A STUDY OF FLUTTER IN
ONE DEGREE OF FREEDOM

Thesis by

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In Partial Fulfillment of the Requirements for the Professional Degree in Aeronautical Engineering.

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1942

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I. ## ACKNOWLEDGMENT

The authors wish to express their appreciation to Drs. M. A. Biot and C. B. Millikan for their helpful suggestions and guidance in the preparation of this paper, and to Drs. E. E. Sechler and L. G. Dunn for their generous assistance in the design of the model equipment. They also wish to thank Mr. A. A. Merrill for his full cooperation in the use of the P.J.C-Merrill wind tunnel.

II. SUMMARY

The authors have attempted to develop a theoretical means of determining the amount of aerodynamic damping, positive or negative, existing on a symmetrical airfoil having freedom to oscillate only about an axis parallel to the span and on the zero camber line. Variables considered were oscillating frequency, wind speed, and hinge position (location of axis of oscillation.)

Having developed what they considered a proper theoretical criterion for determining the value of aerodynamic damping, the authors have conducted an experimental investigation of an airfoil oscillating system having one degree of freedom, in an attempt to verify the theoretical criterion.

It was concluded that for small oscillations the theoretical values of aerodynamic damping moment are correct except for hinge positions near the leading edge. Contrary to the theory, it was considered that the experimental investigation established the fact that negative damping does not occur for hinge positions at or near the leading edge.

A general conclusion was made by the authors that an airfoil oscillating with only one degree of freedom will always have a positive aerodynamic damping moment provided the oscillations are small.

It was found that an airfoil restricted to one degree of freedom (rotation) and pivoted about the 0.75 chord line will flutter when deflected to an amplitude approximating the stall angle. Flutter will be self-induced when the aerodynamic forces are sufficient to deflect the airfoil to the stall angle.

III. INTRODUCTION

The airfoil theory of non-uniform motion has been developed by many authors, in particular Wagner[1], Glauert[2], Theodorsen[3], and Von Karman[4]. This theory has in turn been applied to the problem of aerodynamic instability or "flutter" of airfoils. The mechanism of aerodynamic instability has been analyzed in detail; Theodorsen[3] covers fully the cases of binary and ternary flutter. Also many authors such as Pugsley[5] have developed theories for wing flutter for the practical engineer.

On the other hand it is not believed that sufficient data have been obtained regarding the instability of a thin airfoil having a single degree of freedom (rotational oscillation). It was therefore decided to make a complete investigation of the problem of flutter in one degree of freedom.

All theories of non-stationary flow around airfoils having been studied, it was decided that the theories as outlined by Glauert[2] and Lombard[6] would be used for the theoretical part of this investigation. Having applied these two theories to the problem of flutter in one degree of freedom and compared the one against the other, it was planned that as thorough as possible a practical wind tunnel investigation would be conducted to verify the theoretical results.

IV. THEORETICAL INVESTIGATION

Glauert[2] developed a very thorough theory of the force and moment on an oscillating airfoil. The assumed conditions on which his theory is developed are:

a) The airfoil is thin with a sharp trailing edge, the vorticity of the wake being generated at the sharp trailing edge.

b) The velocity at the sharp trailing edge remains finite (Joukowski's hypothesis).

c) The amplitude of the oscillation is assumed to be small.

d) For purposes of calculation the thin airfoil is replaced by a straight line. However, to avoid infinite velocities at the leading edge the nose is considered to be rounded.

e) The wake is assumed to extend backwards in a straight line prolonged on the chord.

f) Any change in circulation about the airfoil requires that vorticity must be developed in the wake in order that the regional circulation remain constant.

GLAUERT SYMBOLS

General Symbols Used in Discussion of Glauert's Work

t = time.

U = linear velocity of airfoil through fluid.

V = linear velocity perpendicular to U.

Ω = angular velocity of airfoil.

ρ = density.

a = half chord.

x and y = coordinates corresponding to U and V respectively, measured from origin O at mid-chord.

Γ = circulation around airfoil.

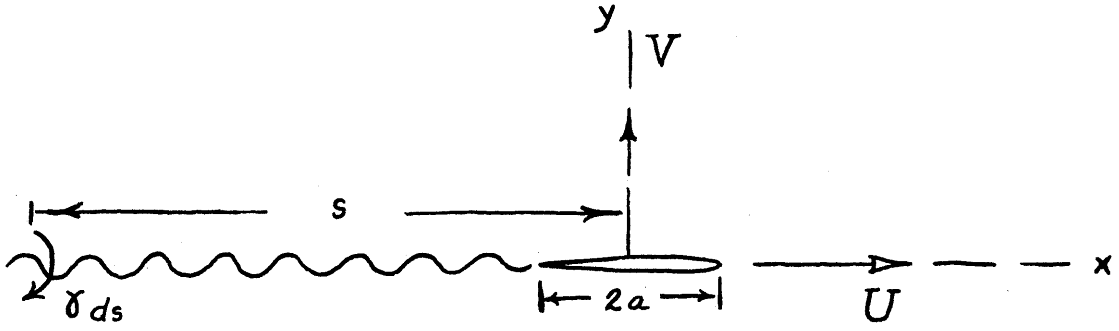
k = vortex strength on airfoil.

S = distance of any point in the wake from the origin O.

$\gamma\,ds$ = strength of vortex element.

M = pitching moment.

Other symbols are explained in the discussion.



The airfoil is defined by the coordinate $x = a \cdot \cos\theta$

Note: The interference of vortex wake is investigated with the circulation being adjusted to give finite velocity at the trailing edge of the airfoil.

Circulation

$$\Gamma = \pi^2 a^2 \Omega - 2\pi a V$$

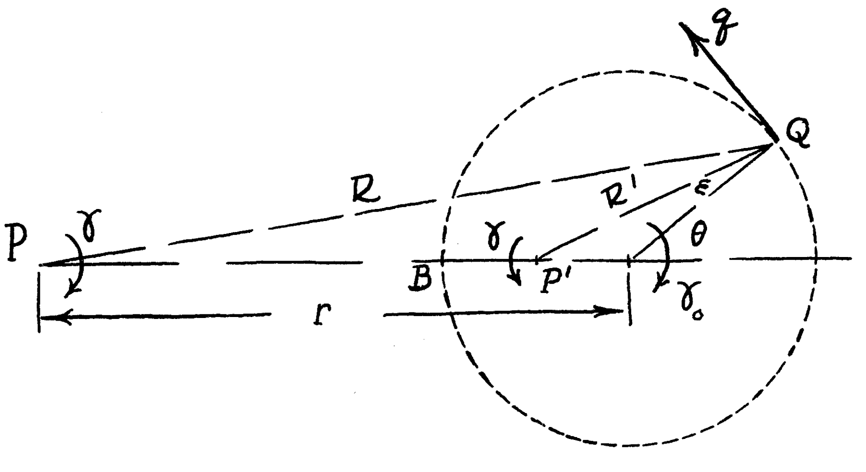
Velocity q, relative to the surface due to the motion of U, V, Ω

$$q_1 \sin\theta = U \sin\theta - V(1 + \cos\theta) + a\Omega \sin^2\theta$$

Vortex strength k_1 along the airfoil

$$k_1 = 2 a\Omega \sin\theta - 2 V \cot \tfrac{1}{2}\theta$$

Consider a circle of radius c which may be converted to the straight line airfoil $\zeta = Z + \frac{c^2}{Z^2}$, the length of the airfoil being 4c.



Tangential velocity q_2 caused by vortex wake at the point Q of the circumference

$$q_2 = \frac{\gamma}{2\pi}\left\{\frac{\cos\varepsilon}{R'} - \frac{\cos(\theta-\varepsilon)}{R}\right\} - \frac{\gamma_0}{2\pi c}$$

$$= \frac{\gamma}{2\pi}\,\frac{r^2-c^2}{rRR'} - \frac{\gamma_0}{2\pi c}$$

where γ_0 is a point vortex at the origin and γ is a point vortex at the inverse point P'.

Now $\dfrac{R'}{R} = \dfrac{c}{r}$; $R^2 = r^2 + c^2 + 2rc\cos\theta$

And at point B which converts into the trailing edge of the airfoil q_2 must be zero.

