HOMOMORPHISMS OF A MODULAR LATTICE

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ABSTRACT

This thesis is an algebraic study of the irreducible ideals of a modular lattice and their application to the characterization of the homomorphisms or congruence relations of the lattice.

First, an arithmetic characterization of modularity is given in terms of the irreducible ideals of the lattice. This is a new structure result for modular lattices which, since it characterizes general modular lattices, is more fundamental in the structure of modular lattices than the Kurosh-Ore Theorem.

Second, through the arithmetic characterization developed, subsets of the irreducible ideals are used to define congruence relations on the lattice and its lattice of ideals. It is then shown that every congruence relation on a modular lattice can be so characterized.

In conclusion, a generalization of the theorem that the congruence relations of a finite dimensional modular lattice form a Boolean algebra is given by proving that the congruence relations on the lattice of ideals of a modular lattice form a Boolean algebra if and only if the lattice is finite dimensional.

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1. Lattice Theory Foundations

In the following paragraphs will be compiled the essentials of lattice theory as they relate to the results to be presented in this manuscript. The terminology and notation are essentially those of Garrett Birkhoff (1).

A partially ordered set is a set P and a binary relation $\leq$ on the elements of P satisfying the postulates

P1: $a \leq a$ for all a in P,

P2: $a \leq b$ and $b \leq c$ imply that $a \leq c$.

For a and b in P, $a = b$ implies and is implied by $a \leq b$ and $b \leq a$. $a < b$ shall mean $a \leq b$ but not $b = a$. An element c in P is said to be the union or join of a subset X of P, when it exists, if $x \leq c$ for each x in X and if $x \leq d$ for each x in X implies $c \leq d$. An element c in P is said to be the intersection or meet of a subset X of P, when it exists, if $c \leq x$ for each x in X and if $d \leq x$ for each x in X implies $d \leq c$.

A lattice is a partially ordered set L in which each two elements have a union and an intersection in L. The union of a and b is designated $a \cup b$ and the intersection, $a \cap b$. The unions and intersections satisfy the following identities

L1: $a \cap a = a \cup a = a$,

L2: $a \cap b = b \cap a$ and $a \cup b = b \cup a$,

L3: $a \cap (b \cap c) = (a \cap b) \cap c$ and $a \cup (b \cup c) = (a \cup b) \cup c$,

L4: $a \cap (a \cup b) = a$ and $a \cup (a \cap b) = a$.

A lattice is said to be complete if every non-empty set has a union and an intersection. An element I is said to be the unit element of the lattice L if it is the union of all the elements of L. Dually, an element 0 is said to be the null element of the lattice if it is the intersection of all elements of the lattice. A lattice with unit and null element is said to be complemented if for each a in L, there exists an a' such that $a \cap a' = 0$ and $a \cup a' = I$. In a lattice an element a is said to cover an element b ($a \succ b$) if $a \geq p > b$ implies $p = a$.

A modular lattice is a lattice L which satisfies the postulate

M: $b \leq a$ implies $a \cap (b \cup c) = (a \cap c) \cup b$ for all c in L.

In a modular lattice $a \cup b \succ b$ implies $a \succ a \cap b$ and conversely.

A distributive lattice is a lattice L which satisfies the postulates

D_1: $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ and

D_2: $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$

for all a, b, and c in L. It is immediate that every distributive lattice is modular, but the converse is false. A Boolean algebra is a complemented, distributive lattice.

A single-valued mapping Θ from a lattice L onto a lattice L' is a homomorphism, if for every a and b in L the two relations

$$\Theta(a \cup b) = \Theta(a) \cup \Theta(b) \text{ and}$$

$$\Theta(a \cap b) = \Theta(a) \cap \Theta(b)$$

are valid. If the mapping is one-to-one, the homomorphism is referred to as an isomorphism.

A congruence relation Θ on a lattice L is an equivalence relation on L with the properties that $a = b$ in L implies $a \equiv b(\Theta)$ and that $a \equiv b(\Theta)$ implies $a \cap c \equiv b \cap c(\Theta)$ and $a \cup c \equiv b \cup c(\Theta)$ for all c in L. The notion of a homomorphism and that of a congruence relation are equivalent in that there exists a one-to-one correspondence between them. If θ is a homomorphism of L onto a lattice L', then the relation $a \equiv b(\theta)$ defined by $a \equiv b(\theta)$, if and only if $\theta(a) = \theta(b)$, is a congruence relation on L. If θ is a congruence relation and a_θ, the equivalence class generated by element a of L, the set of all equivalence classes forms a lattice, and the mapping $\theta(a) \equiv a_\theta$ is a homomorphism of L. For convenience in this manuscript the congruence notation will be utilized. The set of all congruence relations on a lattice L forms a complete, distributive lattice $(\Theta(L))$ under the partial order $\varphi \leqslant \theta$, if and only if $a \equiv b(\varphi)$ implies $a \equiv b(\theta)$. If Γ is a non-void subset of $\Theta(L)$, $a \equiv b(\cap \Gamma)$ is characterized by $a \equiv b(\theta)$ for every θ in Γ, and $a \equiv b(\cup \Gamma)$ is characterized by the existence of x_1, x_2, ..., x_{k+1} in L and θ_i in Γ for $i = 1, 2, \ldots, k$ such that $x_1 = a$, $x_{k+1} = b$, and $x_i \equiv x_{i+1}(\theta_i)$. The 0 and I of $\Theta(L)$ are characterized as $a \equiv b(0)$ if and only if $a = b$ and $a \equiv b(I)$ for all a, b in L.

A subset A of a lattice is said to be an ideal if it has the property that $(a \cup b) \cap c$ is in A for every a and b in A, and c in L. A set with the dual property is a dual ideal. The set of all ideals of a lattice L forms a complete lattice, denoted L_σ, in which the order $\subseteq$ is set inclusion ($\subset$, proper inclusion). As a result of this order the intersection of two ideals is the set intersection of the ideals; the union of two ideals, A and B, is the set of all x in L such that there exist a in A and b in B with $x \leqslant a \cup b$; and the union of a chain of ideals is the set of all elements which are in one of the members of the chain. The set $(a) \equiv \{x \text{ in } L \mid x \leqslant a\}$ is an ideal called a principal ideal. The set of all principal ideals of L forms a sublattice of L_σ which is isomorphic to the lattice L; in such a way any lattice can be embedded in its lattice of ideals. Further, L_σ has been shown to be modular or distributive as the lattice L is modular or distributive.

2. Irreducible Ideals of a Modular Lattice

Irreducible ideals in one form or another have played a significant role in the structure of special types of modular lattices. The fact that the irreducible ideals of a distributive lattice are prime ideals has enabled Stone to study these lattices as topological spaces with the set of all prime ideals being the underlying space (2). By showing that the maximal ideals satisfy the axioms of a projective geometry, Frink was able to study complemented modular lattices (3). A successful study along the same lines for a modular lattice has not yet been conducted; nor has the structure of modular lattices been discovered. However, the results of this section will display an arithmetical property of the set of irreducible and completely irreducible ideals which characterizes the modularity of a lattice. Since this property characterizes modularity without restriction, it is the most general structure result now known for modular lattices.

An ideal P is said to be a proper ideal if it is not the ideal consisting of all of the elements of the lattice. An ideal P is said to be irreducible if it is a proper ideal such that when $P = A \cap B$ in L_σ, then either $P = A$ or $P = B$. An ideal P is completely irreducible if P is a proper ideal such that when $P = \bigcap_{\alpha \in \Gamma} Q_\alpha$, Q_α in L_σ, then there exists α_0 in Γ such that $P = Q_{\alpha_0}$. $\underline{J}$ denotes the set of irreducible ideals of

L and J*, the set of completely irreducible ideals of L. Since every completely irreducible ideal is an irreducible ideal, then $J^* \subseteq J$.

An ideal P is a prime ideal if when $a \cap b$ is in P then either a is in P or b is in P. This definition is equivalent to the stronger statement that $A \cap B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$. A maximal ideal is a proper ideal which is contained in no other proper ideal. Immediately every proper prime ideal is an irreducible ideal and every maximal ideal is a completely irreducible ideal.

The existence of irreducible and completely irreducible ideals is guaranteed by the following lemma of Stone (2).

Lemma S: If V is a non-empty subset of L closed under intersections and A is an ideal of L such that $A \wedge V = \emptyset$; then

(1) there exists P in J such that $P \supseteq A$, $P \wedge V = \emptyset$, and $P' \supset P$ implies $P' \wedge V \neq \emptyset$;

(2) if $\bigcap_{x \in V} V$ is in V then P is in J*.

