QUANTUM PHASE TRANSITIONS IN DISORDERED BOSE SYSTEMS

Thesis by

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Abstract

We study the nature of various quantum phase transitions corresponding to the onset of superfluidity, at zero temperature, of bosons in a quenched medium. Particle-hole symmetry plays an essential role in determining the universality class of the transitions. To obtain a model with an exact particle-hole symmetry it is necessary to use the Josephson junction array Hamiltonian, which may include disorder in the Josephson couplings between phases at different sites. The functional integral formulation of this problem in d spatial dimensions yields a $(d+1)$-dimensional classical XY-model with extended disorder, constant along the extra imaginary time dimension – the so-called *random rod problem*. Particle-hole symmetry may then be broken by adding nonzero chemical potentials or site energies, which may also be site dependent and random. We may then distinguish three cases: (i) exact particle-hole symmetry, in which the site energies all vanish, (ii) *statistical* particle-hole symmetry in which the site energy distribution is symmetric about zero and hence vanishes on average, and (iii) complete absence of particle-hole symmetry in which the distribution is generic. We explore in each case the nature of the excitations in the nonsuperfluid Bose glass phase. We find, for example, that the compressibility, which has the interpretation of a *temporal* spin stiffness or superfluid density, is positive in cases (ii) and (iii), but that it vanishes with an essential singularity as full particle-hole symmetry is restored. We then focus on the critical point and discuss the validity of various scaling arguments. In particular, we argue that the dynamical exponent z could be different from d, and the arguments leading to their equality are incorrect. We then discuss the relevance of a type (ii) particle-hole symmetry breaking perturbation to the random rod critical behavior, identifying a nontrivial crossover exponent. This exponent cannot be calculated exactly but is argued to be positive and the symmetry breaking perturbation therefore relevant. We argue next that a perturbation of type (iii) is irrelevant to the resulting type (ii) critical behavior: the statistical symmetry is *restored* on large scales close to the critical point, and case (ii) therefore describes the dirty boson fixed point. Using various duality transformations we verify all of these ideas in

one dimension. To study higher dimensions we attempt, with partial success, to generalize the Dorogovtsev-Cardy-Boyonovsky double epsilon expansion technique to this problem. We find that when the dimension of time $\epsilon_\tau < \epsilon_\tau^c \simeq \frac{8}{29}$ is sufficiently small the symmetry breaking perturbation of type (ii) is *irrelevant*, but that for sufficiently large $\epsilon_\tau > \epsilon_\tau^c$ this is a *relevant* perturbation and a new stable commensurate fixed point appears. We speculate that this new fixed point becomes the dirty boson fixed point when $\epsilon_\tau = 1$. We also show that for $\epsilon_\tau \ll 1$, there exists a particle-hole asymmetric fixed point of type (iii), but we provide evidence that it merges with the commensurate fixed point for some finite $\epsilon_\tau \approx \frac{2}{3}$. This tends to confirm symmetry restoration at the physical $\epsilon_\tau = 1$.

Contents

Chapter 1

Introduction

There has been much excitement in recent years with the discovery of novel *zero temperature* quantum phase transitions in condensed matter systems. These transitions are driven entirely by quantum fluctuations in the ground state wavefunction. In many cases a crucial requirement is the presence of quenched disorder. Examples include random magnets with various kinds of site, bond [1], or field [2] disorder; the transitions between plateaux in the two-dimensional quantized Hall effects [3]; metal-insulator [4], metal-superconductor, and superconductor-insulator [5] transitions in disordered electronic systems; and the onset of superfluidity of ^4He in porous media [6].

Here we will be studying the zero-temperature superfluid-insulator transition for bosons, with short range repulsion, moving in both periodic and random external potentials. This was motivated by the problem of the onset of superfluidity in a porous media, such as Vycor Glass. However, as is often the case in the study of critical phenomena, the universality class of this transition, or a straightforward generalization of it, also includes other physical phenomena, including many aspects of quantum magnetism and superconductivity.

1.1 Helium in Vycor Glass

Although the basic bulk superfluid transition has been studied for several decades, it is only comparatively recently that we have been looking at the superfluid transition under more exotic conditions. Motivated to some extent by experiments on the Kosterlitz-Thouless transition in two-dimensional thin films [7], experimentalists began to explore the behavior of ^4He absorbed in various porous media, especially Vycor glass [8]. One interest was in very low coverages of helium, with the expectation that the resulting thermodynamic behavior might appear two-dimensional. The great surprise was that over a wide range of coverage of helium in Vycor, superfluid density data demonstrated clear bulk three-dimensional behavior. Even more surprising was the behavior at ultra-low coverages, where the interatomic spacing is of the order 50 to 100Å, 20-30 times the helium effective hard-core diameter, a. There is aways a localized/frozen "inert" monolayer or so, of density ρ_c and we define the effective density, $\bar{\rho}$, is the total density minus the density of this inert layer. In this regime of ultra-low $\bar{\rho}$ (of order 1% of ρ_c), the superfluid density profiles resembled those of a an ideal Bose gas, vanishing nearly linearly as $T_\lambda(\bar{\rho})$ is approached.

The interpretation of these observations is that the pores are interconnected and the atoms move throughout a three-dimensional volume. Hence the three-dimensional nature of the phase transition. The characteristic size of an atomic wavepacket is set by the thermal de Broglie wavelength $\Lambda_T = h/(2\pi m k_B T)^{1/2}$, where m is the mass of the helium atom. At ultra-low coverages, and low temperatures, Λ_T becomes of the order the pore size. In this limit, one could imagine helium behaving like a dilute Bose gas in which the only role of the porous medium could be to yield an effective mass m_{eff}, and some effective interatomic scattering potential [9]. A key effect of the porous medium must be to screen out the long range attractive tail of the interatomic potential in order to preempt condensation. The superfluid transition in an ideal Bose gas takes place when the interatomic separation, $d_s \sim \Lambda_T$, hence we may estimate $T_\lambda(\bar{\rho}) \sim \bar{\rho}^{2/3}$. Thus at such low coverages T_λ is strongly suppressed, and at the superfluid transition the dilute Bose gas picture should hold. This picture seems to agree reasonably well with experiments.

However, the problem with the above effective medium picture is that it does not account for the disorder or randomness inherent in porous media. The effect of disorder near a critical point can be very serious, sometimes completely changing the universality class of the transition. There is a simple criterion, called the Harris criterion [10], which determines when this happens: if the specific heat exponent, α, of the pure nonrandom transition is positive, then disorder is relevant and changes the nature of the transition, else it is irrelevant. For bulk ^{4}He, $\alpha \approx -.0126$, is negative. However, for low coverages the critical behaviour crosses over to that of an ideal Bose gas, with specific heat exponent $\alpha = 1/2$, a strongly positive value. Thus we expect disorder to have an increasingly stronger effect at low coverages. For Vycor, the disorder on long length scales is believed to be extremely weak, making it hard to look for the effect of disorder. However, trying to understand its effect takes us into the realm of localization effects and quantum phase transitions in interacting boson systems.

Quantum phase transitions are those that occur at zero temperature, where fluctuations are entirely due to the Heisenberg uncertainty principle, i.e. the fact that certain observables do not commute with the Hamiltonian. For Helium in Vycor glass, the superfluid transition temperature decreases with coverage, and vanishes at some critical coverage, ρ_c. Below this coverage superfluidity ceases to exist even at T=0. Alternatively, at T=0, if we vary the coverage ρ the system passes from an insulating localized phase at $\rho < \rho_c$ to a superfluid phase at higher values of ρ. This insulator-superfluid transition at T=0 is a quantum phase transition.

1.2 General theory

Localization transitions in fermions have received a lot of attention [4]. There, for a given external potential, it is believed that one has low energy localized states separated from higher energy extended states by a mobility edge. Therefore if free fermions are added to such a system at T=0, two per state, one first fills up the localized levels until a critical density ρ_c is reached, at which point all states below the mobility edge are filled, then any excess density ρ_c goes into the extended states. These excess electrons are free to move about the entire system, and contribute to the conductivity. This Anderson metal-insulator transition is another example of a quantum phase transition. As we aproach the transition from the insulating side, the localization length $\xi \sim |\bar{\rho}|^{-\nu}$ for $\bar{\rho} < 0$, and measures the diverging linear extent of the localized states as the mobility edge is approached. The exponent ν is highly nontrivial.

Building upon the above picture, we now consider bosons instead of fermions. It is immediately apparent that repulsive interactions are crucial: without them the ground state of the system would consist of all particles occupying the lowest energy single-particle localized state. Any kind of short range repulsion would preclude behavior of this type. It is essential to have short range repulsion to get a superfluid onset transition. The insulator-superfluid transition is a consequence of the competition between the kinetic energy, which tries to delocalise the particles and reduce phase fluctuations, and the random potentials which try to pin down the particles and hence increase phase fluctuations. This competition, in conjunction with short range repulsion, plays an essential role in the analysis of this transition. A crude picture of what happens is the following: as particles are added to the system they occupy the lower-lying localised states. As the density is increased, short range repulsion become important, and beyond a critical density, ρ_c, excess particles go into extended states, just as in the free fermion case. This picture provides a framework for a microscopic description of the formation of the frozen "inert" layer in Vycor, and an intuitive argument for the existence of a critical density ρ_c.

To obtain a more precise picture, it is convenient to begin with a simple lattice boson model with nearest neighbor hopping and an onsite repulsion $U > 0$. Disorder could be incorporated in both the random site energies ε_i and the hopping J_{ij}. Thus we consider the second-quantized Hamiltonian, sometimes called the boson Hubbard model,

$$\begin{aligned} \mathcal{H}_B &= -\frac{1}{2}\sum_{i,j} J_{ij}[a_i^\dagger a_j + a_j^\dagger a_i] + \sum_i (\varepsilon_i - \mu)\hat{n}_i \\ &+ \frac{1}{2}\sum_{i,j} V_{ij}\hat{n}_i(\hat{n}_j - \delta_{ij}), \end{aligned} \tag{1.1}$$

where $\hat{n}_i = a^\dagger{}_i a_i$ is the site number operator and μ the overall chemical potential. The index i is assumed to label the lattice sites which could be thought of as idealized pores. For this model there exist three different phases: the Mott insulating phase, the Bose glass phase and the superfluid phase. For the Mott phase, each site has an integer number of particles, n, per site. This is a localized, incompressible phase. In the Bose glass phase, quasiparticle and quasihole excitaions are trapped by disorder. This is a localized but compressible phase, the Bose equivalent of the Fermi glass. The superfluid phase is characterized by long-range order. We will discuss the phase diagrams in Chapter 2. It is believed that in the presence of disorder, the onset to superfluidity always occurs from the Bose glass phase. Only in the absence of disorder can one get a direct Mott to superfluid transition.

Considerable progress in the understanding of transition has been made [6]. Various scaling arguments have been proposed. Many of the detailed results come from Monte Carlo simulations [11]. They provide the best numerical estimates of the critical exponents in 2 dimensions. The basic phase diagrams have been verified in 1 and 2 dimensions (though there is some controversy about whether the Bose glass phase mediates between the Mott and superfluid phases, at certain special commensurate points). Recently it has also been demonstrated that in mean field theory one can can obtain a Bose glass phase, though only as a *line* separating the Mott and superfluid phases [14].

The main motivation for the present work is an attempt to find a satisfactory dimensionality expansion about the upper critical dimension for the transition (analogous to the epsilon expansion about $d = 4$ for classical spin problems). A previous attempt [12], although yielding very reasonable answers, encountered technical difficulties which were never satisfactorily resolved. In this paper we shall resolve these difficulties, demonstrating that the situation is far more complex than was assumed in [12]. In particular, we shall argue that the approach [13] based on a simultaneous expansion in both the dimension, ϵ_τ, of imaginary time (physically equal to unity), and the deviation, $\epsilon = 4 - D$, of the total space-time dimensionality, $D = d + \epsilon_\tau$, from four does not yield a perturbatively accessible renormalization group fixed point, and therefore does not produce a systematic expansion for the dirty boson problem. Nevertheless it does produce nonsystematic estimates for the fixed point properties and we shall argue that certain basic features, such as the mechanism by which mean field theory becomes valid above $d = 4$, were correctly predicted by the original calculation [12]. A key issue that has to be taken into account is particle-hole symmetry, which plays a major role in determining the universality class of the transition. We provide support for statistical particle-hole restoration at the critical point for the generic transition.

1.3 Applications in superconductivity

It would be worth mentioning possible examples of disordered Bose systems besides helium. These include, in particular, two-dimensional granular and amorphous superconductors, and magnetic flux phases of high T_c superconductors.

Superconductor-insulator transitions are observed in granular films, prepared by depositing soft metals such as Sn, Pb, Ga, Al, and In on insulating sustrates [5]. The starting point for comparisons between such granular superconductors and superfluidity is the Josephson junction array Hamiltonian

$$\begin{aligned}\mathcal{H}_J &= -\sum_{i,j} \tilde{J}_{ij} cos(\hat{\phi}_i - \hat{\phi}_j) + \sum_i (\tilde{\varepsilon}_i - \tilde{\mu})\tilde{n}_i \\ &+ \frac{1}{2}\sum_{i,j} U_{ij}\tilde{n}_i\tilde{n}_j, \qquad (1.2)\end{aligned}$$

where $\hat{\phi}_i$ is the phase operator on grain i, and $\tilde{n}_i$ is the conjugate number operator which measures the deviation of the number of Cooper pairs on grain i from some reference value

N_0. The idea is that there exists the usual Josephson coupling between the superconducting grains, giving rise to the $\tilde{J}_{ij} cos(\hat{\phi}_i - \hat{\phi}_j)$ term. The $\tilde{U}$ term is the so-called charging energy which disfavors large fluctuations in the $\tilde{n}_i$. We will talk more about the detailed connection between the Josephson Hamiltonian and the boson Hamiltonian in Chapter 2. The general structure of the phase diagram is similar, with Mott, Bose glass, and superfluid phases present in both systems. The critical behavior is also believed to be the same.

The essence of the equivalence between granular superconductors and superfluids is the assumption of the existence of well defined Cooper pairs well before actual superconductivity occurs. Thus the operators $\hat{\phi}_i$, $\tilde{n}_i$ are well defined even above the critical temperature. In granular systems this assumption is valid because individual grains are usually sufficiently large that they behave like small pieces of bulk superconductor, and order at the bulk transition temperature, T_c^0. Ordering between grains, mediated by the Josephson coupling $\tilde{J}$, occurs at a much lower temperature $T_c(\tilde{J}) \ll T_c^0$. Thus well defined Cooper pairs exist within each grain and may be treated to a good approximation as bosons. To the extent that all excitations of a fermionic character, such as pair-breaking and residual interactions with normal electrons are separated from the bosonic excitations by a finite energy scale $\Delta E \gg T_c(\tilde{J})$ this treatment should be exact near the critical point.

For amorphous systems, such as Bismuth films, without well defined grains, the validity of the boson model is less clear. However, it can be argued [15] that in dirty systems, the role of grains is played by localized states in which it is favourable to put pairs of electrons. Nearby localized states are then assumed to interact via some effective Josephson coupling, eventually leading to bulk superconductivity. An experimental signal of this would again be the existence of a well defined energy gap between hopping of localized Cooper pairs and single electron-type excitations.

As a final example of boson physics in electronic systems, we mention the exotic magnetic fluxes of high-temperature superconductors. Using the Feynman path-integral formulation of boson statistical mechanics, one may view the flux lines in the mixed phases of high T_c compounds as boson world lines, with time progressing parallel to the applied field [16]. The sample thickness then represents the effective inverse temperature of a two-dimensional system of interacting bosons. For columnar disorder, superconducting crystals (e.g. YBCO) are bombarded with energetic heavy ions of tin, iodine or lead to produce columnar pins for the flux lines. For the magnetic field aligned in the direction of the columns, the system can be mapped onto the disordered boson model, and one expects similar phases in this system. The superfluid phase corresponds to an entangled flux fluid phase, while in the Bose glass phase the flux lines get pinned by the columns [17]. The Meissner phase corresponds to a Mott insulating phase. This provides us with a more visual picture of the transitions.

There has also been a lot of attention devoted to the case of "point" impurities in the form of, say, oxygen vacancies. Here the bosons see a time-varying as well as spatially varying random potential [18]. So, instead of the conventional Bose glass, one has instead a "vortex glass." Very little is understood about this phase, or the superfluid–vortex glass transition. One can easily write down a standard functional integral representation of this system. However we find that in the limit of short range (onsite) repulsion this model does not show a disorder mediated transition at all for a three-dimensional sample. So, the nature of this transition, perhaps dominated by longer range interactions, is an open question.

1.4 Outline

The outline of the remainder of this thesis is as follows. In Chap 2 we discuss the role of particle-hole symmetry in the context of the two Hamiltonians (1.1) and (1.2). We discuss the phase diagrams and introduce various useful functional integral formulations for the thermodynamics. We begin in Chap. 3 by considering the role of particle-hole symmetry in the nature of the excitation spectra of the glassy phases. Using a phenomenological model in which we view the structure of the Bose glass phase as a set of random sized, randomly placed isolated superfluid droplets, we focus on the density and compressibility and examine how they vanish as full particle-hole symmetry is restored. In Chap. 4 we begin focusing on the critical point through various phenomenological scaling arguments. In particular we identify a new crossover exponent that describes the relevance of particle-hole symmetry breaking perturbations to the random rod critical behavior. We also discuss the asymptotic restoration of statistical particle-hole symmetry at the dirty boson critical point. In Chap. 5 we illustrate all of these ideas using an exactly soluble one-dimensional model. The analysis is very similar to that of the Kosterlitz-Thouless transition in the classical two-dimensional XY-model. In Chap. 6 we generalize the previous analyses to general $\epsilon_\tau \neq 1$, observing along the way some apparent pathologies that make $\epsilon_\tau = 1$ very special, leading one to question how smooth the limit $\epsilon_\tau \rightarrow 1$ might be. We then introduce the Dorogovtsev-Cardy-Boyanovsky double epsilon expansion formalism, the calculations support asymptotic particle-hole restoration at the critical point. Finally, two appendices outline the derivations of various path integral formulations and duality transformations used in the body of the thesis.

Chapter 2

Lattice models and particle-hole symmetry

It will transpire that an essential ingredient that is necessary in order to correctly understand the physics is an extra "hidden" symmetry, which we call *particle-hole* symmetry, that is present at the critical point, but not away from it. To make this notion precise, we compare the following two lattice models of superfluidity: the first is the usual lattice boson Hamiltonian,

$$\begin{aligned}\mathcal{H}_B &= -\frac{1}{2}\sum_{i,j} J_{ij}[a_i^\dagger a_j + a_j^\dagger a_i] + \sum_i (\varepsilon_i - \mu)\hat{n}_i \\ &+ \frac{1}{2}\sum_{i,j} V_{ij}\hat{n}_i(\hat{n}_j - \delta_{ij}),\end{aligned} \tag{2.1}$$

where $J_{ij} = J_{ji}$ is the hopping matrix element between sites i and j, which we will allow to have a random component; μ is the chemical potential whose zero we fix by choosing the diagonal components, J_{ii}, of J_{ij} in such a way that $\sum_j J_{ij} = 0$ for each i; ε_i is a random site energy with *mean zero*; $V_{ij} = V_{ji}$ is the pair interaction potential, assumed for simplicity to be *non*random; the only nonzero commutation relations are $[a_i, a_j^\dagger] = \delta_{ij}$, and $\hat{n}_i \equiv a_i^\dagger a_i$ is the number operator at site i. The second is the Josephson junction array Hamiltonian,

$$\begin{aligned}\mathcal{H}_J &= -\sum_{i,j} \tilde{J}_{ij} cos(\hat{\phi}_i - \hat{\phi}_j) + \sum_i (\tilde{\varepsilon}_i - \tilde{\mu})\tilde{n}_i \\ &+ \frac{1}{2}\sum_{i,j} U_{ij}\tilde{n}_i\tilde{n}_j,\end{aligned} \tag{2.2}$$

with analogous parameters, but now the commutation relations $[\hat{\phi}_i, \tilde{n}_j] = i\delta_{ij}$. These two Hamiltonians are, in fact, very closely related. It is easy to check that if N_0 is any positive integer then

$$\begin{aligned}a_i^\dagger &= (N_0 + \tilde{n}_i)^{\frac{1}{2}} e^{i\hat{\phi}_i} \\ a_i &= e^{-i\hat{\phi}_i}(N_0 + \tilde{n}_i)^{\frac{1}{2}}\end{aligned} \tag{2.3}$$

satisfy the correct Bose commutation relations, and we identify $\hat{n}_i = N_0 + \tilde{n}_i$. Note, however, that the commutation relations between $\hat{\phi}_i$ and $\tilde{n}_i$ permit $\tilde{n}_i$ to have any integer eigenvalue, positive or negative, whereas the eigenvalues of $\hat{n}_i$ must be non-negative. Therefore it is only when N_0 is large, and the fluctuations in $\tilde{n}_i$ are small compared to N_0, that $\mathcal{H}_J$ and $\mathcal{H}_B$ may be compared quantitatively: inside the hopping term we may then approximate $a_i^\dagger \approx N_0^{\frac{1}{2}} e^{i\hat{\phi}_i}$, $a_i \approx N_0^{\frac{1}{2}} e^{-i\hat{\phi}_i}$ and make the identifications

$$\tilde{J}_{ij} = N_0 J_{ij};\ U_{ij} = V_{ij};\ \tilde{\varepsilon}_i = \varepsilon_i;\ \tilde{\mu} = \mu - N_0 \hat{V}_0 - \frac{1}{2} V_0, \tag{2.4}$$

where $V_0 = V_{ii}$ and $\hat{V}_0 = \sum_j V_{ij}$, and there exists an overall additive constant term $E_0 N = (\frac{1}{2} N_0 \hat{V}_0 + V_0 - \mu) N_0 N$, where N is the number of lattice sites.

Despite this asymptotic equivalence at large N_0, the Josephson Hamiltonian, (2.2), has an exact discrete symmetry which the boson Hamiltonian lacks. Thus the constant shift, $\tilde{n}_i' = \tilde{n}_i + n_0$, where n_0 is any integer, has no effect on the commutation relations or the eigenvalue spectrum of the $\tilde{n}_i$. The Hamiltonian correspondingly transforms as

$$\mathcal{H}_J\{\tilde{n}_i'\} = \mathcal{H}_J\{\tilde{n}_i\} + n_0 \hat{U}_0 \sum_i \tilde{n}_i + N\varepsilon^0(n_0, \tilde{\mu}), \tag{2.5}$$

where $\varepsilon^0(n_0, \tilde{\mu}) = n_0(\frac{1}{2} n_0 \hat{U}_0 - \tilde{\mu})$, and $\hat{U}_0 = \sum_j U_{ij}$. The free energy density, $f_J = -\frac{1}{\beta N} \ln \left[tr\ e^{-\beta \mathcal{H}_J} \right]$, where $\beta = 1/k_B T$, transforms as

$$f_J(\tilde{\mu}) = f_J(\tilde{\mu} - n_0 \hat{U}_0) + \varepsilon^0(n_0, \tilde{\mu}) \tag{2.6}$$

independent of the $\tilde{J}_{ij}$ and $\tilde{\varepsilon}_i$. This implies that the only effect of a shift $n_0 \hat{U}_0$ in the chemical potential is a trivial additive term in the free energy which is linear in $\tilde{\mu}$. This term serves only to increase the overall density, $n = -\frac{\partial f_J}{\partial \tilde{\mu}}$, by n_0 but otherwise has no effect whatsoever on the phase diagram, which therefore will be precisely periodic in $\tilde{\mu}$, with period $\hat{U}_0$.

Consider next the transformation $\tilde{n}_i' = -\tilde{n}_i$, $\hat{\phi}_i' = -\hat{\phi}_i$. The Hamiltonian transforms as

$$\mathcal{H}_J[\tilde{n}_i', \hat{\phi}_i', \tilde{\varepsilon}_i - \tilde{\mu}] = \mathcal{H}_J[\tilde{n}_i, \hat{\phi}_i, -(\tilde{\varepsilon}_i - \tilde{\mu})], \tag{2.7}$$

so that

$$f_J(\tilde{\mu}, \{\tilde{\varepsilon}_i\}) = f_J(-\tilde{\mu}, \{-\tilde{\varepsilon}_i\}). \tag{2.8}$$

Combining the two symmetries (2.6) and (2.8) we see that if all $\tilde{\varepsilon}_i = 0$, then for $\tilde{\mu} = \tilde{\mu}_k \equiv \frac{1}{2}k\hat{U}_0$, where k is any integer, the Hamiltonian possesses a special *particle-hole symmetry*, namely invariance under the transformation $\tilde{n}_i' = k - \tilde{n}_i$, $\hat{\phi}' = -\hat{\phi}$. At $\tilde{\mu} = \tilde{\mu}_k$ the density is precisely $\frac{1}{2}k$ per site, and the thermodynamics is symmetric under addition and removal of particles (the removal of particles being synonymous with the addition of holes). If the ε_i are nonzero, but have a symmetric probability distribution, $p\{\tilde{\varepsilon}_i\} = p\{-\tilde{\varepsilon}_i\}$, then the exact particle-hole symmetry is lost, but there still exists a *statistical* particle-hole symmetry at the same special values $\tilde{\mu}_k$ of $\tilde{\mu}$: self averaging will ensure that $f_J(\tilde{\mu}_k, \{\tilde{\varepsilon}_i\}) = f_J(\tilde{\mu}_k, \{-\tilde{\varepsilon}_i\})$. The lattice boson Hamiltonian (2.1) clearly can never possess either form of particle-hole symmetry since the hopping term mixes the number and phase in an inextricable fashion.

2.1 Phase diagrams

In Fig. 2.1 we plot the zero temperature phase diagram in the $\tilde{\mu}$-J_0 plane, where J_0 is a measure of the overall strength of the hopping matrix, e.g. $J_0 = \frac{1}{N}\sum_{i \neq j} \tilde{J}_{ij}$, in the simplest case of onsite repulsion only: $U_{ij} = U_0\delta_{ij}$. [Further neighbor interactions substantially increase the complexity of the phase diagram in the absence of the random site energies, ϵ_i. One can, in principle, generate Mott insulating phases with arbitrary rational densities ("charge density wave" states). Generically, only the integer fillings are stable against small disorder since the fractional fillings necessarily break the lattice translation symmetry, leading to multiply degenerate ground states related by a discrete translation. It is not hard to see that arbitrarily small random ϵ_i will always generate rare regions where it is energetically favorable to form two such states with a domain wall between (R. Shankar, public communication). If one allows *negative* further neighbor hopping matrix elements, J_{ij}, one can also generate *supersolid* phases which break *both* lattice translational symmetry and XY-phase symmetry, i.e. superfluid charge density wave states [19].] This phase diagram has been discussed in detail previously [6, 11] for the lattice boson Hamiltonian, $\mathcal{H}_B$. Here we emphasize the features unique to $\mathcal{H}_J$, namely the periodicity in $\tilde{\mu}$, and the special points $\tilde{\mu}_k$ corresponding to local extrema in the phase boundaries. In Fig. 2.1(a) we show the phase diagram in the absence of all disorder. For $J_{ij} \equiv 0$ the site occupancies are good quantum numbers and each site has precisely n_0 particles for $n_0 - \frac{1}{2} < \tilde{\mu}/U_0 < n_0 + \frac{1}{2}$. The points $\tilde{\mu}/U_0 = k + \frac{1}{2}$ for integer k are 2^N fold degenerate with either k or $k+1$ particles placed independently on each site. For $J_{ij} > 0$ communication between sites occurs and the effective wavefunction for each particle spreads to neighboring sites. We denote by $\xi(\tilde{\mu}, J_0)$ (to be defined carefully later) the range of this spread. One can show within perturbation theory [6], however, that for sufficiently small, short ranged J_{ij}, the overall density remains *fixed* at n_0. Consider first $\tilde{\mu} \neq 0$. Then only at a critical value $J_{0,c}(\tilde{\mu})$ of J_0 does the system favor adding extra particles (or holes). Equivalently, for a given J_0 there is an interval $\tilde{\mu}_-(J_0) < \tilde{\mu} < \tilde{\mu}_+(J_0)$ of fixed density n_0. These extra particles may be thought of as a dilute Bose fluid moving atop the essentially inert background density, n_0. The physics is identical to that of a dilute Bose gas in the continuum, and is well described by the Bogoliubov model [20], [21]. From this one concludes that the system immediately becomes superfluid (recall that we assume $T = 0$) with a superfluid density $\rho_s \sim n - n_0 \sim J_0 - J_{0,c}$, and an order parameter $\psi_0 \equiv \langle e^{i\hat{\phi}_i} \rangle \sim (n - n_0)^{\frac{1}{2}} \sim (J - J_{0,c})^{\frac{1}{2}}$. The characteristic length in this phase is $\xi_0 = J_0^{\frac{1}{2}}/[\tilde{\mu} - \tilde{\mu}_\pm(J_0)]^{\frac{1}{2}} \sim (n - n_0)^{-\frac{1}{2}} \sim (J_0 - J_{0,c})^{-\frac{1}{2}}$ and represents the distance between "uncondensed" particles: $n - n_0 - |\psi_0|^2 \sim \xi_0^{-d}$. This zero-temperature superfluid onset transition is therefore trivial, in the sense that all exponents are mean-field-like. In fact, historically this onset has never been viewed as an example of a phase transition. Furthermore, although all quantities vary continuously as J_0 decreases toward $J_{0,c}$, the actual onset point is entirely noncritical. For a given value of J_0 within a Mott lobe, the interval $\tilde{\mu}_-(J_0) < \tilde{\mu} < \tilde{\mu}_+(J_0)$ represents a *single* unique (incompressible) thermodynamic state. Thus incompressibility implies that for the given (integer) density, the value of the chemical potential is ambiguous. One might just as well set $\tilde{\mu} = n_0 U_0$, its value at the center of the lobe. The correlation length, $\xi(\tilde{\mu}, J_0)$, is then *independent* of $\tilde{\mu}$, and remains perfectly finite in the Mott phase at $J_{0,c}(\tilde{\mu})$. In this sense the transition has some elements of a first order phase transition.

The more important transition is the one occuring at fixed density, $n = n_0$, at $\tilde{\mu} = n_0 U_0$ through the tip of the Mott lobe at $J_{0,c}(0)$. At this transition $\xi(J_0) \sim (J_{0,c} - J_0)^{-\nu}$ diverges continuously with a characteristic exponent, ν. One may show (see Ref. [6] and below) that the transition is precisely in the universality class of the *classical* $(d+1)$-dimensional XY-model. What distinguishes this transition from the previous ones is precisely particle-hole symmetry: superfluidity is achieved *not* by adding a small density of particles or holes atop

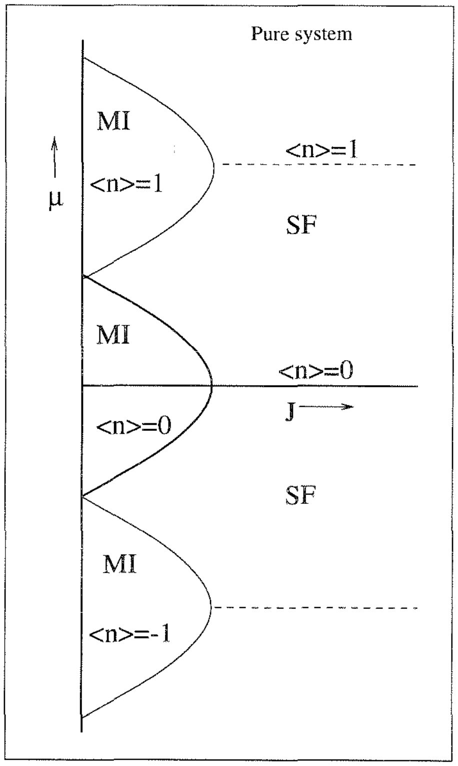

FIG 2.1(a)

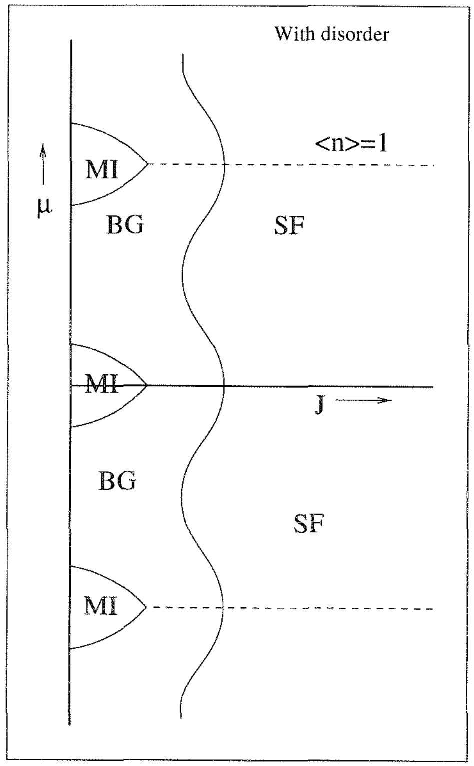

FIG 2.1(b)

Figure 2.1: Zero temperature phase diagram for the Josephson junction model. (a) The model without disorder, showing the periodic sequence of Mott lobes (MI) and the superfluid phase (SF). The transitions to superfluidity at the tips of the Mott lobes are special and are in the universality class of the $(d+1)$-dimensional XY-model. The points μ_k have an exact particle-hole symmetry. (b) The model with disorder, showing the now shrunken (or even absent, if the disorder is sufficiently strong) Mott lobes, and the new Bose glass phase that now intervenes between them and the superfluid phase. The transition to superfluidity now takes place only from the Bose glass phase (BG), and it is believed that the nature of this transition is independent of where it occurs, even at the special points μ_k which now have only a statistical particle-hole symmetry.

the inert background, but by the buildup of superfluid fluctuations *within* the background, to the point where particles and holes *simultaneously* overcome the potential barrier U_0 and hop coherently without resistance.

