SOME INVESTIGATIONS IN THE BUCKLING OF THIN RODS

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ABSTRACT

The general case of a non-uniform, straight rod under varying axial load is investigated, and several methods of solution are indicated or described. The case of a uniform rod with constant axial load is investigated by means of its deflection curve, and the direct determination of the stability with general end restraints is made possible by the use of a graph. The correlation between the end fixities of a rod and its behavior as a beam is given.

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I. Introduction

Quite an extensive body of work has been done in recent times on problems of the buckling of rods. Dinnick, Timoshenko, and others have developed the theory for rods with simple end conditions that are non-uniform or that have simply distributed axial loads (Ref. 1, 2, and 3). Mr. H. B. Howard has given a graphical method for the solution of a non-uniform rod with a constant axial load and pin-ended conditions (Ref. 4). However, with a few exceptions such as the last reference above, the discussions are limited to cases where the parameters of the problem are connected by analytic functions, and very little has been said of the general case. Cases discussed have mostly been limited to those with simple end conditions, and the author has been unable to find an explicit solution of the uniform rod compressed by a constant axial load both of whose ends are restrained elastically, the amount of restraint not being necessarily symmetrical.

It is the purpose of this paper to discuss and to present practical methods of solution for the general buckling case, and to present in a useable form the solution of the simple rod with elastically restrained end conditions.

In the analysis following, the assumptions are made that:

1. The material in the rod considered is elastic.
2. As indicated by the title, the rod is thin enough to buckle below the proportional limit of the material.
3. Conventional beam theory may be applied.
4. Deflections due to shear are negligible.
5. No side loads or moments independent of deflection need be considered. I.e., no "beam-column" action. The column is initially straight.
6. Buckling is uniplanar.
7. All end conditions are position-fixed ($y = 0$ at $x = 0$, $x = L$).
8. End conditions are either pin-ended, or free to rotate; direction-fixed, or unable to rotate; or elastically restrained. The latter means that there will be a restoring moment at the end which is proportional to the angular deflection.

II. General case with pin ends

The column to be considered lies between the points $x = 0$ and $x = L$, and has variable stiffness, axial load, and for the most general case, lies in a resisting medium. The shear considered is that perpendicular to the unbuckled column, the x axis. Following is the notation used:

$y(x)$ deflection of the rod

$P(x)$ axial load

$V(x)$ shear

$M(x)$ bending moment

$EI(x)$ stiffness

$k(x)$ constant of the resisting medium

$P_1'(x)$ distribution of tangent axial load

$P_2'(x)$ distribution of parallel axial load

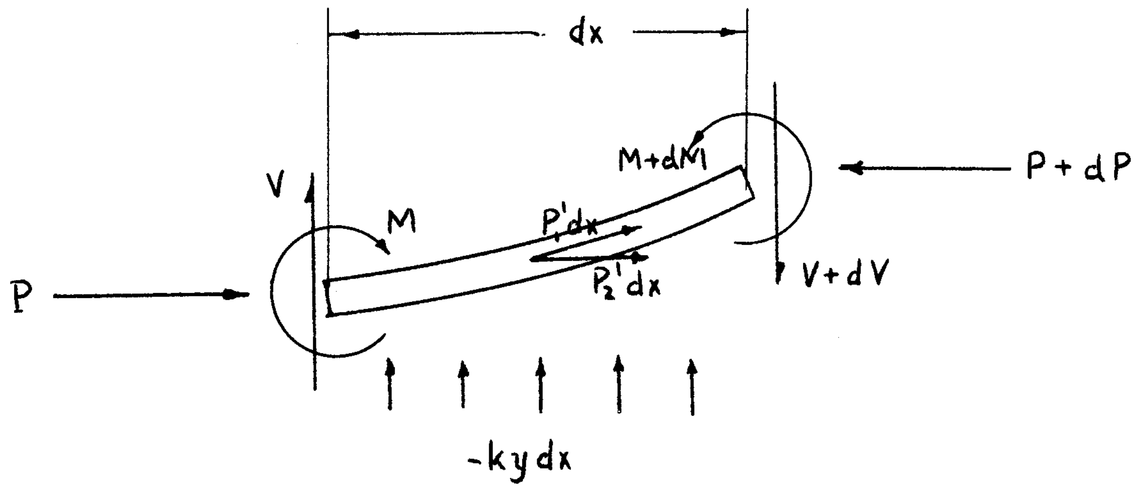
also $P_o = P(0)$

$V_o = V(0)$

$$\frac{dP}{dx} = P_1' + P_2'$$

The beam convention taken is such that positive bending moment, shear, and side loads are those that correspond to positive values of the corresponding derivatives of the deflection.

Distributed tangent axial load is distributed axial load whose direction remains tangent to the direction of the rod at the point of application. Distributed parallel axial load is distributed axial load whose direction remains parallel to the direction of the unbuckled rod, the x axis. The distinction between these types seems not to be commonly recognized, but is essential.



Equating vertical forces and moments in the above gives the equations

$$\frac{dV}{dx} = P_1' \frac{dy}{dx} - ky, \tag{1}$$

$$\frac{dM}{dx} = V - P \frac{dy}{dx}. \tag{2}$$

Differentiating (2), substituting (1), and using the relation

$$\frac{dP}{dx} = P_1' + P_2',$$

$$\frac{d^2M}{dx^2} + P \frac{d^2y}{dx^2} + P_2' \frac{dy}{dx} + ky = 0 \tag{3}$$

The conventional beam equation holds

$$M = EI \frac{d^2y}{dx^2}. \qquad (4)$$

Combining (3) and (4), dividing by EI, gives

$$\frac{d^4y}{dx^4} + 2\frac{d \log EI}{dx}\frac{d^3y}{dx^3} + \frac{P + \frac{d^2EI}{dx^2}}{EI}\frac{d^2y}{dx} + \frac{P_2'}{EI}\frac{dy}{dx} + \frac{k}{EI}y = 0 \qquad (5)$$

To find out whether a thin rod will buckle under given conditions, it is necessary to solve the equation above with the boundary conditions

$$\text{at } x = 0,\ y = 0,\ \frac{d^2y}{dx^2} = 0,\ \frac{dy}{dx} \neq 0, \text{ and}$$

$$\text{at } x = L,\ y = 0.$$

If at $x = L$, $\frac{d^2y}{dx^2} = 0$, the rod is in a critical state. If y everywhere has the same sign as the initial slope, and if $\frac{d^2y}{dx^2}$ at $x = L$ has the opposite sign to the initial slope, the rod will not buckle.

