\left(\frac{\lambda_0}{\lambda_k}\right)^2\right]}$$

on the other hand, $\lambda_0/\lambda_k < 1/3$

hence $$\left|r\alpha\sum_{k=1}^{\infty}\frac{\beta_k}{\lambda_k^2}\sin\lambda_k\xi\,\phi_{k_0}(\tau)\right| < \sum_{k=1}^{\infty}\left|B_k'\right|\frac{2}{1-\left(\frac{\lambda_0}{\lambda_k}\right)^2}$$

Noting the values of B_k' in this expression, we see that it is comparatively small. We can write,

$$\left|\frac{d\eta}{d\xi}\right| < \tau\left|C_0'\sin\lambda_0\right| + \frac{1}{2}$$

In this case the stress is never higher than

$$j_0M\left[\tau\left|C_0'\sin\lambda_0\right| + \frac{1}{2}\right]$$

The values of $\tau\left|C_0'\sin\lambda_0\right| + \frac{1}{2}$ are plotted in fig. 7

In case of resonance with an harmonic of higher order

$$\left|\bar{\phi}_{ki}(\tau)\right| < \frac{2\lambda_i}{\lambda_i^2-\lambda_k^2} \qquad k < i,$$

$$\left|\bar{\phi}_{ki}(\tau)\right| < \frac{2\lambda_k}{\lambda_k^2-\lambda_i^2} \qquad k > i,$$

hence

$$\left|\frac{d\eta_i}{d\xi}\right| < \left|C_i'\right|\tau + \frac{1}{2}\left|B_i'\right| + 4\sum_{k=0}^{i-1}C_k'\frac{\lambda_i}{\lambda_i^2-\lambda_k^2} + 4\sum_{k=i+1}^{\infty}C_k'\frac{\lambda_k}{\lambda_k^2-\lambda_i^2}$$

§5 ACTION OF EARTHQUAKE ACCELERATIONS, SPECTRUM OF A SISMOGRAM

Most of the sismograms show the random character of earthquake accelerations, with, very often, series of quasi periodic oscillations. It is therefore difficult to identify the action of an earthquake with that of the simple acceleration curves studied till now. We might, of course, apply equation (3) and calculate the displacement corresponding to a given sismogram.

But it is much more convenient to divide the problem, and to analyze separately the elastic properties of the building and the frequency distribution of the earthquake. We need therefore a general theorem which we shall here prove in the special case of building oscillations. As we are more interested in the stresses, we will not give the corresponding equations of the displacement; the method being exactly the same as before, we get the stresses from the displacement by a simple derivation.

Equation (3) may be written

$$\frac{\partial \eta}{\partial \xi}(\xi\tau) = 2\alpha \sum_{0}^{\infty} \frac{\beta_k \sin \lambda_k \xi}{\lambda_k^2}\left[\sin \lambda_k \tau \int_0^\tau \cos \lambda_k \theta\, \psi(\theta)\, d\theta - \cos \lambda_k \tau \int_0^\tau \sin \lambda_k \theta\, \psi(\theta)\, d\theta\right] \tag{4}$$

Putting $j(t) = g\,\psi(\theta)$ the acceleration

and

$$F_1(\omega) = \frac{g\,t_0}{\pi}\int_0^\tau \cos \lambda\theta\, \psi(\theta)\, d\theta = \frac{1}{\pi}\int_0^t \cos \omega t\, j(t)\, dt$$

$$F_2(\omega) = \frac{g\,t_0}{\pi}\int_0^\tau \sin \lambda\theta\, \psi(\theta)\, d\theta = \frac{1}{\pi}\int_0^t \sin \omega t\, j(t)\, dt$$

Considering then the Fourier integral

$$j(t) = \frac{1}{\pi}\int_0^\infty d\omega \int_0^t j(\theta) \cos \omega(\theta - t)\, d\theta,$$

where j(τ) represents an acceleration record located in the time interval (0t), we get

$$j(t) = \int_0^\infty F_1(\omega) \cos \omega t\, d\omega + \int_0^\infty F_2(\omega) \sin \omega t\, d\omega$$