Proof: (1) Let $\mathfrak{a} \equiv \{B \text{ in } L_r \mid B \supseteq A \text{ and } B \wedge V = \emptyset\}$ and consider as a partially ordered set under set inclusion. If $\mathfrak{C}$ is a chain in $\mathfrak{a}$ and $C \equiv \bigcup \{B \mid B \text{ in } \mathfrak{C}\}$, then $C = \{x \text{ in } L \mid x \text{ in some } B \text{ of } \mathfrak{C}\}$. Hence $C \wedge V = \emptyset$ and C is in $\mathfrak{a}$. $\mathfrak{a}$ is non-empty since A is in $\mathfrak{a}$ and further every chain in $\mathfrak{a}$ has an upper bound in $\mathfrak{a}$; therefore, by the maximal principle there exist maximal elements of $\mathfrak{a}$. If P is a maximal element of $\mathfrak{a}$ and $P' \supset P$, then $P' \wedge V \neq \emptyset$. P is a proper ideal since V is non-empty. If $P = Q \cap R$, $P \subset Q$, and $P \subset R$; there exist q in Q and r in R such that both are in V. Then $q \cap r$ is in P and V contradicting their void intersection. Therefore either $P = Q$ or $P = R$ and P is in J.

(2) Suppose $P = \bigcap_{\alpha} Q_{\alpha}$ and each $Q_{\alpha} \supset P$. Then $Q_{\alpha} \wedge V \neq \emptyset$ and $\bigcap_{x \in V} V$ is

in Q_α, hence, in $\bigcap_\alpha Q_\alpha = P$, contrary to the assumption that $\bigcap_{x \in V} V$ is in V and $P \wedge V = \emptyset$. Therefore $P = Q_\alpha$ for some α and P is in J*.

Corollary: Every proper ideal of L is an intersection of completely irreducible ideals of L.

Proof: If A is proper, there exists x not in A, and Lemma S implies there exists P in J* such that $P \supseteq A$. Therefore let A' equal $\bigcap \{P \in J^* \mid P \supseteq A\}$, then $A \subseteq A'$. If x is in A' but not in A, then from Lemma S there exists Q in J* such that $Q \supseteq A$ with Q not containing x. Then x is not in A', contrary to assumption. Therefore x in A' implies x is in A and therefore $A' = A$.

In general $J^* \subseteq J$; however, there are cases in which $J^* = J$. In particular the following lemma shows that this condition is true for a finite lattice.

Lemma F: If L is finite dimensional, $J = J^*$.

Proof: Since L satisfies the ascending chain condition, every ideal is a principal ideal and L_σ is isomorphic to L. Therefore, since L satisfies the descending chain condition, L_σ satisfies the descending chain condition. Then let P be in J and $P = \bigcap_\alpha Q_\alpha$. Since L_σ satisfies the descending chain condition, the arbitrary intersection may be replaced by a finite one. Hence $P = Q_\alpha \cap Q_\beta \cap \dots \cap Q_\gamma$, and by a finite number of applications of the irreducibility condition $P = Q_\nu$ for some ν. Thus P is completely irreducible and in J*. Therefore $J = J^*$.

Lemma 1: P in J* implies there exists a unique ideal P* such that $P^* \succ P$.

Proof: Define $P^* = \bigcap \{Q \text{ in } L_\sigma \mid Q \supset P\}$. P* is well defined for P is a proper ideal which implies the set is not empty and the intersection exists since L_σ is a complete lattice. Since each $Q \supset P$, then $P^* \supseteq P$; however since P is completely irreducible and each $Q \supset P$, must $P^* \supset P$. Now if $P^* \supseteq K \supset P$, $K \supset P$ implies that K is in the set defining P^* and consequently that $K \supseteq P^*$. Hence $K = P^*$, which implies that $P^* \succ P$. To show that P^* is unique, let $Q \succ P$; then $Q \supset P$; hence $Q \supseteq P^*$; and $Q \supseteq P^* \supset P$, which by definition implies that $Q = P^*$.

<u>Lemma</u> 2: If P is in L_σ and $\{Q_\alpha \mid \alpha \text{ in } \Gamma\}$ is a chain in L_σ, then $P \cap \bigcup_{\alpha\in\Gamma} Q_\alpha = \bigcup_{\alpha\in\Gamma}(P \cap Q_\alpha)$.

<u>Proof</u>: If x is in $P \cap \bigcup_{\alpha\in\Gamma} Q_\alpha$, then x is in P and x is in $\bigcup_{\alpha\in\Gamma} Q_\alpha$. Since the set of Q_α forms a chain, x is in Q_α for some α in Γ. Therefore x is in $P \cap Q_\alpha$ for some α and hence in $\bigcup_{\alpha\in\Gamma}(P \cap Q_\alpha)$. Consequently $P \cap \bigcup_{\alpha\in\Gamma} Q_\alpha \subseteq \bigcup_{\alpha\in\Gamma}(P \cap Q_\alpha)$; however in a complete lattice the converse is always true. Therefore $P \cap \bigcup_{\alpha\in\Gamma} Q_\alpha = \bigcup_{\alpha\in\Gamma}(P \cap Q_\alpha)$.

<u>Lemma</u> 3: If $\{Q_\alpha\}$ and $\{R_\alpha\}$ are chains in L_σ with the same index set Γ and are related such that $Q_\alpha \subseteq Q_\beta$, if and only if $R_\alpha \subseteq R_\beta$, then $\bigcup_{\alpha\in\Gamma}(Q_\alpha \cap R_\alpha) = (\bigcup_{\alpha\in\Gamma} Q_\alpha) \cap (\bigcup_{\alpha\in\Gamma} R_\alpha)$.

<u>Proof</u>: If x is in $(\bigcup_{\alpha\in\Gamma} Q_\alpha) \cap (\bigcup_{\alpha\in\Gamma} R_\alpha)$, then x is $\bigcup_{\alpha\in\Gamma} Q_\alpha$ and $\bigcup_{\alpha\in\Gamma} R_\alpha$. Since $\{Q_\alpha\}$ and $\{R_\alpha\}$ are chains, there exist α and β in Γ such that x is in Q_α and R_β. If x is in Q_β, then x is in $Q_\beta \cap R_\beta$ and hence in $\bigcup_{\alpha\in\Gamma}(Q_\alpha \cap R_\alpha)$. If x is not in Q_β, x in Q_α implies $Q_\alpha \nsubseteq Q_\beta$; and since $\{Q_\alpha\}$ is a chain then $Q_\beta \subseteq Q_\alpha$. Consequently from the hypothesis of the lemma $R_\beta \subseteq R_\alpha$; and therefore x is in R_α, $Q_\alpha \cap R_\alpha$, and $\bigcup_{\alpha\in\Gamma}(Q_\alpha \cap R_\alpha)$. Hence $(\bigcup_{\alpha\in\Gamma} Q_\alpha) \cap (\bigcup_{\alpha\in\Gamma} R_\alpha) \subseteq \bigcup_{\alpha\in\Gamma}(Q_\alpha \cap R_\alpha)$; however the converse inequality is immediately true; and therefore $\bigcup_{\alpha\in\Gamma}(Q_\alpha \cap R_\alpha) = (\bigcup_{\alpha\in\Gamma} Q_\alpha) \cap (\bigcup_{\alpha\in\Gamma} R_\alpha)$.

The previous results are true for any lattice; however they are applied in that which follows to modular lattices. <u>It is assumed in</u>

<u>all that follows that L is a modular lattice.</u>

<u>Lemma 4</u>: If P, A, and B in L_σ are such that $A \cap B \subseteq P$, then there exist in L_σ $Q \supseteq A$ and $R \supseteq B$ such that $P \cap Q = P \cap R = Q \cap R$.

<u>Proof</u>: Define $Q = A \cup (P \cap B)$ and $R = B \cup (P \cap A)$; then $Q \cap R =$ $[A \cup (P \cap B)] \cap [B \cup (P \cap A)] = (P \cap A) \cup \{B \cap [A \cup (P \cap B)]\} =$ $(P \cap A) \cup \{(B \cap A) \cup (P \cap B)\} = (P \cap B) \cup (P \cap A) = P \cap [B \cup (P \cap A)] =$ $P \cap [A \cup (P \cap B)]$. Therefore $Q \cap P = P \cap R = Q \cap R$.

