INVESTIGATION OF GENERALIZED
CONICAL FLOW FIELDS

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ABSTRACT

The conical transformation of variables of M. D. Haskind and S. V. Falkovich is applied to the steady-state problem of thin delta wings with subsonic leading edges in a supersonic flow. It is shown that solutions may be obtained, in terms of elliptic functions, for lifting wings of zero thickness with prescribed angle-of-attack distribution, and for symmetric non-lifting where the perturbation pressure is prescribed on the wing surface (thickness case, mean-surface assumption); the wing boundary conditions are assumed to be given in terms of polynomials in the space variables in the plane of the wing. Some previously known results are obtained to illustrate the method of analysis.

CONTENTS

I. INTRODUCTION

The aerodynamic characteristics of a thin wing placed in a uniform supersonic flow, such that the flow deviations caused by the wing are sufficiently small, can be determined to a satisfactory degree of approximation by means of the linearized compressible flow theory. In particular, conical flows, in which the perturbation velocity components are constant along rays through the wing apex (i.e., the perturbation potential is a homogeneous function of degree one in the physical space variables) have been investigated thoroughly by Lagerstrom (Ref. 1), and the results of the theory have been applied to obtain the solutions to many practical problems involving supersonic wings. An evident generalization of conical flow is that in which the potential is homogeneous of degree greater than one. Flow fields of this type have been investigated by a number of different methods; Germain (Ref. 2), and Hayes, Roberts, and Haaser (Ref. 3) have used the methods of function theory, while Lomax and Heaslet (Ref. 4) have applied integral equation techniques. In Reference 5, Lampert has extended the Lamé function approach to supersonic wing theory, introduced by Robinson and used by Squire (Ref. 6), to obtain solutions to a number of thickness problems for triangular wings with blunt leading edges.

In Reference 7, Haskind and Falkovich presented a series solution for the problem of a delta wing, with subsonic leading edges, oscillating harmonically in a supersonic flow. A conical transformation of variables was introduced, and the problem was reduced to finding a solution to

Laplace's two-dimensional equation, with prescribed boundary conditions, for each term of the series. If the analysis of Reference 7 is reduced to the steady-state case, then it is equivalent to the superposition of homogeneous flow fields of degree one and higher. In the present paper this approach is applied to the steady-state problems of thin lifting delta wings, and symmetric non-lifting delta wings, having the leading edges swept behind the Mach cone, with the boundary conditions on the wing given as homogeneous polynomials in the coordinate plane containing the wing. The problems reduce to finding solutions to Laplace's equation in two dimensions, with prescribed boundary conditions. The method of conformal transformation is used to obtain the solutions in terms of elliptic functions, similar to the type of analysis introduced in Reference 8. Some previously known results are obtained to illustrate the method of analysis.

II. ANALYSIS

A. Equations of Linear Theory

The steady-state, compressible potential flow of a perfect gas is described by the differential equation

$$a^2 \nabla^2 \Phi = (\nabla \Phi) \nabla \left(\tfrac{1}{2} \nabla \Phi^2 \right) \tag{1}$$

where a is the local speed of sound in the gas; $a = \frac{\partial p}{\partial \rho}$, with constant entropy. If the flow is uniform with velocity V_0 in the x direction, except for small perturbations due to the presence of a body with surface tangents varying little from the free-stream direction, equation (1) reduces to

$$\beta^2 \phi_{xx} - \phi_{yy} - \phi_{zz} = 0, \tag{2}$$

the well-known Prandtl-Glauert equation, where $\beta = \sqrt{M_0^2 - 1}$ and $M_0 = \dfrac{V_0}{a_0}$, the free-stream Mach number. (See Figure 1, showing the (x, y, z) coordinate system.) The potential ϕ in equation (2) is associated with the perturbation velocities u, v, w in the x, y, z directions, respectively, and is connected with Φ by the relation $\Phi = V_0 x + \phi(x, y, z)$.

The difference between the local and free-stream pressures is

$$p - p_0 = -\rho_0 \left[V_0 u + \tfrac{1}{2}(v^2 + w^2) \right] \tag{3}$$

If it is assumed that the boundary conditions on the surface of the body may be satisfied on a "mean surface," in the present paper the xy plane, equation (3) may be written

$$p - p_0 = -\rho_0 V_0 u$$

The local slope of the wing surface is connected with the velocity component w on the wing by

$$\frac{dz}{dx} = \frac{w}{V_0}$$

and, by the mean-surface assumption, may be prescribed as a function of x and y.

B. Haskind-Falkovich Transformation

The Prandtl-Glauert equation is also satisfied by the perturbation velocity components u, v, and w, or by a state property of the gas, such as pressure; i.e.,

$$\beta^2 P_{xx} - P_{yy} - P_{zz} = 0 \tag{4}$$

where P denotes the particular velocity or state property. In his treatment of conical flows, Stewart (Ref. 8) applied the transformation of variables

$$\left.\begin{aligned} r &= \sqrt{x^2 - \beta^2(y^2 + z^2)} \\ \mu &= \frac{x}{r} \\ \theta &= \tan^{-1}\left(\frac{y}{z}\right) \end{aligned}\right\} \tag{5}$$

to equation (4), yielding

$$r^2 P_{rr} + 2rP_r + \frac{\partial}{\partial\mu}\left[(1-\mu^2)P_\mu\right] + \frac{1}{1-\mu^2}P_{\theta\theta} = 0 \qquad (6)$$

If the function P is independent of r, equation (6) reduces to

$$(\mu^2-1)\frac{\partial}{\partial\mu}\left[(1-\mu^2)P_\mu\right] + P_{\theta\theta} = 0$$

and setting $s = \sqrt{\frac{\mu-1}{\mu+1}}$ yields

$$s\frac{\partial}{\partial s}\left(s\frac{\partial P}{\partial s}\right) + \frac{\partial^2 P}{\partial\theta^2} = 0$$

which is the normal form of Laplace's equation in two-dimensional polar coordinates. Hence P may be written as the real part of an analytic function of the complex variable $\mathcal{S} = se^{i\theta}$ (see Figure 2), and the methods of function theory may be employed to find P, subject to suitable boundary conditions.

The Haskind-Falkovich transformation of variables (Ref. 7) is

$$\xi = \frac{r}{x+\beta y} \; ; \quad \eta = \frac{\beta z}{x+\beta y} \qquad (7)$$

or

$$x = \frac{r}{2\xi}(1+\xi^2+\eta^2) \; ; \; y = \frac{r}{2\beta\xi}(1-\xi^2-\eta^2) ; \; z = \frac{r\eta}{\beta\xi} \qquad (7a)$$

and it has been pointed out by Dr. H. J. Stewart that equations (7) are equivalent to the bilinear transformation

$$\epsilon = \frac{1+i\mathcal{S}}{1-i\mathcal{S}} \qquad (8)$$

where $\epsilon = \xi + i\eta$. In (x, y, z) (see Figure 3), the surfaces ξ = constant are elliptic cones with apexes at the origin, and containing the line $z = 0$, $m_0 x + y = 0$; the Mach circle corresponds to $\xi = 0$. The surfaces η = constant are planes containing the line $z = 0$, $m_0 x + y = 0$; the plane $\eta = 0$ coincides with the xy plane, $z = 0$, in which the boundary conditions are to be specified. The ϵ plane is shown in Figure 4. It is seen that the Mach circle $|\mathcal{S}| = 1$ is mapped into the imaginary axis; the right half-plane corresponds to the region inside the Mach circle, while the left half-plane corresponds to the region outside the Mach circle. The segment $Rl\, \mathcal{S} = 0, |Im\, \mathcal{S}| \leq s_0$, corresponding to the wing, is mapped into the line $\eta = 0$, $a \leq \xi \leq b$, and the reflection of the wing in $|\mathcal{S}| = 1$ $\left(Rl\, \mathcal{S} = 0, |Im\, \mathcal{S}| \geq \frac{1}{s_0}\right)$ is mapped into $\eta = 0, -b \leq \xi \leq -a$.

Here

$$a = \frac{1 - s_0}{1 + s_0}, \quad b = \frac{1 + s_0}{1 - s_0}, \quad 0 < a < 1, \; b > 1, \; ab = 1.$$

In equation (4), the transformed Prandtl-Glauert equation, let $P \equiv \phi$, and assume solutions in the form

$$\phi = r^n \chi_n(\xi, \eta)$$

where in this paper n is a positive integer. ($n = 1$ corresponds to conical flow.) This yields

$$\xi^2 \left(\frac{\partial^2 \chi_n}{\partial \xi^2} + \frac{\partial^2 \chi_n}{\partial \eta^2} \right) - n(n+1) \chi_n = 0$$

Let $\chi_n = \xi^{n+1} H_n(\xi,\eta)$. Then

$$\frac{\partial^2 H_n}{\partial \xi^2} + \frac{2(n+1)}{\xi}\frac{\partial H_n}{\partial \xi} + \frac{\partial^2 H_n}{\partial \eta^2} = 0$$

which is Darboux's equation (Ref. 9); solutions for H_n may be written

$$H_n = \left(\frac{1}{\xi}\frac{\partial}{\partial \xi}\right)^{n+1} U_n(\xi,\eta)$$

where $U_n(\xi,\eta)$ is a solution to Laplace's equation in two-dimensional rectangular coordinates,

$$\frac{\partial^2 U_n}{\partial \xi^2} + \frac{\partial^2 U_n}{\partial \eta^2} = 0.$$

Therefore $\chi_n(\xi,\eta)$ is

$$\chi_n = \xi^{n+1}\left(\frac{1}{\xi}\frac{\partial}{\partial \xi}\right)^{n+1} U_n(\xi,\eta). \qquad (9)$$

Since $\nabla^2 U_n = 0$, U_n may be written as the real part of an analytic function of ϵ,

$$W(\epsilon) = U_n + i V_n$$

$$\frac{\partial U_n}{\partial \xi} = \frac{\partial V_n}{\partial \eta}; \quad \frac{\partial U_n}{\partial \eta} = -\frac{\partial V_n}{\partial \xi}$$

and the method of conformal mapping may be used to find the function $W(\epsilon)$ (and its derivatives) which satisfy the given boundary conditions.