The expression $\Delta(\omega) = F_1^2(\omega) + F_2^2(\omega)$ represents what might be called the density of energy as a function of ω, or energy spectrum. Coming back to equation (4), we note that $\frac{\partial \eta}{\partial \xi}$ is the resultant of free oscillations of amplitude...(1). The stress at a certain moment is the sum of the stresses due to each free oscillation existing at that moment. Each of these stresses may be written $Mg \frac{d\eta_K}{d\xi}$ or

$$M \cdot \frac{2\alpha\beta_K \sin\lambda_K \xi}{\lambda_K^2} \cdot \frac{\pi}{t}\sqrt{\Delta(\omega_K)} \cdot \frac{t}{t_0}$$

This resultant can be expressed in a more complete manner as follows:

When an acceleration j(t) acts upon a building during a time interval (Ot), the maximum total stress at the final moment is the sum of the stresses due to every free oscillation existing at that moment. Each of those stresses is a product of three factors; an effective mass,

$$M_K(\xi) = M \cdot \frac{2\alpha\beta_K \sin\lambda_K \xi}{\lambda_K^2} = 2MC_K' \sin\lambda_K \xi,$$

an effective acceleration $\frac{\pi}{t}\sqrt{\Delta(\omega_K)}$

and a coefficient t/t_0

The first factor depends only on the elastic properties of the building. The second factor, which has the dimension of an acceleration, depends essentially on the shape or spectrum of the sismogram. The third one is the relative length of the earthquake. The problem is thus divided in two parts,

1) Calculation of the effective masses for a given building,
2) Calculation of the function $\Delta(\omega)$ or spectrum for a certain number of sismograms of the region. The use of a Fourier

---

(1) $\frac{2\alpha\beta_K \cos\lambda_K \xi}{\lambda_K^2} \cdot \frac{\pi}{g t_0}\sqrt{F_1^2(\omega) + F_2^2(\omega)}$

analyzer will make this part of the work very easy.

A great advantage of this method is the fact that the effective acceleration may be derived in a very simple way from a ground displacement record. Let d(t) be the displacement.

Putting

$$F_{1d}(\omega) = \frac{1}{\pi}\int_0^t \cos\omega t \cdot d(t)\, dt,$$

$$F_{2d}(\omega) = \frac{1}{\pi}\int_0^t \sin\omega t\, d(t)\, dt,$$

we get

$$d(t) = \int_0^\infty F_{1d}(\omega)\cos\omega t\, d\omega + \int_0^\infty F_{2d}(\omega)\sin\omega t\, d\omega$$

The acceleration takes the form

$$j(t) = d''(t) = -\int_0^\infty \omega^2 F_{1d}(\omega)\cos\omega t\, d\omega + \int_0^\infty \omega^2 F_{2d}(\omega)\sin\omega t\, d\omega,$$

hence

$$\omega^2 F_{1d}(\omega) = F_1(\omega)$$

$$\omega^2 F_{2d}(\omega) = F_2(\omega)$$

Putting then $\Delta_d(\omega) = F_{1d}^2(\omega) + F_{2d}^2(\omega)$, we find the required formula

$$\sqrt{\Delta(\omega)} = \omega^2\sqrt{\Delta_d(\omega)}$$

This enables us to express very simply the function $\Delta(\omega)$ corresponding to the acceleration record, when we know $\Delta_d(\omega)$ calculated from the displacement record.

The analysis of a sismogram gives a function density of energy $\Delta(\omega)$ or effective acceleration $\frac{\pi}{t}\sqrt{\Delta(\omega)}$ having one or more maxima as shown in fig. 9. In a given region there generally exists certain characteristic frequencies of the soil which appear in many sismograms. These frequencies will cause the mentioned maxima to be located in the vicinity of fixed values of ω in different records. If three critical values ω_1, ω_2, ω_3 for the building are in the region of those maxima we should, if possible, place them between the peaks. The maximum stresses are given by

$$\pi\frac{t}{t_0}\left[M_1(\xi)\frac{\sqrt{\Delta(\omega_1)}}{t} + M_2(\xi)\frac{\sqrt{\Delta(\omega_2)}}{t} + M_3(\xi)\frac{\sqrt{\Delta(\omega_3)}}{t}\right]$$

$$= \frac{\ell\pi}{t_0}M\left[C_1'\sin\lambda_1\xi\sqrt{\Delta(\omega_1)} + C_2'\sin\lambda_2\xi\sqrt{\Delta(\omega_2)} + C_3'\sin\lambda_3\sqrt{\Delta(\omega_3)}\right]$$