<u>Theorem 1</u>: If P is in J, and A and B in L_σ such that $A \nsubseteq P$, $B \nsubseteq P$, and $A \cap B \subseteq P$; then there exist A' and B' in J such that $A' \supseteq A$, $B' \supseteq B$, and $A' \cap B' = A' \cap P = B' \cap P$.

<u>Proof</u>: Define $\mathcal{C} \equiv \{(Q,R) \mid Q,R \in L \ni Q \supseteq A,\ R \supseteq B, \text{ and } Q \cap R = Q \cap P = R \cap P\}$ and define $(Q',R') \leq (Q'',R'')$ if $Q' \subseteq Q''$ and $R' \subseteq R''$. $\leq$ is immediately a partial order on $\mathcal{C}$, and $A \cap B \subseteq P$ plus Lemma 4 implies that $\mathcal{C}$ is not empty. Let $C \equiv \{(Q_\alpha, R_\alpha) \mid \alpha \text{ in } \Gamma\}$ be a chain of $\mathcal{C}$ and define $Q \equiv \bigcup_{\alpha \in \Gamma} Q_\alpha$ and $R \equiv \bigcup_{\alpha \in \Gamma} R_\alpha$. Obviously $Q \supseteq A$ and $R \supseteq B$. $\{Q_\alpha\}$ and $\{R_\alpha\}$ are chains of L_σ satisfying the hypothesis of Lemma 3; therefore $(\bigcup_{\alpha \in \Gamma} Q_\alpha) \cap (\bigcup_{\alpha \in \Gamma} R_\alpha) = \bigcup_{\alpha \in \Gamma}(Q_\alpha \cap R_\alpha) = \bigcup_{\alpha \in \Gamma}(Q_\alpha \cap P) = \bigcup_{\alpha \in \Gamma}(R_\alpha \cap P)$. Hence by applying Lemma 2, $Q \cap R = Q \cap P = R \cap P$; and (Q,R) is in $\mathcal{C}$. On the other hand $Q \supseteq Q_\alpha$ and $R \supseteq R_\alpha$ for each α in Γ imply $(Q,R) \geq (Q_\alpha, R_\alpha)$ for each α in Γ, and therefore (Q,R) is an upper bound for C in $\mathcal{C}$. Applying the maximal principle, there exist maximal elements in $\mathcal{C}$. Let (A',B') be a maximal element of $\mathcal{C}$, then A' and B' will be shown to have the property of the theorem. First, A' and B' are proper ideals. For if $B \subseteq A'$, then $B \subseteq B'$ and $A' \cap B' \subseteq P$ imply $B \subseteq P$, contrary to assumption. Therefore A' is proper and similarly B' is proper. Second, if $A' = Q_1 \cap Q_2$, it can be shown that $A' = Q_1$ or Q_2. For by computation

$[P \cup (B' \cap Q_1)] \cap [P \cup (B' \cap Q_2)] = P \cup \{Q_1 \cap B' \cap [P \cup (B' \cap Q_2)]\} =$

$P \cup \{Q_1 \cap [(B' \cap P) \cup (B' \cap Q_2)]\} = P \cup \{Q_1 \cap B' \cap Q_2\} = P \cup (B' \cap A') = P.$

Now P irreducible implies $P = P \cup (B' \cap Q_i)$, for $i = 1$ or 2. But $P = P \cup (B' \cap Q_i)$ implies $B' \cap Q_i \subseteq P$ and Lemma 4 implies that there exist $B'' \supseteq B'$ and $Q' \supseteq Q_i$ such that $B'' \cap Q' = B'' \cap P = Q' \cap P$. However $B \subseteq B' \subseteq B''$ and $A \subseteq A' \subseteq Q_i \subseteq Q'$ imply (Q', B'') is in $\mathfrak{C}$ and that $(Q', B'') \geqslant (A', B')$. Therefore the maximality of (A', B') in $\mathfrak{C}$ implies in part $A' = Q'$ and hence $A' = Q_i$. Similarly B' can be shown to have this property and the theorem is established.

Corollary 1: P in J^* implies A' and B' are in J^*.

Proof: A' and B' are proper ideals since they are irreducible; therefore it suffices to show that they possess the second property of completely irreducible ideals. Suppose $A' = \bigcap_{\alpha \in \Gamma} Q_\alpha$; then, since for each α $P \cup (B' \cap Q_\alpha) \supseteq P$, $\bigcap_{\alpha \in \Gamma} [P \cup (B' \cap Q_\alpha)] \supseteq P$. If $\bigcap_{\alpha \in \Gamma} [P \cup (B' \cap Q_\alpha)] \supset P$, then Lemma 1 implies $\bigcap_{\alpha \in \Gamma} [P \cup (B' \cap Q_\alpha)] \supseteq P^*$ or $P \cup (B' \cap Q_\alpha) \supseteq P^*$ for each α in Γ. Then $P^* = P^* \cap [P \cup (B' \cap Q_\alpha)] = P \cup (P^* \cap B' \cap Q_\alpha) \succ P$ for each α in Γ. Since L is a modular lattice then $P^* \cap B' \cap Q_\alpha \succ P \cap B' \cap Q_\alpha = P \cap B'$. However $B \nsubseteq P$ implies $B' \cup P \supset P$; hence $P^* = P^* \cap (P \cup B') = P \cup (P^* \cap B') \succ P$, which then implies $P^* \cap B' \succ P \cap B'$. But $P^* \cap B' \supseteq P^* \cap B' \cap Q_\alpha$; therefore $P^* \cap B' = P^* \cap B' \cap Q_\alpha$ and $Q_\alpha \supseteq P^* \cap B'$ for each α in Γ. Therefore $A' = \bigcap_{\alpha \in \Gamma} Q_\alpha \supseteq P^* \cap B'$; hence $A' \cap B' \supseteq P^* \cap B'$. However since $A' \cap B' = P \cap B'$, this is a contradiction of $P^* \cap B' \succ P \cap B'$. Therefore $\bigcap_{\alpha \in \Gamma} [P \cup (Q_\alpha \cap B')] = P$ and since P is completely irreducible this implies $P = P \cup (Q_\alpha \cap B')$ for some α in Γ. For this α, by the same argument as in the theorem, maximality of the pair (A', B') in $\mathfrak{C}$ implies $A' = Q_\alpha$. In a similar manner B' may be shown to be completely irreducible.

Corollary 2: If a and b are in L and P is an irreducible ideal of L, with $a \cap b$ in P and neither a nor b in P; then there exist irreducible ideals A' and B' of L such that a is in A', b is in B', and $A' \cap B' = A' \cap P = B' \cap P$.

Proof: $a \cap b$ in P implies $(a \cap b) \subseteq P$. a not in P, b not in P, imply $(a) \nsubseteq P$ and $(b) \nsubseteq P$. $(a \cap b) = (a) \cap (b)$ implies $(a) \cap (b) \subseteq P$; therefore applying Theorem 1, the result follows.

Corollary 3: A necessary and sufficient condition that a lattice L be modular is that every completely irreducible ideal P of L has the property that when $a \cap b$ is in P with a and b not in P, there exist completely irreducible ideals $A' \geq a$ and $B' \geq b$ such that $A' \cap B' = A' \cap P = B' \cap P$.

Proof: If L is modular, the condition follows from Corollaries 1 and 2. Conversely, suppose L satisfies the condition and is not modular; then there exist a, b, and c in L such that $a \geqslant b$ and $a \cap (b \cup c) > b \cup (a \cap c)$. Then the corollary to Lemma S implies there exists P in J* such that $b \cup (a \cap c)$ is in P but $a \cap (b \cup c)$ is not. Now $b \cup (a \cap c)$ in P implies $a \cap c$ in P, and $a \cap (b \cup c)$ not in P implies neither a nor c in P. Therefore the condition of the corollary implies that there exist A' and C', completely irreducible ideals of L, such that a is in A', c is in C', and $A' \cap C' = A' \cap P = C' \cap P$. Then $b \leqslant a$ implies b is in A'; b in P then implies b in $A' \cap P \subseteq C'$. Therefore $b \cup c$ is in C' and $a \cap (b \cup c)$ is in $A' \cap C' \subseteq P$, which is a contradiction. Therefore L must be modular.

It may be noted that similar results are valid for dual ideals from the dual nature of a lattice.

The relationship just proved for modular lattices implies that there are two possibilities for arithmetical properties of irreducible ideals of the lattice. For either the ideal is prime or there exist a and b not in the ideal with $a \cap b$ in the ideal. The above result shows what must happen in the latter situation. Though this relationship appears strange, it may be interpreted as one of a linear nature. In fact, it was just such a dual property, $P \cup Q = P \cup R = R \cup Q$, for maximal dual ideals that enabled Frink to define the notion of a line generated by two maximal dual ideals and thus to get a projective geometry out of a complemented modular lattice (3). This relationship will now be used to define a perspectivity and projectivity for irreducible ideals of a modular lattice.

