LIFTING LINE THEORY IN LINEARLY VARYING FLOW

Thesis by

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Acknowledgment

The present paper is based on the solution of the general problem of a wing in non-uniform flow by Th. von Kármán and H. S. Tsien (Ref. 1).

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Notation

a	distance of the wing from the origin of coordinate system, as shown in Figs. 2 and 11.
b	wingspan.
c	chord of wing at a span station y.
C_o	wing chord at the center of the wing span.
C_L	lift coefficient at span station y.
C_D	induced drag coefficient.
$L(y)$	lift force at span station y.
ℓ_n	arbitrary constant of the n-U particular solution of the differential equation.
m	slope of the lift curve of wing profile.
p	pressure above the free stream pressure on the wing surface.
u,v,w	components of the disturbance velocity due to presence of wing. at the wing, in disection of x, y, z.
v',w'	disturbance velocities due to the wing, in the Trefftz plane.
U	free stream velocity, linearly varying along the span.
U_o	free stream velocity at the center of the wing span.
q_o	dynamic head at the center of the wing.
x,y,z	Cartesian coordinates, oriented as shown in Fig. 1.
α	geometrical angle of attack at span station y.
λ	parameter indicating the magnitude of the variation of the free stream velocity across the span.
θ	transformed coordinate related to y and z as indicated in Eq. 2.6 (Part I).
φ	function analogous to potential function in irrotational flow.
ρ	density of fluid.
ξ	transformed coordinate related to y according to Eq. 1.25 (Part II).

Introduction.

The problem of a wing placed in a flow that is nonuniform in the spanwise direction is of interest to the aerodynamicist as well as to the designer of rotating machinery. The portion of a wing located in the slip stream of the propeller, the blades of propellers and fans, wind-tunnel models in nonuniform airstreams are some of the examples. Prandtl's three dimensional wing theory assumes a uniform undisturbed flow and thus it is not valid in these cases. Some authors tried to modify Prandtl's theory in order to make it applicable to such problems. The work of F. Vandrey (Ref. 1) and K. Bausch (Ref. 2) may be mentioned in this connection. A discussion of their work is given in Section 7 of Part I of this analysis. Lately Th. von Kármán and H. S. Tsien presented a general solution of the problem of a wing placed in a flow with the velocity varying in both directions normal to and parallel with the wing span (Ref. 3). In the following the author will utilize the results of von Kármán and Tsien for the simpler case of the velocity varying linearly and in the direction of the span only and solve the "third problem of airfoil theory" (finding the lift distribution for an airfoil of given shape) for a finite wing in an infinite fluid and for the case of a wing between two parallel walls. The wing in an infinite fluid is treated in Part I while Part II contains the solution of the problem of the wing between walls.

Part I. Linearly Varying Flow of an Infinite Fluid Around a Finite Wing.

1) The General Theory of von Kármán and Tsien.

The fundamental concepts of this theory are identical with those originally formulated by Prandtl: a) the wing span is sufficiently large compared with the chord, so that the flow in each plane normal to the span can be considered as two dimensional; b) the wing is replaced by a lifting line having the same spanwise lift distribution as the wing; c) the disturbance caused by the wing is small compared to the velocity of the undisturbed flow.

Let the x-axis be parallel to the direction of the undisturbed flow, the y-axis coincide with the lifting line and the z-axis be normal to x and y (Fig. 1) and denote the pressure by p, the density by ρ and by v_1, v_2 and v_3 the velocity components in the direction of x, y, z. Then the equations of motion of an inviscid, incompressible fluid without external forces are

$$
\begin{aligned}
v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y} + v_3 \frac{\partial v_1}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} + v_3 \frac{\partial v_2}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad 1.1 \\
v_1 \frac{\partial v_3}{\partial x} + v_2 \frac{\partial v_3}{\partial y} + v_3 \frac{\partial v_3}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned}
$$

and the equation of continuity is

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0 \qquad 1.2$$

For the present problem

$$v_1 = U + u, \qquad v_2 = v, \qquad v_3 = w \qquad 1.3$$

Here U is the free stream velocity, assumed to be a function of y only and u, v and w are the disturbance velocities due to the presence of the wing. As mentioned before, u, v and w are small compared with U.

The values of v_1, v_2 and v_3 given by Eq. 1.3 can now be substituted into Eq. 1.1 resulting in a set of linear equations for u, v and w after terms of second order magnitude have been dropped.

$$\begin{aligned} U\frac{\partial u}{\partial x} + v\frac{\partial U}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ U\frac{\partial v}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ U\frac{\partial w}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned} \qquad 1.4$$

Differentiating these equations with respect to x, y and z, respectively, adding the results and combining the sum with Eq. 1.2 the following equation is obtained:

$$\frac{1}{U^2}\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{1}{U^2}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{U^2}\frac{\partial p}{\partial z}\right) = 0 \qquad 1.5$$

giving an equation for the pressure only. The pressure of the undisturbed stream is chosen as zero and the lifting force of the lifting line is represented as a pressure force on the "lower surface" and an equal negative pressure force on the "upper surface", i.e.,

$$\begin{aligned} &p = 0 \text{ at } |x| \to \infty,\ |y| \to \infty,\ |z| \to \infty \\ &\pm\frac{1}{2}L(y) = \int_{-\varepsilon}^{\varepsilon} p\,dx \quad \text{for } z = \pm 0 \end{aligned} \qquad 1.6$$

Here L(y) is the lift per unit span of the lifting line, at the span station y.

If the solution of Eq. 1.5 in terms of the pressure p is known, the induced velocities v and w can be obtained by integration from the corresponding Eq. 1.4, i.e.,

$$v(x,y,z) = v(0,y,z) - \frac{1}{\rho U}\int_0^x \frac{\partial p}{\partial y}\,dx \qquad 1.7$$

and

$$w(x,y,z) = w(0,y,z) - \frac{1}{\rho U}\int_0^x \frac{\partial p}{\partial z}\,dx$$

It should be noted that p is an even function in x, consequently from above

$$\begin{aligned} v(x,y,z) + v(-x,y,z) &= 2\,v(0,y,z) \\ w(x,y,z) + w(-x,y,z) &= 2\,w(0,y,z) \end{aligned} \qquad 1.8$$

considering the induced velocities at $x = +\infty$ and $x = -\infty$

$$\begin{aligned} v(-\infty,y,z) &= 0 \\ w(-\infty,y,z) &= 0 \end{aligned} \qquad 1.9$$

as there can be no disturbance far ahead of the lifting line and therefore Eq. 1.8 gives

$$\begin{aligned} v(\infty,y,z) &= 2v(0,y,z) \\ w(\infty,y,z) &= 2w(0,y,z) \end{aligned} \qquad 1.10$$

Thus the induced velocities at the lifting line are one-half of those far downstream. This result is identical with that obtained in the Prandtl wing theory and a treatment of the problem in the Trefftz plane, i.e., far downstream, suggests itself.

Von Kármán and Tsien show that it is possible to simplify the problem in the Trefftz plane considerably by using a "potential function" $\varphi(y,z)$, which is defined in such a manner that its value at $z = 0$ is one-half of the lift per unit span at the wing. It is related to the disturbance velocities v' and w' in the Trefftz plane as follows:

$$\rho U v' = \frac{\partial \varphi'}{\partial y}$$
$$\rho U w' = \frac{\partial \varphi'}{\partial z} \qquad 1.11$$

The differential equation that φ , the "potential function" at the wing, has to satisfy is obtained from Eq. 1.5; it reads

$$\frac{\partial}{\partial y}\left(\frac{1}{U^2}\frac{\partial \varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{U^2}\frac{\partial \varphi}{\partial z}\right) = 0 \qquad 1.12$$

and the boundary conditions in φ are

$$\varphi = 0 \quad \text{at } |y| \rightarrow \infty , \ |z| \rightarrow \infty$$
$$\varphi = \pm \frac{L(y)}{2} \quad \text{for } z = \pm 0 \qquad 1.13$$

2) Solution of the "Potential Problem" in the Trefftz Plane for the Case of Undisturbed Velocity Linear in y.

Let the free stream velocity be expressed in the form

$$U = y \qquad 2.1$$

let the span of the wing be denoted by b, and let the center of the wing be located a distance a from the origin of the coordinate system (see Fig. 2). The ratio of the velocities at the wing tips, i.e., the severity of the velocity gradient can then be expressed in terms of a parameter λ , where

$$\lambda = \frac{b}{2a} \qquad 2.2$$

$\lambda = 0$ corresponds to the limiting case of constant velocity across the span while a value of $\lambda = 1$ is obtained for the other extreme case — zero velocity at one of the wing tips.

Introducing U = y into Eq. 1.12 the following differential equation is obtained:

$$\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} - \frac{2}{y}\frac{\partial\varphi}{\partial y} = 0 \qquad 2.3$$

The solution of this equation can be simplified by expressing in terms of a new function f(x,y) which satisfies Laplace's equation, i.e.,

$$\nabla^2 f(x,y) = 0 \qquad 2.4$$

Then

$$\varphi(x,y) = f(x,y) - y\frac{\partial f(x,y)}{\partial y} = -y^2\frac{\partial}{\partial y}\left[\frac{f(x,y)}{y}\right] \qquad 2.5$$

A coordinate transformation appears to be useful:

$$\begin{aligned} y &= \frac{b}{4}\left(r+\frac{1}{r}\right)\cos\theta + a \\ z &= \frac{b}{4}\left(r-\frac{1}{r}\right)\sin\theta \end{aligned} \qquad 2.6$$

As at the surface of the wing z = 0, i.e., r = 1 and thus, the "potential function" $\varphi\,(r,\theta)$ is there a function of θ only, the notation $\varphi(\theta)$ will be used to denote values of φ at the wing.

It suggests itself to try to determine $\varphi(\theta)$ in a form analagous to the Fourrier series in the Prandtl wing theory. There $\varphi = \sum_{n=1}^{\infty} a_n \sin n\theta$ where the a_n's are constant coefficients and sin n is the particular solution of the differential equation at the wing. In the present problem a similar form for $\varphi(\theta)$ is used.

$$\varphi(\theta) = \sum_{n=1}^{\infty} l_n\,\varphi_n(\theta) \qquad 2.7$$

where

$$\varphi_n(\theta) = \frac{1}{4}\rho c_o U_o^2\left[\left(\frac{U}{U_o}\right)^2 \sin n\theta\right] = \frac{1}{4}\rho\, c_o\, a^2\left[\left(\frac{a+\frac{b}{2}\cos\theta}{a}\right)^2 \sin n\theta\right] \qquad 2.8$$

The function $f_n(r, \theta)$, related to $\varphi_n(r, \theta)$ according to Eq. 2.5, is taken to be of the form

$$f_n(r,\theta) = \frac{1}{4}\rho c_0 a^2 \sum_{m=1}^{\infty} a_{n,m} \sin m\theta \, \frac{1}{r^m} \qquad 2.9$$

Substituting Eq. 2.8 and 2.9, expressing y in terms of r and θ, and integrating, one obtains for r = 1,

$$\begin{aligned} f_n(\theta) &= \frac{1}{4}\rho c_0 a^2 \frac{b}{2}\left(a + \frac{b}{2}\cos\theta\right)\int_0^{\theta} \frac{\left(1+\frac{b}{2a}\cos\theta'\right)^2 \sin n\theta' \sin\theta' \, d\theta'}{\left(a+\frac{b}{2}\cos\theta'\right)^2} \\ &= \frac{1}{2}\rho c_0 a^2 \sum_{m=1}^{\infty} a_{n,m} \sin m\theta \end{aligned} \qquad 2.10$$

Here $0 \le \theta \le \pi$ at $z = +0$, $a - \frac{b}{2} \le y \le a + \frac{b}{2}$. For negative values of θ, i.e., at z = -0, $a - \frac{b}{2} \le y \le a + \frac{b}{2}$ $\quad f_n(-\theta) = -f_n(\theta)$

Introducing into Eq. 2.10 the parameter $\lambda = \frac{b}{2a}$, one obtains

$$\sum_{m=1}^{\infty} a_{n,m} \sin m\theta = \frac{\lambda^2}{2}\left(\cos\theta + \frac{1}{\lambda}\right)\int_0^{\theta} \sin n\theta' \sin\theta' \, d\theta' \qquad 2.11$$

and carrying out the integration finally

$$\begin{aligned} \sum_{m=1}^{\infty} a_{n,m} \sin m\theta &= \frac{\lambda^2}{4}\left(\cos\theta + \frac{1}{\lambda}\right)\left[\frac{\sin(n-1)\theta}{(n-1)} - \frac{\sin(n+1)\theta}{(n+1)}\right] = \\ &= \frac{\lambda^2}{4}\left\{\left[\frac{\sin(n-1)\theta}{(n-1)} - \frac{\sin(n+1)\theta}{(n+1)}\right]\frac{1}{\lambda} + \left[\sin n\theta + \sin(n-2)\theta\right]\frac{1}{2(n-1)} \right. \\ &\quad \left. - \left[\sin(n+2)\theta + \sin n\theta\right]\frac{1}{2(n+1)}\right\} \end{aligned} \qquad 2.12$$

This relation determines all coefficients $a_{n,m}$ for a given value of n. Thus the particular solutions of Eq. 2.3 are known and it remains only to determine the constants ℓ_n in Eq. 2.7. However, it should be noted that this method of solution breaks down in the case of the first particular solution, i.e., for n = 1. The solution for n = 1 is somewhat lengthy and for this reason it will be discussed separately in the next section.

In order to determine the constants ℓ_n, the relation between the lift per unit span and the effective angle of attack is needed. This relation is the same for the case of varying velocity as for constant velocity and it can be written in the well-known form

$$L(y) = \frac{1}{2}\rho U^2 c m\left[\alpha_o + \frac{w(y)}{U}\right] \qquad 2.13$$

L(y) = lift per unit span at a point y on the span, C is the local wing chord, m the slope of the lift curve, c_o the geometrical angle of attack, w(y) the downwash velocity at the wing.

According to Eq. 1.13, L(y) = 2 φ and thus, using Eq. 2.8

$$L(y) = \sum_{n=1}^{\infty} \frac{\ell_n}{2}\rho c_o a^2\left[\frac{a+\frac{b}{2}\cos\theta}{a}\right]^2 \sin n\theta \qquad 2.14$$

Rewriting Eq. 2.14 in dimensionless form

$$c_L = \frac{L(y)}{\frac{1}{2}\rho c\left[a+\frac{b}{2}\cos\theta\right]^2} = \frac{c_o}{c}\sum_{n=1}^{\infty}\ell_n \sin n\theta \qquad 2.15$$

If one sets also

$$\frac{w(y)}{U} = \sum_{n=1}^{\infty}\ell_n\frac{w_n}{U} \qquad 2.16$$

and substitutes Eq. 2.15 and 2.16 into the dimensionless form of Eq. 2.13, one obtains

$$c_L = \frac{c_o}{c}\sum_{n=1}^{\infty}\ell_n \sin n\theta = m\left[\alpha_o + \sum_{n=1}^{\infty}\ell_n\frac{w_n}{U}\right] \qquad 2.17$$

Eq. 2.17 can be considered as the solution of the problem. It remains only to express $\frac{w_n}{U}$ in terms of $f_n(\theta)$, i.e., in terms of the coefficients $a_{n,m}$ that have already been determined. According to Eq. 1.10, 1.11 and 2.5

$$w_n = -\frac{1}{2\rho y}\left[\frac{\partial f_n}{\partial z} - y\frac{\partial^2 f_n}{\partial y \partial z}\right] \qquad 2.18$$

If the partial derivatives with respect to y and z are expressed in terms of those with respect to r and θ, as carried out in appendix A, the following relation is obtained for the induced angle at the wing, i.e., for r = 1

$$\begin{aligned}\frac{w_n}{U} = -\frac{c_0 a^2}{4\left[a+\frac{b}{2}\cos\theta\right]^2}\Bigg\{&\frac{1}{\frac{b}{2}\sin\theta}\sum_{m=1}^{\infty} m\, a_{n,m}\sin m\theta - \\ &-\left[a+\frac{b}{2}\cos\theta\right]\frac{\cos\theta}{\frac{b^2}{4}\sin^3\theta}\sum_{m=1}^{\infty} m\, a_{n,m}\sin m\theta + \\ &+\left[a+\frac{b}{2}\cos\theta\right]\frac{1}{\frac{b^2}{4}\sin^2\theta}\sum_{m=1}^{\infty}\frac{d}{d\theta}\left[m\, a_{n,m}\sin m\theta\right]\Bigg\}\end{aligned} \qquad 2.19$$

This equation can be rewritten in terms of λ as follows:

$$\begin{aligned}\frac{w_n}{U} = -\frac{c_0}{b}\frac{1}{2\lambda^2}\frac{1}{\left(\frac{1}{\lambda}+\cos\theta\right)^2}\Bigg\{&\sum_{m=1}^{\infty} m\, a_{n,m}\frac{\sin m\theta}{\sin\theta} - \\ &-\left[\frac{1}{\lambda}+\cos\theta\right]\frac{\cos\theta}{\sin^3\theta}\sum_{m=1}^{\infty} m\, a_{n,m}\sin m\theta + \\ &+\left[\frac{1}{\lambda}+\cos\theta\right]\frac{1}{\sin^2\theta}\sum_{m=1}^{\infty}\frac{d}{d\theta}\left[m\, a_{n,m}\sin m\theta\right]\Bigg\}\end{aligned} \qquad 2.20$$

Replacing $\sum_{m=1}^{\infty} m a_{n,m}\sin m\theta$ by $\frac{d}{d\theta}\sum_{m=1}^{\infty} a_{n,m}\cos m\theta$ Eq. 2.20 can be rewritten in the form

$$\frac{w_n}{U} = -\frac{1}{\sin\theta}\frac{c_0}{b}\frac{1}{2\lambda^2}\frac{d}{d\theta}\left\{\frac{\frac{d}{d\theta}\sum\limits_m^{\infty} a_{n,m}\cos m\theta}{\sin\theta\left(\cos\theta+\frac{1}{\lambda}\right)}\right\} \qquad 2.21$$

The series $\sum\limits_m a_{n,m}\cos m\theta$ can be obtained with the help of Eq. 2.12. Replacing in that equation the sine functions by cosines

$$\sum_{m=1}^{\infty} a_{n,m}\cos m\theta = \frac{\lambda^2}{4}\left\{\left[\frac{\cos(n-1)\theta}{(n-1)} - \frac{\cos(n+1)\theta}{(n+1)}\right]\frac{1}{\lambda} + \right. \qquad 2.22$$

$$\left. + \left[\frac{\cos n\theta+\cos(n-2)\theta}{2(n-1)}\right] - \left[\frac{\cos(n+2)\theta+\cos n\theta}{2(n+1)}\right]\right\}$$

Transforming one obtains

$$\sum_{m=1}^{\infty} a_{n,m}\cos m\theta = \frac{\lambda^2}{4}\sin\theta\left(\cos\theta+\frac{1}{\lambda}\right)\left[\frac{\cos(n-1)\theta}{(n-1)} - \frac{\cos(n+1)\theta}{(n+1)}\right] \qquad 2.23$$

and thus

$$\frac{d}{d\theta}\sum_{m=1}^{\infty} a_{n,m}\cos m\theta = \frac{\lambda^2}{4}\left[-\frac{\cos(n-1)\theta}{(n-1)} + \frac{\cos(n+1)\theta}{(n+1)} + 2\left(\cos\theta+\frac{1}{\lambda}\right)\cos n\theta\right] \qquad 2.24$$

Substituting, finally, Eq. 2.24 into Eq. 2.21

$$\frac{w_n}{U} = -\frac{1}{\sin\theta}\frac{c_0}{b}\frac{1}{8}\frac{d}{d\theta}\left[2\cos n\theta + \frac{\frac{\cos(n+1)\theta}{n+1} - \frac{\cos(n-1)\theta}{n-1}}{\left(\cos\theta+\frac{1}{\lambda}\right)}\right] \qquad 2.25$$

Carrying out the differentiation and rearranging the final expression for the downwash angle is obtained.

$$\frac{w_n}{U} = \frac{c_0}{b}\left[\frac{n\sin n\theta}{4\sin\theta} + \frac{\frac{\cos(n-1)\theta}{n-1} - \frac{\cos(n+1)\theta}{n+1}}{8\left(\cos\theta+\frac{1}{\lambda}\right)^2} + \frac{\cos n\theta}{4\left(\cos\theta+\frac{1}{\lambda}\right)}\right] \qquad 2.26$$

Eq. 2.26 together with Eq. 2.17 represents the solution of the problem as the coefficients ℓ_n can be determined by well-known methods as discussed in section 4. After the ℓ_n's have been determined the spanwise distributions of the lift angle of attack and induced drag can be calculated using Eq. 2.15 and 2.16 and the relation for the induced drag coefficient $C_{Di} = C_L\, w/U$, respectively.

3) Determination of the First Particular Solution.

As mentioned in the previous section, Eq. 2.26 cannot be applied to the case of n = 1. In the following a solution of the problem for this case is presented.

Recapitulating, the "potential function" in the Trefftz plane has to satisfy Eq. 2.3 and it is assumed to be of the following form:

$$\varphi_1 = \frac{1}{4}\rho c_0 a^2 \left[\frac{a + \frac{b}{2}\cos\theta}{a}\right]^2 \sin\theta = \frac{1}{4}\rho c_0 y^2 \sqrt{1 - \left(\frac{y-a}{b/2}\right)^2} \qquad 3.1$$

for z = +0 and $a - \frac{b}{2} \leq y \leq a + \frac{b}{2}$. And at z = -0 $\varphi_1(-0) = -\varphi_1(0)$
Furthermore, $\varphi_1 = 0$ for $z = 0$, $y > a + \frac{b}{2}$ and $y < a + \frac{b}{2}$ and also for $z \to \infty$

According to Eq. 1.10 and 1.11, the downwash velocity at the wing is given by

$$w_1 = -\frac{1}{2\rho y}\left(\frac{\partial \varphi_1}{\partial z}\right)_{z=0} \qquad 3.2$$

The suggested solution* of Eq. 3.1 is of the form

$$\frac{\varphi_1}{\frac{1}{2}\rho c_0} = \frac{a^2}{2} e^{-u} \sin v + \frac{ab}{4} e^{-2u} \sin 2v + \frac{b^2}{8} \cosh u\, e^{-2u} \cos^2 v \sin v + \qquad 3.3$$
$$+ \frac{b}{2} \sinh u \sin v \sum_{n=2}^{\infty} A_n e^{-nu} \cos nv$$

*This solution was suggested by Prof. H. Bateman.

Here the following coordinate transformation was used

$$y-a = \frac{b}{2}\cosh u \cos v$$
$$z = \frac{b}{2}\sinh u \sin v \qquad 3.4$$

The A_n's are constant coefficients that have to be determined. Eq. 3.3 is seen to satisfy all boundary conditions as shown below.

a) The disturbance at infinity: If $z \to \infty$ i.e. $u \to \infty$, thus $\varphi_1 = 0$

b) At the wing:

$z = +0$ and $a - \frac{b}{2} \leq y \leq a + \frac{b}{2}$ thus $u = +0$

and from Eq. 3.4, $y - a = \frac{b}{2}\cos v$ where $0 \leq v \leq \pi$

Therefore, at the wing

$$\varphi_1 = \frac{1}{2}\left(a^2 + ab\cos v + \frac{b^2}{4}\cos^2 v\right)\sin v = \frac{y^2}{2}\sqrt{1-\left(\frac{y-a}{b/2}\right)^2}$$

as desired.

Furthermore, for $-\pi \leq v \leq 0$ $\varphi_1(-v) = -\varphi_1(v)$

c) At the wing tips:

$v = 0$ $y - a = \frac{b}{2}\cosh u$ $\varphi_1 = 0$

$v = \pi$ $y - a = -\frac{b}{2}\cosh u$ $\varphi_1 = 0$

It remains to determine the coefficients A_n in such a manner that φ_1 will satisfy the differential equation 2.3. Using the coordinate transformation of Eq. 3.4 the differential equation 2.3 can be expressed, as shown in Appendix B, in the following form:

$$\sinh u \cos v \frac{\partial \varphi}{\partial u} - \cosh u \sin v \frac{\partial \varphi_1}{\partial v} = \frac{1}{b}\left(a + \frac{b}{2}\cosh u \cos v\right)\left(\frac{\partial^2 \varphi_1}{\partial u^2} + \frac{\partial^2 \varphi_1}{\partial v^2}\right) \qquad 3.5$$

Using the expression for φ_1 from Eq. 3.3 one finds

$$\frac{\partial^2\varphi_1}{\partial u^2}+\frac{\partial^2\varphi_1}{\partial v^2}=-\frac{\rho c_0}{2}\left\{\frac{b}{4}\left(\sinh u\cos v\, e^{-2u}\sin 2v+\cosh u\sin v\, e^{-2u}\cos 2v\right)+\right.$$
$$\left.+\cosh u\sin v\sum_{n=2}^{\infty} nA_n e^{-nu}\cos nv+\sinh u\cos v\sum_{n=2}^{\infty} nA_n e^{-nu}\sin nv\right\} \qquad 3.6$$

Similarly one obtains

$$\sinh u\cos v\frac{\partial\varphi_1}{\partial u}+\cosh u\sin v\frac{\partial\varphi_1}{\partial v}=-\sinh u\cos v\left[\frac{a^2}{2}e^{-u}\sin v+\frac{ab}{2}e^{-2u}\sin 2v-\right.$$
$$-\frac{b^2}{8}\sinh u\, e^{-2u}\cos^2 v\sin v+\frac{b^2}{4}\cosh u\, e^{-2u}\cos^2 v\sin v-$$
$$\left.-\frac{b}{2}\cosh u\sin v\sum_{n=2}^{\infty}A_n e^{-nu}\cos nv+\frac{b}{2}\sinh u\sin v\sum_{n=2}^{\infty}nA_n e^{-nu}\cos nv\right] \qquad 3.7$$
$$-\cosh u\sin v\left[\frac{a^2}{2}e^{-u}\cos v+\frac{ab}{2}e^{-2u}\cos 2v+\frac{b^2}{8}\cosh u\, e^{-2u}\cos^3 v-\right.$$
$$-\frac{b^2}{4}\cosh u\, e^{-2u}\cos v\sin^2 v+\frac{b}{2}\sinh u\cos v\sum_{n=2}^{\infty}A_n e^{-nu}\cos nv-$$
$$\left.-\frac{b}{2}\sinh u\sin v\sum_{n=2}^{\infty}nA_n e^{-nu}\sin nv\right]$$

After substituting these expressions into the differential equation (Eq. 3.5) and collecting the terms containing the series, the following relation is obtained:

$$\frac{a^2}{4}\sin v\cos v+\frac{ab}{4}e^{-2u}\left[\sinh u\cos v\sin 2v+\cosh u\sin v\cos 2v\right]+$$
$$+\frac{b^2}{8}e^{-2u}\sin v\cos v\,(\cos^2 v-\cosh^2 u)=\frac{a}{2}\sum_{n=2}^{\infty}nA_n e^{-nu}\left[\cosh u\sin v\cos nv+\right. \qquad 3.8$$
$$\left.+\sinh u\cos v\sin nv\right]+\frac{b}{4}\sin 2v\sum_{n=2}^{\infty}nA_n e^{-nu}\cos nv+\frac{b}{4}\sinh 2u\sum_{n=2}^{\infty}nA_n e^{-nu}\sin nv$$

Transforming Eq. 3.8

$$\frac{a^2}{4}\sin 2v+\frac{ab}{8}e^{-u}\sin 3v-\frac{ab}{8}e^{-3u}\sin v+\frac{b^2}{16}e^{-2u}\left(\frac{1}{2}-\cosh^2 u\right)\sin 2v+\frac{b^2}{64}e^{-2u}\sin 4v=$$
$$=\frac{a}{2}\sum_{n=2}^{\infty}nA_n e^{-(n-1)u}\sin(n+1)v-\frac{a}{2}\sum_{n=2}^{\infty}nA_n e^{-(n+1)u}\sin(n-1)v+ \qquad 3.9$$
$$+\frac{b}{8}\sum_{n=2}^{\infty}nA_n e^{-nu}\sin(n+2)v-\frac{b}{8}\sum_{n=2}^{\infty}nA_n e^{-nu}\sin(n-2)v+\frac{b}{8}\sum_{n=2}^{\infty}nA_n e^{-(n-2)u}\sin nv-$$
$$-\frac{b}{8}\sum_{n=2}^{\infty}nA_n e^{-(n+2)u}\sin nv$$

The expressions containing the series can be combined for values of $n \geq 3$ resulting in

$$-\frac{ab}{8}e^{-3u}\sin v + \left(\frac{a^2}{4}-\frac{b^2}{64}\right)\sin 2v - \frac{b^2}{64}e^{-4u}\sin 2v + \frac{ab}{8}e^{-u}\sin 3v + \frac{b^2}{64}e^{-2u}\sin 4v =$$

$$= \frac{b}{4}A_2 \sin 2v + \left(aA_2 + \frac{3b}{8}A_3\right)e^{-u}\sin 3v - \left(aA_2 + \frac{3b}{8}A_3\right)e^{-3u}\sin v +$$

$$+ \sum_{n=3}^{\infty}\left\{\left[\frac{b}{4}(n-1)A_{n-1} + anA_n + \frac{b}{4}(n+1)A_{n+1}\right]\left[\sinh u \sin nv \cos v + \cosh u \cos nv \sin v\right]e^{-nu}\right. \qquad 3.10$$

The values of the coefficients A_n can now be determined by comparing the coefficients of the corresponding terms of Eq. 3.10. Thus

$$\frac{b}{4}A_2 = \frac{a^2}{4} - \frac{b^2}{64}$$

$$aA_2 + \frac{3b}{8}A_3 = \frac{ab}{8} \qquad 3.11$$

$$\frac{b}{4}A_2 + \frac{3a}{2}A_3 + \frac{b}{2}A_4 = \frac{b^2}{64}$$

and for $n \geq 4$

$$\frac{b(n-1)}{4}A_{n-1} + anA_n + \frac{b(n+1)}{4}A_{n+1} = 0 \qquad 3.12$$

introducing again $\lambda = \frac{b}{2a}$ and setting $B_n = \frac{A_n}{b/2}$ one obtains by solving Eq. 3.11

$$B_2 = \frac{1}{2\lambda^2} - \frac{1}{8}$$

$$B_3 = \frac{1}{\lambda}\left[\frac{1}{2} - \frac{2}{3\lambda^2}\right] \qquad 3.13$$

$$B_4 = \frac{1}{8} - \frac{1}{\lambda^2} + \frac{1}{\lambda^4}$$

and for $n \geq 4$

$$(n+1)B_{n+1} + \frac{2n}{\lambda}B_n + (n-1)B_{n-1} = 0 \qquad 3.14$$

Eq. 3.14 is a difference equation of the second order. Let its fundamental solutions be $h_1(n)$ and $h_2(n)$. Then

$$B_n = m_1 h_1(n) + m_2 h_2(n) \qquad 3.15$$

where m_1 and m_2 are constants.