In order to illustrate this method we will calculate the function $\Delta(\omega)$ for the acceleration given by a finite sine-wave of total length T:

$$j(t) = 0 \qquad t \leq -\frac{T}{2} \qquad t > \frac{T}{2}$$

$$j(t) = j_0 \sin\Omega t \qquad -\frac{T}{2} < t < \frac{T}{2}$$

$$F_1(\omega) = \frac{j_0}{\pi}\int_{-\frac{T}{2}}^{+\frac{T}{2}} \cos\omega t \sin\Omega t \, dt = 0$$

$$F_2(\omega) = \frac{j_0}{\pi}\int_{-\frac{T}{2}}^{+\frac{T}{2}} \sin\omega t \sin\Omega t \, dt = \frac{j_0}{\pi}\left[\frac{\sin(\omega-\Omega)\frac{T}{2}}{\omega-\Omega} - \frac{\sin(\omega+\Omega)\frac{T}{2}}{\omega+\Omega}\right]$$

Near the maximum we have approximately

$$F_2(\omega) = \frac{j_0}{\pi}\frac{\sin(\omega-\Omega)\frac{T}{2}}{\omega-\Omega}$$

$$\sqrt{\Delta(\omega)} = \frac{j_0}{\pi}\left|\frac{\sin(\omega-\Omega)\frac{T}{2}}{\omega-\Omega}\right|$$

This function is plotted in fig. 10. The stresses corresponding to different free oscillations are

$$M_K(\xi)\,\frac{j_0}{t_0}\left|\frac{\sin(\omega_K-\Omega)\frac{T}{2}}{\omega_K-\Omega}\right|$$

In case of resonance with the harmonic of order k the corresponding stress is

$$M_K(\xi)\,\frac{j_0}{t_0}\cdot\frac{T}{2}$$

The stress may be put in the form already obtained by a previous method

$$M\,j_0\,C_K'\sin\lambda_K\xi\,\frac{T}{t_0}.$$

§6 GENERALIZED METHOD FOR BUILDINGS HAVING VARIABLE MASS AND RIGIDITY AT THE DIFFERENT FLOORS

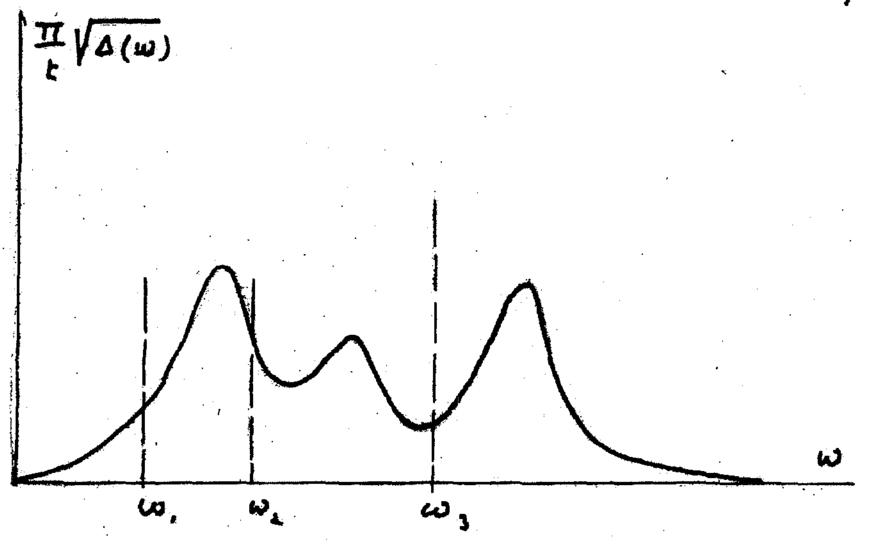
The method is absolutely general and could be used for any type of building considering both bending and shearing deformations. We will restrict ourselves to the shear in case of variable $\mu(x)$ and m(x). The equation of motion takes the form

$$\frac{\partial}{\partial x}\left[\mu(x)\frac{\partial u}{\partial x}\right]=m(x)\frac{\partial^2 u}{\partial t^2}+m(x)\,j(t)$$

If the acceleration is an harmonic function of time $j=j_0\sin\omega t$ the solution is $u=y\sin\omega t$, and