If the linear differential operator θ is defined by $\theta \equiv \xi D \equiv \xi \frac{\partial}{\partial \xi}$, equation (9) may be written in the form

$$\chi_n = \frac{1}{\xi^{n+1}} \theta(\theta-2)(\theta-4)\cdots(\theta-2n) U_n(\xi,\eta). \tag{10}$$

An equivalent expression is

$$\chi_n = \xi^{-n} \int \xi d\xi \cdots \int \xi D^{2n+1} U_n(\xi,\eta)\, d\xi \tag{11}$$

(n integrals)

if $D^{2n+1} U_n(\xi,\eta) \not\equiv 0$, and the arbitrary functions of integration are all taken to be zero.

In (ξ, η, r) coordinates, the velocity components are

$$u = \frac{1}{2} r^{n-1} \left[\frac{n}{\xi}(1+\xi^2+\eta^2)\chi_n + (1-\xi^2+\eta^2)\frac{\partial \chi_n}{\partial \xi} - 2\xi\eta \frac{\partial \chi_n}{\partial \eta} \right] \tag{12a}$$

$$v = -\frac{\beta}{2} r^{n-1} \left[\frac{n}{\xi}(1-\xi^2-\eta^2)\chi_n + (1+\xi^2-\eta^2)\frac{\partial \chi_n}{\partial \xi} + 2\xi\eta \frac{\partial \chi_n}{\partial \eta} \right] \tag{12b}$$

$$w = \beta r^{n-1} \left[-\eta\left(\frac{n}{\xi}\chi_n + \frac{\partial \chi_n}{\partial \xi} \right) + \xi \frac{\partial \chi_n}{\partial \eta} \right] \tag{12c}$$

The n^{th} derivatives of ϕ are obviously independent of r , so they may be written as the real parts of analytic functions of ϵ .

For n=1 (conical)

$$U_0(\epsilon) = u + i\bar{u}$$
$$V_0(\epsilon) = v + i\bar{v}$$
$$W_0(\epsilon) = w + i\bar{w}$$

Thus, in terms of $W(\epsilon)$,

$$\frac{dU_0}{d\epsilon} = \frac{1}{2}(1-\epsilon^2)\frac{d^4W}{d\epsilon^4} \tag{13a}$$

$$\frac{dV_0}{d\epsilon} = -\frac{1}{2}\beta(1+\epsilon^2)\frac{d^4W}{d\epsilon^4} \tag{13b}$$

$$\frac{dW_0}{d\epsilon} = i\beta\epsilon\frac{d^4W}{d\epsilon^4} \tag{13c}$$

and these equations are of course equivalent to the well-known compatibility relations for conical flow. For n=2, the six distinct second derivatives of ϕ may be written

$$\frac{\partial u}{\partial x} = f_{11} = R/\ F_{11}(\epsilon)\ ; \quad \frac{\partial v}{\partial y} = f_{22} = R/\ F_{22}(\epsilon)$$

$$\frac{\partial w}{\partial z} = f_{33} = R/\ F_{33}(\epsilon)\ ; \quad \frac{\partial u}{\partial y} = f_{12} = R/\ F_{12}(\epsilon)$$

$$\frac{\partial w}{\partial y} = f_{23} = R/\ F_{23}(\epsilon)\ ; \quad \frac{\partial w}{\partial x} = f_{31} = R/\ F_{31}(\epsilon)$$

and in terms of $W(\epsilon)$,

$$\left.\begin{aligned}
\frac{dF_{11}}{d\epsilon} &= \frac{1}{4}(1-\epsilon^2)^2 \frac{d^6W}{d\epsilon^6} \\
\frac{dF_{22}}{d\epsilon} &= \frac{1}{4}\beta^2(1+\epsilon^2)^2 \frac{d^6W}{d\epsilon^6} \\
\frac{dF_{33}}{d\epsilon} &= -\beta^2\epsilon^2 \frac{d^6W}{d\epsilon^6} \\
\frac{dF_{12}}{d\epsilon} &= -\frac{1}{4}\beta(1-\epsilon^4) \frac{d^6W}{d\epsilon^6} \\
\frac{dF_{23}}{d\epsilon} &= -\frac{1}{2}i\beta^2\epsilon(1+\epsilon^2) \frac{d^6W}{d\epsilon^6} \\
\frac{dF_{31}}{d\epsilon} &= \frac{1}{2}i\beta\epsilon(1-\epsilon^2) \frac{d^6W}{d\epsilon^6}
\end{aligned}\right\} \qquad (14)$$

and these equations are equivalent to six compatibility relations connecting the functions $F_{ij}\ (\epsilon)$. Equations (13) and (14) will be used in subsequent sections of this paper to determine the zeros of certain derivatives of $W(\epsilon)$.

The lift is

$$L = 2\rho_0 V_0 \iint_S u\,dx\,dy = \frac{4}{\beta}\rho_0 V_0\left(\frac{2s_1}{m_0}\right)^{n+1}\int_a^b \frac{\chi_n(\xi,0)}{\left(\xi+\frac{1}{\xi}\right)^{n+2}}\cdot\frac{d\xi}{\xi}$$

for a given n, where S is the wing area, s_1 the semispan, and m_0 the tangent of one-half the apex angle. By transforming the lift integral directly to (ξ, η) coordinates,

$$L = \frac{\rho_0 V_0}{\beta} \iint_S \frac{u}{\xi}\left(1+\frac{1}{\xi^2}\right) r\,dr\,d\xi .$$

For a given n , $u = r^{n-1} g(\xi)$ on the wing, so the lift on the area bounded by the hyperbolas $r = r_1$, $r = r_2$ and the leading edges $m_0 x \pm y = 0$ is

$$L_1 = \frac{\rho_0 V_0}{\beta(n+1)}\left(r_2^{\,n+1} - r_1^{\,n+1}\right)\int_a^b g(\xi)\left(1+\frac{1}{\xi^2}\right)\frac{d\xi}{\xi} .$$

Assume that u has a singularity at the leading edges (say $\xi = b$) of the form $\dfrac{1}{(b-\xi)^{m-\frac{1}{2}}}$, where m is a positive integer, i.e., near $\xi = b$, $g(\xi) = \dfrac{g_1(\xi)}{(b-\xi)^{m-\frac{1}{2}}}$, $g_1(b) \neq 0$ or ∞.

The integral for L_1 diverges if $m > 1$, so for finite lift $m = 1$.

The drag due to pressure (wave drag) is

$$D = -2\rho_0 V_0 \iint_S u w\,dx\,dy = -\frac{\rho_0 V_0}{\beta}\iint_S \frac{uw}{\xi}\left(1+\frac{1}{\xi^2}\right) r\,dr\,d\xi$$

and if the wing has blunt leading edges, such that near $\xi = b$,

$w = r^{n-1} h_1(\xi)/(b-\xi)^{m-\frac{1}{2}}$, it is necessary that $m = 1$ for finite drag. (For the additional effect of leading-edge "push" on the total drag, see Reference 10.)

C. Boundary Conditions; Determination of $W(\epsilon)$

Wing problems in supersonic linearized theory may be divided into two classes: (1) Lifting wings of zero thickness, where the angle-of-attack distribution of the wing is given, and it is required to find the pressure on the wing, or conversely; and (2) symmetrical non-lifting wings. The thickness distribution may be given, with the pressure to be determined, or conversely.

The symmetry properties, with respect to the xy plane, of ϕ, u, v, and w for these classes are as follows, where A ≡ anti-symmetric, S ≡ symmetric:

	ϕ	u	v	w
Thin lifting wing:	A	A	A	S
Symmetric non-lifting wing:	S	S	S	A

The potential ϕ for any flow may be written as the sum of an even and an odd function, so that any problem may be reduced to these two cases.

The basic physical principle for determining the boundary conditions is that a discontinuity in u or w may occur only where there is a wing, i.e., over certain portions of the xy plane. Thus for a lifting wing of zero thickness, $u = 0$ off the wing in the xy plane, and for symmetric non-lifting wings, $w = 0$ off the wing in the xy plane.

Further, continuity requires that ϕ, u, v, $w = 0$ on the Mach cone (the disturbance is assumed to be zero upstream of the Mach cone). A full discussion of these considerations is given in Reference 1.

1. Lifting Wings of Zero Thickness

Assume that $(w)_o$ is given as a function of x and y, and expand the potential in the form

$$\phi = \sum_{n=1}^{\infty} r^n \chi_n(\xi, \eta).$$

Then

$$(w)_o = \left(\frac{\partial \phi}{\partial z}\right)_o = \beta \xi \sum_{n=1}^{\infty} r^{n-1} \left(\frac{\partial \chi_n}{\partial \eta}\right)_o = f(\xi, r) \quad (\text{given}).$$

Let

$$f(\xi, r) = \beta \xi \sum_{n=1}^{\infty} r^{n-1} f_n(\xi). \qquad (15)$$

Then

$$\left(\frac{\partial \chi_n}{\partial \eta}\right)_o = f_n(\xi)$$

and from equation (9),

$$\left(\frac{\partial \chi_n}{\partial \eta}\right)_o = -\xi^{n+1} \left(\frac{1}{\xi}\frac{\partial}{\partial \xi}\right)^{n+1} \left(\frac{\partial V_n}{\partial \xi}\right)_o$$

by applying the Cauchy-Riemann equation $\dfrac{\partial U_n}{\partial \eta} = -\dfrac{\partial V_n}{\partial \xi}$.