Assuming $h(n)$ of the form

$$h(n) = \frac{1}{n} \rho^n \qquad 3.16$$

and substituting into the difference equation an algebraic equation for ρ is obtained:

$$\rho^2 + \frac{2}{\lambda}\rho + 1 = 0 \qquad 3.17$$

The roots of this equation are

$$\rho_1 = -\frac{1}{\lambda} + \sqrt{\frac{1}{\lambda^2} - 1} \quad ; \quad \rho_2 = -\frac{1}{\lambda} - \sqrt{\frac{1}{\lambda^2} - 1} \quad ; \qquad 3.18$$

Thus, according to Eq. 3.15 and 3.16 the general solution of Eq. 3.14 is

$$B_n = m_1 \frac{1}{n}\left(-\frac{1}{\lambda} + \sqrt{\frac{1}{\lambda^2} - 1}\right)^n + m_2 \frac{(-1)^n}{n}\left(\frac{1}{\lambda} + \sqrt{\frac{1}{\lambda^2} - 1}\right)^n \qquad 3.19$$

It remains to determine the constants m_1 and m_2. This is done with the help of the known values of B_3 and B_4 given in Eq. 3.13. Combining Eq. 3.13 and 3.19

$$\begin{aligned} B_3 &= \frac{1}{\lambda}\left(\frac{1}{2} - \frac{2}{3\lambda^2}\right) = \frac{m_1}{3}\left(-\frac{1}{\lambda} + \sqrt{\frac{1}{\lambda^2} - 1}\right)^3 - \frac{m_2}{3}\left(\frac{1}{\lambda} + \sqrt{\frac{1}{\lambda^2} - 1}\right)^3 \\ B_4 &= \frac{1}{8} - \frac{1}{\lambda^2} + \frac{1}{\lambda^4} = \frac{m_1}{4}\left(-\frac{1}{\lambda} + \sqrt{\frac{1}{\lambda^2} - 1}\right)^4 - \frac{m_2}{4}\left(\frac{1}{\lambda} + \sqrt{\frac{1}{\lambda^2} - 1}\right)^4 \end{aligned} \qquad 3.20$$

Expanding the binomials and rearranging the terms

$$\frac{1}{\lambda}\left(\frac{1}{2}-\frac{2}{3\lambda^2}\right)=\left(\frac{4}{3\lambda^2}-\frac{1}{3}\right)\sqrt{\frac{1}{\lambda^2}-1}\,(m_1-m_2)-\frac{1}{\lambda}\left(\frac{4}{3\lambda^2}-1\right)(m_1+m_2)$$

3.21

$$\frac{1}{8}-\frac{1}{\lambda^2}+\frac{1}{\lambda^4}=-\frac{1}{\lambda}\left(\frac{2}{\lambda^2}-1\right)\sqrt{\frac{1}{\lambda^2}-1}\,(m_1-m_2)+\left(\frac{2}{\lambda^4}-\frac{2}{\lambda^2}+\frac{1}{4}\right)(m_1+m_2)$$

solving for $(m_1 - m_2)$ and $(m_1 + m_2)$

$$\sqrt{1-\lambda^2}\,(m_1-m_2)=\frac{\left(\frac{\lambda^2}{2}-\frac{2}{3}\right)\left(2-2\lambda^2+\frac{\lambda^4}{4}\right)+\left(\frac{4}{3}-\lambda^2\right)\left(\frac{\lambda^4}{8}-\lambda^2+1\right)}{\left(\frac{4}{3}-\frac{\lambda^2}{3}\right)\left(2-2\lambda^2+\frac{\lambda^4}{4}\right)-\left(\frac{4}{3}-\lambda^2\right)\left(2-\lambda^2\right)}$$

3.22

$$(m_1+m_2)=\frac{\left(\frac{4}{3}-\frac{\lambda^2}{3}\right)\left(\frac{\lambda^4}{8}-\lambda^2+1\right)+\left(2-\lambda^2\right)\left(\frac{\lambda^2}{2}-\frac{2}{3}\right)}{\left(\frac{4}{3}-\frac{\lambda^2}{3}\right)\left(2-2\lambda^2+\frac{\lambda^4}{4}\right)-\left(\frac{4}{3}-\lambda^2\right)\left(2-\lambda^2\right)}$$

Carrying out the multiplications one obtains finally

$$m_1-m_2=0\,,\quad m_1+m_2=\frac{1}{2}\quad \text{i.e.,}\quad m_1=m_2=\frac{1}{4} \tag{3.23}$$

Thus Eq. 3.19 becomes

$$B_n=\frac{1}{4n}\left(-\frac{1}{\lambda}+\sqrt{\frac{1}{\lambda^2}-1}\right)^n+\frac{(-1)^n}{4n}\left(\frac{1}{\lambda}+\sqrt{\frac{1}{\lambda^2}-1}\right)^n;\quad n\geq 3 \tag{3.24}$$

The "potential" φ_1 is consequently determined, and the downwash velocity w_1 at the wing, corresponding to φ_1 can now be obtained. As

$$\frac{w_1}{U}=\frac{1}{2\rho y^2}\left(\frac{\partial\varphi_1}{\partial z}\right)_{z=0}=\frac{1}{2\rho y^2}\left[\frac{\partial u}{\partial z}\frac{\partial\varphi_1}{\partial u}+\frac{\partial v}{\partial z}\frac{\partial\varphi_1}{\partial v}\right]$$

3.25

$$=\frac{1}{2\rho y^2}\;\frac{\cosh u\sin v\,\frac{\partial\varphi_1}{\partial u}+\sinh u\cos v\,\frac{\partial\varphi_1}{\partial v}}{\frac{b}{2}\left(\sinh^2 u+\sin^2 v\right)}$$

one obtains by substituting for the partial derivatives of φ_1 and setting $u = 0$ at the wing

$$\left(\frac{\partial \varphi_1}{\partial z}\right)_{z=0} = \frac{\rho c_0}{b \sin v}\left(\frac{\partial \varphi_1}{\partial u}\right)_{u=0} = -\frac{\rho c_0}{b}\left[\frac{a^2}{2} + ab\cos v + \frac{b^2}{4}\cos^2 v + \frac{b}{2}\sum_{n=2}^{\infty} A_n \cos nv\right] \qquad 3.26$$

Introducing λ , replacing A_n by $\frac{b}{2}B_n$ and v by θ

$$\left(\frac{w_1}{U}\right)_{z=0} = -\frac{c_0}{2b}\,\frac{1}{\left(\frac{1}{\lambda}+\cos\theta\right)^2}\left[\frac{1}{2\lambda^2} + \frac{2}{\lambda}\cos\theta + \cos^2\theta - \sum_{n=2}^{\infty} B_n \cos n\theta\right] \qquad 3.27$$

Rewriting this relation in a more convenient form

$$\left(\frac{w_1}{U}\right)_{z=0} = -\frac{c_0}{2b}\left[\frac{1}{2} + \frac{\cos\theta}{\left(\frac{1}{\lambda}+\cos\theta\right)} - \frac{\cos^2\theta}{2\left(\frac{1}{\lambda}+\cos\theta\right)^2} - \frac{\sum_{n=2}^{\infty} B_n \cos n\theta}{\left(\frac{1}{\lambda}+\cos\theta\right)^2}\right] \qquad 3.28$$

The expression for the series can be obtained by combining Eq. 3.13 and 3.24.

$$\sum_{n=2}^{\infty} B_n \cos n\theta = \left(\frac{1}{2\lambda^2} - \frac{1}{8}\right)\cos 2\theta + \frac{1}{4}\sum_{n=3}^{\infty}\frac{\cos n\theta}{n}\left[\left(\sqrt{\frac{1}{\lambda^2}-1} - \frac{1}{\lambda}\right)^n + (-1)^n\left(\sqrt{\frac{1}{\lambda^2}-1} + \frac{1}{\lambda}\right)^n\right] \qquad 3.29$$

Eq. 3.29 can be rewritten as

$$\sum_{n=2}^{\infty} B_n \cos n\theta = \frac{\cos\theta}{2\lambda} + \frac{\cos 2\theta}{8} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos n\theta}{n}\left[\left(\sqrt{\frac{1}{\lambda^2}-1} - \frac{1}{\lambda}\right)^n + (-1)^n\left(\sqrt{\frac{1}{\lambda^2}-1} + \frac{1}{\lambda}\right)^n\right] \qquad 3.30$$

This series can be summed with the help of the well-known series for log (1 + z), where z is a complex number. Taking the real part of the logarithm

$$-\operatorname{Re}\log(1-z) = \operatorname{Re}\sum_{n=1}^{\infty}\frac{(-1)^{n-1} z^n}{n} \qquad 3.31$$

Thus

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sqrt{\frac{1}{\lambda^2}-1}-\frac{1}{\lambda}\right)^n\cos n\theta = Re\sum_{n=1}^{\infty}\frac{\left(\sqrt{\frac{1}{\lambda^2}-1}-\frac{1}{\lambda}\right)^n e^{in\theta}}{n} = -Re\log\left[1-\left(\sqrt{\frac{1}{\lambda^2}-1}-\frac{1}{\lambda}\right)e^{i\theta}\right] \qquad 3.32$$

and from this, using the relation $e^{i\theta} = \cos\theta + i\sin\theta$

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sqrt{\frac{1}{\lambda^2}-1}-\frac{1}{\lambda}\right)^n\cos n\theta = \frac{1}{2}\log\left[\frac{\lambda^2}{2(1-\sqrt{1-\lambda^2})(1+\lambda\cos\theta)}\right] \qquad 3.33$$

Similarly

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sqrt{\frac{1}{\lambda^2}-1}+\frac{1}{\lambda}\right)^n\cos n\theta = \frac{1}{2}\log\left[\frac{\lambda^2}{2(1+\sqrt{1-\lambda^2})(1+\lambda\cos\theta)}\right] \qquad 3.34$$

Substituting Eq. 3.33 and 3.34 into 3.30 and combining terms

$$\sum_{n=2}^{\infty}B_n\cos n\theta = \frac{1}{2\lambda}\cos\theta + \frac{1}{8}\cos 2\theta - \frac{1}{4}\log\left[2\left(\frac{1}{\lambda}+\cos\theta\right)\right] \qquad 3.35$$

Using this relation in connection with Eq. 3.28 the downwash at the wing is obtained in the following, final form:

$$\left(\frac{w_1}{U}\right)_{z=0} = -\frac{c_0}{2b}\left\{\frac{1}{2}+\frac{\cos\theta}{2\left(\frac{1}{\lambda}+\cos\theta\right)}-\frac{\cos 2\theta}{8\left(\frac{1}{\lambda}+\cos\theta\right)^2}+\frac{\log\left[2\left(\frac{1}{\lambda}+\cos\theta\right)\right]}{4\left(\frac{1}{\lambda}+\cos\theta\right)^2}\right\} \qquad 3.36$$

Eq. 3.36 and 3.1 represent the first particular solution of the problem and the complete solution is obtained by combining these results with the particular solutions obtained in section 2.

4) Calculation of the Lift Distribution by Means of the Least Square Method.

Based on the results of the previous sections the problem of a wing in linearly varying flow can be formulated in the following manner:

The velocity of the free stream is given by the relation

$$U = U_0(1+\lambda\cos\theta)\,, \quad 0 \leq \theta \leq \pi \tag{4.1}$$

The local lift coefficient C_L at a given station along the span is expressed in a series of the form

$$c_L = \frac{L}{\frac{1}{2}\rho U^2 c} = \sum_{n=1}^{\infty} l_n \sin n\theta \tag{4.2}$$

and the connection between the lift coefficient and the effective angle of attack is stated as

$$\sum_{n=1}^{\infty} l_n\left[\sin n\theta - \frac{c}{c_0}\, m\, \frac{w_n}{U}\right] = \frac{c}{c_0}\, m\alpha \tag{4.3}$$

In sections 3 and 4 it was found that for $n \geq 2$

$$\frac{w_n}{U} = \frac{c_0}{b}\left[\frac{n\sin n\theta}{4\sin\theta} + \frac{1}{8(\cos\theta+\frac{1}{\lambda})^2}\left(\frac{\cos(n-1)\theta}{n-1} - \frac{\cos(n+1)\theta}{n+1}\right) + \frac{\cos n\theta}{4(\cos\theta+\frac{1}{\lambda})}\right] \tag{4.4}$$

while for n = 1

$$\frac{w_1}{U} = \frac{c_0}{b}\left[\frac{1}{2} + \frac{\cos\theta}{2(\cos\theta+\frac{1}{\lambda})} - \frac{\cos 2\theta}{8(\cos\theta+\frac{1}{\lambda})^2} + \frac{\log[2(\cos\theta+\frac{1}{\lambda})]}{4(\cos\theta+\frac{1}{\lambda})^2}\right]$$

Equations 4.3 and 4.4 are the basic relations for the solution of the present problem, representing a set of simultaneous equations, linear in the unknown l_n. An exact solution would require, of course, an infinite number of equations, however, an approximation of any desired degree of accuracy can be obtained by replacing the infinite series by finite series with a sufficiently large number of terms.

In the case of a wing in a flow with constant velocity a set of similar simultaneous equations has to be solved, and there are two methods of solution that have been used quite frequently. The first, developed by Glauert (ref. 4) consists of satisfying Eq. 4.3 at a limited number of points along the span and taking as many terms in the series as the number of points. The second method is known as the method of least squares and it was applied to the constant velocity case among others by Gates (ref. 5). This method requires that the square of the error remaining in satisfying equation 4.3, averaged over the span, shall be a minimum. The average of the square of the error, denoted by E, can be expressed on basis of Eq. 4.3 and the relation $dy = -\frac{b}{2}\sin\theta\, d\theta$ as follows:

$$E = \frac{1}{\pi}\int_0^{\pi} \sin\theta \left\{ \sum_{n=1}^{\infty} l_n \left[\sin n\theta - \frac{c}{c_0} m \frac{w_n}{U} \right] - \frac{c}{c_0} m\alpha \right\}^2 d\theta \qquad 4.5$$

And minimizing it

$$\frac{\partial E}{\partial l_s} = \frac{2}{\pi}\int_0^{\pi} \sin\theta \left\{ \sum_{n=1}^{\infty} l_n \left[\sin n\theta - \frac{c}{c_0} m \frac{w_n}{U} \right] - \frac{c}{c_0} m\alpha \right\} \left\{ \sin s\theta - \frac{c}{c_0} m \frac{w_s}{U} \right\} d\theta \qquad 4.6$$

Again the number of equations will be equal to the number of coefficients to be calculated. The set of simultaneous equations obtained by either method is linear in the unknown coefficients l_n and thus the values of the l_n's can be determined without difficulty. The least square method affords, however, very likely a better approximation than the first one, for an equal number of terms and for that reason it was used in working out the solution of the present problem.

Introducing the notation

$$\sin\theta\left[\sin n\theta - \frac{c}{C_o} m \frac{w_n}{U}\right] = G_{n\theta} \qquad 4.7$$

Eq. 4.6 can be rewritten

$$\frac{\partial E}{\partial \ell_s} = \frac{2}{\pi}\int_0^{\pi}\sum_{n=1}^{\infty}\left[\ell_n G_{n\theta} - \frac{c}{C_o} m\alpha \sin\theta\right] G_{s\theta}\, d\theta = 0 \qquad 4.8$$

Here, according to Eq. 4.4 and 4.7

$$G_{n\theta} = \sin n\theta \sin\theta + \frac{mC_o}{4b} n \sin n\theta + \frac{mC_o}{4b}\frac{\cos n\theta \sin\theta}{\frac{1}{\lambda}+\cos\theta} + \frac{\frac{\cos(n-1)\theta}{n-1} - \frac{\cos(n+1)\theta}{n+1}}{\frac{1}{\lambda}\left(\frac{1}{\lambda}+\cos\theta\right)^2}\frac{mC_o}{4b}\sin\theta \quad , \text{ for } n \geq 2$$

and 4.9

$$G_{1\theta} = \sin^2\theta + \frac{mC_o}{4b}\left[\sin\theta + \frac{\sin 2\theta}{\frac{1}{\lambda}+\cos\theta} - \frac{\sin\theta\cos^2\theta}{\left(\frac{1}{\lambda}+\cos\theta\right)^2} - \frac{\sin 2\theta}{2\lambda\left(\frac{1}{\lambda}+\cos\theta\right)^2} - \frac{\sin\theta\cos 2\theta}{4\left(\frac{1}{\lambda}+\cos\theta\right)^2} + \frac{\sin\theta \log\left[2\left(\frac{1}{\lambda}+\cos\theta\right)\right]}{2\left(\frac{1}{\lambda}+\cos\theta\right)^2}\right]$$

Eq. 4.8 represents a set of simultaneous equations that can be written as follows:

$$\ell_1\int_0^{\pi} G_{1\theta}^2\, d\theta + \ell_2\int_0^{\pi} G_{2\theta}G_{1\theta}\, d\theta + \cdots + \ell_n\int_0^{\pi} G_{n\theta}G_{1\theta}\, d\theta = \int_0^{\pi}\frac{c}{C_o} m\alpha\sin\theta\, G_{1\theta}\, d\theta$$

$$\ell_1\int_0^{\pi} G_{1\theta}G_{2\theta}\, d\theta + \ell_2\int_0^{\pi} G_{2\theta}^2\, d\theta + \cdots + \ell_n\int_0^{\pi} G_{n\theta}G_{2\theta}\, d\theta = \int_0^{\pi}\frac{c}{C_o} m\alpha\sin\theta\, G_{2\theta}\, d\theta$$

$$\vdots \qquad 4.10$$

$$\ell_1\int_0^{\pi} G_{1\theta}G_{n\theta}\, d\theta + \ell_2\int_0^{\pi} G_{2\theta}G_{n\theta}\, d\theta + \cdots + \ell_n\int_0^{\pi} G_{n\theta}^2\, d\theta = \int_0^{\pi}\frac{c}{C_o} m\alpha\sin\theta\, G_{n\theta}\, d\theta$$

The next step is the evaluation of the above integrals, representing the coefficients of the unknowns ℓ_n. These integrals are polynomials containing terms of the following general form:

a) $$J_{n,\alpha} = \int_0^{\pi}\frac{\cos n\theta\, d\theta}{\left(\frac{1}{\lambda}+\cos\theta\right)^{\alpha}}$$

b) $$J'_{n,\alpha} = \int_0^{\pi}\frac{\sin n\theta\, d\theta}{\left(\frac{1}{\lambda}+\cos\theta\right)^{\alpha}}$$

c) $$H_{n,\alpha} = \int_0^\pi \frac{\cos n\theta \log[2(\frac{1}{\lambda}+\cos\theta)]\,d\theta}{(\frac{1}{\lambda}+\cos\theta)^\alpha}$$

d) $$H'_{n,\alpha} = \int_0^\pi \frac{\sin n\theta \log[2(\frac{1}{\lambda}+\cos\theta)]\,d\theta}{(\frac{1}{\lambda}+\cos\theta)^\alpha} \qquad 4.11$$

e) $$K = \int_0^\pi \frac{[\log 2(\frac{1}{\lambda}+\cos\theta)]^2 \sin^2\theta\,d\theta}{(\frac{1}{\lambda}+\cos\theta)^4}$$

Introducing this notation and using Eq. 4.9 the integrals of Eq. 4.13 are expressed with the help of suitable trigonomatic relations in the following form:

$$\int_0^\pi G_{n\theta} G_{s\theta}\,d\theta = \left[\frac{1}{4} + \left(\frac{mc_0}{4b}\right)^2 \frac{ns}{2}\right]\left[J_{n-s,0} - J_{n+s,0}\right] + \frac{1}{8}\Big[J_{n+s-2,0} + J_{n+s+2,0} -$$

$$- J_{n-s-2,0} - J_{n-s+2,0}\Big] + \frac{mc_0}{16b}(n+s)\left[J'_{n+s-1,0} - J'_{n+s+1,0} - J'_{n-s-1,0} + J'_{n-s+1,0}\right] +$$

$$+ \frac{mc_0}{32b}\Big\{-2J'_{n+s+2,1} - 2J'_{n+s-2,1} + 4J'_{n+s,1} + \frac{mc_0}{2b}(n-s)\left[J_{n-s-1,1} - J_{n-s+1,1}\right] +$$

$$+ \frac{mc_0}{2b}(n+s)\left[J_{n+s-1,1} - J_{n+s+1,1}\right]\Big\} + \frac{mc_0}{64b}\Big\{\left[\frac{2}{n-1} + \frac{1}{n+1} + \frac{2}{s-1} + \frac{1}{s+1}\right] J'_{n+s-1,2} +$$

$$+ \left[\frac{1}{n-1} + \frac{2}{n+1} + \frac{2}{s-1} + \frac{1}{s+1}\right] J'_{n-s+1,2} - \left[\frac{2}{n-1} + \frac{1}{n+1} + \frac{1}{s-1} + \frac{2}{s+1}\right] J'_{n-s-1,2} -$$

$$- \left[\frac{1}{n-1} + \frac{2}{n+1} + \frac{1}{s-1} + \frac{2}{s+1}\right] J'_{n+s+1,2} - \left[\frac{1}{n-1} + \frac{1}{s-1}\right] J'_{n+s-3,2} + \left[\frac{1}{n-1} + \frac{1}{s+1}\right] J'_{n-s-3,2}$$

$$- \left[\frac{1}{n+1} + \frac{1}{s-1}\right] J'_{n-s+3,2} + \left[\frac{1}{n+1} + \frac{1}{s+1}\right] J'_{n+s+3,2}\Big\} + \frac{1}{2}\left(\frac{mc_0}{8b}\right)^2 \Big\{\Big[2 -$$

$$- \frac{n}{s-1} - \frac{n}{s+1} - \frac{s}{n-1} - \frac{s}{n+1}\Big] J_{n+s,2} + \left[2 + \frac{n}{s-1} + \frac{n}{s+1} + \frac{s}{n-1} + \frac{s}{n+1}\right] J_{n-s,2} - \qquad 4.12$$

$$- \left[1 - \frac{n}{s-1} - \frac{s}{n-1}\right] J_{n+s-2,2} - \left[1 + \frac{n}{s+1} + \frac{s}{n-1}\right] J_{n-s-2,2} - \left[1 + \frac{n}{s-1} + \frac{s}{n+1}\right] J_{n-s+2,2}$$

$$- \left[1 - \frac{n}{s+1} - \frac{s}{n+1}\right] J_{n+s+2,2}\Big\} + \left(\frac{mc_0}{16b}\right)^2 \Big\{\left[\frac{2}{n-1} + \frac{1}{n+1} + \frac{2}{s-1} + \frac{1}{s+1}\right] J_{n+s-1,3} +$$

$$+ \left[\frac{2}{n-1} + \frac{1}{n+1} - \frac{1}{s-1} - \frac{2}{s+1}\right] J_{n-s-1,3} - \left[\frac{1}{n-1} + \frac{1}{s-1}\right] J_{n+s-3,3} + \left[\frac{1}{s+1} - \frac{1}{n-1}\right] J_{n-s-3,3} -$$

$$- \left[\frac{1}{n-1} + \frac{2}{n+1} - \frac{2}{s-1} - \frac{1}{s+1}\right] J_{n-s+1,3} - \left[\frac{1}{n-1} + \frac{2}{n+1} + \frac{1}{s-1} + \frac{2}{s+1}\right] J_{n+s+1,3} +$$

$$+\left[\frac{1}{n+1}-\frac{1}{s-1}\right] J_{n-s+3,3}+\left[\frac{1}{n+1}+\frac{1}{s+1}\right] J_{n+s+3,3}\Bigg\} + \left(\frac{mc_0}{8b}\right)^2\frac{1}{8}\Bigg\{\Bigg[\frac{2}{(s-1)(n-1)}+\frac{1}{(s-1)(n+1)}+$$

$$+\frac{1}{(s+1)(n-1)}\Bigg] J_{n+s-2,4}+\left[\frac{2}{(s-1)(n-1)}+\frac{1}{(s-1)(n+1)}+\frac{1}{(n-1)(s+1)}+\frac{2}{(s+1)(n+1)}\right] J_{n-s,4} -$$

$$-\left[\frac{1}{(s-1)(n-1)}\right] J_{n+s-4,4}-\left[\frac{1}{(s-1)(n-1)}+\frac{1}{(s+1)(n+1)}+\frac{2}{(s+1)(n-1)}\right] J_{n-s-2,4}-\Bigg[\frac{1}{(s-1)(n-1)}+ \qquad 4.12$$

$$+\frac{2}{(s-1)(n+1)}+\frac{1}{(s+1)(n+1)}\Bigg] J_{n-s+2,4}-\left[\frac{1}{(s-1)(n-1)}+\frac{2}{(s-1)(n+1)}+\frac{1}{(s+1)(n+1)}+\frac{2}{(s+1)(n-1)}\right] J_{n+s,4}+$$

$$+\frac{1}{(s-1)(n+1)} J_{n-s+4,4}+\left[\frac{1}{(s-1)(n+1)}+\frac{1}{(s+1)(n-1)}+\frac{2}{(s+1)(n+1)}\right] J_{n+s+2,4}+\frac{1}{(s+1)(n-1)} J_{n-s-4,4}-\frac{1}{(s+1)(n+1)} J_{n+s+4,4}\Bigg\}$$