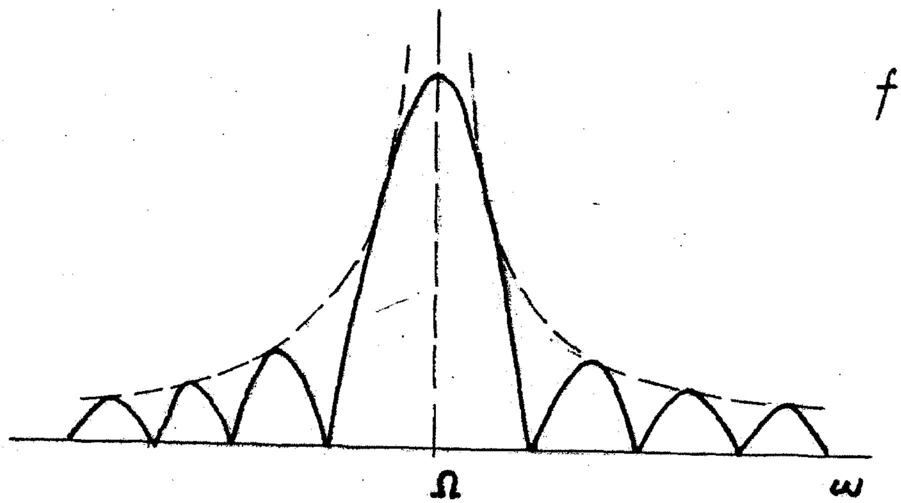
$$\frac{d}{dx}\left[\mu(x)\frac{dy}{dx}\right]+m(x)\omega^2 y=m(x)j_0\ .$$

fig 9



$\frac{\pi}{t}\sqrt{\Delta(\omega)}$
ω
ω_1
ω_2
ω_3

fig 10



Ω
ω

Developing the solution in terms of the orthogonal functions corresponding to the free oscillations and defined by the equations

$$\frac{d}{dx}\left[\mu(x)\frac{dy_i}{dx}\right] + m(x)\omega_i^2 y_i = 0 , \qquad (4')$$

we get $$y = \sum_0^\infty A_i y_i ,$$

$$\sum_0^\infty A_i\left[\omega^2 - \omega_i^2\right] y_i m = j_0$$

After multiplying both sides by y_i and integrating along the total height of the building,

$$A_i = \frac{j_0 C_i}{\omega^2 - \omega_i^2} \quad \text{with} \quad C_i = \frac{\int_0^h y_i dx}{\int_0^h y_i^2 m(x)\, dx}$$

Finally, $$y = j_0 \sum_0^\infty \frac{C_i}{\omega^2 - \omega_i^2} y_i$$

The displacement corresponding to a sudden acceleration is

$$u = j_0 \sum_0^\infty \frac{C_i}{\omega_i^2} y_i(x)\left[\cos\omega_i t - 1\right]$$

The amplitude of each free oscillation is given as a function of the density of energy $\Delta(\omega)$ by

$$u_i = j_0 \frac{C_i}{\omega_i^2} y_i(x)\, \pi\sqrt{\Delta(\omega_i)} ,$$

and the stress at the coordinate x due to that oscillation by

$$\mu(x)\frac{\partial u_i}{\partial x} = j_0 \frac{C_i}{\omega_i^2} \cdot \frac{dy_i}{dx} \cdot \pi\sqrt{\Delta(\omega_i)}$$

The total maximum stress is the sum of each of these expressions.

The preceding considerations show that the problem is solved whenever we know the set of orthogonal functions y_i multiplied by any factor of proportionality, i.e. when we know the shape of the free oscillations.

From y_i we easily deduce the frequency or the corresponding w_i by simple energy considerations. Consider the building oscillating with that frequency. The motion is supposed to be free. When the amplitude is maximum the kinetic energy is equal to zero and the potential energy is

$$\frac{1}{2}\int_0^h \mu(x)\left(\frac{dy_i}{dx}\right)^2 dx$$

On the other hand, the potential energy passes through zero when the strain disappears; at that moment the kinetic energy is maximum and has the value

$$\frac{1}{2}\int_0^h m(x)\,\omega_i^2 y_i^2\,dx$$

Equating those two expressions, we get the value of

$$(5)\qquad \omega_i^2 = \frac{\int_0^h \mu(x)\left(\frac{dy_i}{dx}\right)^2 dx}{\int_0^h m(x)\,y_i^2\,dx}$$

which is independent of any arbitrary constant multiplying y_i.

