Chapter 2

Projection Filters

Dynamics of open quantum systems take place in the space of density matrices, which can be a very high dimensional space, particularly when photon fields are involved. Strictly speaking, density matrices including photons are infinite, but it is common in practice to introduce a cutoff at some high Fock state, and work with finite, but large, density matrices. In simulations of cavity QED with the Jaynes-Cummings model, desired accuracy commonly requires one to keep track of 100 or more Fock states, in addition to the two-level atom. This results in a density matrix that is at least 200×200, indicating that the dynamics take place in a space that is nominally 39,999-dimensional (an $N \times N$ density matrix has $N^2 - 1$ real degrees of freedom, taking Hermiticity into account). However, we know that the dynamics of a particular system do not fully explore this space, and as a result we would like to define a smaller (lower-dimensional) space, and study the system dynamics within that space alone. To do this, we project the equations of motion onto the lower-dimensional space. In this chapter I give a short overview of how to calculate these projected equations of motion, put this projection in the context of stochastic filtering equations, and then derive the form of the projected equations for the cavity QED master equation for a particular form of the lower-dimensional space: a linear density matrix space.

2.1 Filter projection in general

In this section, I will give a brief overview of the process of projecting equations of motion from a high-dimensional manifold onto a lower-dimensional one. I draw heavily upon the excellent description of this process given by van Handel and Mabuchi [11], adapting their derivation to the case of density matrices (instead of Q functions). Let us denote the space of all possible density matrices for a quantum system of interest by M, and the smaller subspace by S. Let our example stochastic dynamical system take the form

$$d\rho_t = A\left[\rho_t\right] dt + B\left[\rho_t\right] dW_t. \tag{2.1}$$

From a geometric standpoint, we would like to think of the right-hand side of this equation as a vector in the tangent space to M at a particular point θ_M. When we project the equation onto S, we would like to keep the components which are in the tangent space to S at θ, denoted $T_\theta S$, and discard the components in the orthogonal complement to this tangent space, denoted $T_\theta S^\perp$. For the right-hand side of Eqn. (2.1) to be treated as a vector, it must transform like one, which means we must interpret it as a Stratonovich stochastic differential equation, rather than an Itô equation. For now I will simply change the notation to reflect this, but in a practical situation (like that following, in Sec. 2.2), one would calculate the appropriate Itô correction term, which would change the forms of A and B. The new equation takes the form

$$d\rho_t = A\left[\rho_t\right] dt + B\left[\rho_t\right] \circ dW_t. \tag{2.2}$$

If we assume that we have a local coordinate system on S so that $\theta = (\theta_1, \theta_2, \ldots)$, then we can write

$$T_\theta S = \mathrm{Span}\left[\frac{\partial\rho(\theta)}{\partial\theta_1}, \frac{\partial\rho(\theta)}{\partial\theta_2}, \cdots\right]. \tag{2.3}$$

We define an inner product on the space of density matrices

$$\langle \rho_A, \rho_B\rangle = \mathrm{Tr}[\rho_A \rho_B], \tag{2.4}$$

which allows us to calculate the metric tensor in this basis:

$$\left\langle \frac{\partial\rho(\theta)}{\partial\theta^i}, \frac{\partial\rho(\theta)}{\partial\theta^j}\right\rangle = \mathrm{Tr}\left[\frac{\partial\rho(\theta)}{\partial\theta^i}\frac{\partial\rho(\theta)}{\partial\theta^j}\right] = g_{ij}(\theta). \tag{2.5}$$

If the basis defined in Eqn. (2.3) is orthonormal, g will simply be the Identity; otherwise it accounts for the non-orthonormality. With an inner product and a metric, we can define orthogonal projection of a vector field $X[\theta]$:

$$\Pi_\theta X[\theta] = \sum_i \sum_j g^{ij}(\theta)\left\langle X[\theta], \frac{\partial\rho(\theta)}{\partial\theta^j}\right\rangle \frac{\partial\rho(\theta)}{\partial\theta^i}, \tag{2.6}$$

where g^{ij} denotes the (i,j) component of the inverse of the metric g defined in Eqn. (2.5).