Furthermore

$$\int_0^\pi G_{n\theta} G_{1\theta}\, d\theta = \left[\frac{3}{8}+\left(\frac{mc_0}{4b}\right)^2\frac{n}{2}\right]\left\{J_{n-1,0}-J_{n+1,0}\right\}-\frac{1}{8}\left\{J_{n-3,0}-J_{n+3,0}\right\}+\frac{mc_0}{16b}(n+1)\Big\{2J'_{n,0}-$$

$$-J'_{n+2,0}-J'_{n-2,0}\Big\}+\frac{mc_0}{32b}\left\{5J'_{n+1,1}-J'_{n-1,1}-3J'_{n+3,1}-J'_{n-3,1}\right\}+\left(\frac{mc_0}{8b}\right)^2\Big\{2J_{n,1}-(1+2n)J_{n+2,1}-$$

$$-(1-2n)J_{n-2,1}\Big\}+\frac{1}{8}\left(\frac{mc_0}{4b}\right)^2\Big\{-\left(\frac{1}{n-1}+2+\frac{3n}{2}\right)J_{n-3,2}+\left(\frac{2}{n-1}+\frac{1}{n+1}+2-\frac{n}{2}\right)J_{n-1,2}-$$

$$-\left(\frac{1}{n-1}+\frac{2}{n+1}-2-\frac{n}{2}\right)J_{n+1,2}+\left(\frac{1}{n+1}-2+\frac{3n}{2}\right)J_{n+3,2}-\frac{2n}{\lambda}J_{n-2,2}+\frac{2n}{\lambda}J_{n+2,2}\Big\}+$$

$$+\frac{1}{8}\left(\frac{mc_0}{4b}\right)\Big\{\frac{1}{2}\left(\frac{3}{n-1}+\frac{3}{n+1}-1\right)J'_{n,2}+\frac{1}{4}\left(3+\frac{2}{n+1}\right)J'_{n+4,2}+\frac{1}{4}\left(3+\frac{2}{n-1}\right)J'_{n-4,2}-\frac{1}{2}\Big(1+$$

$$+\frac{1}{n-1}+\frac{3}{n+1}\Big)J'_{n+2,2}-\frac{1}{2}\left(1+\frac{3}{n-1}+\frac{1}{n+1}\right)J'_{n-2,2}-\frac{1}{\lambda}\left(J'_{n+1,2}+J'_{n-1,2}-J'_{n-3,2}-J'_{n+3,2}\right)\Big\}$$

$$+\frac{mc_0}{32b}\left\{2H'_{n,2}-H'_{n+2,2}-H'_{n-2,2}+\frac{mc_0}{2b}n\left(H_{n-1,2}-H_{n+1,2}\right)\right\}+\frac{1}{32}\left(\frac{mc_0}{4b}\right)^2\Big\{\Big(3-$$

$$-\frac{4}{n-1}\Big)J_{n-4,3}+\left(\frac{4}{n-1}+\frac{4}{n+1}-2\right)J_{n-2,3}+\left(\frac{4}{n-1}-\frac{4}{n+1}-2\right)J_{n,3}-\Big(\frac{4}{n-1}+\frac{4}{n+1}+$$

$$+2\Big)J_{n+2,3}+\left(3+\frac{4}{n+1}\right)J_{n+4,3}+\frac{4}{\lambda}\left(J_{n-3,3}-J_{n-1,3}-J_{n+1,3}+J_{n+3,3}\right)\Big\}+ \qquad 4.13$$

$$+\frac{1}{8}\left(\frac{mc_0}{4b}\right)^2\left\{2H_{n,3}-H_{n+2,3}-H_{n-2,3}\right\}+\left(\frac{mc_0}{32b}\right)^2\Big\{\frac{4}{\lambda(n-1)}J_{n-4,4}-$$

$$-\frac{4}{\lambda(n+1)}J_{n+4,4}-\left[\frac{4}{\lambda(n-1)}+\frac{4}{\lambda(n+1)}\right]J_{n-2,4}-\left[\frac{4}{\lambda(n-1)}-\frac{4}{\lambda(n+1)}\right]J_{n,4}+$$

$$+\left[\frac{4}{\lambda(n-1)}+\frac{4}{\lambda(n+1)}\right]J_{n+2,4}-\left[\frac{2}{(n-1)}-\frac{2}{(n+1)}\right]J_{n+1,4}-\left[\frac{2}{n-1}+\frac{3}{n+1}\right]J_{n-3,4}+\Big[\frac{3}{n-1}+$$

$$+\frac{2}{n+1}\Big]J_{n+3,4}-\left[\frac{2}{n-1}-\frac{2}{n+1}\right]J_{n-1,4}+\frac{3}{n-1}J_{n-5,4}-\frac{3}{n+1}J_{n+5,4}\Big\}+\left(\frac{mc_0}{b}\right)^2\Big\{\Big(\frac{2}{n-1}+$$

$$+\frac{1}{n+1}\Big)H_{n-1,4}-\left(\frac{1}{n-1}+\frac{2}{n+1}\right)H_{n+1,4}-\frac{1}{n-1}H_{n-3,4}+\frac{1}{n+1}H_{n+3,4}\Big\}$$

And

$$\begin{aligned}
\int_0^{\pi} G_{1\theta}^2 \, d\theta = {} & \frac{1}{8} J_{4,0} - \frac{1}{2}\left[1+\left(\frac{mc_0}{4b}\right)^2\right] J_{2,0} + \frac{3mc_0}{8b} J'_{1,0} - \frac{mc_0}{8b} J'_{3,0} + \left[\frac{3}{8}+\frac{1}{2}\left(\frac{mc_0}{4b}\right)^2\right] J_{0,0} + \\
& + \frac{mc_0}{8b}\left\{2J'_{2,1} - J'_{4,1} + \frac{2mc_0}{b}\left[J_{1,1} - J_{3,1}\right]\right\} + \frac{mc_0}{128b}\left\{\frac{3mc_0}{b} J_{0,2} - \frac{mc_0}{b} J_{4,2} + \frac{4}{\lambda} J_{3,2} - \right. \\
& \left. - \frac{2mc_0}{b} J_{2,2} - \frac{4}{\lambda} J_{1,2} + 6 J'_{5,2} - \frac{16}{\lambda} J'_{2,2} - 10 J'_{3,2} + 8 J'_{4,2}\right\} + \frac{mc_0}{4b}\left\{\frac{1}{4} H'_{1,2} - \right. \\
& \left. - \frac{1}{4} H'_{3,2} + \frac{mc_0}{8b}\left[H_{0,2} - H_{2,2}\right]\right\} + \frac{1}{2}\left(\frac{mc_0}{4b}\right)^2 \left\{-\frac{1}{\lambda} J_{0,3} + \frac{1}{\lambda} J_{4,3} + \frac{1}{4} J_{3,3} + \frac{3}{4} J_{5,3} - \right. \\
& \left. - J_{1,3} + \frac{1}{2} H_{1,3} - \frac{1}{2} H_{3,3}\right\} + \left(\frac{mc_0}{32b}\right)^2 \left\{-\frac{9}{2} J_{6,4} - \frac{12}{\lambda} J_{5,4} - \left(3+\frac{8}{\lambda^2}\right) J_{4,4} - \right. \\
& \left. - \frac{4}{\lambda} J_{3,4} + \frac{5}{2} J_{2,4} + \frac{16}{\lambda} J_{1,4} + \left(5+\frac{8}{\lambda^2}\right) J_{0,4}\right\} - \left(\frac{mc_0}{16b}\right)^2 \left\{H_{0,4} + 2H_{2,4} - 3H_{4,4} + \right. \\
& \left. + \frac{4}{\lambda} H_{1,4} - \frac{4}{\lambda} H_{3,4}\right\} + \left(\frac{mc_0}{8b}\right)^2 K \, .
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
\int_0^{\pi} G_{n,\theta} \sin\theta \, d\theta = {} & \frac{1}{4}\left[2J'_{n,0} - J'_{n+2,0} - J'_{n-2,0}\right] + \frac{mc_0}{8b} n \left[J_{n-1,0} - J_{n+1,0}\right] + \\
& + \frac{mc_0}{16b}\left[2J_{n,1} - J_{n+2,1} - J_{n-2,1}\right] + \frac{mc_0}{32b}\left(\frac{1}{n-1}\right)\left[2J_{n-1,2} - J_{n+1,2} - J_{n-3,2}\right] - \\
& - \frac{mc_0}{32b}\left(\frac{1}{n+1}\right)\left[2J_{n+1,2} - J_{n+3,2} - J_{n-1,2}\right] .
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
\int_0^{\pi} G_{1\theta} \sin\theta \, d\theta = {} & \frac{3}{4} J'_{1,0} - \frac{1}{4} J'_{3,0} + \frac{mc_0}{8b}\left[J_{0,0} - J_{2,0}\right] + \frac{mc_0}{8b}\left[J_{1,1} - J_{3,1}\right] - \\
& - \frac{mc_0}{32b}\left[J_{0,2} - J_{4,2}\right] + \frac{mc_0}{16b}\left[H_{0,2} - H_{2,2}\right] - \frac{mc_0}{16b}\frac{1}{\lambda}\left[J_{1,2} - J_{3,2}\right] - \frac{mc_0}{64b}\left[2J_{2,2} - J_{4,2} - J_{0,2}\right]
\end{aligned} \tag{4.16}$$

With the help of Eqs. 4.12 - 4.16 it is possible to calculate the constants in the terms of Eq. 4.10 provided that the integrals appearing in Eq. 4.12 - 4.16 are known.

If the coefficients l_n have been determined, then the problem is solved as C_L, lift force, induced drag, downwash, etc. can easily be calculated on basis of the relations given in section 2. This will be discussed in more detail in connection with the numerical example in section 6.

5) Evaluation of the Integrals $I_{n,\alpha}$, $I'_{n,\alpha}$, $H'_{n,\alpha}$, $H_{n,\alpha}$ and K.

Five types of integrals appear in the simultaneous equations representing the solution of the present problem. The notation used in connection with these integrals is given in Eq. 4.11.

In this section each of the groups of integrals will be taken up and the method of their solution indicated.

a) The Integrals $I_{n,1}$, $I_{n,2}$, $I_{n,3}$ and $I_{n,4}$.

The evaluation of these integrals, in closed form, for different values of n does not cause any difficulties as shown in Appendix C. However, the calculations can be greatly facilitated by the use of a recurrence relation. This recurrence relation is obtained in the following manner:

$$I_{n-1,\alpha} + I_{n+1,\alpha} = \int_0^\pi \frac{\cos(n-1)\theta + \cos(n+1)\theta}{(\frac{1}{\lambda} + \cos\theta)^\alpha} d\theta = 2\int_0^\pi \frac{\cos n\theta \cos\theta \, d\theta}{(\frac{1}{\lambda} + \cos\theta)^\alpha} \qquad 5.1$$

Eq. 5.1 can be written as

$$I_{n-1,\alpha} + I_{n+1,\alpha} = 2\int_0^\pi \frac{\cos n\theta\,(\cos\theta + \frac{1}{\lambda} - \frac{1}{\lambda})}{(\frac{1}{\lambda} + \cos\theta)^\alpha} d\theta =$$

$$= 2\int_0^\pi \frac{\cos n\theta \, d\theta}{(\frac{1}{\lambda} + \cos\theta)^{\alpha-1}} - \frac{2}{\lambda}\int_0^\pi \frac{\cos n\theta \, d\theta}{(\frac{1}{\lambda} + \cos\theta)^\alpha} \qquad 5.2$$

Replacing n by n - 1 in Eq. 5.2 the following recurrence relation is obtained

$$I_{n,\alpha} = 2 I_{n-1,\alpha-1} - \frac{2}{\lambda} I_{n-1,\alpha} - I_{n-2,\alpha} \qquad 5.3$$

It can be shown easily that this relation is valid also for the integrals of the type $I'_{n,\alpha}$, $H'_{n,\alpha}$ and $H_{n,\alpha}$. It should be noted that one of the terms of this recurrence relation has $\frac{2}{\lambda}$ as a factor. As a consequence, the integrals evaluated by Eq. 5.3 are obtained for small values of λ, as small differences of large terms, resulting in inaccuracy, unless the integrals decrease rapidly with increasing values of n. For this reason additional recurrence relations had to be developed for some of the integrals under discussion as shown in the subsequent portions of this section.

Considering now the integrals $I_{n,1}$ a recurrence relation is obtained by setting $\alpha = 1$ in Eq. 5.3. Thus

$$I_{n,1} = 2\, I_{n-1,0} - \frac{2}{\lambda} I_{n-1,1} - I_{n-2,1} \qquad 5.4$$

If the integrals $I_{n,o}$, $I_{n,1}$ and $I_{1,1}$ are known, then it is possible to determine the integrals $I_{n,1}$ for $n \geq 2$ from Eq. 5.4.

Now

$$I_{n,0} = \int_0^{\pi} \cos n\theta \, d\theta = 0 \qquad \text{for } n \neq 0 \qquad 5.5$$

and

$$I_{0,0} = \int_0^{\pi} d\theta = \pi$$

Furthermore,

$$I_{0,1} = \int_0^{\pi} \frac{d\theta}{\frac{1}{\lambda} + \cos\theta} = \pi \frac{\lambda}{\sqrt{1-\lambda^2}} \qquad 5.6$$

and, according to Appendix C:

$$I_{1,1} = \int_0^{\pi} \frac{\cos\theta \, d\theta}{\frac{1}{\lambda} + \cos\theta} = -\pi \frac{\lambda}{\sqrt{1-\lambda^2}} \left[\frac{1-\sqrt{1-\lambda^2}}{1+\sqrt{1+\lambda^2}} \right]^{1/2} \qquad 5.7$$

Then

$$I_{2,1} = -\frac{2}{\lambda} I_{1,1} - I_{0,1}$$

$$I_{3,1} = -\frac{2}{\lambda} I_{2,1} - I_{1,1} \qquad 5.8$$

$$\vdots$$

With the use of Eq. 5.5 - 5.8 all integrals of the group $I_{n,1}$ can be determined.

The next group of integrals to be considered is $I_{n,2}$. Substituting $\alpha = 2$ into Eq. 5.3

$$I_{n,2} = 2\, I_{n-1,1} - \frac{2}{\lambda} I_{n-1,2} - I_{n-2,2} \tag{5.9}$$

The integrals $I_{n,1}$ have been determined above, and according to Appendix D

$$\begin{aligned} I_{1,2} &= -\pi \frac{\lambda^2}{1-\lambda^2} \left[\frac{1-\sqrt{1-\lambda^2}}{1+\sqrt{1-\lambda^2}}\right]^{1/2} \left[1+\frac{1}{\sqrt{1-\lambda^2}}\right] \\ I_{2,2} &= \pi \frac{\lambda^2}{1-\lambda^2} \left[\frac{1-\sqrt{1-\lambda^2}}{1+\sqrt{1-\lambda^2}}\right] \left[2+\frac{1}{\sqrt{1-\lambda^2}}\right] \end{aligned} \tag{5.10}$$

Using these relations the integrals $I_{n,2}$ can all be determined from Eq. 5.9.

Similarly for $\alpha = 3$

$$I_{n,3} = 2\, I_{n-1,2} - \frac{2}{\lambda} I_{n-1,3} - I_{n-2,3} \tag{5.11}$$

The values of $I_{n,2}$ are again known, and according to Appendix C

$$\begin{aligned} I_{1,3} &= -\frac{\pi}{2} \left[\frac{\lambda}{\sqrt{1-\lambda^2}}\right]^3 \left[\frac{1-\sqrt{1-\lambda^2}}{1+\sqrt{1-\lambda^2}}\right]^{1/2} \left[6 - 9\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right) + 3\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right)^2\right] \\ I_{2,3} &= \frac{\pi}{2} \left[\frac{\lambda}{\sqrt{1-\lambda^2}}\right]^3 \left[\frac{1-\sqrt{1-\lambda^2}}{1+\sqrt{1-\lambda^2}}\right] \left[12 - 12\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right) + 3\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right)^2\right] \end{aligned} \tag{5.12}$$

Finally, for $\alpha = 4$

$$I_{n,4} = 2\, I_{n-1,3} - \frac{2}{\lambda} I_{n-1,4} - I_{n-2,4} \tag{5.13}$$

and

$$\begin{aligned} I_{1,4} &= -\frac{\pi}{6} \left[\frac{\lambda}{\sqrt{1-\lambda^2}}\right]^4 \left[\frac{1-\sqrt{1-\lambda^2}}{1+\sqrt{1-\lambda^2}}\right]^{1/2} \left[24 - 72\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right) + 60\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right)^2 - 15\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right)^3\right] \\ I_{2,4} &= \frac{\pi}{6} \left[\frac{\lambda}{\sqrt{1-\lambda^2}}\right]^4 \left[\frac{1-\sqrt{1-\lambda^2}}{1+\sqrt{1-\lambda^2}}\right] \left[60 - 120\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right) + 75\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right)^2 - 15\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right)^3\right] \end{aligned} \tag{5.14}$$

b) The Integrals $I'_{n,1}$ and $I'_{n,2}$.

Whenever Eq. 5.3 does not lead to satisfactory results in determining the integrals belonging to this group, some alternate relations can be substituted. Thus $I'_{n,1}$ can be calculated by combining the recurrence relation 5.3 with a power series in λ, while a second recurrence relation can be used in the case of the integrals $I'_{n,2}$.

$I'_{n,1}$ will be considered first. Eq. 5.3 gives

$$I'_{n,1} = 2 I'_{n-1,0} - \frac{2}{\lambda} I'_{n-1,1} - I'_{n-2,1} \qquad 5.15$$

and

$$I'_{n,0} = \int_0^{\pi} \sin n\theta \, d\theta = \begin{cases} \frac{2}{n} & \text{for odd values of } n \\ 0 & \text{for even values of } n \end{cases} \qquad 5.16$$

Furthermore,

$$I'_{0,1} = 0$$

$$I'_{1,1} = \int_0^{\pi} \frac{\sin\theta \, d\theta}{\frac{1}{\lambda} + \cos\theta} = \log\left(\frac{1+\lambda}{1-\lambda}\right) \qquad 5.17$$

Expanding the logarithm into a power series

$$I'_{1,1} = 2\left[\lambda + \frac{\lambda^3}{3} + \frac{\lambda^5}{5} + \frac{\lambda^7}{7} + \cdots + \frac{\lambda^{2n-1}}{2n-1}\right] \qquad 5.18$$

Substituting Eqs. 5.17 and 5.18 into the recurrence relation 5.15 for $I'_{2,1}$

$$I'_{2,1} = 2 I'_{1,0} - \frac{2}{\lambda} I'_{1,1} - I'_{0,1} = 4 - 4 - 4\left[\frac{\lambda^2}{3} + \frac{\lambda^4}{5} + \cdots + \frac{\lambda^{2n}}{2n+1}\right] \qquad 5.19$$

Similarly

$$I'_{3,1} = 2 I'_{2,0} - \frac{2}{\lambda} I'_{2,1} - I'_{1,1} = 8\left[\frac{\lambda}{12} + \frac{7\lambda^3}{60} + \cdots + \frac{\lambda^{2n-1}(6n-5)}{16n^2-4}\right] \qquad 5.20$$

Expressions of this form were evaluated for the integrals $I'_{n,1}$ covering the range of n from $n = 1$ to $n = 12$, and powers of λ up to λ^{15}. The results are given in Appendix D.

Next $I'_{n,2}$ will be considered. A recurrence relation can be obtained for this group of integrals by partial integration.

$$J'_{n,\alpha} = \int_0^{\pi} \frac{\sin n\theta \, d\theta}{\left(\frac{1}{\lambda} + \cos\theta\right)^{\alpha}} \tag{5.21}$$

Setting

$$u = \frac{1}{\left(\frac{1}{\lambda} + \cos\theta\right)^{\alpha}} \,, \qquad du = \frac{\alpha \sin\theta \, d\theta}{\left(\frac{1}{\lambda} + \cos\theta\right)^{\alpha+1}} \tag{5.22}$$

and

$$dv = \sin n\theta \, d\theta \,, \qquad v = -\frac{\cos n\theta}{n}$$

Thus

$$J'_{n,\alpha} = \int_0^{\pi} u\,dv = \left[uv\right]_0^{\pi} - \int_0^{\pi} v\,du \tag{5.23}$$

Substituting Eqs. 5.22

$$J'_{n,\alpha} = \left[\frac{-\cos n\theta}{n\left(\frac{1}{\lambda} + \cos\theta\right)^{\alpha}}\right]_0^{\pi} + \frac{\alpha}{n} \int_0^{\pi} \frac{\sin\theta \cos n\theta \, d\theta}{\left(\frac{1}{\lambda} + \cos\theta\right)^{\alpha+1}} \tag{5.24}$$

Thus

$$J'_{n,\alpha} = \left[\frac{(-1)^{n-1}}{n\left(\frac{1}{\lambda} - 1\right)^{\alpha}} + \frac{1}{n\left(\frac{1}{\lambda} + 1\right)^{\alpha}}\right] + \frac{\alpha}{n}\left[J'_{n+1,\alpha+1} - J'_{n-1,\alpha+1}\right] \tag{5.25}$$

Replacing n by $n - 1$ and α by $\alpha - 1$ and substituting the following recurrence relation is obtained

$$J'_{n,2} = J'_{n-2,2} + 2(n-1)\, J'_{n-1,1} - \frac{4}{\lambda}\left[\frac{\lambda^2}{1-\lambda^2}\right] \qquad \text{for even } n\text{'s}$$

and (5.26)

$$J'_{n,2} = J'_{n-2,2} + 2(n-1)\, J'_{n-1,1} + 4\left[\frac{\lambda^2}{1-\lambda^2}\right] \qquad \text{for odd } n\text{'s}$$

$I'_{n,1}$ has been determined above, $I'_{0,2} = 0$, and

$$J'_{1,2} = \int_0^{\pi} \frac{\sin\theta\, d\theta}{\left(\frac{1}{\lambda}+\cos\theta\right)^2} = -\int_{\frac{1}{\lambda}+1}^{\frac{1}{\lambda}-1} t^{-2}\,dt = \frac{2\lambda^2}{1-\lambda^2} \qquad 5.27$$

Consequently all the integrals $I'_{n,2}$ can now be determined.

c) The Integrals $H'_{n,2}$.

Again a suitable recurrence relation is obtained by partial integration.

$$H'_{n,\alpha} = \int_0^{\pi} \frac{\log\left[2\left(\frac{1}{\lambda}+\cos\theta\right)\right]}{\left(\frac{1}{\lambda}+\cos\theta\right)^{\alpha}}\,d\theta \qquad 5.28$$

Setting

$$u = \frac{\log\left[2\left(\frac{1}{\lambda}+\cos\theta\right)\right]}{\left(\frac{1}{\lambda}+\cos\theta\right)^{\alpha}}\;; \qquad du = \frac{-\sin\theta\, d\theta}{\left(\frac{1}{\lambda}+\cos\theta\right)^{\alpha+1}} + \frac{\alpha\sin\theta\log\left[2\left(\frac{1}{\lambda}+\cos\theta\right)\right]d\theta}{\left(\frac{1}{\lambda}+\cos\theta\right)^{\alpha+1}}\;; \qquad 5.29$$

$$dv = \sin n\theta\, d\theta\;; \qquad v = -\frac{1}{n}\cos n\theta\;;$$

the following relation is obtained by partial integration

$$H'_{n,\alpha} = \frac{1}{n}\left[\frac{(-1)^{n+1}\log\left[2\left(\frac{1}{\lambda}-1\right)\right]}{\left(\frac{1}{\lambda}-1\right)^{\alpha}} + \frac{\log\left[2\left(\frac{1}{\lambda}+1\right)\right]}{\left(\frac{1}{\lambda}+1\right)^{\alpha}}\right] - \frac{1}{2n}\left[J'_{n+1,\alpha+1} - J'_{n-1,\alpha+1}\right] + \qquad 5.30$$

$$+ \frac{\alpha}{2n}\left[H'_{n+1,\alpha+1} - H'_{n-1,\alpha+1}\right].$$

Replacing n by $n-1$ and α by $\alpha - 1$, substituting $\alpha = 2$ and solving for $H'_{n,2}$

$$H'_{n,2} = H'_{n-2,2} + 2(n-1)H'_{n-1,1} - \left[\frac{(-1)^n\log 2\left(\frac{1}{\lambda}+1\right)}{\frac{1}{\lambda}-1} + \frac{\log 2\left(\frac{1}{\lambda}+1\right)}{\frac{1}{\lambda}+1}\right] + \frac{J'_{n,2} - J'_{n-2,2}}{2} \qquad 5.31$$

The integrals $H'_{n,2}$ can be calculated using Eq. 5.31 if the integrals $H'_{n,1}$ are known. These integrals can be determined with the aid of the recurrence relation 5.3 and of the following expression for the evaluation of the integrals $H'_{n,o}$:

$$H'_{n,o} = \frac{1}{n}\left[(-1)^{n+1}\log\left[2\left(\tfrac{1}{\lambda}-1\right)\right] + \log\left[2\left(\tfrac{1}{\lambda}+1\right)\right]\right] - \frac{1}{2n}\left[J'_{n+1,1} - J'_{n-1,1}\right] \qquad 5.32$$

This relation follows directly from Eq. 5.30.

d) The Integrals $H_{n,2}$, $H_{n,3}$, $H_{n,4}$.