We have already observed that the lattice boson Hamiltonian (2.1) *never* has an exact particle-hole symmetry. Nevertheless one still has Mott lobes (now asymmetric and decreasing in size with n_0) for each integer density, and a unique extremal point, $[J_{0,c}(n_0), \mu_c(n_0)]$, at which one exits the Mott lobe at fixed density $n = n_0$. One may show [6] that the transition through these extremal points is still in the $(d+1)$-dimensional XY universality class, and that particle-hole symmetry must therefore be asymptotically restored *at* the critical point. The difference now is that there is a nontrivial balance between the densities of particle and hole excitations, and the interactions between them. The position of the critical point is no longer fixed by an explicit symmetry, but must be located by carefully tuning both the hopping parameter and the chemical potential.

This phenomenon of "asymptotic symmetry restoration" at a critical point is actually fairly common (and we shall see it again below). For example, though the usual Ising model of magnetism has an up-down spin symmetry, the usual liquid vapor or binary liquid critical points do not. However the Ising model correctly describes the universality class of the transition, and one concludes that the up-down symmetry must be restored near the critical point. Similarly, the p-state clock model with Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} cos \left[\frac{2\pi}{p}(q_i - q_j) \right], \quad q_i = 1, 2, \ldots, p, \tag{2.9}$$

which may be thought of as a kind of discrete XY-model, has for sufficiently large p (specifically, $p > 4$ in $d = 2$; clearly $p = 2$ corresponds to the Ising model and $p = 3$ to the three-state Potts model) a transition precisely in the XY-model universality class. Note, however, that in the *ordered* phase, corresponding to the zero temperature fixed point, the order parameter will (for $d > 2$) spontaneously align along one of the p equivalent directions, q_i, breaking the XY-symmetry and generating a mass for the spin-wave spectrum. This latter property is not relevant in the present case since breaking particle-hole symmetry does not break the symmetry of the order parameter: the nature of the superfluid phase is unaffected.

Consider now the phase diagram, Fig. 2.1(b), in the presence of disorder. We see that the Mott lobes have shrunk (and may in fact disappear altogether for sufficiently strong disorder), and a new phase, which we call the Bose glass phase, separates these lobes from the superfluid phase [6]. In this new phase the compressibility is now finite, but the particles do not hop large distances due to localization effects: particles on top of the inert background still see a residual random potential, whose lowest energy states will be localized. As particles are added to the system, bosons will tend to fill these states until the residual random potential has been smoothed out sufficiently that extended states can form, finally yielding superfluidity [6].

2.2 Criticality and particle-hole symmetry

We will consider two types of disorder: (i) disorder in the hopping parameters, $\tilde{J}_{ij}$, and (ii) onsite disorder in the $\tilde{\varepsilon}_i$. If $\tilde{\mu} \neq \tilde{\mu}_k$ for any k, that is if particle-hole symmetry is broken, we expect the two types of disorder to yield the same type of phase transition. In renormalization group language, each in isolation will generate the other under renormalization. If $\tilde{\mu} = \tilde{\mu}_k$ and the $\tilde{\varepsilon}_i$ have a symmetric distribution about zero so that the Hamiltonian possesses a statistical particle-hole symmetry, the obvious question is whether or not the transition in this case is different from the one in the presence of generic nonsymmetric disorder. We shall argue below that it is *not*, i.e. that breaking particle-hole symmetry *locally* is not substantially different from breaking it *globally*, and that in fact statistical particle-hole symmetry is asymptotically restored at the critical point. (This idea has been used to explain the vanishing of the Hall conductivity at magnetic field-tuned superconducting transitions [22].) Only if $\tilde{\mu} = \tilde{\mu}_k$ and $\tilde{\varepsilon}_i \equiv 0$ does the disorder fully respect particle-hole symmetry. We shall see that in this case the transition is entirely different, lying in the same universality class as the classical $(d+1)$-dimensional XY-model with *columnar* bond disorder, precisely the kind of system addressed in Ref. [13].

The significance of the restoration of a statistical particle-hole symmetry at the critical point is the following. In Ref. [12] only the boson Hamiltonian (2.1) was considered, and a fixed point sought only in the space of parameters accessible to this Hamiltonian. Such a fixed point can therefore never possess a statistical particle-hole symmetry. However if the true fixed point does possess this symmetry, it is clear that it must then lie outside the space of boson Hamiltonians of the form (2.1). Accessing this fixed point requires an enlargement of the parameter space so that both particle-hole symmetric and asymmetric problems may be treated within the same model. The Josephson junction array Hamiltonian, (2.2), satisfies this requirement. We shall see that, indeed, a new statistically particle-hole symmetric dirty boson fixed point can be identified, and that all technical problems encountered in Ref [12] may then be avoided. Specifically, certain diagrams that were ignored in Ref. [12] are in fact important and accomplish the required enlargement of the parameter space. There is, however, a price: this new fixed point is not perturbatively accessible within the double epsilon expansion. Thus, the random rod problem may be analyzed for small ϵ in $d = 4 - \epsilon - \epsilon_\tau$ dimensions if the dimension, ϵ_τ, of time is also small [13]. We find, however, that for small ϵ and ϵ_τ, site disorder is *irrelevant* at the random rod fixed point, and that *full* particle-hole symmetry is therefore restored on large scales close to criticality. This remains true for sufficiently small $\epsilon_\tau < \epsilon_\tau^c(D)$, with $\epsilon_\tau^c(D) = O(1)$. Only for $\epsilon_\tau > \epsilon_\tau^c(D)$ does a new fixed point appear which breaks full particle-hole symmetry. Extrapolation of the critical behavior associated with this new fixed point, though uncontrolled, shows significant similarities to some features of the known behavior at $\epsilon_\tau = 1$. To lowest nontivial order in ϵ_τ we find $\epsilon_\tau^c(D = 4) \simeq \frac{8}{29}$ ($D = 4$, hence $\epsilon = 0$, corresponding to $d = 3$ at $\epsilon_\tau = 1$). This value of ϵ_τ^c is significantly less than unity, and leads us to hope that estimates based on these extrapolations from small ϵ_τ are not too unreasonable. Also, particle-hole asymmetry, which is relevant for small values of ϵ_τ, seems to become *irrelevant* at $\epsilon_\tau > \epsilon_{\tau 1}$, and for $\epsilon = 0$, we obtain an estimate $\epsilon_{\tau 1} \approx 2/3$. This supports our argument that at criticality statistical particle-hole symmetry is restored.

2.3 Functional integral formulations

In order to obtain a formulation of the problem more amenable to analytic treatment, we turn to functional integral representations of the partition function. It will turn out to be important to have an *exact* representation. Representations which involve dividing the Hamiltonian into two pieces, $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, then using the Kac-Hubbard-Stratanovich transformation to decouple $\mathcal{H}_1$, generate effective classical actions with an infinite number of terms, which must then be truncated at some finite order [6]. In addition, such representations work only when $\tilde{\mu}$ lies within a Mott phase when $J_0 = 0$, and hence break down for unbounded, e.g. Gaussian, distributions of site energies. We turn instead to representations obtained from the Trotter decomposition (see App. A). For the lattice boson model, the coherent state representation is most appropriate, and yields a classical Lagrangian

$$\mathcal{L}_B = \int_0^\beta d\tau \left[\sum_i \psi_i^*(\tau)\partial_\tau \psi_i(\tau) - \mathcal{H}_B\{\psi_i^*(\tau), \psi_i(\tau)\} \right], \tag{2.10}$$

where $\mathcal{H}\{\psi_i^*(\tau), \psi_i(\tau)\}$ is obtained by substituting the classical complex variable $\psi_i(\tau)$ for the boson site annihilation operator a_i (and $\psi_i^*(\tau)$ for the creation operator $a_i^\dagger$) wherever it appears in the (normally ordered form of the) quantum Hamiltonian. The partition function is given by $Z = tr^\psi[e^{\mathcal{L}}]$, where $tr^\psi[\cdot]$ is an unrestricted integral over all complex fields, $\psi_i(\tau)$. Notice that the only term that couples different time slices is the "Berry's phase" $\psi^*\partial_\tau\psi$ term which arises from the overlap of two coherent states at neighboring times. This should be contrasted with the spatial coupling, $\frac{1}{2}\sum_{i,j} J_{ij}\psi_i^*\psi_j$ (essentially a discrete version of $\psi^*\nabla^2\psi$), which appears in $\mathcal{H}_B$. The imaginary time dimension is therefore highly anisotropic. This anisotropy is increased further if disorder is present since the ε_i and J_{ij} are τ-independent: the disorder appears in perfectly correlated *columns*, rather than as point-like defects, in $(d+1)$-dimensional space-time.

If the $\psi^*\partial_\tau\psi$ term were replaced by $\psi^*\partial_\tau^2\psi$, only the disorder would contribute to the anisotropy (the fact that the coefficients of $\psi^*\partial_\tau^2\psi$ and $\psi^*\nabla^2\psi$ are different is not important, and may be cured by a simple rescaling). The model then becomes precisely a special case in the family classical models with rod-like disorder treated in Ref. [13]. Clearly the linear time derivative term in (2.10) is more singular than a term with a second derivative in time. It should not be too surprising that its presence leads to different critical behavior.

Yet another crucial property of the $\psi^*\partial_\tau\psi$ term is that it is purely imaginary:

$$\left[\int_0^\beta d\tau \psi^*\partial_\tau\psi\right]^* = -\int_0^\beta d\tau\psi^*\partial_\tau\psi, \tag{2.11}$$

where integration by parts and periodic boundary conditions have been used. Therefore the statistical factor, $e^{\mathcal{L}_B}$, used to compute the thermodynamics is in general a complex number, and leads to interference between different configurations of the $\psi_i(\tau)$. Unlike that with coupling $\psi^*\partial_\tau^2\psi$, this model therefore does not correspond to any classical model with a well defined Hamiltonian in one higher dimension. In fact, it is precisely this property that reflects the particle-hole asymmetry in the model. Interchanging particles and holes is equivalent to interchanging $\psi_i^*(\tau)$ and $\psi_i(\tau)$. The $\psi^*\partial_\tau\psi$ term changes sign under this operation, while $\mathcal{H}_B[\psi^*,\psi]$ is unaffected. Thus although $\mathcal{L}_B$ is invariant under the combination (known as *time reversal*) of complex conjugation and $\tau\to-\tau$, the boson model always violates each separately.

Consider now the canonical coordinate Lagrangian for the Josephson array model (see App. A for a derivation):

$$\begin{aligned}\mathcal{L}_J &= \int_0^\beta d\tau\Big[\sum_{i,j}\tilde{J}_{ij}cos[\phi_i(\tau)-\phi_j(\tau)] \\ &+ \frac{1}{2}\sum_{i,j}(U^{-1})_{ij}(i\dot{\phi}_i(\tau)+\tilde{\mu}-\tilde{\epsilon}_i)(i\dot{\phi}_j(\tau)+\tilde{\mu}-\tilde{\epsilon}_j)\Big],\end{aligned} \tag{2.12}$$

with partition function $Z=tr^\phi e^{\mathcal{L}_J}$. Notice that the linear time derivative, $\dot{\phi}_i$, now appears in a much more symmetric looking fashion. If $\tilde{\mu}-\tilde{\epsilon}_i\equiv 0$ we see that $\mathcal{L}_J$ is particle-hole symmetric and real:

$$\begin{aligned}\mathcal{L}_J[\tilde{\mu}+\tilde{\epsilon}_i\equiv 0] &= \int_0^\beta d\tau\Big[\sum_{i,j}\tilde{J}_{ij}cos[\phi_i(\tau)-\phi_j(\tau)] \\ &- \frac{1}{2}\sum_{i,j}(U^{-1})_{ij}\dot{\phi}_i(\tau)\dot{\phi}_j(\tau)\Big],\end{aligned} \tag{2.13}$$

and takes precisely the form of a classical XY-model in $(d+1)$ dimensions. The periodicity, (2.6), of the phase diagram is a consequence of the periodicity of the ϕ_i in τ. Substituting $\tilde{\mu}-n_0\hat{U}_0$ for $\tilde{\mu}$ and multiplying out the $(U^{-1})_{ij}$ term yields

$$\begin{aligned}\mathcal{L}_J[\tilde{\mu}-n_0\hat{U}_0] &= \mathcal{L}_J[\tilde{\mu}]-in_0\int_0^\beta d\tau\sum_i\dot{\phi}_i(\tau) \\ &+ \beta N\varepsilon^0(n_0,\tilde{\mu}).\end{aligned} \tag{2.14}$$

However $\int_0^\beta d\tau\dot{\phi}_i(\tau)=2\pi m_i$ and $e^{i2\pi m_i n_0}=1$, so the second term simply drops out of the statistical factor, $e^{\mathcal{L}_J}$, and we recover (2.6). Notice that if $\tilde{\mu}-\tilde{\epsilon}_i\equiv\frac{1}{2}\hat{U}_0$, we obtain a statistical factor

$$e^{\mathcal{L}_J[\tilde{\mu}-\tilde{\epsilon}_i=\frac{1}{2}\hat{U}_0]}=(-1)^{\sum_i m_i}e^{\mathcal{L}_J[\tilde{\mu}-\tilde{\epsilon}_i=0]+\beta N\varepsilon^0(\frac{1}{2},0)}, \tag{2.15}$$

which, though real, is not always positive. Although this Lagrangian is also particle-hole symmetric, it too does not correspond to the Hamiltonian of any classical model. This model is very different from that with $\tilde{\mu} - \tilde{\epsilon}_i \equiv 0$. For example it always has superfluid order at $T = 0$, for arbitrarily small J_{ij}, as opposed to (2.13) which orders only for sufficiently large J_{ij} [6].

For later reference, note that, in contrast to (2.10), (2.12) has an obvious generalization to noninteger dimensions of time. (There are however not so obvious generalizations of (2.10)). If ϵ_τ is the dimension of time, we simply write

$$\begin{aligned} \mathcal{L}_J^{(\epsilon_\tau)} &= \int_0^\beta d^{\epsilon_\tau}\tau \Big[\sum_{i,j} \tilde{J}_{ij} cos[\phi_i(\vec{\tau}) - \phi_j(\vec{\tau})] \\ &+ \frac{1}{2}\sum_{i,j}(U^{-1})_{ij}(i\nabla_\tau\phi_i(\vec{\tau}) + \vec{\mu} - \vec{\epsilon}_i) \cdot (i\nabla_\tau\phi_j(\vec{\tau}) \\ &+ \vec{\mu} - \vec{\epsilon}_j)\Big], \end{aligned} \tag{2.16}$$

where $\vec{\tau}$, $\vec{\mu}$ and $\vec{\epsilon}_i$ are ϵ_τ-dimensional vectors: $\vec{\tau} = (\tau_1, \ldots, \tau_{\epsilon_\tau})$, etc.

Renormalization group calculations are performed most conveniently on Lagrangians, such as (2.10), which are polynomials in unbounded, continuous fields and their gradients. Therefore we would like to convert (2.15) to such a model, while retaining the essential physics. If we write $\psi_i(\vec{\tau}) = e^{i\phi_i(\vec{\tau})}$, then (2.16) may be written

$$\begin{aligned} \mathcal{L}_J^{(\epsilon_\tau)} &= \int_0^\beta d^{\epsilon_\tau}\tau \Big[\sum_{i,j} \tilde{J}_{ij} [\psi_i^*(\vec{\tau})\psi_j(\vec{\tau}) + c.c.] \\ &+ \frac{1}{2}\sum_{i,j}(U^{-1})_{ij}\psi_i^*(\vec{\tau})(\nabla_\tau + \vec{\mu} - \vec{\epsilon}_i)\psi_i(\vec{\tau}) \cdot \psi_j^*(\vec{\tau}) \\ &\times (\nabla_\tau + \vec{\mu} - \vec{\epsilon}_j)\psi_j(\vec{\tau})\Big]. \end{aligned} \tag{2.17}$$

For onsite interactions only, $U_{ij} = U_0\delta_{ij}$, the second term simplifies to

$$\int d^{\epsilon_\tau}\tau \frac{1}{2U_0} \sum_i \psi_i^*(\vec{\tau})(\nabla_\tau + \vec{\mu} - \vec{\epsilon}_i)^2\psi_i(\vec{\tau}), \tag{2.18}$$

which is conveniently quadratic in ψ. We now relax the assumption $|\psi_i| = 1$, employing instead the usual $r|\psi|^2 + v|\psi|^4$ Landau-Ginzburg-Wilson weighting factor, obtaining finally

$$\mathcal{L}_\psi^{(\epsilon_\tau)} = \int d^{\epsilon_\tau}\tau \Big\{ \frac{1}{2}\sum_{i,j} \tilde{J}_{ij} [\psi_i^*(\vec{\tau})\psi_j(\vec{\tau}) + c.c]$$

$$+ \frac{1}{2}U_0 \sum_i \psi_i^*(\vec{\tau})(\nabla_\tau + \vec{\mu} - \vec{\epsilon}_i)^2 \psi_i(\vec{\tau})$$
$$- \sum_i \left[r|\psi_i(\vec{\tau})|^2 + v|\psi_i(\vec{\tau})|^4 \right] \Big\}. \tag{2.19}$$

This model retains the exact particle-hole symmetry at $\vec{\mu} + \vec{\epsilon}_i \equiv 0$, but loses the precise periodicity of the phase diagram when $\epsilon_\tau = 1$: thus the second term in (2.13) now becomes [compare (2.10)]

$$n_0 \int_0^\beta d\tau \sum_i \psi_i^* \partial_\tau \psi_i, \tag{2.20}$$

which reduces to the previous form if $|\psi_i| = 1$. However if $|\psi_i|$ fluctuates, as in (2.19), this term is no longer a perfect time derivative and therefore will not yield a simple integer result. We will therefore only use (2.19) near $\tilde{\mu} = 0$ when we study the role of particle-hole symmetry near the phase transition.

Chapter 3

Particle-hole symmetry and the excitation spectrum of the glassy phases

In this chapter we shall consider the nature of the non-superfluid phases in the presence of the two types of disorder, ε_i and J_{ij} (to simplify the notation we henceforth drop the tildes on the Josephson junction model parameters). Recall that for the boson problem the ε_i yield a glassy phase, which we called the Bose glass phase [6], with a finite compressibility and a finite density of excitation states at zero energy. We shall contrast this with the case of the particle-hole symmetric, random J_{ij} model which we shall show has a vanishing compressibility and an excitation spectrum with an exponentially small density of states, $\rho(\varepsilon) \sim e^{-\varepsilon_0/\varepsilon}$, i.e. a "soft gap." We shall find that the compressibility is precisely the spin-wave stiffness in the time direction, which therefore vanishes in the "symmetric glass," but is finite in the Bose glass. This yields an upper bound $z_0 \leq d$ for the dynamical exponent at the particle-hole symmetric transition. An effective *lower bound* on z_0 may be obtained by demanding that particle-hole asymmetry be a *relevant* operator at the symmetric transition. This is a necessary condition in order that the particle-hole asymmetric transition be in a different universality class from the symmetric one. We shall obtain estimates for this lower bound within the double ϵ-expansion in Chap 6.

3.1 Superfluid densities or helicity modulii

Let us begin by defining the superfluid density, called the "helicity modulus," or "spin wave stiffness" in classical spin models. The idea is to compute the change in free energy under a change in boundary conditions. Consider a box shaped system with sides L_α, $\alpha = 1, \ldots, D$. We shall be interested in $D = d + 1$ and $L_D = \beta$. We shall say that ψ obeys θ_α-boundary conditions if

$$\psi(x_1, \ldots, x_\alpha + L_\alpha, \ldots, x_D) \quad = \quad e^{i\theta}\psi(x_1, \ldots, x_\alpha, \ldots, x_D)$$

$$\psi(x_1,\ldots,x_\beta+L_\beta,\ldots,x_D) = \psi(x_1,\ldots,x_\beta,\ldots,x_D),$$
$$\beta\neq\alpha \tag{3.1}$$

i.e., a twist angle θ is imposed on the α direction, while periodic boundary conditions are maintained in all other directions. Let

$$f^{\theta_\alpha} = -\frac{1}{V_D} ln\ tr\left(e^{\mathcal{L}^{\theta_\alpha}}\right),\ V_D\equiv\prod_{\beta=1}^{D} L_\beta, \tag{3.2}$$

where $\mathcal{L}^{\theta_\alpha}$ is the Lagrangian, be the free energy obtained using θ_α-boundary conditions. We may define

$$\tilde{\psi}(x_1,\ldots,x_D) = e^{-i\theta x_\alpha/L_\alpha}\psi(x_1,\ldots,x_D), \tag{3.3}$$

which then obeys periodic boundary conditions in all directions. In all cases of interest one may then write

$$\mathcal{L}^{\theta_\alpha}[\psi] = \mathcal{L}^0[\tilde{\psi}] + \delta\mathcal{L}[\tilde{\psi};\theta/L_\alpha] \tag{3.4}$$

where $\delta\mathcal{L}$ is a Taylor series in powers of θ/L_α, and superscript "0" denotes periodic boundary conditions. Thus

$$\delta f^{\theta_\alpha} \equiv f^{\theta_\alpha} - f^0 = -\frac{1}{V_D}\left[\langle\delta\mathcal{L}\rangle + \frac{1}{2}\langle\delta\mathcal{L}^2\rangle_c + \ldots\right], \tag{3.5}$$

where $\langle\delta\mathcal{L}^2\rangle_c = \langle\delta\mathcal{L}^2\rangle - \langle\delta\mathcal{L}\rangle^2$, and the averages are with respect to $\mathcal{L}^0[\tilde{\psi}]$. Equation (3.5) yields a series of terms in powers of θ/L_α, and we use the notation

$$\delta f^{\theta_\alpha} = -\frac{i\theta}{L_\alpha}\rho_\alpha + \frac{1}{2}\left(\frac{\theta}{L_\alpha}\right)^2\Upsilon_\alpha + \ldots \tag{3.6}$$

to define the coefficients in this series. This only makes sense for $|\theta|\leq\pi$, since it is clear from the definition that f^{θ_α} is periodic in θ_α with period 2π. We shall see that the first term, which in most previous cases was completely *absent*, arises from particle-hole asymmetry. The second term defines the *helicity modulus*, Υ_α, in the direction α.

Let us now turn to specific cases with Lagrangians defined by (2.10) and (2.12). In both cases, when α is a spatial coordinate ($\alpha = 1,\ldots,d$) the sensitivity to boundary conditions comes only from the hopping term [see (2.1) and (2.10)] so that

$$\begin{aligned}
\delta\mathcal{L}_B &= \frac{1}{2}\int_0^\beta d\tau \sum_{i,j} J_{ij}\left[\tilde{\psi}_i^*(e^{i\theta(x_i^\alpha - x_j^\alpha)/L_\alpha} - 1)\tilde{\psi}_j + c.c.\right] \\
&= \frac{i\theta}{2L_\alpha}\int_0^\beta d\tau \sum_{i,j} J_{ij}(x_i^\alpha - x_j^\alpha)[\tilde{\psi}_i^*\tilde{\psi}_j - c.c.] \\
&- \frac{\theta^2}{4L_\alpha^2}\int_0^\beta d\tau \sum_{i,j} J_{ij}(x_i^\alpha - x_j^\alpha)^2[\tilde{\psi}_i^*\tilde{\psi}_j + c.c.] \\
&+ O(\theta^3/L_\alpha^3),
\end{aligned} \tag{3.7}$$

which yields $\rho_\alpha = 0$, $\alpha = 1, \ldots, d$, and

$$\begin{aligned}
\Upsilon_\alpha &= \frac{1}{4}\int_0^\beta d\tau \sum_{i,j,k} \langle\langle J_{ij}(x_i^\alpha - x_j^\alpha) J_{k0} x_k^\alpha \langle [\psi_i^*(\tau)\psi_j(\tau) \\
&- c.c.][\psi_i^*(0)\psi_j(0) - c.c.]\rangle\ \rangle\rangle \\
&+ \frac{1}{2}\sum_i \langle\langle J_{i0}(x_i^\alpha)^2 \langle \psi_i^*(0)\psi_0(0) + c.c.\rangle\ \rangle\rangle, \\
&\quad \alpha = 1, \ldots, d,
\end{aligned} \tag{3.8}$$

where $\langle\langle\cdot\rangle\rangle$ denotes an average over the disorder (we assume self averaging). For nonrandom J_{ij}, with nearest neighbor hopping, J, this reduces to

$$\begin{aligned}
\Upsilon_\alpha &= \frac{1}{4}J^2 a^2 \int_0^\beta d\tau \sum_i \langle [\psi_i^*(\tau)\partial_\alpha\psi_i(\tau) - \psi_i(\tau)\partial_\alpha\psi_i^*(\tau)] \\
&\times [\psi_0^*(0)\partial_\alpha\psi_0(0) - \psi_i(0)\partial_\alpha\psi_i^*(0)]\rangle \\
&+ \frac{1}{2}Ja^2\langle \psi_{\hat{x}_\alpha}^*\psi_0 + \psi_0^*\psi_{\hat{x}_\alpha}\rangle,\ \alpha = 1, \ldots, d,
\end{aligned} \tag{3.9}$$

where $\partial_\alpha\psi_i \equiv \psi_{i+\hat{x}_\alpha} - \psi_i$, and a is the lattice spacing. We recognize this as the discrete version of the usual definition of Υ_α in terms of the current-current correlation function.

The Josephson Lagrangian yields precisely the same expressions if one identifies $\psi_i(\tau) = e^{i\phi_i(\tau)}$. Thus, (3.8) becomes

$$\begin{aligned}
\Upsilon_\alpha &= -\int_0^\beta d\tau \sum_{i,j,k} \langle\langle J_{ij}(x_i^\alpha - x_j^\alpha) J_{k0} x_k^\alpha \langle \\
&\times \sin[\phi_j(\tau) - \phi_i(\tau)] \sin[\phi_0(0) - \phi_k(0)]\rangle\ \rangle\rangle \\
&+ \sum_i J_{i0}(x_i^\alpha)^2 \langle\langle\ \langle \cos[\phi_i(0) - \phi_0(0)\rangle\ \rangle\rangle
\end{aligned}$$

$$\alpha = 1, \ldots, d \tag{3.10}$$

and (3.9) becomes

$$\begin{aligned} \Upsilon_\alpha &= -J^2 a^2 \int_0^\beta d\tau \sum_i \langle \sin[\phi_i(\tau) - \phi_{i+\hat{x}_\alpha}(\tau)] \\ &\times \sin[\phi_0(0) - \phi_{\hat{x}_\alpha}(0)] \rangle \\ &+ Ja^2 \langle cos[\phi_{\hat{x}_\alpha}(0) - \phi_0(0)\rangle, \alpha = 1, \ldots, d. \end{aligned} \tag{3.11}$$

Consider now the stiffness in the time direction. We will show that it is precisely the compressibility, $\kappa \equiv -\frac{\partial^2 f}{\partial \mu^2}$. To see this, note that only terms with time derivatives are sensitive to θ_τ-boundary conditions. In the Bose case, (2.10), we have

$$\delta\mathcal{L} = i\frac{\theta}{\beta}\int_0^\beta d\tau \tilde{\psi}_i^*(\tau)\tilde{\psi}_i(\tau) \tag{3.12}$$

which corresponds precisely to an imaginary shift, $\mu' = \mu + i\frac{\theta}{\beta}$, in the chemical potential. Similarly, in (2.12) we define $\tilde{\phi}_i(\tau) = \phi_i(\tau) - \frac{\theta}{\beta}\tau$, leading to the exactly the same chemical potential shift. Thus in both $\mathcal{L}_B$ and $\mathcal{L}_J$ the time derivatives appear with the chemical potential in just the right way to give rise to what amounts to the Josephson relation between the time derivative of the phase and changes in the chemical potential. We immediately conclude that the series (3.6) takes the form

$$\begin{aligned} \delta f^{\theta_\tau} &= \frac{i\theta}{\beta}\frac{\partial f^0}{\partial \mu} + \frac{1}{2}\left(\frac{i\theta}{\beta}\right)^2 \frac{\partial^2 f^0}{\partial \mu^2} + \ldots \\ &= -\frac{i\theta}{\beta}\rho + \frac{1}{2}\left(\frac{\theta}{\beta}\right)^2 \kappa + \ldots \end{aligned} \tag{3.13}$$

where $\rho = -\frac{\partial f^0}{\partial \mu}$ is the number density, and we identify $\Upsilon_\tau = \kappa$.

Our classical intuition would tell us that Υ_α should be nonzero only when the model has long range order in the phase of the order parameter, i.e. only in the superfluid phase. Although this statement is true for the spatial directions, $\alpha = 1, \ldots, d$, this is not necessarily true for $\alpha = \tau$. Our intuition about ^4He in porous media would lead us to be very surprised if the system were incompressible, $\kappa = 0$, throughout the nonsuperfluid phase. Thus there is no barrier to the continuous addition of particles to the system, even when it is completely localized (only Mott phases, in which disorder is unimportant, are incompressible because the density is pinned at special values commensurate with the lattice [6]). The Bose glass phase is therefore rather special in that the order parameter phase has a *temporal stiffness*, $\Upsilon_\tau = \kappa > 0$, even when there is no spatial stiffness, $\Upsilon_x \propto \rho_s = 0$. Our classical intuition breaks down because the Lagrangian is typically not real and, as discussed earlier, does not have a proper classical interpretation.

The particle-hole symmetric model described by (2.13), however, *does* have a classical interpretation, and despite the fact that the J_{ij} are random and the model anisotropic (the disorder being fixed in time) it would be suprising if the disordered phase possessed long range order in time. Thus we expect κ to vanish when ρ_s does, so that the disordered phase is incompressible. This is permitted because the particle-hole symmetry now dictates that the density be an integer. What distinguishes this disordered phase from the Mott phase, however, is that κ is not zero for an entire interval of μ, but vanishes only for the special value $\mu = 0$ where particle-hole symmetry holds.

3.2 Droplet model of the glassy phases

Let us now understand in detail how these two different behaviors merge with each other in the full phase diagram. Consider therefore the particle-hole symmetric model (2.13) with, for concreteness, $J_{ij} = J(1 + \delta J_{ij}) > 0$ on nearest neighbor bonds only, with all δJ_{ij} *independent* random variables. Let J_c be its critical point, and let J_c^0 be the critical point when all $\delta J_{ij} = 0$ (note that it is entirely possible that $J_c < J_c^0$ since random fluctuations can sometimes enhance superfluid order [11]). In the latter, nonrandom case, the transition is from a Mott insulating phase for $J < J_c^0$ to a superfluid phase for $J > J_c^0$. Suppose now that $-1 \leq \delta J_{ij} \leq \Delta J$ is bounded from above (as well as, trivially, from below) with ΔJ the essential supremum (i.e., the largest value of δJ_{ij} achievable with finite probability density). Then for $J(1 + \Delta J) < J_c^0$ all J_{ij} are smaller than J, and the system must have a Mott gap. However for $J_c > J > J_c^0/(1 + \Delta J)$ one will form, via probabilistic fluctuations, exponentially rare, but arbitrarily large regions of bonds in which all $J_{ij} > J_c^0$. These regions therefore represent finite droplets of superfluid. It is here that the τ-independence of the J_{ij} is crucial – in the classical interpretation these droplets are one-dimensional cylinders with arbitrarily large cross-section, made of material that would be ferromagnetically ordered in the bulk. The fact that these regions are already infinite along one dimension clearly enhances magnetic ordering more than would finite (zero-dimensional) pieces of magnet. We shall see now that these droplets close the Mott gap.

3.2.1 Correlations and excitations in the symmetric glass

Consider such a droplet of volume V, which will occur roughly with density $e^{-p_0 V}$, for some constant p_0. The behavior of $V \times \infty$ cylinders of magnet has been discussed in detail by Fisher and Privman [23], who were concerned with finite size scaling theory of magnets with a continuous $O(n)$ symmetry below their bulk critical points. Their main result was that the correlation length, $\xi_{||}$ along the cylinder is governed by the bulk helicity modulus along the same direction:

$$\xi_{||} = 2\Upsilon(T)V/(n-1)k_B T. \tag{3.14}$$

In our case, $k_B T = 1$, $n = 2$ and $\Upsilon(T) \equiv \Upsilon_\tau(J)$. The correlation function, $G_0(\tau)$, along the cylinder then varies as

$$G_0(\tau) \sim e^{-|\tau|/\xi_{||}}, \quad |\tau| \gg \xi_{||}. \tag{3.15}$$

There is some ambiguity in what we should take for V and $\Upsilon_\tau(J)$ in (3.14): the droplets are neither perfectly spherical, nor is J_{ij} uniform throughout the droplet. Thus V should be some effective volume, while $\Upsilon_\tau(J)$ should be the bulk temporal helicity modulus associated with some effctive uniform $J > J_c^0$, say roughly the average of J_{ij} over the droplet. None of these ambiguities change the order of magnitude estimates we make below.

