# Chapter 5

# Conclusion

In this thesis, we have explored two possible techniques for developing simple models for the dynamics of quantum systems: Proper Orthogonal Decomposition (POD) and Local Tangent Space Alignment (LTSA). Through the lens of two particular regimes in cavity QED, we have evaluated these techniques and explored their advantages and disadvantages for quantum dynamical systems in general.

The main advantage of POD is that the subspace produced by the process is linear, which allows us to easily project the full master equation onto the space and develop filtering equations for the density matrix as a whole. However, this same linearity proves to limit the reduced models, as they are unable to capture the large deviations from the mean that characterize the nonlinear systems. Local Tangent Space Alignment produces a point-wise constructed nonlinear manifold, which can capture the nonlinear behavior. I was able to discover low-dimensional manifolds which capture the essential structure of the system dynamics. However, the lack of a functional form for the map between the manifold space and the density matrix space complicates the search for a reduced filter. In the case of phase bistability, I was able to fit the manifold coordinates by a linear combination of (linear and nonlinear) system observables, in order to derive a system-specific set of "Maxwell-Bloch" equations, which make a very good filter. This sort of fit, however, depends on having a good basis set of functions of observables with which to fit the data, which proved to be missing in the case of absorptive bistability. A structured way to create a much larger basis set might lead to success here, and in models of other quantum systems.

The model reduction techniques discussed here may also be applied in a wide range of systems, including other regimes from cavity QED as well as other nonequilibrium systems. The semi-classical Maxwell-Bloch equations display a range of behaviors in addition to the phase and absorptive bistabilities discussed in this thesis. As discussed by Armen and Mabuchi [22], they also have limit cycles related to Hopf bifurcations. POD and LTSA could be applied to the quantum trajectories which correspond to this semi-classical situation, and produce useful reduced models. The mean (and steady) state of the limit cycle takes the shape of a ring. The POD eigenstates are then a progression of Fourier decompositions on the ring: First, a state whose Q function is a single positive zone and a single negative zone across from it. Then, a similar state whose features are rotated 90 degrees relative to the first state. Following this is a pair or states with two positive and two negative regions, rotated 45 degrees with respect to each other, then a pair with three positive and three negative, rotated 30 degrees, etc. Appropriate addition and subtraction of these sine- and cosine-like states can build a localized state, provided the basis is large enough. The projected filter on this basis, when driven by a white noise innovation, reproduces oscillation around the limit cycle. LTSA applied to the limit cycle produces a ring-shaped manifold, which requires at least two dimensions to contain. LTSA does not handle self-intersecting manifolds well, and it fundamentally cannot recreate a one-dimensional ring with $d = 1$.

The cavity QED system is only one of a large class of non-equilibrium systems where complex topology meets dynamics; other examples include chemical reaction dynamics and protein folding. Provided simulations (or experimental data) with sufficient resolution and coverage of the full space explored by these dynamics, POD or LTSA could draw out features of the underlying dynamical space. Nonlinear manifold learning algorithms such as LTSA are particularly exciting in this context, as these problems are generally interesting precisely because of their nonlinear features. (Understanding protein folding would be a very different problem if linear analysis were sufficient.) If we could construct a global functional form for the maps between the LTSA-generated submanifold and the large original data-space, projecting the dynamical equations to build simple filtering equations might provide significant insight into the dominant and critical features of these dynamics.

Future work to extend POD or LTSA (or adapt other algorithms) to build better filters for quantum systems might benefit by taking advantage of additional information present in the trajectory. Both POD and LTSA use the quantum trajectory as an unordered data set (although the zoned POD algorithm developed here is, in effect, time dependent). Fundamentally, however, we might expect that an algorithm which used the time ordering of the data would have an advantage. For example, a manifold learning algorithm might ensure that points which were nearby in time are also nearby on the manifold (provided the sampling is fine enough). This might also require a different definition of distance, which the failures of POD imply might be necessary in any event.

We have evaluated the performance of these model reduction techniques in part based on their ability to provide a simple filtering equation, which could be used in a control context. However, a controlled system might behave quite differently from the autonomous systems we have observed to build reduced models. In particular, I have not studied the input characteristics of the cavity QED or other quantum models, only their outputs. There is an extensive literature on balanced truncation [43] and other balancing transformations, such as [31], which make use of the input-output characteristics of the system, rather than (or in addition to) its internal structure and dynamics, to build reduced models.

Filtering equations are useful outside of a control context as well, because they allow us to simulate the dynamics of a system by driving it with white noise as the innovation. An exact filter, such as the full master equation, allows us to simulate the full quantum system; approximate filters give us insight into the system by giving us a simpler model which shows the same behaviors. If the approximate model has captured the essence of the exact system, we should be able to drive it with a white noise innovation process and see the approximate system reproducing the behavior of the exact one. We can then analyze the simple model for insight into the more complicated exact system. For example, the success of the "Maxwell-Bloch" equations derived in Section 4.3.1 indicates that these three (or five) equations contain the same kinds of interactions between nonlinearity and noise which also determine the dynamics of a quantum system in the phase bistable regime.

Filters produced by projecting the exact filter onto a lower-dimensional manifold (linear or nonlinear) may bring with them multiplicative noise terms, particularly if those terms are also present in the exact dynamics. Dynamical systems with multiplicative noise have two different kinds of special points, if they exist: points where the deterministic dynamics are stable, and points where the noise terms vanish (or reach a minimum in amplitude). Recall that van Handel and Mabuchi [11] showed that the phase bistable system reduces to the Wonham filter [44] in one limit. The stochastic terms of the Wonham filter vanish at the two points between which the system is observed to switch, while the deterministic terms have a stable point at the middle, where the system spends almost no time. Quantum bistable systems with multiplicative noise, and their high quality filters, may show similar behavior. By constructing multiplicative noise terms in approximate filters directly from the exact dynamics, the projected filter can correctly reproduce switching behavior which *ad hoc* or additive noise models cannot.

A complete understanding of the dynamics of open quantum systems will require understanding a variety of regimes and behaviors. The utility of open quantum systems depends in large part on the dynamics, such as switching, limit cycles, or bifurcation, which occur in nonlinear models. We have chosen to focus on these features of nonlinear systems, rather than chaotic behavior or other hallmarks of nonlinearity, because of their connection to the engineering of useful devices. Proper Orthogonal Decomposition and Local Tangent Space Alignment are just two from a large class of model reduction strategies applicable to these systems, and this thesis has evaluated them only in two parameter regimes of a single example system. However, we have been able to produce simple models which accurately capture interesting dynamics, while building a scaffold on which future work may build.

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