Again partial integration is used to obtain a recurrence formula in addition to that in Eq. 5.3.

$$H_{n,\alpha} = \int_0^{\pi} \frac{\log\left[2\left(\frac{1}{\lambda}+\cos\theta\right)\right]\cos n\theta \, d\theta}{\left(\frac{1}{\lambda}+\cos\theta\right)^{\alpha}} \qquad 5.33$$

with

$$u = \frac{\log 2\left(\frac{1}{\lambda}+\cos\theta\right)}{\left(\frac{1}{\lambda}+\cos\theta\right)^{\alpha}} \; ; \quad du = \frac{-\sin\theta\, d\theta}{\left(\frac{1}{\lambda}+\cos\theta\right)^{\alpha+1}} + \frac{\alpha \sin\theta \log 2\left(\frac{1}{\lambda}+\cos\theta\right)}{\left(\frac{1}{\lambda}+\cos\theta\right)^{\alpha+1}} \; ; \qquad 5.34$$

$$dv = \cos n\theta \, d\theta \; ; \quad v = \frac{1}{n}\sin n\theta \, d\theta$$

integrating by parts

$$H_{n,\alpha} = \frac{2(n-1)}{\alpha-1} H_{n-1,\alpha-1} + H_{n-2,\alpha} + \frac{1}{\alpha-1}\left[J_{n,\alpha} - J_{n-2,\alpha}\right] \qquad 5.35$$

an additional relation is given by Eq. 5.3

$$H_{n,\alpha} = 2 H_{n-1,\alpha-1} - H_{n-2,\alpha} - \frac{2}{\lambda} H_{n-1,\alpha} \qquad 5.36$$

Eliminating from Eqs. 5.35 and 5.36 $H_{n-1,\,\alpha-1}$

$$H_{n,\alpha} = -H_{n-2,\alpha}\frac{\alpha+n-2}{n-\alpha} - \frac{2}{\lambda}\frac{n-1}{n-\alpha} H_{n-1,\alpha} + \frac{1}{n-\alpha}\left[J_{n-2,\alpha} - J_{n,\alpha}\right] \qquad 5.37$$

This is the relation that will be used to evaluate most of the integrals $H_{n,\alpha}$.

Considering first $H_{n,2}$, Eq. 5.37 is rewritten for $\alpha = 2$

$$H_{n,2} = -\frac{n}{n-2} H_{n-2,2} - \frac{2}{\lambda}\frac{n-1}{n-2} H_{n-1,2} + \frac{1}{n-2}\left[J_{n-2,2} - J_{n,2}\right] \qquad 5.38$$

Eq. 5.38 can be used for n 3, as n = 1 gives an identity due to the fact that $H_{-1,2} = H_{1,2}$ and for n = 2 the equation breaks down. Thus $H_{1,2}$ and $H_{2,2}$ have to be found in a different manner.

Considering $H_{1,2}$ Eq. 5.36 gives for n = 1 $\alpha = 2$

$$H_{1,2} = H_{0,1} - \frac{2}{\lambda} H_{0,2} \qquad 5.39$$

This relation introduces two additional unknowns $H_{0,2}$ and H They can be calculated in terms of a power series in λ. $H_{0,1}$ can be transformed in the following way

$$H_{0,1} = \int_0^{\pi} \frac{\log[2(\frac{1}{\lambda}+\cos\theta)]\,d\theta}{\frac{1}{\lambda}+\cos\theta} = \lambda(\log 2 - \log\lambda)\int_0^{\pi}\frac{d\theta}{1+\lambda\cos\theta} + \lambda\int_0^{\pi}\frac{\log(1+\lambda\cos\theta)\,d\theta}{1+\lambda\cos\theta} \qquad 5.40$$

Integrating the first term and introducing well-known power series for $\log(1 + \lambda\cos\theta)$ and for $(1 + \lambda\cos\theta)^{-1}$

$$H_{0,1} = \frac{\lambda\pi}{\sqrt{1-\lambda^2}}\log\frac{2}{\lambda} + \lambda\int_0^{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(\lambda\cos\theta)^n\sum_{m=1}^{\infty}(-1)^m(\lambda\cos\theta)^m d\theta \qquad 5.41$$

Writing now one of the series explicitly and interchanging the order of integration and summation

$$H_{0,1} = \frac{\lambda\pi}{\sqrt{1-\lambda^2}}\log\frac{2}{\lambda} - \lambda\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\pi}\left[1-\lambda\cos\theta+\lambda^2\cos^2\theta+--+\lambda^n\cos^n\theta\right]\lambda^n\cos^n\theta\,d\theta\right. \qquad 5.42$$

The definite integrals can be obtained term for term using the relations

$$\int_0^{\pi}\cos^{(2m-1)}\theta\,d\theta = 0$$

and 5.43

$$\int_0^{\pi}\cos^{2m}\theta\,d\theta = 2\int_0^{\pi/2}\cos^{2m}\theta\,d\theta = \pi\left[\frac{(2m)!}{2^{2m}\,m!\,m!}\right]$$

Substituting Eqs. 5.43 into Eq. 5.42 one obtains

$$H_{0,1} = \frac{\lambda\pi}{\sqrt{1-\lambda^2}} \log\frac{2}{\lambda} - \pi\sum_{m=1}^{\infty} \lambda^{2m+1}(-1)^{2m}\left[1+\frac{1}{2}+\cdots\frac{1}{2m}\right]\frac{(2m)!}{2^{2m}\,m!\,m!} \qquad 5.44$$

This can be written in the simple form

$$H_{0,1} = \frac{\lambda\pi}{\sqrt{1-\lambda^2}} \log\frac{2}{\lambda} - \pi\sum_{m=1}^{\infty} (-1)^{2m}\lambda^{2m+1} P_m \qquad 5.45$$

The **numerical** constants P_m are tabulated in Table I for the range of values of m from m = 1 to m = 13.

Similarly it can be shown that

$$H_{0,2} = J_{0,2}\log\frac{2}{\lambda} - \pi\lambda^2\sum_{m=1}^{\infty}(-1)^{2m}\lambda^{2m}\left[\frac{2m}{1}+\frac{2m-1}{2}+\cdots\frac{1}{2m}\right]\frac{(2m)!}{2^{2m}\,m!\,m!} \qquad 5.46$$

This again is written in the form

$$H_{0,2} = J_{0,2}\log\frac{2}{\lambda} - \pi\lambda^2\sum_{m=1}^{\infty}(-1)^{2m}\lambda^{2m} R_m \qquad 5.47$$

where the constants R_m are also given in Table I.

Using the values of $H_{o,2}$ and $H_{o,1}$ evaluated in this manner it is possible to calculate $H_{1,2}$ from Eq. 5.39.

$H_{2,2}$ is calculated as follows:

Using Eq. 5.36

$$H_{2,2} = 2H_{1,1} - \frac{2}{\lambda}H_{1,2} - H_{0,2} \qquad 5.48$$

and also

$$H_{1,1} = 2H_{0,0} - \frac{2}{\lambda}H_{0,1} - H_{1,1} \qquad 5.49$$

Eliminating $H_{1,1}$ from Eq. 5.48 and 5.49

$$H_{2,2} = 2H_{0,0} - \frac{2}{\lambda}H_{0,1} - \frac{2}{\lambda}H_{1,2} - H_{0,2} \qquad 5.50$$

and using

$$H_{0,0} = \int_0^{\pi}\log\left[2\left(\frac{1}{\lambda}+\cos\theta\right)\right]d\theta = \pi\log\left[\frac{1}{\lambda}+\sqrt{\frac{1}{\lambda^2}-1}\right] \qquad 5.51$$

$H_{2,2}$ can thus be calculated and using the values of $H_{1,2}$ and $H_{2,2}$ all other integrals of the group $H_{n,2}$ can now be determined. To calculate the integrals $H_{n,3}$ and $H_{n,4}$, using the values of the integrals $I_{n,2}$ and $H_{n,2}$ that are already known, a recurrence relation is obtained by adding and subtracting Eqs. 5.35 and 5.36 and eliminating $H_{n-1,\alpha}$ between the two resulting equations. The following relation results:

$$2H_{n,\alpha}\left[1-\frac{1}{\lambda^2}\right](n-1) = 2(n+\alpha-2)H_{n-1,\alpha-1} + \frac{2}{\lambda}(n-\alpha+1)H_{n,\alpha-1} + J_{n,\alpha} - J_{n-2,\alpha} - \frac{1}{\lambda}\left(J_{n-1,\alpha} - J_{n+1,\alpha}\right) \qquad 5.52$$

e) <u>The Integral K.</u>

The last of the integrals that have to be evaluated is

$$K = \int_0^{\pi} \frac{\left[\log 2\left(\frac{1}{\lambda}+\cos\theta\right)\right]^2 \sin^2\theta \, d\theta}{\left(\frac{1}{\lambda}+\cos\theta\right)^4} \qquad 5.53$$

Substituting $2\left(\frac{1}{\lambda}+\cos\theta\right) = t$, $-2\sin\theta\, d\theta = dt$ Eq. 5.53 can be rewritten

$$K = -2^3 \int_0^{\pi} \frac{(\log t)^2 \sin\theta \, dt}{t^4} \qquad 5.54$$

by partial integration, setting

$$du = -\frac{[\log t]^2 dt}{t^4}; \qquad u = \frac{[\log t]^2}{3t^3} + \frac{2\log t}{9t^3} + \frac{2}{27t^3} \qquad 5.55$$

$$v = \sin\theta \qquad dv = \cos\theta \, d\theta$$

one obtains

$$K = -\frac{2^3}{3} M - \frac{2}{9} H_{1,3} - \frac{2}{27} J_{1,3} \qquad 5.56$$

where

$$M = \frac{1}{8}\int_0^{\pi} \frac{\left[\log 2\left(\frac{1}{\lambda} + \cos\theta\right)\right]^2 \cos\theta\, d\theta}{\left(\frac{1}{\lambda} + \cos\theta\right)^3} = \frac{\lambda^3}{8}\int_0^{\pi} \frac{\left[\log \frac{2}{\lambda} + \log(1 + \lambda\cos\theta)\right]^2 \cos\theta\, d\theta}{(1 + \lambda\cos\theta)^3} \qquad 5.57$$

Squaring the binomial it is possible to transform Eq. 5.57

$$M = -\left[\log \frac{2}{\lambda}\right]^2 \frac{1}{8} J_{1,3} + \frac{\log \frac{2}{\lambda}}{4} H_{1,3} + \frac{\lambda^3}{8}\int_0^{\pi} \frac{\left[\log(1 + \lambda\cos\theta)\right]^2 \cos\theta\, d\theta}{(1 + \lambda\cos\theta)^3} \qquad 5.58$$

It remains thus to find

$$\int_0^{\pi} \frac{\left[\log(1 + \lambda\cos\theta)\right]^2 \cos\theta\, d\theta}{(1 + \lambda\cos\theta)^3}$$

Substituting for $\left[\log(1 + \lambda\cos\theta)\right]^2$ the square of the well-known power series

$$\log(1 + \lambda\cos\theta) = \lambda\cos\theta - \frac{\lambda^2\cos^2\theta}{2} + \frac{\lambda^3\cos^3\theta}{3} + \cdots \qquad 5.59$$

the integral can be written in the following form

$$\int_0^{\pi} \frac{\left[\log(1 + \lambda\cos\theta)\right]^2 \cos\theta\, d\theta}{(1 + \lambda\cos\theta)^3} = \int_0^{\pi} \sum_{n=1}^{\infty} a_n \lambda^n \cos^{n+1}\theta\, (1 + \lambda\cos\theta)^{-3}\, d\theta \qquad 5.60$$

where a_n are numerical constants.

Expressing $(1 + \lambda\cos\theta)^{-3}$ in a binomial series, multiplying it with the series in Eq. 5.60 and integrating term by term, using the relations given in Eq. 5.43, one obtains

$$\int_0^{\pi} \frac{\left[\log(1 + \lambda\cos\theta)\right]^2 \cos\theta\, d\theta}{(1 + \lambda\cos\theta)^3} = \sum_{n=2}^{\infty} \frac{\pi}{\lambda}\left[\frac{(2n)!}{2^{2n}\, n!\, n!}\, b_n \lambda^n\right]$$

Setting $\frac{(2n)!}{2^{2n}\, n!\, n!}\, b_n = S_n$ the integral is expressed in the final form

$$\int_0^{\pi} \frac{[\log(1+\lambda\cos\theta)]^2 \cos\theta \, d\theta}{(1+\lambda\cos\theta)^3} = -\frac{\pi}{\lambda} \sum_{n=1}^{\infty} S_n \lambda^{2n} \tag{5.61}$$

The constants S_n are tabulated in Table I. Using this integral and the values of $I_{1,3}$ and $H_{1,3}$ M and finally K can be evaluated according to the relations 5.58 and 5.53.

6) Numerical Example.

As an example, the case of a rectangular wing of constant profile and of constant angle of incidence along the span is worked out. The wing has an aspect ratio $c/b = 6$, the slope of the lift curve for the profile was chosen as $m = 5.67$ per unit radian, the angle of attack selected was one radian. The velocity at one of the wing tips was chosen as twice the velocity at the other, i.e., using the notation introduced in Section 2.

$$U_1 = a - \frac{b}{2} \quad ; \quad U_2 = a + \frac{b}{2} \quad ; \quad U_0 = a$$

and as $U_2 = 2U_1$ i.e.,

$$\frac{2U_1}{U_0} = 2 - \frac{2b}{2a} = 2(1-\lambda) \quad ; \quad \frac{U_2}{U_0} = 1 + \frac{b}{2a} = 1 + \lambda$$

thus $\lambda = \frac{1}{3}$

The example is calculated by the least square method, following the outline of Section 4, using eight terms in the series for C_L, i.e., eight simultaneous equations of the form given in Eq. 4.10.

The first step in the calculations is the evaluation of the integrals $I_{n,\alpha}$, $I'_{n,\alpha}$, $H_{n,\alpha}$, $H'_{n,\alpha}$ and K for $\lambda = \frac{1}{3}$ using the relations given in the previous section. The result of these calculations for the required range of n and α is summarized in Table II.

Using the values in Table II it becomes now possible to evaluate the constants $\int_0^\pi G_{n\theta} G_{s\theta}\, d\theta$ and $\int_0^\pi m\alpha \sin\theta\, G_{n\theta}\, d\theta$ for values of $n = 1$ and $s = 1$ to $n = 8$ and $s = 8$ with the aid of relations 4.12 - 4.16.

The following set of simultaneous equations is the result of the calculations:

$$
\begin{array}{llllllll}
1.9524\, \ell_1 & -.02021\, \ell_2 & -.66778\, \ell_3 & +.00433\, \ell_4 & -.05488\, \ell_5 & -.00880\, \ell_6 & -.02373\, \ell_7 & -.00073\, \ell_8 = a_1 \\
-.02021\, \ell_1 & +2.1670\, \ell_2 & +.00411\, \ell_3 & -.84386\, \ell_4 & +.00504\, \ell_5 & -.09677\, \ell_6 & -.00033\, \ell_7 & -.04354\, \ell_8 = a_2 \\
-.66778\, \ell_1 & +.00411\, \ell_2 & +3.0642\, \ell_3 & +.00883\, \ell_4 & -1.0104\, \ell_5 & -.01794\, \ell_6 & -.13434\, \ell_7 & -.00016\, \ell_8 = a_3 \\
.00433\, \ell_1 & -.84386\, \ell_2 & +.00883\, \ell_3 & +4.1350\, \ell_4 & +.01019\, \ell_5 & -1.1729\, \ell_6 & +.00500\, \ell_7 & -.16988\, \ell_8 = a_4 \\
-.05488\, \ell_1 & +.00504\, \ell_2 & -1.0104\, \ell_3 & +.01019\, \ell_4 & +5.3879\, \ell_5 & +.01081\, \ell_6 & -1.3337\, \ell_7 & +.00477\, \ell_8 = a_5 \\
-.00880\, \ell_1 & -.09677\, \ell_2 & -.01794\, \ell_3 & -1.1729\, \ell_4 & +.01081\, \ell_5 & +6.8192\, \ell_6 & +.14655\, \ell_7 & -1.4936\, \ell_8 = a_6 \\
-.02373\, \ell_1 & -.00033\, \ell_2 & -.13434\, \ell_3 & +.00500\, \ell_4 & -1.3337\, \ell_5 & +.14655\, \ell_6 & +8.4274\, \ell_7 & +.01375\, \ell_8 = a_7 \\
-.00073\, \ell_1 & -.04354\, \ell_2 & -.00016\, \ell_3 & -.16988\, \ell_4 & +.00477\, \ell_5 & -1.4936\, \ell_6 & +.01375\, \ell_7 & +10.212\, \ell_8 = a_8
\end{array}
\qquad 6.1
$$

Here $a_n = \int_0^\pi \alpha m \sin\theta\, G_{n,\theta}\, d\theta$

$$
\begin{array}{llll}
a_1 = 9.8505 & a_3 = 1.4795 & a_5 = -.21557 & a_7 = -.07204 \\
a_2 = -.37915 & a_4 = -.00363 & a_6 = -.00011 & a_8 = +.00001
\end{array}
\qquad 6.2
$$

Equations 6.1 can be solved easily by inverting the matrix of the coefficients of ℓ_n. The method used for inverting the matrix is the inversion by submatrices (Ref. 6). A brief outline of this method is given in Appendix E. The result of the inversion is the following set of equations:

$$
\begin{array}{llllllll}
.55876\, a_1 & +.00508\, a_2 & +.13298\, a_3 & +.00037\, a_4 & +.03282\, a_5 & +.00101\, a_6 & +.00887\, a_7 & +.00019\, a_8 = \ell_1 \\
.00508\, a_1 & +.50643\, a_2 & +.00003\, a_3 & +.11168\, a_4 & -.00087\, a_5 & +.02820\, a_6 & +.00067\, a_7 & +.00814\, a_8 = \ell_2 \\
.13298\, a_1 & +.00003\, a_2 & +.38174\, a_3 & +.00140\, a_4 & +.07758\, a_5 & +.00092\, a_6 & +.01872\, a_7 & +.00011\, a_8 = \ell_3 \\
.00037\, a_1 & +.11168\, a_2 & +.00140\, a_3 & +.28009\, a_4 & -.00126\, a_5 & +.05260\, a_6 & -.00127\, a_7 & +.01283\, a_8 = \ell_4 \\
.03282\, a_1 & -.00087\, a_2 & +.07758\, a_3 & -.00126\, a_4 & +.20901\, a_5 & -.00113\, a_6 & +.03443\, a_7 & -.00033\, a_8 = \ell_5 \\
.00100\, a_1 & +.02820\, a_2 & +.00092\, a_3 & +.05260\, a_4 & -.01126\, a_5 & +.16156\, a_6 & -.00304\, a_7 & +.02463\, a_8 = \ell_6 \\
.00887\, a_1 & -.00067\, a_2 & +.01872\, a_3 & -.00127\, a_4 & +.03443\, a_5 & -.00304\, a_6 & +.12449\, a_7 & -.00065\, a_8 = \ell_7 \\
.00019\, a_1 & +.00814\, a_2 & +.00011\, a_3 & +.01283\, a_4 & -.00033\, a_5 & +.02463\, a_6 & -.00065\, a_7 & +.10177\, a_8 = \ell_8
\end{array}
\qquad 6.3
$$

Substituting the values of a_n from 6.2, and carrying out the indicated operations, one obtains

$$\begin{aligned} \ell_1 &= 5.2977 & \ell_5 &= .1613 \\ \ell_2 &= -.1423 & \ell_6 &= .0002 \\ \ell_3 &= .7270 & \ell_7 &= .0435 \\ \ell_4 &= -.04143 & \ell_8 &= -.0013 \end{aligned} \qquad 6.3$$

Using these values of the coefficients the local lift coefficients can be determined, according to Eq. 2.5, i.e.,

$$C_L = \frac{C_0}{C} \sum_{n=1}^{\infty} \ell_n \sin n\theta \qquad 6.4$$

Substituting for θ the value corresponding to the span station using the following relation based on Eqs. 2.2 and 2.6

$$\eta = \frac{y-a}{b/2} = \cos\theta \quad , \quad 0 \leq \theta \leq \pi \qquad 6.5$$

Here η is the distance along the span, measured from the center of the span, and expressed as a fraction of the span. The lift force referred to $q_0 C_0 = \frac{1}{2} \rho C_0 U_0^2$

$$\frac{L}{\frac{1}{2}\rho C_0 U_0^2} = C_L \frac{U^2}{U_0^2} \frac{C}{C_0} = C_L \left(1 + \lambda \cos\theta\right)^2 \frac{C}{C_0} \qquad 6.6$$

The induced angle of attack can be determined according to Eq. 2.17

$$\frac{w}{U} = \frac{C_L}{m} - \alpha \qquad 6.7$$

The induced drag coefficient is given by

$$C_D = C_L \frac{w}{U} \qquad 6.8$$

and the ratio of the drag force per unit span to the square of the velocity at the center

$$\frac{D}{\frac{1}{2}\rho C_0 U_0^2} = C_D \left(1 + \lambda \cos\theta\right)^2 \frac{C}{C_0} \qquad 6.9$$

The quantities given by Eqs. 6.4 - 6.8 were evaluated, using the coefficients ℓ_k from Eq. 6.3, at 10 stations along the span. The results are given in Table III and plotted in Fig. 3 - 6.

For the sake of comparison the case of constant velocity across the span was also calculated for the wing under consideration.

The following set of simultaneous equations was obtained in this case

$$\begin{aligned} 1.8958\ \ell_1 - .6447\ \ell_3 - .0540\ \ell_5 - .02400\ \ell_7 &= a_1 = 9.6641 \\ -.64470\ \ell_1 + 3.0324\ \ell_3 - .9927\ \ell_5 - .1336\ \ell_7 &= a_3 = -1.5120 \\ -.0540\ \ell_1 - .9927\ \ell_3 - 5.3636\ \ell_5 - 1.3179\ \ell_7 &= a_5 = -.2160 \\ -.02400\ \ell_1 - .1336\ \ell_3 - 1.3179\ \ell_5 + 8.4058\ \ell_7 &= a_7 = -.07200 \end{aligned} \qquad 6.10$$

And the solution, obtained by inversion,

$$\begin{aligned} .5789\ a_1 + .1480\ a_3 + .0356\ a_5 + .0096\ a_7 &= \ell_1 \\ .1480\ a_1 + .4272\ a_3 + .0856\ a_5 + .0206\ a_7 &= \ell_3 \\ .0355\ a_1 + .0856\ a_3 + .2111\ a_5 + .0346\ a_7 &= \ell_5 \\ .0096\ a_1 + .0206\ a_3 + .0346\ a_5 + .1247\ a_7 &= \ell_7 \end{aligned} \qquad 6.11$$

The values of the coefficients ℓ_n are given below. The corresponding coefficients for $\lambda = 1/3$ are also shown, for sake of comparison.

	$\lambda = 0$	$\lambda = 1/3$		$\lambda = 0$	$\lambda = 1/3$
$\ell_1 =$	5.3361	5.2977	$\ell_5 =$	.1507	.1613
$\ell_2 =$	0	-.1423	$\ell_6 =$	0	.0002
$\ell_3 =$	.6872	.7270	$\ell_7 =$	.0412	.0435
$\ell_4 =$	0	-.04143	$\ell_8 =$	0	-.0013

Again distributions of C_L, C_D, w/U, etc., were calculated. The results are tabulated in Table IV and plotted in Fig. 3 - 6.

The total lift, obtained by integrating the lift over the span was found to be L = 4.28 , the total drag D = 0.550 , and thus L/D = 7.78 indicating an increase in the L/D ratio due to the non-uniformity of the flow as the corresponding value for the constant velocity case is $L/D = 8.25$

Fig. 3 indicates that the lift coefficient is practically unchanged due to the nonuniformity of the air stream, as there is only a slight decrease in C_L at the high speed end of the span and a slight increase in C_L at the low speed end.

7) Comparison of Present Results with Those of K. Bausch and of F. Vandrey.

F. Vandrey (Ref. 1) treats the problem of a wing in a spanwise varying flow by replacing the continuous function of the velocity versus spanwise distance by a step function. He takes n strips of constant velocity separated by plane surfaces of discontinuity, i.e., by vortex sheets normal to the span. The flow within each of the strips is irrotational and by considering the flow far downstream from the wing Vandrey reduces the problem to n 2-dimensional potential problems. Let the potential in strips i and j be denoted by Φ_i and Φ_j and the free stream velocities, taken constant across the strips, by V_i and V_j. Then the boundary conditions along the surfaces of discontinuity separating the strips can be written according to C. Koning (Ref. 7) as follows:

a) No pressure drop across the boundary $V_i \Phi_i = V_j \Phi_j$

b) No flow across the boundary $\dfrac{V_i}{V_j} = \dfrac{\partial\Phi_i/\partial n}{\partial\Phi_j/\partial n}$ 7.1

The wing equation (Eq. 2.13) for each of the strips is given in the form

$$\frac{1}{2}\left(\frac{\partial\Phi_i}{\partial z}\right)_{z=0} + \frac{4\Phi_i}{cm} = \alpha V_i \qquad 7.2$$

Setting $V_i = V_0(1 + v_i)$ where V_0 is the mean value of the free stream velocity, and noticing from the first of Eq. 7.1 that the velocities V_i are inversely proportional to the potential Φ_i, Vandrey assumes the potential to be of the form $\Phi_i = (1 - v_i)\Phi_0$ where Φ_0 represents a potential corresponding to the mean value of the free stream velocity V_o. Dividing Eq. 7.2 by $(1 - v_i)$ and expressing Φ_i in terms of Φ_o one obtains

$$\frac{1}{2}\frac{\partial\Phi_o}{\partial z} + 4\frac{\Phi_o}{cm} = \alpha(1+2v_i)V_o \qquad 7.3$$

This equation can be considered as representing the flow over a fictitious wing with angle of attack $\alpha(1 + 2 v_i)$ in a flow of constant velocity V_o, a problem that can be solved with the help of the Prandtl wing theory. As $\Gamma = 2\Phi$

$$\Gamma_i = (1-v_i)\Gamma_o \qquad 7.4$$

and neglecting second order terms

$$\rho V_o \Gamma_o = \rho V_i \Gamma_i \qquad 7.5$$

Thus the lift distribution over the wing in nonuniform flow is, according to Vandrey, the same as that over the fictitious wing in uniform flow.

By assuming an infinite number of strips, i.e., replacing effectively $V_i = V_o(1+v_i)$ by $V(y) = V_o[1+v(y)]$ Vandrey's method can also be applied to the case of continuously varying velocity.

Vandrey calls attention to the fact that the second of the boundary conditions 7.1 can only be satisfied by Φ_i if first order terms in the induced velocities are neglected. This is due to the assumption $\Phi_i = \Phi_o(1-v_i)$. Vandrey tries to estimate the error caused by this inaccuracy, however, his reasoning seems to be rather speculative. In order to obtain a more factual evaluation of the merits of Vandrey's work, the present example was also worked out using his method. The results are summarized in Table V and plotted in Figs. 7 - 10.

The solution proposed by K. Bausch (Ref. 2) is based on the assumption that the Prandtl wing equation remains valid for spanwise varying free stream velocity. Bausch writes the wing equation in the form

$$\frac{1}{2}\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} + 4\frac{\Phi}{c_m} = V(y)\,\alpha \qquad 7.6$$

where V(y) is the varying free stream velocity. Bausch reasons further that according to Eq. 7.6 Φ is only a function of the product of $V(y)\alpha$ rather than of $V(y)$ and α separately, thus the value of Φ as determined from 7.6 is unchanged if the following substitution is made

$$V(y)\,\alpha = V_0\,\alpha(y) \qquad 7.7$$

where V_0 is a now constant free stream velocity and $\alpha(y)$ a fictitious twist. Thus, according to Bausch

$$[\Phi]_{V(y)\alpha} = [\Phi]_{V_0\alpha(y)} \qquad 7.8$$

and consequently the lift coefficient

$$[C_L]_{V(y)\alpha} = \frac{V_0}{V(y)}\,[C_L]_{V_0\alpha(y)} \qquad 7.9$$

Based on these relations Bausch calculates the lift coefficient-distribution for a fictitious wing with twist $\alpha(y)$ in a flow of constant velocity V_0 and converts it with the help of relation 7.9 into the lift coefficient distribution across the wing without twist in a flow of varying velocity V(y).