THE CALCULATION OF THE ORTHOGONAL FUNCTIONS

The orthogonal functions y_i may be found by two methods. One is semi-empirical and very simple. We note that those functions are defined by equation (4′), which by a change of the independent variable

$$z = \int \frac{dx}{\mu(x)}$$

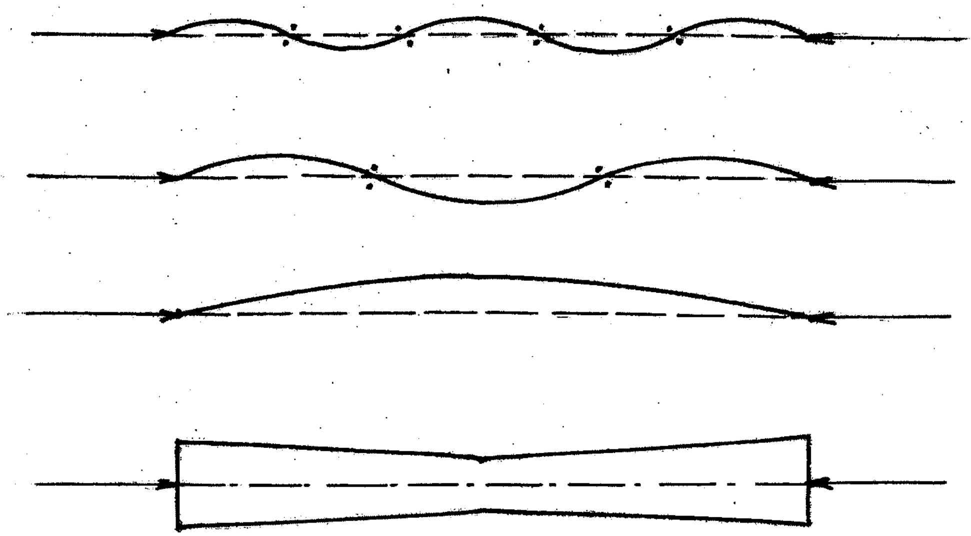


fig.11.

becomes

$$\frac{1}{\mu[x(z)]}\frac{d^2}{dz^2}[y_i(z)] + m[x(z)]\omega^2 y_i = 0$$

This is the equation of buckling under a load P of a beam of moment of inertia

$$I(z) = \frac{P}{Em\mu\omega^2}.$$

If we consider an elastic strip of uniform thickness h, its moment of inertia will have the value I under the condition that the variable width b satisfies the equation ,

$$\frac{bh^3}{12} = I = \frac{P}{Em\mu\omega^2} = \frac{A}{\mu m}$$

where A is an arbitrary constant. We may choose for z any scale convenient, for we are only interested in the shape of the functions y_i.

In order to realize the given limit conditions, the strip will be repeated symmetrically around a point representing the top of the building. (fig. 11). The deformation of the half strip under different loads will give the different functions y_i. Only the first deformation is stable, so that the others will have to be stabilized by a special but very simple device. We then compute by the energetic method the corresponding values of the frequencies. (equation 5).

Another method is analytical and might be useful in case of only slight deviations from the case of constant rigidity and mass previously investigated. It is known in atomic physics as "perturbation calculus".

Putting
$$m(x) = m_0 + \epsilon(x)$$
$$\mu(x) = \mu_0 + \gamma(x)$$
$$y_i(x) = y_{0i} + \delta_i(x)$$
$$\omega_i^2 = \omega_{i0}^2 + \rho_i$$

where $m_o, \mu_o, y_{oi}, \omega_{io}^2$ are known and $\epsilon, \gamma, \delta_i, \rho_i$ small variations.