Therefore $\gamma_0 = \gamma\,\dfrac{r+c}{r-c}$

Then $$q_2 = \frac{\gamma}{\pi}\left(\frac{r+c}{r-c}\right)\frac{r(1+\cos\theta)}{r^2+c^2+2rc\cos\theta}$$

Now $$k_2 = 2\frac{q}{d\zeta/dz} = \frac{q}{\sin\theta}$$

$$= -\frac{\gamma}{\pi}\left(\frac{r+c}{r-c}\right)\frac{r(1+\cos\theta)}{(r^2+c^2+2rc\cos\theta)\sin\theta}$$

Since $r^2+c^2 = rs$; $2c = a$; $\dfrac{r+c}{r-c} = \sqrt{\dfrac{s+a}{s-a}}$

$$k = k_1 + k_2$$

$$= 2a\Omega\sin\theta - 2V\cot\frac{\theta}{2} - \frac{1}{\pi}\int_a^\infty \sqrt{\frac{s+a}{s-a}}\,\frac{(1+\cos\theta)\,\gamma\,ds}{(s+a\cos\theta)\sin\theta}$$

h of the chord from the leading edge, the point advancing with a linear velocity W.

$$\Gamma = 2\pi a W\alpha + 4\pi a^2 \Omega\left(\tfrac{3}{4}-h\right)\int_0^\infty \left\{\sqrt{\frac{s+a}{s-a}} - 1\right\} \gamma \, ds$$

$$\Gamma = \Gamma_0 + \Gamma_1 \sin pt + \Gamma_2 \cos pt$$

Let $s - a = 2a\theta$

$$\frac{2ap}{W} = \lambda$$

$$C = \int_0^\infty \left\{\sqrt{\frac{\theta+1}{\theta}} - 1\right\} \cos \lambda\theta \, d\theta$$

$$S = \int_0^\infty \left\{\sqrt{\frac{\theta+1}{\theta}} - 1\right\} \sin \lambda\theta \, d\theta$$

Then

$$\Gamma_0 = 2\pi a W \alpha_0$$

$$\Gamma_1 = 2\pi a W\alpha_1 - \lambda S \Gamma_1 + \lambda C \Gamma_2$$

$$\Gamma_2 = 2\pi a W\left(\tfrac{3}{4}-h\right)\lambda\alpha_1 - \lambda C \Gamma_1 - \lambda S \Gamma_2$$

Let

$$\xi_1 = A_1 + B_1\left(\tfrac{3}{4}-h\right)$$

$$\xi_2 = A_2 + B_2\left(\tfrac{3}{4}-h\right)$$

where

$$A_1 = \frac{1+\lambda S}{(1+\lambda S)^2 + (\lambda C)^2}$$

$$A_2 = \frac{-\lambda C}{(1+\lambda S)^2 + (\lambda C)^2}$$

$$B_1 = -\lambda A_2$$

$$B_2 = \lambda A_1$$

Then $\Gamma_1 = 2\pi a W \alpha_1 \xi_1$

$\Gamma_2 = 2\pi a W \alpha_1 \xi_2$

This gives

$$\frac{M_o}{4a^2\rho W^2} = \frac{\pi}{128}\lambda^2\alpha_1 \sin pt$$

$$\frac{Y_o}{2a\rho W^2} = \frac{\pi}{2}\lambda\alpha_1 \cos pt - \frac{\pi}{4}\left(\tfrac{1}{2}-h\right)\lambda^2\alpha_1 \sin pt$$

$$\frac{Y_p}{\rho} = 2\pi a W\left\{W\alpha + a\Omega(1-2h) - aW\int_a^\infty \frac{\gamma\, ds}{\sqrt{s^2-a^2}}\right.$$

Let $P = \int_0^\infty \frac{\cos\lambda\theta}{\sqrt{\theta(\theta+1)}}\,d\theta$

$$Q = \int_0^\infty \frac{\sin\lambda\theta}{\sqrt{\theta(\theta+1)}}\,d\theta$$

Then $$\frac{Y_m}{2a\rho W^2} = \pi\alpha_o + \pi\left\{\tfrac{1}{2} - \tfrac{\lambda}{2}(Q\xi_1 - P\xi_2) - \tfrac{1}{4}\left(\tfrac{1}{2}-h\right)\lambda^2\right\}\alpha_1 \sin pt$$
$$+ \pi\left\{(1-h) - \tfrac{1}{2}(P\xi_1 + Q\xi_2)\right\}\lambda\alpha_1 \cos pt$$

$$\frac{M_h}{4a^2\rho W^2} = -\pi\left(\tfrac{1}{4}-h\right)\alpha_o$$
$$- \pi\left[\left(\tfrac{1}{4}-h\right)\left\{1 - \tfrac{\lambda}{2}(Q\xi_1 - P\xi_2)\right\} - \tfrac{1}{4}\left(\tfrac{1}{2}-h\right)^2\lambda^2 - \frac{\lambda^2}{128}\right]\alpha_1 \sin pt$$
$$- \pi\left[\left(\tfrac{1}{2}-h\right)\left(\tfrac{3}{4}-h\right) - \tfrac{1}{2}\left(\tfrac{1}{4}-h\right)(P\xi_1 + Q\xi_2)\right]\lambda\alpha_1 \cos pt$$

Let $$\mu = \pi\left\{\left(\tfrac{1}{2}-h\right)\left(\tfrac{3}{4}-h\right) - \tfrac{1}{2}\left(\tfrac{1}{4}-h\right)\left(P\xi_1 + Q\xi_2\right)\right\}$$

$$\nu = \pi\left\{(1-h) - \tfrac{1}{2}\left(P\xi_1 + Q\xi_2\right)\right\}$$

Taking those parts which are proportional to the angular velocity

$$M_h = -\mu\left(\frac{2a\Omega}{W}\right)\left(4a^2\rho W^2\right)$$

$$Y_T = \nu\left(\frac{2a\Omega}{W}\right)\left(2a\rho W^2\right)$$

Redefining the factors in the above equations -

Ω = angular frequency = ω

2a = chord = C

W = linear velocity of point h = V (or U)

μ = damping factor; μ positive means oscillation damped.

μ negative means oscillation diverges.

ν = lifting factor; positive means increas in lift.

negative means decrease in lift.

Now P, Q, C and S may be calculated making use of Bessel Functions. (See Appendix I).

Let $\lambda = \dfrac{c\omega}{V}$

Then $$P = \frac{\pi}{2}\left\{J_0\left(\frac{\lambda}{2}\right)\sin\frac{\lambda}{2} - Y_0\left(\frac{\lambda}{2}\right)\cos\frac{\lambda}{2}\right\}$$

$$Q = \frac{\pi}{2}\left\{J_0\left(\frac{\lambda}{2}\right)\cos\frac{\lambda}{2} + Y_0\left(\frac{\lambda}{2}\right)\sin\frac{\lambda}{2}\right\}$$

$$C = \frac{\pi}{4}\left\{\left[J_0\left(\frac{\lambda}{2}\right) - Y_1\left(\frac{\lambda}{2}\right)\right]\sin\frac{\lambda}{2} - \left[J_1\left(\frac{\lambda}{2}\right) + Y_0\left(\frac{\lambda}{2}\right)\right]\cos\frac{\lambda}{2}\right\}$$

$$S = \frac{\pi}{4}\left\{\left[J_0\left(\frac{\lambda}{2}\right) - Y_1\left(\frac{\lambda}{2}\right)\right]\cos\frac{\lambda}{2} + \left[J_1\left(\frac{\lambda}{2}\right) + Y_0\left(\frac{\lambda}{2}\right)\right]\sin\frac{\lambda}{2}\right\} - \frac{1}{\lambda}$$

Tables showing the variation of μ with h and λ have been computed using the formula for μ and the auxiliary formulae for P, Q, C and S. See tables 1 and 2. It will be noted that these tables are for a range of h from 0 to 0.25 and for a value of h of 0.75. The reason for concentrating on these values is that Glauert[(2] showed that these are the only two hinge positions for which the aerodynamic damping may become small or negative.

Having obtained from the Glauert theory the variation of the damping moment parameter μ , the theory of Lombard[(6] was developed along similar lines for purpose of comparison.