Bausch's method was also applied to the present example and the results are given in Table V and plotted in Figs. 7 - 10.

A comparison of the approximate methods of Vandrey and Bausch with the present solution, also plotted in Figs. 7 - 10, seems to indicate that both of these approximations are reasonably good, at least for values of $\lambda \leq 1/3$. It has to be expected, of course, that the approximations break down if the nonuniformity of the flow becomes excessive, i.e., as λ approaches unity. Concerning the relative merits of the methods of Vandrey and Bausch it should be noticed, that Vandrey's results are in good agreement at the high speed end of the span, while Bausch's data coincides with the present calculations on the low speed side. Thus it seems that the error introduced into Vandrey's calculations by the assumption $\Phi_i = \Phi_o(1-v_i)$ is of the same order of magnitude as that in Bausch's method due to the application of the results of the Prandtl wing theory to non-uniform flow.

Part II. Lifting Line Theory for a Wing in Linearly Varying Flow Between Two Parallel Walls.

1) Formulation of the Problem and Solution in the Trefftz Plane.

The flow over a wing located between two parallel walls normal to the wing span is of interest in connection with the design of axial flow fans and compressors, two-dimensional windtunnel tests, etc. The problem can be stated in the following manner: A wing of span b (Fig. 11) is located between two solid walls parallel to the x - z plane, situated at $y = a$ and $y = a + b$. The velocity of the air-stream is given by $U = y$. The degree of variation of U across the span is expressed, as it was done in the problem discussed in Part I, in terms of the parameter λ. According to Fig. 2

$$\lambda = \frac{U_2 - U_1}{2U_0} = \frac{U_2 - U_1}{U_2 + U_1} \qquad 1.1$$

Expressing this ratio in terms of the notation used in Fig. 11

$$\lambda = \frac{b}{2a+b} \qquad 1.2$$

By considering the flow at $x = +\infty$, i.e., in the Trefftz plane, one can apply the relations obtained in Section 1 of Part I to the present problem. Thus a solution has to be found to the following differential equation (Eq. 2.2):

$$\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} - \frac{2}{y}\frac{\partial \varphi}{\partial y} = 0 \qquad 1.3$$

This equation determines the "potential function" φ, which was defined in such a manner that at $z = +0$

$$\varphi(y,0) = \frac{1}{2}\, L(y) \qquad 1.4$$

where L(y) is the lift per unit span.

Equation 1.3 has to satisfy the following boundary conditions:

a) the "potential" φ has to vanish as z approaches infinity.

b) the components of the disturbance velocity in the direction normal to the walls has to vanish at the wall, i.e., according to Eq. 1.11, Part I.

$$(v)_{y=a} = \frac{1}{\rho y}\left(\frac{\partial \varphi}{\partial y}\right)_{y=a} = 0 \ ; \ (v)_{y=a+b} = \frac{1}{\rho y}\left(\frac{\partial \varphi}{\partial y}\right)_{y=a+b} = 0 \ . \qquad 1.5$$

To satisfy these boundary conditions φ is expressed in the following form:

$$\varphi(y,z) = l_\mu e^{-\mu z} Y_\mu(y) \qquad 1.6$$

where $Y_\mu(y)$ is an unknown function.

By substituting Eq. 1.6 into Eq. 1.3 the following differential equation for $Y_\mu(y)$ is obtained

$$\frac{\partial^2 Y_\mu(y)}{\partial y^2} + \mu^2 Y_\mu(y) - \frac{2}{y}\frac{\partial Y_\mu(y)}{\partial y} = 0 \qquad 1.7$$

Eq. 1.7 is one of the equivalent forms of Bessel's differential equation and its solution is well-known (E.g. cf. von Kármán and Biot, ref. 8). It can be shown that

$$Y_\mu(y) = y^{3/2} Z_{3/2}(\mu y) \qquad 1.8$$

provided that $Z_{3/2}$ denotes the general solution of Bessel's equation of 3/2 order, that can be expressed in the following form

$$Z_{3/2}(\mu, y) = \frac{A}{\sqrt{\frac{\pi \mu y}{2}}}\left[\frac{\sin \mu y}{\mu y} - \cos \mu y\right] + \frac{B}{\sqrt{\frac{\pi \mu y}{2}}}\left[\sin \mu y + \frac{\cos \mu y}{\mu y}\right] \qquad 1.9$$

where A and B are arbitrary constants. With the help of this relation

Eq. 1.8 $Y_\mu(y)$ can be rewritten as follows:

$$Y_\mu(y) = A'(\sin\mu y - \mu y\cos\mu y) + B'(\mu y\sin\mu y + \cos\mu y) \qquad 1.10$$

where the constants A' and B' have to be evaluated from the boundary conditions. As $\partial\varphi/\partial\varphi = 0$ at y = a and y = a + b, it follows from Eq. 1.6 using the relation

$$\frac{\partial Y_\mu(y)}{\partial(\mu y)} = A'\mu y\sin\mu y + B'\mu y\cos\mu y \qquad 1.11$$

that

$$\begin{aligned} A'\sin\mu a + B'\cos\mu a &= 0 \\ A'\sin\mu(a+b) + B'\cos\mu(a+b) &= 0 \end{aligned} \qquad 1.12$$

A' and B' are constants different from zero, thus the determinant of their factors has to vanish, i.e.,

$$\sin\mu a\cos\mu(a+b) = -\cos\mu a\sin\mu(a+b) \qquad 1.13$$

This equation is satisfied if

$$\mu b = n\pi \qquad 1.14$$

Substituting this result into Eq. 1.12, one obtains

$$B' = -A'\tan\left[\frac{n\pi a}{b}\right] \qquad 1.15$$

and finally, by substituting Eqs. 1.14 and 1.15 into Eq. 1.10 and introducing a new constant C

$$Y_n(y) = C\left[\sin\frac{n\pi(y-a)}{b} - \frac{n\pi y}{b}\cos\frac{n\pi(y-a)}{b}\right] \qquad 1.16$$

It is now convenient to set

$$C = \frac{1}{4}\rho c_0 V_0^2 \ell_n \qquad 1.17$$

where the subscript zero denotes conditions at the center of the span. Using Eqs. 1.16 and 1.17 a particular solution of Eq. 1.3 can be expressed in the following form

$$\varphi_n = \frac{l_n}{4}\rho c_o U_o^{\,2} e^{-\frac{n\pi z}{b}}\left[\sin\frac{n\pi(y-a)}{b} - \frac{n\pi y}{b}\cos\frac{n\pi(y-a)}{b}\right] \qquad 1.18$$

According to Eq. 1.4 the lift force per unit span is given by twice the value of the "potential" at $z = 0$, i.e.,

$$L(y) = 2\varphi(y,0) = \varphi_o + \sum_{n=1}^{\infty}\varphi_n(y,0) \qquad 1.19$$

The particular solution of zero order φ_o = const., and thus it satisfies automatically the differential equation and the boundary conditions.

Substituting Eq. 1.18 into Eq. 1.8 a dimensionless expression for the lift distribution can be obtained:

$$\frac{L(y)}{\frac{1}{2}\rho U_o^2 c_o} = L_o + \sum_{n=1}^{\infty} l_n\left[\sin\frac{n\pi(y-a)}{b} - \frac{n\pi y}{b}\cos\frac{n\pi(y-a)}{b}\right] \qquad 1.20$$

The downwash velocity at the wing is connected with φ as given in Eq. 1.11 of Part I, i.e.,

$$w_n = \frac{1}{2\rho y}\frac{\partial\varphi}{\partial z} \qquad 1.21$$

This expression can be transformed with the help of relations 1.18 and 1.21 into the following relation, giving the induced angle at the wing (z = 0).

$$\left(\frac{w}{U}\right)_{z=0} = \sum_{n=1}^{\infty}\frac{w_n}{U} = \sum_{n=1}^{\infty} -\frac{l_n c_o}{8}\left(\frac{U_o}{U}\right)^2\frac{n\pi}{b}\left[\sin\frac{n\pi(y-a)}{b} - \frac{n\pi y}{b}\cos\frac{n\pi(y-a)}{b}\right] \qquad 1.22$$

With the help of Eqs. 1.21 and 1.22 the relation between the lift per unit span and effective angle of incidence can be easily expressed. Again, as in the problem discussed in Part I, the value of the coefficients l_n will be evaluated from this relation.

Thus, combining Eqs. 1.20 and 1.22 with the equation

$$L = \frac{1}{2} \rho V^2 c m \left[\alpha + \frac{w}{V} \right] \qquad 1.23$$

one obtains

$$L_0 + \sum_{n=1}^{\infty} \ell_n \left[\sin \frac{n\pi(y-a)}{b} - \frac{n\pi y}{b} \cos \frac{n\pi(y-a)}{b} \right] \left[1 + \frac{cm}{8} \frac{n\pi}{b} \right] = m\alpha \frac{c}{c_0} \left(\frac{V}{V_0} \right)^2 \qquad 1.24$$

Introducing the notation

$$\frac{\pi(y-a)}{b} = \xi \qquad 1.25$$

i.e.,

$$\frac{V}{V_0} = \frac{y}{a + \frac{b}{2}} = \frac{\xi + \frac{\pi a}{b}}{1 + \frac{2a}{b}} \qquad 1.26$$

into Eqs. 1.20 and 1.24, one obtains the following final relations representing the solution of the problem:

$$\frac{L}{\frac{1}{2} \rho V_0^2 c_0} = L_0 + \sum_{n=1}^{\infty} \ell_n \left[\sin n\xi - n\left(\xi + \frac{a\pi}{b}\right) \cos n\xi \right] \qquad 1.27$$

and

$$L_0 + \sum_{n=1}^{\infty} \ell_n \left[\sin n\xi - n\left(\xi + \frac{a\pi}{b}\right) \cos n\xi \right] \left[1 + \frac{c}{b} \frac{n\pi m}{8} \right] = \alpha m \frac{c}{c_0} \frac{4}{\pi^2} \left[\frac{\xi + \frac{a}{b}\pi}{1 + \frac{2a}{b}} \right]^2 \qquad 1.28$$

2) The Calculation of the Lift Distribution by the Least Square Method.

The discussion in Section 4 of Part I concerning the method to be used in calculating the coefficients ℓ_n appearing in Eq. 4.2 of Part I applies also to the present problem. Thus, in the light of those remarks the least square method will be used to obtain a set of simultaneous equations, linear in the unknowns ℓ_n.

Using the notation

$$G_{n\xi} = \left[\sin n\xi - n\left(\xi + \frac{a\pi}{b}\right)\cos n\xi\right]\left[1 + \frac{c}{b}\,\frac{n\pi m}{8}\right] \qquad 2.1$$

and following the outline given in Section 4 of Part I, the following expressions are obtained:

$$\frac{\partial E}{\partial \ell_s} = \frac{2}{\pi}\int_0^\pi \sum_{n=1}^{\infty}\left[L_o + \ell_n G_{n\xi} - m\alpha\frac{c}{c_o}\frac{4}{\pi^2}\left(\frac{\xi + \frac{a}{b}\pi}{1 + \frac{2a}{b}}\right)^2\right] G_{s\xi}\, d\xi = 0 \qquad 2.2$$

and

$$\frac{\partial E}{\partial L_o} = \frac{2}{\pi}\int_0^\pi \sum_{n=1}^{\infty}\left[L_o + \ell_n G_{n\xi} - m\alpha\frac{c}{c_o}\frac{4}{\pi^2}\left(\frac{\xi + \frac{a}{b}\pi}{1 + \frac{2a}{b}}\right)^2\right] d\xi = 0$$

The integrals appearing in the simultaneous equation represented by Eq. 2.2 will now be determined:

$$\int_0^\pi G_{n\xi} G_{s\xi}\, d\xi = \left[1 + \frac{nm\pi}{8}\frac{c}{b}\right]\left[1 + \frac{sm\pi}{8}\frac{c}{b}\right]\int_0^\pi\left[\sin n\xi - n\xi\cos n\xi - \right. \qquad 2.3$$

$$\left. - \frac{na\pi}{b}\cos n\xi\right]\left[\sin s\xi - s\xi\cos s\xi - \frac{sa\pi}{b}\cos s\xi\right] d\xi$$

Carrying out the indicated multiplication, integrating and rearranging the terms, one obtains for $n \neq m$, using the notation

$$\lambda = \frac{b}{2a+b} \qquad 2.4$$

$$\int_0^\pi G_{n\xi} G_{s\xi}\, d\xi = \left[1 + \frac{nm\pi}{8}\frac{c}{b}\right]\left[1 + \frac{sm\pi}{8}\frac{c}{b}\right]\frac{ns\pi}{2\lambda}\left[\frac{1}{(n+s)^2} + \frac{1}{(n-s)^2}\right]\left[\lambda - 1 + (-1)^{n+s}\right] \qquad 2.5$$

and similarly, for $n = m$

$$\int_0^\pi G_{n\xi}^2\, d\xi = \left[1 + \frac{nm\pi}{8}\frac{c}{b}\right]^2\left[\frac{5\pi}{4} + \frac{n^2\pi^3}{2}\left(\frac{1-\lambda^2}{4\lambda^2} + \frac{1}{3}\right)\right] \qquad 2.6$$

Furthermore

$$\frac{4m\alpha}{\pi^2}\int_0^{\pi}\left(\frac{\xi+\frac{a}{b}\pi}{1+\frac{2a}{b}}\right)^2 G_{n\xi}\,d\xi = \frac{16}{\pi^2}\,m\alpha\lambda^2\left[1+\frac{nm\pi}{8}\,\frac{c}{b}\right]\left\{\frac{1}{n^3}\left[2(-1)^n-1\right]+ + \frac{\pi^2}{4n}\left[\left(1-\frac{1}{\lambda}\right)^2-\left(1+\frac{1}{\lambda}\right)^2(-1)^n\right]\right\} \qquad 2.7$$

and

$$\int_0^{\pi}\left(\frac{\xi+\frac{a}{b}\pi}{1+\frac{2a}{b}}\right)^2 d\xi = \frac{\pi^3}{24}\,\frac{1}{\lambda}\left[(1+\lambda)^3-(1-\lambda)^3\right] \qquad 2.8$$

Finally

$$\int_0^{\pi} G_{n\xi}\,d\xi = \frac{2}{n}\left[1+\frac{nm\pi}{8}\,\frac{c}{b}\right]\left[1-(-1)^n\right] \qquad 2.9$$

With the help of Eqs. 2.5 - 2.9 the simultaneous equations can be solved for the coefficients ℓ_n. If the ℓ_n's are known, the lift distribution, the downwash angle, and the induced drag can be determined with the help of relations 1.20 and 1.22.

3) Numerical Example.

The distribution of lift, drag and induced angle of incidence on a wing between two parallel walls is determined using the results of the previous sections.

The wing selected for this example is rectangular in plan form and has an aspect ratio of 6, i.e., $c = c_o$ = const. and $b/c = 6$. The slope of the lift curve for the wing profile was chosen as $m = 5.67$ per unit radian. The parallel walls are located, in terms of the notation used in Fig. 11, at $y = a$ and $y = a + b$, and the value of was chosen as 1/3, corresponding to a ratio of the free stream velocities at the two walls of 2:1.

Using these numerical data the integrals given by Eqs. 2.5 - 2.9

were determined for values of n and s ranging from 1 to 8. The results are presented in Table VII. Substituting these values into Eq. 2.2 the following simultaneous equations were obtained:

$$\begin{array}{llllllllll}
3.1416\,L_o & +5.4844\,\ell_1 & +\;0 & +2.8177\,\ell_3 & +\;0 & +2.2844\,\ell_5 & +\;0 & +2.05583\,\ell_7 & +\;0 & = 18.473 \\
5.4844\,L_o & +75.386\,\ell_1 & -41.691\,\ell_2 & +4.2670\,\ell_3 & -16.171\,\ell_4 & +2.7736\,\ell_5 & -12.594\,\ell_6 & +2.3541\,\ell_7 & -11.199\,\ell_8 & = 64.902 \\
0 & -41.691\,\ell_1 & +451.12\,\ell_2 & -180.44\,\ell_3 & +15.109\,\ell_4 & -51.388\,\ell_5 & +8.2876\,\ell_6 & -36.118\,\ell_7 & +6.5725\,\ell_8 & = -26.119 \\
2.8177\,L_o & +4.2670\,\ell_1 & -180.44\,\ell_2 & +1471.5\,\ell_3 & -504.93\,\ell_4 & +37.768\,\ell_5 & -119.01\,\ell_6 & +18.183\,\ell_7 & -76.304\,\ell_8 & = 35.263 \\
0 & -16.171\,\ell_1 & +15.109\,\ell_2 & -504.93\,\ell_3 & +3596.6\,\ell_4 & -1128.1\,\ell_5 & +78.572\,\ell_6 & -234.64\,\ell_7 & +34.418\,\ell_8 & = 18.742 \\
2.2844\,L_o & +2.7736\,\ell_1 & -51.388\,\ell_2 & +37.768\,\ell_3 & -1128.1\,\ell_4 & +7406.0\,\ell_5 & -2188.8\,\ell_6 & +145.12\,\ell_7 & -416.66\,\ell_8 & = 28.713 \\
0 & -12.594\,\ell_1 & +8.2876\,\ell_2 & -119.01\,\ell_3 & +78.572\,\ell_4 & -2188.8\,\ell_5 & +13599\,\ell_6 & -3851.9\,\ell_7 & +246.31\,\ell_8 & = 16.247 \\
2.0558\,L_o & +2.3541\,\ell_1 & -36.118\,\ell_2 & +18.183\,\ell_3 & -234.64\,\ell_4 & +145.12\,\ell_5 & -3851.9\,\ell_6 & +22993\,\ell_7 & -6307.9\,\ell_8 & = 25.871 \\
0 & -11.199\,\ell_1 & +6.5725\,\ell_2 & -76.304\,\ell_3 & +34.418\,\ell_4 & -416.66\,\ell_5 & +246.31\,\ell_6 & -6307.9\,\ell_7 & +36529\,\ell_8 & = 14.994
\end{array} \qquad 3.1$$

If the matrix formed by the coefficients of the ℓ_n's in the above equation is denoted by $[G_{n\xi} G_{s\xi}]$ these equations can be written in the abbreviated form

$$\sum_s \ell_s [G_{n\xi} G_{s\xi}] = a_n \qquad 3.2$$

where

$$a_n = m\alpha \frac{c}{c_o} \frac{4}{\pi^2} \int_0^\pi \left[\frac{\xi + \frac{a}{b}\pi}{1 + \frac{2a}{b}}\right]^2 G_{s\xi}\, d\xi \qquad 3.3$$

According to the method outlined in Appendix E, the unknown ℓ_n's can be obtained from Eq. 3.2 using the following relation

$$\ell_n = \sum_s [G_{n\xi} G_{s\xi}]^{-1} a_s \qquad 3.4$$

where $[G_{n\xi} G_{s\xi}]^{-1}$ is the inverse of the matrix $[G_{n\xi} G_{s\xi}]$. Using the method of inversion by sub-matrices (see Appendix E) $[G_{n\xi} G_{s\xi}]^{-1}$ was determined.

$$\left[G_{n\xi}G_{s\xi}\right]=\begin{bmatrix} -.36944 & -.0286112 & -.0031065 & -.0012240 & -.000334192 & -.00019364 & -.00007896 & -.00005541 & -.00002148 \\ -.0286112 & .016245 & .0015994 & .00024356 & .00011198 & .000037059 & .00002426 & .00001014 & .000007105 \\ -.0031066 & .0015994 & .0024959 & .00032482 & .000050604 & .000026228 & .0000084170 & .0000060778 & .000001964 \\ -.0011224 & .00024356 & .00032482 & .00075975 & .00011190 & .000018727 & .0000099317 & .0000032819 & .000002211 \\ -.0003341 & .00011198 & .000050604 & .00011190 & .00030956 & .000049431 & .0000082422 & .0000036741 & .000001110 \\ -.0001936 & .000037059 & .000026228 & .000018727 & .000049431 & .00015022 & .000025298 & .0000044692 & .0000023138 \\ -.0000789 & .000024260 & .000008417 & .000009931 & .0000082422 & .000025299 & .000098393 & .0000142197 & .0000022124 \\ -.0000554 & .000010138 & .0000060778 & .0000032819 & .0000036741 & .000004469 & .000014219 & .000048165 & .0000082777 \\ -.00002148 & .000007105 & .000001964 & .000002211 & .000001110 & .000002313 & .000002212 & .000008277 & .000028822 \end{bmatrix} \qquad 3.5$$

Substituting the terms of this matrix into Eq. 3.4 the following values of ℓ_n are obtained:

$$\begin{matrix} L_o = 5.0065 & \ell_3 = .01171 & \ell_6 = -.00045 \\ \ell_1 = .49133 & \ell_4 = -.00072 & \ell_7 = .00054 \\ \ell_2 = -.00753 & \ell_5 = .00186 & \ell_8 = -.00012 \end{matrix} \qquad 3.6$$

The dimensionless lift distribution can now be calculated using the relation given in Eq. 1.20. The lift coefficient can be determined with the aid of the relation

$$c_L = \frac{L}{\frac{1}{2}\rho c_o U_o^2}\left(\frac{U_o}{U}\right)^2 \qquad 3.7$$

and the induced angle is found according to Eq. 1.23 from

$$\frac{w}{U} = \frac{c_L}{m} - \alpha \qquad 3.8$$

while the drag coefficient follows from

$$c_D = c_L \frac{w}{U} \qquad 3.9$$

These results are given in Table VIII for ten stations along the span. Spanwise plots of the results are given in Figs. 12 - 16, showing also the corresponding results for the same wing in a flow of constant velocity. The abscissa of the curves is the distance along the span in fractions of the span, i.e.,

$$\eta = \frac{y-a}{b} = \frac{\xi}{\pi} \quad , \quad 0 \leq \eta \leq 1 . \qquad 3.10$$

The total lift, represented by the area under the lift distribution curve, was found to be 5.63, the total drag was 0.18 and the L/D = 31,3

It can be noted from Fig. 12 that the spanwise variation of the free stream velocity causes only a slight increase in the lift coefficient at the low speed end of the span and a slight decrease of C_L at the high speed end. These results are consistent with those obtained in the case of the wing without parallel walls discussed in Part I.

Appendix A.

Using the transformation given in Eq. 2.6, i.e.,

$$y = \frac{b}{4}\left(r + \frac{1}{r}\right)\cos\theta + a$$

$$z = \frac{b}{4}\left(r - \frac{1}{r}\right)\sin\theta \qquad \text{A.1}$$

it is desired to express the partial derivatives $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial y \partial z}$ in terms of the variables r, θ. Now

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} \qquad \text{A.2}$$

and

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z}$$

But also,

$$\frac{\partial r}{\partial r} = 1 = \frac{\partial r}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial r} \qquad \text{A.3}$$

and

$$\frac{\partial r}{\partial \theta} = 0 = \frac{\partial r}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial \theta}$$

Therefore, eliminating from A.3 first $\frac{\partial r}{\partial z}$ then $\frac{\partial r}{\partial y}$ the following relations result

$$\frac{\partial r}{\partial y} = \frac{\frac{\partial z}{\partial \theta}}{\frac{\partial y}{\partial r}\frac{\partial z}{\partial \theta} - \frac{\partial y}{\partial \theta}\frac{\partial z}{\partial r}} \qquad \text{A.4}$$

and

$$\frac{\partial r}{\partial z} = \frac{-\frac{\partial y}{\partial \theta}}{\frac{\partial y}{\partial r}\frac{\partial z}{\partial \theta} - \frac{\partial y}{\partial \theta}\frac{\partial z}{\partial r}}$$

Similarly

$$\frac{\partial \theta}{\partial r} = 0 = \frac{\partial \theta}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial z}{\partial r} \qquad \text{A.5}$$

$$\frac{\partial \theta}{\partial \theta} = 1 = \frac{\partial \theta}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial \theta}{\partial z}\frac{\partial z}{\partial \theta}$$

and consequently

$$\frac{\partial \theta}{\partial y} = \frac{-\frac{\partial z}{\partial r}}{\frac{\partial y}{\partial r}\frac{\partial z}{\partial \theta} - \frac{\partial y}{\partial \theta}\frac{\partial z}{\partial r}} \qquad \text{A.6}$$

and

$$\frac{\partial \theta}{\partial z} = \frac{\frac{\partial y}{\partial r}}{\frac{\partial y}{\partial r}\frac{\partial z}{\partial \theta} - \frac{\partial y}{\partial \theta}\frac{\partial z}{\partial r}}$$

Now the expressions A.4 and A.6 can be easily calculated using Eq. A.1.

$$\frac{\partial y}{\partial r} = \frac{b}{4}\left(1 - \frac{1}{r^2}\right)\cos\theta \; ; \qquad \frac{\partial y}{\partial \theta} = -\frac{b}{4}\left(r + \frac{1}{r}\right)\sin\theta \; ;$$

$$\frac{\partial z}{\partial r} = \frac{b}{4}\left(1 + \frac{1}{r^2}\right)\sin\theta \; ; \qquad \frac{\partial z}{\partial \theta} = \frac{b}{4}\left(r - \frac{1}{r}\right)\cos\theta \; ; \qquad \text{A.7}$$

Hence

$$\frac{\partial y}{\partial r}\frac{\partial z}{\partial \theta} - \frac{\partial y}{\partial \theta}\frac{\partial z}{\partial r} = \frac{b^2 r}{16}\left[\left(1 - \frac{1}{r^2}\right)^2 \cos^2\theta + \left(1 + \frac{1}{r^2}\right)^2 \sin^2\theta\right] \qquad \text{A.8}$$

and consequently

$$\frac{\partial r}{\partial y} = \frac{\left(r - \frac{1}{r}\right)\cos\theta}{\frac{br}{4}\left[\left(1 - \frac{1}{r^2}\right)^2\cos^2\theta + \left(1 + \frac{1}{r^2}\right)^2\sin^2\theta\right]}$$

$$\frac{\partial \theta}{\partial y} = \frac{-\left(1 + \frac{1}{r^2}\right)\sin\theta}{\frac{br}{4}\left[\left(1 - \frac{1}{r^2}\right)^2\cos^2\theta + \left(1 + \frac{1}{r^2}\right)^2\sin^2\theta\right]} \qquad \text{A.9}$$

$$\frac{\partial r}{\partial z} = \frac{\left(r + \frac{1}{r}\right)\sin\theta}{\frac{br}{4}\left[\left(1 - \frac{1}{r^2}\right)^2\cos^2\theta + \left(1 + \frac{1}{r^2}\right)^2\sin^2\theta\right]}$$

$$\frac{\partial \theta}{\partial z} = \frac{\left(1 - \frac{1}{r^2}\right)\cos\theta}{\frac{br}{4}\left[\left(1 - \frac{1}{r^2}\right)^2\cos^2\theta + \left(1 + \frac{1}{r^2}\right)^2\sin^2\theta\right]}$$

The partial derivatives in question can be easily obtained by substituting the relations A.9 into Eq. A.2.