This may be written

$$\theta(\theta-2)\cdots(\theta-2n)(DV_n)_0 = -\xi^{n+1} f_n(\xi) \qquad (16)$$

which is an ordinary differential equation for $(DV_n)_0$. The solution is

$$(DV_n)_0 = C_0 + C_2\xi^2 + \cdots + C_{2n}\xi^{2n} + p_n(\xi)$$

where $p_n(\xi)$ is the particular solution corresponding to $f_n(\xi)$. Assume that $(w)_0$ is given as a homogeneous polynomial of degree m in x and y . Then $n = m+1$, and the potential is

$$\phi = r^n \chi_n(\xi, \eta).$$

From equations (7a), $\xi f_n(\xi)$ is a homogeneous polynomial of degree $(n-1)$ in $(\xi \pm \frac{1}{\xi})$, so $\xi^{n+1} f_n(\xi)$ is an odd polynomial in ξ of degree $(2n-1)$; from the form of the left-hand side of equation (16), the particular solution $p_n(\xi)$ is an odd polynomial in ξ, of degree $(2n-1)$. The solution of equation (16) for this type of boundary condition may therefore be written

$$(DV_n)_0 = C_0 + C_1\xi + \cdots + C_{2n}\xi^{2n} \qquad (17)$$

with $(n+1)$ constants of integration $C_0, C_2, \cdots C_{2n}$.

The function $\frac{dW}{d\epsilon}$ will necessarily contain terms of the form $i\epsilon^{2\lambda}$ (λ = positive integer, $\leq n$), corresponding to the even-powered terms in equation (17); by direct substitution into the equation defining χ_n as a function of DU_n , it is found that the contribution of these terms to

$\chi_n(\xi,\eta)$ is everywhere zero, so the integration constants $C_0, C_2, \cdots C_{2n}$ may be put equal to zero, and

$$(DV_n)_0 = C_1\xi + C_3\xi^3 + \cdots + C_{2n-1}\xi^{2n-1} \tag{18}$$

where the constants C_i are all known.

On the Mach cone $(\xi = 0)$, ϕ, u, v, and w = 0. From the equations (12) defining u, v, w in terms of (ξ, η, r), these conditions are all satisfied if

$$\left(\frac{\partial U_n}{\partial \xi}\right)_{\xi=0} = 0 .$$

Therefore the function $\frac{dW}{d\epsilon}$ may be continued analytically over the left half-plane by the principle of reflection. This gives

$$\frac{d}{d\epsilon}W(-\bar{\epsilon}) = -\overline{\frac{dW(\epsilon)}{d\epsilon}} \tag{19}$$

so DU_n is an odd function of ξ, and DV_n is an even function of ξ. From equations (12) and the symmetry properties of u, v, and w, DU_n is an odd function of η and DV_n is an even function of η, so that $\frac{dW}{d\epsilon}$ is even in ϵ.

Let

$$G_n(\epsilon) \equiv \frac{d^{2n+1}W}{d\epsilon^{2n+1}} = D^{2n+1}U_n + i\,D^{2n+1}V_n = g_n + i\,h_n .$$

From equations (18) and (19), $G_n(\epsilon)$ is pure real on the real and reflected wings; the other boundary conditions on $G_n(\epsilon)$ (obtained from the symmetry properties of $W(\epsilon)$) are shown in Figure 5. Since $G_n(\epsilon) = Rl$ on both top and bottom of the real and reflected wings, it may be continued across them by reflection; this gives

$$G_n(\bar{\epsilon}) = \overline{G_n(\epsilon)}$$

which contradicts the known symmetry properties of g_n and h_n. Hence a contour crossing the real or reflected wings must pass into a second Riemann sheet, and the ϵ plane is necessarily double-sheeted; the sheets are connected along the slits $\eta = 0$, $a \leq \xi \leq b$, and $\eta = 0$, $-b \leq \xi \leq -a$, corresponding respectively to the real and reflected wings.

Apply the conformal transformation

$$\epsilon = b\,dn(\epsilon_1, k)$$
$$k^2 = 1 - \frac{a^2}{b^2}$$

Then the entire two-sheeted ϵ plane is mapped into a rectangle in the ϵ_1 plane, with corners at the points $\pm K \pm 2iK'$. (See Figure 6.) The points are lettered to correspond with points in the ϵ plane, Figure 4. The unprimed Roman numerals refer to the upper sheet, the primed numerals to the lower sheet. It is seen that the top and bottom sides of the real wing are mapped into the real axis, $-K \leq \xi_1 \leq K$. The points $\epsilon_1 = nK + 2in'K'$ $(n, n' = 0, \pm 1, \pm 2, \ldots)$ correspond to the leading-edge points $\epsilon = \pm a, \pm b$.

Transform $G_n(\epsilon)$ to the ϵ_1 plane, $G_n(\epsilon) \rightarrow G_n(\epsilon_1)$ at corresponding points. The boundary conditions on $G_n(\epsilon_1)$ and its derivative are shown in Figure 7. If $G_n(\epsilon_1)$ is continued outside the basic rectangle by reflection, it is found that

$$G_n(\epsilon_1 + 2K) = G_n(\epsilon_1)$$

$$G_n(\epsilon_1 + 4iK') = G_n(\epsilon_1)$$

i.e., $G_n(\epsilon_1)$ and its derivative are elliptic functions with periods $2K$, $4iK'$. The singularities of $G_n(\epsilon_1)$ must occur at the leading-edge points; in the period cell $-2K' \leq Im\ \epsilon_1 < 2K'$, $-K \leq Rl\ \epsilon_1 < K$ these are $\epsilon_1 = -K,\ 0,\ -K-2iK',\ -2iK'$. For the lift to be finite, the maximum order of the poles of $G_n'(\epsilon_1)$ is $(2n-1)$, so the maximum possible order R of $G_n'(\epsilon_1)$ is $R \leq 8n$.

The boundary conditions on $G_n'(\epsilon_1)$ require zeros at the points $-K \pm iK'$ and $\pm iK'$ in the period cell; the latter points correspond to $\epsilon = \infty$, and for $|\epsilon| >>> 1$,

$$\frac{dW}{d\epsilon} = i\left(\frac{b_2}{\epsilon^2} + \frac{b_4}{\epsilon^4} + \cdots\right)$$

so

$$G_n(\epsilon) = O\left(\frac{1}{\epsilon^{2n+2}}\right) \qquad |\epsilon| >>> 1$$

and $G_n'(\epsilon_1)$ has zeros of order at least $(2n+1)$ at $\pm iK'$ and the conjugate points. The use of the compatibility equations in determining the orders and locations of all the zeros of $G'_n(\epsilon_1)$ is shown in the section on applications; the arbitrary multiplying constant associated with $G'_n(\epsilon_1)$ can be determined by the condition that the even derivatives of $W(\epsilon)$ are real on the Mach circle $(\epsilon = i\eta,\ \epsilon_1 = \xi_1 + iK')$ by equation (19).

If u (i.e., the pressure distribution) is prescribed, then $(DU_n)_0$ is a known polynomial in ξ (see equation (20), page 19, symmetrical non-lifting wings, with pressure prescribed), and it is necessary to find a certain derivative of $(DV_n)_0$ to obtain the angle of attack distribution supporting the given loading. The solution for $(DU_n)_0$ contains one constant of integration, which can be evaluated by using the symmetry properties of the velocity components.

2. Symmetric Non-Lifting Wings

Consider a thin wing, symmetric about the xy plane, and assume that the velocity u (i.e., the perturbation pressure) on the wing is given as a homogeneous polynomial in x and y of degree m. Then $n = m + 1$, and

$$(u)_0 = \frac{1}{2} r^{n-1} f_n(\xi)$$

where $f_n(\xi)$ is a given homogeneous polynomial in $(\xi \pm \frac{1}{\xi})$, of degree $(n-1)$. By equation (12a)

$$(1-\xi^2)\frac{d\chi_n}{d\xi} + \frac{n}{\xi}(1+\xi^2)\chi_n = f_n(\xi)$$

so

$$\chi_n(\xi, 0) = c_0\left(\xi - \frac{1}{\xi}\right)^n + (-1)^n\left(\xi - \frac{1}{\xi}\right)^n \int \frac{\xi^n f_n(\xi)}{(1-\xi^2)^{n+1}}\, d\xi$$

on the wing, where c_0 is a constant of integration. The real part of $\frac{dW}{d\epsilon}$ on the wing is

$$(DU_n)_0 = c_0(a_0 + a_2\xi^2 + \cdots + a_{2n}\xi^{2n}) + (b_0 + b_2\xi^2 + \cdots + b_{2n}\xi^{2n}) + (d_1\xi + d_3\xi^3 + \cdots + d_{2n-1}\xi^{2n-1})$$

where the a_i and b_i are known constants, and the d_i are constants of integration. Associated with the odd polynomial, the function $\frac{dW}{d\epsilon}$ will contain terms of the form $\epsilon^{2\lambda-1}$, $\lambda \leq n$, and it can be shown directly from the equation defining $\chi_n(\xi,\eta)$ that the contribution of such terms to $\chi_n(\xi,\eta)$ is everywhere zero. Hence put $d_1 = d_3 = \cdots = d_{2n-1} = 0$, and

$$(DU_n)_0 = c_0(a_0 + a_2\xi^2 + \cdots + a_{2n}\xi^{2n}) + (b_0 + b_2\xi^2 + \cdots + b_{2n}\xi^{2n}). \tag{20}$$

The conditions ϕ, u, v, w = 0 on the Mach cone give

$$\frac{d}{d\epsilon}W(-\bar{\epsilon}) = -\overline{\frac{dW(\epsilon)}{d\epsilon}} \tag{21}$$

so by the symmetry properties of u, v, and w, $\frac{dW}{d\epsilon}$ is an odd function of ϵ. Let

$$G_n(\epsilon) \equiv \frac{d^{2n+2}W}{d\epsilon^{2n+2}} = g_n + i h_n .$$

By virtue of equations (20) and (21), Rl $G_n(\epsilon) = 0$ on the real and reflected wings; the boundary conditions on $G_n(\epsilon)$ are shown in Figure 8. The function $G_n(\epsilon)$ may be continued across the wing slits by reflection, which gives

$$G_n(\bar{\epsilon}) = -\overline{G_n(\epsilon)}$$

and this contradicts the known symmetry properties of g_n and h_n. Therefore, as in the case of a lifting wing, the ϵ plane is a two-sheeted Riemann surface, connected along the slits $\eta = 0$, $a \leq |\xi| \leq b$, and $G_n(\epsilon)$ may be mapped to the ϵ_1 plane (Figure 6) to obtain the solution to the problem. The boundary conditions on $G_n(\epsilon_1)$ and its derivative are shown in Figure 9. If $G_n(\epsilon_1)$ is continued outside the basic rectangle by reflection, it is found that

$$G_n(\epsilon_1 + 2K) = G_n(\epsilon_1)$$

$$G_n(\epsilon_1 + 4iK') = G_n(\epsilon_1)$$

so $G_n(\epsilon_1)$, and therefore its derivatives, are elliptic functions with periods $2K$, $4iK'$. The singularities of $G_n(\epsilon_1)$ must occur at the points corresponding to the leading edges of the wing; for finite drag the maximum order of the poles of $G_n(\epsilon_1)$ is $(2n + 1)$, so the order R of $G_n(\epsilon_1)$ is $R \leq 4(2n+1)$.