3. Perspectivity and Projectivity in J

Two irreducible ideals P and Q of L are said to be perspective ($P \sim Q$) if

(1) $P = Q$ or

(2) $P \nsubseteq Q$ and $Q \nsubseteq P$, and there exists R in J such that $P \cap Q = P \cap R = R \cap Q$.

Two irreducible ideals P and Q of L are said to be projective ($P \approx Q$) if there exist $P_1, P_2, \ldots, P_{k+1}$ in J such that $P_1 = P$, $P_{k+1} = Q$, and $P_i \sim P_{i+1}$ for $i = 1, 2, \ldots, k$.

Corollary:

(1) $P \sim Q$ implies $P \approx Q$,

(2) $\sim$ is symmetric and reflexive, and

(3) $\simeq$ is an equivalence relation on J.

Proof: The results of this corollary are immediate consequences of the definitions.

Lemma 5: J* is closed under projectivity.

Proof: It is necessary to show that P in J* and $P \simeq Q$ imply that Q is in J*. It suffices to consider $P \sim Q$ and $P \neq Q$. Therefore $P \nleq Q$, $Q \nleq P$; and there exists R in J such that $P \cap Q = P \cap R = Q \cap R$. If $R \leq P$, then $P \cap Q = R \cap P = R$; but R in J implies $R = Q$ or $R = P$. $R = Q$ and $R \leq P$ contradict $Q \nleq P$. $R = P$ and $P \cap Q = R$ imply $Q \geq P$ which is contrary to $P \nleq Q$; thus $R \nleq P$. Therefore Q, R, and P satisfy the hypothesis of Theorem 1 with $Q \cap R \leq P$, $Q \nleq P$ and $R \nleq P$. Consequently, by Corollary 1 of the theorem, there exist Q' and R' in J* such that $R \leq R'$, $Q \leq Q'$, and $R' \cap Q' = P \cap R' = P \cap Q'$. Then $P \cap R = P \cap R' \cap R = Q' \cap R' \cap R = Q' \cap R$, but $Q \geq P \cap R = Q' \cap R$ implies $Q = Q \cup (Q' \cap R) = Q' \cap (R \cup Q)$. If $Q = R \cup Q$, then $R \leq Q$; but just as $R \nleq P$, $R \nleq Q$. Hence $Q \neq Q \cup R$ and the irreducibility of Q implies $Q = Q'$; thus Q is in J*.

Theorem 2: If N is a subset of J, define for A and B in L_σ

$A \equiv B(N)$ if when P is in J such that $P \geq A \cap B$ and

$P \ngeq A \cup B$, then $P \simeq Q$ for some Q in N;

then $A \equiv B(N)$ is a congruence relation on L_σ.

Proof: From the definition it is obvious that the relation is reflexive and symmetric. For transitivity, let $A \equiv B(N)$, $B \equiv C(N)$, $P \geq A \cap C$, and $P \ngeq A \cup C$ with P in J.

Case 1: Assume $P \geq B$. Then $P \geq B \cap A$ and $P \geq B \cap C$; however $P \ngeq A \cup C$ implies that either one or both of the relations $P \ngeq B \cup A$, $P \ngeq B \cup C$ hold. Since $A \equiv B(N)$ and $B \equiv C(N)$, in either eventuality there exists some Q in

N such that $P \approx Q$.

Case 2: Assume $P \not\geq B$ and that either $P \geq A$ or $P \geq C$. Then either $P \geq A \cap B$ and $P \not\geq A \cup B$ or $P \geq C \cap B$ and $P \not\geq C \cup B$; $A \equiv B(N)$ and $B \equiv C(N)$ imply there exists some Q in N such that $P \approx Q$.

Case 3: Assume $P \not\geq B$ and that neither $A \subseteq P$ nor $C \subseteq P$. Since $A \nsubseteq P$, $C \nsubseteq P$, and $A \cap C \subseteq P$; Theorem 1 implies that there exist A' and C' in J such that $A \subseteq A'$, $C \subseteq C'$, and $A' \cap C' = A' \cap P = C' \cap P$. Then $B \nsubseteq P$ implies B is not contained in both A' and C'. Therefore either $A' \geq A \cap B$ and $A' \not\geq A \cup B$ or $C' \not\geq B \cup C$ and $C' \geq B \cap C$; since $A \equiv B(N)$ and $C \equiv B(N)$, either $A' \approx Q'$ with Q' in N or $C' \approx Q''$ with Q'' in N. However $A \nsubseteq P$ and $C \nsubseteq P$ imply $A' \nsubseteq P$ and $C' \nsubseteq P$. If $P \subseteq A'$ or $P \subseteq C'$, then $P = A' \cap C'$ which, since P is irreducible, implies $P = A'$ or $P = C'$, contradicting the assumption that neither A nor C is contained in P. Therefore $P \nsubseteq A'$, $P \nsubseteq C'$, $C' \nsubseteq P$, $A' \nsubseteq P$, and $A' \cap C' = P \cap C' = P \cap A'$ imply $P \sim A'$ and $P \sim C'$. Therefore $P \approx A'$ and $P \approx C'$; $P \approx Q'$ or $P \approx Q''$; and $P \approx Q$ for some Q in N. Since these cases are exhaustive, $A \equiv C(N)$. Therefore this relation is transitive and is an equivalence relation. To show that the relation is a congruence relation, it remains to show that unions and intersections are preserved.

Let $A \equiv B(N)$, P be an irreducible ideal, $P \geq (A \cup C) \cap (B \cup C)$, and $P \not\geq A \cup C \cup B$. Then since $(A \cup C) \cap (C \cup B)$ contains C and $A \cap B$, P contains C and $A \cap B$. Now $P \geq C$, $P \not\geq C \cup A \cup B$ imply $P \not\geq A \cup B$; therefore $A \equiv B(N)$ implies $P \approx Q$ for some Q in N. Hence $A \cup C \equiv B \cup C(N)$.

Let $A \equiv B(N)$, P be an irreducible ideal, $P \geq A \cap C \cap B$, and $P \not\geq (A \cap C) \cup (B \cap C)$. Since $C \geq (A \cap C) \cup (B \cap C)$ and $A \cup B \geq (A \cap C) \cup (B \cap C)$, $P \not\geq (A \cap C) \cup (B \cap C)$ implies $C \nsubseteq P$ and $A \cup B \nsubseteq P$. Therefore if $P \geq A \cap B$, $A \equiv B(N)$ would imply that there exists Q in N such that $P \approx Q$. If $A \cap B \nsubseteq P$, since

$C \not\subseteq P$ and $A \cap B \cap C \subseteq P$, Theorem 1 implies there exist C' and $(A \cap B)'$ in J such that $C \subseteq C'$, $A \cap B \subseteq (A \cap B)'$, and $(A \cap B)' \cap C' = C' \cap P = (A \cap B)' \cap P$. Since $C \cap (A \cup B) \supseteq (A \cap C) \cup (B \cap C)$, if $A \cup B$ is in $(A \cap B)'$, then $C \cap (A \cup B)$ would be contained in $C' \cap (A \cap B)'$ and hence in P, contradicting the fact that $(A \cap C) \cup (B \cap C) \not\subseteq P$. Therefore $A \cup B \not\subseteq (A \cap B)'$; then $A \equiv B(N)$ implies $(A \cap B)' \simeq Q$ for some Q in N. But $A \cap B \not\subseteq P$ implies $(A \cap B)' \not\subseteq P$. If $P \subseteq (A \cap B)'$, then $P = (A \cap B)' \cap C'$; however in view of $C \not\subseteq P$ and $A \cap B \not\subseteq P$, this would contradict the irreducibility of P. Therefore $P \not\subseteq (A \cap B)'$, $(A \cap B)' \not\subseteq P$, and $(A \cap B)' \cap C' = C' \cap P = (A \cap B)' \cap P$ imply $P \sim (A \cap B)'$. Since $(A \cap B)' \simeq Q$, then $P \simeq Q$ and therefore $A \cap C \equiv B \cap C(N)$.

Corollary: If N is a subset of J, for a and b in L define

$a \equiv b(N)$ if when P is in J such that $P \supseteq a \cap b$ and $P \not\supseteq a \cup b$, then $P \simeq Q$ for some Q in N;

then $a \equiv b(N)$ is a congruence relation on L.