The equation becomes ,

$$\frac{d}{dx}\left[(\mu_o+\gamma(x))\frac{d}{dx}(y_{io}(x)+\delta(x))\right]+(m_o+\epsilon(x))(\omega_{io}^2+\rho_i)(y_{io}+\delta(x))=0$$

Taking into account the identity

$$\frac{d}{dx}\left[\mu_o\frac{dy_{oi}}{dx}\right]+m_o\omega_{io}^2 y_i=0$$

and neglecting the small terms of higher order, we finally get

$$\frac{d}{dx}\left[\mu_o\frac{d}{dx}\delta(x)\right]+m_o\omega_{io}^2\delta(x)=-\frac{d}{dx}\left[\gamma(x)\frac{d}{dx}(y_{io}(x))\right]-y_{io}\left[\epsilon(x)\omega_{io}^2+\rho m_o\right]$$

This equation has only finite solutions if the function of the right side is orthogonal to the characteristic functions of the equation obtained by equating the left side to zero[(1)].

This condition may be written ,

$$\int_0^h y_{io}\frac{d}{dx}\left[\gamma(x)\frac{d}{dx}y_{io}(x)\right]+\omega_{io}^2\int_0^h y_{io}^2\,\epsilon(x)+\rho_i m_o\int_0^h y_{io}^2\,dx=0$$

and gives the values of ρ_i .

The second side of the equation is then entirely known and δ_i is determined by

$$\frac{d}{dx}\left[\mu_o\frac{d}{dx}\delta(x)\right]+m_o\omega_{io}^2\delta(x)=f_i(x)$$

Developing the solution in terms of the orthogonal functions

$$\delta(x)=\sum A_K y_{Ko} ,$$

putting this expression back into the equation

$$\sum A_K m_o\left[\omega_{oi}^2-\omega_{oK}^2\right]y_{Ko}=f_i(x) ,$$

multiplying then both sides by y_{Ko} , and integrating from 0 to h, we get

$$A_K=\frac{C_K}{\omega_{io}^2-\omega_{Ko}^2} \qquad i\neq k$$

---

(1) Hilbert Courant Chap. V, p. 277.

where

$$C_k = \frac{\int_0^h y_{k0} f_i(x)\, dx}{\int_0^h m_0 y_{k0}^2(x)\, dx}$$

Hence

$$\delta_i(x) = \sum_0^\infty \frac{C_k}{\omega_{i0}^2 - \omega_{0k}^2} y_{k0} \qquad k \neq i$$

and the required orthogonal functions may finally be written

$$y_i = y_{0i} + \sum_0^\infty \frac{C_k}{\omega_{i0}^2 - \omega_{0k}^2} y_{k0} \qquad k \neq i.$$

CHAPTER III

CRITICAL TORSIONAL VIBRATIONS OF A ROTATING ACCELERATED SHAFT

CRITICAL OSCILLATIONS BY TORSION OF A ROTATING ACCELERATED SHAFT

The method previously developed for the calculation of the transient shearing vibration of a building may be applied to the study of critical torsional vibrations in a shaft rotating with a constant angular acceleration. These amplitudes will be functions of the angular acceleration and depend on the way the external moment is distributed on the shaft. The equation of oscillation may be written

$$\mathcal{M} + \frac{\partial}{\partial x}\left[K\frac{\partial\theta}{\partial x}\right] = I\frac{\partial^2\theta}{\partial t^2} \tag{1}$$

where θ is the angular coordinate of a cross section.

The equation of the free oscillations is

$$\frac{\partial}{\partial x}\left[K(x)\frac{\partial\theta}{\partial x}\right] = I(x)\frac{\partial^2\theta}{\partial t^2} \tag{2}$$

Which has an infinite number of solutions of the type $\theta = \Theta_i(x)e^{i\omega_i t}$ and where $\Theta_i(x)$ verifies the equations

$$\frac{d}{dx}\left[K(x)\frac{d\Theta_i}{dx}\right] + \omega_i^2 I(x)\Theta_i = 0 \tag{3}$$

If the moment is harmonic $\mathcal{M} = \mathcal{M}_0(x)e^{i\omega t}$ the solution of (1) may be expanded in a series of orthogonal functions. According to (1) and (3) we get

$$\theta(xt) = e^{i\omega t}\sum A_i\Theta_i(x)$$

$$\mathcal{M}_0(x) = I(x)\sum A_i(\omega^2 - \omega_i^2)\Theta_i(x)$$

Multiplying both sides by $\Theta_i(x)$ and integrating along the shaft,

$$A_i = \frac{C_i}{\omega^2 - \omega_i^2} \qquad C_i = \frac{\int_0^l \mathcal{M}_0(x)\Theta_i(x)\,dx}{\int_0^l I(x)\Theta_i^2(x)\,dx}$$