Lombard[(6] develops formulae for the forces and moments on an airfoil performing steady-state oscillations. He carries out this development by several different methods; his use of the theory of Von Karman and Sears[(4] is considered to be particularly clear and based on the most modern procedure, so it was used in this paper to verify the damping moment criterion μ as found by Glauert.

Lombard starts this part of his work with the expression for lift as given by Von Karman and Sears[(4] :

$L = L_0 + L_1 + L_2$ where

$$L_0 = \rho V \int_{-c/2}^{c/2} \gamma_0(x)\,dx \qquad \text{(Quasi-steady lift)}$$

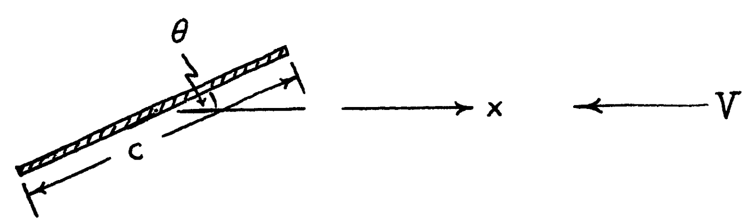
$$L_1 = \rho \frac{d}{dt}\left[\int_{-c/2}^{c/2} \gamma_0(x)\,dx\right] \qquad \text{(Apparent mass lift)}$$

$$L_2 = \frac{\rho V c}{2} \int_{c/2}^{\infty} \frac{\gamma(\xi)\,d\xi}{\sqrt{\xi^2 - \frac{c^2}{4}}} \qquad \text{(Lift from wake effects)}$$

The following Lombard notations will be used in this paper in developing a damping moment criterion similar to Glauert's:

- c = chord
- x = distance in line of motion, positive forward
- Γ = circulation
- $d\Gamma$ = circulation about any element dx.
- $\gamma(x)$ = vorticity on airfoil
- $\gamma(\xi)$ = vorticity in wake

The values of L_0 and L_1 are developed first.



Noting that

$$x = (c/2)\cos\theta$$

$$dx = -(c/2)\sin\theta\ d$$

the equations for L_0 and L_1 can be changed to

$$L_0 = \frac{\rho V c}{2} \int_0^{\pi} \gamma_o \sin\theta \, d\theta$$

$$L_1 = \frac{\rho c^2}{4} \frac{d}{dt} \left[\int_0^{\pi} \gamma_o \sin\theta \cos\theta \, d\theta \right]$$

For a value for $\gamma_o(x)$ Lombard uses

$$\gamma_o = c\dot{\phi}\sin\theta + 2V\left(\phi - \frac{y_m}{V}\right)\left(\frac{1+\cos\theta}{\sin\theta}\right)$$

$\dot{\phi}$ = angular velocity

Y_m = vertical velocity of the chord midpoint.

ϕ = angle of airfoil to wing in free stream.

Then $$\int_0^{\pi} \gamma_o \sin\theta \, d\theta = \frac{\pi c \dot{\phi}}{2} + 2\pi V\left(\phi - \frac{\dot{y}_m}{V}\right)$$

$$\int_0^{\pi} \gamma_o \sin\theta\cos\theta \, d\theta = \pi V\left(\phi - \frac{\dot{y}_m}{V}\right)$$

This leads to

$$L_0 = \frac{\pi\rho c^2}{4} V\dot{\phi} + \pi\rho c V^2\left(\phi - \frac{\dot{y}_m}{V}\right)$$

$$L_1 = \frac{\pi\rho c^2}{4}\left(V\dot{\phi} - \ddot{y}_m\right)$$

For developing L_2, account must be taken of the effects of the wake vorticity under a steady state oscillation of ν radians per second. Lombard uses the complex notation to introduce the oscillation into y and ϕ.

$$\bar{y} = \bar{C}_1 e^{i\nu t}$$

$$\bar{\phi} = \bar{C}_2 e^{i\nu t}$$

The instantaneous values of y and ϕ are represented by the real parts of $\bar{y}$ and $\bar{\phi}$.

The wake vorticity is represented by

$$\bar{\gamma}(\xi) = \bar{g}\, e^{i\nu[t-(\xi/v)]}$$

where

$\bar{g}$ is a constant which may be complex.

v is the constant mean horizontal velocity.

$$L_2 + iL_2 = \bar{L}_2 = \rho\frac{vc}{2}\int_{c/2}^{\infty}\frac{\bar{\gamma}(\xi)\,d\xi}{\sqrt{\xi^2 - c^2/4}}$$

Inserting the complex wake vorticity,

$$\bar{L}_2 = \rho\frac{vc}{2}\,\bar{g}\, e^{i\nu t}\int_{\frac{2\xi}{c}=1}^{\infty}\frac{e^{-\left(\frac{i\nu c}{2v}\right)\cdot\left(\frac{2\xi}{c}\right)}}{\sqrt{\left(\frac{2\xi}{c}\right)^2 - 1}}\, d\left(\frac{2\xi}{c}\right)$$

Using the Von Karman and Sears (4 modified Hankel function method, Lombard writes

$$\bar{L}_2 = \frac{vc}{2}\,\bar{g}\,K_o\, e^{i\nu t}$$

To satisfy the conditions of (a) equality of vorticity in the wake and vorticity shed by the wing and (b) tangential flow at the trailing edge, it is found that

$$\frac{c\bar{g}}{2} = -\frac{\bar{G}_o}{K_o + K_1}$$

where K_o and K_1 are modified Hankel functions and $\bar{G}_o$ is found by

$$L_o + iL_o = \bar{L}_o = \rho v \bar{\Gamma}_o = \rho v \bar{G}_o\, e^{i\nu t}$$

Therefore

$$\bar{L}_2 = -\bar{L}_o\left(\frac{K_o}{K_o + K_1}\right)$$

Since the modified Hankel functions K_0 and K_1 are both complex, there is a phase difference between $\bar{L}_2$ and $\bar{L}_0$. Also $\bar{L}_2$ acts at the quarter-chord point.

Substituting the equation for $\bar{L}_0$,

$$\bar{L}_0 + \bar{L}_2 = \pi \rho c V^2 \left[\phi - \frac{\dot{y}_m}{V} + \frac{\dot{\phi} c}{4V}\right]\left[\frac{K_1}{K_0 + K_1}\right]$$

$$\left[\phi - \frac{\dot{y}_m}{V} + \frac{\dot{\phi} c}{4V}\right] = \alpha_h =$$ angle of attack at the 75 percent point.

Using the Kassner-Fingado notation,

$$\frac{K_1}{K_0 + K_1} = \bar{P} = A_1 - iB_1 = \frac{K_1\left(\frac{i\nu c}{2V}\right)}{K_0\left(\frac{i\nu c}{2V}\right) + K_1\left(\frac{i\nu c}{2V}\right)}$$

where A and B are real and positive, and are functions of $\left(\frac{i\nu c}{2V}\right)$.

λ in the Glauert notation is seen to be $\frac{2}{i}\left(\frac{i\nu c}{2V}\right)$.