For $|\epsilon| >>> 1$,

$$\frac{dW}{d\epsilon} = \frac{b_1}{\epsilon} + \frac{b_3}{\epsilon^3} + \cdots$$

where the b_i are real constants; so

$$G_n(\epsilon) = O\left(\frac{1}{\epsilon^{2n+2}}\right), \quad |\epsilon| >>> 1$$

and $G_n(\epsilon_1)$ has zeros of order at least $(2n + 2)$ at the points $\epsilon_1 = \pm i K'$ in the period cell.

If $(w)_0$ (i.e., the thickness distribution) is prescribed, then $(DV_n)_0$ is a given polynomial in ξ (see equation (18), page 15, lifting wings) and the derivatives of $(DU_n)_0$ must be found to obtain the pressure distribution. As in the lifting case, the solution does not involve any constants of integration.

III. APPLICATIONS

A. Flat Lifting Delta Wing (Conical)

Consider a thin lifting wing with the boundary conditions on the wing $(w)_0 = w_0$ (constant). From equation (15)

$$f_n(\xi) = \frac{w_0}{\beta\xi} \qquad n = 1$$

$$= 0 \qquad n > 1$$

$n = 1, \ \phi = r\chi_1 .$

and the imaginary part of $\dfrac{dW}{d\epsilon}$ on the wing is

$$\left(\frac{\partial V_1}{\partial \xi}\right)_0 = \frac{w_0}{\beta}|\xi| .$$

Let

p_1 = order of zeros of $G_1'(\epsilon_1)$ at $-K \pm iK'$ $\qquad p_1 \geq 1$

p_2 = order of zeros of $G_1'(\epsilon_1)$ at $\pm iK'$. $\qquad p_2 \geq 3$

p_0 = number of other zeros of $G_1'(\epsilon_1)$, if any (multiple zeros counted according to their multiplicity).

Then

$$R = p_0 + 2p_1 + 2p_2 \leq 8$$

$$2p_1 + 2p_2 \geq 8$$

so $p_0 = 0, \ p_1 = 1, \ p_2 = 3, \ R = 8$, and the elliptic function satisfying all the conditions is

$$G_1'(\epsilon_1) = C \frac{dn\,\epsilon_1}{sn^2\epsilon_1\, cn^2\epsilon_1}$$

where C is a real constant, to be determined. Integration gives

$$G_1(\epsilon_1) = -C \frac{cn^2\epsilon_1 - sn^2\epsilon_1}{sn\,\epsilon_1\, cn\,\epsilon_1}$$

$$\frac{d^2W}{d\epsilon^2} = -Cb\left[(1+k'^2)\epsilon_1 - 2E(\epsilon_1)\right] + i c_1$$

and imposing the condition that $D^2 V_1 = 0$ on $\epsilon = i\eta$ (Mach circle) gives

$$C = \frac{w_0}{\beta b(2E' - k^2K')}.$$

From equations (11) and (12a) the velocity u on the wing is

$$u = \frac{w_0}{\beta} \cdot \frac{(1-k')^2}{2\sqrt{k'}(2E' - k^2K')} \cdot \frac{1+\xi^2}{\sqrt{(\xi^2 - a^2)(b^2 - \xi^2)}}$$

which is in agreement with the result given in Reference 8.

B. Quasisteady Pitching Wing

The boundary condition on the surface of a flat wing pitching about its apex with angular velocity q may be approximated by $(w)_0 = -qx$.

From equation (15)

$$f_n(\xi) = -\frac{q}{2\beta}\left(1+\frac{1}{\xi^2}\right), \quad n = 2$$

$$= 0 \qquad\qquad n \neq 2$$

so

$$\phi = r^2 \chi_2$$

and

$$(DY_2)_0 = -\frac{q}{6\beta}|\xi|(1-\xi^2).$$

From equation (14),

$$\frac{dF_{23}}{d\epsilon} = \frac{\partial f_{23}}{\partial \xi} - i\frac{\partial f_{23}}{\partial \eta} = -\frac{1}{2}i\beta^2\epsilon(1+\epsilon^2)\, G_2'(\epsilon) \qquad (22)$$

By symmetry, $\frac{\partial w}{\partial y} \equiv f_{23} = 0$ on $|\epsilon| = 1$, and by the boundary conditions, $f_{23} = 0$ on the real and reflected wings. It follows from equation (22) that $G_2'(\epsilon) = 0$ at $\epsilon = \pm 1$, so $G_2'(\epsilon_1)$ has zeros of order at least one at the points $\pm K/2$, $\pm K/2 - 2iK'$ in the period cell. Let

p_1 = order of zeros of $G_2'(\epsilon_1)$ at $-K \pm iK'$ $\quad p_1 \geq 1$

p_2 = order of zeros of $G_2'(\epsilon_1)$ at $\pm iK'$ $\quad p_2 \geq 5$

p_3 = order of zeros of $G_2'(\epsilon_1)$ at $\pm\frac{K}{2}, \pm\frac{K}{2} - 2iK'$ $\quad p_3 \geq 1$

p_0 = number of other zeros of $G_2'(\epsilon_1)$ (multiple zeros counted according to their multiplicity).

Then

$$R = p_0 + 2p_1 + 2p_2 + 4p_3 \leq 16$$
$$2p_1 + 2p_2 + 4p_3 \geq 16$$

so $p_0 = 0,\ p_1 = 1,\ p_2 = 5,\ p_3 = 1, R = 16$. The function satisfying all the conditions is therefore

$$G_2'(\epsilon_1) = C\frac{dn\,\epsilon_1(1 - b^2 dn^2\epsilon_1)}{sn^4\epsilon_1\, cn^4\epsilon_1} \quad ; C \text{ real.}$$

Integration yields

$$G_2 = \frac{C}{3}(b^2 - 1)\left[\frac{cn^2\epsilon_1 + k' sn^2\epsilon_1}{sn^3\epsilon_1\, cn^3\epsilon_1} + 4(1-k')\frac{cn^2\epsilon_1 - sn^2\epsilon_1}{sn\,\epsilon_1\, cn\,\epsilon_1}\right] \quad (23)$$

and from equation (11)

$$\chi_2(\xi, 0) = -\frac{C}{3}b^2(1-k')\left(1 + \frac{1}{\xi^2}\right)\sqrt{(\xi^2 - a^2)(b^2 - \xi^2)}$$

on the wing ($\eta = 0+,\ a \leq \xi \leq b$).

Integration of equation (23) and application of the condition $D^4 V_2 = 0$ on $\epsilon_1 = \xi_1 + iK'$ yields

$$C = -\frac{6q}{\beta}\cdot\frac{k'^2\sqrt{k'}}{(1-k')^2}\cdot\frac{1}{k^2(1-3k')K' - (k'^2 - 6k' + 1)E'}$$

so χ_2 is known. In terms of (x, y, z) coordinates, equation (12a) yields for the pressure coefficient

$$\frac{\Delta p}{q} = \frac{4k_1^2 q x}{\sqrt{m_0^2 - y^2/x^2}} \cdot \frac{2m_0^2 - y^2/x^2}{k_1'^2 K(k_1) + (1-2k_1'^2) E(k_1)}$$

$$k_1 \equiv \frac{2\sqrt{k'}}{1+k'}$$

as given in Reference 11.

C. Elliptic Cone (Conical)

Consider a thin symmetric wing, with the pressure specified as constant on the surface; $(u)_0 = u_0$. It follows that $n = 1$ (conical) and

$$(\chi_1)_0 = c_0\left(\xi - \frac{1}{\xi}\right) + u_0 \xi \; ; \quad \left(\frac{\partial \chi_1}{\partial \xi}\right)_0 = c_0\left(1 + \frac{1}{\xi^2}\right) + u_0$$