In most practical cases $k = 0$, and $P_1' = 0$, $P_2' = 0$, or $P_1' = P_2' = 0$. In any of these cases a great simplification of the equations governing the deflection will result.

Case 1. $k = P_1' = P_2' = 0$.

In this case (1) integrates to $V = V_o$, and equating moments on the rod as a whole (pin-ended case) gives $V_o = 0$. Also $P = P_o$, and

$$M = -P_o y, \tag{6}$$

which combined with (3) and (4) gives

$$\frac{d^2 M}{dx^2} + \frac{P_o}{EI} M = 0, \text{ or} \tag{7}$$

$$\frac{d^2 y}{dx^2} + \frac{P_o}{EI} y = 0 \tag{8}$$

Case 2. $k = P_2' = 0$.

In this case (3) and (4) give directly

$$\frac{d^2 M}{dx^2} + \frac{P}{EI} M = 0. \tag{9}$$

Case 3. $k = P_1' = 0$.

In this case (1) integrates to $V = V_o$, and equating moments on the rod as a whole (pin-ended case) gives the relation

$$V_o = -\frac{1}{L}\int_0^L P_2' \; y \; dx = \frac{1}{L}\int_0^L \frac{dy}{dx}\int_0^x P_2'(\xi)\, d\xi \; dx \tag{10}$$

Equations (2) and (4) give

$$\frac{d^3 y}{dx^3} + \frac{d \log EI}{dx}\frac{d^2 y}{dx^2} + \frac{P}{EI}\frac{dy}{dx} = \frac{V_o}{EI} \tag{11}$$

and differentiating (2) and substituting (2) gives

$$\frac{d M}{dx^2} - \frac{P_2'}{P}\frac{dM}{dx} + \frac{P}{EI} M = -\frac{P_2'}{P} V_o \tag{12}$$

Equations (11) and (12) are not truly non-homogeneous, due to the form of relation (10). In the analysis of a fixed-free column buckling under its own weight, not here considered since the upper end is not position fixed, $V_o = 0$, simplifying the equations.

To find out whether a rod in one of the above cases will buckle, it is necessary to solve equation (7), (9), or (12) with the boundary conditions

$$\text{at } x = 0,\ M = 0,\ \frac{dM}{dx} \neq 0.$$

If at $x = L$, $M = 0$, the rod is in a critical state. If M everywhere has the same sign as its initial derivative, the rod will not buckle.

No attempt will be made here to analyze any of the analytical cases; the reader will find interesting examples of analytic cases in Reference 1, 2, and 3. Methods of solution will be indicated for cases where the parameters are not analytically known but are, say, given by curves on graph paper.

Methods of solution of the buckling problem may be classed into several types:

1. Graphical Methods. The only practical graphical method available is based upon the polar graphical methods for beam columns and may be applied to cases 1 and 2 above. This method is given by H. B. Howard in Reference 4 and proceeds as follows: The rod is considered as a beam column with the boundary conditions $M = 0$, $\frac{dM}{dx} \neq 0$ at $x = 0$,

solved graphically considering P/EI to vary stepwise, and the stability found from the value of M at $x = L$. An excellent American exposition of the basic graphical methods for beam columns is to be found in Reference 5.

2. Energy Methods. The simplest form of an energy method, the Raleigh method, consists of assuming a form for the buckled deflection curve, and equating the column shortening energy to the total elastic energy less that for the unbuckled column. This method gives quick estimates for critical loads and is developed by Timoshenko in Reference 2. Assuming more general forms for the deflection curve with linear parameters increases the accuracy, and involves Raleigh-Ritz methods. None of these energy methods are based on the deflection equations derived above.

3. Method of Power Series. In this method the variable coefficients of the differential equation considered (5, 7, 9, 11, or 12) are first expressed in polynomial approximation by standard statistical fitting methods. A solution in power series fitting the appropriate boundary conditions is obtained (for equation 5 this involves evaluating two independent solutions at $x = L$ in order to fit the condition $y = 0$ there). Buckling of the rod will then depend upon the position of the first root of the power series obtained by dividing the series for $\frac{d^2y}{dx^2}$ or M by its first term. This series will be of the form

$$1 + a_1 x + a_2 x^2 + a_3 x^3 \; \ldots\ldots = 0, \qquad (13)$$

and will have an infinite radius of convergence (be an entire function), due to the properties of the differential equation. This latter property permits the expression of the infinite sums of the inverse n'th powers of the roots in terms of the coefficients of the series. Thus, if r_i are the roots,

$$\begin{aligned} \sum r_i^{-1} &= -a_1 \\ \sum r_i^{-2} &= a_1^2 - 2a_2 \\ \sum r_i^{-3} &= -a_1^3 + 3a_1a_2 - 3a_3 \\ &\text{etc.} \end{aligned} \tag{14}$$

Let $\sum r_i^{-n} = R_n$. For n sufficiently large, the first term will be dominant, and the first root desired may be approximated by

$$r_1 \cong (R_n)^{-\frac{1}{n}} \tag{15}$$

The proper expression for R_n is obtained by the methods of symmetric functions. This method is apt to be very long and onerous.

It should be noted that for equation (11), the solution will come out in the form (boundary conditions are $y = \frac{d^2y}{dx^2} = 0$ at $x = 0$):

$$y = y_1 + V_o y_2, \tag{16}$$

which with relation (10) gives

$$V_o = -\frac{\int_0^L P_2' y_1 dx}{L + \int_0^L P_2' y_2 dx} \tag{17}$$

which will give y explicitly. Equation (12) may be treated similarly.

4. Continuous Numerical Solutions. The equations may be treated by any of a number of standard methods for the numerical solution of differential equations.

5. Ritz Methods. In a Ritz method the variable y (or M) is assumed to be of the form

$$y = \sum_{i=1}^{n} c_i y_i(x) \qquad (18)$$

where the c_i are parameters to be determined, and the y_i are a finite number of functions which must satisfy the boundary conditions but must not interfere with the freedom of the other end conditions and are chosen according to the judgement of the person using the method. A subsidiary condition must be specified to insure $\frac{dy}{dx} \neq 0$ at $x = 0$:

$$\sum_{i=1}^{n} c_i a_i = K, \text{ any constant.} \qquad (19)$$

Letting the differential equation be expressed in the symbolic form

$$O\left[y\right] = 0, \qquad (20)$$

variational reasoning leads to the system of equations

$$\sum_{i=1}^{n} c_i \int_0^L O\left[y_i\right] y_j \, f(x)dx = \lambda \, a_j \qquad (21)$$

where $f(x)$ is a weighting function which may be unity and which is at the discretion of the analyst, and λ is a parameter whose value is not important. Equations (21) may be solved for the c_i, giving an approximation for the function y (or of course M), which may be plotted to investigate its roots. For equation (5), of course, the roots of $\frac{d^2y}{dx^2}$ are investigated.