We now wish to constrain the dynamics of Eqn. (2.2) to evolve on S:

$$d\rho(\theta_t) = \Pi_{\theta_t} A\left[\rho(\theta_t)\right] dt + \Pi_{\theta_t} B\left[\rho(\theta_t)\right] \circ dW_t, \tag{2.7}$$

which is just a stochastic differential equation for the parameters θ_t. Next, note that, in the Stratonovich calculus,

$$d\rho(\theta_t) = \sum_i \frac{\partial\rho(\theta_t)}{\partial\theta_t^i} \circ d\theta_t^i. \tag{2.8}$$

If we insert the definition of the orthogonal projection into Eqn. (2.2), we see that

$$\begin{aligned} d\rho(\theta_t) &= \sum_i \sum_j g^{ij}(\theta) \left\langle A\left[\rho(\theta_t)\right], \frac{\partial \rho(\theta)}{\partial \theta^j} \right\rangle \frac{\partial \rho(\theta)}{\partial \theta^i} dt \\ &+ \sum_i \sum_j g^{ij}(\theta) \left\langle B\left[\rho(\theta_t)\right], \frac{\partial \rho(\theta)}{\partial \theta^j} \right\rangle \frac{\partial \rho(\theta)}{\partial \theta^i} \circ dW_t. \end{aligned} \tag{2.9}$$

Comparing this expression with Eqn. (2.8), we can pull out the equations for $d\theta_t^i$:

$$d\theta_t^i = \sum_j g^{ij}(\theta) \left\langle A\left[\rho(\theta_t)\right], \frac{\partial \rho(\theta)}{\partial \theta^j} \right\rangle dt + \sum_j g^{ij}(\theta) \left\langle B\left[\rho(\theta_t)\right], \frac{\partial \rho(\theta)}{\partial \theta^j} \right\rangle \circ dW_t. \tag{2.10}$$

Note that in order to apply this procedure, we need to know a functional form for $\rho(\theta)$, meaning we need a map from the smaller space (spanned by θ) to the larger space (where ρ lives). This is in addition to knowing the form of the projection from the larger space to the smaller, facilitated by Eqn. (2.6) and the like. (The manifold learning algorithms discussed in Chapter 4 provide only point-wise maps, so projecting the filters onto them will be a challenge.)

When we want to use these projected equations as a filter, the measurement photocurrent driving them is still the same as that which drives the full-space SME (thought of as a filter). The innovation process, dW, however, is different because it is defined as the difference between the measurement result and the filter's current estimate, which differs for each filter. Assuming that we can construct the map which reverses the projection Π, giving us a θ_M from each θ, we can directly compare the state generated by the projected equations of motion (2.10) with the corresponding trajectory. We should be careful to note that the projected equations will often be generated from an SME which does not correspond to measuring every output from the system, whereas a quantum trajectory simulation necessarily requires measurement of all outputs so as to allow the creation of a stochastic Schrödinger equation. In order to reduce the difference between these two cases for cavity QED, in trajectory simulations I have consciously chosen to measure the atomic spontaneous emission in the quadrature which gives the least additional information about the system in its measurement record (for both absorptive and phase bistability, this is the σ_y quadrature). It is possible that differences persist, but they ought to be minor because the trajectories are required to average (over long times or many runs) to the same mean as for the unmeasured-atom situation reflected in Eqn. (1.3).

2.2 Projecting onto a linear density matrix space

2.2.1 The stochastic master equation

The stochastic master equation we are concerned with is, as before, that for a two-level atom interacting with a single harmonic mode in an optical cavity, with measurement performed on the field leaking out of the cavity. This is the normalized Itô form of the equation, for homodyne measurement of the phase quadrature:

$$\begin{aligned} \mathcal{D}[\rho] &= -i[H,\rho]dt + \kappa\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right)dt \\ &\quad +\gamma\left(2\sigma\rho\sigma^\dagger - \sigma^\dagger\sigma\rho - \rho\sigma^\dagger\sigma\right)dt \\ &\quad +\sqrt{2\kappa}\left(\rho a^\dagger - a\rho - i\mathrm{Tr}[\rho\left(a^\dagger - a\right)]\rho\right)dW. \end{aligned} \tag{2.11}$$

If we wanted to measure the amplitude quadrature instead, we would replace a with ia everywhere outside of the Hamiltonian. If we removed the nonlinear term $\mathrm{Tr}[\rho\left(a^\dagger - a\right)]\rho$, we would have the un-normalized version of the SME. It has the distinct advantage of being linear, but will allow the trace of the density matrix to differ from 1. In a stochastic simulation, we can use the unnormalized equation, and simply renormalize ρ after each time step. However, for completeness, and because the filters from the normalized equation seem to be better "behaved," I chose to use the normalized form, with its attendant complications resulting from nonlinearity.