By symmetry, $v = 0$ at $\epsilon = \pm 1$, and therefore $\left(\frac{\partial \chi_1}{\partial \xi}\right)_0 = 0$ at $\epsilon = \pm 1$. Hence $c_0 = -\frac{u_0}{2}$, so

$$(\chi_1)_0 = \frac{u_0}{2}\left(\xi + \frac{1}{\xi}\right) \qquad a \leq |\xi| \leq b$$

$$\left(\frac{\partial U_1}{\partial \xi}\right)_0 = \frac{u_0}{2}(\xi^2 - 1) \qquad a \leq \xi \leq b$$

$$= -\frac{u_0}{2}(\xi^2 - 1) \qquad -b \leq \xi \leq -a.$$

On $|\epsilon| = 1$, $v = 0$, and by the boundary conditions $\mathrm{Rl}\; G_1(\epsilon) = 0$ on the wing. But

$$\frac{dV_0}{d\epsilon} = \frac{\partial v}{\partial \xi} - i\frac{\partial v}{\partial \eta} = -\frac{1}{2}\beta(1+\epsilon^2)\, G_1(\epsilon)$$

and it therefore follows that $G_1(\pm 1) = 0$. Hence $G_1(\epsilon_1)$ has zeros at the points $\epsilon_1 = \pm K/2, \ \pm K/2 - 2iK'$ in the period cell. Let p_1 be the order of these zeros, let p_2 be the order of the zeros at $\epsilon_1 = \pm iK'$, and let p_0 be the number of other zeros, if any, of $G_1(\epsilon_1)$. Then

$$R = 4p_1 + 2p_2 + p_0 \leq 12$$
$$4p_1 + 2p_2 \geq 12$$
$$p_1 \geq 1, \ p_2 \geq 4, \ p_0 \geq 0$$

so

$$p_0 = 0, \ p_1 = 1, \ p_2 = 4, \ R = 12.$$

The elliptic function satisfying the conditions is therefore

$$G_1(\epsilon_1) = iC \frac{1 - b^2 dn^2 \epsilon_1}{sn^3 \epsilon_1 \, cn^3 \epsilon_1} \tag{24}$$

where C is a real constant. Integration of equation (24) yields

$$\frac{d^3 W}{d\epsilon^3} = u_0 + iCbk^2\left(\frac{1-k'}{k'}\right)\Big[(1-k')\epsilon_1 + \frac{1-k'}{k'}E(\epsilon_1) - \frac{dn\,\epsilon_1}{sn\,\epsilon_1\, cn\,\epsilon_1}\left(sn^2\epsilon_1 + k' cn^2 \epsilon_1\right)\Big]$$

and C is determined by the condition that $\mathrm{Rl}\, \dfrac{d^3W}{d\epsilon^3} = 0$ on the Mach cone $(\epsilon_1 = \xi_1 + iK')$:

$$C = \frac{k'^2 u_0}{bk^2(1-k')^2} \cdot \frac{1}{(1+k')K' - E'}$$

From the equations for χ_n and w, it is found that the surface slope is

$$\frac{dz}{dx} = \pm\left(\frac{u_0}{V_0}\right) \cdot \frac{2k'}{1+k'} \cdot \frac{1}{(1+k')K' - E'} \cdot \frac{1}{\sqrt{m_0^2 - y^2/x^2}}$$

and the surface is an elliptic cone, in agreement with the result given by Squire and others.

IV. LIST OF SYMBOLS

a	Coordinate in ϵ plane, corresponding to wing leading edge.
A	Antisymmetric.
b	Coordinate in ϵ plane, corresponding to wing leading edge.
C	Real constant.
$cn(u,k)$	Jacobi elliptic function, argument u and modulus k.
D	Operator $\partial/\partial\xi$; wing drag.
$dn(u,k)$	Jacobi elliptic function, argument u and modulus k.
dz/dx	Airfoil surface slope.
E, E'	Complete elliptic integrals, second kind, moduli k and k', respectively.
F_{ij}	$f_{ij} + i\overline{f_{ij}}$
f_{ij}	A second derivative of ϕ in (x, y, z).
G_n	$\dfrac{d^{2n+1}W}{d\epsilon^{2n+1}}$ or $\dfrac{d^{2n+2}W}{d\epsilon^{2n+2}}$
g_n	Real part of G_n.
h_n	Imaginary part of G_n.
Im	Imaginary part.
i	$\sqrt{-1}$
K, K'	Complete elliptic integrals, first kind, moduli k and k', respectively.
k	$\sqrt{1 - a^2/b^2}$
k'	$\sqrt{1 - k^2}$
L	Wing lift.

M_0	Free-stream Mach number.
m_0	Slope of wing leading edge, in xy plane.
n	Degree of a homogeneous flow (positive integer).
0	(subscript) Value on wing; constant value.
p	Local static pressure.
p_0	Free-stream static pressure.
p_i	Order of a zero of an elliptic function.
q	Dynamic pressure; angular velocity of pitch.
r	$\sqrt{x^2 - \beta^2(y^2+z^2)}$
Rl	Real part.
R	Order of an elliptic function.
S	Wing area; symmetric.
s_1	Wing semispan.
s	$\sqrt{\frac{\mu-1}{\mu+1}}$
s_0	Coordinate in ζ plane corresponding to an airfoil leading edge.
$sn(u,k)$	Jacobi elliptic function, argument u and modulus k.
U_0	$u + i\bar{u}$
u	Perturbation velocity component in x direction: $\frac{\partial\phi}{\partial x}$
V_0	$v + i\bar{v}$; free-stream velocity.
v	Perturbation velocity component in y direction: $\frac{\partial\phi}{\partial y}$
W	$U_n + iV_n$
W_0	$w + i\bar{w}$

w	Perturbation velocity component in z direction: $\dfrac{\partial\phi}{\partial z}$
x, y, z	Cartesian coordinates.
β	$\sqrt{M_0^2 - 1}$
ϵ	$\xi + i\eta$
ϵ_1	$\xi_1 + i\eta_1$
ς	$s e^{i\theta}$
ξ	$\dfrac{r}{x + \beta y}$
η	$\dfrac{\beta z}{x + \beta y}$
θ	$\tan^{-1}(y/z)$; operator $\xi \dfrac{\partial}{\partial \xi}$
μ	x/r
ρ_0	Free-stream density.
ϕ	Perturbation potential.