Proof: Replacing the use of Theorem 1 by Corollary 2 to the theorem, the proof is the same as that of the theorem above.

4. Characterization of the Congruence Relations of L.

In the previous section it was shown that an arbitrary subset of irreducible ideals of a modular lattice was capable of defining a congruence relation on the lattice. It remains to be shown that every congruence relation on L can be so generated. This will be done by studying the relationship between the congruence relations on L and L_σ. There exist means of proceeding from a congruence relation on L to a congruence relation on L_σ and of proceeding from a congruence relation on L_σ to one on L.

Let θ be a congruence relation on L and define a congruence relation $\hat{\theta}$ on L_σ by

$A \equiv B(\hat{\theta})$ if for each a in A there exists b in B such that $a \equiv b(\theta)$, and conversely.

<u>Lemma 6</u>: The relation $A \equiv B(\hat{\theta})$ is a congruence relation on L_σ.

<u>Proof</u>: From the definition $A \equiv B(\hat{\theta})$ is immediately an equivalence relation on L_σ. Let $A \equiv B(\hat{\theta})$ and let a be in $A \cap C$, then since a is in A and $A \equiv B(\hat{\theta})$ there exists b in B such that $a \equiv b(\theta)$. Then $a \equiv a \cap b(\theta)$, and a in C implies $a \cap b$ is in $C \cap B$. The converse follows in the same way; hence $A \cap C \equiv B \cap C(\hat{\theta})$. Let p be in $A \cup C$, then there exist a in A and c in C such that $p \leq a \cup c$. a in A, $A \equiv B(\hat{\theta})$ imply there exist b in B such that $a \equiv b(\theta)$; hence $a \cup c \equiv b \cup c(\theta)$. Therefore $p \equiv p \cap (a \cup c) \equiv p \cap (b \cup c)(\theta)$ and $p \cap (b \cup c)$ is in $B \cup C$. Since the converse follows in the same way, $A \cup C \equiv B \cup C(\hat{\theta})$, and $A \equiv B(\hat{\theta})$ is a congruence relation on L_σ.

If θ is a congruence relation on L_σ, define for a and b in L

$a \equiv b(\breve{\theta})$ if and only if $(a) \equiv (b)(\theta)$.

<u>Lemma 7</u>: $a \equiv b(\breve{\theta})$ is a congruence relation on L.

<u>Proof</u>: $a \equiv b(\breve{\theta})$ is immediately an equivalence relation since θ is an equivalence relation on L_σ. Since $(x) \cap (y) = (x \cap y)$ and $(x) \cup (y) = (x \cup y)$, $a \equiv b(\breve{\theta})$ implies $(a \cap c) \equiv (a) \cap (c) \equiv (b) \cap (c) \equiv (b \cap c)(\theta)$ and similarly $(a \cup c) \equiv (b \cup c)(\theta)$. Hence $a \cap c \equiv b \cap c(\breve{\theta})$ and $a \cup c \equiv b \cup c(\breve{\theta})$; thus $a \equiv b(\breve{\theta})$ is a congruence relation on L.

<u>Lemma 8</u>: If A and B are principal ideals of L, $A \equiv B(\hat{\breve{\theta}})$ if and only if $A \equiv B(\theta)$.

<u>Proof</u>: Let $A = (x)$ and $B = (y)$, then $(x) \equiv A \equiv B \equiv (y)(\theta)$ implies

$x \equiv y(\breve{\theta})$. Let $a \leq x$, then $a \equiv a \cap x \equiv a \cap y(\breve{\theta})$ and $a \cap y$ is in (y). Similarly for b in (y) there exists a in (x) such that $b \equiv a(\breve{\theta})$ and therefore $(x) \equiv (y)(\hat{\breve{\theta}})$. If $(x) \equiv (y)(\hat{\breve{\theta}})$, since x is in (x) there exists b in (y) such that $x \equiv b(\breve{\theta})$; likewise there exists a in (x) such that $y \equiv a(\breve{\theta})$. But $a \leq x$ and $b \leq y$ imply $x \equiv a \cup x \equiv y \cup b \equiv y(\breve{\theta})$ and therefore $(x) \equiv (y)(\theta)$.

In order to characterize the congruence relations in terms of subsets of J a "collapsed" or "characteristic" subset is defined. Thus if θ is a congruence relation on L_σ, define

$$N(\theta) \equiv \{P \text{ in } J \mid \text{there exists } A \text{ in } L_\sigma \text{ such that } A \supset P \text{ and } A \equiv P(\theta)\},$$

$$E(\theta) \equiv \{P \text{ in } J \mid \text{if } A \supseteq P \text{ and } A \equiv P(\theta), \text{ then } A = P\}.$$

Immediately $N(\theta) \vee E(\theta) = J$ and $N(\theta) \wedge E(\theta) = \emptyset$, where $\vee$ and $\wedge$ denote set union and set intersection respectively in J.

<u>Lemma</u> 9: $N(\theta)$ is closed under projectivity.

<u>Proof</u>: It is necessary to show that $P \approx Q$ and P in $N(\theta)$ imply Q is in $N(\theta)$. It suffices to consider $P \sim Q$ and $P \neq Q$; then $P \nleq Q$, $Q \nleq P$, and there exists R in J such that $P \cap R = P \cap Q = Q \cap R$. Now P in $N(\theta)$ implies that there exists $A \supset P$ such that $A \equiv P(\theta)$. Define $A' = Q \cup (A \cap R)$, then $A' \supseteq Q$. If $A' = Q$, then $A \cap R \subseteq Q$; and $A \cap R \subseteq Q \cap R \subseteq P$. Therefore $P = (A \cap R) \cup P = A \cap (P \cup R)$; then P irreducible and $P \neq A$ imply $P = P \cup R$ or $R \subseteq P$. But this implies $R = P \cap Q$; and R, an irreducible ideal, implies $R = P$ or $R = Q$. Now $R = Q$ implies $Q \subseteq P$, and $R = P$ implies $P \subseteq Q$; each of which is a contradiction of the non-comparability of P and Q. Therefore $Q \neq A'$. $A \equiv P(\theta)$ implies $A \cap R \equiv P \cap R(\theta)$ and $Q \cup (A \cap R) \equiv Q \cup (Q \cap R)(\theta)$; therefore $A' \equiv Q(\theta)$ and Q is in $N(\theta)$.

<u>Lemma 10</u>: Let $\subseteq$ denote set inclusion among the subsets of J, then:

(1) $\varphi \leq \theta$ implies $N(\varphi) \subseteq N(\theta)$,

(2) $N(\varphi \cap \theta) = N(\varphi) \wedge N(\theta)$,

(3) $N(\varphi \cup \theta) = N(\varphi) \vee N(\theta)$,

(4) $N(\theta) = \emptyset$ if and only if $\theta = 0$, and

(5) $N(I) = J$.

<u>Proof</u>: (1) P in $N(\varphi)$ implies there exists $A \supset P$ such that $A \equiv P(\varphi)$. Then $\varphi \leq \theta$ implies $A \equiv P(\theta)$ and therefore P is in $N(\theta)$.

(2) From part (1) $N(\varphi \cap \theta) \subseteq N(\varphi) \wedge N(\theta)$. Let P be in $N(\varphi)$ and $N(\theta)$, then there exist A and A' such that $A \supset P$, $A' \supset P$, $A \equiv P(\varphi)$, and $A' \equiv P(\theta)$. Then $A \supseteq A \cap A' \supseteq P$ and $A \equiv P(\varphi)$ imply $A \cap A' \equiv P(\varphi)$, and likewise $A \cap A' \equiv P(\theta)$; therefore $A \cap A' \equiv P(\varphi \cap \theta)$. $A \supset P$ and $A' \supset P$ imply $A \cap A' \supseteq P$, and the irreducibility of P implies $P \neq A \cap A'$. Hence $A \cap A' \supset P$ and P is in $N(\varphi \cap \theta)$. Therefore $N(\varphi \cap \theta) = N(\varphi) \wedge N(\theta)$.