The full temporal correlation function, $G(\tau)$, is obtained by averaging $G_0(\tau)$ over all droplets. We estimate this as

$$G(\tau) \approx \int dV \int d\Upsilon_\tau p(V, \Upsilon_\tau) G_0(\tau), \tag{3.16}$$

where $p(V, \Upsilon_\tau)$ is the probability density for droplets of volume V and bulk helicity modulus Υ_τ,

$$p(V, \Upsilon_\tau) \sim e^{-V/V_0(\Upsilon_\tau)}. \tag{3.17}$$

The coefficient $V_0(\Upsilon_\tau)$, which we interpret as the "typical" droplet size for a given Υ_τ, will depend on the detailed shape of the tail of the probability distribution for $J_{ij} > J_c^0$. Using (3.15), for large τ we may do the integral over V using the saddle point method. The integration will be dominated by V near the solution of $\frac{d}{dV}(V/V_0 + \tau/2\Upsilon_\tau V) = 0$. This yields

$$G(\tau) \sim \int d\Upsilon_\tau e^{-\sqrt{2\tau/\Upsilon_\tau V_0(\Upsilon_\tau)}}, \quad \tau \to \infty. \tag{3.18}$$

The coefficient $\Upsilon_\tau V_0(\Upsilon_\tau)$ will have a minimum at some value, $\bar{\Upsilon}_\tau(J)$, corresponding to the most probable large droplets, and this will govern the asymptotic behavior of the integral (3.18) to yield finally,

$$G(\tau) \sim e^{-\sqrt{\tau/\tau_0(J)}}, \quad \tau_0(J) = \frac{1}{2}\bar{\Upsilon}_\tau V_0(\bar{\Upsilon}_\tau). \tag{3.19}$$

The droplets therefore yield a *stretched* exponential behavior, to be contrasted with the purely exponential behavior in the Mott phase. From (3.19) we may derive the quantum mechanical single-particle density of states [6],[20], $\rho_1(\epsilon)$, defined as the inverse Laplace transform of $G(\tau)$:

$$G(\tau) = \int_0^\infty d\epsilon \rho_1(\epsilon) e^{-\epsilon|\tau|}. \tag{3.20}$$

It is easy to see that exponential decay in $G(\tau)$ requires a *gap* in $\rho_1(\epsilon)$,

$$\rho_1(\epsilon) = 0, \epsilon < \epsilon_c \quad \Leftrightarrow \quad G(\tau) \sim e^{-\epsilon_c|\tau|}, \tag{3.21}$$

while slower than exponential decay permits $\rho_1(\epsilon) > 0$ for all $\epsilon > 0$. The form (3.19) yields

$$\rho_1(\epsilon) \sim e^{-\frac{1}{4\tau_0(J)|\epsilon|}}, \tag{3.22}$$

a "soft gap."

3.2.2 Correlations, excitations and compressibility in the Bose glass

Now consider the compressibility. Its computation requires the addition of a small uniform chemical potential, μ. As alluded to earlier, we expect $\rho_1(\epsilon)$ to be *finite* at $\epsilon = 0$ in the presence of μ, implying power law behavior for $G(\tau)$:

$$G(\tau) \approx \rho_1(0;\mu)/\tau, \quad \tau \to \infty, \tag{3.23}$$

though we shall see that $\rho_1(0;\mu)$ will be exponentially small in $\frac{1}{\mu}$. Such behavior lies far outside any classical intuition. To see how this behavior comes about we must generalize the ideas of Fisher and Privman to this case. Fortunately this is relatively straightforward: a compact statement of the Fisher-Privman result is that long time correlations along $V \times \infty$ cylinders (for $n = 2$) are governed by an effective one dimensional classical action

$$S^0_{eff} = \frac{1}{2} V \int_0^\beta d\tau \Upsilon_\tau \dot{\phi}(\tau)^2, \tag{3.24}$$

where $\phi(\tau)$ is a coarse-grained phase. This immediately yields

$$G_0(\tau) \equiv \langle e^{i[\phi(\tau)-\phi(0)]} \rangle = e^{-\frac{1}{2}\langle[\phi(\tau)-\phi(0)]^2\rangle}, \tag{3.25}$$

which, upon using

$$\frac{1}{2}\langle[\phi(\tau) - \phi(0)]^2\rangle = \int_{-\infty}^{\infty} \frac{1 - e^{i\omega\tau}}{\omega^2} \frac{d\omega}{\Upsilon_\tau V} = \frac{|\tau|}{2\Upsilon_\tau V} \tag{3.26}$$

yields (3.14) and (3.15).

Now we must generalize (3.24) to finite μ. This is accompished using (3.6): effective long wavelength, long time "hydrodynamic" fluctuations in the phase ϕ are governed by precisely the same elastic moduli that govern equilibrium twists in the phase. Thus in (3.6) one simply replaces $\frac{\theta}{L_\alpha}$ by $\partial_\alpha \phi$ and integrates over all space. If, as in the present case, the twists in different directions, α, superimpose without interacting (this may be checked directly from (3.7), where now one defines $\tilde{\psi} = e^{-i\sum_\alpha \theta_\alpha x_\alpha / L_\alpha}\psi$, with ψ obeying θ_α-boundary conditions simultaneously in each direction), one simply sums over all directions α to obtain the final answer:

$$S_{eff} = \sum_{\alpha=1}^{D} \int d^d x d\tau \left[-i\rho_\alpha \partial_\alpha \phi + \frac{1}{2} \Upsilon_\alpha (\partial_\alpha \phi)^2 \right]. \tag{3.27}$$

In the case where the interactions are spatially isotropic one has $\rho_\alpha = 0$ and $\Upsilon_\alpha = \Upsilon$ for $\alpha = 1, \ldots, d$. With the identifications (3.13) for $\alpha = \tau$ we then have

$$S_{eff} = \int d^d x d\tau \left[-i\rho\dot{\phi} + \frac{1}{2}\kappa\dot{\phi}^2 + \frac{1}{2}\Upsilon|\nabla\phi|^2 \right]. \tag{3.28}$$

For $V \times \infty$ cylindrical geometries, the effective one-dimensional Fisher-Privman result is obtained by assuming that for each τ, $\phi(\mathbf{x}, \tau)$ is essentially constant in space, and hence that only the temporal fluctuations are important. More formally, the finiteness of V implies a gap of order $V^{-2/d}$ in the spatial spin-wave spectrum between uniform $\phi(\mathbf{x})$ and the next excited state in which ϕ twists by 2π from one side of the system to the other. The temporal spectrum has no such gap (the frequency, ω, in (3.26) is continuous), and therefore the asymptotic long time, large distance behavior may be obtained by assuming $\phi(\mathbf{x},\tau) = \phi(\tau)$ only. The $|\nabla\phi|^2$ term in (3.28) then drops out, and we find the proper generalization of (3.24):

$$S_{eff}^{(1)} = V\int_0^\beta d\tau \left[\frac{1}{2}\kappa\dot{\phi}^2 - i\rho\dot{\phi}\right]. \tag{3.29}$$

All the effects of particle-hole asymmetry are in the ρ term.

Let us now study the consequences of (3.29). First, when ρV is an integer (i.e. the density in the bulk is commensurate with the volume, V) the 2π-periodic boundary conditions on ϕ imply that the ρ term simply drops out of the statistical factor, $e^{S_{eff}^{(1)}}$. This then implies that only ρV mod 1 (the fractional part) matters in (3.29). Recall that the values of ρ and κ are appropriate to a bulk superfluid system with some effective $J > J_c^0$. The bulk compressibility, $\kappa_0(J)$, of such a system is finite and nonzero. When $\mu = 0$ the density is $\rho = 0$ (or, more generally, some integer), so for small μ we must have

$$\begin{aligned} \rho &= \kappa_0(J)\mu + O(\mu^2) \\ \kappa &= \kappa_0(J) + O(\mu). \end{aligned} \tag{3.30}$$

We must be careful to distinguish ρ and κ from the *actual* density and compressibility of the droplet of volume V. The latter must be computed from (3.29) as follows: the free energy density is given by

$$f = f_0 - \frac{1}{\beta V} tr^\phi \left[e^{-S_{eff}^{(1)}}\right], \tag{3.31}$$

in which f_0 is the *bulk* free energy density corresponding to the input parameters, κ and ρ. Thus, for example, at a given value of the chemical potential, $\mu = \mu_0$, we have

$$-\left(\frac{\partial f_0}{\partial \mu}\right)_{\mu=\mu_0} = \rho(\mu_0) \equiv \rho^0, \; -\left(\frac{\partial^2 f}{\partial \mu^2}\right) = \kappa(\mu_0) \equiv \kappa^0, \tag{3.32}$$

and hence, correct to quadratic order in $\mu - \mu_0$, we may take

$$f_0(\mu) = f_0(\mu_0) - \rho^0(\mu - \mu_0) - \frac{1}{2}\kappa^0(\mu - \mu_0)^2. \tag{3.33}$$

The effective action must also be correct to quadratic order, therefore for the purposes of computing the full free energy, consistency requires that in $S^{(1)}_{eff}$ we take $\kappa \equiv \kappa^0$ and $\rho = \rho^0 + \kappa^0(\mu - \mu_0)$. Therefore, for the purposes of computing derivatives with respect to μ, the only μ-dependence in the fluctuation part of the free energy is in ρ. We have then

$$\begin{aligned} f &= f_0 - \frac{1}{\beta V} ln\left[\sum_{m=-\infty}^{\infty} tr^{\phi^m}\left\{e^{-S^{(1)}_{eff}}\right\}\right] \\ &= f_0 - \frac{1}{\beta V} ln\left[\sum_{m=-\infty}^{\infty} e^{2\pi m\rho V} tr^{\phi^m}\left\{e^{-S^{(0)}_{eff}}\right\}\right], \end{aligned} \tag{3.34}$$

where tr^{ϕ^m} means that we impose the temporal boundary condition $\phi(\beta) = \phi(0) + 2\pi m$. Now define $\tilde{\phi}(\tau) = \phi(\tau) - 2\pi m\tau/\beta$, so that $\tilde{\phi}(\beta) = \tilde{\phi}(0)$, to obtain

$$\begin{aligned} f &= f_0 - \frac{1}{\beta V} ln\Big[\sum_{m=-\infty}^{\infty} e^{i2\pi m\rho V} e^{-2\pi^2 m^2 \kappa^0/\beta} \\ &\times \ \mathrm{tr}^{\tilde{\phi}}\left\{e^{-S^{(0)}_{eff}[\tilde{\phi}]}\right\}\Big] \\ &= f_0 + f_{00}[\kappa^0] - \frac{1}{\beta V} ln\left[\sum_{l=-\infty}^{\infty} e^{-\frac{\beta}{2\kappa^0 V}(\rho V - l)^2}\right], \end{aligned} \tag{3.35}$$

where we have used (see App. B)

$$\sum_{m=-\infty}^{\infty} e^{i2\pi m x} \frac{e^{-m^2/2K}}{\sqrt{2\pi K}} = \sum_{l=-\infty}^{\infty} e^{-2\pi^2 K(x-l)^2}, \tag{3.36}$$

with $K = \beta/4\pi^2\kappa^0 V$, and $f_{00}[\kappa^0] = ln\left[\sqrt{\beta/2\pi\kappa^0 V} tr^{\tilde{\phi}}\left\{e^{-S^{(0)}_{eff}[\tilde{\phi}]}\right\}\right]$ is independent of $\mu - \mu_0$. In the limit $\beta \to \infty$ only the term with minimal $(x-l)^2$, i.e. $-\frac{1}{2} \leq x - l \leq \frac{1}{2}$, contributes (at the boundaries, two terms are degenerate). Let $l_0(\kappa^0, \mu)$ be this minimizing value of l. We then obtain finally,

$$f = f_0 + f_{00} + \frac{1}{2\kappa^0 V}(\rho V - l_0)^2. \tag{3.37}$$

The density is therefore

$$-\frac{\partial f}{\partial \mu} = \rho - \frac{1}{V}(\rho V - l_0) = \frac{l_0}{V}. \tag{3.38}$$

There are then exactly l_0 particles in the droplet for the interval of μ such that $|\rho(\mu)V - l_0| < \frac{1}{2}$, and we have established the desired result that the droplet is then incompressible on this same interval.

Consider next the temporal correlation function which is given by

$$\begin{aligned} G_\rho(\tau-\tau') &= \langle e^{i[\phi(\tau)-\phi(\tau')]}\rangle \\ &= \frac{\mathrm{tr}^\phi\left[e^{-S^{(1)}_{eff}}e^{i[\phi(\tau)-\phi(\tau')]}\right]}{\mathrm{tr}^\phi\left[e^{-S^{(1)}_{eff}}\right]} \\ &= \frac{\sum_{m=-\infty}^{\infty} e^{i2\pi m\rho V}\mathrm{tr}^{\phi^m}\left[e^{-S^{(0)}_{eff}}e^{i[\phi(\tau)-\phi(\tau')]}\right]}{\sum_{m=-\infty}^{\infty} e^{i2\pi m\rho V}\mathrm{tr}^{\phi^m}\left[e^{-S^{(0)}_{eff}}\right]}. \end{aligned} \quad (3.39)$$

Defining the same periodic field, $\tilde{\phi}(\tau)$, we obtain

$$\begin{aligned} G_\rho(\tau-\tau') &= \langle e^{i[\phi(\tau)-\phi(\tau')]}\rangle_{S^{(0)}_{eff}} \\ &\times \frac{\sum_{m=-\infty}^{\infty} e^{i2\pi m(\rho V+\frac{\tau-\tau'}{\beta})}e^{-2\pi^2 m^2\kappa^0 V/\beta}}{\sum_{m=-\infty}^{\infty} e^{i2\pi m(\rho V)}e^{-2\pi^2 m^2\kappa^0 V/\beta}} \\ &= e^{-|\tau-\tau'|/2\kappa^0 V}e^{-(\tau-\tau')(\rho V \bmod 1)/\kappa^0 V}, \quad \beta\to\infty, \end{aligned} \quad (3.40)$$

where we have used (3.36). Once again, in the limit $\beta\to\infty$ only the term with $-\frac{1}{2}\leq \rho V - l \equiv \rho V \mathrm{mod}\ 1 \leq \frac{1}{2}$, contributes and the final line of (3.40) results. We see that $G_\rho(\tau)$ decays exponentially for both $\tau\to\pm\infty$, but at different rates:

$$\begin{aligned} G_\rho(\tau) &= e^{-(1\pm\gamma)|\tau|/2\kappa^0 V}, \quad \tau\to\pm\infty \\ -1<\gamma &\equiv 2(\rho V \bmod 1)\leq 1. \end{aligned} \quad (3.41)$$

This exponential decay signifies an energy gap, proportional to $\frac{1}{\kappa^0 V}$, for adding a particle, and is equivalent to the incompressibility result above. However, for large V this gap is very small, and we need only increase μ (and hence ρ) by a small amount to add a single particle to the droplet. For given μ the number of particles in the drop will be $l=[\rho V]$, the greatest integer less than or equal to ρV. Since there exist arbitrarily large droplets, an arbitrarily small change in μ will then add particles to the system in precisely those droplets with volume $V\geq\frac{1}{\rho}\approx\frac{1}{\kappa^0\mu}$. Focussing on μ near zero (where $\kappa^0=\kappa_0$), we may estimate the total density as

$$\begin{aligned}\rho_{tot} &\sim \int dV d\kappa_0 p(V,\kappa_0)\frac{[\kappa_0\mu V]}{V} \\ &\sim \bar{\kappa}_0\mu\int_{V>\frac{1}{\kappa_0\mu}} dV e^{-V/V_0(\bar{\kappa}_0)} \\ &\sim \bar{\kappa}_0\mu e^{-1/\bar{\kappa}_0\mu V_0(\bar{\kappa}_0)},\end{aligned} \tag{3.42}$$

where $\bar{\kappa}_0$ is defined analogously to $\bar{\Upsilon}_\tau$ in (3.19). In the derivation of this formula we have assumed that $\mu > 0$, but the result is valid also for $\mu < 0$ if μ is replaced by $|\mu|$ in the exponent (only). The total compressibility may then be estimated as

$$\kappa_{tot} \sim \bar{\kappa}_0 e^{-1/\bar{\kappa}_0 V_0(\bar{\kappa}_0)|\mu|}. \tag{3.43}$$

Finally we may use the above results to estimate the total temporal correlation function and to exhibit the finite density of states, (3.20), at $\bar{\epsilon} = 0$. Once again, the total correlation function, $G_\rho(\tau)$ is the average of $G_\rho^{(0)}(\tau)$ over all droplets:

$$G_\rho(\tau) = \int dV d\kappa_0 p(V,\kappa_0) G_\rho^{(0)}(\tau;\kappa_0,V). \tag{3.44}$$

For large τ and small $\mu > 0$, only large volumes contribute to the integral. It is clear from (3.41) that $G_\rho^{(0)}(\tau)$ decays most slowly when ρV is close to half-integer, and those droplets with such "resonant" values of V will contribute the leading large τ-dependence. The smallest resonant volume (into which a single particle will be added) is precisely $V = \frac{1}{2\rho}$, and contributions from higher order resonances, $V = \frac{3}{2\rho}, \frac{5}{2\rho}, \ldots$, will be exponentially smaller in $\frac{1}{\rho}$. Thus

$$\begin{aligned}G_\rho(\tau) &\sim \int dV d\kappa_0 p(V,\kappa_0) e^{-|\tau|/2\kappa_0 V} e^{-\gamma\tau/2\kappa_0 V} \\ &\sim e^{-1/2\bar{\kappa}_0\mu V_0(\bar{\kappa}_0)}\int_0^\delta \frac{dx}{\bar{\kappa}_0\mu} e^{-\frac{1}{4}\mu x|\tau|},\end{aligned} \tag{3.45}$$

where $x = |\rho V - \frac{1}{2}|$ and $\delta < \frac{1}{2}$ is a cutoff and we have replaced V by its smallest resonant value, $\frac{1}{2\rho} \approx \frac{1}{2\kappa_0\mu}$, everywhere except in $\gamma = 2(\rho V \bmod 1)$. The integration is now trivial, and we obtain

$$G_\rho(\tau) \sim \frac{4}{\bar{\kappa}_0\mu^2|\tau|}\left[1 - e^{-\frac{1}{4}\mu\delta|\tau|}\right] e^{-1/2\bar{\kappa}_0\mu V_0(\bar{\kappa}_0)}. \tag{3.46}$$

This reproduces the $\frac{1}{\tau}$ behavior, (3.20), predicted for the Bose glass phase with

$$\rho_1(\epsilon = 0) \sim e^{-1/2\bar{\kappa}_0 \mu V_0(\bar{\kappa}_0)}. \tag{3.47}$$

All results are valid for $\mu < 0$ if μ is replaced by $|\mu|$ in the exponent. Note that the power law prefactors of the exponential must not be taken seriously because we have made a very crude estimate for the probability function $p(V, \kappa_0)$. Recall that $\bar{\kappa}_0$ is the "most probable" compressibility for large droplets.

To summarize, we have seen that for the particle-hole symmetric model the correlation function, $G(\tau)$, has stretched exponential behavior coming from large rare regions in which $J > J_c^0$. This is known as a Griffiths singularity [24], and this kind of effect is ubiquitous in random systems. Since $G(\tau)$ still decays faster than any power law, the effects of these singularities are obviously physically rather subtle. In contrast, when $\mu \neq 0$ the model no longer has a classical interpretaion, and the behavior is far more singular: for given μ, *finite* droplets of size $V \approx \frac{1}{2}\bar{\kappa}_0 \mu$ give rise to power law decay of $G_\rho(\tau)$ – no longer do the singularities occur only in the limit $V \to \infty$. Quantum mechanically we understand this as being a consequence of the existence of arbitrarily low energy single particle excitations, arising from superfluid droplets with very small energy gaps for the addition of an extra particle. It is interesting to see this derived explicitly from the interference terms in the Lagrangian [see (3.36)-(3.41)].

3.2.3 Statistical particle-hole symmetry

The above results were derived by assuming a random J_{ij} model with a small uniform μ. Suppose instead that we maintain $\mu = 0$, but include instead the random site energies ε_i with a symmetric distribution. The model is then statistically particle-hole symmetric (see the discussion in Chap 2). Does this change any of the results? If we assume the ε_i to be statistically independent of the J_{ij} (or, more specifically, with statistics such that the ε_i do not automatically vanish in large superfluid droplets) it is clear that the answer must be no: If the scale of the ε_i is $\delta\mu$ (e.g., $\delta\mu = \langle\langle \varepsilon_i^2 \rangle\rangle^{\frac{1}{2}}$) then it is clear that we will be able to find superfluid droplets of arbitrary size in which all $\varepsilon_i > \delta\mu$. The previous analysis then goes through precisely as before with the results (3.46) and (3.47), but now with μ replaced by $\delta\mu$, and a different volume scale, $V_0(\bar{\kappa}_0, \delta\mu)$, now depending on $\delta\mu$. Thus, at least at this qualitative level, statistical particle-hole symmetry is the same as generic particle-hole *asymmetry.* It seems very unlikely then that the superfluid transition in this case would be any different either. We shall address this issue in the next chapter.

Chapter 4

Particle-hole symmetry and scaling near criticality

In order to discuss scaling it is convenient (but by no means necessary) to use the ψ^4 Lagrangian, (2.18), further simplified by taking the continuum limit and dropping all unnecessary dimensionful coefficients. For now we take $\epsilon_\tau = 1$ only. Consider then the Lagrangian

$$\begin{aligned}\mathcal{L}_c &= -\int d^dx \int d\tau \Big[\frac{1}{2}|\partial_\tau\psi|^2 + \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}r(\mathbf{x})|\psi|^2 \\ &+ u|\psi|^4 + g(\mathbf{x})\psi^*\partial_\tau\psi\Big]. \end{aligned} \tag{4.1}$$

The random coefficient $r(\mathbf{x})$ is equivalent to the random J_{ij}, while the random coefficient $g(\mathbf{x})$ is equivalent to the random site potential, $-\mu_i \equiv -\mu + \varepsilon_i$. We write $r(\mathbf{x}) = r_0 + \delta r(\mathbf{x})$, and the phase transition occurs when r_0 becomes sufficiently negative. When $g \equiv 0$ the particle-hole symmetric problem is recovered. If the $|\partial_\tau\psi|^2$ term is dropped and we take $g \equiv 1$ we obtain the closest approximation to the boson coherent state Lagrangian, (2.10), which was the starting point for the work in Ref. [12]. We shall see that the $|\partial_\tau\psi|^2$ term, which was ignored in Ref. [12], is actually crucial for a correct understanding of the critical behavior. The model (4.1) with $g \equiv 0$ is precisely the model studied in Ref. [13], which we shall henceforth refer to as the "classical random rod" problem.

The superfluid density, ρ_s, and compressibility, κ, are related to twists, spatial and temporal respectively, in the superfluid order parameter (see Sec. 3.1). Thus we can introduce

$$\tilde{\psi}(\mathbf{x},\tau) = e^{-(i\mathbf{k}_0.\mathbf{x}+i\omega_0\tau)}\psi(\mathbf{x},\tau), \tag{4.2}$$

and impose periodic boundary conditions on $\tilde{\psi}(\mathbf{x},\tau)$. This is equivalent to imposing a twist θ_α in the boundary condition, with $\omega_0 = \theta_0/\beta$ and $bfk_0 = (\theta_1/L_1, .., \theta_d/L_d)$. In (4.1), if we replace $\psi(\mathbf{x},\tau)$ by $\tilde{\psi}(\mathbf{x},\tau)$, then

$$|\partial_\tau\psi|^2 \rightarrow |\partial_\tau\tilde{\psi}|^2 + 2i\omega_0[\tilde{\psi}^*\partial_\tau\tilde{\psi} - \tilde{\psi}\partial_\tau\tilde{\psi}^*] - \omega^2|\tilde{\psi}|^2,$$

and

$$|\nabla_x\psi|^2 \rightarrow |\nabla_x\tilde{\psi}|^2 + 2i\mathbf{k}_0[\tilde{\psi}^*\nabla_x\tilde{\psi} - \tilde{\psi}\nabla_x\tilde{\psi}^*] - k_0^2|\tilde{\psi}|^2.$$

The free energy can be expanded in powers of k_0 and ω_0, giving

$$f(k_0, \omega_0) = f(0,0) + \frac{1}{2}\Upsilon k_0^2 + \rho\omega_0 + \frac{1}{2}\kappa\omega_0^2 + ... \tag{4.3}$$

For the Lagrangian (4.1) we can write down especially simple expressions for the helicity moduli Υ and κ, [(3.6) and (3.8)]. We find

$$\begin{aligned}
\Upsilon_\alpha &= \langle\langle\ \langle|\psi|^2\rangle\ \rangle\rangle + \int d^dx \int d\tau \langle\langle\ \langle\psi^*\partial_\alpha\psi(\mathbf{x},\tau) \\
&\times\ \psi^*\partial_\alpha\psi(\mathbf{0},0)\rangle\ \rangle\rangle, \quad \alpha = 1,\ldots,d \\
\kappa &\equiv \Upsilon_\tau = \langle\langle\ \langle|\psi|^2\rangle\ \rangle\rangle + \int d^dx \int d\tau \langle\langle\ \langle\psi^*[\partial_\tau\psi \\
&-\ g|\psi|^2](\mathbf{x},\tau)[\psi^*\partial_\tau\psi - g|\psi|^2](\mathbf{0},0)\rangle\ \rangle\rangle \\
\rho &\equiv \rho_\tau = \langle\langle\ \langle|\psi|^2\rangle g(\mathbf{x})\rangle\rangle.
\end{aligned} \tag{4.4}$$

We shall ultimately require these expression only when $g \equiv 0$.

Let us first consider the scaling of ρ_s and κ for the classical random rod problem, which corresponds to $\mathbf{g}(\mathbf{x}) \equiv 0$. Note that $\mathbf{k}_0\tilde{\psi}^*\nabla_x\tilde{\psi}$ and $\omega_0\tilde{\psi}^*\partial_\tau\tilde{\psi}$ are symmetry breaking perturbations (they break the $x \leftrightarrow -x$, and $\tau \leftrightarrow -\tau$ symmetries of the Lagrangian). It is proposed that the singular part of the free energy varies as

$$f_s(k_0, \omega_0) \approx A|\delta|^{2-\alpha}\Phi(k_0\xi, \omega_0\xi_\tau), \tag{4.5}$$

where $\xi \approx \xi_0|\delta|^{-\nu_0}$ and $\xi_\tau \approx \xi_{\tau,0}|\delta|^{-\nu_{\tau 0}}$ are the correlation lengths in the spatial and temporal directions, respectively, and the *dynamical exponent* is defined by $z_0 = \nu_\tau/\nu_0$. The subscript 0 on the exponents indicate that they are those appropriate to the classical random rod problem, and the generic auxillary parameter, δ is $r_0 - r_{0,c}$ in (4.1), but more generally is any parameter such as chemical potential, pressure, strength of disorder, film thickness, or magnetic field which moves the system through the phase transition at $T = 0$, defined to occur at $\delta = 0$. We assume that $\delta > 0$ corresponds to the disordered phase and $\delta < 0$ to the ordered (superfluid or superconducting) phase. There are actually two distinct sets scaling functions and amplitudes for $\delta > 0$ and $\delta < 0$, but for notational simplicity we shall not make this distinction explicit. ρ_s (and κ in this case) are nonzero only in the superfluid phase.

More properly, the boundary condition dependence appears in a finite size scaling ansatz for the free energy

$$\Delta f^\theta \approx \beta^{-1} L^{-d} \Phi_0^\theta (A\delta L^{1/\nu_0}, B\delta\beta^{1/z_0\nu_0}), \tag{4.6}$$

where A and B are nonuniversal scale factors (no nonuniversal scale factor is needed due to quantum hyperuniversality). The existence of a nonzero stiffness (in the ordered phase), i.e., a leading finite-size correction of order L^{-2} or β^{-2}, now requires that the scaling function $\Phi_0^\theta(x,y) \approx x^{d\nu} y^{z\nu} (\Phi_1^\theta x^{-2\nu} + \Phi_2^\theta y^{-2z\nu})$ for large x,y, (and $\delta > 0$), yielding

$$\begin{aligned} \Upsilon &\approx A^{(d-2)\nu_0} B^{z_0\nu_0} (\Phi_1^\theta/\theta^2) \delta^{\upsilon_0} \\ \kappa &\approx A^{d\nu_0} B^{-z_0\nu_0} (\Phi_2^\theta/\theta^2) \delta^{\upsilon_{\tau 0}} \end{aligned} \tag{4.7}$$

implying the Josephson scaling relations $\upsilon_0 = (d + z_0 - 2)\nu_0 = 2 - \alpha_0 - 2\nu_0$ and $\upsilon_{\tau 0} = (d - z_0)\nu_0 = 2 - \alpha_0 - 2z_0\nu_0$, and requiring in addition $\Phi_{1,2}^\theta \propto \theta^2$. The crucial assumption is that the leading boundary condition dependence is all in the singular, i.e., finite size scaling part of the free energy. We emphasize here that we can make this assumption only because k_0 and ω_0 introduce relevant perturbations which fundamentally alter the symmetry of the Lagrangian. Further evidence for this is that we expect all stiffness to vanish identically in the disordered phase of the classical model: thus Υ and κ can have no analytic part at all.

Now we can try to extend the above arguments for nonzero $\mathbf{g}(\mathbf{x})$, following Ref. [6]. In the same way, the singular free energy could be written as (incorrectly, as it turns out),

$$f_s(k_0, \omega_0) \approx A|\delta|^{2-\alpha} \Phi(k_0\xi, \omega_0\xi_\tau). \tag{4.8}$$

This then implies that the compressibility scales as

$$\kappa \sim \delta^{(d-z)\nu}. \tag{4.9}$$

All exponents now refer to the dirty boson critical point. For $\mathbf{g}(\mathbf{x}) \neq 0$, both the Bose glass and superfluid phases are compressible, so it is expected that the compressibility remains finite right through the transition. This leads to the prediction that $z = d$.

The arguments leading to $z = d$ turn out to be incorrect. For $\mathbf{g}(\mathbf{x}) \neq 0$, the density of the Bose glass phase varies smoothly with μ. A temporal twist only perturbs slightly a term, $g(\mathbf{x})\psi^* \partial_\tau \psi$, that is already present in the Lagrangian, and we *do not* expect it to produce a new relevant perturbation. Thus the scaling variable *cannot* be $\omega_0 \xi_\tau$. Rather ω_0 produces only an infinitesimal shift $\mu \to \mu - i\omega_0$ which through the above analyicity argument, alters the free energy in a completely predictable fashion unrelated to scaling. Spatial twists still represent a relevant pertubation, and therefore we predict a scaling form at $\omega_0 = 0$:

$$\Delta f^\theta = \beta^{-1} L^{-d} \Phi^\theta (A\delta L^{1/\nu}, B\delta\beta^{1/z\nu}), \tag{4.10}$$

with $\Phi(x,y) \approx \Phi_1 x^{(d-2)\nu} y^{z\nu}$ for large x,y, yielding $\Upsilon \approx A^{(d-2)\nu} B^{z\nu} (\Phi_1^\theta / \theta^2 \delta^v$, $v = (d + z - 2)\nu = 2 - \alpha - 2\nu$ as before. All exponents refer to the dirty boson critical point. We expect small subleading corrections in $y^{z\nu}$. Now, if we include a finite ω_0, the basic change in (4.10) is that $\mu \to \mu - i\omega_0$ everywhere, in addition we must include changes arising from boundary condition dependence of the analytic part of the free energy. Thus

$$\begin{aligned} \Delta f^\theta &= \beta^{-1} L^{-d} \Phi^\theta (A\delta_\theta L^{1/\nu}, B\delta_\theta \beta^1 / z\nu) \\ &\quad + f_a(J, \mu - i\omega_0) - f_a(J, \mu), \end{aligned} \tag{4.11}$$

where f_a is analytic, $\delta_\theta = J - J_c(\mu - i\omega_0) \approx \delta + i\omega_0 J_c'(\mu)$. Most importantly, Φ^θ is the same function as that in (4.11), and therefore produces small corrections in $(\beta/\xi_\tau)^{-1}$. The scaling function itself therefore contributes nothing to κ, which must therefore arise

(a) from the analytic part of the free energy,

(b) from the ω_0 dependence of δ_θ, and

(c) from other subleading terms in the nonanalytic free energy.

Part (b) couples derivatives with respect to μ, or equivalently ω_0, to derivatives with respect to δ, producing the most singular part of the compressibility, $\kappa_s \sim |\delta|^{-\alpha}$. There is a famous theorem [25] that requires $\nu \geq \frac{2}{d}$, which implies $\alpha = 2 - (d+z)\nu$ is negative. The application of this theorem to the disordered boson problem has been questioned recently [26], however it seems very unlikely that α will be positive. So we expect the singular part of κ to vanish at criticality. The main contribution comes from the analytic part of the free energy.

$$f_a(J,\mu) = -\rho_c(J)[\mu - \mu_c(J)] - \frac{1}{2}\kappa_c(J)[\mu - \mu_c(J)]^2 + ..., \tag{4.12}$$

expanded for convenience about the transition line $\mu_c(J)$. We immediately then obtain a finite compressibility right through the transition, with the exponent z nevertheless completely undetermined. The fact that the leading correction is linear in ω_0 also leads to a incorrect predictions for the scaling of the total density, $\rho = -\frac{\partial f}{\partial \mu}$. If the density comes from $\omega_0 \xi_\tau$ dependence in the singular free energy, it follows that

$$\rho = \frac{\partial f_s}{\partial \omega_0} \sim \delta^{d\nu}. \tag{4.13}$$

However the density has to be finite at the transition, and the finite piece cannot come from the singular free energy. So the original scaling hypothesis for temporal twists, ω_0, is not valid.