V. FIGURES

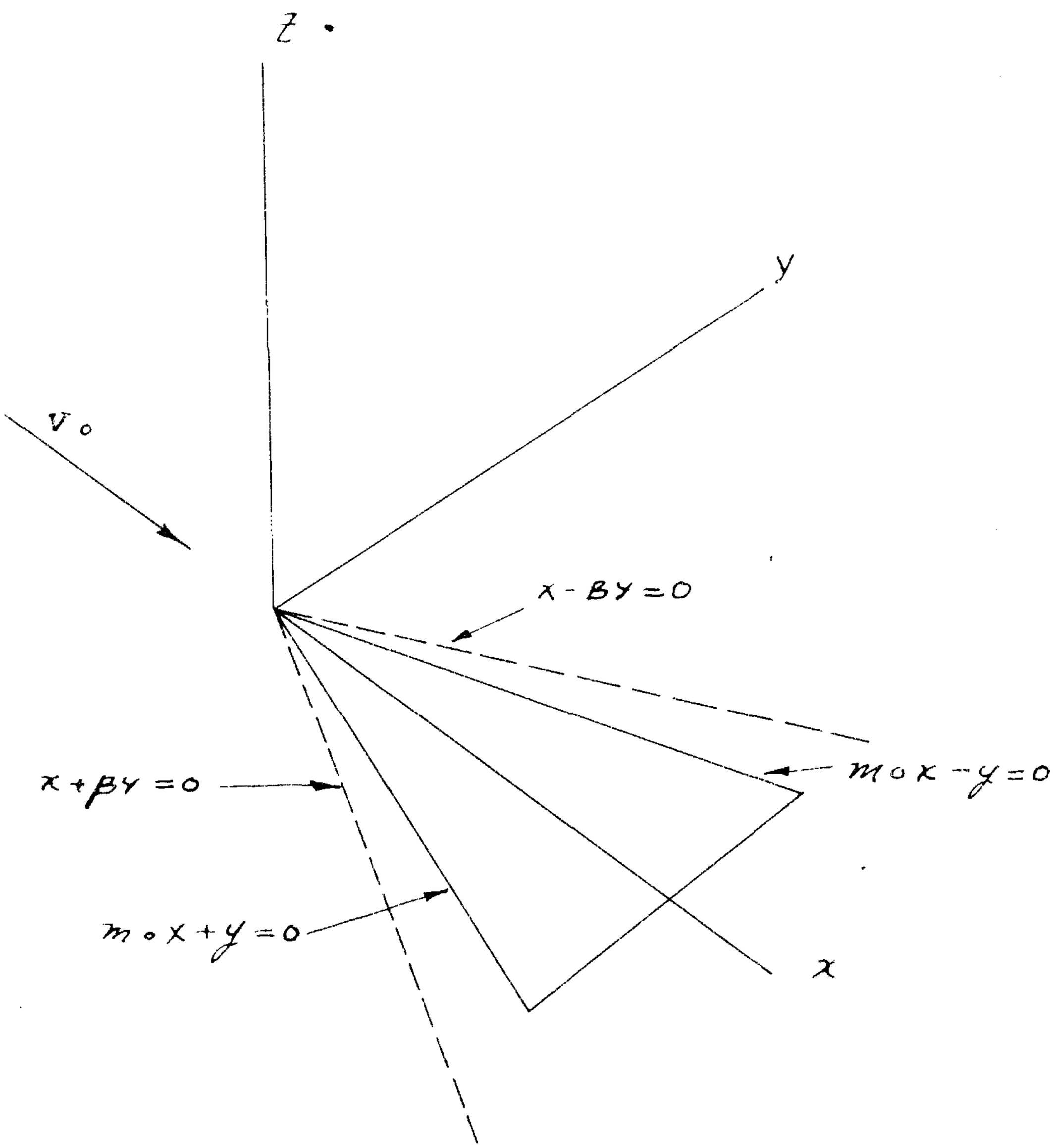

Fig. (1). (x,y,z) COORDINATE SYSTEM

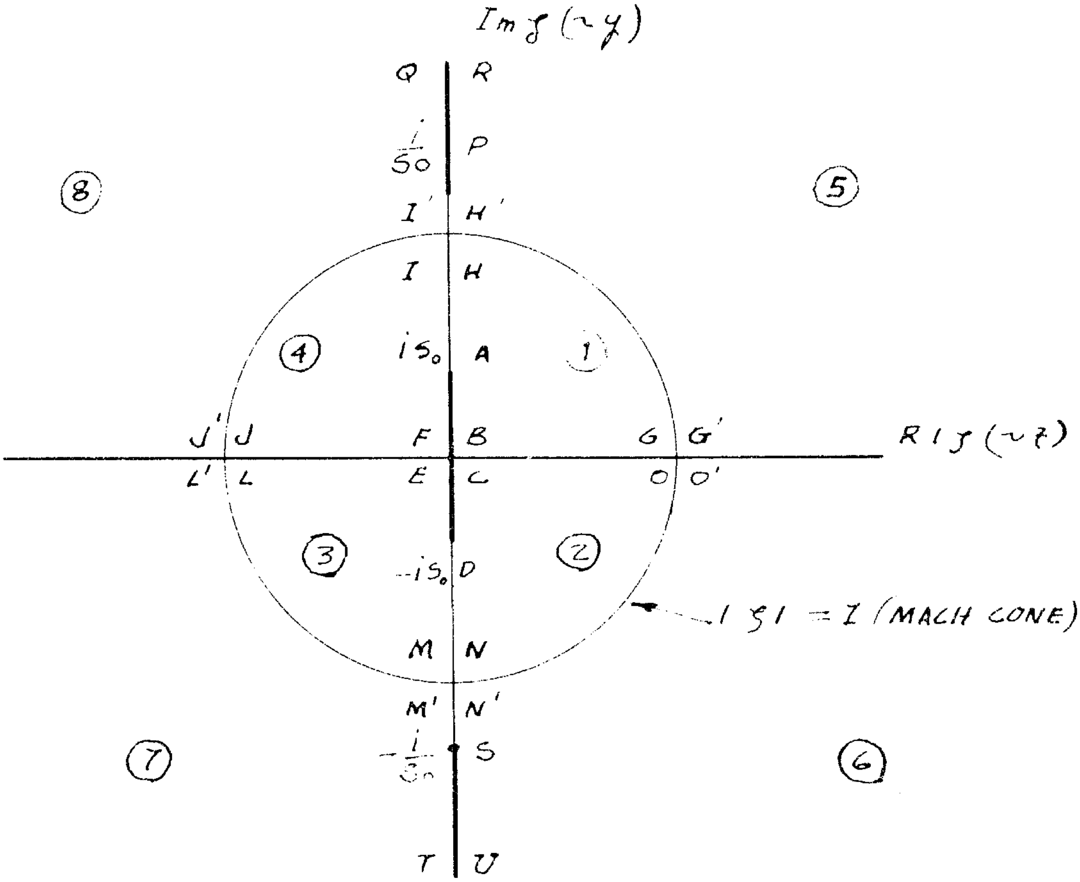
Im ζ (~y)
Q
R
P
I'
H'
I
H
is₀
A
J'
J
F
B
G
G'
Rl ζ (~z)
L'
L
E
C
O
O'
-is₀
D
M
N
M'
N'
S
T
U
1
2
3
4
5
6
7
8
| ζ | = 1 (MACH CONE)

Fig. (2). ζ PLANE

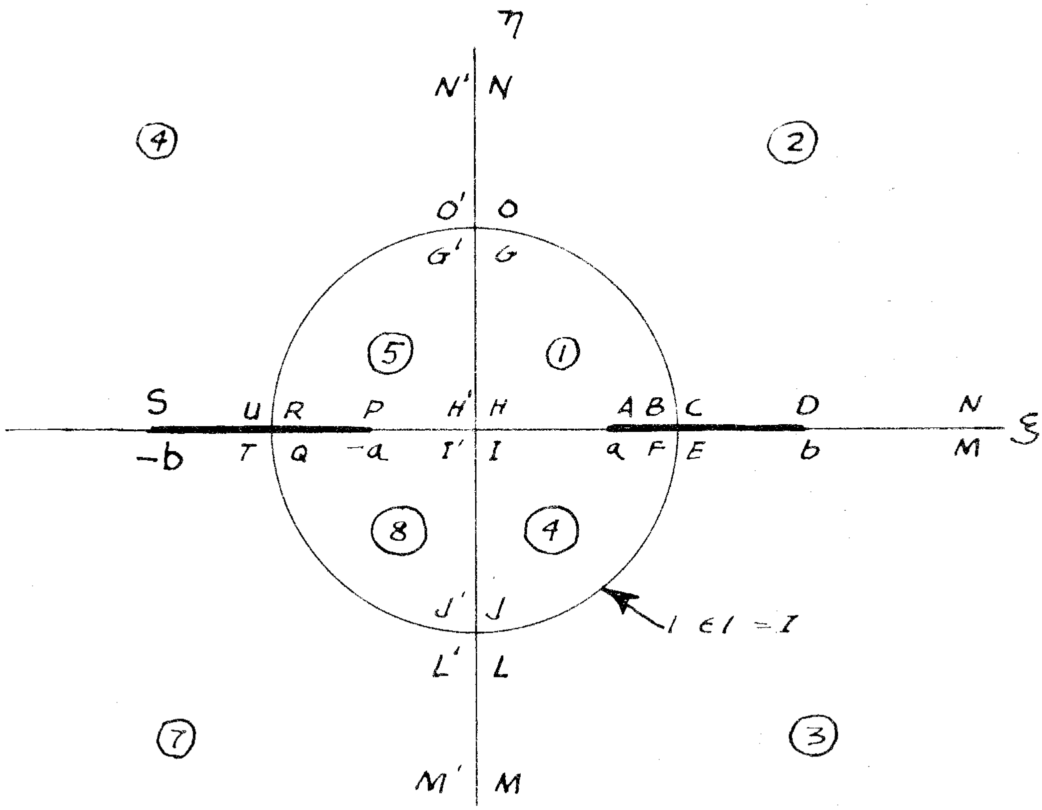

Fig (2 a) ε PLANE

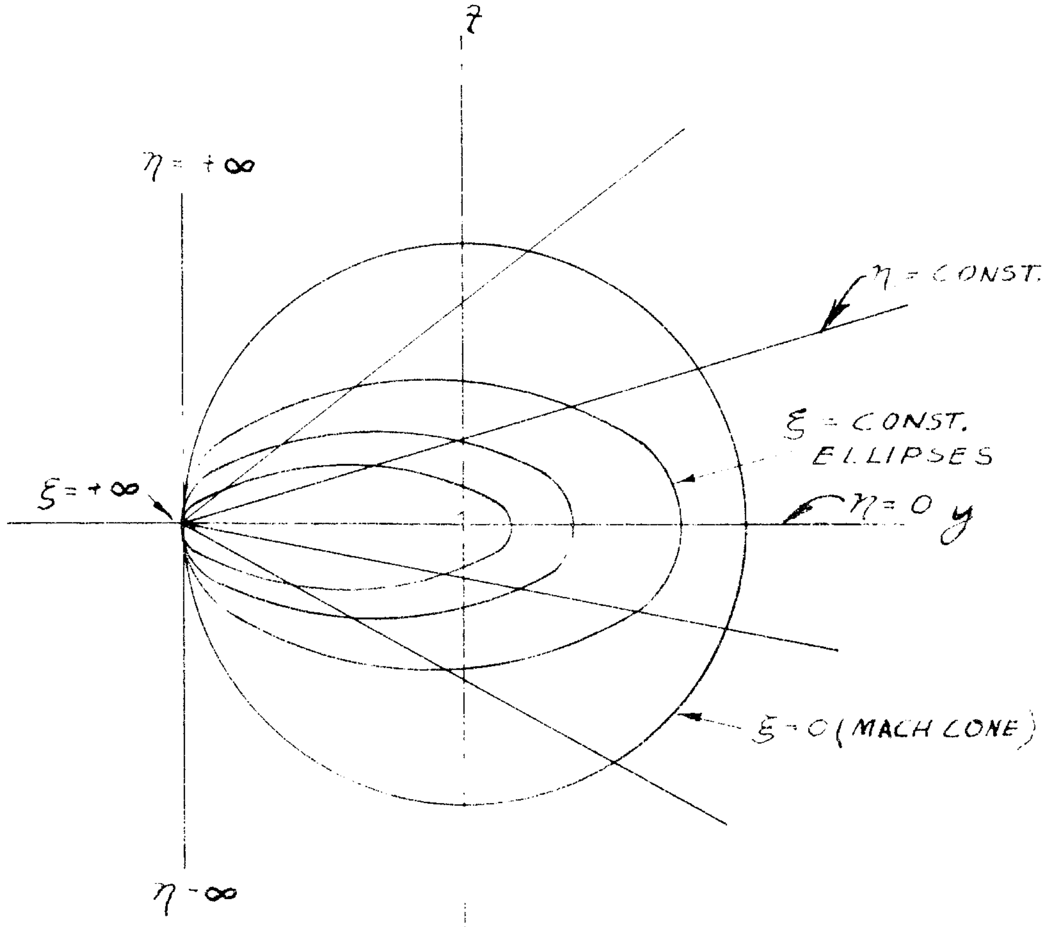

Fig (3) TRACE OF SURFACES ξ = CONST.
η = CONST., in TRANSVERSE PLANE x = CONST.