(3) Part (1) implies $N(\varphi \cup \theta) \supseteq N(\varphi) \vee N(\theta)$. Let P be in $N(\varphi \cup \theta)$, then there exists $A \supset P$ such that $A \equiv P(\varphi \cup \theta)$. Then from the nature of the union of two congruence relations there exist ideals $P_1, P_2, \ldots, P_{k+1}$ and γ_i in $\{\varphi, \theta\}$ for $i = 1, 2, \ldots, k$ such that $P_{k+1} = A$, $P_1 = P$, and $P_i \equiv P_{i+1}(\gamma_i)$. $P_{k+1} = A \supset P$ implies that not all $P_i \subseteq P$; therefore let j be the first index such that $P_j \nsubseteq P$. Since $P_1 = P$, then $j > 1$ and $P_{j-1} \subseteq P$. Then $P_{j-1} \equiv P_j(\gamma_{j-1})$ implies $P \equiv P \cup P_{j-1} \equiv P \cup P_j(\gamma_{j-1})$, and $P_j \nsubseteq P$ implies $P \cup P_j \supset P$. Therefore P in $N(\gamma_{j-1})$ and γ_{j-1} in $\{\varphi, \theta\}$ imply P is in $N(\varphi) \vee N(\theta)$. Hence $N(\varphi \cup \theta) = N(\varphi) \vee N(\theta)$.

(4) Since 0 is equality, it is immediate that there exists no P in $N(0)$ and hence that $N(0) = \emptyset$. Let $N(\theta) = \emptyset$ and $A \equiv B(\theta)$. Suppose $A \neq B$,

then the corollary of Lemma 8 implies there exists $P \in J^*$ such that $P \supseteq A \cap B$ and $P \not\supseteq B \cup A$. Therefore $P \equiv P \cup (A \cap B) \equiv P \cup A \cup B(\Theta)$ and P is in $N(\Theta)$ contrary to assumption. Therefore $A = B$ and $\Theta = 0$.

(5) Since I identifies everything, the result is obvious.

Lemma 11: If $N(\varphi \vee \Theta) = J$ and $\varphi \wedge \Theta = 0$, then $N(\Theta) = E(\varphi)$.

Proof: Let P be in $N(\Theta)$. Lemma 10 implies that $N(\varphi) \wedge N(\Theta) = N(\varphi \wedge \Theta) = N(0) = \emptyset$. Therefore P in $N(\Theta)$ implies P is not in $N(\varphi)$ and hence P is in $E(\varphi)$. Hence $N(\Theta) \subseteq E(\varphi)$. Let P be in $E(\varphi)$, then P is not in $N(\varphi)$. But P in $J = N(\varphi \vee \Theta) = N(\varphi) \vee N(\Theta)$, and hence in $N(\Theta)$. Therefore $N(\Theta) = E(\varphi)$.

Theorem 3: $a \equiv b(\Theta)$ if and only if $a \equiv b(\, N(\hat{\Theta})\,)$.

Proof: Let $a \equiv b(\Theta)$, $a \cap b$ be in P, and $a \cup b$ not be in P. Then $A \equiv (a \cup b) \cup P \supset P$. Now it can be shown that $A \equiv P(\hat{\Theta})$. For p in P there exists an element in A congruent to p, namely p itself suffices. For x in $A = (a \cup b) \cup P$, there exists p in P such that $x \leqslant (a \cup b) \cup p$. Then $y = x \cap [(a \cap b) \cup p]$ is in P and $y \equiv x \cap [(a \cap b) \cup p] \equiv x \cap [(a \cap a) \cup p] \equiv x \cap [(a \cup a) \cup p] \equiv x \cap [(a \cup b) \cup p] \equiv x(\Theta)$. Therefore $A \equiv P(\hat{\Theta})$, P is in $N(\hat{\Theta})$, and $a \equiv b(\, N(\hat{\Theta})\,)$.

Conversely, let $a \equiv b(\, N(\hat{\Theta})\,)$ and define

$$S = \{x \text{ in } L \mid x \cup (a \cap b) \equiv a \cap b(\Theta)\}.$$

S is an ideal of L, for let x and y be any two elements of S and w be any element of L, then $x \cup (a \cap b) \equiv a \cap b \equiv y \cup (a \cap b)(\Theta)$. Then computing, $[(x \cup y) \cap w] \cup (a \cap b) \equiv [(x \cup y) \cap w] \cup [x \cup (a \cap b)] \equiv [(x \cup y) \cap w] \cup [x \cup y \cup (a \cap b)] \equiv x \cup y \cup (a \cap b) \equiv x \cup (a \cap b) \equiv a \cap b(\Theta)$, and hence $(x \cup y) \cap w$ is in S. Further $(a \cap b) \cup (a \cap b) \equiv a \cap b(\Theta)$ implies that $a \cap b$ is in S. Define $V \equiv \{x \text{ in } L \mid x \equiv a \cup b(\Theta)\}$. $a \cup b$ in V implies $V \neq \emptyset$. Then V

is closed under intersections, for if x and y are in V, then $x \cap y \equiv (a \cup b) \cap (a \cup b) \equiv a \cup b(\Theta)$ implies $x \cap y$ is in V. If $S \wedge V = \emptyset$, then from Lemma 8 there exists an ideal $P \in J$, maximal under the conditions $P \supseteq S$ and $P \wedge V = \emptyset$. Since $a \cup b$ is in V, P is an irreducible ideal of L such that $a \cap b$ is in P and $a \cup b$ is not in P. But since $a \equiv b(\ N(\hat{\theta})\ )$, $P \approx Q$ for some Q in $N(\hat{\theta})$; and Lemma 9 implies that P is in $N(\hat{\theta})$. However if P is in $N(\hat{\theta})$, there exists $A \supset P$ such that $A \equiv P(\hat{\theta})$. $A \supset P$ implies that $A \wedge V \neq \emptyset$ and there exists x in A such that $x \equiv a \cup b(\theta)$. $A \equiv P(\hat{\theta})$ implies there exists p in P such that $x \equiv p(\theta)$, and then $p \equiv x \equiv a \cup b(\theta)$ implies p is in V contrary to the fact that $P \wedge V$ is empty. Hence the assumption that $S \wedge V$ is empty leads to a contradiction; therefore $a \equiv b(\ N(\hat{\theta})\ )$ implies $S \wedge V \neq \emptyset$ and there exists x in S such that $x \equiv a \cup b(\Theta)$. But x in S implies $x \cup (a \cap b) \equiv a \cap b(\Theta)$; therefore $a \equiv a \cup (a \cap b) \equiv a \cup x \cup (a \cap b) \equiv a \cup (a \cup b) \cup (a \cap b) \equiv a \cup b \equiv (a \cup b) \cup (a \cap b) \cup b \equiv x \cup (a \cap b) \cup b \equiv (a \cap b) \cup b \equiv b(\Theta)$ and the theorem is proved.

The following theorem is an outgrowth of an attempt to generalize the theorem that the congruence relations of a finite modular lattice form a Boolean algebra (1). Though it is a theorem about the congruence relations on L_σ, it is in particular a statement about the congruence relations of L in the finite case since here L and L_σ are isomorphic.

<u>Theorem 4</u>: $J = J^*$ if and only if for every Θ in $\Theta(L_\sigma)$ there exists φ, such that $\Theta \cap \varphi = 0$ and $N(\ \Theta \cup \varphi\ ) = J$.

<u>Proof</u>: Let $J = J^*$ and let Θ be a congruence relation on L_σ. Then by Theorem 2 $E(\Theta)$ defines a congruence relation on L_σ. $N(\ E(\Theta)\ ) = E(\Theta)$. For if P is in $N(\ E(\Theta)\ )$, there exists $A \supset P$ such that $A \equiv P(\ E(\Theta)\ )$; then P is in J such that $P \supseteq A \cap P = P$ and $P \nsupseteq A \cup P = A$. Thus $A \equiv P(\ E(\Theta)\ )$ im-

plies $P \approx Q$ for some Q in $E(\Theta)$. Lemma 9 implies P is in $E(\Theta)$ and therefore $N(\ E(\Theta)\ ) \subseteq E(\Theta)$. However since $J = J^*$ for P in $E(\Theta)$ there exists $P^* \succ P$. If R is any irreducible ideal of L such that $R \supseteq P \cap P^* = P$ and $R \not\supseteq P \cup P^* = P^*$; then $P^* \succ P$ implies $P = R \cap P^*$, and $P \neq P^*$ with P irreducible implies $P = R$. Therefore for such R, $R \approx P$ and P is $E(\Theta)$; thus $P^* \equiv P(\ E(\Theta)\ )$ and P is in $N(\ E(\Theta)\ )$. Therefore $N(\ E(\Theta)\ ) = E(\Theta)$. Hence $N(\Theta \cup E(\Theta)\ ) = N(\Theta) \vee N(\ E(\Theta)\ ) = N(\Theta) \vee E(\Theta) = J$ and $N(\ \Theta \cap E(\Theta)\ ) = N(\Theta) \wedge N(\ E(\Theta)\ ) = N(\Theta) \wedge E(\Theta) = \emptyset$ from Lemma 10. However $N(\Theta \cap E(\Theta)\ ) = \emptyset$ implies $\Theta \cap E(\Theta) = 0$ from Lemma 10.