Using a complete complex notation, Lombard writes for the three lift values:

(a) At 25 percent chord

$$\bar{L} = \pi \rho c V^2 \bar{\alpha}_h \bar{P}$$

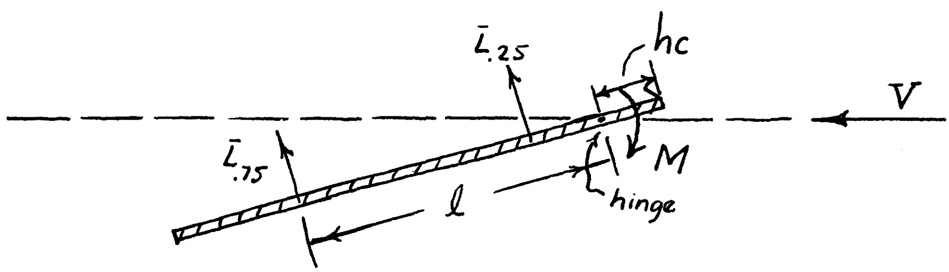
(b) At 50 percent chord

$$\bar{L} = -\frac{\pi \rho c^2}{4} \ddot{y}_m$$

(c) At 75 percent chord

$$\bar{L} = \frac{\pi \rho c^2}{4} V \dot{\phi}$$

Equation (b) above is not needed for the investigation desired in this paper, since $\ddot{y}_m$ is zero for freedom in oscillation only. Equations (a) and (c) will now be used to develop a damping moment criterion for comparison with the μ found by the Glauert method.



$$\bar{L}_{.25} = \pi \rho c V^2 \bar{\alpha}_h \bar{P}$$

$$\alpha_h = \phi + \frac{\dot{\phi} \ell}{V} = \phi + \frac{i\nu\phi\ell}{V} = \phi\left(1 + \frac{i\nu\ell}{V}\right)$$

= effective angle of attack at 0.75 chord.

$$\bar{L}_{.75} = \frac{\pi\rho c^2}{4} V \dot{\phi} = \frac{\pi\rho c^2}{4} V i \nu \phi$$

Arm of $L_{.25}$ = 0.25 c - hc = c(0.25-h)

Arm of $L_{.75}$ = 0.75c - hc = c(0.75-h)

Moment $$M = (\pi\rho c V^2)\phi\left(1 + \frac{i\nu\ell}{V}\right) c\,(0.25 - h)(A_1 - i B_1) + \frac{\pi\rho c^2}{4} V i \nu \phi c\,(0.75 - h)$$

From the general oscillation formula

$$(-A\nu^2 + i\nu B + C)\phi = M_1 + i M_2$$

It is seem that

$$i \nu B \Phi = i M_2$$

or $\nu B \Phi$ = the imaginary part of the moment.

This gives

$$B = \pi \rho c^2 V^2 (0.25-h) \left[\frac{A_1 c\,(0.75-h)}{V} - \frac{B_1}{\nu} \right] + \frac{\pi}{4} \rho c^3 V (0.75-h)$$

$$= \pi \rho c^3 V \left[(0.25-h) \left\{ A_1 (0.75-h) - B_1 \frac{V}{c\nu} \right\} + \frac{1}{4} (0.75-h) \right]$$

The critical damping condition is B = 0. If B is positive, the aerodynamic forces will dampen the motion. If B is negative, the aerodynamic forces will be such as to cause the airfoil to flutter.

For the development of flutter criterion values for comparison with the μ values found by the Glauert method, the following change is made:

$$B_2 = \frac{B}{\rho c^3 V} = \pi \left[\left(\tfrac{1}{4}-h\right) \left\{ A_1 \left(\tfrac{3}{4}-h\right) - \frac{B_1}{\lambda} + \frac{1}{4} \left(\tfrac{3}{4}-h\right) \right] \right.$$

This is done because the criterion is based on whether the factor B is positive or negative, hence it will be just as satisfactory to determine whether B_2 is positive or negative.

Using the method of computation as indicated above and outlined in Appendix II, tables of B_2 were drawn up. Se Tables 3 and 4. Comparison shows that these agree very closely with the values of μ found by the Glauert method and included in Tables 1 and 2.

The μ or B_2 values found for the various values of hinge position h and reduced frequency $\left(\frac{c\nu}{V}\right)$ are useful only to determine whether or not the aerodynamic damping is positive or negative. A change is needed to make available the amount of positive or negative damping existing for a given oscillating system. The results of the Lombard method will be used for this purpose.

In determining the true damping moment, it is best to make the equation dimensionless. For this purpose, the function g is introduced.

$$g = \frac{B}{I\nu} = \frac{\rho c^3 V}{I\nu} B_2$$

$$= \frac{\pi \rho c^4}{I} \cdot \frac{V}{c\nu} \cdot \frac{B_2}{\pi}$$

$$= \frac{\frac{\pi \rho c^2}{4}}{m} \left(\frac{2c}{k}\right)^2 \frac{1}{\lambda} \frac{B_2}{\pi}$$

Analyzing this expression,

$\frac{\pi \rho c^2}{4}$ = mass of air in a cylinder of air of diameter equal to the chord and unit length along the airfoil span.

m = mass of the oscillating system per unit length of airfoil.

$\frac{2c}{k}$ = ratio of twice the chord to the radius of gyration of the system.

λ = reduced frequency modulus

B_2 = the flutter criterion as developed previously.

For a given hinge position of an airfoil system it is seen that all terms are constant except λ and B_2, the latter being a function of λ and the hinge position. This gives

$$g' = \frac{B_2}{\lambda} \qquad \text{for one hinge position}$$

The significance of using g' instead of B_2 for a flutter criterion is that it is properly magnified by division by λ. In other words in addition to determining the value of B_2, λ also has a direct effect on the amount of positive or negative damping, giving an increased amount of damping for very small values of λ and a decreased amount for very large values of λ.

The calculation of B_2 showed that it approaches zero for the hinge at three-quarters chord at high values of λ, but never becomes negative. The effect of dividing B_2 by λ for this hinge position obviously cannot give a negative value of g' but merely makes g' nearer to zero than B_2. This would tend to give zero damping at the three-quarter chord hinge position.

For hinge positions near the leading edge conditions are different. Previous calculations of B_2 showed that negative damping occurs for hinge positions forward of 0.25

chord and for λ values less than 0.10. Dividing the B_2 values by λ to obtain g', it is seen that the true magnitude of the damping factor is greatly multiplied for these low values of λ.

From the theoretical calculations it is concluded that for a thin symmetrical airfoil of infinite aspect ratio the following facts hold:

(1) For hinge positions at or greater than 0.25 chord there is no possibility of negative damping or flutter.

(2) For hinge positions at or close to 0.75 chord the damping moment tends to zero as the reduced frequency approaches infinity.

(3) For hinge positions between zero and 0.25 chord there is definitely negative damping at low values of the reduced frequency. In general, higher values of the reduced frequency require hinge positions farther forward if negative damping or flutter is to exist.

(4) The flutter conditions at forward hinge positions are essentially low frequency, high airstream velocity oscillations.

IV. PRACTICAL INVESTIGATION

After obtaining theoretical means of determining whether or not negative damping or flutter should occur for a given condition of hinge position, airstream velocity and frequency of the oscillating system, it was desired to design and construct a model apparatus by which the theory could be verified in a wind tunnel.

The equipment which it was felt was needed was an airfoil reasonably rigid against bending and torsion, having on each end of it a shaft about which the airfoil would be restrained from performing any other movement. Means were needed for shifting the hinge position to any desired position along the chord; it was decided that this could best be accomplished by inserting in the shaft at each end of the airfoil a pair of knuckle joints.

In order to reduce the friction in oscillating movement to a minimum, it was decided that the suspension of the shaft should be on crossed fine steel wires. By having these wires attached to the shaft structure at some distance from the center of the shaft axis, and then placing the wires under tension, it was planned to be able to place any desired amount of linear restraint against oscillating movement on the shaft and hence on the airfoil.

It was also realized that means had to be provided for static balance of the system; to this end fittings were designed by means of which weights could be held rigid with reference to the shaft at any desired distance from the shaft. These were also to serve to vary the moment of inertia of the system.

The functional requirements of the test model having been decided, the necessary parts were designed and fabricated. The assembled apparatus is shown in Figures 1, 2, 3 and 4. The airfoil was made of pine, rectangular in planform, with a span of 36", a chord of 6", and an 0009 airfoil section. To the ends of the airfoil were fastened endplates which could be removed as desired during tests. By means of suitable fittings there were fitted to each end of the airfoil a section of shaft 6" long and another section 12" long. These shafts were of aluminum tubing, outer diameter 1 1/8 inches and wall thickness 0.040". The steel wires were placed in tension by hanging weights as shown. Means were provided for fastening them at distances of either 3" or 6" from the axis of the shaft; this was done by utilizing clamp studs on an 0.030" aluminum plate clamped to the shaft as shown. Also the wires could be fastened at a distance equal to the radius of the shaft by using clamping studs immediately outside the shaft. The balancing arms and counterweights are also shown in the pictures.

When the test equipment was first tried in the tunnel, it was found that there was insufficient restraint against bending of the airfoil and fittings; the center of the airfoil tended to vibrate up and down. To prevent this movement, which would not limit the airfoil to the one degree of freedom desired, a fine wire was passed through the airfoil vertically, and fastened to each side of the airfoil by means of clamping studs. Care of course had to be taken to insure this wire passing exactly through the hinge line.