The required solution is

$$\theta(x\,t) = e^{i\omega t} \sum \frac{C_i\,\Theta_i(x)}{\omega^2 - \omega_i^2}$$

The theory of integral equations shows that this method is quite general as well for flexion as for torsion. We must now calculate the deformation cause by a distributed moment suddently applied to the shaft. Noting the value of the integral

$$\int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\omega}\,d\omega = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

this moment may be written,

$$M_o(x)\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega}\,d\omega$$

to which corresponds the solution

$$\theta_1(x\,t) = \sum C_i\Theta_i(x)\int_{-\infty}^{+\infty}\frac{e^{i\omega t}\,d\omega}{\omega(\omega^2-\omega_i^2)} = -\sum\frac{C_i}{\omega_i^2}\Theta_i(x)\left[\cos\omega_i t - 1\right]$$

If the applied moment varies with the time and has the form $M_o(x)\,\psi(t)$ it may be considered as composed of an infinite number of small jumps $M_o(x)\frac{d\psi}{dt}\,dt$ and the corresponding solution is given by integration

$$\theta_2(x\,t) = \int_0^t \theta_1(x,\,t-\tau)\,\frac{d}{d\tau}\,\psi(\tau)\,d\tau$$

Integrating by parts and assuming $\psi(0)=0$

$$\theta_2(x\;t) = +\int_0^t \frac{\partial}{\partial t}\left[\theta_1(x,\,t-\tau)\right]\psi(\tau)\,d\tau$$

$$= \sum\frac{C_i}{\omega_i}\,\Theta_i(x)\left[\sin\omega_i t\int_0^t\psi(\tau)\cos\omega_i\tau\,d\tau - \cos\omega_i t\int_0^t\psi(\tau)\sin\omega_i\tau\,d\tau\right]$$

By putting

$$f_1(\omega_i) = \int_0^t \psi(\tau) \cos \omega_i \tau \, d\tau$$

$$f_2(\omega_i) = \int_0^t \psi(\tau) \sin \omega_i \tau \, d\tau$$

the amplitudes of the free oscillations composing the motion are

$$\frac{C_i}{\omega_i} \Theta_i(x) \sqrt{f_1^2(\omega_i) + f_2^2(\omega_i)}$$

This formula gives the complete solution of the problem. Consider a shaft rotating with a uniform acceleration. In most cases the amplitudes of the torque variations will be constant. We call ν the angular acceleration and put $\beta = \frac{\nu}{2}$. The variable part of the moment is, per unit length:

$$M_0(x) \sin \beta t^2.$$

Applying the preceding formula we get

$$f_1(\omega_i) = \int_0^t \cos \omega_i \tau \sin \beta \tau^2 \, d\tau$$

Put

$$\frac{\omega_i}{2\sqrt{\beta}} = \alpha \qquad , \qquad t\sqrt{\beta} = x \, ,$$

$$2 \cos \omega_i \tau \sin \beta \tau^2 = \sin\left[(x+\alpha)^2 - \alpha^2\right] + \sin\left[(x-\alpha)^2 - \alpha^2\right]$$

$$2\sqrt{\beta} f_1(\omega_i) = \int_0^x \sin\left[(x+\alpha)^2 - \alpha^2\right] dx + \int_0^x \sin\left[(x-\alpha)^2 - \alpha^2\right] dx$$

$$= \int_0^{x+\alpha} \sin\left[y^2 - \alpha^2\right] dy + \int_{-\alpha}^{x-\alpha} \sin\left[y^2 - \alpha^2\right] dy$$

$$= \cos \alpha^2 \int_\alpha^{x+\alpha} \sin y^2 \, dy - \sin \alpha^2 \int_\alpha^{x+\alpha} \cos y^2 \, dy$$

$$+ \cos \alpha^2 \int_{-\alpha}^{x-\alpha} \sin y^2 \, dy - \sin \alpha^2 \int_{-\alpha}^{x-\alpha} \cos y^2 \, dy$$