6. Methods of Successive Approximation. These methods fall into a few subclasses, which will be considered separately.

(a). Method of integral equations. This method is applicable only to equations (7), (9), or (12). The substitution is first made (in the case of equations 7 or 9), assuming P never vanishes,

$$dx = j\,d\theta, \text{ where } \theta = 0 \text{ at } x = 0,$$

$$j^2 = \sqrt{\frac{EI}{P}}, \text{ and } \theta = \theta_0 \text{ at } x = L.$$

This substitution reduces the equation to

$$\frac{d^2M}{d\theta^2} + p(\theta)\,\frac{dM}{d\theta} + M = 0, \tag{22}$$

where

$$p(\theta) = -\frac{dj}{dx} = -\frac{d \log j}{d\theta} \tag{23}$$

The function M under the boundary conditions satisfies the integral equation

$$M(\theta) = A \sin\theta - \int_0^\theta \sin(\theta - t)p(t)M'(t)dt \tag{24}$$

This may be solved by letting

$$M(\theta) = M_o(\theta) + M_1(\theta) + M_2(\theta) + \ldots\ldots , \qquad (25)$$

where

$$M_o(\theta) = A \sin \theta$$

$$M_{n+1}(\theta) = - \int_0^\theta \sin (\theta - t) p(t) M_n'(t) dt \qquad (26)$$

$$= \int_0^\theta M_n(t) \left[p'(t) \sin(\theta - t) - p(t) \cos(\theta - t) \right] dt$$

If this solution is sufficiently rapidly convergent, it may be plotted to investigate the roots. For equation (12), equation (22) becomes

$$\frac{d^2M}{d\theta^2} + p(\theta) \frac{dM}{d\theta} + M = - q(\theta) V_o, \quad \text{where}$$

$$p(\theta) = - \frac{q(\theta)}{j} - \frac{d \log j}{d\theta}; \quad q(\theta) = \frac{EIP_2'}{P^2}, \text{ and equation (24)}$$

becomes

$$M(\theta) = A \sin \theta - \int_0^\theta \sin(\theta - t) \left[p(t) M'(t) - 2V_o q(0) \cos t + V_o q(t) \right] dt.$$

This may be solved similarly, but seems quite troublesome.

(b). Reduction of equation to first order. This applies only to equations (7) and (9). In equation (22), a solution is assumed of the form

$$M(\theta) = A(\theta) \sin \left[\theta + \delta(\theta) \right], \text{ such that}$$

$$\frac{dM}{d\theta} = A(\theta) \cos \left[\theta + \delta(\theta) \right], \text{ with the boundary} \qquad (27)$$

conditions $\delta(0) = 0,\ A(0) = M'(0)$.

This leads, from equation (22), to the conditions

$$A' \sin(\theta + \delta) + A\,\delta' \cos(\theta + \delta) = 0,$$

$$A' \cos(\theta + \delta) + pA \cos(\theta + \delta) - A\,\delta' \sin(\theta + \delta) = 0,$$

which give $$A' + pA \cos^2(x + \delta) = 0$$

$$pA \sin(x + \delta) \cos(x + \delta) - A\delta' = 0.$$

The above equations are solvable under the boundary conditions, leaving, since $A \neq 0$ from $\frac{dy}{dx} \neq 0$ at $x = 0$,

$$\frac{d\delta}{d\theta} = \frac{p(\theta)}{2} \sin 2(\theta + \delta) \qquad (28)$$

This first order non-linear differential equation may be solved by standard methods for numerical solution but is very easily solved by successive approximation, giving

$$\delta(\theta) = \lim_{n \to \infty} \delta_n(\theta), \text{ where}$$

$$\delta_o(\theta) = 0, \qquad (29)$$

$$\delta_{n+1}(\theta) = \frac{1}{2} \int_0^\theta p(t) \sin 2\left[t + \delta_n(t)\right] dt$$

The function $A(\theta)$ has no roots, so the first root of $M(\theta)$ will occur at

$$\theta + \delta(\theta) = \pi \qquad (30)$$

The substitution $z = \tan(\theta + \delta)$ changes (28) into

$$\frac{dz}{d\theta} = 1 + z^2 + p(\theta)z, \tag{31}$$

which has $z = 0$ at $\theta = 0$, can be solved to about $z = 1$, transformed to the equation for $1/z$, solved to about $1/z = -1$, transformed back. The first root of $M(\theta)$ will occur when z again reaches 0.

(c). Methods of Iteration. As these methods can be readily used to calculate critical loads, this is the application which will be described here. In any case where $k = 0$ the procedure is direct, while when $k \neq 0$ a method of including the k term must be devised. It is assumed that the proportional relations between the various axial loads is fixed, so that they may be expressed

$$\left.\begin{aligned} P &= \mu \, \underline{P_0} \\ P &= \mu \, \underline{P} \\ P_1' &= \mu \, \underline{P_1'} \\ P_2' &= \mu \, \underline{P_2'} \\ V_0 &= \mu \, \underline{V_0} \end{aligned}\right\} \tag{32}$$

where μ is a parameter which will determine the critical load. Equation (5) may be rewritten

$$\frac{d^2}{dx^2}\left(EI \frac{d^2y}{dx^2}\right) = -\mu\left(P \frac{d^2y}{dx^2} + P_2 \frac{dy}{dx} + \frac{k}{\mu} y\right), \tag{33}$$

and let

$$f_n(x) = -\left(P \frac{d^2y_n}{dx^2} + P_2 \frac{dy_n}{dx} + \frac{k}{\mu_n} y_n\right). \tag{34}$$

Then the iteration process to be used may be expressed

$$y_{n+1}(x) = \mu_{n+1}\left\{A_{n+1}\int_0^x \frac{(x-\xi)\,\xi\,d\xi}{EI(\xi)} + B_{n+1}x + \int_0^x \frac{(x-\xi)}{EI(\xi)}\int_0^\xi (\xi - t)\ f_n(t)\ dt\,d\xi\right\}, \tag{35}$$

which satisfies the condition $y_{n+1}(0) = y''_{n+1}(0) = 0$. The iteration is started with an approximation to the deflection curve $y_o(x)$, and is continued through equation (35), adjusting A_{n+1} and B_{b+1} at each step to satisfy $y_{n+1}(L) = y''_{n+1}(L) = 0$. An estimation of μ_{n+1} at each step is only necessary when $k = 0$, as μ_n only then occurs in (34).