It was found that the use of wire to provide the necessary controlled restraint against rotation and at the same time prevent other than infinitesimal movement in translation worked very well. Of course, there was little restraint provided against movement in the direction of the span of the airfoil, but there was not much force found to be exerted in this direction.

The proposed plan of investigation in the wind tunnel was as follows:

(a) Conduct tests with the hinge position at and near the 0.75 chord point. Frequency of oscillation and tunnel airstream speed were to be varied through as wide a range as possible to determine whether or not there exists any condition of negative damping or flutter with these hinge positions. The tests were to be conducted both with and without endplates, in order to attempt to detect the effect of the use of endplates on flutter.

(b) Conduct tests with the hinge position varying from the 0.25 chord point to the leading edge. This is the range of hinge positions in which theory shows that flutter should occur. In this part of the test it was proposed to use much finer wire than in part (a), since for forward hinge positions flutter is theoretically possible for low reduced frequency which means that high moment of inertia and low restraint against oscillation are required. As in (a), tests were to be carried out at all possible frequencies of the oscillating system and all tunnel speeds available.

The tests were run in the following manner. A desired hinge position was set up on the equipment. A series of runs was made for this position, varying the tension in the wires and the speed of the tunnel. For each state of wire tension and tunnel speed, the frequency of the system was found by means of either a Strobotac instrument or visual observation using a stop watch, the tunnel airspeed was found by means of a static pressure manometer properly calibrated, and the amount of damping was determined as accurately as possible by observing the time required for an externally forced amplitude to decrease to one-half that amplitude.

VI. DISCUSSION OF RESULTS OF EXPERIMENTAL INVESTIGATION

Tests were conducted in the P.J.C.-Merrill wind tunnel. Because of a lack of instrumentation facilities, the tests were conducted from a qualitative viewpoint, attempting for the most part to determine whether positive or negative aerodynamic damping existed. Turbulence was kept at a reasonably low value by using a fairly fine mesh wire screen; hot-wire turbulence measurements indicated a turbulence value of about 0.7 percent both along and across the direction of air flow.

(A) Leading-edge hinge positions

The first part of the wind tunnel experimentation consisted of an investigation of the damping existing for hinge positions near the leading edge, with values of reduced frequency varied over as wide a range as possible with the equipment used.

Hinge positions of 0.10 c, 0.05 c, and the leading edge were used. For each hinge position the frequency was varied from about 0.7 cycles per second to 4 cycles per second, and the wind velocity from about 15 feet per second to 50 feet per second. The limits of the reduced frequency values used were:

λ minimum = 0.072

λ maximum = 0.838

As was to be expected for forward hinge positions, the oscillating frequency of the system increased with the airstream velocity for constant torsional restraint and moment of inertia.

In all the runs, covering the full range of reduced frequency for all three hinge positions, the only motion detected was a slight vibration which was considered caused by the turbulence and the fact that the airfoil supporting system was not entirely rigid. Sample data for this part of the investigation will be found in Table A.

The theoretical calculations showed that for the forward hinge positions used and the values of reduced frequency at which the tests were conducted the μ factor reaches a negative value of 0.10 and the g' factor or μ/λ a negative value of 1.4. These negative values seemed to indicate that the test conditions should have resulted in definitely negative damping and a condition of flutter. As was stated above, however, the only motion was a slight vibration.

From the failure of the wind tunnel test to verify the theoretical calculated values of the damping factor, it was concluded that the theoretical method of developing a criterion for flutter cannot be applied to the case of an airfoil having a hinge position near the leading edge.

(B) Quarter-Chord Hinge Position

The second part of the experimental investigation consisted of tests of damping conditions with a hinge position at the quarter-chord point. Table B furnishes sample data for the tests conducted.

No tendency toward flutter was observed in this part of the investigation. The theory that the only portion of the lift on an airfoil which varies with speed is that which acts at the quarter chord point was confirmed by the fact that data taken showed the frequency to be independent of the wind velocity.

It was concluded that the theoretical values of damping moment for a quarter-chord hinge line are correct and that there is no possibility of flutter under these conditions.

(C) Three-Quarter Chord Hinge Position

The third part of the experimental investigation consisted of tests of oscillations about a hinge position at the 0.75 chord point. Tests were made at both high frequency and low frequency conditions.

For the high frequency part of the tests, the inertia of the system was kept as low as possible and the wires were rigged with a 12-inch radius and 80 pounds of weight on each of eight wires. This gave a frequency which with equipment available was estimated to be 25 cycles per second.

For the lowest velocity used, 15 feet per second, the reduced frequency was 5.24. This value of reduced frequency is high enough so that according to theoretical calculations the aerodynamic damping should be practically zero. Obviously with the high mechanical damping which was placed on the system for this part of the investigation (to produce high frequency oscillations), it would have required an extremely high negative aerodynamic damping moment to produce flutter. The tests, conducted both with and without end plates, were made in order to determine whether the aerodynamic damping moment might tend to negative values instead of merely approaching zero. Since no flutter occurred, it was concluded that no high values of negative damping existed.

The 0.75 chord hinge position was then tested for low values of reduced frequency. This part of the investigation was an attempt to verify the results reported by Biot[7]. Biot's tests were conducted for a very small model, and it was desired to conduct a similar test at a larger Reynolds Number.

This part of the test was made using values of reduced frequency varying from .0545 to .544. At low velocities it was found that divergent oscillations could be induced by deflecting the airfoil to an initial amplitude which approximated the angle at stall. For higher wind velocities it was observed that the lift at the 25 percent chord point was sufficient to deflect the airfoil in rotation to an angle approximating the stall angle at which time the condition of divergent oscillation was set up. The flutter in the case of both low and high velocity was sufficient to be catastrophic if allowed to continue. The action of the airfoil for this part of the test was similar to that found by Bollay and may be considered as corroboration to his findings. Data as obtained is recorded in Table C.

It was concluded from the investigation involving oscillation about the 0.75 chord line that, as concerns the small oscillation theory, there is no possibility of flutter. On the other hand there is a definite danger of flutter when the angle of attack of the airfoil reaches the stall. Moreover there is a definite range of velocity at which flutter is induced provided the system has insufficient restraint to prevent the stall angle being attained. It is possible that an airplane may experience such a phenomenon if the airplane is subjected to sudden heavy gusts. And, too, an airplane approaching for a landing in severe weather may be subject to this type of flutter.

After concluding the aforementioned tests a series of runs was made varying the value of reduced frequency from the high value of the first tests to the low value of the second tests. At medium values of reduced frequency violent flutter was set up. It was found that this flutter was the result of extremely small vertical oscillation in the system the frequency of which coincided with the torsional frequency of the system. In other words a perfect case for two dimensional flutter existed. The violent flutter could easily be prevented by only a slight damping of the vertical motion. This experience is recorded for the sole purpose of cautioning others who may endeavor to carry out experiments in one degree of freedom flutter. Fortunately for the authors the frequency of the small vertical oscillation had no effect at the very low and very high values of reduced frequency at which the important tests were run.

VII. RECOMMENDATIONS

The authors were not completely satisfied with the equipment which they designed and used in their experimental investigation, because it required using artificialities to prevent an additional degree of freedom being introduced by bending or vertical translation of the airfoil. The tendency of the airfoil to bend was due to the fact that it was constructed of wood, and the tendency to translate vertically was due to the low restraint against bending offered by the fittings used on the shafts extending from the airfoil to the wire supporting system.

It was therefore considered by the authors that a further investigation of the problem would be of definite value, and that the equipment should be constructed in keeping with the following considerations:

(a) The airfoil to have considerable restraint against bending; since the weight need not be kept at a minimum, the airfoil might well be made of solid aluminum.

(b) Fittings for varying the hinge position to be designed to permit no bending.

(c) Tension wires to be used to provide a bearing for oscillation, but to be connected as near as possible to the axis of oscillation so that they can be kept under high tension to prevent translation of the airfoil and at the same time not cause a high torsional restraint.

(d) Torsional restraint to be accomplished by springs acting on arms attached to the oscillation shaft of the airfoil.