Appendix B.

The differential Eq. 2.3, i.e.,

$$\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}-\frac{2}{y}\frac{\partial\varphi}{\partial y}=0 \qquad \text{B.1}$$

shall be expressed in terms of new variables u, v where

$$\begin{aligned} y-a &= \frac{b}{2}\cosh u\cos v \\ z &= \frac{b}{2}\sinh u\sin v \end{aligned} \qquad \text{B.2}$$

Now
$$\frac{\partial\varphi}{\partial y}=\frac{\partial u}{\partial y}\frac{\partial\varphi}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial\varphi}{\partial v}$$
and
$$\frac{\partial\varphi}{\partial z}=\frac{\partial u}{\partial z}\frac{\partial\varphi}{\partial u}+\frac{\partial v}{\partial z}\frac{\partial\varphi}{\partial v} \qquad \text{B.3}$$

Using relations identical to those derived in Appendix A (Eqs. A.4 and A.6)

$$\frac{\partial u}{\partial y}=\frac{\frac{\partial z}{\partial v}}{\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}}$$

$$\frac{\partial u}{\partial z}=\frac{-\frac{\partial y}{\partial v}}{\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}}$$

$$\frac{\partial v}{\partial y}=\frac{-\frac{\partial z}{\partial u}}{\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}} \qquad \text{B.4}$$

$$\frac{\partial v}{\partial z}=\frac{\frac{\partial y}{\partial u}}{\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}}$$

Calculating the partial derivatives using Eq. B.2

$$\frac{\partial y}{\partial u}=\frac{b}{2}\sinh u\cos v; \qquad \frac{\partial y}{\partial v}=-\frac{b}{2}\cosh u\sin v$$

$$\frac{\partial z}{\partial v}=\frac{b}{2}\sinh u\cos v; \qquad \frac{\partial z}{\partial u}=\frac{b}{2}\cosh u\sin v \qquad \text{B.5}$$

And
$$\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}=\frac{b^2}{4}\left[\sin^2 v+\sinh^2 u\right] \qquad \text{B.6}$$

Using relations B.5, B.6 and B.4

$$\frac{\partial u}{\partial y} = \frac{\sinh u \cos v}{\frac{b}{2}\left[\sinh^2 u + \sin^2 v\right]}; \qquad \frac{\partial v}{\partial y} = \frac{-\cosh u \sin v}{\frac{b}{2}\left[\sinh^2 u + \sin^2 v\right]} \tag{B.7}$$

$$\frac{\partial u}{\partial z} = \frac{\cosh u \sin v}{\frac{b}{2}\left[\sinh^2 u + \sin^2 v\right]}; \qquad \frac{\partial v}{\partial z} = \frac{\sinh u \cos v}{\frac{b}{2}\left[\sinh^2 u + \sin^2 v\right]}$$

Thus

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial z}; \qquad \frac{\partial u}{\partial z} = -\frac{\partial v}{\partial y};$$

and

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0; \qquad \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0 \tag{B.8}$$

Using relations B.3, B.7, B.8 it can easily be shown that

$$\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]\left[\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial v^2}\right]$$

and

$$\frac{2}{y}\frac{\partial \varphi}{\partial y} = \frac{2}{\left[a + \frac{b}{2}\cosh u \cos v\right]} \frac{1}{\left[\sinh^2 u + \sin^2 v\right]\frac{b}{2}} \left[\sinh u \cos v \frac{\partial \varphi}{\partial u} - \cosh u \sin v \frac{\partial \varphi}{\partial v}\right] \tag{B.9}$$

Furthermore

$$\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{1}{\left(\frac{b}{2}\right)^2\left(\sinh^2 u + \sin^2 v\right)} \tag{B.10}$$

Combining Eqs. B.9 and B.10, the following differential equation is obtained:

$$\left[\sinh u \cos v \frac{\partial \varphi}{\partial u} - \cosh u \sin v \frac{\partial \varphi}{\partial v}\right] = \left[\frac{a}{b} + \frac{1}{2}\cosh u \cos v\right]\left[\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial v^2}\right] \tag{B.11}$$

B.11 is the transformed form of Eq. B.1 where the transformation used is given in B.2

Appendix C.

The following integral is to be solved for values of $\alpha = 1, 2, 3, 4$

$$I_{n,\alpha} = \int_0^{\pi} \frac{\cos n\theta \, d\theta}{(\frac{1}{\lambda} + \cos\theta)^{\alpha}} \qquad \text{C.1}$$

Multiplying numerator and denominator by K^{α} and letting

$$\frac{K}{\mu} = \lambda \quad ; \quad K = \sqrt{\mu^2 - 1}$$

i.e., C.2

$$K = \frac{1}{\sqrt{\frac{1}{\lambda^2} - 1}} \quad ; \quad \mu = \frac{1}{\sqrt{1-\lambda^2}}$$

$I_{n,\alpha}$ can be rewritten as

$$I_{n,\alpha} = \left[\frac{1}{\sqrt{\frac{1}{\lambda^2} - 1}}\right]^{\alpha} \int_0^{\pi} \frac{\cos n\theta \, d\theta}{(\mu + \sqrt{\mu^2 - 1}\cos\theta)^{\alpha}} \qquad \text{C.3}$$

According to Hobson: Spherical Harmonics, p. 24

$$\int_0^{\pi} \frac{\cos n\theta \, d\theta}{(\mu + \sqrt{\mu^2-1}\cos\theta)^{\alpha}} = (-1)^n \pi \frac{(\alpha - 1 + n)!}{(\alpha-1)!} P_{\alpha-1}^{-n}\left(\frac{1}{\sqrt{1-\lambda^2}}\right) \qquad \text{C.4}$$

where P is the associated Legendre function as defined by Hobson.

Hobson also shows that this P function can be expressed in a hypergeometric series as follows:

$$P_{\alpha-1}^{-n}\left(\frac{1}{\sqrt{1-\lambda^2}}\right) = \frac{1}{n!}\left[\frac{\frac{1}{\sqrt{1-\lambda^2}} - 1}{\frac{1}{\sqrt{1-\lambda^2}} + 1}\right]^{\frac{n}{2}} F\left(\alpha, 1-\alpha; n+1; \frac{1 - \frac{1}{\sqrt{1-\lambda^2}}}{2}\right) \qquad \text{C.5}$$

The notation $F(a,b;c;z)$ signifies the following hypergeometric series:

$$F(a,b;c;z) = 1 + \frac{ab}{1c} z + \frac{a(a+1)b(b+1)}{1.2.c(c+1)} z^2 + \frac{a(a+1)(a+2)\,b(b+1)(b+2)}{1.2.3\,c(c+1)(c+2)} z^3 + \cdots \qquad \text{C.6}$$

This hypergeometric series has been evaluated for values of $\alpha = 2, 3, 4$.

It was found

$$\alpha = 2: \quad F\left(2, -1; n+1; \frac{1 - \frac{1}{\sqrt{1-\lambda^2}}}{2}\right) = \frac{n + \frac{1}{\sqrt{1-\lambda^2}}}{n+1}$$

C.7

$$\alpha = 3: \quad F\left(3, -2; n+1; \frac{1 - \frac{1}{\sqrt{1-\lambda^2}}}{2}\right) = \left\{(n+1)(n+2) - 3(n+2)\left[1 - \frac{1}{\sqrt{1-\lambda^2}}\right] + 3\left[1 - \frac{1}{\sqrt{1-\lambda^2}}\right]^2\right\} \frac{1}{(n+1)(n+2)}$$

$$\alpha=4:\quad F\left(4,-3;\,n+1;\,\frac{1-\frac{1}{\sqrt{1-\lambda^2}}}{2}\right)=\frac{1}{(n+1)(n+2)(n+3)}\Bigg\{(n+1)(n+2)(n+3)-$$

$$-6(n+2)(n+3)\left[1-\frac{1}{\sqrt{1-\lambda^2}}\right]+15(n+3)\left[1-\frac{1}{\sqrt{1-\lambda^2}}\right]^2+15\left[1-\frac{1}{\sqrt{1-\lambda^2}}\right]^3\Bigg\} \qquad \text{C.7}$$

Substituting these relations into 6.4, the following expressions are obtained:

$$J_{n,1}=(-1)^n\pi\frac{\lambda}{\sqrt{1-\lambda^2}}\left[\frac{1-\sqrt{1-\lambda^2}}{1+\sqrt{1-\lambda^2}}\right]^{n/2};$$

$$J_{n,2}=(-1)^n\pi\frac{\lambda^2}{1-\lambda^2}\left[n+\frac{1}{\sqrt{1-\lambda^2}}\right]\left[\frac{1-\sqrt{1-\lambda^2}}{1+\sqrt{1-\lambda^2}}\right]^{n/2};$$

$$J_{n,3}=(-1)^n\frac{\pi}{2}\left(\frac{\lambda}{\sqrt{1-\lambda^2}}\right)^3\left[\frac{1-\sqrt{1-\lambda^2}}{1+\sqrt{1-\lambda^2}}\right]^{n/2}\left[(n+1)(n+2)-3(n+2)\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right)+3\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right)^2\right]; \qquad \text{C.8}$$

$$J_{n,4}=(-1)^n\frac{\pi}{6}\left(\frac{\lambda}{\sqrt{1-\lambda^2}}\right)^4\left[\frac{1-\sqrt{1-\lambda^2}}{1+\sqrt{1-\lambda^2}}\right]^{n/2}\Bigg[(n+1)(n+2)(n+3)-6(n+2)(n+3)\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right)+$$

$$+15(n+3)\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right)^2-15\left(1-\frac{1}{\sqrt{1-\lambda^2}}\right)^3\Bigg];$$

Appendix D.

Using the method outlined in Section 5 b, the integrals $I'_{n,1}$ have been evaluated in terms of a power series in λ. The results are given below; for $n = 0$ to $n = 20$ and including terms in λ up to the 15th power.

$$I'_{o,1} = 0$$

$$I'_{1,1} = 2(\lambda + .333333\,\lambda^3 + .20\,\lambda^5 + .142857\,\lambda^7 + .111111\,\lambda^9 + .0909091\,\lambda^{11} + .0769231\,\lambda^{13} + .0666667\,\lambda^{15})$$

$$I'_{2,1} = -4(.333333\,\lambda^2 + .20\,\lambda^4 + .142857\,\lambda^6 + .111111\,\lambda^8 + .0909091\,\lambda^{10} + .0769231\,\lambda^{12} + .0666667\,\lambda^{14} + .0588235\,\lambda^{16})$$

$$I'_{3,1} = 8(.0833333\,\lambda + .116667\,\lambda^3 + .0928571\,\lambda^5 + .0753968\,\lambda^7 + .0631313\,\lambda^9 + .0541958\,\lambda^{11} + .0474359\,\lambda^{13} + .0421569\,\lambda^{15})$$

$$I'_{4,1} = -8(.0666667\,\lambda^2 + .0857143\,\lambda^4 + .0793651\,\lambda^6 + .0707071\,\lambda^8 + .0629371\,\lambda^{10} + .0564103\,\lambda^{12} + .0509804\,\lambda^{14} + .0464396\,\lambda^{16})$$

$$I'_{5,1} = 8(.050\,\lambda + .0547619\,\lambda^3 + .0658730\,\lambda^5 + .0660173\,\lambda^7 + .0627428\,\lambda^9 + .0586241\,\lambda^{11} + .0545249\,\lambda^{13} + .0507224\,\lambda^{15})$$

$$I'_{6,1} = -8(.0428571\,\lambda^2 + .0460317\,\lambda^4 + .0526695\,\lambda^6 + .0547786\,\lambda^8 + .0543123\,\lambda^{10} + .0526395\,\lambda^{12} + .0504644\,\lambda^{14} + .0481350\,\lambda^{16})$$

$$I'_{7,1} = 8(.0357142\,\lambda + .0373015\,\lambda^3 + .0394660\,\lambda^5 + .0435397\,\lambda^7 + .0458818\,\lambda^9 + .0466543\,\lambda^{11} + .0464039\,\lambda^{13} + .0455476\,\lambda^{15})$$

$$I'_{8,1} = -8(.0317460\,\lambda^2 + .0329004\,\lambda^4 + .0344100\,\lambda^6 + .0369852\,\lambda^8 + .0389962\,\lambda^{10} + .0401682\,\lambda^{12} + .0406309\,\lambda^{14} + .0405745\,\lambda^{16})$$

$$I'_{9,1} = 8(.0277777\,\lambda + .0284992\,\lambda^3 + .0293539\,\lambda^5 + .0304306\,\lambda^7 + .0321106\,\lambda^9 + .0336822\,\lambda^{11} + .0348580\,\lambda^{13} + .0356014\,\lambda^{15})$$

$$I'_{10,1} = -8(.0252525\,\lambda^2 + .0258075\,\lambda^4 + .0264513\,\lambda^6 + .0272361\,\lambda^8 + .0283682\,\lambda^{10} + .0295478\,\lambda^{12} + .0305719\,\lambda^{14} + .0313608\,\lambda^{16})$$

$$I'_{11,1} = 8(.0227272\,\lambda + .0231157\,\lambda^3 + .0235486\,\lambda^5 + .0240416\,\lambda^7 + .0246257\,\lambda^9 + .0254134\,\lambda^{11} + .0262858\,\lambda^{13} + .0271202\,\lambda^{15})$$

$$I'_{12,1} = -8(.0209790\,\lambda^2 + .0212897\,\lambda^4 + .0216319\,\lambda^6 + .0220153\,\lambda^8 + .0224586\,\lambda^{10} + .0230238\,\lambda^{12} + .0236686\,\lambda^{14} + .0243261\,\lambda^{16})$$

$$I'_{13,1} = 8(.0192307\lambda+.0194637\lambda^3+.019715\lambda^5+.0199890\lambda^7+.0202915\lambda^9+.0206341\lambda^{11}+.0210514\lambda^{13}+.0215319\lambda^{15})$$

$$I'_{14,1} = -8(.0179485\lambda^2+.0181407\lambda^4+.0183461\lambda^6+.0185677\lambda^8+.0188097\lambda^{10}+.0190790\lambda^{12}+.0193953\lambda^{14}+.0197576\lambda^{16})$$

$$I'_{15,1} = 8(.0166664\lambda+.0168177\lambda^3+.0169770\lambda^5+.0171464\lambda^7+.0173279\lambda^9+.0175239\lambda^{11}+.0177392\lambda^{13}+.0179834\lambda^{15})$$

$$I'_{16,1} = -8(.0156869\lambda^2+.0158132\lambda^4+.0159466\lambda^6+.0160881\lambda^8+.0162381\lambda^{10}+.0163993\lambda^{12}+.0165714\lambda^{14}+.0167668\lambda^{16})$$

$$I'_{17,1} = 8(.0147074\lambda+.0148087\lambda^3+.0149162\lambda^5+.0150299\lambda^7+.0151483\lambda^9+.0152747\lambda^{11}+.0154037\lambda^{13}+.0155502\lambda^{15})$$

$$I'_{18,1} = -8(.0139305\lambda^2+.0140192\lambda^4+.0141132\lambda^6+.0142085\lambda^8+.0143113\lambda^{10}+.0144081\lambda^{12}+.0145289\lambda^{14}+.0146408\lambda^{16})$$

$$I'_{19,1} = 8(.0131537\lambda+.0132298\lambda^3+.0133101\lambda^5+.0133872\lambda^7+.0134743\lambda^9+.0135415\lambda^{11}+.0136541\lambda^{13}+.0137314\lambda^{15})$$

$$I'_{20,1} = -8(.0125290\lambda^2+.0126010\lambda^4+.0126612\lambda^6+.0127400\lambda^8+.0127718\lambda^{10}+.0129001\lambda^{12}+.0129338\lambda^{14}+.0130523\lambda^{16})$$

Appendix E.

Outline of the Solution of a Set of Simultaneous Linear Equations by Matrix Inversion Using the Method of Submatrices. (Ref. 6)

The following set of simultaneous equations to be considered

$$\begin{aligned} y_1 &= a_{11} x_1 + a_{12} x_2 + \cdots\cdots a_{1n} x_n \\ y_2 &= a_{21} x_1 + a_{22} x_2 + \cdots\cdots a_{2n} x_n \\ &\vdots \\ y_n &= a_{n1} x_1 + a_{n2} x_2 + \cdots\cdots a_{nn} x_n \end{aligned} \qquad \text{E.1}$$

This set of equations can be written as a simple matrix equation

$$y = a x \qquad \text{E.2}$$

where x and y are column matrices, i.e.,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad \text{E.3}$$

and a is the square matrix formed by the coefficients of the x's. The above matrix equation E.2 can be solved for x, i.e.,

$$x = y\, a^{-1} \qquad \text{E.4}$$

where a^{-1} is the reciprocal of the matrix a, defined by

$$a\, a^{-1} = I \qquad \text{E.5}$$

and I is the unit matrix.

Thus the problem of solving the Eqs. E.1 is reduced to finding the matrix a^{-1}. This can be accomplished by use of several methods. In the following one of them, the reciprocation by the method of submatrices, will be outlined.

Considering the matrix a, i.e.,

$$a = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \text{E.6}$$

one can form a sequence of square matrices S_1 S_2 - - S_4 by progressively including more and more terms as shown below:

$$S_1 = [a_{11}] \qquad S_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \text{E.7}$$

$$S_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{S_2} & \begin{matrix} a_{13} \\ a_{23} \end{matrix} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] \qquad \text{E.8}$$

Commencing with the second of these, S_2, one can successively find the inverse of S_2, S_3, etc. The inverse of S_2, i.e., S_2^{-1} can be found immediately from

$$S_2^{-1} = \frac{}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \qquad \text{E.9}$$

From this S_3 can be determined by applying the scheme of computations given below:

$$S_3^{-1} = \left[\begin{array}{c|c} S_2^{-1} + X\theta^{-1}Y & -X\theta^{-1} \\ \hline -\theta^{-1}Y & \theta^{-1} \end{array}\right] \qquad \text{E.10}$$

The symbols of Eq. E.10 are explained in the following arrangement suggested for the numerical work:

	$\alpha_{21} = [a_{31} \;\; a_{32}]$	$\alpha_{22} = a_{33}$
$X = S_2^{-1} \alpha_{12}$	S_2^{-1}	$\alpha_{12} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$
$\theta^{-1} = \frac{1}{\theta}$	$Y = \alpha_{21} S_2^{-1}$	$\theta = \alpha_{22} - Y \alpha_{12}$

E.11

The relations E.10 and E.11 can easily be extended for the calculation of S_4^{-1} and by increasing in such a manner the order of the inverted matrix step by step the final matrix a_n^{-1} can be determined.

For a more detailed outline of the method, see Ref. 6 .

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TABLE I

$$H_{0,1} = \frac{\lambda\pi}{\sqrt{1-\lambda^2}} \log\frac{2}{\lambda} - \pi \sum_{m=1}^{\infty} (-1)^{2m} \lambda^{2m+1} P_m$$

$$H_{0,2} = \left(\log\frac{2}{\lambda}\right) J_{0,2} - \pi\lambda^2 \sum_{m=1}^{\infty} (-1)^{2m} \lambda^{2m} R_m$$

$$\int_0^{\pi} \frac{[\log(1+\lambda\cos\theta)]^2 \cos\theta \, d\theta}{(1+\lambda\cos\theta)^3} = -\frac{\pi}{\lambda} \sum_{m=1}^{\infty} \lambda^{2m} S_m$$

m =	1	2	3	4	5	6	7	8	9	10	11	12	13
P_m	.75000	.781250	.765625	.743164	.720801	.700041	.681113	.663910	.648240	.633911	.620751	.608610	.597362
R_m	1.2500	2.40625	3.48437	4.50098	5.46787	6.39350	7.28408	8.14437	8.97808	9.78819	10.5771	11.3469	12.0993
S_m	0	1.5000	6.11979	14.5332	26.4879	44.0237	65.4454	91.4489	122.091	157.409	197.425	242.154	291.602

TABLE II

Values of Integrals $I_{n,\alpha}$, $I'_{n,\alpha}$, $H_{n,\alpha}$, $H'_{n,\alpha}$ for $\lambda = \frac{1}{3}$

n	$I_{n,1}$	$I_{n,2}$	$I_{n,3}$	$I_{n,4}$	$H_{n,2}$	$H_{n,3}$	$H_{n,4}$	$I'_{n,0}$	$I'_{n,1}$	$I'_{n,2}$	$H'_{n,2}$
0	1.1107	.4165	.1649	.0683	.6855	.2630	.1062	0	0	0	0
1	-.1906	-.1388	-.0781	-.0401	-.1650	-.1036	-.0555	2.0	.6931	.250	.4233
2	.0327	.0354	.0260	.0163	.0308	.0286	.0198	0	-.1589	-.1137	-.1374
3	-.0056	-.0080	-.0073	-.0054	-.0050	-.0067	-.0058	.6667	.2601	.1145	.1686
4	.0010	.0017	.0019	.0016	.0008	.0014	.0015	0	-.0687	-.0528	-.0596
5	-.0002	-.0003	-.0004	-.0004	-.0003	-.0002	-.0004	.4000	.1520	.0649	.0971
6	0	.0001	.0001	.0001	0	0	.0002	0	-.0433	-.0329	-.0376
7	0	0	0	0	0	0	0	.2857	.1078	.0454	.0687
8	0	0	0	0	0	0	0	0	-.0319	-.0241	-.0277
9	0	0	0	0	0	0	0	.2222	.0836	.0351	.0532
10	0	0	0	0	0	0	0	0	-.0253	-.0192	-.0221
11	0	0	0	0	0	0	0	.1818	.0683	.0286	.0435
12	0	0	0	0	0	0	0	0	-.0210	-.0159	-
13	0	0	0	0	0	0	0	.1538	.0578	.0242	-
14	0	0	0	0	0	0	0	0	-.0180	-.0137	-
15	0	0	0	0	0	0	0	.1333	.0501	.0210	-
16	0	0	0	0	0	0	0	0	-.0157	-.0120	-
17	0	0	0	0	0	0	0	.1176	.0442	.0185	-
18	0	0	0	0	0	0	0	0	-.0139	-.0106	-
19	0	0	0	0	0	0	0	.1053	.0395	.0166	-
20	0	0	0	0	0	0	0	0	-.0125	-.0100	-

TABLE III

Lift, Drag, and Induced Angle Distributions for $\lambda = \frac{1}{3}$

θ	0	20	40	60	80	100	120	140	160	180
η	.5000	.4698	.3830	.2500	.0868	-.0868	-.2500	-.3830	-.4698	-.5000
$(U/U_o)^2$	1.7778	1.7246	1.5759	1.3610	1.1191	.88761	.69446	.55453	.47165	.4444
C_L	0	2.4960	3.7826	4.3984	4.6543	4.6984	4.5732	4.0912	2.7604	0
L/q_oC_o	0	4.3046	5.9610	5.9862	5.2086	4.1703	3.1759	2.2687	1.3019	0
w/U	-1.000	-.5598	-.3329	-.2242	-.1791	-.1713	-.1934	-.2784	-.5131	-1.000
C_D	0	1.3973	1.2592	.9863	.8337	.8050	.8846	1.1392	1.4165	0
D/q_oC_o	0	2.4096	1.9842	1.3424	.9330	.7145	.6143	.6317	.6681	0

TABLE IV

Lift, Drag, and Induced Angle Distributions for $\lambda = 0$

θ	0	20	40	60	80	100	120	140	160	180
$L/q_oC_o = C_L$	0	2.5950	3.9330	4.5262	4.7427	4.7427	4.5262	3.9330	2.5950	0
w/U	-1.00	- .54232	-.30634	-.20171	-.16353	-.16353	-.20171	-.30634	-.54232	1.000
$D/q_oC_o = C_D$	0	1.4073	1.2048	.91298	.72557	.77557	.91298	1.2048	1.4073	0

TABLE V

Lift, Drag, and Induced Angle Distributions ($\lambda = \frac{1}{3}$) According to Vandrey

θ	0	20	40	60	80	100	120	140	160	130
C_L	0	2.4222	3.7817	4.4244	4.7246	4.7204	4.3347	3.4819	2.0344	0
$L/q_o c_o$	0	4.1773	5.9596	6.0216	5.2873	4.1899	3.0103	1.9308	.9595	0
w/U	-1.000	-.5728	-.330	-.2197	-.1667	-.1675	-.2355	-.3857	-.6412	-1.000
C_D $D/q_o c_o$	0	2.3928	1.9845	1.3229	.8814	.7018	.7089	.7451	.6153	0

TABLE VI

Lift, Drag, and Induced Angle Distributions ($\lambda = \frac{1}{3}$) According to Bausch

θ	0	20	40	60	80	100	120	140	160	180
C_L	0	2.5684	3.9452	4.5157	4.7385	4.7389	4.5159	3.9452	2.5684	0
$L/q_o c_o$	0	4.4295	6.2172	6.1459	5.3029	4.2063	3.1361	2.1877	1.2114	0
w/U	-1.000	-.5470	-.3042	-.2036	-.1643	-.1643	-.2036	-.3042	-.5470	-1.000
C_D $D/q_o c_o$	0	2.4229	1.8913	1.2513	.8713	.6911	.6385	.6655	.6626	

TABLE VII

Values of the Integrals Appearing in Eq. 2.2 for $\lambda = \frac{1}{3}$

n	1	2	3	4	5	6	7	8
$\int_0^\pi G_1 G_n d\xi$	75.386	-41.691	4.2670	-16.171	2.7736	-12.594	2.3541	-11.199
$\int_0^\pi G_2 G_n d\xi$	-41.691	451.12	-180.44	15.109	-51.389	8.2876	-36.118	6.5725
$\int_0^\pi G_3 G_n d\xi$	4.2670	-180.44	1471.52	-504.93	37.768	-119.01	18.183	-76.304
$\int_0^\pi G_4 G_n d\xi$	-16.171	15.109	-504.93	3596.6	-1128.1	78.572	-234.64	34.418
$\int_0^\pi G_5 G_n d\xi$	2.7736	-51.389	37.768	-1128.1	7406.0	-2188.8	145.12	-416.66
$\int_0^\pi G_6 G_n d\xi$	-12.594	8.2876	-119.01	78.572	-2188.8	13599	-3851.9	246.31
$\int_0^\pi G_7 G_n d\xi$	2.3541	-36.118	18.183	-234.64	145.12	-3851.9	22993	-6307.9
$\int_0^\pi G_8 G_n d\xi$	-11.199	6.5725	-76.304	34.418	-416.66	246.31	-6307.9	36529
$\int_0^\pi G_n d\xi$	5.4844	0	2.8177	0	2.2844	0	2.0558	0

TABLE VIII

ξ	0	20	40	60	80	100	120	140	160	180
η	.0	.111°	.222°	.333°	.444°	.555°	.666°	.777°	.888°	.999°
$(U/U_o)^2$	.444	.54870	.66392	.79012	.92730	1.0754	1.2346	1.4047	1.5857	1.7778
C_L	7.6030	6.4844	8.9456	5.7326	5.4954	5.3747	5.2958	5.1920	5.0881	4.7992
$L/q_o C_o$	3.3791	3.5580	3.9747	4.5290	5.0959	5.7802	6.5381	7.2932	8.0682	8.5320
w/U	+.3410	+.1436	+.0486	+.0110	-.0308	-.0521	-.0660	-.0843	-.1026	-.1536
C_D	-2.5926	-.93116	-.28896	-.06305	.16926	.28002	.34952	.43769	.52204	.73716
D	-1.1523	-.51093	-.19317	-.04982	.156954	.30115	.43151	.61482	.82780	1.3105