In order for the geometry of projection to make sense, we need the components in this equation to transform like vectors, which means it needs to be a Stratonovich equation. We have two options for undertaking this transformation: 1) calculate the Itô correction term for Eqn. (2.11) or 2) use the much simpler (linear) un-normalized equation, transform it to Stratonovich form, and then normalize. I choose to do the first. This is the correct normalized Stratonovich form of the equation, calculated directly from Eqn. (2.11), for homodyne measurement of the phase quadrature:

$$\begin{aligned} \mathcal{D}[\rho] &= -i[H,\rho]dt + \kappa\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right)dt \\ &\quad +\gamma\left(2\sigma\rho\sigma^\dagger - \sigma^\dagger\sigma\rho - \rho\sigma^\dagger\sigma\right)dt \\ &\quad -\kappa\Big(2a\rho a^\dagger - a^2\rho - \rho\left(a^\dagger\right)^2 + 2\mathrm{Tr}[(a^\dagger - a)\rho](\rho a^\dagger - a\rho) \\ &\quad -2\rho\left(\mathrm{Tr}[(a^\dagger - a)\rho]\right)^2 + \rho\left[\mathrm{Tr}[(a^\dagger - a)(\rho a^\dagger - a\rho]\right]\Big)dt \\ &\quad +i\sqrt{2\kappa}\left(\rho a^\dagger - a\rho - \mathrm{Tr}[\rho\left(a^\dagger - a\right)]\rho\right)\circ dW, \end{aligned} \tag{2.12}$$

where the Hamiltonian H is as in Eqn. (1.1), and dW is the innovation.

2.2.2 The density matrix

With a master equation in hand, we now turn to the possible forms of the space onto which we would like to project it. The dynamics of Eqn. (2.12) take place in the space of all density matrices (positive Hermitian operators with trace 1), but we expect that the dynamics of the system limit the fraction of this space which a physical system will explore. Proper Orthogonal Decomposition (see Chapter 3) generates a linear subspace directly from the dynamics, so we will now examine the detailed form of such a space and the mechanics of the projection.

Imagine an N-dimensional linear density matrix space. Density matrices in this space have the following form:

$$\rho(v) = \rho_0 + \sum_{i=1}^{N} v_i \rho_i \tag{2.13}$$

where the ρ_is are trace-0 Hermitian matrices (directions in density-matrix space), and ρ_0 is a positive, trace-1 Hermitian matrix (a valid density matrix), which serves as the origin in our linear space. The coefficients v_i are real, to maintain Hermiticity. There is nothing that forces $\rho(v)$ to remain positive, so it might cease to be a valid density matrix. However, when acting as part of a filter we expect it to stay positive almost all the time, except when presented with a measurement record which it is unable to do a good job of accommodating.

The partial derivatives of ρ are

$$\frac{\partial \rho(v)}{\partial v^i} = \rho_i. \tag{2.14}$$

We recall the definition of the inner product between matrices/operators as the trace of the product, and so we define the metric in this space

$$g_{ij} = \mathrm{Tr}[\rho_i \rho_j] \quad (i, j > 0). \tag{2.15}$$

We assume that the ρ_is have been orthonormalized so that $g_{ij} = g^{ij} = \delta_{ij}$ ($g = \mathrm{Id}$).

If we had not used the normalized form of the stochastic master equation, we would have extended the dimension of the linear space by 1 to include a coefficient on ρ_0. Then we would redefine the state to be the ratio of each coefficient to v_0, which would complicate the equations to be evolved. Alternatively, in simulations, we would simply rescale all of the coefficients at each time step, setting $v_i = \tilde{v}_i/\tilde{v}_0$, $i \geq 0$. In practice, filtering using the normalized equations seems to be somewhat more robust, and it has the advantage of providing us with exact, nonlinear equations directly.

2.2.3 Projection

Following the general derivation given in Section 2.1, and specializing to our particular space S, spanned by the states ρ_i, we have that the orthogonal projection of (2.12) is

$$\Pi_v \mathcal{D}[\rho(v)] = \sum_{i=1}^{N} \sum_{j=1}^{N} g^{ij} \left\langle \mathcal{D}[\rho(v)], \frac{\partial \rho(v)}{\partial v^j} \right\rangle \frac{\partial \rho(v)}{\partial v^i}. \tag{2.16}$$

Simplifying because we know that $g_{ij} = g^{ij} = \delta_{ij}$, we see that

$$\Pi_v \mathcal{D}[\rho(v)] = \sum_{i=1}^{N} \langle \mathcal{D}[\rho(v)], \rho_i \rangle \, \rho_i. \tag{2.17}$$