In most physical cases there will be a least value of μ for which the boundary conditions may be satisfied, and the iteration process will converge, assuming there is no degeneracy. This method will yield any desired accuracy, and the Raleigh method (Energy Methods) will give a very accurate value for μ from a reasonably accurate deflection curve from the iteration.

Equations (7), (8), and (9) use simpler iterations than that above. Equation (11) may be expressed

$$\frac{d}{dy}\left(EI\,\frac{d^2y}{dx^2}\right) = \mu\left(\underline{V_o} - \underline{P}\,\frac{dy}{dx}\right), \tag{36}$$

whence letting

$$\underline{V_{o_n}} = -\frac{1}{L}\int_0^L \underline{P_2'}\ y_n\,dx, \tag{37}$$

the iteration is given by

$$y_{n+1} = \mu_{n+1} \left\{ A_{n+1}x + \int_0^x \frac{(x - \xi)}{EI(\xi)} \int_0^\xi (\underline{V_{o_n}} - \underline{P} \frac{dy_n}{dt}) dt d\xi \right\} \qquad (38)$$

where A_{n+1} must be chosen such that $y_{n+1}(L) = 0$. Equation (12) may be treated similarly, but M_n must be integrated in order to find $\underline{V_{o_n}}$, and the use of equation (11) is much less troublesome.

Discussion of the Methods. Methods 2 and 6c are the only methods which yield critical loads directly. The other methods only determine whether a rod is stable or unstable.

Methods 2 and 5 do not lead to exact calculations of the problem, the other methods all giving increasing accuracy with increase of labor while in methods 2 and 5 larger numbers of assumed functions lead to impractical algebraic problems.

Most of the methods are applications of standard mathematical methods to the particular problem, excepting methods 1 and 6b. Method 1 was independently discovered by the author, who later found himself preceded by Mr. Howard. Method 6b is due to the author alone.

Methods 1, 6a, and 6b may be applied only to the simplified cases, the other methods being generally applicable.

In practical computations, if stability of one of the simple cases is to be investigated, methods 1 or 6b can be applied very rapidly. For any case, method 2 is very rapid in its simplest form, and if close accuracy is required, method 6c supplemented by 2 will probably be best.

The effect of discontinuities in the various parameters and the effect of discrete axial loads and side restraining forces applied in the span of the rod must be considered. Method 1 implies in the practical plotting the approximation of a discontinuous equivalent rod, and is thus unaffected by discontinuities. Method 3 will not work, while the other methods will, it being necessary to consider integration by parts of the various integrals across the discontinuities in methods 5 and 6, and to stop and restart the solution of method 4 with new boundary conditions at a discontinuity.

III. Uniform rod with elastic end conditions.

The case now to be considered is that of a uniform rod with no forces or moments on it except at the ends, with the two ends position fixed and elastically restrained. The deflection curve is set up in coordinates that simplify the geometry without original reference to the location of the ends of the rod. The following is the notation introduced in this section:

c	column fixity
L_o	half wave length of deflection curve
δ_o	amplitude of deflection curve
δ	deflection of rod at center
ϕ	angle of unbuckled rod with x axis

The following appear with the subscript $_1$ or $_2$, referring to the left or right hand end of the rod, respectively:

θ	angle of rod at end with unbuckled position
L	distance of end from origin
y	ordinate at end
M	moment at end
K	coefficient of end restraint = M/θ

Further, the quantities are defined:

$$\alpha = \sqrt{c} = \frac{L}{L_o} = \frac{L_1 + L_2}{L_o}, \text{ and}$$

$$\epsilon = \frac{L_1 - L_2}{L_o}, \ \theta = \theta_1 + \theta_2$$

The general solution to equation (7) with constant stiffness is

$$y = A + Bx + C \cos \frac{x}{j} + D \sin \frac{x}{j}. \qquad (39)$$

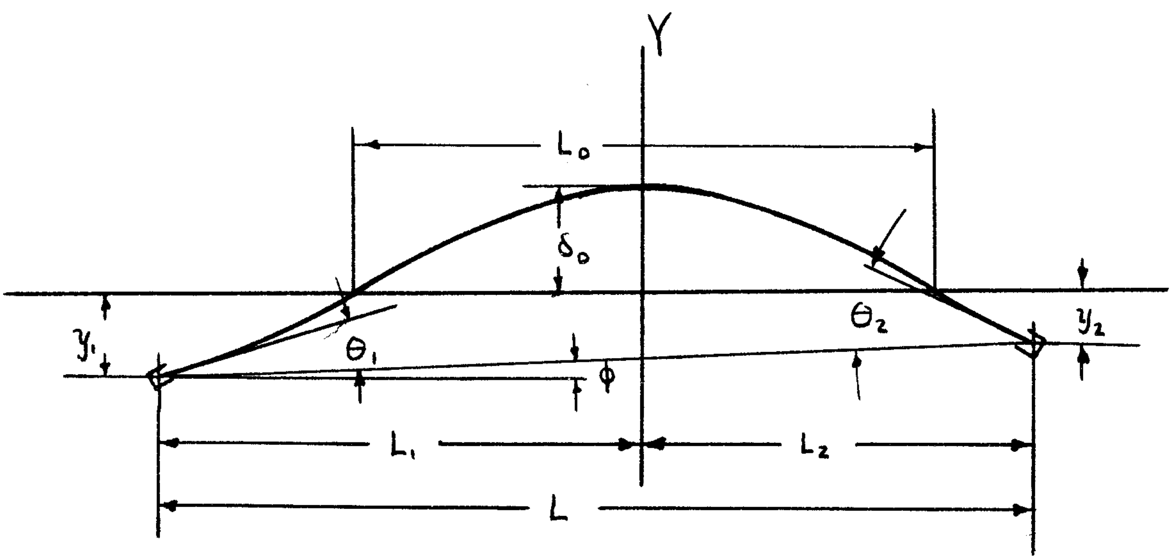
Translating and rotating any solution will bring it to the form