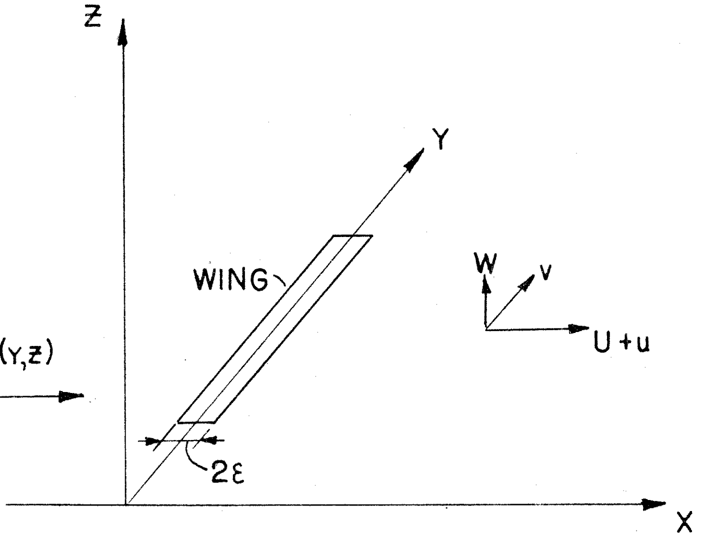

Z
Y
WING
W
v
U + u
U (Y, Z)
2ε
X

FIG. 1

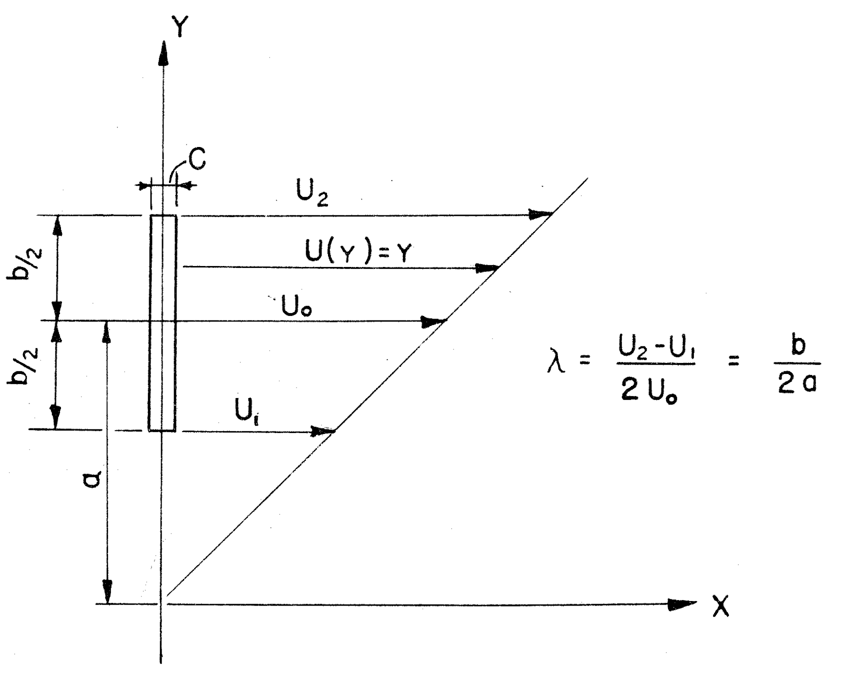

Y
C
U_2
$U(Y) = Y$
U_o
U_1
b/2
b/2
a
X
$\lambda = \frac{U_2 - U_1}{2 U_o} = \frac{b}{2a}$

FIG. 2

C_L – DISTRIBUTION OVER A WING IN LINEARLY VARYING FLOW

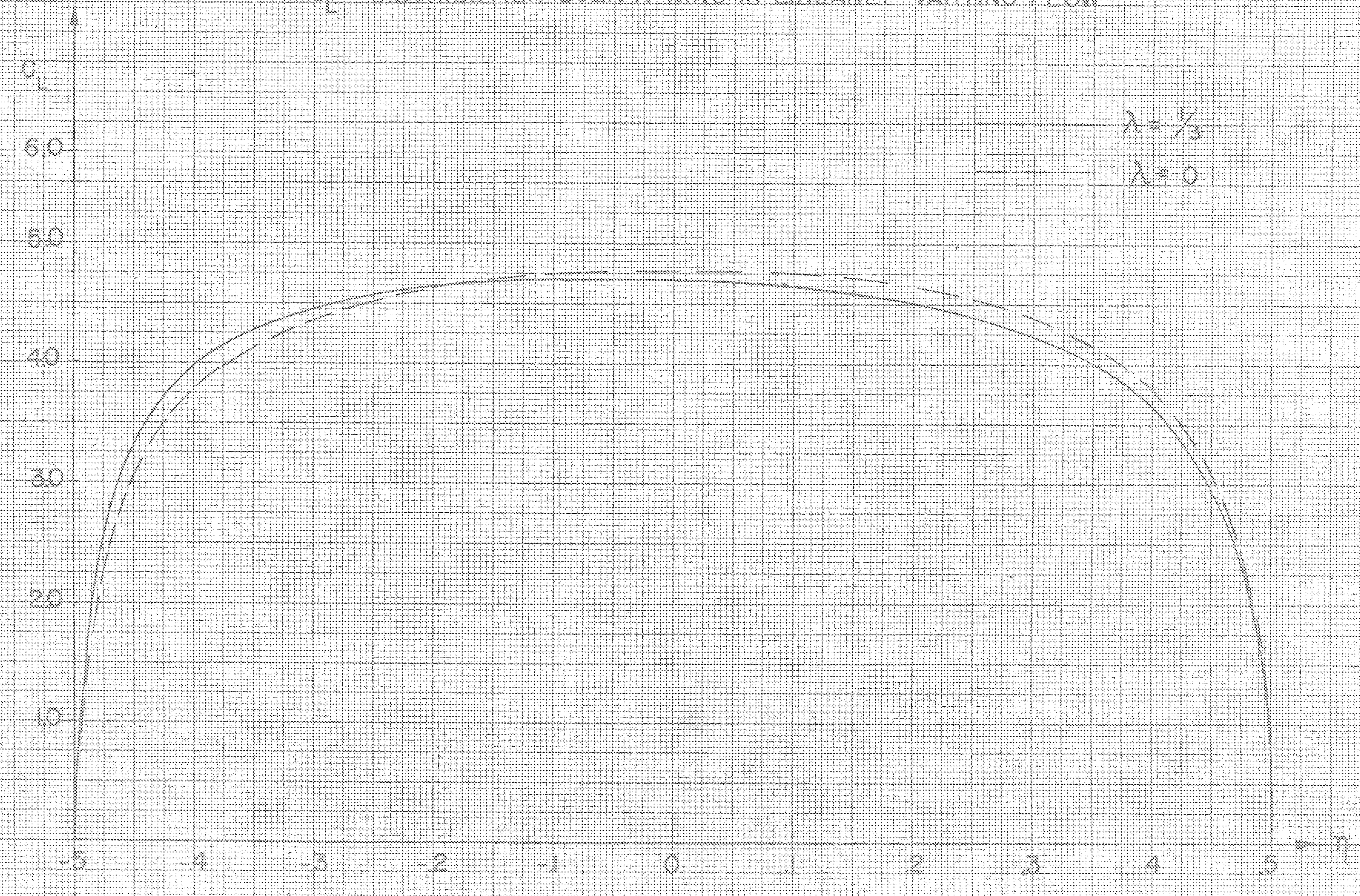


FIG. 3

LIFT DISTRIBUTION OVER A WING IN LINEARLY VARYING FLOW

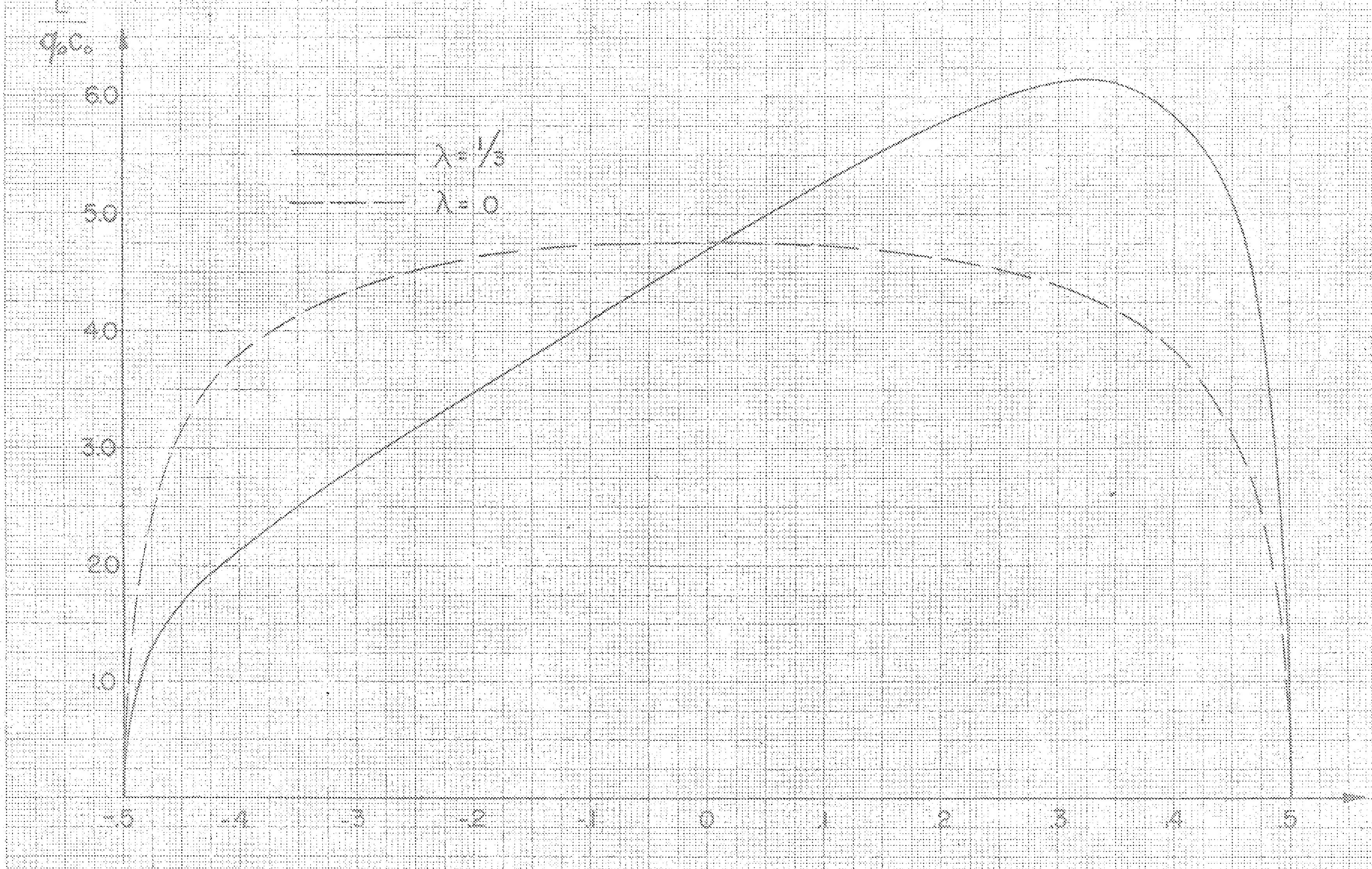
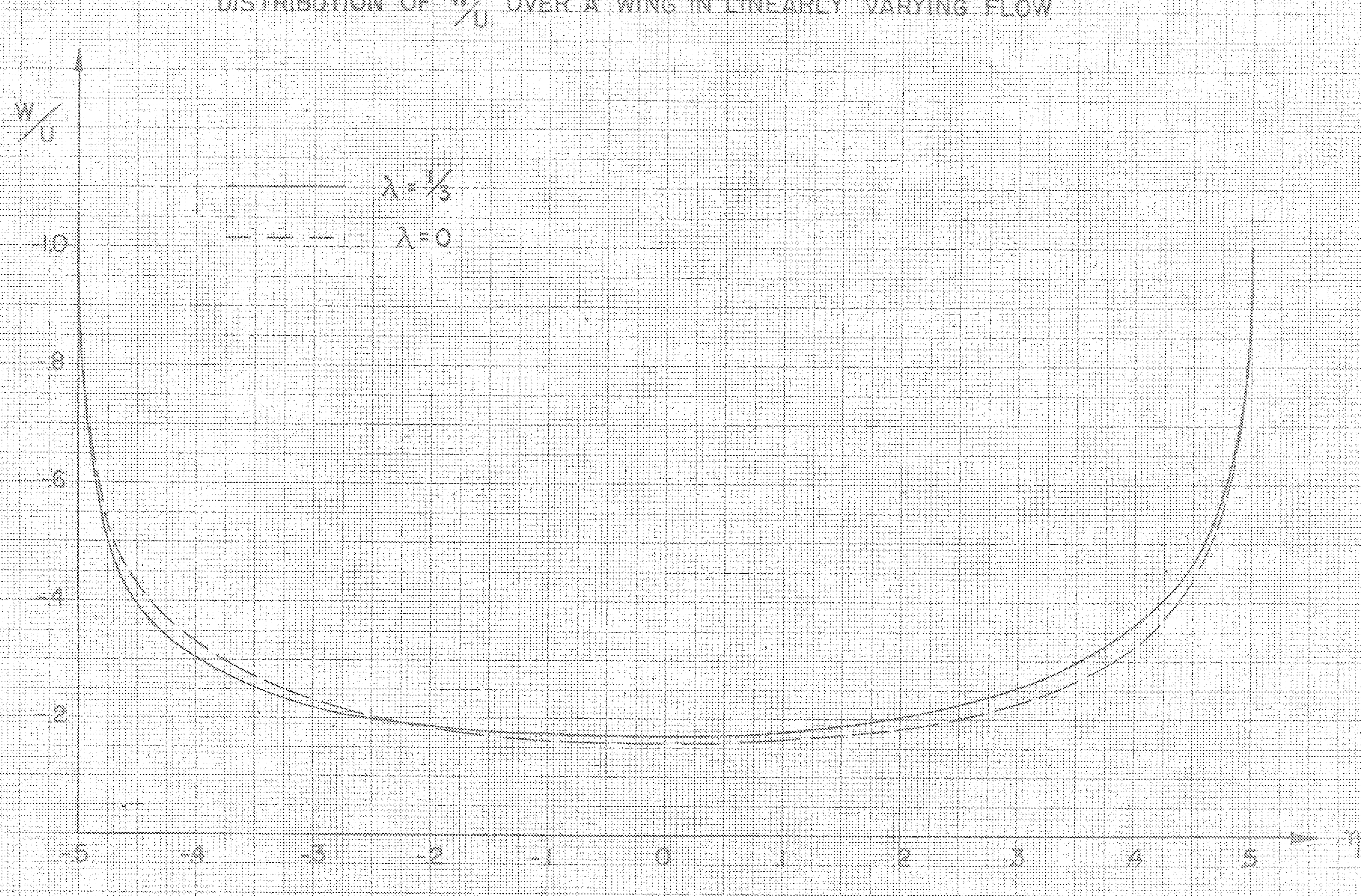


FIG. 4



DISTRIBUTION OF W/U OVER A WING IN LINEARLY VARYING FLOW
W/U
$\lambda = 1/3$
$\lambda = 0$
10
8
6
4
2
-5
-4
-3
-2
-1
0
1
2
3
4
5
η

FIG. 5

DRAG FORCE DISTRIBUTION OVER A WING IN LINEARLY VARYING FLOW

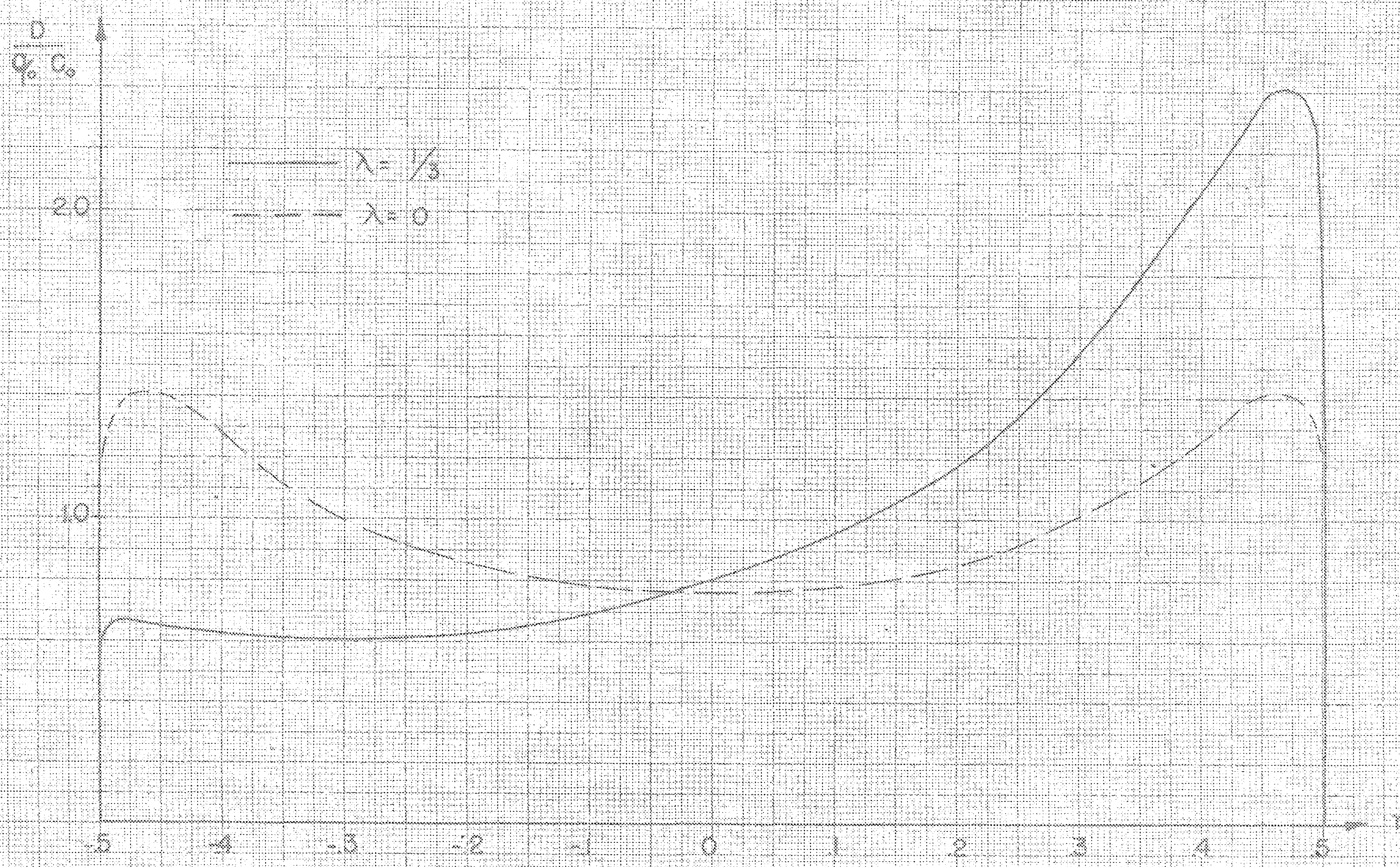
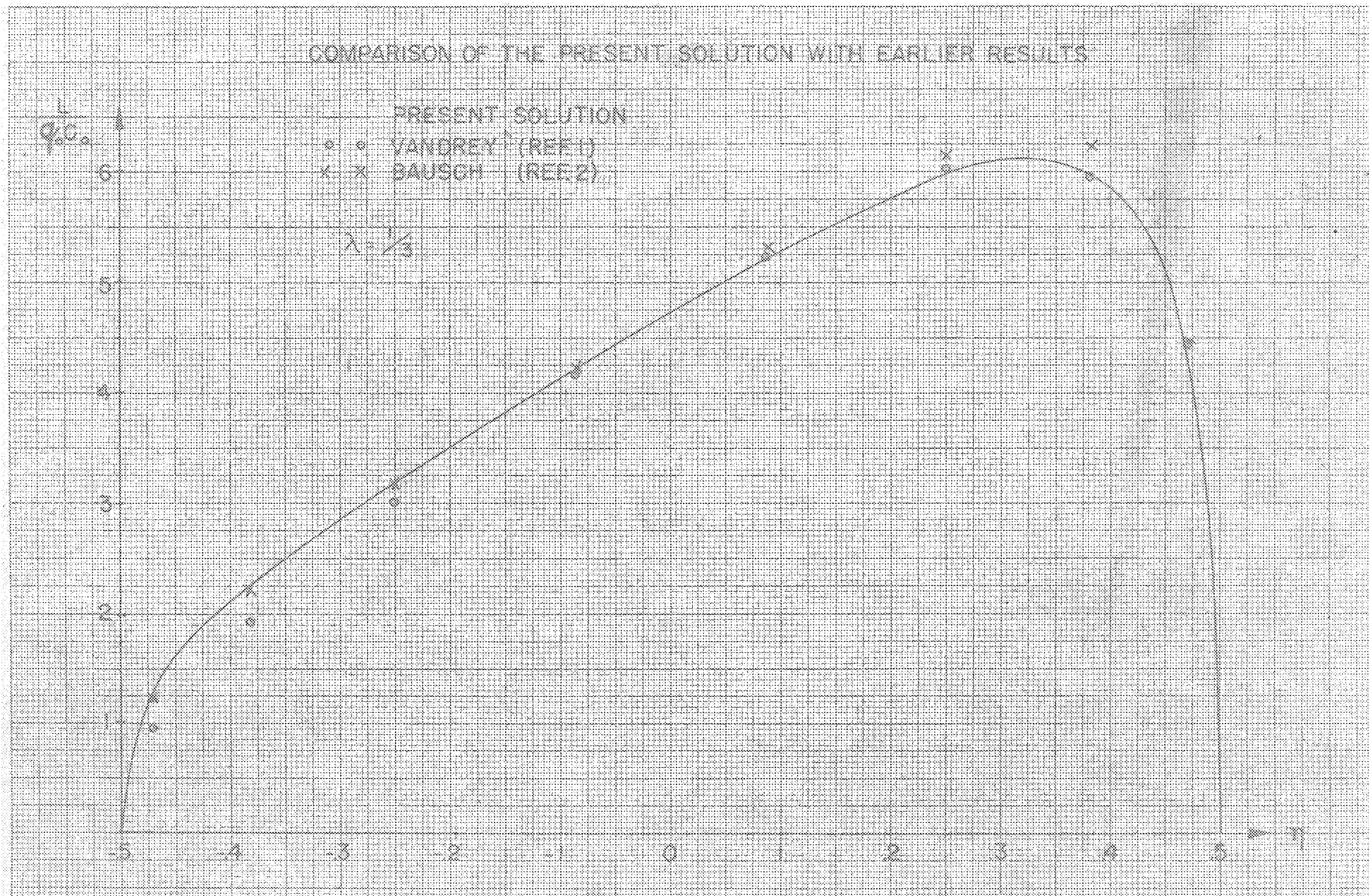


FIG. 6



COMPARISON OF THE PRESENT SOLUTION WITH EARLIER RESULTS
PRESENT SOLUTION
VANDREY (REF.1)
BAUSCH (REF.2)
λ = 1/3
6
5
4
3
2
1
-.5
-.4
-.3
-.2
-.1
0
.1
.2
.3
.4
.5
η