For an arbitrary filtering SME of the form (2.2), we constrain the filter to evolve in our space of density matrices, and combine Eqn. (2.17) with Eqn. (2.7) to find that

$$dv_i = \langle A[\rho(v)], \rho_i \rangle \, dt + \langle B[\rho(v)], \rho_i \rangle \circ dW. \tag{2.18}$$

We have split the master equation, Eqn. (2.12), into deterministic and stochastic parts, as in Eqn. (2.2), to clarify calculation:

$$\begin{aligned} A[\rho] &= -i[H,\rho] + \kappa\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right) \\ &\quad +\gamma\left(2\sigma\rho\sigma^\dagger - \sigma^\dagger\sigma\rho - \rho\sigma^\dagger\sigma\right) \\ &\quad -\kappa\Big(2a\rho a^\dagger - a^2\rho - \rho\left(a^\dagger\right)^2 + 2\mathrm{Tr}[(a^\dagger - a)\rho](\rho a^\dagger - a\rho) \\ &\quad -2\rho\left(\mathrm{Tr}[(a^\dagger - a)\rho]\right)^2 + \rho\left[\mathrm{Tr}[(a^\dagger - a)(\rho a^\dagger - a\rho)]\right]\Big) && (2.19) \\ B[\rho] &= i\sqrt{2\kappa}\left(\rho a^\dagger - a\rho - \mathrm{Tr}[\rho\left(a^\dagger - a\right)]\rho\right). && (2.20) \end{aligned}$$

We will now project each of these terms, in order to derive the detailed form of Eqn. (2.18).

2.2.4 The deterministic terms $\mathbf{A}[\rho]$

Let us start calculating the terms in (2.18). First, we define

$$\begin{aligned} \mathcal{L}_H(\rho) &\equiv -i[H,\rho] && (2.21) \\ \mathcal{L}_a(\rho) &\equiv \kappa\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right) && (2.22) \\ \mathcal{L}_\sigma(\rho) &\equiv \gamma\left(2\sigma\rho\sigma^\dagger - \sigma^\dagger\sigma\rho - \rho\sigma^\dagger\sigma\right) && (2.23) \\ \mathcal{L}_{ISL}(\rho) &\equiv \kappa\left(2a\rho a^\dagger - a^2\rho - \rho\left(a^\dagger\right)^2\right) && (2.24) \\ \mathcal{L}_{ISN}(\rho) &\equiv \kappa\Big(2\mathrm{Tr}[(a^\dagger - a)\rho](\rho a^\dagger - a\rho) - 2\rho\left(\mathrm{Tr}[(a^\dagger - a)\rho]\right)^2 \\ &\quad +\rho\left[\mathrm{Tr}[(a^\dagger - a)(\rho a^\dagger - a\rho)]\right]\Big). && (2.25) \end{aligned}$$

Then

$$\begin{aligned} \langle A[\rho(v)], \rho_i \rangle &= \langle \mathcal{L}_H(\rho(v)) + \mathcal{L}_a(\rho(v)) + \mathcal{L}_\sigma(\rho(v)) \\ &\quad -\mathcal{L}_{ISL}(\rho(v)) - \mathcal{L}_{ISN}(\rho(v)), \rho_i \rangle && (2.26) \\ &= \langle \mathcal{L}_H(\rho(v)), \rho_i \rangle + \langle \mathcal{L}_a(\rho(v)), \rho_i \rangle + \langle \mathcal{L}_\sigma(\rho(v)), \rho_i \rangle \\ &\quad - \langle \mathcal{L}_{ISL}(\rho(v)), \rho_i \rangle - \langle \mathcal{L}_{ISN}(\rho(v)), \rho_i \rangle . && (2.27) \end{aligned}$$

Expanding the form of $\rho(v)$, we have

$$\langle \mathcal{L}_H(\rho(v)), \rho_i \rangle = \left\langle \mathcal{L}_H(\rho_0 + \sum_{j=1}^{N} v_j \rho_j), \rho_i \right\rangle \tag{2.28}$$

$$= \langle \mathcal{L}_H(\rho_0), \rho_i \rangle + \sum_{j=1}^{N} v_j \langle \mathcal{L}_H(\rho_j), \rho_i \rangle , \tag{2.29}$$

and the same for $\mathcal{L}_a$, $\mathcal{L}_\sigma$, and $\mathcal{L}_{ISL}$, because they are all linear in ρ.