$$y = \delta_o \cos \frac{x\pi}{L_o}, \text{ where} \qquad (40)$$

$$\frac{\pi}{L_o} = \frac{1}{j}, \quad \text{and} \quad P = \frac{\pi^2 EI}{L_o^2},$$

which with the definition for $\sqrt{c}$ above gives

$$P = \frac{c\pi^2 EI}{L^2} \qquad (41)$$



$$y = \delta_o \cos \frac{x\pi}{L_o} \tag{40}$$

$$\frac{dy}{dx} = -\frac{\delta_o \pi}{L_o} \sin \frac{x\pi}{L_o} \tag{42}$$

$$\frac{d^2y}{dx^2} = -\frac{\delta_o \pi^2}{L_o^2} \cos \frac{x\pi}{L_o} \tag{43}$$

$$\left.\begin{aligned} \theta_1 &= \left(\frac{dy}{dx}\right)_1 - \phi \\ \theta_2 &= \phi - \left(\frac{dy}{dx}\right)_2 \end{aligned}\right\} \tag{44}$$

$$\theta = \frac{\delta_o \pi}{L_o} \left(\sin \frac{L_1 \pi}{L_o} + \sin \frac{L_2 \pi}{L_o}\right)$$

$$\theta = \frac{2\pi\delta_o}{L_o} \sin \frac{\alpha\pi}{2} \cos \frac{\epsilon\pi}{2} \tag{45}$$

$$\delta = y_c - \frac{y_1 + y_2}{2}, \quad y_c = y\left(\frac{L_2 - L_1}{2}\right).$$

$$= \delta_o \cos \frac{(L_1 - L_2)\pi}{2L_o} - \frac{\delta_o}{2}\left(\cos \frac{L_1\pi}{2} + \cos \frac{L_2\pi}{2}\right)$$

$$= \delta_o \cos \frac{\epsilon\pi}{2} \left(1 - \cos \frac{\alpha\pi}{2}\right) \tag{46}$$

Combining (46) and (47), there results

$$\frac{L\theta}{\delta} = \frac{2\pi\alpha \sin \alpha\pi/2}{1 - \cos \alpha\pi/2}, \quad \text{or}$$

$$\frac{L\theta}{o} = 2\pi\sqrt{c} \quad \cot \sqrt{c}\, \frac{\pi}{4}. \tag{47}$$

The significant point about expression (47) is that the quantity expressed is independent of the quantity ϵ, which is a measure of the asymmetry of the rod. The relation will hold for any bent compressed rod irrespective of the stability if c is considered the ratio of actual to the Euler load for a pin ended rod. Equation (47) is plotted in Figure 1.

$$\phi = \frac{y_2 - y_1}{L} = \frac{\delta_o}{L}\left(\cos\frac{L_2\pi}{L} - \cos\frac{L_1\pi}{L}\right)$$

$$\phi = \frac{2\delta_o}{L}\sin\frac{\alpha\pi}{2}\sin\frac{\epsilon\pi}{2} \qquad (48)$$

From equations (44),

$$\left.\begin{aligned} \theta_1 &= \frac{\delta_o}{L}\left[\pi\alpha\sin\frac{L_1\pi}{L_o} - 2\sin\frac{\alpha\pi}{2}\sin\frac{\epsilon\pi}{2}\right] \\ \theta_2 &= \frac{\delta_o}{L}\left[\pi\alpha\sin\frac{L_2\pi}{L_o} + 2\sin\frac{\alpha\pi}{2}\sin\frac{\epsilon\pi}{2}\right] \end{aligned}\right\} \qquad (49)$$

From equation (4),

$$\left.\begin{aligned} M_1 &= -\frac{EI\pi^2\delta_o}{L_o^2}\cos\frac{L_1\pi}{L_o} \\ M_2 &= -\frac{EI\pi^2\delta_o}{L_o^2}\cos\frac{L_2\pi}{L_o} \end{aligned}\right\} \qquad (50)$$

From $K = M/\theta$,

$$\left.\begin{aligned} \frac{K_1L}{EI} &= \frac{-\pi^2\alpha^2\cos\frac{L_1\pi}{L_o}}{\pi\alpha\sin\frac{L_1\pi}{L_o} - 2\sin\alpha\frac{\pi}{2}\sin\frac{\epsilon\pi}{2}} \\ \frac{K_2L}{EI} &\quad \frac{-\pi^2\alpha^2\cos\frac{L_2\pi}{L_o}}{\pi\alpha\sin\frac{L_2\pi}{L_o} + 2\sin\alpha\frac{\pi}{2}\sin\frac{\epsilon\pi}{2}} \end{aligned}\right\} \qquad (51)$$

Expressions (51) may be expressed

$$\left.\begin{aligned}
\frac{K_1L}{EI} &= \frac{-\pi^2 c\left(\cos\frac{\sqrt{c}\,\pi}{2}\cos\frac{\epsilon\pi}{2} - \sin\frac{\sqrt{c}\,\pi}{2}\sin\frac{\epsilon\pi}{2}\right)}{\pi\sqrt{c}\left(\sin\frac{\sqrt{c}\,\pi}{2}\cos\frac{\epsilon\pi}{2} + \cos\frac{\sqrt{c}\,\pi}{2}\sin\frac{\epsilon\pi}{2}\right) - 2\sin\frac{\sqrt{c}\,\pi}{2}\sin\frac{\epsilon\pi}{2}} \\
\frac{K_2L}{EI} &= \frac{-\pi^2 c\left(\cos\frac{\sqrt{c}\,\pi}{2}\cos\frac{\epsilon\pi}{2} + \sin\frac{\sqrt{c}\,\pi}{2}\sin\frac{\epsilon\pi}{2}\right)}{\pi\sqrt{c}\left(\sin\frac{\sqrt{c}\,\pi}{2}\cos\frac{\epsilon\pi}{2} - \cos\frac{\sqrt{c}\,\pi}{2}\sin\frac{\epsilon\pi}{2}\right) + 2\sin\frac{\sqrt{c}\,\pi}{2}\sin\frac{\epsilon\pi}{2}}
\end{aligned}\right\} \quad (52)$$

or

$$\left.\begin{aligned}
\frac{K_1L}{EI} &= \frac{\pi^2 c\,(\cos\sqrt{c}\,\pi + \cos\epsilon\pi)}{(1-\cos\sqrt{c}\,\pi)(1-\cos\epsilon\pi) - \pi\sqrt{c}\,(\sin\sqrt{c}\,\pi + \sin\epsilon\pi) + \sin\sqrt{c}\,\pi\sin\epsilon\pi} \\
\frac{K_2L}{EI} &= \frac{\pi^2 c\,(\cos\sqrt{c}\,\pi + \cos\epsilon\pi)}{(1-\cos\sqrt{c}\,\pi)(1-\cos\epsilon\pi) - \pi\sqrt{c}\,(\sin\sqrt{c}\,\pi - \sin\epsilon\pi) - \sin\sqrt{c}\,\pi\sin\epsilon\pi}
\end{aligned}\right\} \quad (53)$$

Equations (52) or (53) may be used for computation. Since ϵ is only a measure of the asymmetry, it is desired to eliminate ϵ to obtain c in terms of the end restraint coefficients. This may be done by a graph, and such a graph is presented in Figure 2.