These expressions may be reduced to the well known Fresnel integrals

$$\int_0^u \sin y^2 dy = \frac{1}{2}\int_0^{u^2} \frac{\sin z}{\sqrt{z}} dz = \sqrt{\frac{\pi}{2}}\, S(u^2),$$

$$\int_0^u \cos y^2 dy = \frac{1}{2}\int_0^{u^2} \frac{\cos z}{\sqrt{z}} dz = \sqrt{\frac{\pi}{2}}\, C(u^2), \; *$$

We finally get

$$f_1(\omega_i) = \frac{1}{2\sqrt{\beta}}\sqrt{\frac{\pi}{2}}\left\{\left[S(\overline{x+\alpha}^2) + S(\overline{x-\alpha}^2)\right]\cos\alpha^2 - \left[C(\overline{x+\alpha}^2) + C(\overline{x-\alpha}^2)\right]\sin\alpha^2\right\}$$

By a similar treatment

$$f_2(\omega_i) = \frac{1}{2\sqrt{\beta}}\sqrt{\frac{\pi}{2}}\left\{\left[C(\overline{x+\alpha}^2) - C(\overline{x-\alpha}^2) + 2C(\alpha)\right]\cos\alpha^2 + \left[S(\overline{x-\alpha}^2) - S(\overline{x+\alpha}^2) + 2S(\alpha)\right]\sin\alpha^2\right\}.$$

The amplitude of the free oscillations is then calculated by putting into (5) the expression

$$f(\omega_i) = \sqrt{f_1^2(\omega_i) + f_2^2(\omega_i)}$$

This quantity has an upper limit independent of the time

$$f(\omega_i) \leq 1.165\sqrt{\frac{\pi}{2\nu}}$$

After a sufficiently long time and for $\alpha > 5$ the value of f reduces approximately to

$$f(\omega_i) = \sqrt{\frac{\pi}{2\nu}}$$

In short, when a shaft is rotating with a constant angular acceleration the maximum amplitude of a critical torsional oscillation is given by $1.165\sqrt{\frac{\pi}{2}}\frac{C_i}{\omega_i\sqrt{\nu}}\,\Theta_i(x)$

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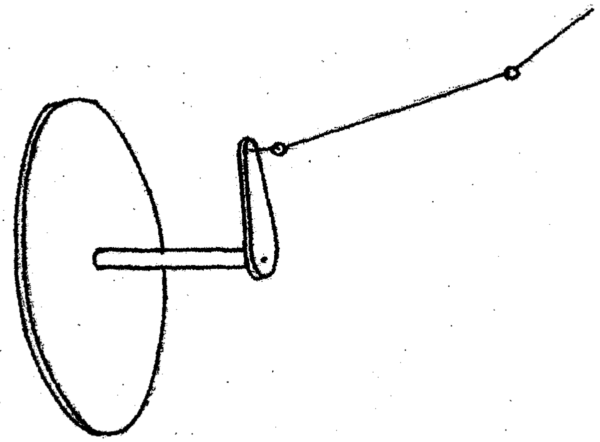
* Tables of functions S & C: Johnke and Emide, Funktionentafiln p. 23.



Application

As an illustration of the theory we shall treat a very simple example. Consider a crank (fig. 1) connected by a shaft to a flywheel. The system is supposed to receive power from a reciprocating engine. If the maximum torque on the shaft is M, the mean torque will be M/2. The variable part of the torque is and is applied with an angular velocity which is twice that of the shaft. If K is the torsional rigidity of the shaft an angular displacement θ of the crank will produce a restoring moment $K\theta$; the characteristic value of ω for the free oscillation is $\omega = \sqrt{\frac{K}{I}}$ where I is the moment of inertia of the crank.

The characteristic function $\Theta_i(x)$ is reduced to a constant that we put equal to 1, we get,

$$C = \frac{\int_0^l M_0(x)\,\Theta_i(x)\,dx}{\int_0^l I(x)\,\Theta_i^2(x)\,dx} = \frac{1}{2}\frac{M}{I}$$

The angular acceleration is supposed to be ν and the maximum torsional amplitude of vibration will be, in radians, smaller then

$$\Theta = 1.165\sqrt{\frac{\pi}{2}}\,\frac{M}{2I}\sqrt{\frac{I}{K}}\,\frac{1}{\sqrt{\nu}}$$

$$\Theta = 0.730\,\frac{M}{\sqrt{\nu K I}}$$