For the classical random rod model, $g(\mathbf{x}) \equiv 0$, we have already seen that $\kappa = 0$ in the disordered phase [notice that (4.11) is no longer valid since the special symmetry at $\mu \equiv 0$ implies that δ and μ are "orthogonal" thermodynamic coordinates and the derivatives with respect to μ that define $\kappa \equiv \kappa_s$ do not mix with derivatives with respect to δ]. We therefore expect κ to rise continuously from zero for $\delta < 0$, with the exponent $\zeta_\tau > 0$. This implies that $z \leq d$ in this case [equality is still permitted and would imply a discontinuity in κ at $\delta = 0$, which indeed is the case in $d = 1$ (see Chap. 5)]. Note that for homogeneous classical disorder, where the coefficient r in (4.1) depends on *both* $\mathbf{x}$ and τ, we will have isotropic scaling, $z = 1$. The rod disorder should increase z.

We now ask the following question. Since we expect the presence of $g(\mathbf{x})$ to change the universality class of the the phase transition, there must be an associated positive crossover exponent, ϕ_g, which quantifies the instability of the classical random rod fixed point with respect to this term. What is the value of ϕ_g, and what conditions does it place on the values of the classical fixed point exponents?

To begin to answer this question, let us write $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_g$, where

$$\mathcal{L}_g = -\int d^d x \int d\tau g(\mathbf{x}) \psi^* \partial_\tau \psi(\mathbf{x}, \tau). \tag{4.14}$$

We assume (see below) that $\langle\langle g(\mathbf{x}) \rangle\rangle = 0$, $\langle\langle g(\mathbf{x}) g(\mathbf{x}') \rangle\rangle = \Delta_g \phi(\mathbf{x} - \mathbf{x}')$, which implies a statistical particle-hole symmetry, and that $g(\mathbf{x})$ and $r(\mathbf{x})$ are *independent.* The correlation function $\phi(\mathbf{x}) = \delta(\mathbf{x})$ for uncorrelated disorder, but for reasons that will become evident below, we shall allow more general long-range power-law correlated disorder with $\phi(\mathbf{x}) \approx |\mathbf{x}|^{-(d+a)}$ for large $|\mathbf{x}|$ and some exponent, a. The crossover exponent, ϕ_g, which in general will be a function of a, is defined by the scaling form, valid for small Δ_g,

$$f_s \approx A|\delta_g|^{2-\alpha}\Phi_g\left(\frac{B\Delta_g}{|\delta_g|^{\phi_g}}\right) \tag{4.15}$$

where the subscript on δ_g is to serve as a reminder that Δ_g will also generate a shift in the position of the critical point: $\delta_g = \delta + c_1\Delta_g + \ldots$. The value of ϕ_g may now be inferred from the derivative

$$\begin{aligned}\left(\frac{\partial f_s}{\partial \Delta_g}\right)_{\Delta_g=0} &\approx A|\delta|^{2-\alpha}[B|\delta|^{-\phi_g} - (2-\alpha)c_1|\delta|^{-1}] \\ & \qquad |\delta| \to 0.\end{aligned} \tag{4.16}$$

Note the very singular $|\delta|^{-1}$ term generated by the shift, which may often dominate the $|\delta|^{-\phi_g}$ term of interest. Now, this derivative may also be calculated directly in perturbation theory:

$$f(\Delta_g) - f(0) = \frac{1}{V_D}\left[\langle\mathcal{L}_g\rangle_0 - \frac{1}{2}[\langle\mathcal{L}_g^2\rangle_0 - \langle\mathcal{L}_g\rangle_0^2] + O(g^3)\right] \tag{4.17}$$

where the averages are with respect to $\mathcal{L}_0$. Assuming that $g(\mathbf{x})$ and $r(\mathbf{x})$ self average, we have $\langle\mathcal{L}_g\rangle_0 = \langle\langle\ \langle\mathcal{L}_g\rangle_0\ \rangle\rangle = 0$, and

$$\begin{aligned}f(\Delta_g) &- f(0) = \frac{1}{2V_D}\langle\langle\ \langle\mathcal{L}_g^2\rangle_0\ \rangle\rangle + O(g^4) \\ &= -\frac{1}{2}\Delta_g\int d\tau\int d^dx\phi(\mathbf{x})\langle\langle\ \langle\psi^*\partial_\tau\psi(\mathbf{x},\tau) \\ &\times\ \psi^*\partial_\tau\psi(\mathbf{0},0)\rangle_0\ \rangle\rangle + O(\Delta_g^2),\end{aligned} \tag{4.18}$$

where independence of $g(\mathbf{x})$ and $r(\mathbf{x})$ has been used. Thus

$$-2\left(\frac{\partial f}{\partial \Delta_g}\right)_{\Delta_g=0} = \int d\tau\int d^dx\phi(\mathbf{x})\mathcal{G}_g(\mathbf{x},\tau) \tag{4.19}$$

where we have defined the correlation function

$$\mathcal{G}_g(\mathbf{x},\tau) = \langle\langle\ \langle\psi^*\partial_\tau\psi(\mathbf{x},\tau)\psi^*\partial_\tau\psi(\mathbf{0},0)\rangle_0\ \rangle\rangle. \tag{4.20}$$

Let us define the Fourier transforms

$$\begin{aligned} \hat{\phi}(\mathbf{k}) &= \int d^d x e^{i\mathbf{k}\cdot\mathbf{x}} \phi(\mathbf{x}) \\ \hat{\mathcal{G}}_g(\mathbf{k},\omega) &= \int d\tau \int d^d x e^{i(\mathbf{k}\cdot\mathbf{x}+\omega\tau)} \mathcal{G}(\mathbf{x},\tau). \end{aligned} \tag{4.21}$$

Then when $g(\mathbf{x}) \equiv 0$ we have from (4.4) and (4.7),

$$\Upsilon_\tau = \rho_0 - \hat{\mathcal{G}}_g(\mathbf{0},0) \sim |\delta|^{(d-z)\nu}, \tag{4.22}$$

where $\rho_0 \equiv \langle\langle\ \langle|\psi|^2\rangle_0\ \rangle\rangle$. More generally we expect a scaling form for small $|\mathbf{k}|$ and ω:

$$\Upsilon_\tau(\mathbf{k},\omega) \equiv \rho_0 - \hat{\mathcal{G}}_g(\mathbf{k},\omega) \approx A_1|\delta|^{(d-z)\nu}\mathcal{Y}(k\xi,\omega\xi_\tau) \tag{4.23}$$

where for $\delta > 0$ we have $\lim_{w,s\to 0}\mathcal{Y}(w,s) = 0$ while for $\delta < 0$ we have $\lim_{w,s\to 0}\mathcal{Y}(w,s) = 1$. Now,

$$\begin{aligned} -\quad 2\left(\frac{\partial f}{\partial \Delta_g}\right)_{\Delta_g=0} &= \int_{\mathbf{k}} \hat{\phi}(\mathbf{k})\hat{\mathcal{G}}_g(\mathbf{k},\omega=0) \\ &= \rho_0\phi(\mathbf{x}=\mathbf{0}) - \int_{\mathbf{k}} \hat{\phi}(\mathbf{k})\Upsilon_\tau(\mathbf{k},0), \end{aligned} \tag{4.24}$$

where we have used the convenient shorthand notation $\int_{\mathbf{k}} \equiv \int \frac{d^d k}{(2\pi)^d}$, $\int_\omega \equiv \int \frac{d\omega}{2\pi}$, etc. For uncorrelated disorder, $\hat{\phi}(\mathbf{k}) \equiv 1$, while for power law correlated behavior, $\hat{\phi}(\mathbf{k}) \approx c_a k^a + c_0 + \ldots$ as $k \to 0$. The final k-integral must be treated carefully to extract its δ-dependence. Let us rewrite

$$\tilde{\mathcal{Y}}(w) = w^{z-d}\mathcal{Y}(w,0),\ \ \tilde{\mathcal{Y}}(w\to\infty) \to y_0 > 0. \tag{4.25}$$

Clearly one cannot simply scale ξ out of the integral in (4.24) since the resulting integral over $\hat{\phi}(w/\xi)w^{d-z}\tilde{\mathcal{Y}}_\pm(w)$ will not converge. Rather, one must first subtract out the large w behavior. Let us write

$$\tilde{\mathcal{Y}}_\pm(w) \approx y_0 + y_1 w^{-\omega_1} + y_2 w^{-\omega_2} + y_3 w^{-\omega_3} + \ldots \tag{4.26}$$

where the source of the spectrum of exponents, $0 < \omega_1 < \omega_2 < \omega_3 < \ldots$, will be discussed below. We see then that the term $y_j w^{-\omega_j}$ yields a contribution

$$\begin{aligned} \int_{\mathbf{k}} \hat{\phi}(\mathbf{k}) y_j k^{d-z} (k\xi)^{-\omega_j} &= b_j |\delta|^{\omega_j \nu} \\ b_j \equiv \xi_0^{-\omega_j} y_j \int_{\mathbf{k}} \hat{\phi}(\mathbf{k}) k^{-\omega_j + d - z}&, \end{aligned} \tag{4.27}$$

so long as $\omega_j < \omega_{max}(a) \equiv 2d - z + \min\{a, 0\}$ (i.e., the integral converges at small k – convergence at large k is ensured by the implicit lattice cutoff, $k < k_\Lambda \sim \frac{\pi}{a}$). Let us define the subtracted scaling function,

$$\delta\tilde{\mathcal{Y}}(w) = \tilde{\mathcal{Y}}(w) - \sum_{\omega_j < \omega_{max}(a)} y_j w^{-\omega_j}. \tag{4.28}$$

Then

$$\begin{aligned} \int_{\mathbf{k}} \hat{\phi}(\mathbf{k}) \tilde{\mathcal{Y}}(k\xi) k^{d-z} &\approx \sum_{\omega_j < \omega_{max}(a)} b_j |\delta|^{\omega_j \nu} \\ &+ \int_{\mathbf{k}} k^{d-z} \hat{\phi}(\mathbf{k}) \delta\tilde{\mathcal{Y}}(k\xi), \end{aligned} \tag{4.29}$$

where the last term may be evaluated as

$$\begin{aligned} \int_{\mathbf{k}} k^{d-z} \hat{\phi}(\mathbf{k}) \delta\tilde{\mathcal{Y}}(k\xi) &\approx b_a |\delta|^{(2d-z+a)\nu} + b_0 |\delta|^{(2d-z)\nu}, \\ b_a &\equiv c_a \int_{\mathbf{w}} w^{d-z+a} \delta\tilde{\mathcal{Y}}(w) \\ b_0 &\equiv c_0 \int_{\mathbf{w}} w^{d-z} \delta\tilde{\mathcal{Y}}(w). \end{aligned} \tag{4.30}$$

In fact $b_0 = 0$ in our case because, by time reversal invariance, $\langle \psi^* \partial_\tau \psi \rangle \equiv 0$.

Now, the origin of the exponents ω_j is as follows: the operator

$$P \equiv \int d^d x \int d\tau \int d\tau' \psi^* \partial_\tau \psi(\mathbf{x}, \tau) \psi^* \partial_\tau \psi(\mathbf{x}, \tau')$$

will have an expansion in terms of eigenoperators of a renormalization group transformation near the critical fixed point of interest: $P = h_1 O_1 + h_2 O_2 + h_3 O_3 + \ldots$, and O_i is assumed to have renormalization group eigenvalue λ_i. This implies that $\langle P \rangle \sim h_1 |\delta|^{(d+z-\lambda_1)\nu} + h_2 |\delta|^{(d+z-\lambda_2)\nu} + \ldots$. But since $\langle P \rangle = \int_{\mathbf{k}} \tilde{\mathcal{Y}}(k\xi) k^{d-z}$, comparison with (4.26) and (4.27) implies that $\omega_j = d + z - \lambda_j$. The ω_j therefore reflect the renormalization group transformation properties of P near the fixed point.

We can now understand the crossover exponent, ϕ_g. Comparing (4.16) and (4.29) together with (4.30) (with $b_0 = 0$), we see that

$$\begin{aligned} 2 - \alpha - \phi_g &= \nu \min\{\omega_j, 2d - z + a\} \\ \Rightarrow \lambda_g &\equiv \frac{\phi_g}{\nu} = \max\{\lambda_j, 2z - d - a\}. \end{aligned} \tag{4.31}$$

The crossover exponent is therefore either $\phi_g = \lambda_1 \nu$, or for small enough a, $\phi_g = (2z - d - a)\nu$. Thus

$$\lambda_g = \begin{cases} 2z - d - a, & a < 2z - d - \lambda_1 \\ \lambda_1, & a > 2z - d - \lambda_1, \end{cases} \tag{4.32}$$

implying

$$\phi_g > 0 \Leftrightarrow \begin{cases} z > \frac{d+a}{2}, & a < 2z - d - \lambda_1 \\ \lambda_1 > 0, & a > 2z - d - \lambda_1. \end{cases} \tag{4.33}$$

In particular, for short range correlated disorder, where in effect $a \to \infty$, we require $\lambda_1 > 0$ in order that dirty boson disorder destabilize the random rod fixed point. The exponent λ_1 is a nontrivial exponent, and we shall compute it within the ϵ, ϵ_τ-expansion [13]. A naive estimate for this exponent is obtained by supposing that the equality $2z - d - a = \lambda_1$ should occur when $a \simeq 0$, yielding $\lambda_1 \simeq 2z - d$ which becomes positive for $z > \frac{d}{2}$. Note that this same estimate would have been obtained from the second term in (4.30) if we had assumed $b_0 \neq 0$. As an aside, this estimate is actually exact in the corresponding derivation of the Harris criterion for classical disordered magnets. There the correlation function $\langle |\psi(\mathbf{x})|^2 |\psi(\mathbf{y})|^2 \rangle_0$ appears. Since $\langle \psi^2 \rangle$ does not vanish, neither does the coefficient analogous to b_0. This gives rise to a free energy contibution $\langle |\psi|^2 \rangle_0^2 \sim \delta^{2-2\alpha}$ which leads immediately to the Harris criterion, $\phi_g = \alpha$.

In fact, we shall find that $\lambda_1 > 2z - d$, i.e. a drops out at some negative value, and ϕ_g becomes positive for z somewhat *less* than $\frac{d}{2}$. The random rod result $z = 1$ in $d = 1$ is consistent with this criterion, although this case is somewhat special because the random rod fixed point is the *same* as the pure fixed point in $d = 1$ (i.e., random rod disorder is irrelevant, though boson disorder is relevant). The generalized Harris criterion [13] indicates that rod disorder is irrelevant when $\alpha_{pure} + \nu_{pure} < 0$. Using hyperscaling (valid here for $d < 3$), and the fact that $z = 1$ at the pure fixed point, this requires $\nu_{pure} > \frac{2}{d}$ (compare the less stringent requirement, $\nu_{pure} > \frac{2}{d_{tot}}$ with $d_{tot} = d + 1$, for the usual Harris criterion for point disorder). For $d=1$, $\nu \to \infty$, while for $d=2$, $\nu \approx \frac{2}{3}$, so the pure fixed point becomes unstable to rod disorder somewhere in between $d=1$ and $d=2$. In all cases where rod disorder is irrelevant, one then trivially has $z > \frac{d}{2}$, and dirty boson disorder will certainly be relevant.

For a more direct and formal way to see why an equivalent of the Harris criterion cannot be used to give us the crossover exponent for Δ_g, we can turn to renormalization group arguments. It turns out that the reason why the Harris criterion works is that we are perturbing about the pure fixed point. There we were trying to find the crossover exponent for $\Delta \sum_{\alpha,\beta} \int d^d x d\tau \phi_\alpha^2(x,\tau)\phi_\beta^2(x,\tau)$ about the nondisordered fixed point. We are using a replicated Lagrangian, with α and β being the replica indices. We look at a vertex $\Gamma_\Delta^{(N+M)}$ with N fields with the replica index α and M fields with the index β, $\alpha \neq \beta$, and one insertion of Δ. Since only terms that represent disorder can couple different replica indices, for $\Delta = 0$ there are no terms that couple fields with the index α to fields with index β. Thus we have

$$\Gamma_\Delta^{(N+M)}(p_1, ..., p_N; q_1, ..., q_N) = \Gamma_{\phi_\alpha^2}^N(p_1, ..., p_N)\Gamma_{\phi_\beta^2}^M(q_1, ..q_M). \tag{4.34}$$

The above factorization implies that $Z_\phi^{\frac{1}{2}(N+M)} Z_{\phi^2}^2$ will renormalize $\Gamma_\Delta^{(N+M)}$, and hence the anomalous dimension of $\phi_\alpha^2\phi_\beta^2$ is $2\gamma_{\phi^2}^*$, where $\gamma_{\phi^2}^*$ is the anomalous dimension of ϕ^2. This leads to the Harris criterion. In our case, we want the crossover exponent for Δ_g in the presence of Δ_r. Here since Δ_r is already present, it will couple different indices, even in the absence of Δ_g. So we cannot break up the vertex function, as we did for the Harris criterion, and are unable to relate the anomalous dimension of Δ_g to that of the g_0 term. In summary, we cannot get an equivalent of the Harris criterion because we are perturbing about a disordered fixed point.

We now turn to the question of the relevance of $g_o \equiv \langle\langle g(\mathbf{x})\rangle\rangle$. If one carries through a naive scaling analysis using $\mathcal{L}_{g_0} = g_0 \int d^d x \int d\tau \psi^* \partial_\tau \psi(\mathbf{x},\tau)$ in place of (4.14), one could arrive at (incorrectly),

$$\frac{\partial^2 f}{\partial g_0^2} = -\hat{\mathcal{G}}_g(\mathbf{0},0) = -\rho_0 + \Upsilon_\tau \sim |\delta|^{(d-z)\nu}. \tag{4.35}$$

The singular free energy has a dependence on g_0 or μ_0 that goes as

$$f_s = A'\delta^{(d+z)\nu}\Phi_{\mu_0}(\mu_0\delta^{\phi_{\mu_0}}). \tag{4.36}$$

We could then be lead to identify a crossover exponent via

$$2 - \alpha - 2\phi_{\mu_0} = (d - z)\nu \;\Rightarrow\; \phi_{\mu_0} = z\nu \tag{4.37}$$

which is always positive. Now, if $\delta g(\mathbf{x}) \equiv 0$, this is the correct exponent describing the crossover from the critical behavior at the special commensurate point, to the generic incommensurate transition. However for nonzero $\delta g(\mathbf{x}) \equiv 0$, this is not true, for the same reasons that z may not be d, which we discussed in the beginning of this chapter. In particular, there is no $|\delta|^{(d-z)\nu}$ piece in the compressibility whatsoever. The exponent ϕ_{μ_0} is related to subleading terms in the in the nonanalytic free energy, and is completely undetermined by any scaling analysis. We will obtain ϕ_{μ_0} both in 1D and within the ϵ expansion, and show explicitly that particle-hole asymmetry is irrelevant about the statistically symmetric dirty boson critical point, in agreement with previous arguments.

Chapter 5

Calculations in one dimension

In this chapter we review and expand upon the analysis of one-dimensional versions of the dirty boson problem [6, 28]. In App. B we derive various dual representations for the one-dimensional Lagrangian based on the discrete-time Villain representation, (B.4). We shall analyze the sine-Gordon version, (B.14) with (B.15):

$$\begin{aligned} \mathcal{L}_{SG} &= \frac{1}{2}\sum_{\mathbf{R}}\left[\frac{1}{K_I}(\partial_T S_{\mathbf{R}})^2 + V_0(\partial_I S_{\mathbf{R}})^2\right] \\ &- \sum_{\mathbf{R}}\mu_I(\partial_I S_{\mathbf{R}}) - 2y_0\sum_{\mathbf{R}} cos(2\pi S_{\mathbf{r}}), \end{aligned} \tag{5.1}$$

where we have assumed that $V_{IJ} \equiv V_0\delta_{IJ}$ is diagonal. Here $\mathbf{R} = (I, T)$, where I and T are integers, are points on a discrete space-time (dual) lattice, the $-\infty < S_{\mathbf{R}} < \infty$ are continuous spin variables, and the cosine term represents an external periodic potential which prefers integer values of the $S_{\mathbf{R}}$. This model has the physical interpretation of a fluctuating interface, represented by the "height variables" $S_{\mathbf{R}}$. In the absence of μ_I, which has the interpretation of a random tilt potential, the phase transition in this model is from a flat phase, where $S_{\mathbf{R}}$ has only small fluctuations about some integer value and exponentially decaying correlations, to a rough phase in which the interface wanders and has logarithmically divergent correlations. This rough phase corresponds to the superfluid phase in the boson model, and the renormalized, long wavelength value of y_0 vanishes. In the presence of the random tilting potential, μ_I, the rough phase is qualitatively unchanged, but the flat phase is no longer necessarily quite so flat: see below.

When $y_0 = 0$ it is clear that the μ_I may be removed by the following transformation: define the random walk

$$w_I = \frac{1}{V_0}\sum_{J=0}^{I}\mu_J, \tag{5.2}$$

where for $I < 0$ w_I is actually minus the sum from $J = I$ to 0. Now let

$$\tilde{S}_{\mathbf{R}} = S_{\mathbf{R}} - w_I, \ \mathbf{R} = (I, T). \tag{5.3}$$

Then

$$\begin{aligned} \mathcal{L}_{SG} &= \frac{1}{2}\sum_{\mathbf{R}}\left[\frac{1}{K_I}(\partial_T \tilde{S}_{\mathbf{R}})^2 + V_0(\partial_I \tilde{S}_{\mathbf{R}})^2 - \frac{1}{V_0}\mu_I^2\right] \\ &- 2y_0 \sum_{\mathbf{R}} cos(2\pi \tilde{S}_{\mathbf{R}} + 2\pi w_I), \end{aligned} \tag{5.4}$$

and when $y_0 = 0$ it is clear that the μ_I yield only a trivial additive constant to the Free energy. In this limit the two-point correlation function is given by

$$\begin{aligned} G_{\mathbf{R}\mathbf{R}'} &\equiv \frac{1}{2}\langle \tilde{S}_{\mathbf{R}} - S_{\mathbf{R}'}\rangle_0 \\ &\approx \frac{1}{2\pi}\sqrt{\frac{K_0}{V_0}} ln\left[\frac{\rho(\mathbf{R}-\mathbf{R}')}{\rho_0}\right], \rho \to \infty, \end{aligned} \tag{5.5}$$

where we have taken $K_I \equiv K_0$ fixed,

$$\rho(\mathbf{R}-\mathbf{R}') = \left[\frac{1}{K_0 V_0}(I - I')^2 + K_0 V_0 (T - T')^2\right]^{\frac{1}{2}} \tag{5.6}$$

is the appropriately rescaled distance, and $\rho_0 = O(1)$ is a constant scale factor. When K_I fluctuates one must also average over it as well. The result is still (5.5), but K_0 then becomes a complicated effective parameter. The generalized Harris criterion (Ref. [13] and Chap. 4) tells us that disorder in the coefficient K_0 is an *irrelevant* perturbation at the critical point in $d = 1$, so we will, for the rest of this chapter, simply take $K_I \equiv K_0$.

Let us now consider the y_0 term as a perturbation on the quadratic term in $\mathcal{L}_{SG}$. One may, for example, compute the correlation function

$$C_\lambda(\mathbf{R}-\mathbf{R}') = \langle\langle \ \langle e^{i\lambda(\tilde{S}_{\mathbf{R}} - \tilde{S}_{\mathbf{R}'})}\rangle\rangle \ \rangle \tag{5.7}$$

in powers of y_0. Deep in the superfluid/rough phase, where K_0/V_0 is large, this is a well defined expansion. It is also well defined when $\mu = \langle\langle \mu_I \rangle\rangle$ is large: the cosine term in (5.4) then oscillates very rapidly from site to site, and effectively averages itself out. This corresponds to the region between Mott lobes in Fig. 2.1. In the orginal variables, $S_{\mathbf{R}}$, one then has

$$\langle\langle\ \langle S_{\mathbf{R}} - S_{\mathbf{R}'}\rangle\ \rangle\rangle = -i\frac{\partial}{\partial\lambda}C_\lambda(\mathbf{R}-\mathbf{R}')|_{\lambda=0}$$
$$\approx \left[\frac{\mu}{V_0} - C(\mu, K_0, V_0)y_0^2 + O(y_0^4)\right](I'-I), \tag{5.8}$$

where $c(\mu, K_0, V_0)$ is a positive constant that may be calculated explicitly [29]. The corrugation due to y_0 therefore slows down the rate of climb of the interface from its unperturbed rate, μ/V_0. When K_0/V_0 and μ become small this perturbation theory breaks down – a signal of the phase transition. One can, in fact, infer precisely where this happens from a scaling argument analogous to the one used to derive (4.20).

To this end, define

$$O_{\mathbf{R}} = \cos(2\pi\tilde{S}_{\mathbf{R}} + 2\pi w_I) \tag{5.9}$$

and introduce a "temperature" variable, analogous to δ in Sec. IV, by adding a mass term

$$\frac{1}{2}t\sum_{\mathbf{R}}\tilde{S}_{\mathbf{R}}^2 \tag{5.10}$$

to $\mathcal{L}_{SG}$. By this device we may discuss the relevance of the y_0 term to the critical behavior as $t \to 0$. To this end, we postulate a scaling form for the singular part of the free energy,

$$f_s(y_0) \approx At^{2-\alpha}\Phi\left(\frac{By_0^2}{t^{\phi_y}}\right) \tag{5.11}$$

so that

$$\frac{1}{2}\left(\frac{\partial^2 f_s}{\partial y_0^2}\right)_{y_0=0} \approx ABt^{2-\alpha-\phi_y}\Phi'(0). \tag{5.12}$$

The superfluid phase always occurs at $y_0 = 0$, so there will be no shift in the critical value of $t = 0$. As usual, the y_0 term is relevant if $\phi_y > 0$. Now, the derivative in (5.12) may be computed in terms of the average

$$\frac{\partial^2 f}{\partial y_0^2} = -\frac{4}{\beta L}\langle\langle\ \langle\left(\sum_{\mathbf{R}} O_{\mathbf{R}}\right)^2\rangle_0\ \rangle\rangle$$
$$= -2\sum_{\mathbf{R}}\langle\langle cos(2\pi w_I)\rangle\rangle e^{4\pi^2 G(\mathbf{R},t)}, \tag{5.13}$$

where

$$G(\mathbf{R},t) = \left[\left(-\frac{1}{K_0}\partial_T^2 - V_0\partial_I^2 + t\right)\delta_{\mathbf{RR}'}\right]^{-1}_{\mathbf{R},0}$$
$$\approx \int_{\mathbf{k},\omega} \frac{1-e^{i(kI+\omega T)}}{K_0\omega^2 + \frac{1}{V_0}k^2 + t}, \quad |\mathbf{R}| \to \infty. \tag{5.14}$$

To evaluate this further, let us write

$$\mu_I = \mu + \delta\mu_I, \;\; \langle\langle \delta\mu_I \rangle\rangle = 0, \tag{5.15}$$

and we assume that the $\delta\mu_I$ are independent, with a symmetric distribution. Let us define a measure of the disorder strength, Δ, via

$$\langle\langle e^{i\frac{2\pi}{V_0}\delta\mu_I} \rangle\rangle \equiv e^{-2\pi^2\Delta^2}, \tag{5.16}$$

then

$$\frac{\partial^2 f}{\partial y_0^2} = -2\sum_{\mathbf{R}} cos\left(\frac{2\pi}{V_0}I\right) e^{-2\pi^2\Delta^2|I|} e^{-4\pi^2 G(\mathbf{R},t)}. \tag{5.17}$$

For $t \to 0$, $G(\mathbf{R},t)$ has the logarithmic form (5.5), and $e^{-4\pi^2 G}$ is therefore slowly varying relative to the exponentially decaying prefactor in (5.17). We may therefore do the sum over I by setting $I = 0$ in $G(\mathbf{R},t)$ to obtain

$$\frac{\partial^2 f}{\partial y_0^2} \approx D(\mu, V_0, \Delta) \sum_T e^{-4\pi^2 G(\mathbf{R},t)}, \tag{5.18}$$

where D is some constant, diverging as $\Delta \to 0$. Now one may write

$$e^{-4\pi^2 G(\mathbf{R},t)} \approx [\rho(\mathbf{R})/\rho_0]^{-\omega} E(\rho(\mathbf{R})t^{\frac{1}{2}}/\rho_0), \;\; |\mathbf{R}| \to \infty, \tag{5.19}$$

where $\omega = 2\pi\sqrt{K_0/V_0}$ determines the power law decay of correlations at criticality (i.e. in the superfluid phase) and the scaling function $E(w)$ decays exponentially for large w [this can be seen explicitly by writing $G\mathbf{R},t) = G(\mathbf{R},0) + \delta G(\mathbf{R},t)$ and using (5.14)] and $E(0) = 1$. This exhibits the scaling of the correlations when $y = 0$. Thus the correlation length exponent is $\nu = \frac{1}{2}$, and we may finally compute

$$\begin{aligned} \frac{\partial^2 f}{\partial y_0^2} &\approx 2D \int_1^\infty dT (\sqrt{K_0 V_0}|T|/\rho_0)^{-\omega} E(\sqrt{K_0 V_0}|T|^{\frac{1}{2}}/\rho_0) \\ &\sim t^{\frac{\omega-1}{2}}. \end{aligned} \tag{5.20}$$

From (5.5) and (5.6) we see that, up to scale factors, space-time is isotropic. Thus $z = 1$ and hyperscaling yields $2 - \alpha = 2\nu$, so that from (5.12) we may finally identify

$$\phi_y = \frac{3-\omega}{2} = \frac{3}{2} - \pi\sqrt{K_0/V_0}. \tag{5.21}$$

Hence y_0 becomes relevant when $\sqrt{K_0/V_0} < \frac{3}{2\pi}$. This should be compared to the analogous result, $\sqrt{K_0/V_0} < \frac{1}{\pi}$, for the usual Kosterlitz-Thouless transition where $\mu_I \equiv 0$. Thus, the interface roughens *earlier* (i.e., at smaller K_0), meaning that superfluidity is *more stable,* in the presence of disorder. For $\sqrt{K_0/V_0} > \frac{3}{2\pi}$, y_0 is irrelevant and may be set to zero to calculate universal quantities near the phase transition. At the critical point one has $\omega \equiv \omega_c = 3$, which should be compared to the Kosterlitz-Thouless value, $\omega_c = 4$. One may then, for example, invert the duality transformation in this limit to obtain the actual superfluid correlation function. One finds that (B.4), with (B.16), takes the form

$$\begin{aligned} \tilde{\mathcal{L}}_J(y_0 \to 0) &= \frac{1}{2}\sum_{\mathbf{r}} \Big[K_0(\tilde{\phi}_{\mathbf{r}+\hat{\mathbf{x}}} - \tilde{\phi}_{\mathbf{r}})^2 \\ &+ \frac{1}{V_0}(\tilde{\phi}_{\mathbf{r}+\hat{\tau}} - \tilde{\phi}_{\mathbf{r}})^2 \Big], \end{aligned} \tag{5.22}$$

where $\mathbf{r} = (i, \tau)$ is the direct lattice integer position vector and where now $-\infty < \tilde{\phi}_{\mathbf{r}} < \infty$ is a continuous phase variable [since (B.16) forces $\nabla \times \mathbf{m} \equiv 0$ as $y \to 0$, we may write $\mathbf{m} = \nabla p$, where p is an integer scalar field, then define $\tilde{\phi}_{\mathbf{r}} = \phi_{\mathbf{r}} - 2\pi p_{\mathbf{r}}$]. Thus

$$\mathcal{G}(\mathbf{r}) \equiv \langle e^{i(\phi_{\mathbf{r}} - \phi_0)} \rangle = \langle e^{i(\tilde{\phi}_{\mathbf{r}} - \tilde{\phi}_0)} \rangle \sim \tilde{\rho}(\mathbf{r})^{-\eta},\ \eta = \frac{1}{\omega}, \tag{5.23}$$

where $\tilde{\rho}(\mathbf{r})$ is the same as $\rho(\mathbf{r})$ in (5.6), but with $K_0 V_0$ replaced by $\frac{1}{K_0 V_0}$. The exponent η is defined in such a way that $\mathcal{G}(i, \tau = 0) \sim |i|^{-(d+z-2+\eta)}$ at criticality. Equation (5.23) then follows since $d = z = 1$ and $\tilde{\rho}(i, \tau = 0) \propto |i|$ for large $|i|$. At the critical point we have $\eta = \frac{1}{3}$, which should be compared to the Kosterlitz-Thouless value, $\eta = \frac{1}{4}$.

The above calculation was performed at $y_0 = 0$. When $y_0 > 0$, in the region where it is irrelevant, the parameters K_0 and V_0 in the Lagrangians (5.4) and (5.22) must be *renormalized* to values $K_R(y_0)$ and $V_R(y_0)$ before setting $y_0 = 0$ in the derivation of (5.23). Thus $\omega = 2\pi\sqrt{K_R/V_R}$ and $\phi_y = \frac{3}{2} - \pi\sqrt{K_R/V_R}$, but the relation $\eta = \frac{1}{\omega}$ is still exact. The parameters K_R and V_R are the exact, long wavelength (hydrodynamic) interface stiffness moduli that a bulk experimental probe would measure, and are directly analogous to the superfluid density and compressibility in the superfluid problem – see (3.27). The above analysis shows that when the ratio $\sqrt{V_R/K_R}$ exceeds the universal value $\frac{2\pi}{3}$, y_0 becomes relevant, and simple renormalization of the Gaussian Lagrangian, (5.22), is invalid. We then expect $V_R/K_R \to \infty$, and the interface becomes localized. *At* the critical point separating the localized and delocalized phases, the interface is still delocalized, with the universal parameter values quoted above.