If $K_2 = 0$, the above reduce to

$$\frac{K_1L}{EI} = \frac{\pi^2 c}{\pi\sqrt{c}\,\cot\sqrt{c}\,\pi - 1} \qquad (54)$$

If $K_2 = K_1$,

$$\frac{K_1L}{EI} = -\pi\sqrt{c}\,\cot\frac{\sqrt{c}\,\pi}{2} \qquad (55)$$

If $K_2 = \infty$,

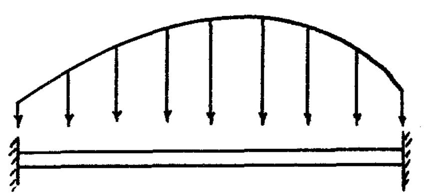
$$\frac{K_1 L}{EI} = \frac{1 - \pi\sqrt{c}\cot\sqrt{c}\,\pi}{1 - \frac{2}{\pi\sqrt{c}}\tan\frac{\sqrt{c}\,\pi}{2}} \tag{56}$$

The equations (54), (55), and (56) above are plotted on Figure 3, and correspond, respectively, to the intercepts, to values on the diagonal, and to the asymptotes of the curves on Figure 2. They are the formulas for a compressed rod which is, respectively, pin-ended at one end, equally restrained at both ends, and rigidly fixed at one end.

The corresponding equations for rods with negative axial loads, or with compression, may be obtained immediately from the above by the substitution $\sqrt{c} = i\sqrt{-c}$, where $-c$ is the ratio of the tensile axial load to the Euler compressive load and i is the imaginary unit. The resulting equations have hyperbolic functions in the place of trigonometric, and no imaginary terms are left. It should be remembered that the coefficients of end restraint may be considered negative.

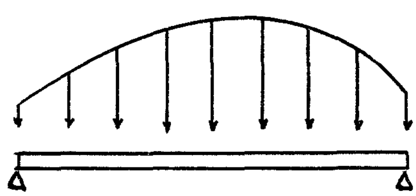
With the information on Figure 2 (expanded to cover a more extensive range) critical loads for continuous beam columns or systems of rigidly jointed columns may be very easily investigated. A loading on the system is assumed, the c's for all the units are calculated on the basis that all are critical. Then considering the units in the same order as they would be considered in a truss analysis for a system of members, the necessary K's for criticality are calculated starting at the ends. At a joint, the K given to the last unit considered is minus the sum of the K's calculated there for the other units. For a simple continuous column the procedure is obvious. The system will then be stable or unstable depending upon the stability of the last member considered. The author believes this method will be very much simpler than the three moment equation method or Lundquist's method described in Reference 5.

The coefficients of end restraint may be directly correlated to the end moments and slopes on the rod when it is loaded as a beam without axial load under any beam loading condition, with the given end restraints. Let the end moments on the beam with the ends rigidly fixed by $\underline{M_1}$ and $\underline{M_2}$, the end slopes on the beam with the ends pin-ended be $\underline{\theta_1}$ and $\underline{\theta_2}$, and the end moments and slopes on the beam with the ends elastically restrained by M_1, M_2, θ_1, and θ_2.



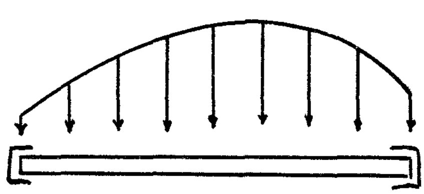
$M_1 = \underline{M_1}$ $\qquad \theta_1 = 0$

$M_2 = \underline{M_2}$ $\qquad \theta_2 = 0$



$M_1 = 0$ $\qquad \theta_1 = \underline{\theta_1}$

$M_2 = 0$ $\qquad \theta_2 = \underline{\theta_2}$



$M_1 = n_1 \underline{M_1}$ $\qquad \theta_1 = (1 - p_1)\,\underline{\theta_1}$

$M_2 = n_2 \underline{M_2}$ $\qquad \theta_2 = (1 - p_2)\,\underline{\theta_2}$

The quantities n_1, n_2, p_1, and p_2 are defined above. The quantities q and r are defined

$$q = M_1/M_2 \qquad\qquad r = \theta_1/\theta_2 \tag{57}$$

Further let $k_1 = \frac{K_1 L}{EI}$, $k_2 = \frac{K_2 L}{EI}$. Then the following relationships will hold, all of which come easily from conventional beam theory and will not be derived:

$$\left.\begin{aligned} q &= \frac{2r - 1}{2 - r} \\ r &= \frac{2q + 1}{2 + q} \end{aligned}\right\} \tag{58}$$

$$\left.\begin{aligned} p_1 &= \frac{2qn_1 + n_2}{2q + 1} = \frac{k_1(k_2 + 4) + \frac{2}{r} k_2}{(k_1 + 4)(k_2 + 4) - 4} \\ p_2 &= \frac{2n_2 + qn_1}{2 + q} = \frac{k_2(k_1 + 4) + 2rk_1}{(k_1 + 4)(k_2 + 4) - 4} \end{aligned}\right\} \tag{59}$$

$$\left.\begin{aligned} n_1 &= \frac{2rp_1 - p_2}{2r - 1} = \frac{3k_1(k_2 + 4 + \frac{2}{q})}{4(k_1 + 3)(k_2 + 3) - k_1 k_2} \\ n_2 &= \frac{2p_2 - rp_1}{2 - r} = \frac{3k_2(k_1 + 4 + 2q)}{4(k_1 + 3)(k_2 + 3) - k_1 k_2} \end{aligned}\right\} \tag{60}$$

$$\left.\begin{aligned} k_1 &= \frac{3n_2}{(1 - n_1) + \frac{1}{2q}(1 - n_2)} = 4\,\frac{p_1 - \frac{1}{2r} p_2}{1 - p_1} \\ k_2 &= \frac{3n_2}{(1 - n_2) + \frac{q}{2}(1 - n_1)} = 4\,\frac{p_2 - \frac{r}{2} p_1}{1 - p_2} \end{aligned}\right\} \tag{61}$$

These equations may be used in the experimental determination of coefficients of end restraint of a rod by loading it as a beam and measuring either the end slopes or end moments.