In fact, if we define

$$\mathcal{L} \equiv \mathcal{L}_H + \mathcal{L}_a + \mathcal{L}_\sigma - \mathcal{L}_{ISL} \tag{2.30}$$

then we have

$$\langle \mathcal{L}(\rho(v)), \rho_i \rangle = \langle \mathcal{L}(\rho_0), \rho_i \rangle + \sum_{j=1}^{N} v_j \langle \mathcal{L}(\rho_j), \rho_i \rangle . \tag{2.31}$$

Plugging this into Eqn. (2.18), we see that the linear, deterministic part of dv is

$$dv_{i(lindet)} = \langle \mathcal{L}(\rho_0), \rho_i \rangle + \sum_{j=1}^{N} v_j \langle \mathcal{L}(\rho_j), \rho_i \rangle \, dt. \tag{2.32}$$

If we think of $d\mathbf{v}$ as a vector, we see that this is just a matrix multiplication, where each entry in the matrix L is simply

$$L_{ij} = \langle \mathcal{L}(\rho_j), \rho_i \rangle + \langle \mathcal{L}(\rho_0), \rho_i \rangle \delta_{ij}. \tag{2.33}$$

Now we need to take a look at $\mathcal{L}_{ISN}$, the nonlinear terms from the Itô to Stratonovich conversion.

$$\mathcal{L}_{ISN}(\rho) \equiv \kappa\Big(2\mathrm{Tr}[(a^\dagger - a)\rho](\rho a^\dagger - a\rho) - 2\rho \left(\mathrm{Tr}[(a^\dagger - a)\rho]\right)^2$$
$$+\rho \left[\mathrm{Tr}[(a^\dagger - a)(\rho a^\dagger - a\rho)]\right] \Big). \tag{2.34}$$

Let us start with the first term, and plug in the approximate form of ρ for our linear space.

$$\mathrm{Tr}\left[(a^\dagger - a)\left(\rho_0 + \sum_{j=1}^{N} v_j \rho_j\right)\right] \left(\left(\rho_0 + \sum_{j=1}^{N} v_j \rho_j\right) a^\dagger - a \left(\rho_0 + \sum_{j=1}^{N} v_j \rho_j\right)\right)$$
$$= \mathrm{Tr}[(a^\dagger - a)\rho_0](\rho_0 a^\dagger - a\rho_0) + \mathrm{Tr}[(a^\dagger - a)(\rho_0)] \left(\left(\sum_{j=1}^{N} v_j \rho_j\right) a^\dagger - a \left(\sum_{j=1}^{N} v_j \rho_j\right)\right)$$
$$+\mathrm{Tr}\left[(a^\dagger - a)\left(\sum_{j=1}^{N} v_j \rho_j\right)\right] \left((\rho_0) a^\dagger - a(\rho_0)\right)$$
$$+\mathrm{Tr}\left[(a^\dagger - a)\left(\sum_{j=1}^{N} v_j \rho_j\right)\right] \left(\left(\sum_{j=1}^{N} v_j \rho_j\right) a^\dagger - a \left(\sum_{j=1}^{N} v_j \rho_j\right)\right)$$

$$
\begin{aligned}
&= \mathrm{Tr}[(a^\dagger - a)\rho_0](\rho_0 a^\dagger - a\rho_0) + \mathrm{Tr}[(a^\dagger - a)\rho_0]\sum_{k=1}^{N} v_k(\rho_k a^\dagger - a\rho_k) \\
&\quad + \sum_{j=1}^{N} v_j \mathrm{Tr}[(a^\dagger - a)\rho_j](\rho_0 a^\dagger - a\rho_0) + \sum_{j=1}^{N} v_j \mathrm{Tr}[(a^\dagger - a)\rho_j]\sum_{k=1}^{N} v_k(a\rho_k + \rho_k a^\dagger) \\
&= \left(\mathrm{Tr}[(a^\dagger - a)\rho_0] + \sum_{j=1}^{N} v_j \mathrm{Tr}[(a^\dagger - a)\rho_j]\right) \times \left((\rho_0 a^\dagger - a\rho_0) + \sum_{k=1}^{N} v_k(\rho_k a^\dagger - a\rho_k)\right). \qquad (2.35)
\end{aligned}
$$

Now, let us do the inner product with ρ_i, noting that the traces are things we have calculated anyway, because they're just the expectation values of $-2iy$ for each ρ_j:

$$
\begin{aligned}
&\left(\mathrm{Tr}[(a^\dagger - a)\rho_0] + \sum_{j=1}^{N} v_j \mathrm{Tr}[(a^\dagger - a)\rho_j]\right) \\
&\times \left\langle (\rho_0 a^\dagger - a\rho_0) + \sum_{k=1}^{N} v_k(\rho_k a^\dagger - a\rho_k), \rho_i \right\rangle \\
&= -2i\left(\langle y_0\rangle + \sum_{j=1}^{N} v_j \langle y_j\rangle\right) \times \left(\langle(\rho_0 a^\dagger - a\rho_0), \rho_i\rangle + \sum_{k=1}^{N} v_k \langle(\rho_k a^\dagger - a\rho_k), \rho_i\rangle\right). \qquad (2.36)
\end{aligned}
$$

It doesn't simplify much because we can't use the orthogonality of the ρ_is once the as are present. For simulations, however, we can pre-calculate the values of everything in the angle brackets.