(e) Instrumentation to be provided to permit accurate measurement of the frequency and the amplitude of oscillations while the system is oscillating.

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TABLE 1

μ VALUES BY GLAUERT METHOD

h ↓ \ λ →	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18
0.20	0.2202	0.2579	0.2859	0.3047	0.3204	0.3362	0.3487	0.3583
0.15	0.0628	0.1332	0.1916	0.2325	0.2670	0.2953	0.3186	0.339
0.10	-0.0792	0.0345	0.1131	0.1737	0.2262	0.2670	0.3016	0.333
0.05	-0.2064	-0.0571	0.0471	0.1294	0.1980	0.2513	0.2985	0.339
0.00	-0.3189	-0.1323	-0.0022	0.0993	0.1820	0.2482	0.3085	0.358
-0.05	-0.4150	-0.1929	-0.0374	0.0836	0.182	0.2639		
-0.10	-0.4964	-0.2384	-0.0575	0.0820	0.198			
-0.15	-0.5632	-0.2692	-0.0641	0.0946				
-0.20	-0.6145	-0.2849	-0.0556					
-0.25	-0.6506	-0.2862	-0.0405					
-0.30	-0.6720	-0.2720	0.0044					
-0.35	-0.6776	-0.2438						
-0.40	-0.6688	-0.2001						
-0.45	-0.6447	-0.1420						
-0.50	-0.6054	-0.0688						
-0.55	-0.5501	0.0188						
-0.60	-0.4798							
-0.65	-0.3955							
-0.70	-0.2956							
-0.75	-0.1803							
-0.80	-0.0503							
-0.85	+0.0952							

For all values of λ, μ for h of 0.25 is 0.3927.

TABLE 2

VALUES FOUND BY GLAUERT METHOD

$h \downarrow$ \ $\lambda \rightarrow$	0.2	0.4	0.6	0.8	1.0	30	200	∞
0.00	0.401	0.647	0.746	0.793	0.822	0.833	0.8836	0.8836
0.10	0.358	0.510	0.573	0.603	0.622	--	--	--
0.20	0.368	0.420	0.442	0.453	0.460	--	--	--
0.25	0.393	0.393	0.393	0.393	0.393	0.393	0.393	0.393
0.30	0.429	0.375	0.352	0.341	0.334	0.320	--	--
0.40	0.545	0.378	0.306	0.269	0.247	0.200	--	--
0.50	0.712	0.424	0.301	0.237	0.197	0.098	0.0982	--
0.75	1.355	0.742	0.469	0.326	0.238	0.0003	0.00001	0.00
1.00	2.330	1.345	0.901	0.661	0.514	0.098	0.0981	0.0981

TABLE 3

B$_2$ VALUES BY LOMBARD THEORY

λ

$h \downarrow \backslash \lambda \rightarrow$	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18
0.20	0.220	0.258	0.285	0.306	0.322	0.336	0.346	0.359
0.10	-0.079	0.033	0.112	0.174	0.225	0.266	0.302	0.335
0	-0.318	-0.133	-0.002	0.100	0.183	0.250	0.309	0.359
-0.05						0.263		
-0.10	-0.495	-0.239	-0.055	0.083	0.197			
-0.15				0.096				
-0.20	-0.613	-0.286	-0.055					
-0.25								
-0.30	-0.670	-0.274	0.005					
0.40	-0.667	-0.202						
0.50	-0.602	-0.071						
0.55		0.016						
0.60	-0.479							
0.70	-0.293							
0.80	-0.048							
0.85	+0.097							

TABLE 4

B_2 VALUES BY LOMBARD THEORY

$h \downarrow$ $\lambda \rightarrow$	0.20	0.40	0.60	0.80	1.00	30	200
0	0.402	0.647	0.746	0.795	0.822	0.883	0.8835
0.10	0.359	0.511	0.573	0.604	0.622	0.663	0.663
0.20	0.369	0.4205	0.4425	0.453	0.460	0.475	0.475
0.25	0.393						
0.30	0.430	0.376	0.353	0.341	0.334	0.318	0.318
0.40	0.544	0.376	0.306	0.269	0.247	0.193	0.1922
0.50	0.710	0.424	0.300	0.235	0.1975	0.0983	0.0977
0.60						0.0358	0.0352
0.70						0.0044	0.0038
0.74						0.0006	0.00030
0.75	1.354	0.7405	0.469	0.324	0.237	0.00047	0.0000126
0.76						0.00063	0.00017
1.00	2.324	1.343	0.900	0.658	0.512	0.0989	0.0991

Hinge Position - leading edge

Frequency radians/second	Velocity feet/second	λ	R	Cycles to damp to $\frac{1}{2}$ amplitude
4.94	0	-	0	12
5.23	15.2	0.172	48600	11
5.36	20.1	0.133	64200	$9\frac{1}{2}$
5.71	24.7	0.115	78900	$7\frac{1}{2}$
5.97	29.9	0.100	95500	$6\frac{1}{2}$
6.28	32.3	0.0972	103200	6
6.67	36.8	0.0906	117600	x
6.81	39.5	0.0862	126200	x
7.21	41.6	0.0866	132800	x
7.55	44.8	0.0842	143200	x
7.60	48.3	0.0786	154400	x

(x - damping could not be measured because at high speed the system was subjected to definite shaking.)

TABLE A

Hinge Position - 0.25 Chord

Frequency radians/second	Velocity feet/second	λ	R	Cycles to damp to ½ amplitude
4.74	16.2	0.1463	51700	10.5
4.74	23.6	0.1030	75300	9
4.74	26.6	0.0891	84900	8
4.74	30.8	0.0769	98300	7
4.74	34.8	0.0680	111300	5.5
4.74	39.6	0.0598	126500	4
4.74	42.0	0.0563	134200	3.5
4.74	45.2	0.0524	144300	2.5
12.33	16.7	0.370	53400	4½
12.33	23.6	0.261	75500	4
12.33	26.4	0.234	84500	4
12.33	31.9	0.193	102000	4
12.33	34.2	0.1805	109200	3.5
12.33	39.7	0.1555	126800	3.5
12.33	41.9	0.1473	133800	3

TABLE B

Hinge position - 0.75 chord.

Freq. rad/sec	V ft/sec	$\lambda = \frac{c.\omega}{V}$	R	Degrees Deflection Required to Start Oscillation
4.48	16.9	0.1325	53950	17
4.33	21.3	0.102	68000	17
3.69	26.4	0.0699	84300	17
3.37	30.9	0.0545	98700	0
6.28	0	-		-
6.21	17.6	0.1765	56200	17°
5.27	21.3	0.1237	68000	17
4.91	27.2	0.0903	86800	15.5
4.71	32.1	0.0734	102500	0
6.28	16.8	0.187	53700	18.5°
6.09	21.1	0.1443	67400	19
5.02	26.0	0.0965	83000	17
-	31.5	-	100600	0
8.97	16.9	0.2655	53950	23
7.56	21.1	0.1793	67400	19
6.82	26.6	0.1282	84900	17
-	31.5	-	100600	0
12.56	16.5	0.380	52700	(x)
7.49	20.7	0.181	66100	18°
11.83	27.1	0.218	86600	0
17.42	16.0	0.544	51100	(x)
15.68	20.0	0.392	63900	(x)
16.98	26.7	0.3175	81200	18°
15.68	31.3	0.2505	100000	18
-	35.8	-	114300	0

0 - starts torsional divergence without initial deflection from neutral position.

x - to start oscillations required deflection that could not be measured.