IV. Elastic end conditions in the general case

Where no simplification may be made, equation (5) holds as before, but must be investigated under more complex boundary conditions:

$$\text{at } x = 0,\ y = 0,\ EI\,\frac{d^2y}{dx^2} = K_1\,\frac{dy}{dx} \neq 0, \text{ and}$$

$$\text{at } x = L,\ y = 0.$$

The stability depends upon the relative values of $EI\,\frac{d^2y}{dx^2}$ and $-K_2\,\frac{dy}{dx}$ at $y = L$. The condition $EI\,\frac{d^2y}{dx^2} = -K_2\,\frac{dy}{dx}$ at $x = L$ may be substituted for the condition $y = 0$ there, and the stability investigated with reference to the roots of y. Methods 2, 3, 4, 5, and 6c may be applied.

For the simplified cases, equations (7), (9), (11) and (12) will remain unchanged. Equation (6) will become:

Case 1.

$$M = V_ox - P_oy + M_1, \text{ where} \tag{62}$$

$$V_o = \frac{M_2 - M_1}{L} \tag{63}$$

$$M_1 = K_1\left(\frac{dy}{dx}\right)_1, \quad M_2 = -K_2\left(\frac{dy}{dx}\right)_2 \tag{64}$$

Equation (8) becomes

$$\frac{d^2y}{dx^2} + \frac{P_o}{EI}\,y = \frac{V_ox}{EI} + \frac{M_1}{EI} \tag{65}$$

Relation (10) is changed, with (64) holding, to

$$V_o = -\frac{1}{L}\left\{ M_1 - M_2 + \int_0^L P_2' \, y \, dx \right\}. \qquad (66)$$

Of methods 1, 6a, and 6b which applied to the simplified case, only method 6a may be applied with end restraint present. The method is the same except that a term B cos θ is included in the integral equation and in the expression for $M_o(\theta)$. The two independent solutions arising must be integrated to get dy/dx and y to be fit to the more general boundary equations described above.

Generally, all the comments relative to the various methods of solution made above in Section II hold for the case with elastically restrained ends.