Let us take the third term, the other quadratic term:

$$
\begin{aligned}
&\left(\rho_0 + \sum_{j=1}^{N} v_j\rho_j\right) \mathrm{Tr}\left[(a^\dagger - a)\left(\left(\rho_0 + \sum_{j=1}^{N} v_j\rho_j\right) a^\dagger - a\left(\rho_0 + \sum_{j=1}^{N} v_j\rho_j\right)\right)\right] \\
&= \rho_0 \mathrm{Tr}[(a^\dagger - a)(\rho_0 a^\dagger - a\rho_0)] + \rho_0 \sum_{k=1}^{N} v_k \left[\mathrm{Tr}[(a^\dagger - a)(\rho_k a^\dagger - a\rho_k)]\right] \\
&\quad + \sum_{j=1}^{N} v_j\rho_j \left[\mathrm{Tr}[(a^\dagger - a)(\rho_0 a^\dagger - a\rho_0)]\right] \\
&\quad + \sum_{j=1}^{N} v_j\rho_j \sum_{k=1}^{N} v_k \left[\mathrm{Tr}[(a^\dagger - a)(\rho_k a^\dagger - a\rho_k)]\right] \\
&= \left(\mathrm{Tr}[(a^\dagger - a)(\rho_0 a^\dagger - a\rho_0)] + \sum_{k=1}^{N} v_k \left[\mathrm{Tr}[(a^\dagger - a)(\rho_k a^\dagger - a\rho_k)]\right]\right) \\
&\quad \times \left(\rho_0 + \sum_{j=1}^{N} v_j\rho_j\right). \qquad (2.37)
\end{aligned}
$$

This does simplify once we do the inner product with ρ_i:

$$\langle \rho_0, \rho_i \rangle + \sum_{j=1}^{N} v_j \langle \rho_j, \rho_i \rangle = G_i + v_i \tag{2.38}$$

where

$$G_i \equiv \langle \rho_0, \rho_i \rangle . \tag{2.39}$$

And now for the cubic term:

$$\left(\rho_0 + \sum_{j=1}^{N} v_j \rho_j \right) \left(\mathrm{Tr}\left[(a^\dagger - a) \left(\rho_0 + \sum_{k=1}^{N} v_k \rho_k \right) \right] \right)^2 . \tag{2.40}$$

Noting the above simplification, we see that after we do the inner product, we have this:

$$\begin{aligned} & (G_i + v_i) \Big(\mathrm{Tr}\left[(a^\dagger - a)\rho_0 \right]^2 + 2 \sum_{k=1}^{N} v_k \mathrm{Tr}\left[(a^\dagger - a)\rho_0 \right] \mathrm{Tr}\left[(a^\dagger - a)\rho_k \right] \\ & + \sum_{j=1}^{N} \sum_{k=1}^{N} v_j v_k \mathrm{Tr}\left[(a^\dagger - a)\rho_j \right] \mathrm{Tr}\left[(a^\dagger - a)\rho_k \right] \Big) \\ = \; & -4 (G_i + v_i) \Big(\langle y_0 \rangle^2 + 2 \sum_{k=1}^{N} v_k \langle y_0 \rangle \langle y_k \rangle + \sum_{j=1}^{N} \sum_{k=1}^{N} v_j v_k \langle y_j \rangle \langle y_k \rangle \Big). \end{aligned} \tag{2.41}$$