Conversely, let P be any irreducible ideal of L and define

$$\Theta \equiv \cup \{ \gamma \text{ in } \Theta(L_\sigma) \mid P \text{ is in } E(\gamma) \}.$$

Then P is in $E(\Theta)$ for if $A \supseteq P$ and $A \equiv P(\Theta)$; then $\Theta = \cup \gamma$ implies there exist $P_1, P_2, \ldots, P_{k+1}$ and γ_i in $\{\gamma \mid P \text{ in } E(\gamma)\}$ for $i = 1, 2, \ldots, k$ such that $P_1 = P$, $P_{k+1} = A$, and $P_i \equiv P_{i+1}(\ \gamma_i)$. By taking the union of each side with P, it is evident that it can be assumed that each $P_i \supseteq P$. Since γ_i in $\{ \gamma \mid P \text{ in } E(\gamma) \}$, P is in $E(\ \gamma_i)$ for each i and $P_1 = P$ implies $P_i = P$ for each i, $i = 1, 2, \ldots, k+1$. Therefore $A = P$ and P in $E(\Theta)$. Now by hypothesis there exists φ such that $N(\ \Theta \cup \varphi\ ) = J$ and $\Theta \cap \varphi = 0$. Then Lemma 11 implies P is in $N(\varphi)$ and there exists $A \supset P$ such that $A \equiv P(\varphi)$. Let $\Gamma \equiv \{ Q \text{ in } J^* \mid Q \supseteq P,\ Q \not\supseteq A \}$. The corollary to Lemma 8 implies Γ is not empty and $P = A \cap [\bigcap_\Gamma Q]$. $A \neq P$ and P in J imply then that $P = \bigcap_\Gamma Q$. Since Γ is a subset of J, it defines by Theorem 2 a congruence relation on L_σ. Suppose P is in $E(\Gamma)$, then as congruence relations $\Gamma \leq \Theta$ and $N(\Gamma) \subseteq N(\Theta)$. But since each Q in Γ is in J^*, as above $Q^* \supset Q$ and $Q^* \equiv Q(\Gamma)$, and each Q is in $N(\Gamma)$. Thus as subsets of J, $\Gamma \subseteq N(\Theta)$. However for each Q in Γ, $Q \not\supseteq A$ implies $Q \cup A \supset Q$; and $A \equiv P(\varphi)$, $P \subseteq Q$ imply

$A \cup Q \equiv P \cup Q \equiv Q(\varphi)$. Hence each Q in Γ is in $N(\varphi)$. $N(\varphi) \wedge N(\theta) = N(\varphi \cap \theta) = N(0) = \emptyset$ implies this is a contradiction and P cannot be in $L(\Gamma)$. Therefore P is in $N(\Gamma)$ and there exists $A' \supset P$ such that $A' \equiv P(\Gamma)$. But $P \supseteq A' \cap P = P$, $P \not\supseteq A' \cup P = A'$, and $A' \equiv P(\Gamma)$ imply $P \backsimeq Q$ for some Q in Γ. Then $Q \in J^*$ and Lemma 5 imply that P is in J^*. Therefore $J = J^*$.

<u>Corollary 1</u>: $\Theta(L_\sigma)$ is a Boolean algebra if and only if $J = J^*$ and $\theta = N(\theta)$ as congruence relations for all θ in $\Theta(L_\sigma)$.

<u>Proof</u>: Assume $\Theta(L_\sigma)$ is a Boolean algebra. Then for each θ in $\Theta(L_\sigma)$ there exists θ' such that $\theta \cap \theta' = 0$ and $\theta \cup \theta' = I$. Since $N(I) = J$ then $N(\theta \cup \theta') = N(I) = J$, and by Theorem 4 $J = J^*$. Let $A \equiv B(\theta)$ and let P be in J such that $P \supseteq A \cap B$ and $P \not\supseteq A \cup B$. Then $P \cup A \cup B \supset P$ and $P \cup A \cup B \equiv P \cup (A \cap B) \equiv P(\theta)$ imply P is in $N(\theta)$. Therefore $A \equiv B(\,N(\theta)\,)$. Hence as congruence relations, $\theta \leq N(\theta)$. Lemma 10 implies then that $N(\theta) \subseteq N(\,N(\theta)\,)$. If P is in $N(\,N(\theta)\,)$, there exists $A \supset P$ such that $A \equiv P(\,N(\theta)\,)$. But $P \supseteq A \cap P = P$ and $P \not\supseteq A \cup P = A$; therefore by the definition of the congruence relation, there exists Q in $N(\theta)$ such that $P \backsimeq Q$. Lemma 9 then implies that P is in $N(\theta)$. Therefore $N(\,N(\theta)\,) = N(\theta)$. Now if $\Theta(L_\sigma)$ is a Boolean algebra, it is relatively complemented and there exists τ in $\Theta(L_\sigma)$ such that $\theta = N(\theta) \cap \tau$ and $\tau \cup N(\theta) = I$. Lemma 10 then implies $N(\theta) = N(\,\tau \cap N(\theta)\,) = N(\tau) \wedge N(\,N(\theta)\,) = N(\tau) \wedge N(\theta)$ and $J = N(I) = N(\,\tau \cup N(\theta)\,) = N(\tau) \vee N(\,N(\theta)\,) = N(\tau) \vee N(\theta)$. Therefore as subsets of J, $N(\theta) \subseteq N(\tau)$ and $N(\tau) = J$. If τ' is the complement of τ, $\tau' \cup \tau = I$, $\tau' \cap \tau = 0$, and Lemma 10 imply $N(\tau') = N(\tau') \wedge J = N(\tau') \wedge N(\tau) = N(\,\tau' \cap \tau\,) = N(0) = \emptyset$; and therefore by Lemma 10 $\tau' = 0$. Hence $\tau = I$ and $\theta = N(\theta)$ as congruence relations. Therefore $A \equiv B(\theta)$ if and only if $A \equiv B(\,N(\theta)\,)$.

Conversely, if the first part of the condition holds, Theorem 4 implies there exists φ for each Θ such that $\varphi \cap \Theta = 0$ and $N(\varphi \cup \Theta) = J$. To show then that $\Theta(L_\sigma)$ is a Boolean algebra it will be sufficient to show that $\varphi \cup \Theta = I$, since it is well known that $\Theta(L_\sigma)$ is a distributive lattice. The congruence relation I is defined by $A \equiv B(I)$ for all A and B in L_σ and it must be shown that $A \equiv B(\varphi \cup \Theta)$ if and only if $A \equiv B(I)$. However it is immediate that $A \equiv B(\varphi \cup \Theta)$ implies $A \equiv B(I)$. Therefore let A and B be any two ideals of L. Since every P in J is contained in $N(\varphi \cup \Theta)$, those P in J such that $P \supseteq A \cap B$ with $P \not\supseteq A \cup B$ are contained in $N(\varphi \cup \Theta)$, and hence $A \equiv B(N(\varphi \cup \Theta))$. Therefore the second part of the condition implies $A \equiv B(\varphi \cup \Theta)$; thus $\varphi \cup \Theta = I$.

<u>Corollary 2</u>: The congruence relations of a finite modular lattice form a Boolean algebra.

<u>Proof</u>: From Lemma F $J = J^*$. Let $A \equiv B(N(\Theta))$; then since the ideals of L are all principal ideals, $A = (x)$ and $B = (y)$ for some x and y. Since all ideals are principal, Lemma 8 implies $\hat{\tilde{\Theta}} = \Theta$. From Theorem 2 $(x) \equiv (y)(N(\Theta))$ implies $x \equiv y(N(\Theta))$ or $x \equiv y(N(\hat{\tilde{\Theta}}))$ and Theorem 3 then implies $x \equiv y(\tilde{\Theta})$; hence $(x) \equiv (y)(\Theta)$. Therefore $A \equiv B(N(\Theta))$ implies $A \equiv B(\Theta)$, but as in Corollary 1 above $A \equiv B(\Theta)$ implies $A \equiv B(N(\Theta))$; hence $A \equiv B(\Theta)$ if and only if $A \equiv B(N(\Theta))$. Thus Corollary 1 above also implies $\Theta(L_\sigma)$ is a Boolean algebra. But since L and L_σ are isomorphic, $\Theta(L)$ is a Boolean algebra.