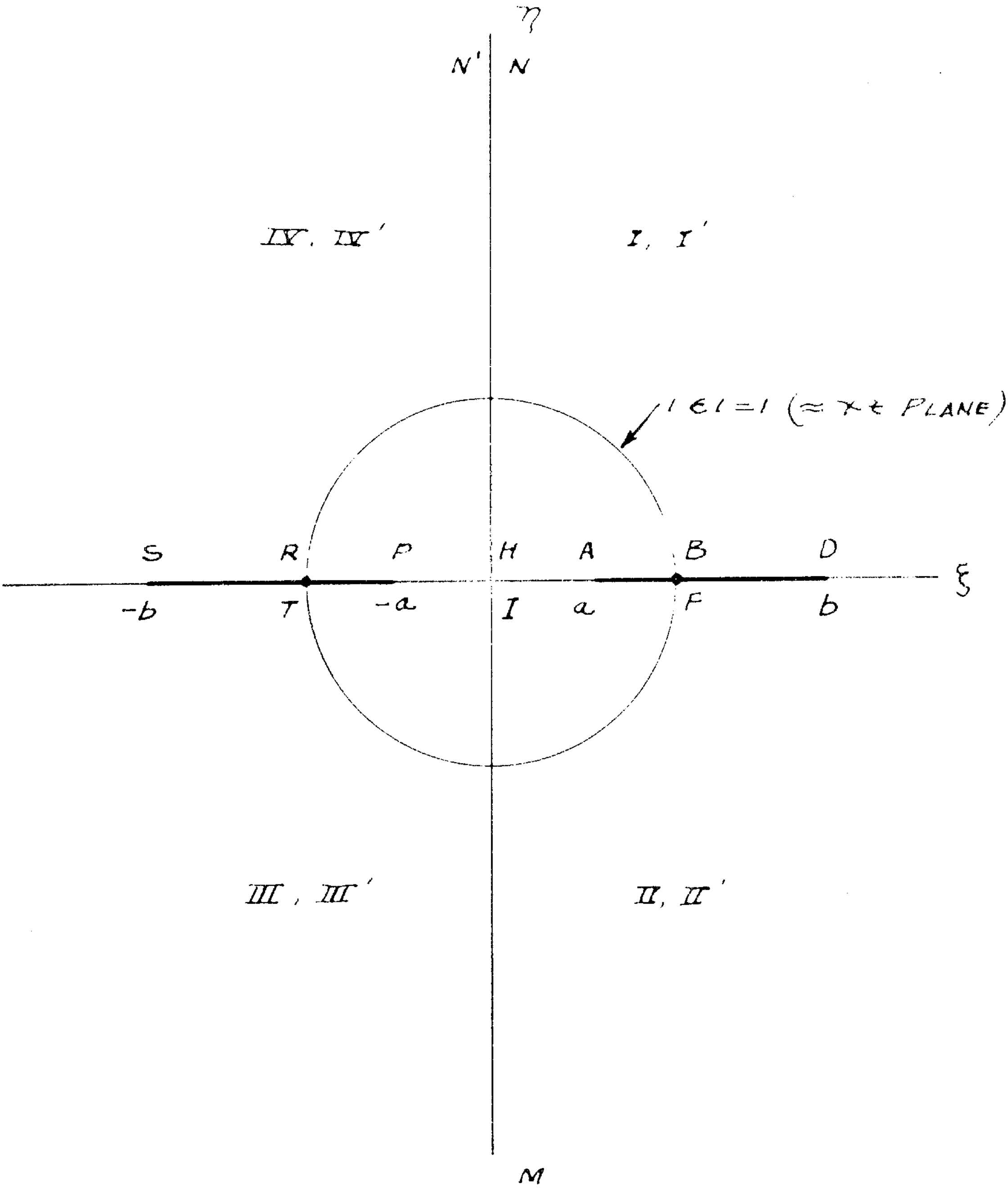

Fig. (4) ε PLANE

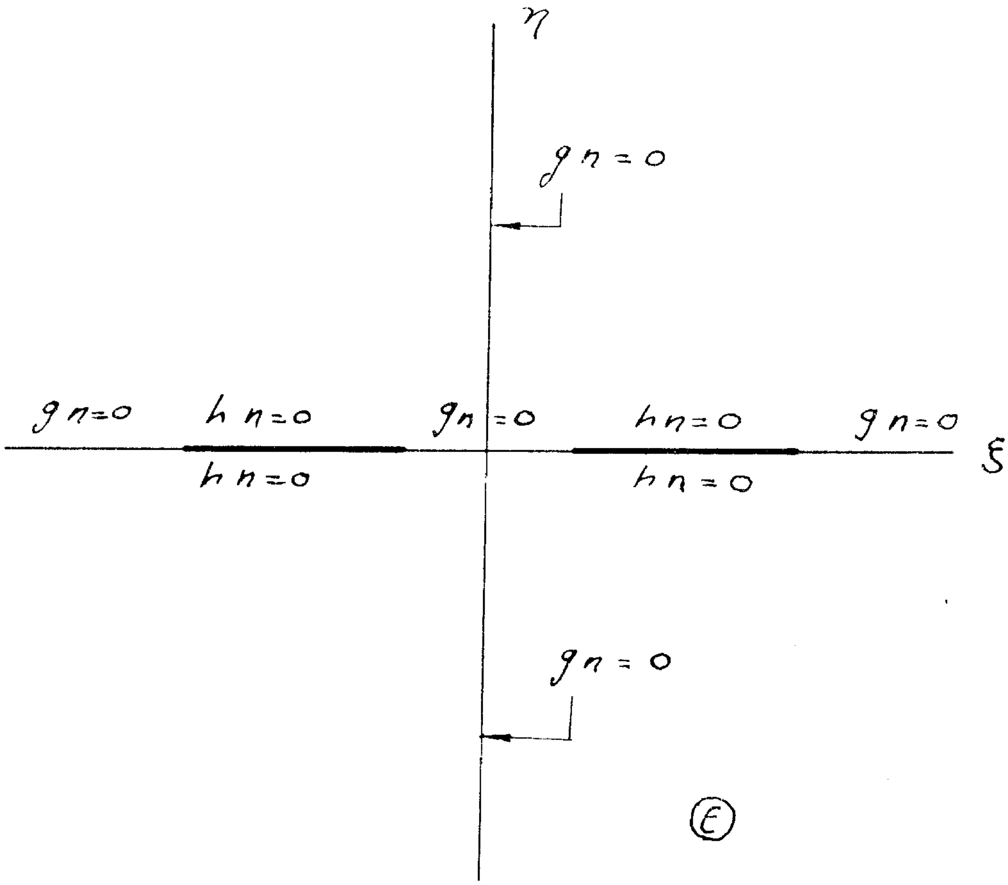

Fig. (5) BOUNDARY CONDITIONS ON $G_n(\epsilon_1) = g_n + i h_n$. LIFTING WING

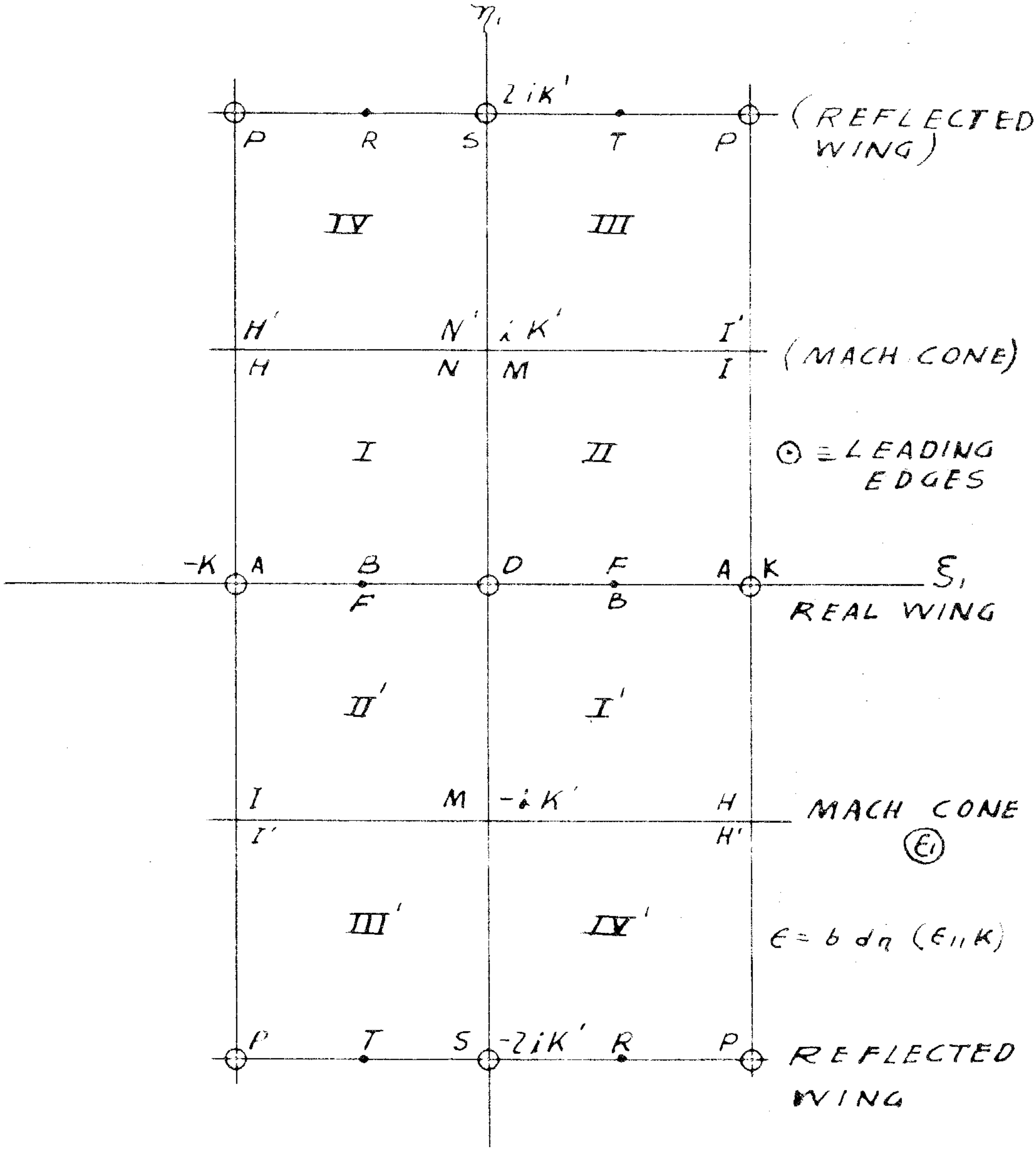

Fig: (6) ϵ_1 Plane. Basic Rectangle

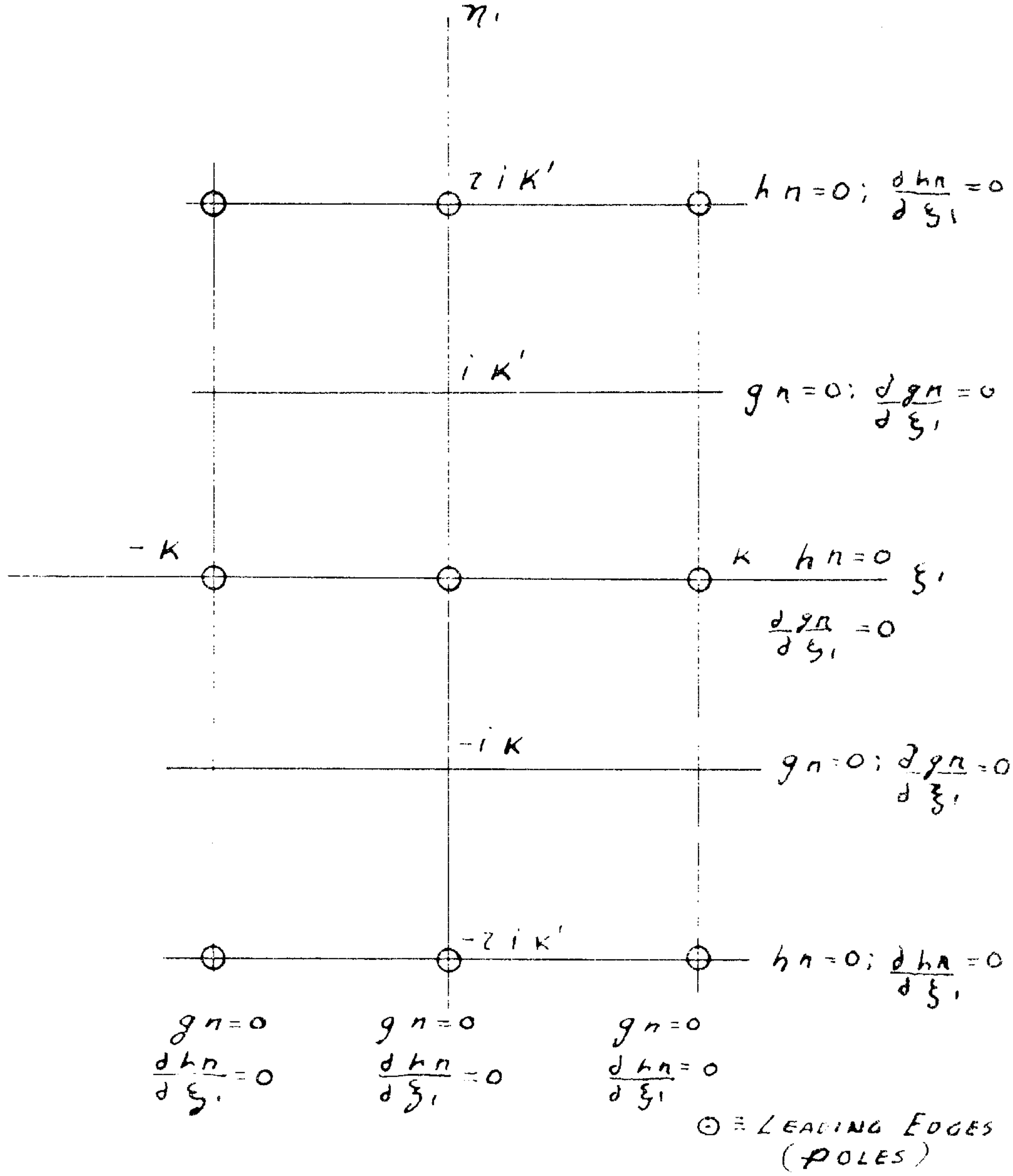