Assembling all of the parts of $\mathcal{L}_{ISN}(\rho)$ together, we have:

$$\begin{aligned} dv_{i(ISN)} &= - \langle \mathcal{L}_{ISN}(\rho), \rho_i \rangle \, dt \\ &= -\kappa \Bigg(-4i \left(\langle y_0 \rangle + \sum_{j=1}^{N} v_j \langle y_j \rangle \right) \times \left(\langle (\rho_0 a^\dagger - a\rho_0), \rho_i \rangle + \sum_{k=1}^{N} v_k \langle (\rho_k a^\dagger - a\rho_k), \rho_i \rangle \right) \\ & \quad +8 (G_i + v_i) \left(\langle y_0 \rangle + \sum_{k=1}^{N} v_k \langle y_k \rangle \right)^2 \\ & \quad +(G_i + v_i) \left(\mathrm{Tr}[(a^\dagger - a)(\rho_0 a^\dagger - a\rho_0)] + \sum_{k=1}^{N} v_k \left[\mathrm{Tr}[(a^\dagger - a)(\rho_k a^\dagger - a\rho_k)] \right] \right) \Bigg) dt. \end{aligned} \tag{2.42}$$

2.2.5 The stochastic terms B[ρ]

Recall that

$$B[\rho] = i\sqrt{2\kappa} \left(\rho a^\dagger - a\rho - \mathrm{Tr}[\rho \left(a^\dagger - a \right)]\rho \right) . \tag{2.43}$$

The linear portion of this $[\mathcal{LS}(\rho) \equiv i\sqrt{2\kappa}\left(\rho a^\dagger - a\rho\right)]$ is just like the deterministic case, with

$$dv_{i(lin)} = \langle \mathcal{LS}(\rho_0), \rho_i \rangle + \sum_{j=1}^{N} v_j \langle \mathcal{LS}(\rho_j), \rho_i \rangle \circ dW, \tag{2.44}$$

and will work out to a simple, constant over time, matrix multiplication by a matrix LS:

$$LS_{ij} = i\sqrt{2\kappa}\left(\left\langle \rho_j a^\dagger - a\rho_j, \rho_i \right\rangle + \left\langle \rho_0 a^\dagger - a\rho_0, \rho_i \right\rangle \delta_{ij}\right), \tag{2.45}$$

Now let us turn to the nonlinear term:

$$i\left\langle \mathrm{Tr}[\rho(v)\left(a^\dagger - a\right)]\rho(v), \rho_i \right\rangle = i\left\langle \mathrm{Tr}\left[\left(\rho_0 + \sum_{j=1}^{N} v_j\rho_j\right)\left(a^\dagger - a\right)\right]\left(\rho_0 + \sum_{k=1}^{N} v_k\rho_k\right), \rho_i \right\rangle. \tag{2.46}$$

This breaks out into 4 chunks: A constant (independent of v_i),

$$i\left\langle \mathrm{Tr}[\rho_0\left(a^\dagger - a\right)]\rho_0, \rho_i \right\rangle, \tag{2.47}$$

two linear terms:

$$i\left\langle \mathrm{Tr}[\rho_0\left(a^\dagger - a\right)]\left(\sum_{k=1}^{N} v_k\rho_k\right), \rho_i \right\rangle \text{ and} \tag{2.48}$$

$$i\left\langle \mathrm{Tr}\left[\left(\sum_{j=1}^{N} v_j\rho_j\right)\left(a^\dagger - a\right)\right]\rho_0, \rho_i \right\rangle, \tag{2.49}$$

and one quadratic term:

$$i\left\langle \mathrm{Tr}\left[\left(\sum_{j=1}^{N} v_j\rho_j\right)\left(a^\dagger - a\right)\right]\left(\sum_{k=1}^{N} v_k\rho_k\right), \rho_i \right\rangle. \tag{2.50}$$

Many of the components of these terms are constants.

The constant term is

$$2G_i \langle y_0 \rangle, \tag{2.51}$$

and the two linear terms are

$$2\langle y_0 \rangle \sum_{k=1}^{N} v_k \langle \rho_k, \rho_i \rangle = 2\langle y_0 \rangle \sum_{k=1}^{N} v_k \delta_{ik} = 2\langle y_0 \rangle v_i \tag{2.52}$$

and

$$2G_i \sum_{j=1}^{N} v_j \langle y_j \rangle. \tag{2.53}$$

The quadratic term is

$$2\sum_{j=1}^{N}\sum_{k=1}^{N} v_k v_j \langle y_j \rangle \langle \rho_k, \rho_i \rangle = 2\sum_{j=1}^{N}\sum_{k=1}^{N} v_k v_j \langle y_j \rangle \delta_{ki} = 2v_i \sum_{j=1}^{N} v_j \langle y_j \rangle . \qquad (2.54)$$