TABLE C

Appendix I

$$P = \int_0^\infty \frac{\cos \lambda\theta}{\sqrt{\theta(\theta+1)}}\, d\theta \qquad Q = \int_0^\infty \frac{\sin \lambda\theta}{\sqrt{\theta(\theta+1)}}\, d\theta$$

$$P + iQ = \int_0^\infty \frac{e^{i\lambda\theta}}{\sqrt{\theta(\theta+1)}}\, d\theta$$

Let $\theta + 1 = \frac{z+1}{2}$; $\theta - 1 = \frac{z-1}{2}$

Then
$$P + iQ = e^{-\frac{i\lambda}{2}} \int_1^\infty \frac{e^{\frac{1}{2} i z}\, dz}{\sqrt{z^2-1}}$$

$$= e^{-\frac{i\lambda}{2}} \left[-\frac{\pi}{2} Y_0\left(\frac{\lambda}{2}\right) + \frac{\pi}{2} J_0\left(\frac{\lambda}{2}\right) i \right]$$

Thus

$$P = \frac{\pi}{2} \left\{ J_0\left(\frac{\lambda}{2}\right) \sin\frac{\lambda}{2} - Y_0\left(\frac{\lambda}{2}\right) \cos\frac{\lambda}{2} \right\}$$

$$Q = \frac{\pi}{2} \left\{ J_0\left(\frac{\lambda}{2}\right) \cos\frac{\lambda}{2} + Y_0\left(\frac{\lambda}{2}\right) \sin\frac{\lambda}{2} \right\}$$

$$C = \int_0^\infty \left\{ \sqrt{\frac{\theta+1}{\theta}} - 1 \right\} \cos \lambda\theta\, d\theta$$

$$S = \int_0^\infty \left\{ \sqrt{\frac{\theta+1}{\theta}} - 1 \right\} \sin \lambda\theta\, d\theta$$

$$C + iS = \int_0^\infty \left\{ \sqrt{\frac{\theta+1}{\theta}} - 1 \right\} e^{i\lambda\theta} d\theta$$

Let $\theta = \frac{x-1}{2}$

Then $$\int_0^\infty \left\{ \sqrt{\frac{\theta+1}{\theta}} - 1 \right\} e^{i\lambda\theta} d\theta =$$

$$\frac{e^{-\frac{i\lambda}{2}}}{2} \int_1^\infty \left\{ \sqrt{\frac{x+1}{x-1}} - 1 \right\} e^{\frac{i\lambda x}{2}} dx =$$

$$\frac{e^{-\frac{i\lambda}{2}}}{2} \int_1^\infty \left\{ \frac{x+1}{\sqrt{x^2-1}} - 1 \right\} e^{\frac{i\lambda x}{2}} dx =$$

$$\frac{e^{-\frac{i\lambda}{2}}}{2} \left\{ \int_1^\infty \frac{e^{\frac{i\lambda x}{2}}}{\sqrt{x^2-1}} dx + \int_1^\infty \left\{ \frac{x}{\sqrt{x^2-1}} - 1 \right\} e^{\frac{i\lambda x}{2}} dx \right\}$$

$$K_n(z) = \frac{\sqrt{\pi}}{\Gamma(n+\frac{1}{2})} \left(\frac{z}{2}\right)^n \int_1^\infty e^{-z\varepsilon} (\varepsilon^2-1)^{n-\frac{1}{2}} d\varepsilon$$

$$K_0(z) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2})} \int_1^\infty e^{-z\varepsilon} (\varepsilon^2-1)^{-\frac{1}{2}} d\varepsilon$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \quad \therefore K_0(z) = \int_1^\infty \frac{e^{-z\varepsilon}}{\sqrt{\varepsilon^2-1}} d\varepsilon$$

And $$\int_1^\infty \frac{e^{\frac{i\lambda x}{2}} dx}{\sqrt{x^2-1}} = K_0\left(\frac{i\lambda}{2}\right) = i\frac{\pi}{2} H_0^{(1)}\left(\frac{\lambda c}{2}\right) \quad (a)$$

$$f\left(\frac{i\lambda}{2}\right) = \int_1^\infty \left\{ \frac{x}{\sqrt{x^2-1}} - 1 \right\} e^{\frac{i\lambda x}{2}} dx$$

$$F\left(\frac{i\lambda}{2}\right) = \int f\left(\frac{i\lambda}{2}\right) d\left(\frac{i\lambda}{2}\right) + C$$

$$= \int_1^\infty \left\{ \frac{x}{\sqrt{x^2-1}} - 1 \right\} \frac{e^{\frac{i\lambda x}{2}}}{x} dx + C$$

$$= \int_1^\infty \frac{e^{\frac{i\lambda x}{2}} dx}{\sqrt{x^2-1}} - \int_1^\infty \frac{e^{\frac{i\lambda x}{2}}}{x} dx + C$$

Let $x = \frac{2z}{\lambda}$

Then $\int_1^\infty \frac{e^{\frac{i\lambda x}{2}}}{x} dx = \int_{\frac{\lambda}{2}}^\infty \frac{e^{iz}}{z} dz$

$$F\left(\frac{i\lambda}{2}\right) = K_0\left(-\frac{i\lambda}{2}\right) - \int_{\frac{\lambda}{2}}^\infty \frac{e^{iz}}{z} dz + C$$

$$F'\left(\frac{i\lambda}{2}\right) = -K_0'\left(-\frac{i\lambda}{2}\right) - \frac{1}{i} \frac{\partial}{\partial \frac{\lambda}{2}} \int_{\frac{\lambda}{2}}^\infty \frac{e^{iz}}{z} dz$$

$$= -K_0'\left(-\frac{i\lambda}{2}\right) - \frac{1}{i} \left\{ \frac{-e^{\frac{i\lambda}{2}}}{\frac{\lambda}{2}} \right\}$$

$$= -K_0'\left(\frac{i\lambda}{2}\right) + \frac{2e^{\frac{i\lambda}{2}}}{i\lambda}$$

$$\int_1^\infty \left\{\frac{x}{\sqrt{x^2-1}} - 1\right\} e^{\frac{i\lambda x}{2}}\,dx = -K_0'\left(-\frac{i\lambda}{2}\right) + \frac{2e^{\frac{i\lambda}{2}}}{i\lambda}$$

$$K_0'\left(-\frac{i\lambda}{2}\right) = -K_1\left(-\frac{i\lambda}{2}\right)$$

$$\int_1^\infty \left\{\frac{x}{\sqrt{x^2-1}} - 1\right\} e^{\frac{i\lambda x}{2}}\,dx = +K_1\left(-\frac{i\lambda}{2}\right) + \frac{2e^{\frac{i\lambda}{2}}}{i\lambda} \quad (b)$$

Thus from (a) & (b)

$$C + iS = \int_0^\infty \left\{\sqrt{\frac{\theta+1}{\theta}} - 1\right\} e^{i\lambda\theta}\,d\theta =$$

$$= \tfrac{1}{2} e^{\frac{i\lambda}{2}} \left\{K_0\left(-\frac{i\lambda}{2}\right) + K_1\left(-\frac{i\lambda}{2}\right) + \frac{2e^{\frac{i\lambda}{2}}}{i\lambda}\right. =$$

$$= \frac{e^{\frac{i\lambda}{2}}}{2}\left\{ i\frac{\pi}{2} H_0^{(1)}\left(\frac{\lambda}{2}\right) - \frac{\pi}{2} H_1^{(1)}\left(\frac{\lambda}{2}\right) + \frac{2e^{\frac{i\lambda}{2}}}{i\lambda}\right\}$$

Now

$$H_0^{(1)}\left(\frac{\lambda}{2}\right) = J_0\left(\frac{\lambda}{2}\right) + iY_0\left(\frac{\lambda}{2}\right)$$

$$H_1^{(1)}\left(\frac{\lambda}{2}\right) = J_1\left(\frac{\lambda}{2}\right) + iY_1\left(\frac{\lambda}{2}\right)$$

Thus

$$C = \frac{\pi}{4}\left\{\left[J_0\left(\frac{\lambda}{2}\right) - Y_1\left(\frac{\lambda}{2}\right)\right]\sin\frac{\lambda}{2} - \left[J_1\left(\frac{\lambda}{2}\right) + Y_0\left(\frac{\lambda}{2}\right)\right]\cos\frac{\lambda}{2}\right\}$$

$$S = \frac{\pi}{4}\left\{\left[J_0\left(\frac{\lambda}{2}\right) - Y_1\left(\frac{\lambda}{2}\right)\right]\cos\frac{\lambda}{2} + \left[J_1\left(\frac{\lambda}{2}\right) + Y_0\left(\frac{\lambda}{2}\right)\right]\sin\frac{\lambda}{2}\right\} - \frac{1}{\lambda}$$

APPENDIX II

Computation of B_2 Values by Lombard Method.