Recall, finally, the discussion of "asymptotic symmetry restoration" at the end of Chap 4. We can see this explicitly in (5.4), where the chemical potential is absorbed completely into the y_0 term. Whenever y_0 is irrelevant (which includes the critical point itself) this term vanishes on long length-scales, and in some renormalized sense is indeed "redundant." Within the scaling analysis, μ appears only in the cosine factor in (5.17). This factor is completely dominated by the exponential decay due to the fluctuating part of the μ_I, and is therefore of no real consequence. The origin of this whole effect can be seen in in (5.9): only the value of $\theta_I \equiv w_I \bmod 2\pi$ is important, and when the variance measure, Δ, of $\delta\mu_I$ is sufficiently large this field is basically uniformly distributed over the interval $[0, 2\pi)$, irrespective of the "mean drift" $\mu I/V_0$.

Following Giamarci and Schulz [6, 28], detailed renormalization group flows may be constructed, to confirm the above results. We start with the sine-Gordon Lagrangian (5.1), and set $\mu_0 = 0$. Then we replace,

$$\tilde{S}_{\mathbf{R}} = S_{\mathbf{R}} - \frac{1}{V_0}\sum_{J=0}^{I} \mu_J = S_{\mathbf{R}} - w_I. \tag{5.24}$$

This transformation gets rid of $\mu_I(\partial_I S_{\mathbf{R}})$, and the cosine term becomes $y_0 cos(2\pi\tilde{S}_{\mathbf{R}} + 2\pi w_I)$. The disorder averaged free energy density, $f = -\frac{1}{\beta V}\langle\langle lnZ\rangle\rangle$, can formally be evaluated by the use of the replica trick: $lnZ \to \frac{(Z^p - 1)}{p}$ as $p \to 0$. We need to take the average, $\langle\langle Z^p\rangle\rangle$, as $p \to 0$. Note that all the disorder is now contained in w_I. For Gaussian distrubition, the average is

$$\langle\langle Z^p\rangle\rangle = \int \prod DS_\alpha(I,T)e^{-\tilde{\mathcal{L}}} \tag{5.25}$$

where

$$\begin{aligned}\tilde{\mathcal{L}} &= \sum_{\mathbf{R}}\sum_{\alpha=1}^{p}[\frac{1}{2K_0}(\partial_T\tilde{S}_{\mathbf{R},\alpha})^2 + \frac{V_0}{2}(\partial_x\tilde{S}_{\mathbf{R},\alpha})] \\ &-y_0^2\sum_{I,J}\sum_{T_1T_2}\sum_{\alpha\beta}e^{-2\pi^2\Delta^2|I-J|}cos[2\pi\tilde{S}_\alpha(I,T_1) - 2\pi\tilde{S}_\beta(J,T_2)],\end{aligned} \tag{5.26}$$

α being the replica index that arises since we are averaging Z^p. Note that the cosine term, which contained all the disorder, couples different replica indices (α,β). This is quite general, in the replicated Lagrangian terms that represent disorder always couple different replica indices.

It is argued that near criticality, long wavelength fluctuations dominate. Hence, in the continuum limit, (and after rescaling time), we can write the Lagrangian in the form

$$\begin{aligned}\tilde{\mathcal{L}} &= \int dkd\omega\sum_{\alpha=1}^{p}\frac{K}{2}[\omega^2\tilde{S}_\alpha(k,\omega)^2 + k^2\tilde{S}_\alpha(k,\omega)] \\ &-D\int dxd\tau_1d\tau_2\sum_{\alpha\beta}cos[2\pi\tilde{S}_\alpha(x,\tau_1) - 2\pi\tilde{S}_\beta(x,\tau_2)].\end{aligned} \tag{5.27}$$

K here really corresponds to $\sqrt{K_0/V_0}$. A Wilson momentum shell renormalization group calculation for this Lagrangian has been carried out by Giamarchi and Schulz [6, 28]. The idea is to integrate out $\tilde{S}(k,\omega)$ with $k_\Lambda/b < k < k_\Lambda$, and rescale as $k' = bk$ and $\omega' = b^z\omega$. k_Λ is the ultraviolet cutoff for k. The parameter z is chosen to keep coefficients of $k^2(\tilde{S}_\alpha)^2$ and $\omega^2(\tilde{S}_\alpha)^2$ remain identical. Working to lowest order in D and $K - \frac{2}{3}$, one obtains the recursion relations, ($l = ln(b)$, is the flow parameter)

$$\begin{aligned}\frac{\partial\bar{D}}{\partial l} &= \frac{9}{2\pi}(K - \frac{2\pi}{3})\bar{D} \\ \frac{\partial K}{\partial l} &= \frac{\bar{D}}{2}\end{aligned} \tag{5.28}$$

with

$$z = 1 + \frac{\bar{D}}{2K}. \tag{5.29}$$

Here $\bar{D} = \frac{48\pi^2}{k_\Lambda^3}D$.

Note that D flows towards zero for $K < \frac{2\pi}{3}$. There is a fixed stable line of the flows for $K < \frac{2\pi}{3}$ at $D = 0$. Points which flow to this stable line represent the superfluid phase. For $K > \frac{2\pi}{3}$, D flows off to infinity, which is considered to be a signature of the disorder-dominated localized phase: the Bose glass phase. $D = 0$, $K = \frac{2\pi}{3}$ corresponds to the critical fixed point. The critical exponents are $z = 1$, and $\eta = K_c/\pi = \frac{2}{3}$. (Note that the the correlation length in the Bose glass phase goes as $\xi \sim e^{1/(K-K_c)^{\frac{1}{2}}}$, unlike that suggested in [6]).

Now we reintroduce the symmetry breaking term, $\mu_0 \partial_T S_\alpha(I,T)$, in the sine-Gordon Lagrangian. It introduces a net tilt of the interface. It can be absorbed into $(\partial_T S_{\mathbf{R}})^2$, by the additional transformation

$$\tilde{S}'_{\mathbf{R}} = \tilde{S}_{\mathbf{R}} - \frac{\mu_0 I}{V_0}. \tag{5.30}$$

This transformation however gives us an extra piece in the Lagrangian, $-\frac{\beta N \mu_0^2}{2V_0^2}$, which gives rise to an analytic term, of the form $A\mu_0^2$, in the free energy. In the replicated Lagrangian, the cosine term now becomes

$$-y_0^2 \int dx_1 dx_2 d\tau_1 d\tau_2 e^{-\pi_2 \Delta^2 |x_1 - x_2|} \sum_{\alpha,\beta} cos(2\pi \tilde{S}'_\alpha(x_1,\tau_1) - 2\pi \tilde{S}'(x_2,\tau_2) + \frac{2\pi\mu_0(x_2 - x_1)}{V_0}). \tag{5.31}$$

Around criticality, we can expand this in powers of $(x_1 - x_2)$. The leading asymmetric term is of the form

$$g_0 \int dx d\tau_1 d\tau_2 \sum_{\alpha,\beta} \partial_x \tilde{S}'(x,\tau_2) cos[2\pi \tilde{S}'_\alpha(x,\tau_1) - 2\pi \tilde{S}'_\beta(x,\tau_2)]. \tag{5.32}$$

The recursion relation for g_0 is

$$\frac{dg_0}{dl} = (2 - \frac{2\pi}{K}) g_0. \tag{5.33}$$

Close to the critical fixed point, it flows as

$$\frac{dg_0}{dl} \simeq -g_0, \tag{5.34}$$

and is explicitly irrelevant. This demonstrates that finite compressibility comes from the analytic part of the free energy, and the singular free energy depends on μ_0 as $\frac{\mu_0}{\xi}$ and not $\mu_0 \xi_\tau$.

We end this section with a brief discussion of the generic Mott insulator-to-superfluid transition as a function of μ in the nonrandom case. The Mott insulator corresponds to the flat phase of the interface model, which exists even in the presence of the random tilt potential so long as the $\delta\mu_I$ are bounded by a sufficiently small number, $\delta\mu_I < \Delta_1$, and μ is not too large. The Mott gap corresponds to a free energy barrier, $\mu_c(K_0, V_0)$, against the formation of steps in the interface. Thus for $\mu < \mu_c$ the chemical potential is insufficient to overcome the corrugation of the cosine potential, and therefore the interface remains flat. Only for $\mu > \mu_c$ does it become advantageous for the interface to have a finite density of steps. For μ close to μ_c this density will be small, and we may treat each step in isolation. If we are well below the tip of the Mott lobe, we may compute the approximate shape of each step via mean field theory: i.e., by simply minimizing the action

$$\begin{aligned}\mathcal{L}_{MF} &= \beta\sum_I[\frac{1}{2}V_0(\partial_I S_I)^2 - 2y_0 cos(2\pi S_I)] \\ &\approx \beta\int dx[\frac{1}{2}V_0(\partial_x S)^2 - 2y_0 cos(2\pi S)] \end{aligned} \tag{5.35}$$

with the boundary conditions $S(x \to \infty) = 0$, $S(x \to -\infty) = 1$, and $S(0) = \frac{1}{2}$. The solution is the soliton

$$S(x) - \frac{1}{2} = \frac{1}{\pi} arctan[sinh(2\pi\sqrt{2y_0/V_0}x)] \tag{5.36}$$

and the total action of such a step is

$$\frac{1}{\beta}\mathcal{L} \approx \varepsilon_s \equiv \frac{4}{\pi}\sqrt{2y_0V_0}. \tag{5.37}$$

If there are N widely separated solitons, the total action is

$$\mathcal{L} \approx \beta N\varepsilon_s - \beta\mu\int dx(\partial_x S) = -\beta N(\varepsilon_s - \mu). \tag{5.38}$$

If $\mu < \varepsilon_s$, $\mathcal{L}$ is minimized by $N = 0$. If $\mu > \varepsilon_s$, N grows until the number of solitons is stabilized by interactions, giving rise to a "soliton compressibility" κ. Thus if

$$\mathcal{L} \approx \beta N[(\varepsilon_s - \mu) + \frac{1}{2}\kappa\rho],\ \rho \equiv N/L, \tag{5.39}$$

then for $\mu > \epsilon_s$ the soliton density will be $\rho = (\mu - \varepsilon)/\kappa$, and the interface will have an average slope ρ. Clearly, in this limit we have $\mu_c \approx \varepsilon_s$. Since ρ then vanishes linearly with $\mu - \varepsilon_s$, this self-consistently validates the low density approximation, (5.36)-(5.38). One may improve somewhat on the above analysis by allowing the solitons to meander in time, τ. Let $x_n(\tau)$ be the center of the nth soliton. Then we expect

$$\begin{aligned} \mathcal{L} &\approx \int_0^\beta d\tau \Big[\sum_n \frac{1}{2} m_{eff} \dot{x}_n(\tau)^2 \\ &+ \frac{1}{2} \sum_{n \neq m} v_{eff}[x_n(\tau) - x_m(\tau)] \Big] \\ &+ \beta N(\varepsilon_s - \mu), \end{aligned} \tag{5.40}$$

where the effective mass $m_{eff} \approx \frac{1}{K_0}$, and the effective interaction potential, $v_{eff}(x)$, could, in principle, be computed from (5.35) by considering two solitons a distance x apart. Equation (5.40) is just the Feynman path integral formulation of a one-dimensional interacting Bose gas. Note that the usual sum over permutations, P, of the endpoints, $x_n(\beta) = x_{P(n)}(0)$, is also included in (5.40) because the field $\partial_I S_I$ satisfies periodic boundary conditions in τ.

We have therefore come full circle back to a boson representation. One knows from studies of the one-dimensional dilute Bose gas at $T = 0$ [30] that the compressibility $\kappa \sim V_0$ is indeed finite in the dilute limit, and hence that ρ indeed rises linearly with $\mu - \varepsilon_s$. We claim also that this soliton gas is a superfluid. This is not completely obvious because even though the interface is tilted, superfluidity occurs only if deviations from the average tilt, $\tilde{S}_{\mathbf{R}} = S_{\mathbf{R}} - \rho I$, diverge logarithmically at large distance, as in (5.5). This means that the soliton positions, $x_n(\tau)$, must be mobile enough to roughen the tilted interface. Since the density is small, roughening can occur only on length scales much larger $\frac{1}{\rho}$, and the effective, long wavelength stiffness, $K_R(\mu)$, will hence be very small, vanishing with ρ. One expects $K_R(\mu) \propto \rho$, corresponding to superfluid density $\rho_s \propto \rho$ in the boson picture. This is indeed found to be the case [30]. Note that the reason the transition occurs at $\sqrt{K_R/V_R} = 0$ (and hence $\eta = \infty$), rather than at the universal Kosterlitz-Thouless value of $\frac{\pi}{2}$ (and $\eta = \frac{1}{4}$) found at the tip of the Mott lobe, is because the y_0 term has been been renormalized to zero identically by the uniform tilt of the interface. The uniformity of μ is, of course, crucial here. In the presence of disorder we have seen the μ_c decreases, but nevertheless, just above μ_c we expect the solitons to be *localized* by the $\delta\mu_I$ and the interface will not roughen immediately. The density of solitons must be increased further, until the Bose glass-superfluid transition occurs, at which $\sqrt{K_R/V_R}$ jumps to the universal value of $\frac{3}{2\pi}$ (and $\eta = \frac{1}{3}$) found earlier.

Chapter 6

The epsilon expansion

Unlike the classical point disorder problem, the classical random rod problem [(2.13), or (4.1) with $g(\mathbf{x}) \equiv 0$ but $r(\mathbf{x})$ random], does not have a simple epsilon expansion about $d = 4$. Rather, as shown in Refs. [13], one must consider also the limit in which the dimension, ϵ_τ of the rods is small, and perform a double expansion in $\epsilon = 4 - D$ and ϵ_τ (recall that $D = d + \epsilon_\tau$ is the total dimensionality). The exponents take mean-field values, $z = 1$, $\nu = \frac{1}{2}$, $\eta = 0$, etc. at $\epsilon = \epsilon_\tau = 0$, and deviations from these values may be computed as two-variable power series in ϵ and ϵ_τ.

Our purpose in this section is to extend this technique to the dirty boson problem. We saw in Chap 4 that a certain nontrivial crossover exponent must be positive if, as expected, particle-hole symmetric disorder is to lead to new critical behavior, different from that of the classical random rod problem. This result was confirmed explicitly for $d = 1$ in Chap 5: there, random rod disorder was found to be an *irrelevant* perturbation on the pure (Kosterlitz-Thouless) critical behavior, whereas dirty boson-type disorder was found to be *relevant*, leading to new critical behavior. We shall find that for *small* ϵ_τ, particle-hole symmetric disorder is an irrelevant perturbation on the random rod problem, and therefore that the crossover exponent *changes sign*, from negative to positive, at a certain value, $\epsilon_\tau = \epsilon_\tau^c(D)$. To first order in ϵ_τ we obtain the estimate $\epsilon_\tau^c(D = 4) = \frac{8}{29}$ ($D = 4$ yielding $d = 3$ at $\epsilon_\tau = 1$). For $\epsilon_\tau > \epsilon_\tau^c$ there are then two fixed points, the stable dirty boson fixed, and the unstable random rod fixed point. This then establishes the nonperturbative nature of the dirty boson fixed point.

6.1 Scaling for general ϵ_τ

Let us now extend the scaling arguments to noninteger ϵ_τ. We consider the following generalization of (4.1):

$$\begin{aligned}\mathcal{L}_c &= -\int d^dx \int d^{\epsilon_\tau}\tau[\frac{1}{2}|(\nabla_\tau - \mathbf{g}(\mathbf{x}))\psi|^2 + \frac{1}{2}|\nabla\psi|^2 \\ &+ \frac{1}{2}r(\mathbf{x})|\psi|^2 + u|\psi|^4],\end{aligned} \tag{6.1}$$

where $\mathbf{g}(\mathbf{x})$ is an ϵ_τ-dimensional vector. This form is based on (2.18), with the same simplifications used in (4.1). We will write $\mathbf{g}(\mathbf{x}) = \mathbf{g}_0 + \delta\mathbf{g}(\mathbf{x})$, and assume that $\delta\mathbf{g}(\mathbf{x})$ is isotropically distributed in $\vec{\tau}$ space. This yields the correct $\epsilon_\tau = 1$ limit, and ensures that the free energy depends only on $g_0 \equiv |\mathbf{g}_0|$. Clearly, $\mathbf{g}_0 = 0$ is the generalization of particle-hole symmetric disorder.

The evaluation of the stiffness constants, (4.4), is slightly more complicated than before: although the spatial stiffnesses, Υ_α, are as before, the temporal stiffness now takes on a tensor character. Consider a θ-boundary condition in the $\vec{\tau}$-subspace:

$$\psi(\mathbf{x}, \vec{\tau} + \beta\hat{\tau}_\mu) = e^{i\theta_\mu}\psi(\mathbf{x}, \vec{\tau}), \;\; \mu = 1, \ldots, \epsilon_\tau. \tag{6.2}$$

Defining the periodic field, $\tilde{\psi} = e^{-i\vec{\theta}\cdot\vec{\tau}/\beta}\psi$, and substituting into (6.1), we find that

$$\mathcal{L}_c[\psi; \mathbf{g}_0] = \mathcal{L}_c[\tilde{\psi}; \mathbf{g}_0 - i\vec{\theta}/\beta]. \tag{6.3}$$

The free energy, $f^{\vec{\theta}} = -\beta^{-\epsilon_\tau}L^{-d}ln\left\{tr\left[e^{\mathcal{L}_c[\psi]}\right]\right\}$, is therefore shifted by

$$\begin{aligned}\delta f^{\vec{\theta}} &\equiv f^{\vec{\theta}} - f^0 = (-i\vec{\theta}/\beta)\cdot\frac{\partial f^0}{\partial \mathbf{g}_0} \\ &+ \frac{1}{2}(-i\vec{\theta}/\beta)\cdot\frac{\partial^2 f^0}{\partial \mathbf{g}_0 \partial \mathbf{g}_0}\cdot(-i\vec{\theta}/\beta) + O[(\theta/\beta)^3].\end{aligned} \tag{6.4}$$

Isotropy implies that $f^{\vec{\theta}}$ is actually a function only of $\vec{\theta}\cdot\hat{\mathbf{g}}_0$. This implies that

$$\begin{aligned}\frac{\partial f^0}{\partial \mathbf{g}_0} &= \hat{\mathbf{g}}_0\frac{\partial f^0}{\partial g_0} \\ \frac{\partial^2 f^0}{\partial \mathbf{g}_0\partial \mathbf{g}_0} &= \frac{1}{g_0}(\mathbf{I} - \hat{\mathbf{g}}_0\hat{\mathbf{g}}_0)\frac{\partial f^0}{\partial g_0} + \hat{\mathbf{g}}_0\hat{\mathbf{g}}_0\frac{\partial^2 f^0}{\partial g_0^2},\end{aligned} \tag{6.5}$$

so that if we define $\rho_0 = \frac{\partial f^0}{\partial g_0}$ and $\kappa_0 = -\frac{\partial^2 f^0}{\partial g_0^2}$, then we have

$$\begin{aligned}\delta f^{\vec{\theta}} &= \frac{i}{\beta}(\vec{\theta}\cdot\hat{g}_0)\rho_0 + \frac{1}{2\beta^2}[|\vec{\theta}|^2 - (\vec{\theta}\cdot\hat{\mathbf{g}}_0)^2]\frac{\rho_0}{g_0} \\ &+ \frac{1}{2\beta^2}\kappa_0 + O[(\theta/\beta)^3]. \end{aligned} \tag{6.6}$$

It is straightforward to write down expressions for ρ_0 and κ_0 analogous to (4.4), but we will require only

$$\begin{aligned}&\kappa_0(g_0 = 0) \equiv \Upsilon_\tau = \langle\langle\ \langle|\psi|^2\rangle\ \rangle\rangle \\ &+ \int d^dx \int d^{\epsilon_\tau}\tau \langle\langle\ \langle\psi^*\partial_{\tau_1}\psi(\mathbf{x},\vec{\tau})\psi^*\partial_{\tau_1}\psi(\mathbf{0},0)\rangle\ \rangle\rangle, \end{aligned} \tag{6.7}$$

where τ_1 is any given direction in $\vec{\tau}$-space. The long wavelength action generalizing (3.27) then takes the form

$$\begin{aligned}S_{eff} &= \int d^dx \int d^{\epsilon_\tau}\tau \Big[\frac{1}{2}\Upsilon|\nabla\phi|^2 \\ &+ \frac{\rho_0}{2g_0}[|\nabla_\tau\phi|^2 - (\hat{\mathbf{g}}_0\cdot\nabla_\tau\phi)^2] \\ &+ \frac{1}{2}\kappa_0(\hat{\mathbf{g}}_0\cdot\nabla_\tau\phi)^2 + i\rho_0(\hat{\mathbf{g}}_0\cdot\nabla_\tau\phi)\Big]. \end{aligned} \tag{6.8}$$

Note that when $g_0 \to 0$ we have $\rho_0/g_0 \to \kappa_0(g_0 = 0) \equiv \Upsilon_\tau$, and S_{eff} reduces to the more familiar form

$$S_{eff}(g_0 = 0) = \int d^dx \int d^{\epsilon_\tau}\tau[\frac{1}{2}\Upsilon|\nabla\phi|^2 + \frac{1}{2}\Upsilon_\tau|\nabla_\tau\phi|^2]. \tag{6.9}$$

Defining the frequency variables $\omega_{||} = \hat{\mathbf{g}}_0\cdot\vec{\omega}$ and $\vec{\omega}_\perp = \vec{\omega} - \omega_{||}\hat{\mathbf{g}}_0$, (6.8) yields a long wavelength, low frequency Green's function

$$\begin{aligned}G(\mathbf{k},\omega) &\equiv \langle|\psi(\mathbf{k},\omega)|^2\rangle_{S_{eff}} \\ &\approx \frac{|\psi_0|^2}{(\rho_0/g_0)|\vec{\omega}_\perp|^2 + \kappa_0\omega_{||}^2 + \Upsilon|\mathbf{k}|^2}, \end{aligned} \tag{6.10}$$

where $\psi_0 = \langle\psi(\mathbf{x},\vec{\tau})\rangle \sim |\delta|^\beta$ is the order parameter, and for slow variations, $\psi(\mathbf{x},\vec{\tau}) \approx \psi_0 e^{i\phi(\mathbf{x},\vec{\tau})}$. In deriving this form we have neglected the surface term, $i\rho_0\hat{\mathbf{g}}_0\cdot\nabla_\tau\phi$, relative to the others. This is valid in the superfluid phase where ϕ has only small fluctuations about the long range ordered value, which we have taken to be $\phi \equiv 0$. In the disordered Bose glass phase, as we saw in Chap 4, this term becomes important (see also below).

Equation (6.10) determines the correlations in the hydrodynamic limit, even near the critical point, so long as new, unforseen low energy excitations do not develop [11](e). One could write down a scaling function of the form:

$$G(\mathbf{k},\vec{\omega}) \approx G_0\xi^{2-\eta} g(k\xi, \omega_\perp \xi_\tau^\perp, \omega_{||}\xi_\tau^{||}). \tag{6.11}$$

Here $\mathbf{k}$, $\vec{\omega}_\perp$ and $\omega_{||}$ appear scaled by their appropriate correlation lengths, ξ, $\xi_\tau^{||} \sim \xi^{z_{||}}$ and $\xi_\tau^\perp \sim \xi^{z_\perp}$, where, for $g_0 \neq 0$, we have allowed for different scalings parallel and perpendicular to $\mathbf{g}_0$. When $g_0 = 0$ we expect $\xi_\tau^{||} = \xi_\tau^\perp \equiv \xi_\tau$ and $z_{||} = z_\perp \equiv z$. Comparison with (6.10) implies that

$$g(x,y,z) \approx \frac{1}{g_x x^2 + g_\perp y^2 + g_{||} z^2}, x, y, z \to 0, \tag{6.12}$$

in which g_x, $g_\perp$ and $g_{||}$ are universal numbers. Thus

$$\begin{aligned} \Upsilon &\approx G_0^{-1}|\psi_0|^2 g_x \xi^\eta \\ \kappa_0 &\approx G_0^{-1}|\psi_0|^2 g_{||}\xi_\tau^{||2}\xi^{-2+\eta} \\ \rho_0/g_0 &\approx G_0^{-1}|\psi_0|^2 g_\perp \xi_\tau^{\perp 2}\xi^{-2+\eta}. \end{aligned} \tag{6.13}$$

The hyperscaling relation is now $2-\alpha = [d + z_{||} + (\epsilon_\tau - 1)z_\perp]\nu$. Along with the usual scaling relations, $\alpha + 2\beta + \gamma = 2$ and $\gamma = (2-\eta)\nu$, this immediately implies that $\Upsilon \sim |\delta|^\zeta$, $\kappa_0 \sim |\delta|^{\zeta_\tau^{||}}$ and $\rho_0/g_0 \sim |\delta|^{\zeta_\tau^\perp}$, where

$$\begin{aligned} \zeta &= [d + (\epsilon_\tau - 1)z_\perp + z_{||} - 2]\nu \\ \zeta_\tau^{||} &= [d + (\epsilon_\tau - 1)z_\perp - z_{||}]\nu \\ \zeta_\tau^\perp &= [d + (\epsilon_\tau - 3)z_\perp + z_{||}]\nu. \end{aligned} \tag{6.14}$$

If $z_\perp = z_{||} = z$, then $\zeta = (d + \epsilon_\tau - 2)\nu$ and $\zeta_\tau^{||} = \zeta_\tau^\perp \equiv \zeta_\tau = [d + (\epsilon_\tau - 2)z]\nu$. Following the argument [6], relating finite compressibility to exponent $z = d$, it can be argued (incorrectly) that if the full compressibility, κ_0, remains finite through the transition even for noninteger ϵ_τ, then we have

$$z = \frac{d}{2-\epsilon_\tau}. \ (?) \tag{6.15}$$

This is the generalization of the result $z = d$ at $\epsilon_\tau = 1$ [6].

The above analysis is actually incorrect for the particle-hole asymmetric transition, as argued in Chapter 4. In particular, for the incommensurate case, g_0 could play the role of the control parameter, and thus the singular part of the compressibility goes as

$$\kappa_s \sim \delta^{-\alpha} \tag{6.16}$$

with $\alpha = 2 - (d + \epsilon_\tau z)\nu$. If this is also the total compressibility, for it to be compatible with the previous analysis, we must have $\alpha = \zeta_\tau$, which implies $z = 1/\nu$. The other option is that the the total compressibility κ_0 is finite. Since the renormalization group analysis, does not agree with either $z = 1/\nu$ or $z = \frac{d}{2-\epsilon_\tau}$, we are forced to conclude that there is no δ^{ζ_τ} piece in the compressibility.

We next calculate the particle-hole symmetric crossover exponent, (4.29), for general ϵ_τ. The calculation is essentially identical to that leading to (4.29). The perturbation, (4.14), now becomes

$$\mathcal{L}_{\mathbf{g}} = -\int d^d x \int d^{\epsilon_\tau}\tau \psi^* [\mathbf{g}(\mathbf{x}) \cdot \nabla_\tau] \psi(\mathbf{x}, \tau). \tag{6.17}$$

We assume $\langle\langle g_\mu(\mathbf{x}) g_\nu(\mathbf{x}) \rangle\rangle = \Delta_g \phi(\mathbf{x} - \mathbf{x}') \delta_{\mu\nu}$, where ϕ is a delta function for short range correlated disorder, and varies as $x^{-(d+a)}$ for large x for power law correlated disorder. Equation (4.19) now becomes

$$\begin{aligned} -2\left(\frac{\partial f}{\partial \Delta_g}\right)_{\Delta_g=0} &= \int d^d x \int d^{\epsilon_\tau}\tau \phi(\mathbf{x}) \\ &\times \langle\langle\ \langle \psi^* \nabla_\tau \psi(\mathbf{x}, \vec{\tau}) \cdot \psi^* \nabla_\tau \psi(\mathbf{0}, \mathbf{0}) \rangle_c\ \rangle\rangle, \end{aligned} \tag{6.18}$$

where the remaining disorder average is over the random rod disorder, $r(\mathbf{x})$, while the statistical average is with respect to the random rod Hamiltonian with $\mathbf{g}(\mathbf{x}) \equiv 0$. From isotropy in $\vec{\tau}$-space, the right hand side of (6.18) is just ϵ_τ times the average for any given direction in $\vec{\tau}$-space [i.e., with ∇_τ replaced by, say, ∂_{τ_1} as in (6.7)]. Extracting the scaling behavior of (6.18) is now straightforward: equation (4.25) becomes

$$\Upsilon_\tau(\mathbf{k}, \mathbf{0}) \approx A_2 k^{d-(2-\epsilon_\tau)z} \tilde{\mathcal{Y}}_\pm(k\xi), \tag{6.19}$$

and as in (4.26), $\tilde{\mathcal{Y}}_\pm(w)$ will have a similar spectrum of exponents, $0 < \omega_1(\epsilon_\tau) < \omega_2(\epsilon_\tau) <$

$\omega_3(\epsilon_\tau) < \ldots$, now depending on ϵ_τ. This leads in the same way to (4.29) and (4.30), still with $b_0 \equiv 0$, and a term $b_a|\delta|^{[2d-(2-\epsilon_\tau)z]\nu}$. We then find

$$\phi_g/\nu = \begin{cases} 2z - d - a, & a < 2z - d - \lambda_1(\epsilon_\tau) \\ \lambda_1(\epsilon_\tau), & a > 2z - d - \lambda_1(\epsilon_\tau) \end{cases} \tag{6.20}$$

essentially as before [see (4.32)], but now with all exponents evaluated in ϵ_τ time dimensions.

We shall compute $\lambda_1(\epsilon_\tau)$ to $O(\epsilon_\tau)$ below. From the naive (and, as we shall see, incorrect) estimate, $\lambda_1 = 2z - d$, we again have $\lambda_1 > 0$ for $z > \frac{d}{2}$. Since $z = 1$ at $\epsilon_\tau = 0$, we expect, as stated earlier, $\lambda_1 < 0$ for small ϵ_τ, becoming positive only for $\epsilon_\tau > \epsilon_\tau^c > 0$. We shall find that for $\epsilon_\tau > \epsilon_\tau^c$ a new stable fixed point with $\Delta_g^* > 0$ bifurcates away from the random rod fixed point (with $\Delta^* = 0$). The exponent z is substantially *smaller* than $\frac{d}{2}$ at this point, violating (6.15) for any $\epsilon_\tau > 0$. Assuming that this new fixed point may indeed be identified with the true dirty boson fixed point when $\epsilon_\tau = 1$, we conclude that (6.15) is incorrect. Either the compressibility, κ, *vanishes* at criticality, or the finite part of the compressibility comes from the analytic free energy. Later we shall generalize the analysis of the excitation spectrum of the Bose glass phase in Chap. 3 to general ϵ_τ. For $\epsilon_\tau < 1$, the issue of whether the Bose glass phase is compressible turns out to be rather subtle.

6.2 Phase diagram for general ϵ_τ

It turns out that $\epsilon_\tau = 1$ is special. For $\epsilon_\tau > 1$, as we will see in the next section, there is no incompressible phase at all we always have superfluidity. To understand the phase diagram for $\epsilon_\tau < 1$ it would be constructive to first consider the infinite range hopping model ($J_{ij} = J/N$ for all i,j, where N is the number of sites). In the absence of disorder the mean-field Lagrangian is given by

$$L_{MF} = \int d^{\epsilon_\tau}\tau[-\frac{J}{N}\sum_{i,j} cos[\phi_i(\tau) - \phi_j(\tau)] + \sum_i \frac{1}{2U}(\partial_\tau\phi_i - i\mu)^2], \tag{6.21}$$

where τ_1, τ_2,... go from 0 to β.

$$\phi_i(\tau + \beta e_\mu) = \phi_i(\tau) + 2\pi n_\mu,$$

n_μ being an integer.

A complex Hubbard-Stratanovich field M(τ) may now be introduced to decouple the hopping term. Using

$$exp(-(\frac{a}{N})z^*z) = \frac{a}{N\pi}\int d\psi d\psi^* exp - (a\psi z^* + a\psi^* z + aN|\psi|^2)$$

the partition function can be written as (without the normalization factor)

$$\begin{aligned} Z &= \int \prod_{\tau}\{dM(\vec{\tau})dM^*(\vec{\tau})\}e^{-\int d\tau NJM^*M}\{Tr^{\{\phi_i\}}exp - \int d\tau[JM(\tau)(\sum e^{-i\phi_i}) \\ &+JM^*(\tau)\sum e^{i\phi_i} + \sum_i \frac{1}{2U}(\partial_{\vec{\tau}}\phi_i - i\mu)^2]\} \end{aligned} \tag{6.22}$$

which is of the form

$$Z = \int \prod_{\tau}\{dM(\vec{\tau})dM^*(\vec{\tau})\}exp[-NS\{M\}]. \tag{6.23}$$

In the thermodynamic limit, a saddle point evaluation of the integral becomes exact. We assume that the lowest-energy saddle point involves a time independent field, $M(\tau) = M$, which can be chosen to be real. Then S(M) can be expanded in powers of M,

$$S(M) = \beta[r(\mu, J)|M|^2 + u|M|^4 + ...]. \tag{6.24}$$

We need to expand 6.22 in powers of m. For this purpose it is convenient to make a change of variables to $\tilde{\phi}_i(\vec{\tau})$, where

$$\tilde{\phi}_i(\vec{\tau}) = \phi_i(\vec{\tau}) - \frac{2\pi\vec{n}.\vec{\tau}}{\beta} \tag{6.25}$$

such that $\tilde{\phi}_i(\vec{\tau} + \beta\hat{e}_i) = \tilde{\phi}(\vec{\tau})$. Then $r(\mu, J)$ is given by

$$\begin{aligned} r(\mu, J) &= J - \frac{J^2}{8Z_0\beta^{\epsilon_\tau}}\int d^{\epsilon_\tau}\tau_1 d^{\epsilon_\tau}\vec{\tau}_2[\int d\tilde{\phi}\{e^{i(\tilde{\phi}(\tau_1)-\tilde{\phi}(\tau_2))}e^{-\frac{1}{2U}\int(\partial_\tau\tilde{\phi})^2}\} \\ &\sum_n\{e^{2\pi n.(\frac{\tau_1}{\beta}-\frac{\tau_2}{\beta}-\mu U^{-1}\beta^{\epsilon_\tau-1})}e^{-2\pi^2 n^2\frac{1}{U\beta^{2-\epsilon_\tau}}}\} + c.c.(with - \mu)], \end{aligned} \tag{6.26}$$

where $Z_0 = \int d\tilde{\phi}e^{-\frac{1}{2U}\int(\partial_\tau\tilde{\phi})^2}$.