The stochastic part of Eqn. (2.18) from the nonlinear (trace) term is therefore

$$\begin{aligned} dv_{i(trace)} &= -\sqrt{8\kappa}\left(\langle y_0 \rangle G_i + \langle y_0 \rangle v_i + G_i \sum_{j=1}^{N} v_j \langle y_j \rangle + v_i \sum_{j=1}^{N} v_j \langle y_j \rangle \right) \circ dW \\ &= -\sqrt{8\kappa}\left(\langle y_0 \rangle G_i + \langle y_0 \rangle v_i + (G_i + v_i) \sum_{j=1}^{N} v_j \langle y_j \rangle \right) \circ dW. \qquad (2.55) \end{aligned}$$

So, putting all of Eqn. (2.18) together we have

$$\begin{aligned} dv_i &= \left(\langle \mathcal{L}(\rho_0), \rho_i \rangle + \sum_{j=1}^{N} v_j \langle \mathcal{L}(\rho_j), \rho_i \rangle \right) dt + dv_{i(ISN)} \\ &\quad + \left(\langle \mathcal{LS}(\rho_0), \rho_i \rangle + \sum_{j=1}^{N} v_j \langle \mathcal{LS}(\rho_j), \rho_i \rangle \right) \circ dW \\ &\quad -\sqrt{8\kappa}\,(G_i + v_i) \left(\langle y_0 \rangle + \sum_{j=1}^{N} v_j \langle y_j \rangle \right) \circ dW. \qquad (2.56) \end{aligned}$$

2.2.6 Stratonovich back to Itô

For numeric simulation with an Itô-Euler integrator, we require Itô equations, so we need to transform our Stratonovich equations back into Itô. Currently our equations of motion for the projected filter have the form

$$\mathcal{D}[\mathbf{v}] = A_v[\mathbf{v}]dt + B_v[\mathbf{v}] \circ dW. \qquad (2.57)$$

The correction term has the form

$$\frac{1}{2}\left(\mathbf{D}B_v[\mathbf{v}]\right) B_v[\mathbf{v}] \qquad (2.58)$$

where $\mathbf{D}(\cdot)$ is the derivative.

The $\mathcal{LS}$ part of the stochastic term is just matrix multiplication by LS, so its derivative is just LS.

The part of B that comes from the normalizing term in the SME has a derivative of

$$\begin{aligned}
\mathbf{D}B_{v(trace)}[\mathbf{v}]_{ij} &= -\frac{\partial}{\partial v_j}\sqrt{8\kappa}\left(\langle y_0\rangle G_i + \langle y_0\rangle v_i + (G_i + v_i)\sum_{k=1}^{N} v_k \langle y_k\rangle\right) \\
&= -\sqrt{8\kappa}\left(\langle y_0\rangle \delta_{ij} + (G_i + v_i)\sum_{k=1}^{N}\langle y_k\rangle \delta_{jk} + \delta_{ij}\sum_{k=1}^{N} v_k \langle y_k\rangle\right) \\
&= -\sqrt{8\kappa}\left(\left(\langle y_0\rangle + \sum_{k=1}^{N} v_k \langle y_k\rangle\right)\delta_{ij} + (G_i + v_i)\langle y_j\rangle\right). \qquad (2.59)
\end{aligned}$$

For notational simplicity, let us call this matrix DB. Let us call the part of B that comes from the normalizing term, which is a vector, $\mathbf{B_n}$ (you can read its elements off of Eqn. (2.55)). Then the full correction term to take us back to Itô form is

$$\frac{1}{2}\left(\mathbf{D}B_v[\mathbf{v}]\right) B_v[\mathbf{v}] = \frac{1}{2}\left(LS + DB\right)\left(\mathbf{LS} + LS\mathbf{v} + \mathbf{B_n}\right) \qquad (2.60)$$

where

$$\mathbf{LS}_i = \langle \mathcal{LS}(\rho_0), \rho_i \rangle\,. \qquad (2.61)$$

Applying this term, we now have the complete Itô stochastic differential equation for the dynamics of the projected filter:

$$\begin{aligned}
d\mathbf{v} &= \left(\mathbf{L} + L\mathbf{v} + d\mathbf{v}_{ISN} + \frac{1}{2}\left(LS + DB\right)\left(\mathbf{LS} + LS\mathbf{v} + \mathbf{B_n}\right)\right) dt \\
&\quad + \left(\mathbf{LS} + LS\mathbf{v} + \mathbf{B_n}\right) dW \qquad (2.62)
\end{aligned}$$

where

$$\mathbf{L}_i = \langle \mathcal{L}(\rho_0), \rho_i \rangle\,. \qquad (2.63)$$

With the machinery in place, we can now turn to generating linear density matrix spaces by Proper Orthogonal Decomposition of quantum trajectories, project the filter onto them, and evaluate their performance.