From the formulae

$$A - iB = \frac{K_1(ix)}{K_0(ix) + K_1(ix)}$$

and

$$K_1(ix) = -i\frac{\pi}{2} H_0^{(2)}(x)$$

$$K_0(ix) = -\frac{\pi}{2} H_1^{(2)}(x)$$

$$H_0^{(2)}(x) = J_0(x) - iY_0(x)$$

$$H_1^{(2)}(x) = J_1(x) - iY_1(x)$$

it follows that

$$A_1 = \frac{J_1(x)\left[Y_0(x) + J_1(x)\right] - Y_1(x)\left[J_0(x) - Y_1(x)\right]}{\left[Y_0(x) + J_1(x)\right]^2 + \left[J_0(x) - Y_1(x)\right]^2}$$

$$B_1 = \frac{Y_1(x)\left[Y_0(x) + J_1(x)\right] + J_1(x)\left[J_0(x) - Y_1(x)\right]}{\left[Y_0(x) + J_1(x)\right]^2 + \left[J_0(x) - Y_1(x)\right]^2}$$

These values, found using $\frac{c\nu}{2V}$ for (x), are used in the equation

$$B_2 = \frac{B}{\rho c V^3} = \pi\left[\left(\frac{1}{4} - h\right)\left\{A_1\left(\frac{3}{4} - h\right) - \frac{B_1}{\lambda}\right\} + \frac{1}{4}\left(\frac{3}{4} - h\right)\right]$$

$$\lambda = \frac{c\nu}{2V}$$

Fig 1.

Fig 2.

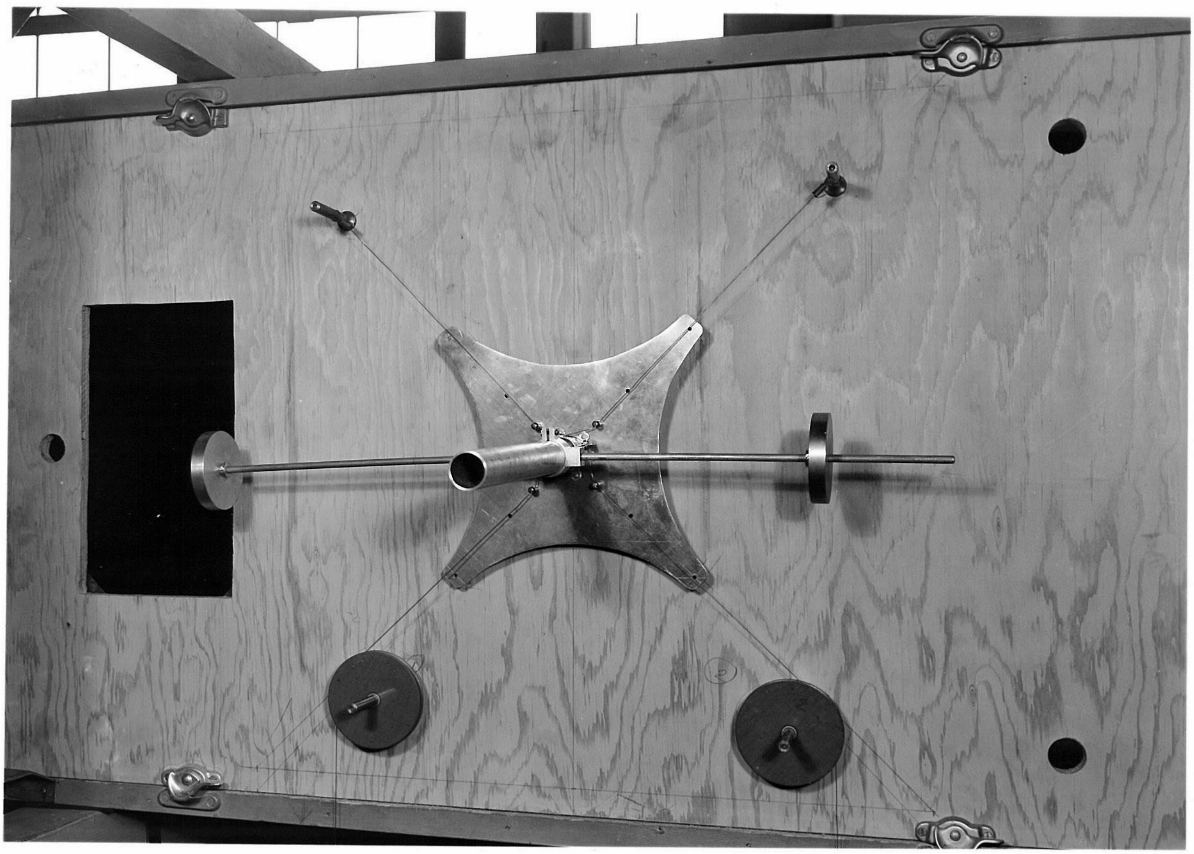

Fig. 3

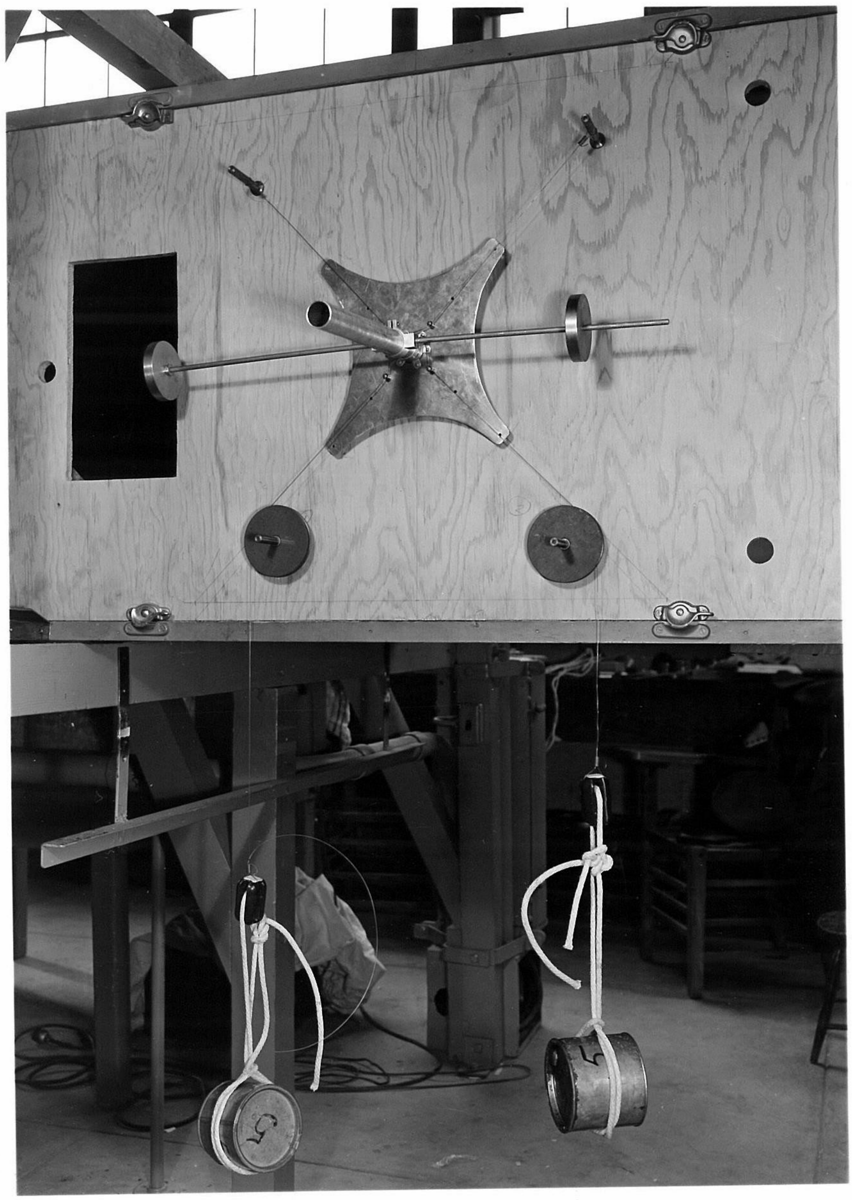

Fig. 4