We will use the identity

$$\sum_{\vec{n}} e^{i2\pi\vec{n}.(\frac{\vec{\tau}_1}{\beta}-\frac{\vec{\tau}_2}{\beta}-\vec{\mu}U^{-1}\beta^{\epsilon_\tau-1})}e^{\frac{-2\pi^2 n^2 U^{-1}}{\beta^{2-\epsilon_\tau}}} = \sum_{\vec{l}} e^{-2\pi^2 Y(\vec{x}-\vec{l})^2}, \tag{6.27}$$

where $\vec{x} = -\vec{\mu}U^{-1}\beta^{1-\epsilon_\tau} + \frac{\vec{\tau_1}-\vec{\tau_2}}{\beta}$; and $Y = \frac{\beta^{2-\epsilon_\tau}U}{4\pi^2}$. In the limit $\beta \to \infty$, all the contribution comes from $\vec{l} = 0$. In this limit

$$r = J - \frac{J^2}{2}\int d^{\epsilon_\tau}\tau e^{-(U/2)|\vec{\tau}|^{2-\epsilon_\tau}+\vec{\tau}\cdot\vec{\mu}}. \tag{6.28}$$

The phase boundary between the Mott phase and the superfluid phase occurs at $r(\mu, J) = 0$ (for $r < 0$ we have the superfluid phase with nonzero magnetization $M \neq 0$, while for $r > 0$ we have the Mott phase). Thus the boundary is given by

$$1/J_c = \frac{1}{2}\int d^{\epsilon_\tau}\tau e^{-(U/2)|\vec{\tau}|^{2-\epsilon_\tau}+\vec{\tau}\cdot\vec{\mu}}. \tag{6.29}$$

We plot the phase boundary for different values of ϵ_τ in 6.1. For large values of μ, this boundary goes as

$$J_c \sim exp[-A\mu'^{(\frac{2-\epsilon_\tau}{1-\epsilon_\tau})}] \tag{6.30}$$

where

$$A = \frac{1-\epsilon_\tau}{(2-\epsilon_\tau)^{\frac{2-\epsilon_\tau}{1-\epsilon_\tau}}}$$

and $\mu' = \mu(2U^{-1})^{1/(2-\epsilon_\tau)}$. Also note that there is only one Mott lobe for $\epsilon_\tau < 1$. The easiest way to convince oneself of this is to use the methods developed in the next section to show that for all values of μ, a single site always has $\rho = 0$. In mean field, at the commensurate point $\mu = 0$ one has the dyanmical exponent $z = 1$ while on the remainder of the line $z = 2$ (this is true even in the presence of disorder).

Since in mean field theory, every site gets effectively decoupled, it is very easy to incorporate the effect of site disorder within this formalism. In the presence of disorder, the phase boundary is at

$$1/J_c = \frac{1}{2}\int d^{\epsilon_\tau}\tau \int d^{\epsilon_\tau}\varepsilon p(\vec{\varepsilon})e^{-(U/2)|\vec{\tau}|^{2-\epsilon_\tau}+\vec{\tau}\cdot(\vec{\mu}-\vec{\varepsilon})}, \tag{6.31}$$

where $p(\vec{\varepsilon})$ is the single site distribution for the site disorder, $\vec{\varepsilon}$. So, in the presence of disorder the Mott lobes shrink, but the mean field critical behavior is unchanged. For finite hopping, we expect a finite width Bose glass phase to appear between the Mott and superfluid phases.

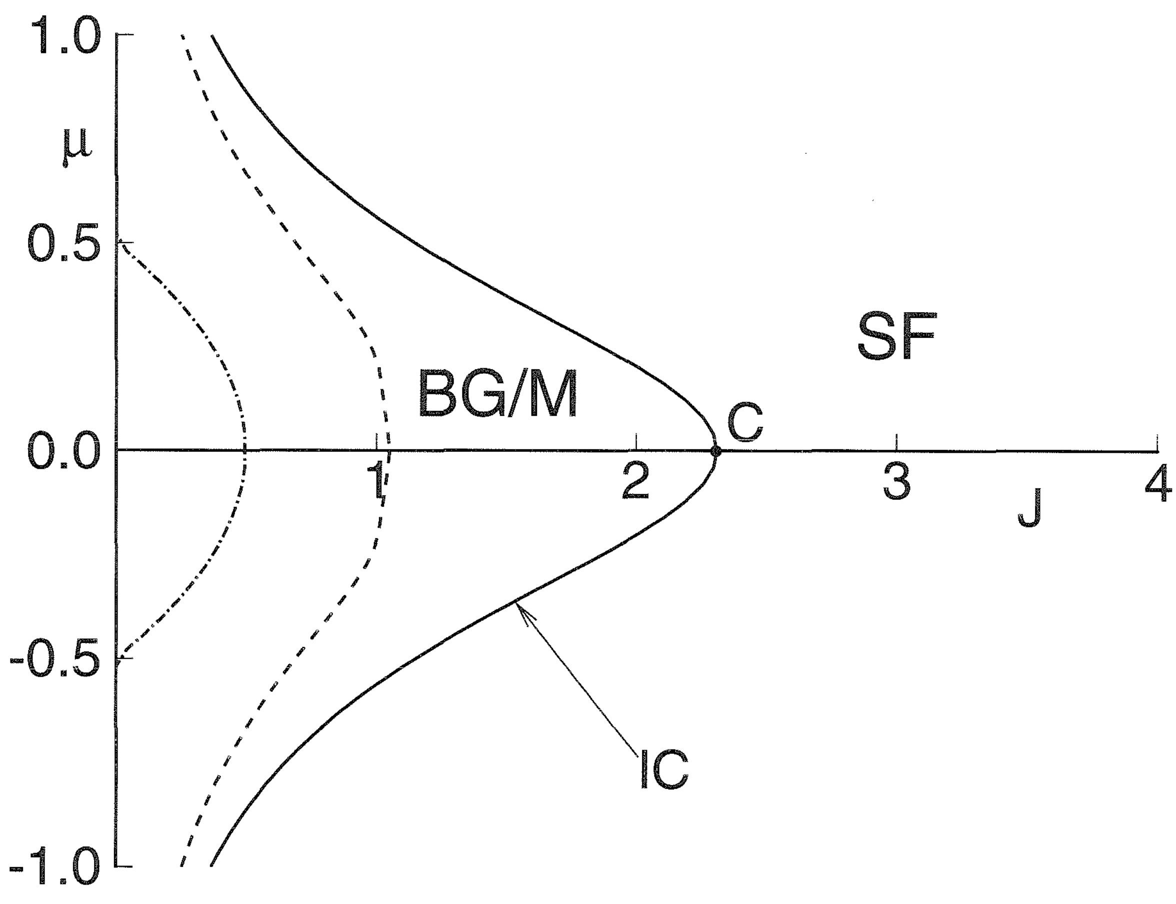

Figure 6.1: The phase diagram for the Josephson junction model, showing Bose glass/Mott (BG/M) and superfluid (SF) phases, with incommensurate line (IC) and commensurate point (C) for $\epsilon_\mathcal{T} = 0.5, 0, 9, 1$, with larger $\epsilon_\mathcal{T}$ corresponding to a smaller Mott lobe. Note that only one Mott lobe persists for $\epsilon_\mathcal{T} < 1$. For $\epsilon_\mathcal{T} = 1$ the Mott lobe shown is actually repeated priodically.

6.3 Droplet model for general ϵ_τ

To establish these results, let us first generalize the results of Sec. III to general ϵ_τ. We consider then the effective action (6.8) in a finite droplet in which ϕ is assumed constant in space [compare (3.29)]:

$$\begin{aligned} S^{(1)}_{eff} &= V\int d^{\epsilon_\tau}\tau[\frac{1}{2}\kappa|\nabla_\tau\phi|^2 + i\vec{\rho}\cdot\nabla_\tau\phi] \\ &\equiv S^{(0)}_{eff} + iV\int d^{\epsilon_\tau}\tau\vec{\rho}\cdot\nabla_\tau\phi. \end{aligned} \tag{6.32}$$

We take $\vec{\mu} = (1,1,\ldots,1)\mu$ and $\vec{\rho} = (1,1,\ldots,1)\rho$ so that $|\vec{\mu}| = \sqrt{\epsilon_\tau}\mu$ and $|\vec{\rho}| = \sqrt{\epsilon_\tau}\rho$. Finite μ, ρ and κ yield the correct normalization both at $\epsilon_\tau = 1$ and small ϵ_τ. Periodic boundary conditions imply $\phi(\vec{\tau} + \beta\hat{\mathbf{e}}_i) = \phi(\vec{\tau}) + 2\pi m_i$, for integer m_i and $i = 1,\ldots,\epsilon_\tau$. There is only one integer for each temporal direction $\hat{\mathbf{e}}_i$ because $\nabla_\tau\phi$ must be periodic in all directions. If m_i were to depend on the time components $\vec{\tau}^\perp$ perpendicular to $\hat{\mathbf{e}}_i$, there would be a nonperiodic contribution, $\nabla_\tau m_i(\vec{\tau}^\perp)$, to $\nabla_\tau\phi$.

Let us first generalize the free energy calculation, (3.31), to general ϵ_τ. First let us define the periodic field

$$\tilde{\phi}(\vec{\tau}) = \phi(\vec{\tau}) + \frac{2\pi}{\beta}\sum_{i=1}^{\epsilon_\tau} m_i\tau_i \equiv \phi(\vec{\tau}) + \frac{2\pi}{\beta}\mathbf{m}\cdot\vec{\tau}. \tag{6.33}$$

The generalization of (3.35) then reads

$$\begin{aligned} f &= f_0 - \frac{1}{\beta^{\epsilon_\tau}V}\ln\left[\sum_{\mathbf{m}} e^{2\pi i\beta^{\epsilon_\tau-1}\mathbf{m}\cdot\vec{\rho}V - 2\pi^2\kappa^0 V\beta^{\epsilon_\tau-2}\mathbf{m}^2}\right. \\ &\times \left.\mathrm{tr}^{\tilde{\phi}}\left\{e^{-S^0_{eff}[\tilde{\phi}]}\right\}\right] \\ &= f_0 + f_{00}[\kappa^0] \\ &- \frac{1}{\beta^{\epsilon_\tau}V}\ln\left[\sum_{\mathbf{l}} e^{-\beta^{2-\epsilon_\tau}|\beta^{\epsilon_\tau-1}\vec{\rho}V - \mathbf{l}|^2/2\kappa^0 V}\right] \end{aligned} \tag{6.34}$$

where (3.36) has been used for each component of $\mathbf{m}$ and the functions f_0 and f_{00} are defined analogously to (3.33) and (3.35): $f_0(|\vec{\mu}|) = f_0(|\vec{\mu}_0|) - |\vec{\rho}^0|(|\vec{\mu}| - |\vec{\mu}_0|) - \frac{1}{2}\kappa^0(|\vec{\mu}| - |\vec{\mu}_0|)^2$, while f_{00} is the free energy associated with $S^{(0)}_{eff}$ with periodic boundary conditions. When $\beta \to \infty$ for $\epsilon_\tau < 2$ only the term with minimal exponent survives. Thus if l_0 is chosen such that

$|\beta^{\epsilon_\tau - 1}\rho V - l_0| \leq \frac{1}{2}$, then

$$\begin{aligned} f(\beta \to \infty) &= f_0 + f_{00} \\ &+ \epsilon_\tau \beta^{2(1-\epsilon_\tau)}(\beta^{\epsilon_\tau - 1}\rho V - l_0)^2/2\kappa^0 V^2. \end{aligned} \tag{6.35}$$

Recalling that $|\vec{\rho}| = \sqrt{\epsilon_\tau}\rho = \sqrt{\epsilon_\tau}\rho^0 + \kappa^0(|\vec{\mu}| - |\vec{\mu}_0|)$, the density is then

$$\begin{aligned} \rho_{drop} &= -\frac{1}{\sqrt{\epsilon_\tau}}\frac{\partial f}{\partial|\vec{\mu}|} \\ &= \rho - (\rho - \beta^{1-\epsilon_\tau} l_0/V) = \beta^{1-\epsilon_\tau} l_0/V. \end{aligned} \tag{6.36}$$

Now, for $0 < \epsilon_\tau < 1$ we have $\beta^{\epsilon_\tau - 1} \to 0$ as $\beta \to \infty$, and hence $l_0 \equiv 0$ for all ρV. Therefore $\rho_{drop} \equiv 0$, and all droplets have zero density, and are completely incompressible, no matter their size. This seems to indicate that the Bose glass phase is incompressible. However there are additional subtleties to worry about here.The point is that since the Mott to superfluid phase boundary has μ dependence, the distribution of droplet sizes would change as we change μ. This would be equivalent to having the volume of the droplets depend on μ, i.e. $V(\mu)$. This would lead to a finite compressibility in the Bose glass phase for $\epsilon_\tau < 1$. For $\epsilon_\tau = 1$ we recover immediately the result (3.38). For $2 > \epsilon_\tau > 1$ we have $\beta^{\epsilon_\tau - 1} \to \infty$ as $\beta \to \infty$, and hence l_0 grows with β, and $l_0/\beta^{\epsilon_\tau} V \to \rho$. The droplet is therefore compressible – the density varies continuously and is equal to the bulk density.

We may similarly compute the temporal correlation function for general ϵ_τ. The result is

$$\begin{aligned} & G_\rho(\tau - \tau') \\ =\; & G_0(\tau - \tau')\frac{\sum_{\mathbf{m}} e^{i2\pi\mathbf{m}\cdot(\vec{\rho}V\beta^{\epsilon_\tau - 1} + \frac{\vec{\tau}-\vec{\tau}'}{\beta})} e^{-2\pi^2\mathbf{m}^2\kappa^0 V\beta^{\epsilon_\tau - 2}}}{\sum_{\mathbf{m}} e^{i2\pi\mathbf{m}\cdot\vec{\rho}V\beta^{\epsilon_\tau - 1}} e^{-2\pi^2\mathbf{m}^2\kappa^0 V\beta^{\epsilon_\tau - 2}}} \\ =\; & G_0(\tau - \tau')\frac{\sum_{\mathbf{l}} e^{-\beta^{2-\epsilon_\tau}|\beta^{\epsilon_\tau - 1}\vec{\rho}V + (\vec{\tau}-\vec{\tau}')/\beta - \mathbf{l}|^2/2\kappa^0 V}}{\sum_{\mathbf{l}} e^{-\beta^{2-\epsilon_\tau}|\beta^{\epsilon_\tau - 1}\vec{\rho}V - \mathbf{l}|^2/2\kappa^0 V}} \end{aligned} \tag{6.37}$$

where

$$G_0(\tau - \tau') = \langle e^{i[\phi(\tau) - \phi(\tau')]}\rangle_{S^{(0)}_{eff}} = e^{-C(\vec{\tau} - \vec{\tau}')/\kappa^0 V} \tag{6.38}$$

and $C(\vec{\tau}) \propto |\vec{\tau}|^{2-\epsilon_\tau}$ is the Coulomb potential in ϵ_τ dimensions. Once again, when $\beta \to \infty$

and $\epsilon_\tau < 2$ only the term in which all $l_i = l_0$ survives and we find

$$G_\rho(\tau - \tau') = G_0(\tau - \tau') \times e^{-\beta^{1-\epsilon_\tau}(\rho V \beta^{\epsilon_\tau - 1} - l_0) \sum_{i=1}^{\epsilon_\tau} (\tau_i - \tau_i')/\kappa^0 V}, \tag{6.39}$$

where, it should be recalled, the l_0 factor in parentheses lies between $-\frac{1}{2}$ and $\frac{1}{2}$. For $2 > \epsilon_\tau > 1$ we have $\beta^{1-\epsilon_\tau} \to 0$ and $G_\rho(\vec{\tau}) = G_0(\vec{\tau})$: the droplet behaves like the bulk superfluid, in agreement with our finding for the density. For $\epsilon_\tau = 1$ we recover (3.41). For $0 < \epsilon_\tau < 1$ we again have $l_0 = 0$ and

$$G_\rho(\vec{\tau}) = G_0(\vec{\tau}) e^{-\vec{\rho}\cdot\vec{\tau}/\kappa^0}. \tag{6.40}$$

6.4 Renormalization group calculations

Let us finally revisit the renormalization group calculation carried out in Ref. [12], but now with explicit attention to issues of particle-hole symmetry. To this end, we use the standard replica trick to average over the disorder [31], and obtain the replicated Lagrangian, $\mathcal{L}_c^{(p)} = \mathcal{L}_1^{(p)} + \mathcal{L}_2^{(p)}$, where $p \to 0$ is the number of replicas, and

$$\begin{aligned} \mathcal{L}_1^{(p)} &= -\sum_{\alpha=1}^{p} \int d^d x \int d^{\epsilon_\tau}\tau [\frac{1}{2} e_\tau |\nabla_\tau \psi_\alpha|^2 \\ &+ \mathbf{g}_0 \cdot (\psi^* \nabla_\tau \psi_\alpha) + \frac{1}{2} e_x |\nabla \psi_\alpha|^2 \\ &+ \frac{1}{2} r_0 |\psi_\alpha|^2 + u |\psi_\alpha|^4] \end{aligned} \tag{6.41}$$

$$\begin{aligned} \mathcal{L}_2^{(p)} &= \frac{1}{2} \sum_{\alpha,\beta=1}^{p} \int d^d x \int d^{\epsilon_\tau}\tau \int d^{\epsilon_\tau}\tau' \\ \times \ & [\tilde{\Delta}_\tau |\psi_\alpha(\mathbf{x},\vec{\tau})|^2 |\psi_\beta(\mathbf{x},\vec{\tau}')|^2 \\ + \ & \Delta_g [\psi^* \nabla_\tau \psi - \mathbf{g}_0 |\psi|^2](\mathbf{x},\vec{\tau}) \cdot [\psi^* \nabla_\tau \psi - \mathbf{g}_0 |\psi|^2](\mathbf{x},\vec{\tau}')]. \end{aligned} \tag{6.42}$$

For simplicity we have taken $\mathbf{g}(\mathbf{x})$ and $\tilde{r}(\mathbf{x}) \equiv r(\mathbf{x}) + \mathbf{g}(\mathbf{x})^2$ to be independent Gaussian random fields with $\mathbf{g}(\mathbf{x}) = \mathbf{g}_0 + \delta g(\mathbf{x})$, $\tilde{r}(\mathbf{x}) = r_0 + \delta\tilde{r}(\mathbf{x})$ and

$$\begin{aligned} \langle\langle \delta\mathbf{g}(\mathbf{x})\rangle\rangle &= 0,\ \langle\langle \delta\mathbf{g}_\mu(\mathbf{x})\delta\mathbf{g}_\nu(\mathbf{x}')\rangle\rangle = \Delta_g \delta(\mathbf{x}-\mathbf{x}')\delta_{\mu\nu} \\ \langle\langle \delta\tilde{r}(\mathbf{x})\rangle\rangle &= 0,\ \langle\langle \delta\tilde{r}(\mathbf{x})\delta\tilde{r}(\mathbf{x}')\rangle\rangle = \tilde{\Delta}_r \delta(\mathbf{x}-\mathbf{x}'). \end{aligned} \tag{6.43}$$

The analysis in Ref. [12] was carried out with $\mathbf{g}_0 = (1, 1, \ldots, 1)$ and $\Delta_g = 0$, but $\tilde{\Delta}_r > 0$. However, considerations of particle-hole symmetry lead us to expect that $\mathbf{g}_0 = 0$ and $\Delta_g > 0$ should yield a more appropriate model. In Fourier space we have

$$\begin{aligned} \mathcal{L}_2^{(p)} &= \frac{1}{2}\sum_{\alpha,\beta=1}^{p} \int_{\mathbf{k}_1}\int_{\mathbf{k}_2}\int_{\mathbf{k}_3}\int_{\mathbf{k}_4} \delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4) \\ &\times \int_{\vec{\omega}}\int_{\vec{\omega}'} [\Delta_r - 2i\Delta_g \mathbf{g}_0 \cdot \vec{\omega} - \Delta_g \vec{\omega}\cdot\vec{\omega}'] \\ &\times \psi_\alpha^*(\mathbf{k}_1, \vec{\omega})\psi(\mathbf{k}_2, \vec{\omega})\psi_\beta^*(\mathbf{k}_3, \vec{\omega}')\psi_\beta(\mathbf{k}_4, \vec{\omega}'), \end{aligned} \tag{6.44}$$

where $\Delta_r \equiv \tilde{\Delta}_r + \mathbf{g}_0^2\Delta_g$. Nominally, by naive power counting, it appears that the leading term at low frequencies is the Δ_r term, and one might expect the other two frequency dependent terms to be strongly irrelevant. This will turn out to be true for small ϵ_τ, where naive power counting is almost valid. However, beccause $\vec{\omega}$ and $\vec{\omega}'$ refer to different replicas, α and β, these terms break particle-hole symmetry, and are not as strongly irrelevant as one might expect (as compared to, for example, ω^2 corrections to the $|\psi|^4$ coeeficient, u), and in fact become relevant for larger ϵ_τ.

We shall begin by setting $\mathbf{g}_0 = 0$. Nominally, this term again appears to multiply more relevant terms than does Δ_g – especially the quadratic $\mathbf{g}_0 \cdot \psi^*\nabla_\tau\psi$ term focused on in Ref. [12]. We shall return later to understand how the irrelevance of this term comes about. We shall perform a standard Wilson momentum shell renormalization group transformation [33] in which successive shells in k-space are integrated out. For each such $\mathbf{k}$ all frequencies, ω, are integrated out. Since the frequency is unbounded, the Brillouin zone is really a hypercylinder. After each integration, we rescale $\mathbf{k}$ and ω in order to maintain the same wavevector cuttoff, k_Λ. The spin rescaling factor and the dynamical exponent, z, are determined in the usual way by setting the coefficients of the quadratic terms, $\psi_\alpha^*\nabla^2\psi_\alpha$ and $\psi_\alpha^*\nabla_\tau^2\psi_\alpha$, equal to unity. We show the diagrams that contribute at lowest nontrivial

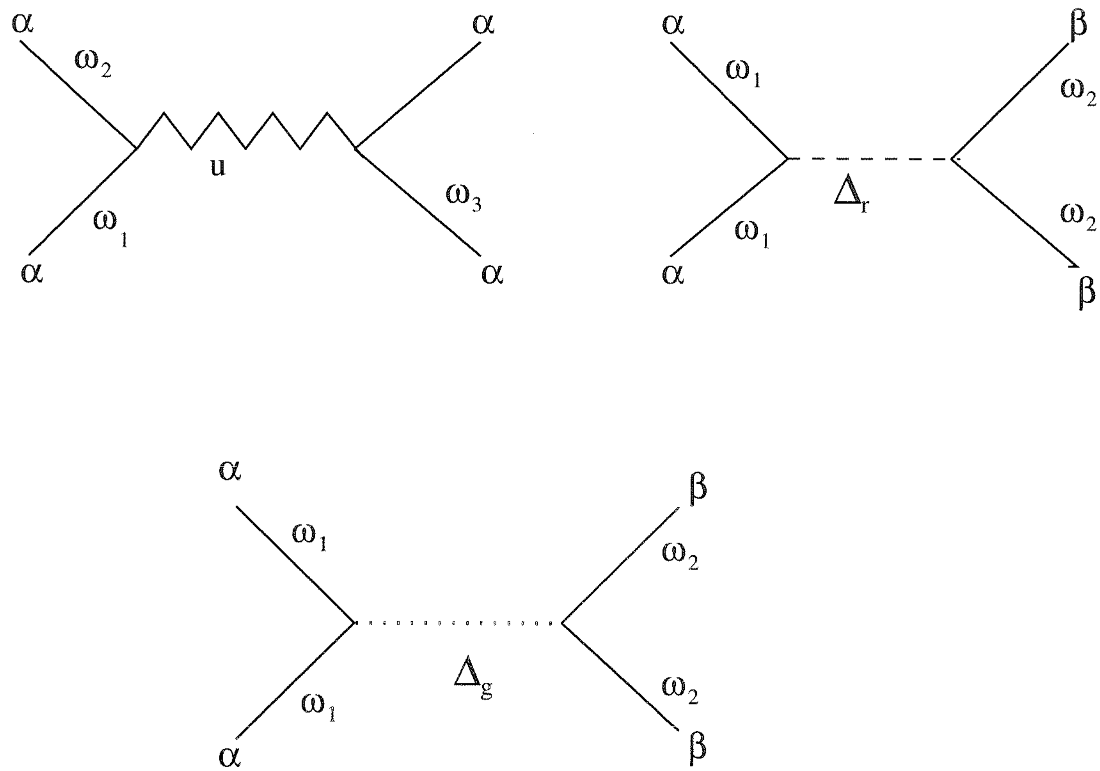

Figure 6.2: Vertices corresponding to u, Δ_r and Δ_g.

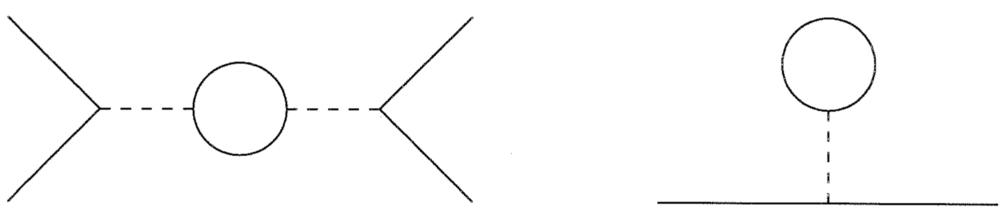

Figure 6.3: Diagrams that do not contribute as the number of replicas $p \to 0$.

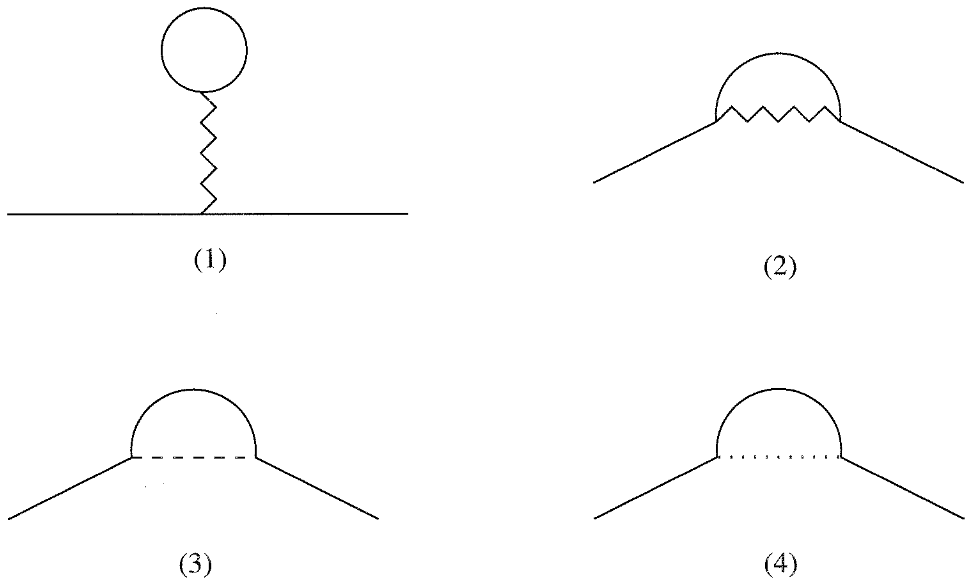

Figure 6.4: Diagrams that contribute to the propagator renormalization.

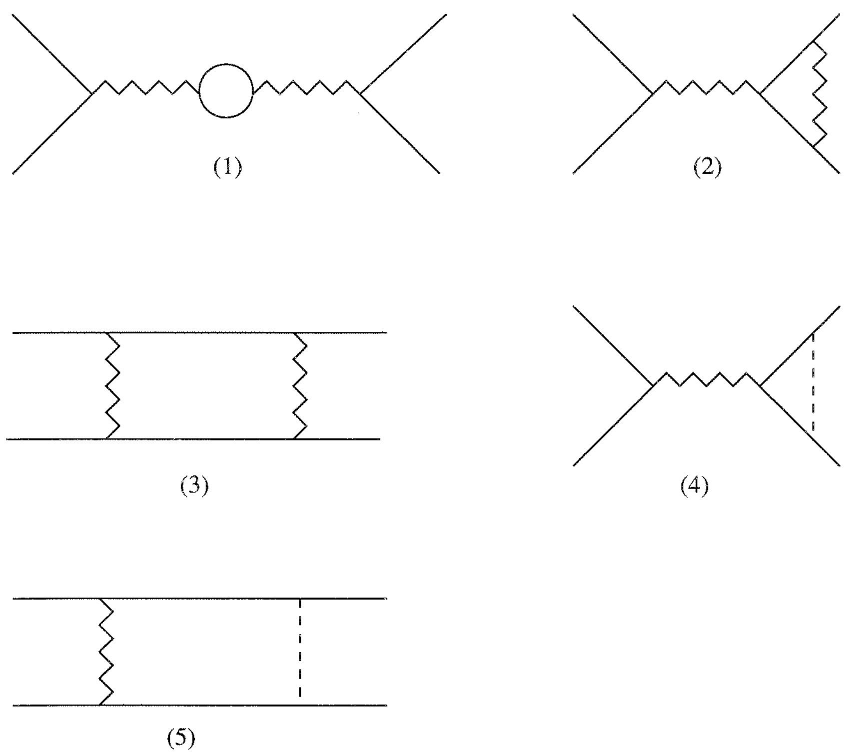

Figure 6.5: Diagrams that contribute to the renormalization of u.

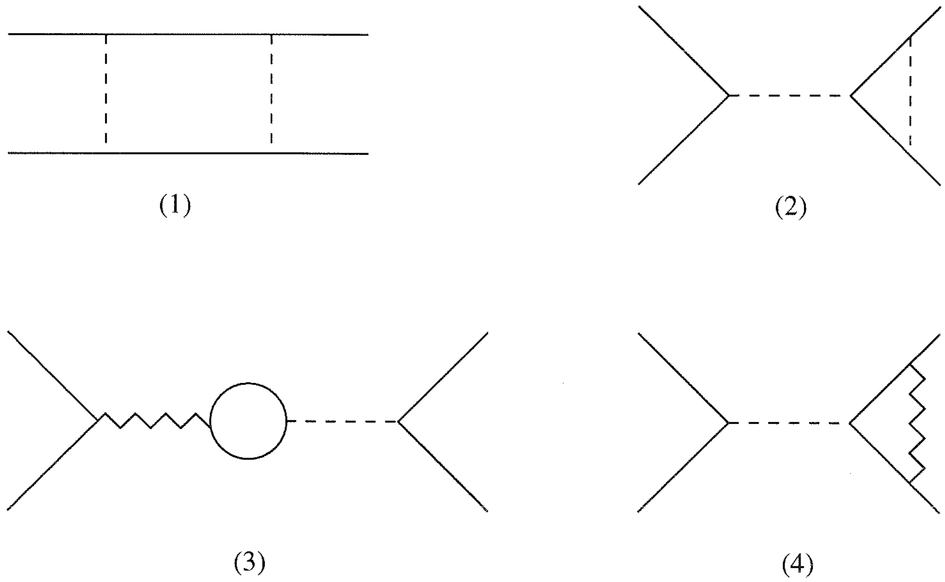

Figure 6.6: Diagrams that contribute to the renormalization of Δ_r.