FIG. 7

COMPARISON OF THE PRESENT SOLUTION WITH EARLIER RESULTS

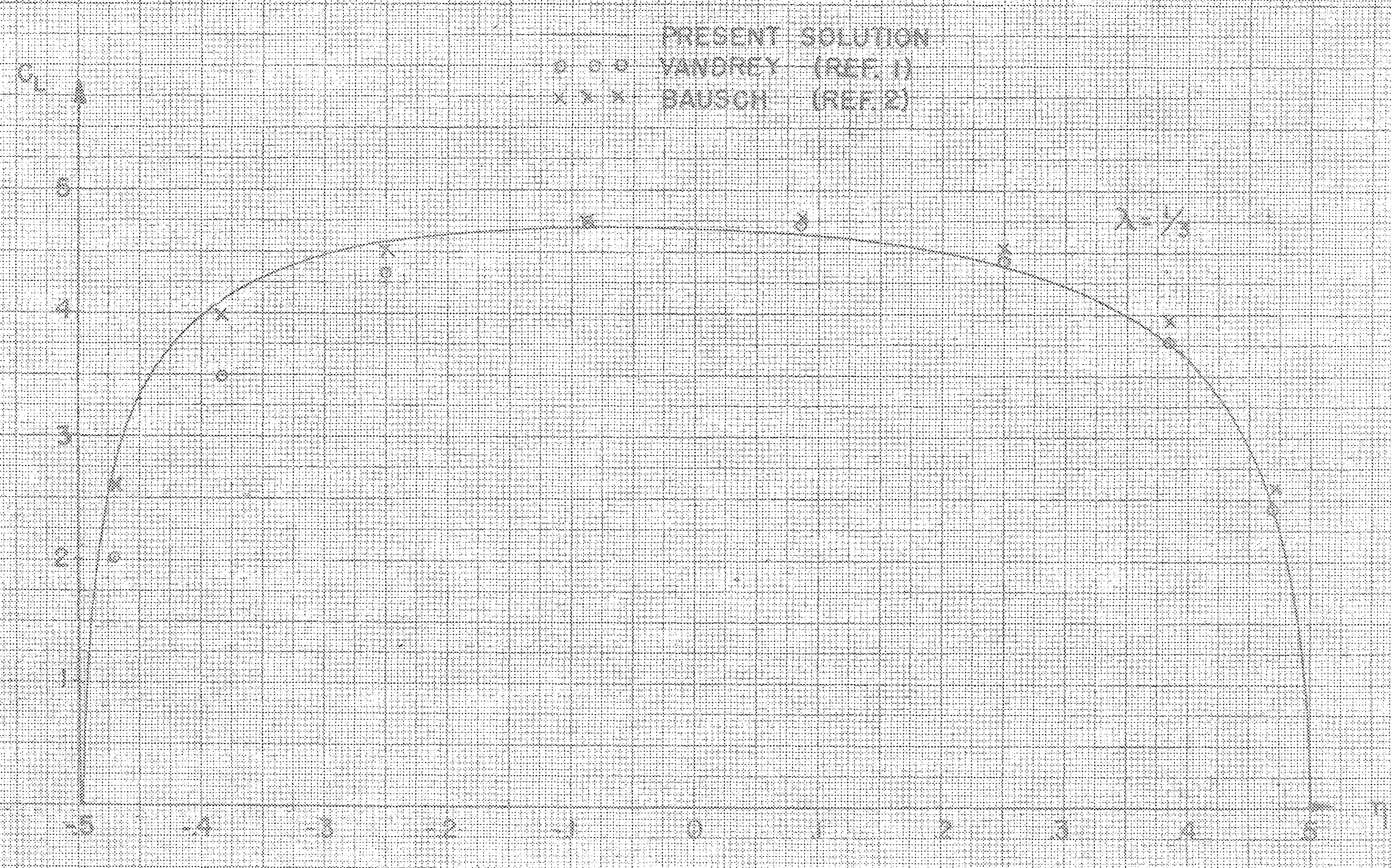


FIG 8

COMPARISON OF THE PRESENT SOLUTION WITH EARLIER RESULTS

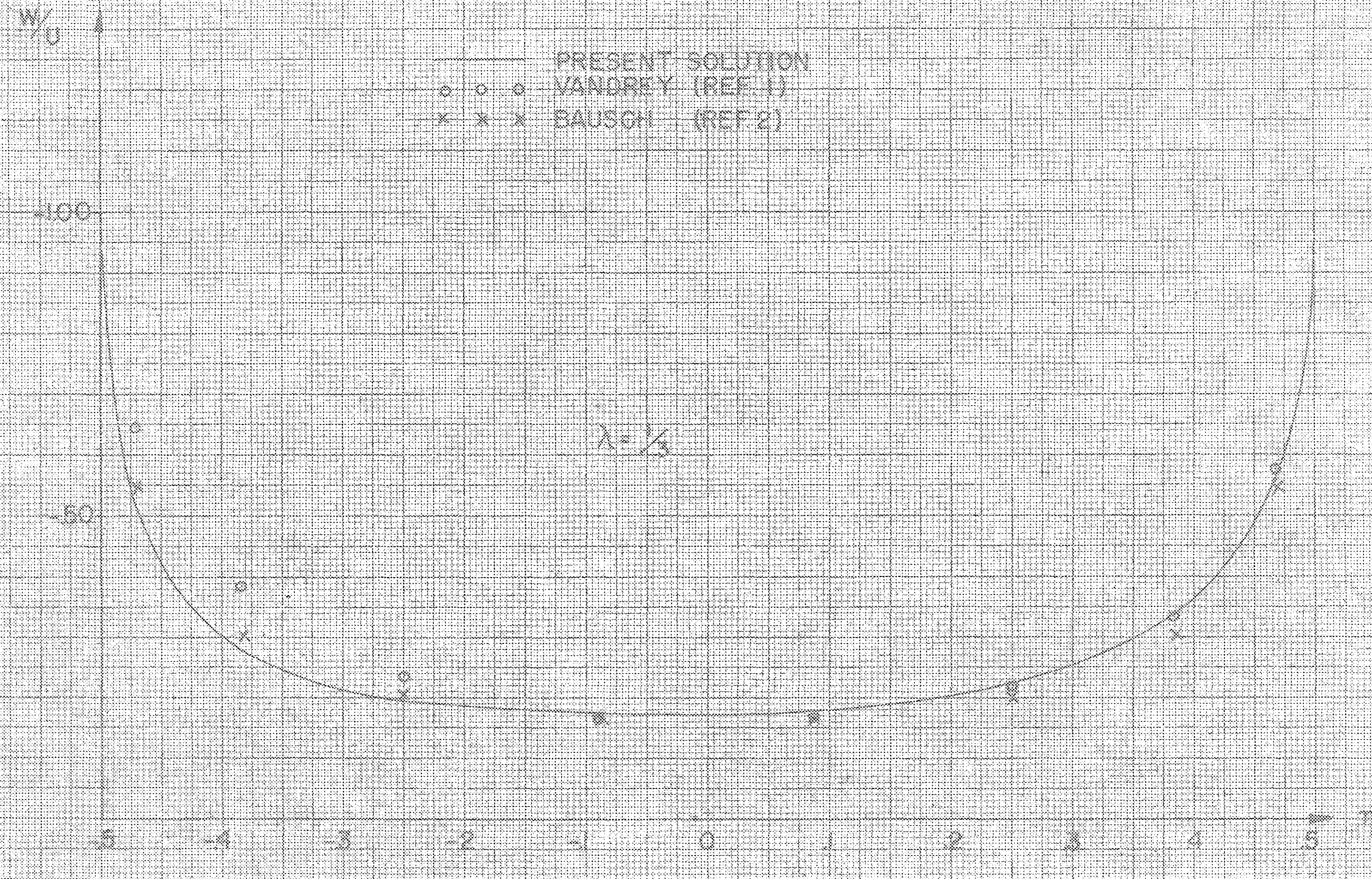


FIG. 9

COMPARISON OF THE PRESENT SOLUTION WITH EARLIER RESULTS

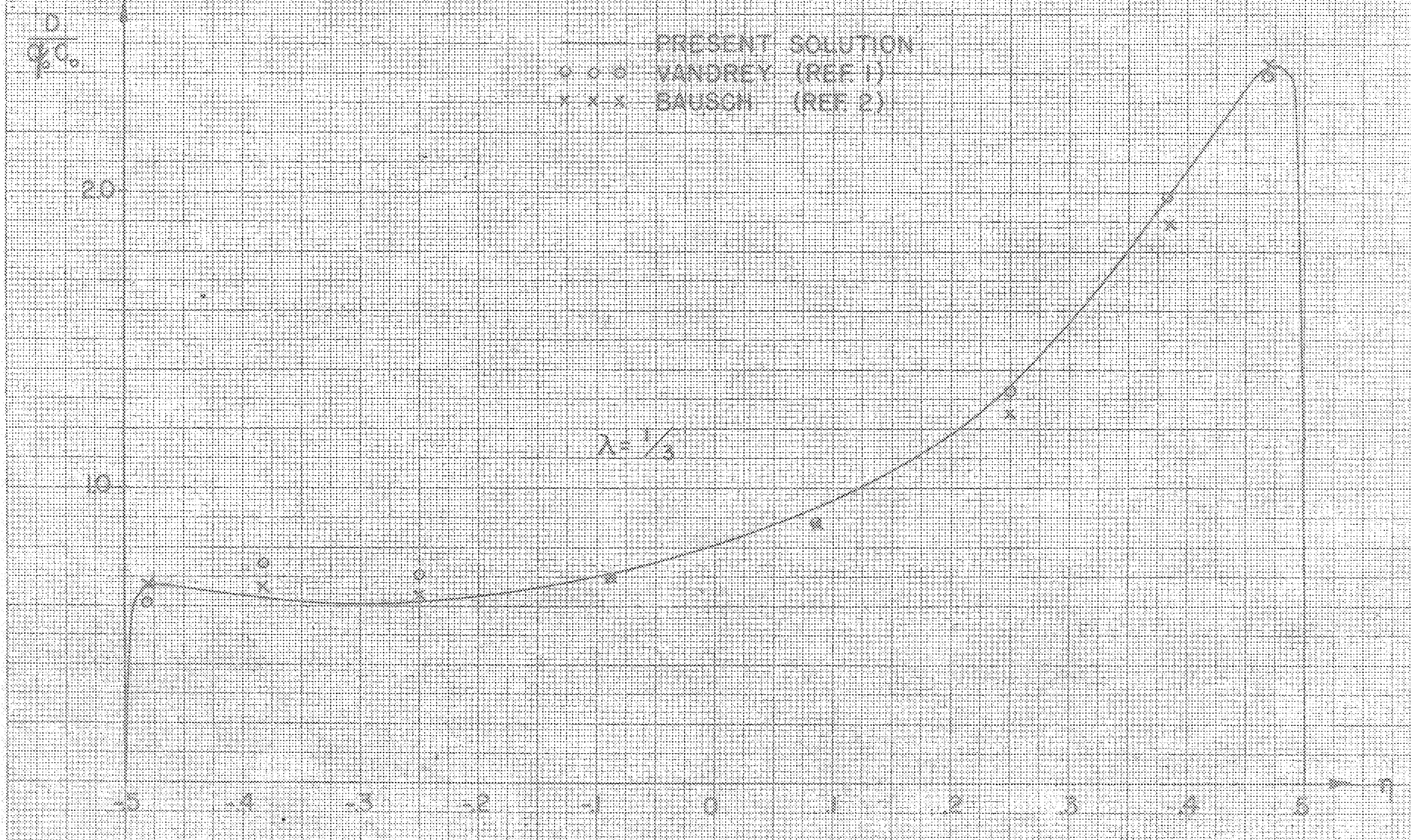
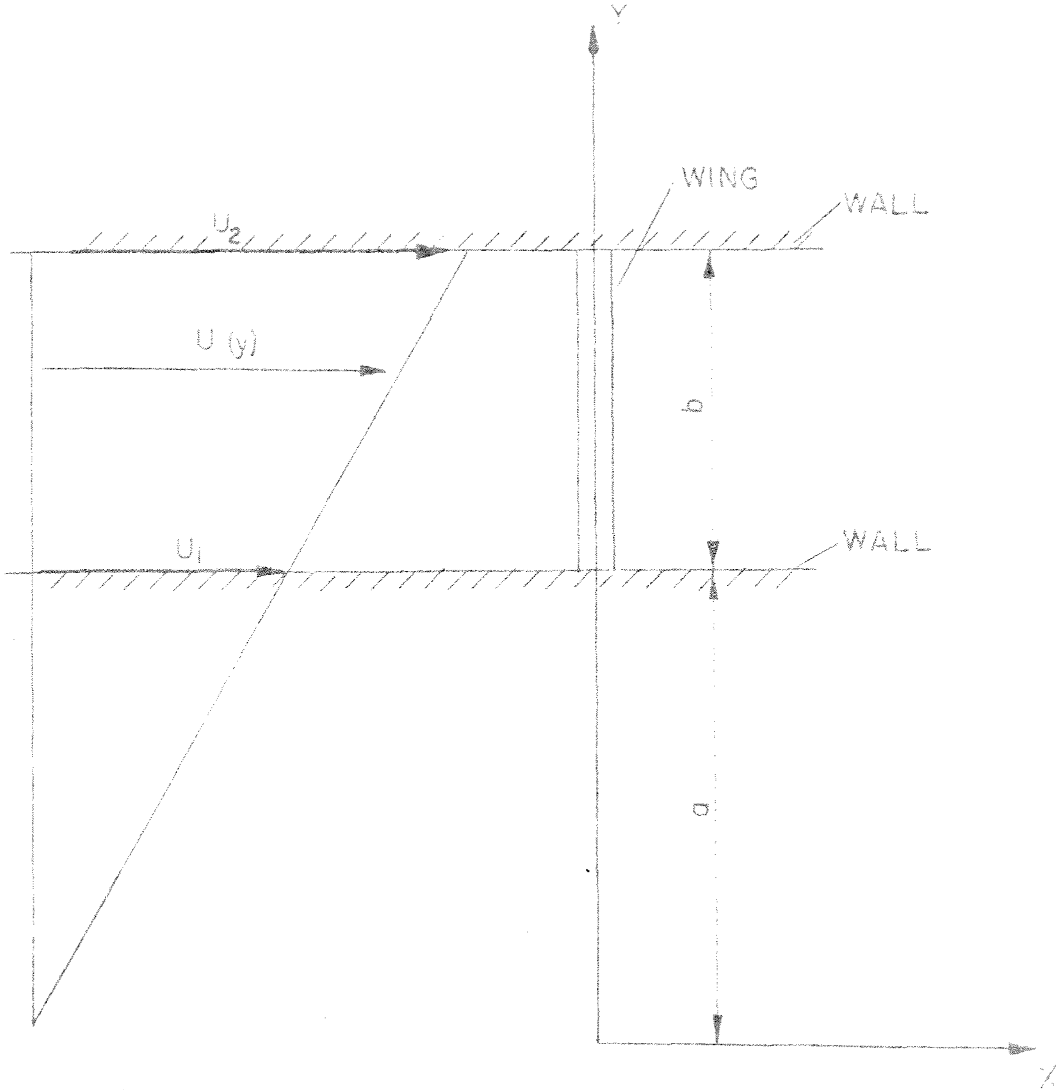


FIG. 10



$$\lambda = \frac{U_2 - U_1}{U_2 + U_1} = \frac{b}{2a+b}$$

FIG. 11

DISTRIBUTION OF C_L OVER A WING BETWEEN PARALLEL WALLS

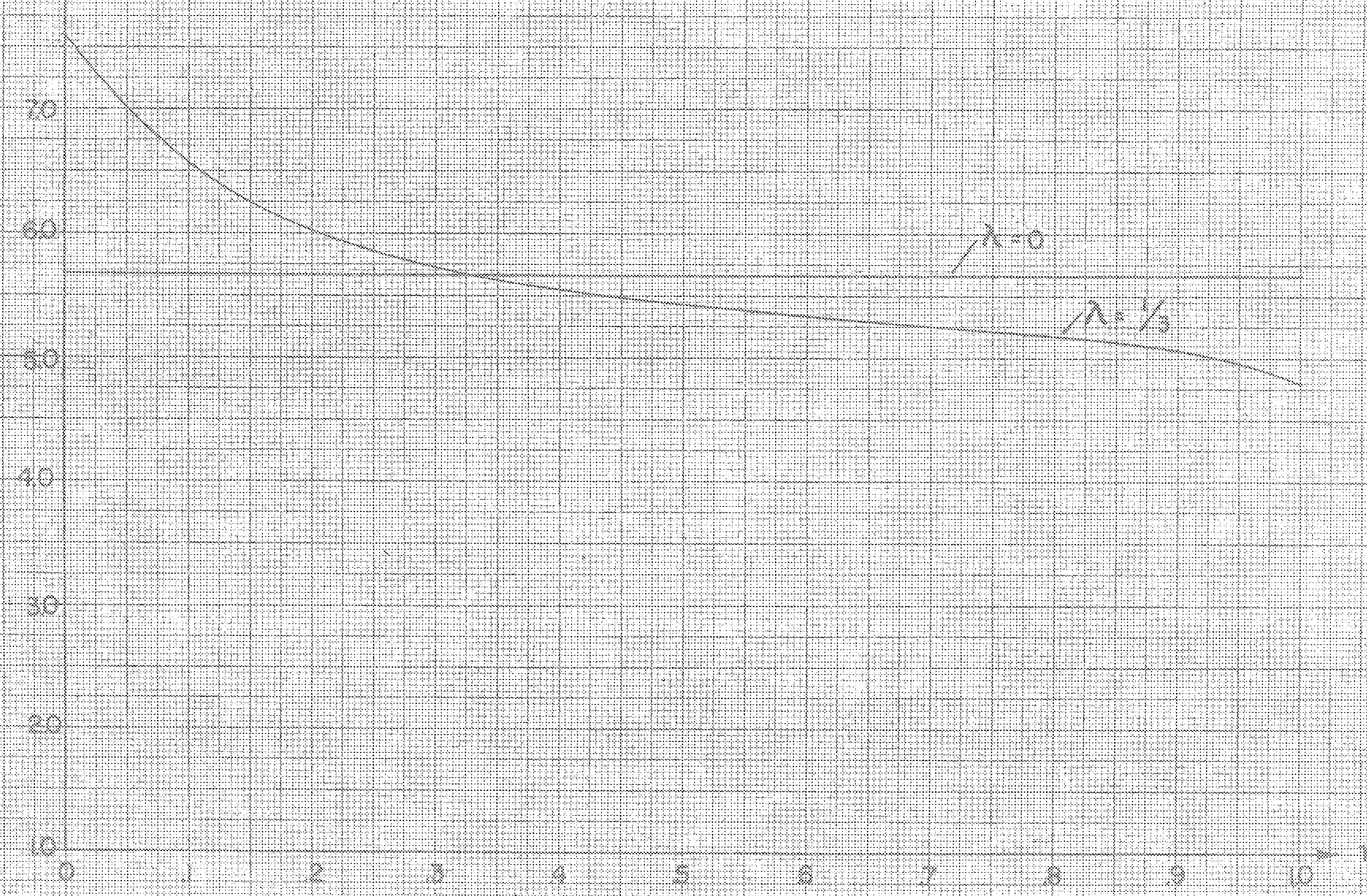
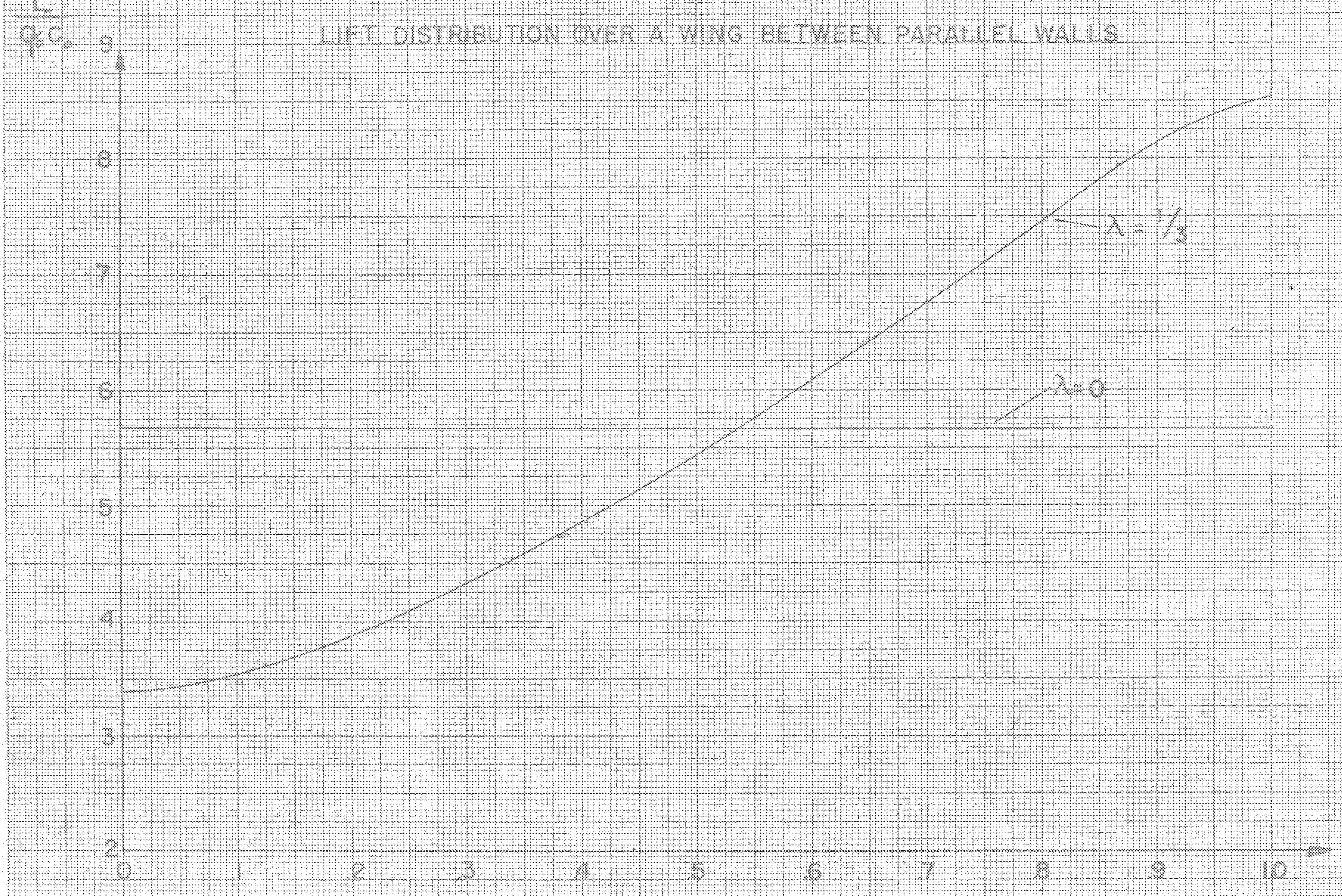


FIG. 12



LIFT DISTRIBUTION OVER A WING BETWEEN PARALLEL WALLS
L
q C
9
8
7
6
5
4
3
2
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
η
λ = 1/3
λ = 0

FIG. 13

DOWNWASH DISTRIBUTION OVER A WING BETWEEN PARALLEL WALLS

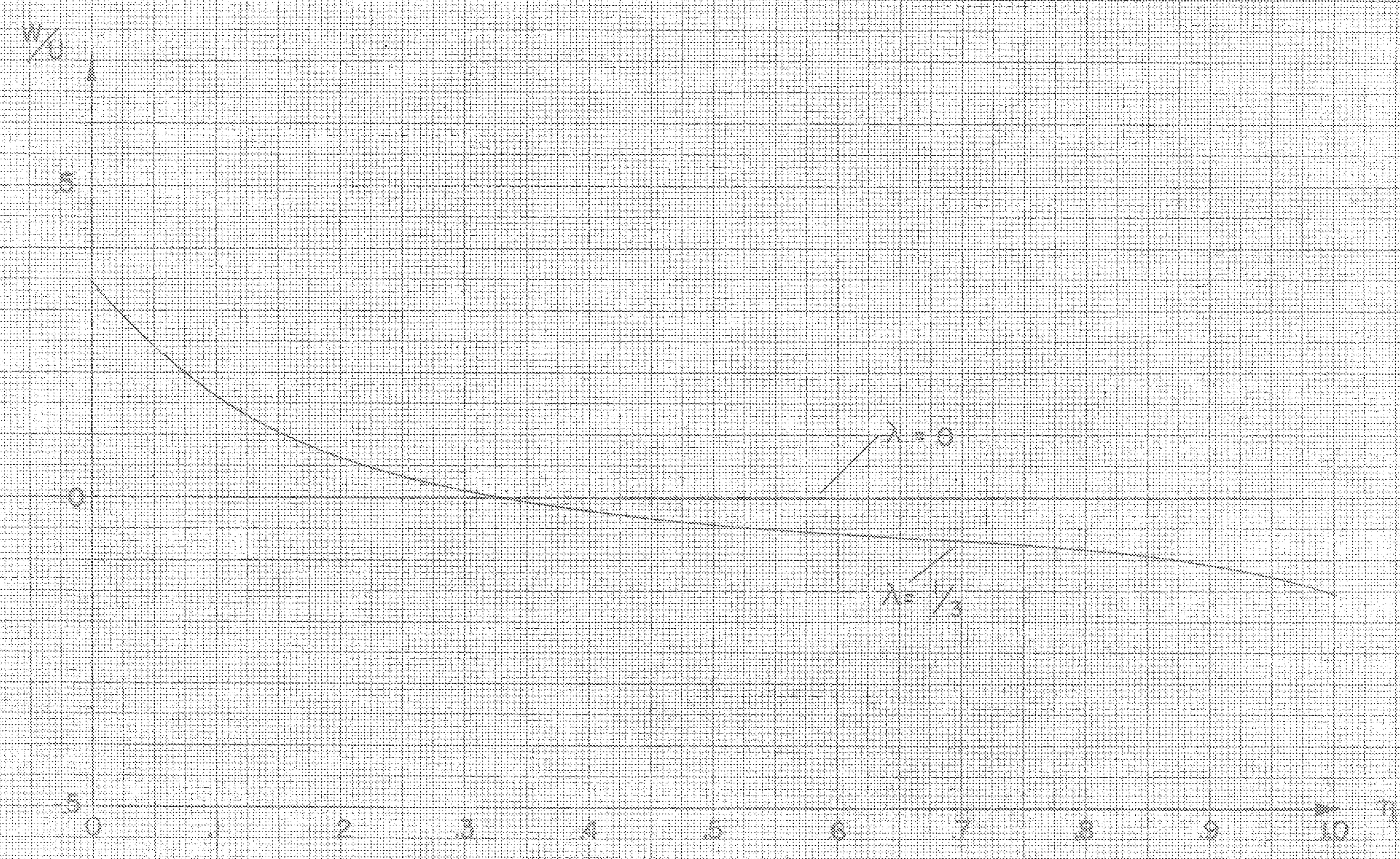


FIG. 14

DISTRIBUTION OF THE DRAG FORCE OVER A WING BETWEEN PARALLEL WALLS

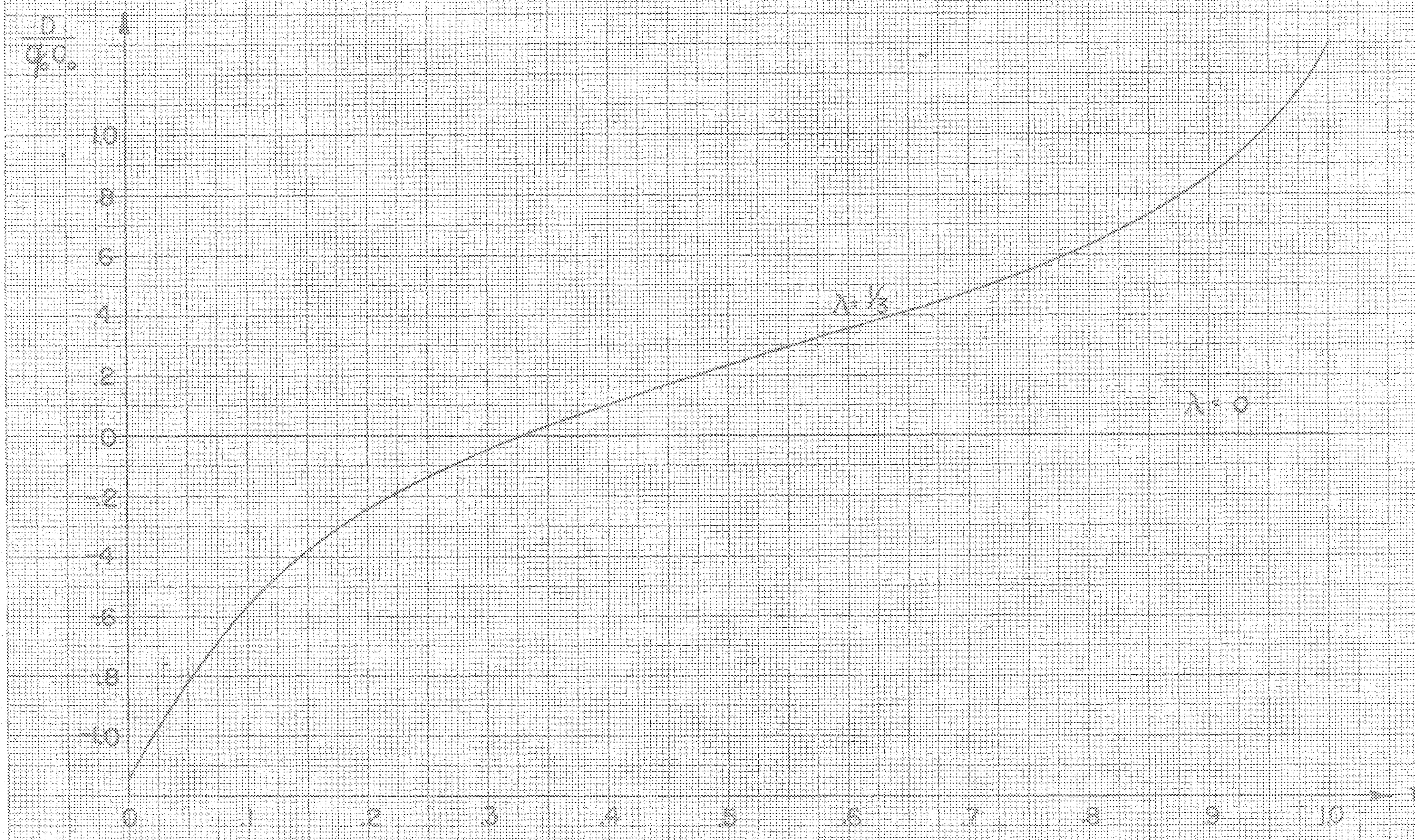


FIG. 15