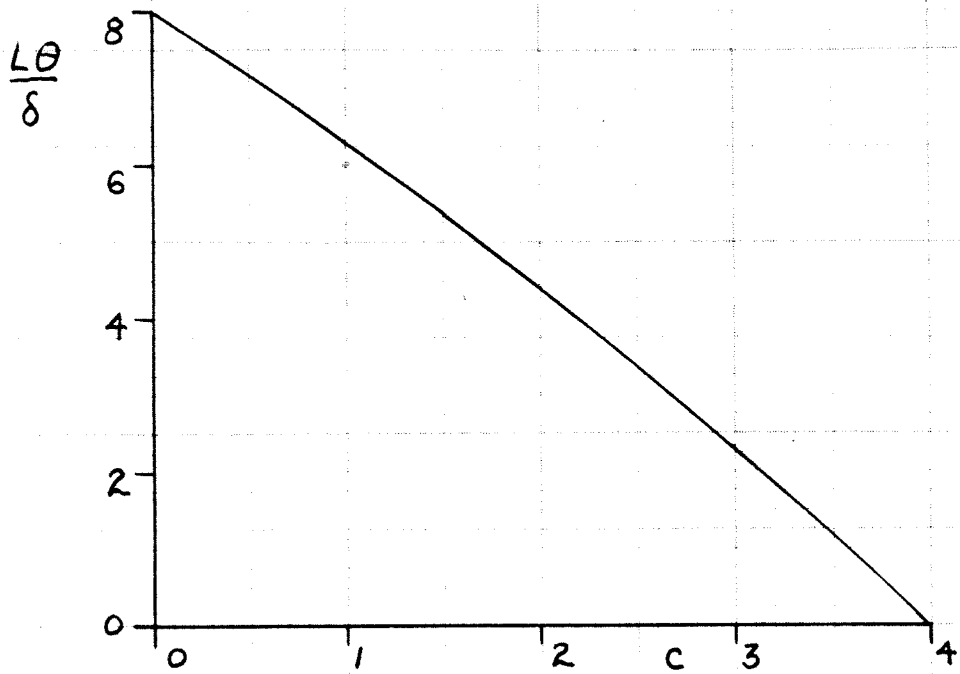


FIG. 1

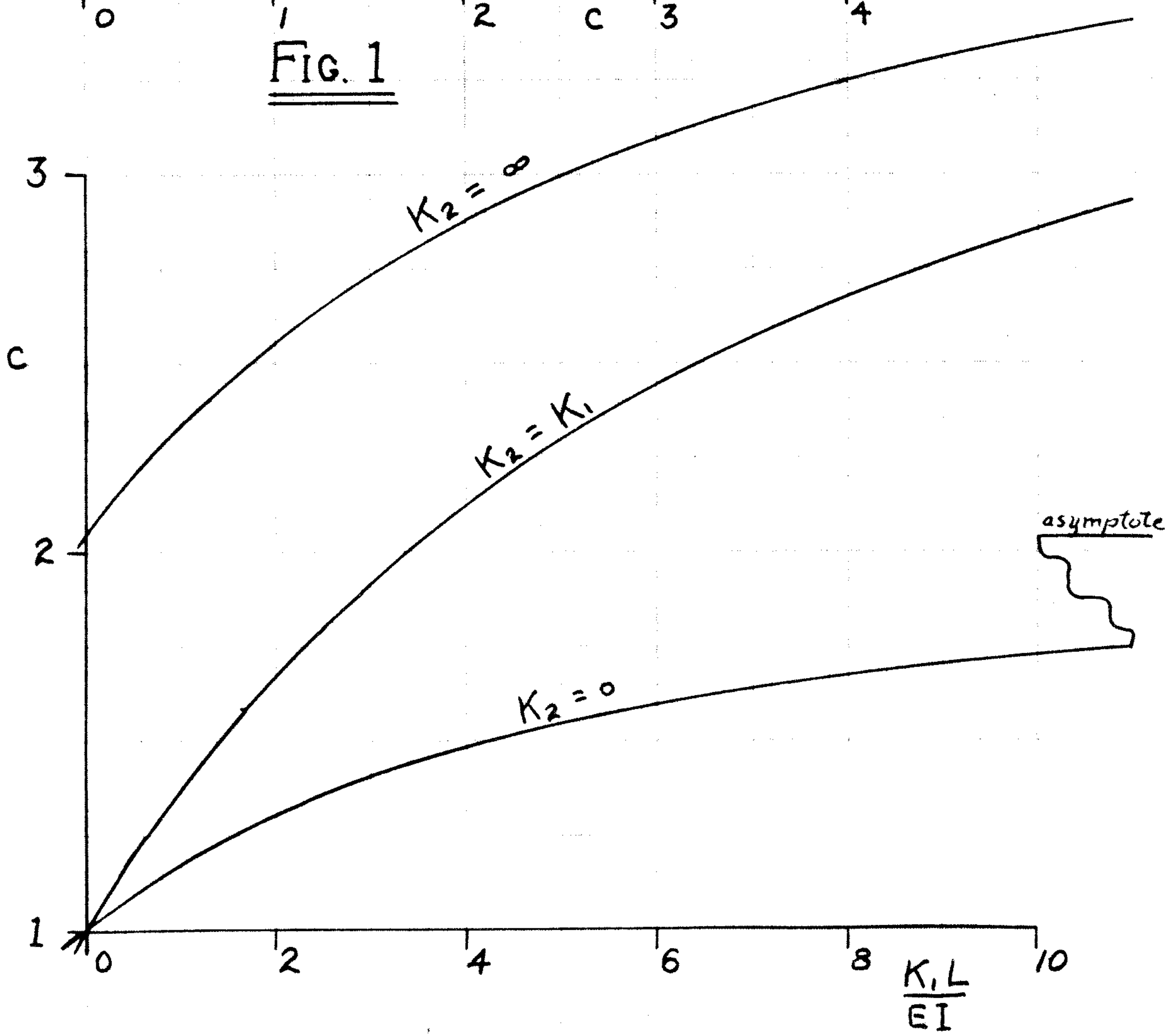
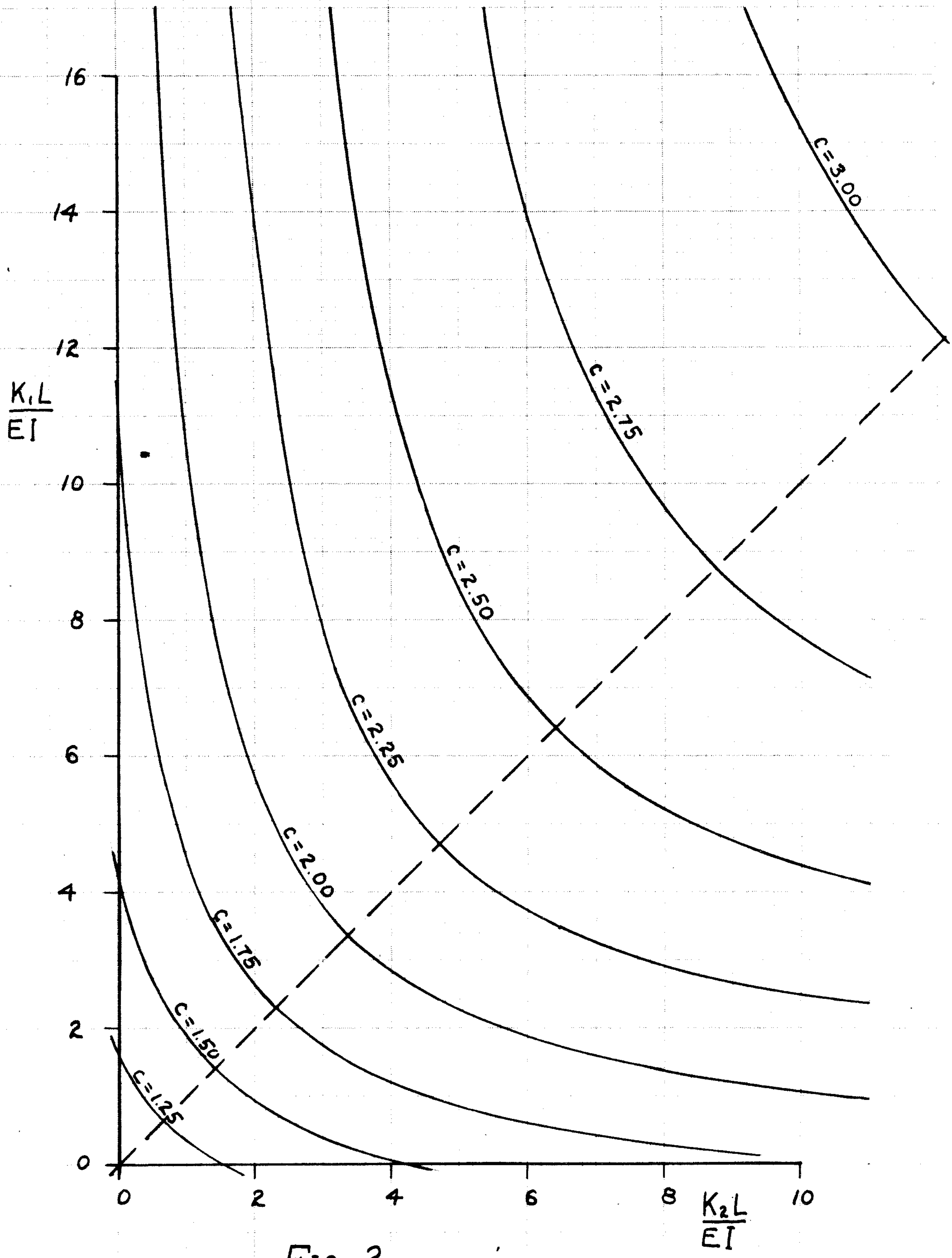


FIG. 3



16
14
12
10
8
6
4
2
0
$\frac{K_1 L}{EI}$
0
2
4
6
8
10
$\frac{K_2 L}{EI}$
c = 1.25
c = 1.50
c = 1.75
c = 2.00
c = 2.25
c = 2.50
c = 2.75
c = 3.00

FIG. 2