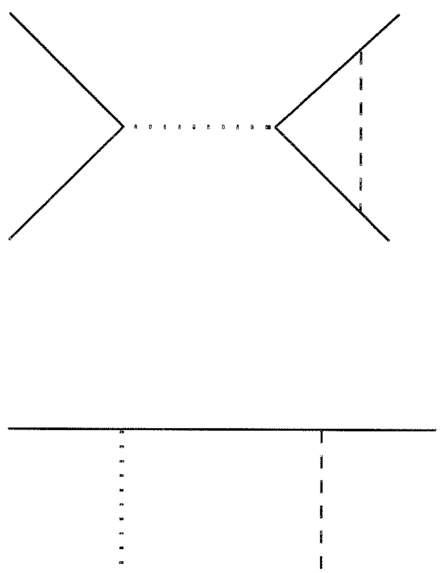

Figure 6.7: Diagrams that contribute to the renormalization of Δ_g.

order. We obtain, then, in a straightforward way the recursion relations:

$$\frac{d\bar{r}}{dl} = 2\bar{r} + \frac{2(m+1)\bar{u}}{1+\bar{r}} - \frac{2\bar{\Delta}_r}{1+\bar{r}} + O(\bar{u}^2, \bar{\Delta}_r^2, \bar{\Delta}_g^2) \tag{6.45}$$

$$\frac{d\bar{u}}{dl} = \epsilon\bar{u} - 2(m+4)\bar{u}^2 + 12\bar{u}\bar{\Delta}_r + O(\bar{u}^3, \ldots) \tag{6.46}$$

$$\frac{d\bar{\Delta}_r}{dl} = (\epsilon + \epsilon_\tau)\bar{\Delta}_r + 8\bar{\Delta}_r^2 - 4(m+1)\bar{u}\bar{\Delta}_r + O(\bar{u}^3, \ldots) \tag{6.47}$$

$$\frac{d\bar{\Delta}_g}{dl} = \bar{\Delta}_g(\epsilon + \epsilon_\tau + 10\bar{\Delta}_r - 2\bar{\Delta}_g - 2) \equiv \lambda_g\bar{\Delta}_g - 2\bar{\Delta}_g^2 \tag{6.48}$$

$$z = 1 + \bar{\Delta}_r + \bar{\Delta}_g; \ \eta = 0, \tag{6.49}$$

where m is the number of boson species ($m = 1$ physically), $\bar{r} = r_0/k_\Lambda^2$, $\bar{u} = K_d u$, $\bar{\Delta}_r = K_d\Delta_r$, $\bar{\Delta}_g = k_\Lambda^2 K_d \Delta_g$ are appropriately rescaled by the cutoff, and $K_d = 2/(4\pi)^{d/2}\Gamma(d/2)$ is $(2\pi)^{-d}$ times the area of the unit sphere in d-dimensions. Note that $\bar{\Delta}_g$ does not enter any recursion relations except its own at this order. If one sets $\bar{\Delta}_g = 0$ one obtains the usual Boyanovsky-Cardy lowest order recursion relations [13], with fixed point

$$\begin{aligned} \bar{r}^* &= -\frac{3m\epsilon + (5m+2)\epsilon_\tau}{8(2m-1)}, \ \bar{u}^* = \frac{\epsilon + 3\epsilon_\tau}{4(2m-1)} \\ \bar{\Delta}_g^* &= 0, \ \bar{\Delta}_r^* = \frac{(2-m)\epsilon + (m+4)\epsilon_\tau}{8(2m-1)}, \end{aligned} \tag{6.50}$$

correct to linear order in ϵ and ϵ_τ. For sufficiently small ϵ and ϵ_τ we see that $\lambda_g^* < 0$ and this fixed point is stable against the perturbation $\bar{\Delta}_g$. However, for

$$\epsilon_\tau > \epsilon_\tau^c \equiv \frac{8(2m-1) - 3(m+2)\epsilon}{13m+16} \tag{6.51}$$

this fixed point becomes unstable. Setting $m = 1$ and $\epsilon = 0$ (so that $\epsilon_\tau = 1$ corresponds to $d = 3$) we find

$$\epsilon_\tau^c = \frac{8}{29}, \tag{6.52}$$

which is actually quite small, and therefore might actually be a reasonable estimate. We may write $\lambda_g = 2z - d + 8\Delta_r$. The last term shows the deviation from the naive result,

$\lambda_g = 2z - d$, which would have led to the estimate, $\epsilon_\tau^c = \frac{8}{9}$, which is uncomfortably close to 1, considering how poorly controlled this expansion is at higher order [13].

For $\epsilon_\tau > \epsilon_\tau^c$, there is now a new stable fixed point with

$$\bar{\Delta}_g^* = \frac{1}{2}(\epsilon + \epsilon_\tau + 10\bar{\Delta}_r^* - 2) = \frac{13m + 16}{4(2m-1)}(\epsilon_\tau - \epsilon_\tau^c). \tag{6.53}$$

Therefore the new fixed point bifurcates continuously away from the random rod fixed point. The dynamical exponent, z, is substantially *larger* than the random rod value:

$$z = 1 + \bar{\Delta}_r^* + \bar{\Delta}_g^* = \frac{(m+4)\epsilon + (7m+10)\epsilon_\tau}{4(2m-1)}, \quad \epsilon_\tau > \epsilon_\tau^c, \tag{6.54}$$

which, for $m = 1$ and $\epsilon = 0$, yields $z = \frac{17}{4}$ at $\epsilon_\tau = 1$ (not far from $z = 3$, considering the crudeness of our estimates). However, this result grows, instead of shrinking as it should if $z = d$, with ϵ at fixed ϵ_τ, but this could change at higher order. The "thermal" eigenvalue, determining ν, is

$$\frac{1}{\nu} = 2 - 2(m+1)\bar{u}^* + 2\bar{\Delta}_r^* = \ldots, \tag{6.55}$$

while, as stated above, $\eta = 0$. These results are both unchanged from their random rod values at this order.

Finally, let us include the $\mathbf{g}_0$ term. To linear order the flow equation for $g_0 = |\mathbf{g}_0|$ is found to be

$$\dot{g}_0 = g_0[1 + \bar{\Delta}_r - \bar{\Delta}_g]. \tag{6.56}$$

This term becomes *irrelevant* at the dirty boson fixed point for $\epsilon_\tau \geq \epsilon_{\tau 1} = \frac{1}{3}\frac{4(2m-1)}{1+m-\epsilon}$. For $m = 1$ and $\epsilon = 0$ one obtains $\epsilon_{\tau 1} = 2/3$. This is again an uncontrolled estimate, but does indeed indicate that the statistical symmetry is restored prior to $\epsilon_\tau = 1$. For $\epsilon_\tau < \epsilon_{\tau 1}$ there is a new incommensurate fixed point at nonzero g_0. In order to locate it one should choose z to keep g_0, rather than e_τ, fixed during the RG flow. The flow equation for Δ_g becomes

$$\dot{\bar{\Delta}}_g = \bar{\Delta}_g[\epsilon + \epsilon_\tau - 4 + 8\bar{\Delta}_r] + 8\bar{\Delta}_r^2, \tag{6.57}$$

while the remaining ones are identical to those in [12]. The fixed point found in that work did not account for Δ_g, but we see that if $\Delta_r^* = O(\epsilon, \epsilon_\tau)$ then $\Delta_g^* = O(\epsilon^2, \epsilon\epsilon_\tau, \epsilon_\tau^2)$, so the results given there are indeed correct to $O(\epsilon, \epsilon_\tau)$. However, as ϵ_τ grows, so does Δ_g^*,

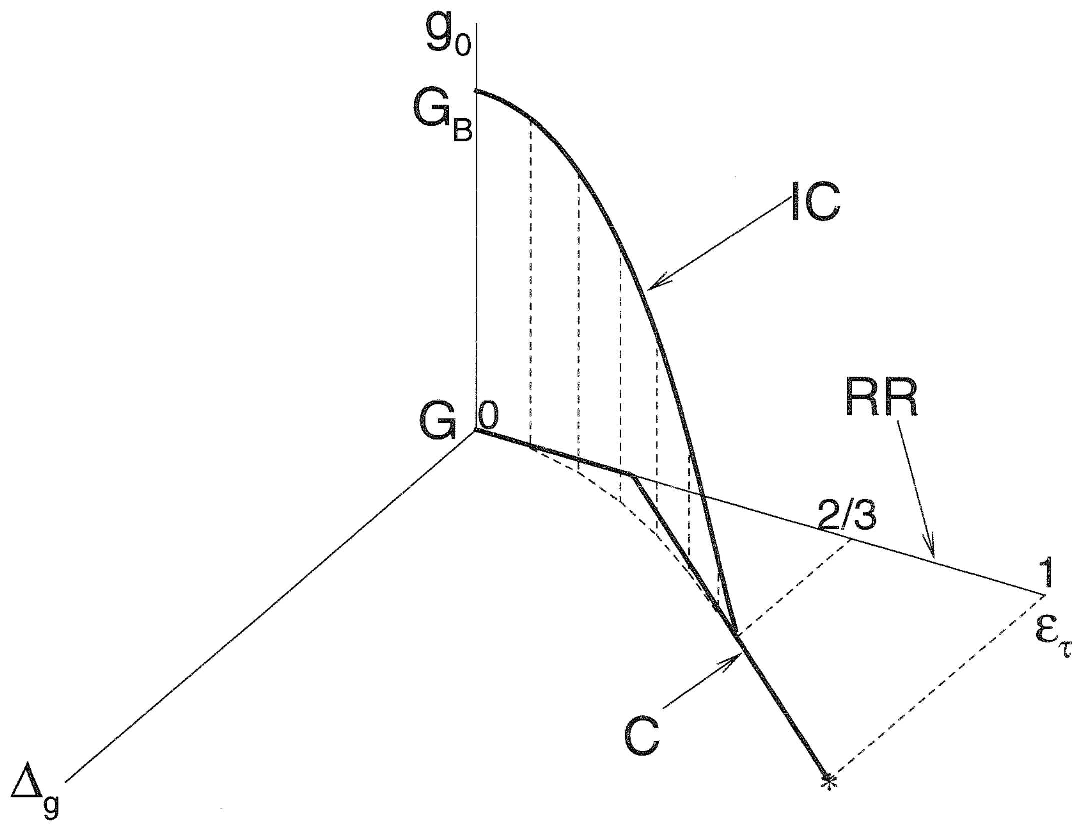

Figure 6.8: Proposed behavior of the random rod(RR), commensurate dirty boson boson(C) and incommensurate(IC) fixed points as a function of ϵ_τ. Here G and G_B are commensurate and incommensurate Gaussian fixed points. We propose that at $\epsilon_\tau = \epsilon_{\tau 1} \approx 2/3$, C and IC merge and at $\epsilon_\tau = 1$, C is the stable fixed point that describes the physical dirty boson problem.

eventually leading to the merging with the dirty boson fixed point.

The technical problems encountered in [12] are also overcome in our analysis. Specifically, the $|\nabla_\tau \psi_\alpha|^2$ term, which was also ignored in [12] has the flow equation $\dot{e}_\tau = -(1+\bar{\Delta}_r)e_\tau + \bar{\Delta}_g$, implying that it is of the same order as $\bar{\Delta}_g^*$ at the fixed point. A nonzero e_τ fixes the convergence problems found in [12], and allows one to remove the unphysical frequency cutoff ω_Λ.

Finally, to see how the two fixed points merge we write down flow eqns in the intermediate region, $\epsilon_{\tau 1} > \epsilon_\tau > \epsilon_\tau^c$, where one must consider both e_τ and $\bar{g}_0 \equiv k_\Lambda g_0$. We will choose z so that $e_\tau + \bar{g}_0 \equiv 1$ remains fixed. The flow equation for g_0 is then $\dot{g}_0 1 = (2+2\bar{\Delta}_r - z)g_0$ with $z = (1+\bar{\Delta}_r\bar{\Delta}_g)+(1+\bar{\Delta}_r)\bar{g}_0$. At the fixed point we therefore find $\bar{g}_0^* = (1+\bar{\Delta}_r^* - \bar{\Delta}_g^*)/(1+\bar{\Delta}_r^*)$, which vanishes precisely when g_0 becomes irrelevant at the dirty boson fixed point.

To summarize, for small ϵ_τ the unstable DBC fixed point and stable incommensurate fixed point exist. For $\epsilon_\tau^c < \epsilon_\tau < \epsilon_{\tau 1}$ there are three fixed points, with the new commensurate dirty boson fixed point being more stable than the DBC fixed point, but less stable than the asymmetric fixed point. Finally, for $\epsilon_\tau > \epsilon_{\tau 1}$ the incommensurate fixed point merges with the dirty boson fixed point, which is then completely stable. This provides a detailed scenario by which statistical particle-hole symmetry is restored (see Fig. 6.2). We caution, however, that due both to the uncontrolled nature of the double ϵ-expansion at the dirty boson fixed point, and the special nature of $\epsilon_\tau = 1$, extrapolation of these results to $\epsilon_\tau = 1$ is not completely justified. The general scenario we propose, however, seems very natural and illuminating.

6.5 Relevance of "point" disorder

In this section we discuss the relevance of "point" disorder for magnetic flux phases of high-tempearture superconductors. As mentioned in the introduction, flux lines could be viewed as boson world lines. We can represent the disorder by a potential $w(\mathbf{x}, \tau)$. The partition function can be written as a functional integral of the form

$$Z = \int \{d\psi\} e^{-\mathcal{L}}, \tag{6.58}$$

$$\mathcal{L} = -\int d^dx d\tau [\frac{1}{2}e_\tau|\partial_\tau\psi|^2 + (g_0 + \delta g_1(\mathbf{x},\tau))(\psi^*\partial_\tau\psi) + \frac{1}{2}e_x|\nabla\psi|^2 + \frac{1}{2}(r_0 + \delta r_1(\mathbf{x},\tau))|\psi|^2 + u_0|\psi|^4]. \quad (6.59)$$

For "point" disorder Δr and Δg are functions of both x and τ (unlike the dirty boson case). The replicated Lagrangian can again be written as $\mathcal{L}^{(p)} = \mathcal{L}_1^{(p)} + \mathcal{L}_2^{(p)}$, with

$$\mathcal{L}_1^{(p)} = -\sum_{\alpha=1}^{p}\int d^dx \int d^{\epsilon_\tau}\tau[\frac{1}{2}e_\tau|\nabla_\tau\psi_\alpha|^2 + \mathbf{g}_0\cdot(\psi^*\nabla_\tau\psi_\alpha) + \frac{1}{2}e_x|\nabla\psi_\alpha|^2 + \frac{1}{2}r_0|\psi_\alpha|^2 + u|\psi_\alpha|^4] \quad (6.60)$$

$$\mathcal{L}_2^{(p)} = \frac{1}{2}\sum_{\alpha,\beta=1}^{p}\int d^dx \int d^{\epsilon_\tau}\tau[\tilde{\Delta}_{r1}|\psi_\alpha(\mathbf{x},\vec{\tau})|^2|\psi_\beta(\mathbf{x},\tau)|^2 + \Delta_{g1}[\psi^*\nabla_\tau\psi - \mathbf{g}_0|\psi|^2](\mathbf{x},\vec{\tau})\cdot[\psi^*\nabla_\tau\psi - \mathbf{g}_0|\psi|^2](\mathbf{x},\vec{\tau})]. \quad (6.61)$$

We investigate the relevance of both Δ_{r1} and Δ_{g1} at $d = 2$. First we consider the commensurate point $g_0 = 0$. Δ_{r1} is the usual classical disorder, and its relevance/irrelevance is linked to the sign of α, as indicated by the Harris criterion. The singular free energy can be written as

$$f_s = \delta^{\nu(d+z)}\bar{f}(\Delta_{r1}\delta^{-\alpha}). \quad (6.62)$$

For the commensurate nondisordered transition at d=2, $\alpha = -.0126$. The negative sign indicates that Δ_{r1} is irrelevant for this case.

Next we investigate the role of Δ_{g1}. Since we are perturbing about the pure fixed point, scaling calculations (similar to those in Chapter 4), would give us the crossover exponent. Thus

$$\partial f/\partial\Delta_{g1} = -\frac{1}{2}\mathcal{G}_g(x=0,\tau=0) \quad (6.63)$$

where

$$\mathcal{G}_g(x,\tau) = \langle\langle\ \langle\psi^*\partial_\tau\psi(x,\tau)\psi^*\partial_\tau\psi(0,0)\rangle\ \rangle\rangle. \quad (6.64)$$

About the pure commensurate fixed point

$$f_s(g_0) = \delta^{\nu(z+d)} f_s(g_0 \delta^{-\nu z}), \tag{6.65}$$

$$\begin{aligned} \frac{\partial^2 f_s}{\partial g_0^2} &= \int d^d x d\tau \mathcal{G}_g(x,\tau) \\ &\sim \delta^{\nu(z+d)-2\nu z}. \end{aligned} \tag{6.66}$$

Using the form

$$\mathcal{G}_g(x,\tau) = \xi^a Y(x/\xi, \tau/\xi^z), \tag{6.67}$$

and rescaling x and τ,

$$\Rightarrow \delta^{-\nu(a+d+z)} \int d^d x' d\tau' Y(x',\tau') \sim \delta^{\nu(d-z)}. \tag{6.68}$$

This implies that the exponent $a = -2d$. If

$$f_s(\Delta_{g1}) \approx \delta^{\nu(z+d)} \bar{f}(\Delta_{g1}/\delta^{\phi_g}) \tag{6.69}$$

then it follows that

$$\Rightarrow \phi_g = z - d. \tag{6.70}$$

Note that this argument works only because we are expanding about the nondisordered fixed point. If $\phi_g > 0$, Δ_{g1} is relevant, else it is irrelevant. Since at the commensurate fixed point without disorder, $z = 1$, Δ_{g1} is irrelevant for $d > 1$.

Next we investigate the effect of disorder on the generic incommensurate transition. Here, either g_0 or r_0 can be chosen as the control parameter, playing the role of δ. For either choice the critical exponents are the same, implying that we can apply the Harris criterion to both Δ_{g1} and Δ_{r1}. So their relevance is related to the sign of α. At $d = 2$, the incommensurate transition is mean-field like and $\alpha = 0$, so disorder is marginally relevant. Above $d = 2$, where the nondisordered transition has mean field exponents, the crossover exponent for disorder is $\phi_\Delta = 2 - (d+z)\nu$, (not α, which is equal to 0). Thus "point" disorder plays no role at the incommensurate transition for $d \geq 2$.

Note that these calculations were done in the limit of short range interactions. For short range interactions, point disorder does not seem to play any role in 2 dimensions (i.e. three-dimensional sample). So one should look for the vortex glass phase in the limit of "unscreened" interactions between the vortex lines. This agrees with work done by Bokil and Young [37], where they find that screening of the interaction between vortices destroys the (2+1) dimensional vortex glass to superfluid transition.

Appendix A

Functional integrals based on the Trotter decomposition

Given a Hamiltonian, $\mathcal{H}$, thermodynamics is obtained from the partition function $Z = tr\left[e^{-\beta\mathcal{H}}\right]$. The Trotter decomposition involves identifying a convenient complete set of states $\{|\alpha\rangle\}$, writing $e^{-\beta\mathcal{H}} = \left[e^{-\Delta\tau\mathcal{H}}\right]^M$, where $M = \beta/\Delta\tau$, and inserting the $|\alpha\rangle$'s between each element of the product:

$$\begin{aligned} Z &= \sum_{\alpha_0}\sum_{\alpha_1}\cdots\sum_{\alpha_{M-1}} \langle\alpha_0|e^{-\Delta\tau\mathcal{H}}|\alpha_1\rangle\langle\alpha_1|e^{-\Delta\tau\mathcal{H}}|\alpha_2\rangle\ldots \\ &\times \langle\alpha_{M-1}|e^{-\Delta\tau\mathcal{H}}|\alpha_0\rangle. \end{aligned} \tag{A.1}$$

Often one can write decompose $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, and choose the $|\alpha\rangle$'s to be eigenstates of $\mathcal{H}_0$. This is especially convenient when $\mathcal{H} = \mathcal{H}[\{\hat{q}_i\},\{\hat{p}_i\}]$ is written in terms of a set of canonically conjugate positions, $\hat{q}_i$, and momenta, $\hat{p}_i$, and takes the special form

$$\mathcal{H} = \mathcal{H}_0[\{\hat{q}_i\}] + \mathcal{H}_1[\{\hat{p}_i\}] \tag{A.2}$$

in which $\mathcal{H}_1$ is a quadratic polynomial in the $\hat{p}_i$:

$$\mathcal{H}_1 = \frac{1}{2}\sum_{i,j} S_{ij}\hat{p}_i\hat{p}_j + \sum_i t_i\hat{p}_i. \tag{A.3}$$

The $|\alpha\rangle$'s are then taken to be product eigenstates of the positions $\hat{q}_i$: $|q\rangle \equiv |q_1,\ldots,q_N\rangle = \otimes_{i=1}^N|q_i\rangle$ with $\hat{q}_i|q\rangle = q_i|q\rangle$. We also define product eigenstates of the momenta, $|p\rangle \equiv |p_1,\ldots,p_N\rangle = \otimes_{i=1}^N|p_i\rangle$, with $\hat{p}_i|p\rangle = p_i|p\rangle$. From the canonical commutation relations, $[\hat{q}_i,\hat{p}_i] = i\delta_{ij}$, which in turn imply

$$\begin{aligned} [\hat{q}_i, e^{-i\lambda\hat{p}_j}] &= \lambda e^{-i\lambda\hat{p}_j}\delta_{ij} \\ [e^{i\lambda\hat{q}_i}, p_j] &= -\lambda e^{i\lambda\hat{q}_i}\delta_{ij}, \end{aligned} \tag{A.4}$$

we immediately infer that

$$\begin{aligned} e^{-i\lambda\hat{p}_j}|q_j\rangle &= |q_j+\lambda\rangle \\ e^{i\lambda\hat{q}_j}|p_j\rangle &= |p_j+\lambda\rangle. \end{aligned} \tag{A.5}$$

From this we obtain the *wavefunctions*

$$\begin{aligned} \langle q_i|p_i\rangle &= \langle q_i=0|e^{iq_i\hat{p}_i}|p_i\rangle = e^{iq_ip_i}\langle q_i=0|e^{ip_i\hat{q}_i}|p_i=0\rangle \\ &= \frac{1}{\mathcal{N}}e^{iq_ip_i}, \end{aligned} \tag{A.6}$$

where $\frac{1}{\mathcal{N}} = \langle q_i=0|p_i=0\rangle$ normalizes the wavefunction. Note that if the q_i are periodic variables, defined only modulo 2π, say, then $|q_i\rangle \equiv |q_i+2\pi\rangle$ are identified. This then requires the same periodicity of the wavefunctions, and determines the possible eigenvalues, p_i (which then must be integers in this case).

We may now compute, for small $\Delta\tau$,

$$\begin{aligned} \langle q|e^{-\Delta\tau\mathcal{H}}|q'\rangle &\approx \langle q|e^{-\Delta\tau\mathcal{H}_0(\hat{q})}e^{-\Delta\tau\mathcal{H}_1(\hat{p})}|q'\rangle \\ &= e^{-\Delta\tau\mathcal{H}_0(q)}\sum_p\langle q|p\rangle\langle p|e^{-\Delta\tau\mathcal{H}_1(\hat{p})}|q'\rangle \\ &= e^{-\Delta\tau\mathcal{H}_0(q)}\sum_p e^{-\Delta\tau\mathcal{H}_1(p)}\langle q|p\rangle\langle p|q'\rangle \\ &= e^{-\Delta\tau\mathcal{H}_0(q)}\sum_p e^{i\sum_i p_i(q_i-q_i')}e^{-\Delta\tau\mathcal{H}_1(p)}, \end{aligned} \tag{A.7}$$

and we are then left only with computing the inverse Fourier transform of the Gaussian function $e^{-\Delta\tau\mathcal{H}_1(p)}$. To do this, we first complete the square:

$$\begin{aligned} \mathcal{H}_1[\hat{p}] &= \frac{1}{2}\sum_{i,j}S_{ij}(\hat{p}_i+\nu_i)(\hat{p}_j+\nu_j) - \frac{1}{2}\sum_{i,j}\nu_i\nu_j \\ \nu_i &\equiv \sum_j(S^{-1})_{ij}\nu_j;\ t_i = \sum_j S_{ij}\nu_j. \end{aligned} \tag{A.8}$$

We specialize now to the case of integer p_i, using the formula

$$\sum_{p_i=-\infty}^{\infty} = \int_{-\infty}^{\infty} dp_i \sum_{n_i=-\infty}^{\infty} \delta(p_i-n_i)$$

$$= \int_{-\infty}^{\infty} \sum_{m_i=-\infty}^{\infty} e^{i2\pi m_i p_i}. \tag{A.9}$$

Then

$$\begin{aligned}
\sum_p \quad e & \quad ^{i\sum_i p_i(q_i-q_i')} e^{-\Delta\tau\mathcal{H}_1(p)} \\
&= \sum_m \int dp e^{ip_i(q_i-q_i'+2\pi m_i)} e^{-\Delta\tau\mathcal{H}_1(p)} \\
&= \sum_m e^{\frac{1}{2}\Delta\tau\sum_{i,j} S_{ij}\nu_i\nu_j - i\sum_i \nu_i(q_i-q_i'+2\pi m_i)} \\
&\times \int d\bar{p} e^{-\frac{1}{2}\Delta_\tau \sum_{i,j} S_{ij}\bar{p}_i\bar{p}_j + i\sum_i \bar{p}_i(q_i-q_i'+2\pi m_i)} \\
&= \frac{1}{\mathcal{N}(\Delta\tau)} \sum_m e^{\frac{1}{2}\Delta\tau\sum_{i,j} S_{ij}\nu_i\nu_j - \sum_i \nu_i(q_i-q_i'+2\pi m_i)} \\
&\times \; e^{-\frac{1}{2\Delta\tau}\sum_{i,j}(S^{-1})_{ij}(q_i-q_i'+2\pi m_i)(q_j-q_j'+2\pi m_i)}
\end{aligned} \tag{A.10}$$

where, in the second line, we have changed variables to $\bar{p}_i = p_i + \nu_i$, and $\mathcal{N}(\Delta\tau) = \det(\Delta\tau S)^{\frac{1}{2}}$.

Now consider the limit $\Delta\tau \to 0$. For given $\{q_i\}$ only a *single* term in the m-sum will survive, namely that which minimizes the exponent. Furthermore, only if $q_i - q_i' + 2\pi m_i = O(\Delta\tau^{\frac{1}{2}})$ will the term contribute to the path integral, (A.1). Therefore, modulo 2π, $q_i(\tau)$ becomes a continuous function in the limit $\Delta\tau \to 0$, and $q_i - q_i' + 2\pi m_i \to -\Delta\tau\dot{q}_i$. Clearly we will nearly always have $m_i = 0$, with $m_i = \pm 1$ every so often when the periodic boundary conditions are enforced (for example, when $q_i = 2\pi^-$ and $q_i' = 0^+$, or vice versa). When $\Delta\tau \to 0$ we may equivalently define a continuous function $q_i(\tau)$, taking arbitrary real values, and then sum over all boundary conditions $q_i(\tau) = q_i(0) + 2\pi m_i$, with m_i running over all integers. Thus, we finally have, as $\Delta\tau \to 0$:

$$\begin{aligned}
\langle q| \quad & e^{-\Delta\tau\mathcal{H}_1(\hat{p})}|q'\rangle = \frac{1}{\mathcal{N}(\Delta\tau)} e^{-\frac{1}{2}\Delta\tau\sum_{i,j}(S^{-1})_{ij}\dot{q}_i\dot{q}_j} \\
\times \quad & e^{-i\Delta\tau\sum_i \nu_i\dot{q}_i + \frac{1}{2}\Delta\tau\sum_{i,j} S_{ij}\nu_i\nu_j} \\
= \quad & \frac{1}{\mathcal{N}(\Delta\tau)} e^{-\frac{1}{2}\Delta\tau\sum_{i,j}(S^{-1})_{ij}(\dot{q}_i+it_i)(\dot{q}_j+it_j)},
\end{aligned} \tag{A.11}$$

and

$$Z = \det(S)^{-\frac{1}{2}} \prod_i \int \mathcal{D}q_i(\tau) \exp\left\{ - \int_0^\beta d\tau \Big[\mathcal{H}_0[q(\tau)] + \frac{1}{2}\sum_{i,j}(S^{-1})_{ij}(\dot{q}_i + it_i)(\dot{q}_j + it_j) \Big] \right\}, \tag{A.12}$$

where $\int \mathcal{D}q_i(\tau)$ is a functional integral is over all paths with a uniform measure.

Equation (A.12) now leads directly to the path integral representation for the Josephson Hamiltonian, (2.12), with $\hat{q}$ replaced by $\hat{\phi}$ and $\hat{p}$ replaced by $\hat{n}$. The term $\mathcal{H}(\hat{q})$ is then just the Josephson cosine coupling term.

If, instead of canonical coordinates, $\mathcal{H} = \mathcal{H}[a^\dagger, a]$, is written instead in terms of raising and lowering operators, another convenient complete set of states are the coherent states,

$$\begin{aligned} |\alpha\rangle &= \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \;\; a|\alpha\rangle = \alpha|\alpha\rangle \\ \langle\alpha|\alpha'\rangle &= \sum_{n=0}^{\infty} \frac{(\alpha^*\alpha)^n}{n!} e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\alpha'|^2} \\ &= e^{\alpha^*\alpha' - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\alpha'|^2}. \end{aligned} \tag{A.13}$$

Thus, for any normally ordered operator, $O(a^\dagger, a)$, we have

$$\langle\alpha|O(a^\dagger, a)|\alpha'\rangle = O(\alpha^*, \alpha')\langle\alpha|\alpha'\rangle, \tag{A.14}$$

and hence for $\Delta\tau \to 0$

$$\begin{aligned} \langle\alpha|e^{-\Delta\tau\mathcal{H}[a^\dagger,a]}|\alpha'\rangle &\approx \langle\alpha|(1 - \Delta\tau\mathcal{H}[a^\dagger, a])|\alpha'\rangle \\ &\approx \langle\alpha|\alpha'\rangle e^{-\Delta\tau\mathcal{H}[\alpha^*,\alpha']}. \end{aligned} \tag{A.15}$$

Recognizing that

$$\begin{aligned} \sum_n \Big[\alpha_n^*\alpha_{n+1} &- \frac{1}{2}|\alpha_n|^2 - \frac{1}{2}|\alpha_{n+1}|^2\Big] \\ &= \Delta\tau \sum_n \alpha_n^* \left(\frac{\alpha_{n+1} - \alpha_n}{\Delta\tau}\right), \end{aligned} \tag{A.16}$$

where the periodic boundary conditions in imaginary time have been used, we arrive at

(2.10) (with ψ replacing α) when $\Delta\tau \to 0$.

Appendix B

Duality transformations in one and two dimensions

In order to obtain a model amenable to analysis in one dimension one must derive a *dual representation* for the Josephson Lagrangian, (2.12). In Ref. [6] this was done in a rather *ad hoc* fashion using the Haldane representation for one-dimensional bosons. Here we perform the duality transformation directly, on a variant of (2.12), in a much more transparent manner, following closely the analogous derivation for the two-dimensional XY-model and its associated Kosterlitz-Thouless transition [34].

The transformation is best carried out in discrete time. The continuous time limit is mathematically well defined but, as we shall see, physically less transparent: one runs into logarithmically divergent coupling constants, just as in the Trotter decomposition of the quantum Ising model in a transverse field [35]. These are consequence of the usual exponential weighting times in continous time, by discrete state, Markov processes. In order to avoid introducing the probabilistic formalism necessary for dealing with the continuum limit we shall maintain a discrete time variable.

We begin by introducting the Villain, or periodic Gaussian form of the XY-coupling [34]:

$$e^{-K_0(1-cos(\phi))} \to \sum_{m=-\infty}^{\infty} e^{-\frac{1}{2}K(\phi-2\pi m)^2} \equiv e^{V_0(\phi,K)}, \tag{B.1}$$

which allows the duality transformation to be carried out *exactly*. We will consider only the case where J_{ij} in (2.12) is nearest neighbor, but possibly random. In (B.1) $K_0 = J_{ij}\Delta\tau$ for the bond $\langle ij \rangle$, and therefore already includes the effect of discretizing the τ-integral in (2.12). There are two limits in which K and K_0 may be quantitatively compared. For large K_0 the variable ϕ will have only small fluctuations about zero, and only the curvature, K_0, near the minimum of the cosine potential at $\phi = 0$ is important. In this limit only the $m = 0$ term contributes to the Villain form, and the two potentials therefore match when $K \approx K_0$. Conversely, when K_0 is small, ϕ fluctuates strongly, and many m contribute to the Villain sum. It is then convenient to use the Fourier representation [34]

$$e^{V_0(\phi,K)} = \sum_{l=-\infty}^{\infty} \frac{e^{-l^2/2K}}{\sqrt{2\pi K}} e^{il\phi}, \tag{B.2}$$

where now, for small K, only the $l = 0, \pm 1$ terms are important. This yields

$$V_0(\phi, K) \approx -\frac{1}{2} ln(2\pi K) + 2e^{-\frac{1}{2K}} cos(\phi), \; K \to 0, \tag{B.3}$$

and hence the correspondence $K \approx 1/2ln(2/K_0)$ in this limit. Now, the continuum limit, $\Delta\tau \to 0$, corresponds to $K_0 \to 0$, and therefore (B.3) is appropriate. This yields $K \approx -1/2ln(\tilde{J}_{ij}\Delta\tau)$ as $\Delta\tau \to 0$. This is the logarithmic behavior alluded to above. We shall keep $\Delta\tau$ small but finite, setting aside the question of its optimal value, which would have to be addressed for *quantitative* comparison of the Josephson and Villain forms of our model. For our purposes it is important only that the discrete and continuous time versions lie in the same universality class.