Fig. (7) BOUNDARY CONDITIONS on $G_n(\epsilon_1)$, $G'_n(\epsilon_1)$: LIFTING WING

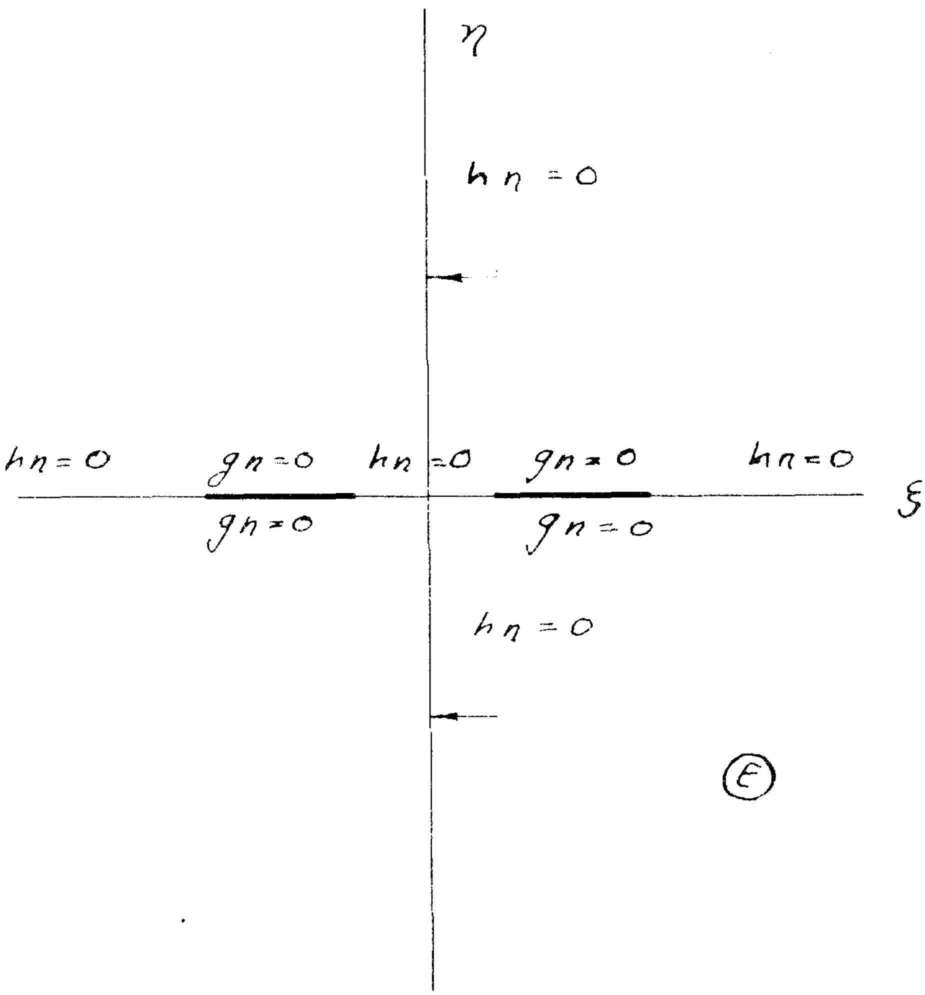

Fig. (8) BOUNDARY CONDITIONS on $G_n(E)$, SYMMETRIC NON-LIFTING WING

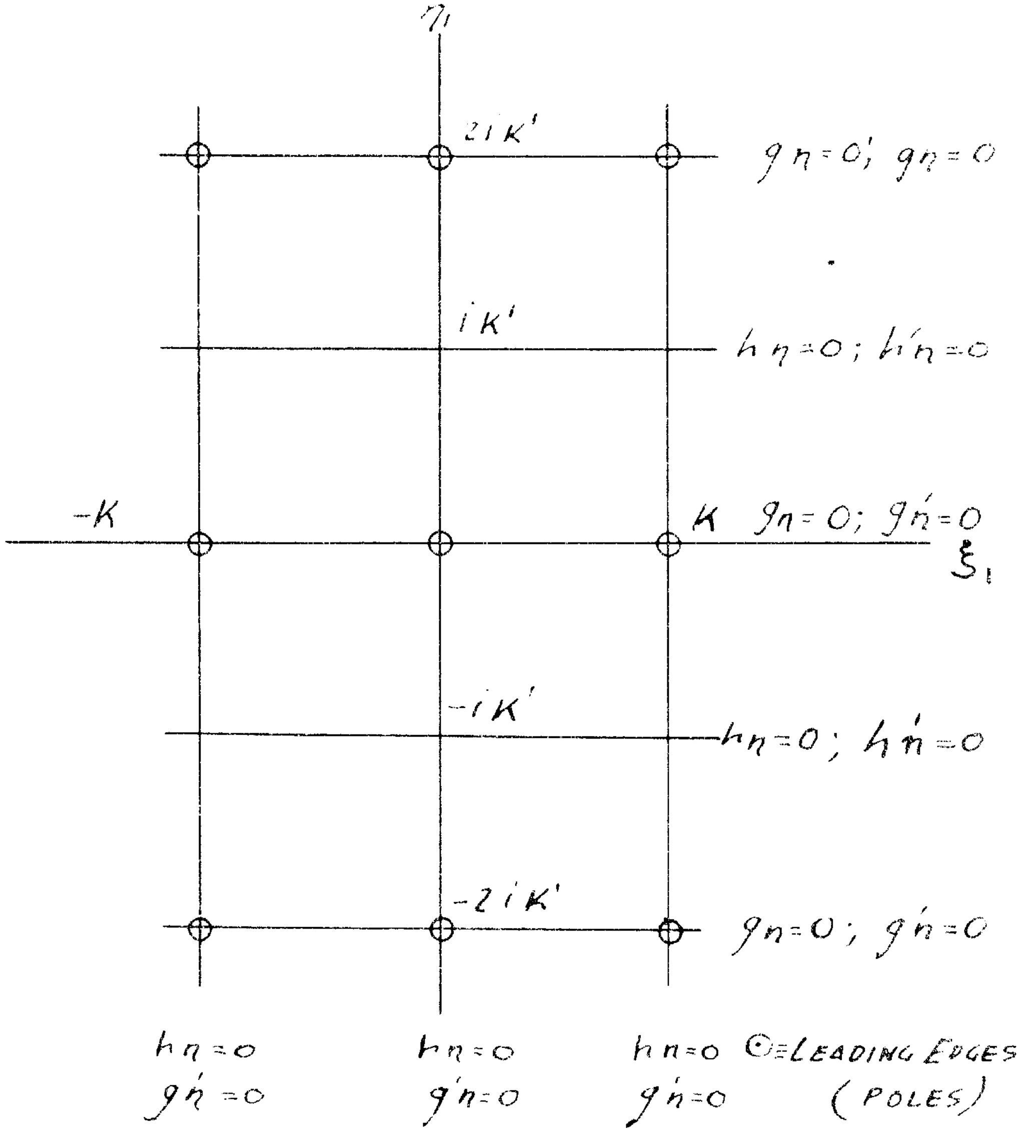

Fig. (9) BOUNDARY CONDITIONS ON $G_n(\epsilon_1)$, $G_n'(\epsilon_1)$: SYMMETRIC WING

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