<u>Theorem 5</u>: $\Theta(L_\sigma)$ is a Boolean algebra if and only if L is finite.

<u>Proof</u>: If L is finite, L_σ is isomorphic to L. Then since $\Theta(L)$ is a Boolean algebra, $\Theta(L_\sigma)$ is a Boolean algebra.

If $\Theta(L_\sigma)$ is a Boolean algebra, Corollary 1 of Theorem 4 implies that $J = J^*$ and that the correspondence between Θ on L_σ and $N(\Theta)$ is one-to-one.

First, if $J = J^*$, then L has a null element. For since the void set is immediately an irreducible ideal, $\bigcap\{(x) \mid x \in L\} = \emptyset$ contradicts $J = J^*$. Thus the intersection is not void and there exists z in L such that $z \leq x$ for all x in L.

Suppose L does not satisfy the ascending chain condition, then there exist $x_1 < x_2 < \ldots < x_n < \ldots$. Let $A = \bigcup_{n=1}^{\infty}(x_n)$; this is clearly not a principal ideal of L. If $((x_n))$ denotes the principal ideal of L_σ generated by (x_n) in L_σ, then $\bigcup_{n=1}^{\infty}((x_n)) \subseteq (A) = (\ \bigcup_{n=1}^{\infty}(x_n)\ )$ in $L_{\sigma\sigma}$. $\bigcup_{n=1}^{\infty}((x_n))$ is not a principal ideal of L_σ; for if $\bigcup_{n=1}^{\infty}((x_n)) = (B)$, there exists m such that B is in $((x_m))$ and $(x_n) \subseteq B$ for each n. Then $B \subseteq (x_m)$ and $A = \bigcup_{n=1}^{\infty}(x_n) \subseteq B \subseteq (x_m)$. Thus $\bigcup_{n=1}^{\infty}(x_n) = (x_m)$ contrary to the fact that A is not principal. Thus $\bigcup_{n=1}^{\infty}((x_n)) \subset (\ \bigcup_{n=1}^{\infty}(x_n)\ )$ and there exist completely irreducible ideals of L_σ containing $\bigcup_{n=1}^{\infty}((x_n))$ but not $(\ \bigcup_{n=1}^{\infty}(x_n)\ )$. Not all of the irreducible ideals can be principal ideals; for if $\bigcup_{n=1}^{\infty}((x_n)) = \bigcap_{\nu}(A_\nu)$ with L_σ complete, then $(\bigcap_{\nu} A_\nu) = \bigcap_{\nu}(A_\nu) = \bigcup_{n=1}^{\infty}((x_n))$, contrary to the fact that $\bigcup_{n=1}^{\infty}((x_n))$ is not principal. Therefore there exist irreducible ideals of L_σ which are not principal, and which contain $\bigcup_{n=1}^{\infty}((x_n))$ but do not contain $(\ \bigcup_{n=1}^{\infty}(x_n)\ )$. Clearly every irreducible ideal of L generates a principal irreducible ideal of L_σ; and since $J = J^*$, these are completely irreducible ideals of L_σ. Thus let H be the set of all principal irreducible ideals of L_σ. From the corollary to Theorem 2, H defines a congruence relation H on L_σ. Clearly $P^* \equiv P(H)$ for each P in J^*, and thus $N(H) = J$. Each (x_n) is in $\bigcup_{n=1}^{\infty}((x_n))$, and thus there exist irreducible ideals of L_σ which contain (x_n), but not $\bigcup_{n=1}^{\infty}(x_n)$, and which are not in H. Thus if it can be shown that $\mathfrak{P} \approx \mathfrak{Q}$ for some $\mathfrak{Q}$ in H implies $\mathfrak{P}$ is in H, it will be clear that $(x_n) \not\equiv \bigcup_{n=1}^{\infty}(x_n)(H)$ and that the congruence relation generated by H is not the trivial congruence relation on L_σ, contrary to the one-to-one nature of the correspondence above. It suffices to consider $\mathfrak{P} \sim \mathfrak{Q}$ with $\mathfrak{Q}$ in H. Then let $\mathfrak{P} \cap \mathfrak{Q} = \mathfrak{Q} \cap \mathfrak{R} = \mathfrak{P} \cap \mathfrak{R}$, and let P, Q, and R respectively denote the union in L_σ of the set of ideals in $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$. Let C denote the union of the ideals of $\mathfrak{P} \cap \mathfrak{R}$, then $P \cap Q = Q \cap R = R \cap P = C$. For immediately $C \subseteq R \cap P$; then if x is in $R \cap P$, x is in R and x is in P. x in R implies x is in $R_1 \cup R_2 \cup \ldots \cup R_n$ with R_i in $\mathfrak{R}$, and x in P implies x is in $P_1 \cup P_2 \cup \ldots \cup P_m$ with P_i in $\mathfrak{P}$. Thus (x) is in $\mathfrak{P} \cap \mathfrak{R}$ and $(x) \subseteq C$; therefore x is in C and $R \cap P = C$. Since $\mathfrak{P} \cap \mathfrak{Q} = \mathfrak{P} \cap \mathfrak{R} = \mathfrak{Q} \cap \mathfrak{R}$, the result follows. Now since $\mathfrak{Q}$ is principal, $\mathfrak{Q} = (Q)$ and $(P) \cap \mathfrak{Q} = (P) \cap (R) = (R) \cap \mathfrak{Q}$. Now $(P) \supseteq \mathfrak{P}$ and $(R) \supseteq \mathfrak{R}$; thus $(P) \cap \mathfrak{R} = \mathfrak{R} \cap \mathfrak{Q} \subseteq \mathfrak{P}$ and $\mathfrak{P} = \mathfrak{P} \cup [(P) \cap \mathfrak{R}] = (P) \cap [\mathfrak{P} \cup \mathfrak{R}]$. Clearly the non-comparability of $\mathfrak{P}$ and $\mathfrak{Q}$ implies $\mathfrak{R} \not\subseteq \mathfrak{P}$; therefore $(P) = \mathfrak{P}$ and $\mathfrak{P}$ is principal. Consequently, the contradiction is proved and L must satisfy the ascending chain condition.

Since L has a null element and satisfies the ascending chain condition, it is complete. Let $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant \ldots$, let $\mathfrak{C}$ be the set of x_n, and assume $\bigcap x_n$ is not in $\mathfrak{C}$. Then $\bigcap x_n \neq u$, the unit element of the lattice. Let $V \equiv \{y \mid y \geqslant x_n, \text{ some } n\}$. Immediately V is a dual ideal and $\bigcap x_n$ is not in V. Then applying the dual to Lemma 5 to the lattice of elements greater than or equal to $\bigcap x_n$, there exists $V' \supseteq V$, a dual ideal, such that $\bigcap x_n$ is not in V', every element of V' is greater than $\bigcap x_n$, and any dual ideal properly containing V', each of whose elements is greater than or equal to $\cap x_n$, contains $\cap x_n$. Since $V' \wedge (\cap x_n) = \emptyset$, Lemma 8 implies there exists P in J such that $P \wedge V' = \emptyset$, $P \supseteq (\cap x_n)$, and when $A \supset P$, $A \wedge V' \neq \emptyset$. Since $J = J^*$ and L satisfies the ascending chain condition, $P^* \succ P$ and there exist x and y such that $P = (y)$, $P^* = (x)$, and $x \succ y$. Then since $P^* \supset P$, there exists s in V' such that $s \leqslant x$ and x is in V'. Since $\cap x_n$ is in P, $\cap x_n \leqslant y$. From the maximality of V', y not in V' implies that the dual ideal containing y and V' contains $\cap x_n$. Thus there exists t in V' such that $y \cap t = \cap x_n$. t in V' implies t is not in P and $(x) = P^* \leq (t) \cup P$ implies $x \leqslant t \cup y$. Then $x = x \cap (t \cup y) = y \cup (t \cap x)$ and $t \cap x \cap y = t \cap y = \cap x_n$. Therefore since x and t are in V', $x \cap t$ is in V'. Further $x = y \cup (x \cap t) \succ y$ implies by modularity that $x \cap t \succ x \cap t \cap y = \cap x_n$. Thus $x \cap t$ is the minimal element of V'; in particular $x \cap t \leqslant x_n$ for all n. Thus $x \cap t \leqslant \cap x_n$, contradicting $x \cap t \succ \cap x_n$. Therefore $\cap x_n$ is in $\mathfrak{C}$. Thus $\cap x_n = x_m$ for some m, and $x_m = x_{m+1} = \ldots$; therefore the descending chain condition is satisfied and L is finite.

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