Using (B.1) we then define the Villain form of the Josephson Lagrangian, (2.12):

$$\begin{aligned} e^{\tilde{\mathcal{L}}_J[\phi]} &\equiv \sum_{\mathbf{m}} e^{\tilde{\mathcal{L}}_J[\phi,\mathbf{m}]} \\ \tilde{\mathcal{L}}_J &\equiv -\frac{1}{2} \sum_{\mathbf{r},\alpha\neq 0} K_i^\alpha (\partial_\alpha \phi_{\mathbf{r}} - 2\pi m_{\mathbf{r}}^\alpha)^2 \\ &- \frac{1}{2} \sum_{i,j,\tau} (\mathbf{V}^{-1})_{ij} (\partial_\tau \phi_{i\tau} - i\nu_i - 2\pi m_{i\tau}^0) \\ &\times (\partial_\tau \phi_{j\tau} - i\nu_j - 2\pi m_{j\tau}^0), \end{aligned} \tag{B.4}$$

where $\partial_\alpha \phi_{\mathbf{r}} \equiv \phi_{\mathbf{r}+\hat{\alpha}} - \phi_{\mathbf{r}}$ and $\mathbf{m}_{\mathbf{r}} = (m_{\mathbf{r}}^0, m_{\mathbf{r}}^1, \ldots, m_{\mathbf{r}}^d)$ is a $(d+1)$-dimensional integer vector field defined at each space-time lattice site $\mathbf{r} \equiv (i, \tau)$, and the index $\alpha = 0, 1, \ldots, d$ runs over the neighboring sites in the $\hat{\alpha} \equiv \hat{\mathbf{x}}_\alpha$ direction, with $\hat{\mathbf{x}}_0 = \hat{\tau}$. Until further notice there is no restriction on the dimensionality, d. The parameters K_i^α, V_{ij} and ν_i are, respectively, the Villain analogues of $J_{ij}\Delta\tau$, $U_{ij}\Delta\tau$ and $\mu_i\Delta\tau$ in (2.12). Since $e^{\tilde{\mathcal{L}}_J}$ is separately periodic in all the differences $\partial_\alpha\phi_{\mathbf{r}}$ we may write it as a Fourier series,

$$e^{\tilde{\mathcal{L}}_J[\phi]} = \frac{1}{\mathcal{M}} \sum_{\mathbf{n}} e^{i\sum_{\mathbf{r},\alpha} n_{\mathbf{r}}^\alpha (\phi_{\mathbf{r}+\hat{\alpha}} - \phi_{\mathbf{r}})} e^{\hat{\mathcal{L}}_J[\mathbf{n}]}, \tag{B.5}$$

where

$$\frac{1}{\mathcal{M}}e^{-\hat{\mathcal{L}}[\mathbf{n}]} = \prod_{\mathbf{r},\alpha}\int_0^{2\pi}\frac{d\theta_{\mathbf{r}}^{\alpha}}{2\pi}e^{-i\sum_{\mathbf{r},\alpha}n_{\mathbf{r}}^{\alpha}\theta_{\mathbf{r}}^{\alpha}}e^{-\tilde{\mathcal{L}}_J[\phi_{\mathbf{r}+\hat{\alpha}}-\phi_{\mathbf{r}}\to\theta_{\mathbf{r}}^{\alpha}]} \tag{B.6}$$

so that

$$\hat{\mathcal{L}}_J[\mathbf{n}] = -\frac{1}{2}\sum_{i,\tau,\alpha\neq 0}\frac{(n_{i\tau}^{\alpha})^2}{K_i^{\alpha}} - \frac{1}{2}\sum_{i,j}V_{ij}n_{i\tau}^0 n_{j\tau}^0 + \sum_{i\tau}\nu_i n_{i\tau}^0, \tag{B.7}$$

and the normalization is

$$\mathcal{M} = \left(\prod_{i,\tau,\alpha\neq 0}\frac{2\pi}{K_i^{\alpha}}\right)^{\frac{1}{2}} det(2\pi\mathbf{V})^{\frac{1}{2}}. \tag{B.8}$$

To derive (B.7) we have used the identity

$$\int_{-\infty}^{\infty}\frac{du}{2\pi}e^{-imu}e^{-\frac{1}{2}K(u+iv)^2} = \frac{1}{\sqrt{2\pi K}}e^{-mv}e^{-m^2/2K} \tag{B.9}$$

and its higher-dimensional generalizations. The partition function now becomes

$$Z = tr^{\phi}e^{\tilde{\mathcal{L}}_J[\phi]} = \frac{1}{\mathcal{M}}\sum_{\mathbf{n}}\prod_{\mathbf{r}}\delta_{\nabla\cdot\mathbf{n}_{\mathbf{r}},0}e^{-\hat{\mathcal{L}}_J[\mathbf{n}]}, \tag{B.10}$$

where $\nabla\cdot\mathbf{n}$ is the discrete space-time divergence,

$$\nabla\cdot\mathbf{n}_{\mathbf{r}} = \sum_{\alpha}(n_{\mathbf{r}}^{\alpha} - n_{\mathbf{r}-\hat{\alpha}}^{\alpha}). \tag{B.11}$$

Note that this formulation is entirely real, and has been used as a basis for Monte Carlo simulations of the dirty boson problem [36].

B.1 One dimension

Let us now restrict attention to $d = 1$. One may then solve the constraint $\nabla\cdot\mathbf{n} = 0$ by introducing a *dual lattice* integer field, $s_{\mathbf{R}}$, such that

$$\mathbf{n}_{\mathbf{r}} = (\nabla\times S)_{\mathbf{r}} \equiv (S_{\mathbf{R}-\hat{\mathbf{x}}} - S_{\mathbf{R}}, S_{\mathbf{R}} - S_{\mathbf{R}-\hat{\tau}}), \tag{B.12}$$

where the dual lattice bond connecting $\mathbf{R}-\hat{\tau}$ to $\mathbf{R}$ is the one that cuts the real lattice bond connecting $\mathbf{r}$ to $\mathbf{r}+\hat{\mathbf{x}}$, while that connecting $\mathbf{R}-\hat{\mathbf{x}}$ to $\mathbf{R}$ crosses the one connecting

$\mathbf{r}$ to $\mathbf{r}+\hat{\tau}$, i.e. $\mathbf{R} \equiv (I,T) = \mathbf{r} + \frac{1}{2}(\hat{\mathbf{x}}+\hat{\tau})$. Thus the α-component of the discrete curl of a scalar is the difference between its values on the two dual sites that border the bond from $\mathbf{r}$ to $\mathbf{r}+\hat{\alpha}$. The field $S_{\mathbf{R}}$ is defined uniquely up to an overall additive constant, and the constrained trace over the $\mathbf{n_r}$ is precisely equivalent to the free trace over the $s_{\mathbf{R}}$. One therefore has

$$Z = \frac{1}{\mathcal{M}} \sum_{S} e^{\hat{\mathcal{L}}_J[\nabla\times S]} \tag{B.13}$$

with

$$\begin{aligned} \hat{\mathcal{L}}_J[\nabla\times S] &= -\frac{1}{2}\sum_{\mathbf{R}} \frac{1}{K_I}(\partial_\tau S_{\mathbf{R}})^2 \\ &- \frac{1}{2}\sum_{I,J,T} V_{IJ}(\partial_x S_{IT})(\partial_x S_{JT}) \\ &+ \sum_{\mathbf{R}} \nu_I(\partial_x S_{\mathbf{R}}). \end{aligned} \tag{B.14}$$

Here $\partial_\tau S_{\mathbf{R}} = S_{\mathbf{R}} - S_{\mathbf{R}-\hat{\tau}}$, etc., K_I is the Villain coupling on the real lattice bond that cuts $(\mathbf{R}-\hat{\tau},\mathbf{R})$, and similarly for ν_I and V_{IJ}. Note that in this representation the Lagrangian is purely real and has a very natural classical interpretation, namely that of a three-dimensional interface model. The field $s_{\mathbf{R}}$ represents the height of a surface over a two-dimensional plane. The first two terms in $\hat{\mathcal{L}}_J$ yield the energy cost for steps in the τ and x directions, respectively. In this case the energy associated with steps in the τ direction is random, but only in the spatial index. The last term represents a random tilting potential which *favors* steps in the x direction with the same sign as ν_I. It is precisely this breaking of the symmetry between left and right steps that reflects the broken particle-hole symmetry in the original quantum Hamiltonian. Note that in this dual model the symmetry being broken is associated with parity ($x \to -x$) rather than time reversal ($\tau \to -\tau$).

The more common sine-Gordon representation is obtained from (B.5) by softening the integer constraint on the $S_{\mathbf{R}}$. Thus $\sum_{S_{\mathbf{R}}=-\infty}^{\infty} = \int_{-\infty}^{\infty} dS_{\mathbf{R}} \sum_{h_{\mathbf{R}}=-\infty}^{\infty} \delta(S_{\mathbf{R}} - h_{\mathbf{R}})$ is replaced by $\int dS_{\mathbf{R}} q(S_{\mathbf{R}})$, where $q(t)$ is periodic with period one, and is peaked around $t=0$. The sine-Gordon model (5.1) results from the choice

$$q(t) = e^{2y_0 cos(2\pi t)} \tag{B.15}$$

where y_0 is called the fugacity. The integer constraint is recovered in the limit $y_0 \to \infty$.

For small y_0 (see below) this term may be obtained directly by including a term

$$ln(y_0)\sum_{\mathbf{R}}(\nabla\times\mathbf{m})^2_{\mathbf{R}} \tag{B.16}$$

in the original Lagrangian, (B.4). The discrete curl of a vector field is a scalar field on the dual lattice obtained by summing the vector field around the dual lattice plaquette,

$$(\nabla\times\mathbf{m})_{\mathbf{R}} = m^1_{\mathbf{r}} + m^0_{\mathbf{r}+\hat{\mathbf{x}}} - m^1_{\mathbf{r}+\hat{\tau}} - m^0_{\mathbf{r}}, \tag{B.17}$$

and is precisely the *vorticity* on that plaquette.

Finally, the Coulomb gas representation is obtained either from the sine-Gordon representation by expanding the exponential in (B.15) and integrating out the $S_{\mathbf{R}}$, or from the discrete version, (B.14), by substituting $\sum_{l_{\mathbf{R}}=-\infty}^{\infty} e^{i2\pi l_{\mathbf{R}} S_{\mathbf{R}}}$ for $\sum_{h_{\mathbf{R}}=-\infty}^{\infty}\delta(S_{\mathbf{R}}-h_{\mathbf{R}})$:

$$\begin{aligned} Z &= tr^l tr^S\left[e^{2\pi i\sum_{\mathbf{R}} l_{\mathbf{R}} S_{\mathbf{R}}} e^{\hat{\mathcal{L}}_J[\nabla\times S]}\right] \\ &= tr^l\left[e^{\mathcal{L}_C[l]}\right], \end{aligned} \tag{B.18}$$

where [we include the term (B.15) for completeness]

$$\begin{aligned} \mathcal{L}_C[l] &= \frac{1}{2}\sum_{\mathbf{R},\mathbf{R}'}\mathcal{G}_{\mathbf{R}\mathbf{R}'}(2\pi l_{\mathbf{R}} + i\partial_x\nu_I)(2\pi l_{\mathbf{R}'} + i\partial_x\nu_{I'}) \\ &+ \ln(y_0)\sum_{\mathbf{R}} l^2_{\mathbf{R}}, \end{aligned} \tag{B.19}$$

and $\mathcal{G}_{\mathbf{R}\mathbf{R}'}$ is the inverse of the quadratic form:

$$(\mathcal{G}^{-1})_{\mathbf{R}\mathbf{R}'} = \frac{1}{K_I}(\partial_T\partial_{T'}\delta_{\mathbf{R}\mathbf{R}'}) + (\partial_I\partial_{I'}V_{II'})\delta_{TT'}. \tag{B.20}$$

For diagonal $V_{IJ} = V_0\delta_{IJ}$ and fixed $K_I \equiv K_0$, $\mathcal{G}_{\mathbf{R}\mathbf{R}'}$ is, modulo a trivial rescaling, the inverse of the two-dimensional lattice Laplacian, and yields the usual logarithmic Coulomb interaction at large distances. So long as V_{IJ} is short ranged and $K_I = K_0 + \delta K_I$ with $\delta K_I/K_I \ll 1$, $\mathcal{G}_{\mathbf{R}\mathbf{R}'}$ will still be Coulomb-like at large distances. Note that $\mathcal{L}_C$ is once again complex, with $\partial_x\nu$ playing the role of complex offset charges.

The sine-Gordon form yields the same result, (B.19), except that the values of $l_{\mathbf{R}}$ are restricted to $0, \pm 1$ only. When y_0 is small large values of $l_{\mathbf{R}}$ are suppressed anyway, and the difference between (B.15) and (B.16) is negligible.

B.2 Two dimensions

What can one say in two dimensions? The constraint $\nabla \cdot \mathbf{n} = 0$ in (B.10) may still be satisfied, now using a three component dual lattice vector field, $\mathbf{a}$, with $\nabla \times \mathbf{a} = \mathbf{n}$. The problem is that $\mathbf{a}$ has the usual gauge freedom: any gauge transformation $\mathbf{a}' = \mathbf{a} + \nabla\lambda$, where λ is any dual lattice scalar field, has no effect on $\mathbf{n}$. The constrained sum on $\mathbf{n}$ is therefore not equivalent to a free sum on $\mathbf{a}$. One may fix the gauge by demanding that $\nabla \cdot \mathbf{a} = 0$, but this is precisely the same constraint that we have been trying eliminate to in the first place!

In order to obtain a more appealing dual representation in $d = 2$, it is convenient to generalize the Lagrangian, (B.4), to include external magnetic fields. In the quantum Hamiltonian, (2.2), this entails a more general hopping term

$$- \sum_{i,\alpha\neq 0} J_i^\alpha cos(\partial_\alpha \hat{\phi}_i - A_i^\alpha), \tag{B.21}$$

where we again assume only nearest neighbor hopping, and the vector potential, $\mathbf{A}_i = (A_i^1, A_i^2)$, is related to the (random, if desired) *static* externally applied dual lattice scalar magnetic field, B_I, via [compare (B.17)]

$$B_I = (\nabla \times \mathbf{A})_I = A^y_{i+\hat{\mathbf{x}}} - A^y_i - A^x_{i+\hat{\mathbf{y}}} + A^x_i, \tag{B.22}$$

where $I = i + \frac{1}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}})$ is the spatial dual lattice site. We introduce here the convenient notation where ∇ includes only the spatial part of the gradient, while $\nabla^{(3)}$ will denote the full space-time gradient. We now treat the following *generalized Villain model*

$$\begin{aligned} \mathcal{L}_V &= -\frac{1}{2} \sum_{\mathbf{r},\alpha\neq 0} K_i^\alpha (\partial_\alpha \phi_{\mathbf{r}} - A_i^\alpha - 2\pi m_{\mathbf{r}}^\alpha)^2 \\ &- \frac{1}{2} \sum_{\mathbf{R},\alpha\neq 0} \frac{(q_{\mathbf{R}}^\alpha)^2}{\kappa_I^\alpha} \\ &- \frac{1}{2} \sum_{i,j,\tau} (V^{-1})_{ij} (\partial_\tau \phi_{i\tau} - i\nu_i - 2\pi m_{i\tau}^0) \\ &\times \ (\partial_\tau \phi_{j\tau} - i\nu_j - 2\pi m_{j\tau}^0), \end{aligned} \tag{B.23}$$

where

$$\begin{aligned}
\mathbf{q_R} &= (\nabla^{(3)} \times \mathbf{m})_{\mathbf{R}} \\
&= (m^y_{\mathbf{r}+\hat{\mathbf{x}}} - m^y_{\mathbf{r}} - m^x_{\mathbf{r}+\hat{\mathbf{y}}} + m^x_{\mathbf{r}}, \\
&\quad m^0_{\mathbf{r}+\hat{\mathbf{y}}} - m^0_{\mathbf{r}} - m^y_{\mathbf{r}+\hat{\tau}} + m^y_{\mathbf{r}}, \\
&\quad m^x_{\mathbf{r}+\hat{\tau}} - m^x_{\mathbf{r}} - m^0_{\mathbf{r}+\hat{\mathbf{x}}} + m^0_{\mathbf{r}}) \qquad \text{(B.24)}
\end{aligned}$$

is the full three-dimensional discrete curl of $\mathbf{m}$, and $\mathbf{R} \equiv (I,T)$ is the center of the cube with one corner at $\mathbf{r} = (i,\tau)$, i.e. $\mathbf{R} = (i + \frac{1}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}}), \tau + \frac{1}{2}) = \mathbf{r} + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The second term has been added for later convenience and, as we shall see, does not change the fundamental 2π-periodicity in the individual $\phi_{\mathbf{r}}$. The partition function is

$$Z = \sum_{\mathbf{m}} \int D\phi \; e^{\mathcal{L}_V[\mathbf{m},\phi]}. \qquad \text{(B.25)}$$

The field $\mathbf{q}$ clearly satisfies the current conservation constraint,

$$\begin{aligned}
(\nabla^{(3)} \cdot \mathbf{q})_{\mathbf{R}} &\equiv (q^0_{IT} - q^0_{I,T-1}) + (q^x_{IT} - q^x_{I-\hat{\mathbf{x}},T}) \\
&+ (q^y_{IT} - q^y_{I-\hat{\mathbf{y}},T}) = 0, \qquad \text{(B.26)}
\end{aligned}$$

and is invariant under the addition of perfect gradients to $\mathbf{m}$, i.e. $\mathbf{m} \to \mathbf{m}' = \mathbf{m} + \nabla^{(3)} n$. We shall use this later to isolate the vortex constibution to $\mathbf{m}$.

Let us first decouple the interaction term by introducing a ficticious *continuous* scalar magnetic field $-\infty < b_{i\tau} < \infty$ and using the identity

$$\begin{aligned}
&e^{-\frac{1}{2}\sum_{i,j}(V^{-1})_{ij} x_i x_j} \\
&= \det(V)^{\frac{1}{2}} \int Db \; e^{-\frac{1}{2}\sum_{i,j} V_{ij} b_i b_j + i\sum_i b_i x_i}. \qquad \text{(B.27)}
\end{aligned}$$

The partition function is then $Z = \sum_{\mathbf{m}} \int D\phi \int Db \; e^{\tilde{\mathcal{L}}_V[\mathbf{m},\phi,b]}$, with

$$\begin{aligned}
\tilde{\mathcal{L}}_V &= -\frac{1}{2} \sum_{\mathbf{r},\alpha\neq 0} K^\alpha_i (\partial_\alpha \phi_{\mathbf{r}} - A^\alpha_i - 2\pi m^\alpha_{\mathbf{r}})^2 \\
&+ i \sum_{\mathbf{r}} b_{\mathbf{r}} (\partial_\tau \phi_{\mathbf{r}} - 2\pi m^0_{\mathbf{r}})
\end{aligned}$$

$$- \frac{1}{2}\sum_{i,j} V_{ij} b_{i\tau} b_{j\tau} + \sum_{\mathbf{r}} \nu_i b_{i\tau}$$
$$- \frac{1}{2}\sum_{\mathbf{R},\alpha\neq 0} \frac{(q^\alpha_{\mathbf{R}})^2}{\kappa^\alpha_I}. \tag{B.28}$$

Now introduce the two component dual lattice ficticious vector potential, $\mathbf{a_R} = (a^x_{\mathbf{R}}, a^y_{\mathbf{R}})$ such that $b_{\mathbf{r}} = (\nabla \times \mathbf{a})_{\mathbf{r}}$. Let us also now identify the vortex part of the vector field $\mathbf{m}$: choose a real lattice scalar field n in such a way that

$$\mathbf{m} = \nabla^{(3)} n + \mathbf{m}_V, \;\; \nabla \cdot (\mathbf{K}\mathbf{m}^\perp_V) = 0, \tag{B.29}$$

where $(\mathbf{K}\mathbf{m}^\perp_V)_{\mathbf{r}} \equiv (K^x_i m^x_{V,\mathbf{r}}, K^y_i m^y_{V,\mathbf{r}})$ is the spatial part of $\mathbf{m}_V$, suitably scaled by the nearest neighbor exchange constants. This amounts to choosing a kind of rescaled Coulomb gauge for $\mathbf{m}_V$. The reason for this choice will become evident below. From (B.24) we have

$$(\nabla \times \mathbf{m})^\perp_{\mathbf{R}} = (\nabla \times \mathbf{m}_V)^\perp_{\mathbf{R}} = q^0_{\mathbf{R}}. \tag{B.30}$$

We may then write

$$(\mathbf{K}\mathbf{m}^\perp_V)_{\mathbf{r}} = (\nabla \times S)_{\mathbf{r}}, \;\; -\nabla \cdot ({}^*\mathbf{K}^{-1}\nabla S)_{\mathbf{R}} = q^0_{\mathbf{R}}, \tag{B.31}$$

where $({}^*\mathbf{K}^{-1}\nabla S)_{\mathbf{R}} \equiv [\frac{1}{K^y_{i+\hat{\mathbf{x}}}}(S_{\mathbf{R}+\hat{\mathbf{x}}} - S_{\mathbf{R}}), \frac{1}{K^x_{i+\hat{\mathbf{y}}}}(S_{\mathbf{R}+\hat{\mathbf{y}}} - S_{\mathbf{R}})$ so that on the dual lattice the divergence is scaled by the inverses of the exchange constants, and with x and y interchanged. Equivalently, we define the dual lattice exchange constants ${}^*\mathbf{K}_I \equiv ({}^*K^x_I, {}^*K^y_I) = (K^y_{i+\hat{\mathbf{x}}}, K^x_{i+\hat{\mathbf{y}}})$ to be those on the direct lattice bonds that *cut* the given dual lattice bonds. This implies that

$$S_{I\tau} = \sum_J {}^*G_{IJ} q^0_{J\tau}; \;\; -\nabla \cdot ({}^*\mathbf{K}^{-1}\nabla){}^*G_{IJ} = \delta_{IJ}, \tag{B.32}$$

where ${}^*G_{IJ}$ is the discrete *dual lattice* two-dimensional Coulomb-type potential, normalized so that ${}^*G_{00} = 0$, say.

It is clear that neither n nor $\mathbf{m}_V$ will be strictly integer fields. However, for two realizations, $\mathbf{m}_1$ and $\mathbf{m}_2$ of $\mathbf{m}$, with the same $\mathbf{m}_V$, the fact that $\mathbf{m}$ is an integer field requires that $\mathbf{m}_1 - \mathbf{m}_2 = \nabla(n_1 - n_2)$ be an integer field. Therefore $(n_1 - n_2)$ can always be chosen integer. As a consequence, for a given $\mathbf{m}_V$ we may alway write $n = n_0 + \delta n_V$, where δn_V

is fixed, equal to the fractional part of $\mathbf{m}-\mathbf{m}_V$ for example, and n_0 runs over all integer fields. It is precisely this freedom in ∇n that guarantees 2π periodicity in the $\phi_{\mathbf{r}}$. The trace over $\mathbf{m}$ is now equivalent to a free trace over n_0 and over $\mathbf{q}$ with the constraint (B.26).

Since ϕ and n_0 occur only in the combination $\phi - 2\pi n_0$ we may combine the two into a single trace over an unbounded field $-\infty < \theta = \phi - 2\pi n < \infty$. Note that we have absorbed δn into θ as well which, for fixed $\mathbf{q}$, amounts to a trivial shift in the integration variable. By the same device we may assume that

$$\begin{aligned}(\nabla\cdot\mathbf{KA})_i &\equiv (K_i^x A_i^x - K_{i-\hat{\mathbf{x}}}^x A_{i-\hat{\mathbf{x}}}^x)\\ &+ (K_i^y A_i^y - K_{i-\hat{\mathbf{y}}}^y A_{i-\hat{\mathbf{y}}}^y) = 0,\end{aligned} \tag{B.33}$$

since any perfect gradient subtracted from $\mathbf{A}$ may also be absorbed into θ. This implies that

$$(\mathbf{KA})_i = (\nabla\times\Phi)_i;\ \Phi_I = \sum_J {}^*G_{IJ}B_J, \tag{B.34}$$

where Φ_I is then the static magnetic potential due to B_I.

The partition function is now $Z = \sum_{\mathbf{q}}' \int Db \int D\theta\ e^{\tilde{\mathcal{L}}_V}$ [the prime on the sum indicating the constraint (B.26)] with

$$\begin{aligned}\tilde{\mathcal{L}}_V &= -\frac{1}{2}\sum_{\mathbf{r},\alpha\neq 0} K_i^\alpha(\theta_{\mathbf{r}+\hat{\alpha}}-\theta_{\mathbf{r}})^2\\ &- \frac{1}{2}\sum_{\mathbf{r},\alpha\neq 0} K_i^\alpha(A_i^\alpha + 2\pi m_{V,\mathbf{r}}^\alpha)^2\\ &+ \sum_{\mathbf{r},\alpha\neq 0} K_i^\alpha(A_i^\alpha + 2\pi m_{V,\mathbf{r}}^\alpha)(\theta_{\mathbf{r}+\hat{\alpha}}-\theta_{\mathbf{r}})\\ &+ i\sum_{\mathbf{r}} b_{\mathbf{r}}(\theta_{\mathbf{r}+\hat{\tau}}-\theta_{\mathbf{r}}) - 2\pi i\sum_{\mathbf{r}} b_{i\tau}m_{\mathbf{r}}^0\\ &- \frac{1}{2}\sum_{i,j} V_{ij}b_{i\tau}b_{j\tau} + \sum_{\mathbf{r}}\nu_i b_{\mathbf{r}} - \sum_{\mathbf{R},\alpha\neq 0}\frac{(q_{\mathbf{R}}^\alpha)^2}{\kappa_I^\alpha}.\end{aligned} \tag{B.35}$$

Now, the gauge choice (B.29) for $\mathbf{m}$ and $\mathbf{A}$ implies that the first term on the second line vanishes after an integration by parts. The first term on the third line may also be integrated by parts to yield

$$-2\pi i\sum_{i,\tau} b_{i\tau}m_{i\tau}^0 = -2\pi i\sum_{i,\tau}\mathbf{a}_{I\tau}\cdot[\mathbf{q}_{I\tau}^\perp - ({}^*\mathbf{K}^{-1}\nabla)\partial_\tau S]. \tag{B.36}$$

If we choose the gauge condition

$$\nabla \cdot ({}^*\mathbf{K}^{-1}\mathbf{a})_{\mathbf{R}} = 0, \tag{B.37}$$

implying

$$\begin{aligned} ({}^*\mathbf{K}^{-1}\mathbf{a})_{\mathbf{R}} &= (\nabla \times \Omega)_{\mathbf{R}}; \quad \Omega_{i\tau} = \sum_j G_{ij} b_{j\tau} \\ -\nabla \cdot (\mathbf{K}\nabla) G_{ij} &= \delta_{ij}, \end{aligned} \tag{B.38}$$

where G_{ij} is then the *direct lattice* Coulomb-type potential. An integration by parts then immediately shows that the second term vanishes. Furthermore, using (B.31) and (B.34) the second term in (B.35) may be integrated by parts to obtain

$$\begin{aligned} &\frac{1}{2} \sum_{\mathbf{r},\alpha\neq 0} K_i^\alpha (A_i^\alpha + 2\pi m_{V,\mathbf{r}}^\alpha)^2 \\ &\quad = 2\pi^2 \sum_{I,J,T} {}^*G_{IJ} \left(q_{IT}^0 + \frac{B_I}{2\pi} \right) \left(q_{JT}^0 + \frac{B_J}{2\pi} \right). \end{aligned} \tag{B.39}$$

Finally, since the θ-dependence is now purely Gaussian, we may integrate it out. Using the correlation function

$$\frac{1}{2} \langle (\theta_{i\tau} - \theta_{j\sigma})^2 \rangle_0 = G_{ij} \delta_{\tau\sigma}, \tag{B.40}$$

where the average is with respect to the very first term in (B.35) only, we obtain the partition function $Z = \sum_{\mathbf{q}}' \int Db \; e^{\mathcal{L}_C}$, where

$$\begin{aligned} \mathcal{L}_C &= 2\pi^2 \sum_{I,J,T} {}^*G_{IJ} \left(q_{IT}^0 + \frac{B_I}{2\pi} \right) \left(q_{JT}^0 + \frac{B_J}{2\pi} \right) \\ &+ \frac{1}{2} \sum_{i,j,\tau} G_{ij} (\partial_\tau b_{i\tau})(\partial_\tau b_{j\tau}) \\ &+ 2\pi i \sum_{\mathbf{R}} \mathbf{a}_{\mathbf{R}} \cdot \mathbf{q}_{\mathbf{R}}^{\perp} - \frac{1}{2} \sum_{i,j} V_{ij} b_{i\tau} b_{j\tau} + \sum_{\mathbf{r}} \nu_i b_{\mathbf{r}} \\ &- \frac{1}{2} \sum_{\mathbf{R},\alpha\neq 0} \frac{(q_{\mathbf{R}}^\alpha)^2}{\kappa_I^\alpha}. \end{aligned} \tag{B.41}$$

Note that the second term may also be written

$$\frac{1}{2}\sum_{i,j,\tau} G_{ij}(\partial_\tau b_{i\tau})(\partial_\tau b_{j\tau}) = -\frac{1}{2}\sum_{\mathbf{R},\alpha\neq 0} \frac{1}{{}^*K_I^\alpha}(\partial_\tau a_{\mathbf{R}}^\alpha)^2. \tag{B.42}$$

Recalling now the form (B.10) for the original Villain model, we may now reverse the transformation (B.5) with $\mathbf{q}$ in place of $\mathbf{n}$. Introducing a new vortex phase field, $0 \leq \varphi_{I\tau} < 2\pi$, we have $Z = \sum_{\mathbf{l}} \int D\varphi \int Db\; e^{\mathcal{L}_D}$, where the dual Villain Lagrangian is

$$\begin{aligned}
\mathcal{L}_D &= -\frac{1}{2}\sum_{\mathbf{R},\alpha\neq 0} \kappa_I^\alpha(\partial_\alpha\varphi_{\mathbf{R}} - 2\pi a_{\mathbf{R}}^\alpha - 2\pi l_{\mathbf{R}}^\alpha)^2 \\
&- \frac{1}{2}\sum_{\mathbf{R}} \frac{1}{(2\pi)^{2*}K_I^\alpha}[\partial_\alpha(\partial_\tau\varphi_{\mathbf{R}} - 2\pi i\Phi_I - 2\pi l_{\mathbf{R}}^0)]^2 \\
&- \frac{1}{2}\sum_{I,\alpha\neq 0} K_i^\alpha (A_i^\alpha)^2 - \frac{1}{2}\sum_{\mathbf{R},\alpha\neq 0}\frac{1}{{}^*K_I^\alpha}(\partial_\tau a_{\mathbf{R}}^\alpha)^2 \\
&- \frac{1}{2}\sum_{i,j} V_{ij} b_{i\tau} b_{j\tau} + \sum_{i,\tau}\nu_i b_{\mathbf{r}}.
\end{aligned} \tag{B.43}$$

Finally, we recognize this as the discrete time Villain form of the *quantum vortex Hamiltonian*

$$\begin{aligned}
\mathcal{H}_V &= 2\pi^2\sum_{I,J} {}^*\Gamma_{IJ}\left(\hat{N}_I + \frac{B_I}{2\pi}\right)\left(\hat{N}_J + \frac{B_J}{2\pi}\right) \\
&- \sum_{I,\alpha\neq 0} \mathcal{J}_I^\alpha cos(\partial_\alpha\hat{\varphi}_I - \hat{a}_I^\alpha) \\
&+ \frac{1}{2}\sum_{i,j} U_{ij}(\nabla\times\hat{\mathbf{a}})_i(\nabla\times\hat{\mathbf{a}})_j - \sum_i \mu_i(\nabla\times\hat{\mathbf{a}})_i \\
&+ \sum_{I,\alpha\neq 0}\frac{(\Pi_I^{(a)\alpha})^2}{2M_I^\alpha},
\end{aligned} \tag{B.44}$$

where $\hat{N}_I$ is the number operator conjugate to the phase operator $\hat{\varphi}_i$ and $\vec{\Pi}^{(a)}$ is the momentum conjugate to the quantum operator $\mathbf{a}$. For small $\Delta\tau$ we have the identifications (see (B.3) and below)

$$\begin{aligned}
{}^*G_{IJ} &= {}^*\Gamma_{IJ}\Delta\tau \\
K_I^\alpha &= M_I^\alpha\Delta\tau \\
\kappa_I^\alpha &= -1/2ln(\mathcal{J}_I^\alpha\Delta\tau).
\end{aligned} \tag{B.45}$$

A less complete version of this duality transformation was first written down in [5](c). Notice that the vortex hopping term arose from the extra κ_I^α term in (B.23) that does not actually appear in the original Hamiltonian. Notice also that the electomagnetic vector potential which was quenched in the original model now fluctuates. However these fluctuations are about a quenched random mean determined by the classical minimum of the first two terms on the second line, and therefore are not expected to constitute a significant difference between the two models. Note finally that the $\Delta\tau \to 0$ limit is incompatible with that for the original model, (2.2). In particular, for fixed $\tilde{J}_{ij}$ we saw from (B.3) that K_I^α vanishes logarithmically with $\Delta\tau$, implying then that M_I^α diverges as $-1/2\Delta\tau ln(\tilde{J}_{ij}\Delta\tau)$: the vector potential fluctuations become extremely massive. More importantly, from (B.32) $^*G_{IJ}$ scales linearly with the K_I^α, implying that $^*\Gamma_{IJ}$ diverges in the same logarithmic fashion: the interaction potential between vortices becomes very strong. The duality transformation is therefore not very clean, and is only intuitively well defined in discrete time. This, however, is not expected to affect the universal properties of the phase transitions governed by the two Hamiltonians. In particular, when the original interaction U_{ij} is logarithmic, the model is approximately self dual, and the universal features are expected to be identical, leading to certain exact predictions in this case [5](c).

Finally, we note that other more space-time symmetric dual models, involving vortex loops that interact via a Biot-Savart-type law, may also be obtained by using other gauge choices and starting with a slightly different version of the Villain model. This description is well-known for the classical three-dimensional XY-model. However this dual description has no natural correspondence with any dual quantum mechanical Hamiltonian.

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