Chapter 1

Introduction and Background

Quantum mechanics forms the basis for our understanding of the behavior of light and matter. While most day-to-day events we observe can be accurately modeled using classical mechanics and electromagnetism, many physical systems require a quantum mechanical description. Whether we are interested in the details of a chemical reaction or the flow of electrons through a semiconductor inside a computer, or even a system as apparently simple as the interaction of the electromagnetic field with a single atom, quantum mechanical modeling is both highly accurate and necessary. As optical networks and computing resources shrink and become faster, they will inevitably encounter limits set by quantum mechanics. While these limits might impose restrictions on the behavior of a system, they may also open the door to other uses for the same system. This thesis is dedicated to examining ways of building simpler models for quantum systems which respect their dynamics, but may eventually also allow those systems to be more easily developed into useful technology.

Quantum mechanics is an inherently probabilistic model for the physical world. In order to model the dynamics of quantum systems, we propagate not a single position and momentum as we might for a classical system, but instead a probability distribution. Undergraduate quantum mechanics usually introduces the Schrödinger equation, a partial differential equation for the wavefunction of an isolated physical system, from which we can calculate the possible results of measurements of certain observable quantities. However, as we work to match this simple model with the physical world, we encounter limits to the model. First, we are forced to reckon with the difficulty of building an isolated system, and then measuring it. By definition, the measurement process is an intrusion from the "outside" into a supposedly isolated system. This is usually dealt with by hand-waving arguments about an omnipotent experimenter suddenly introducing a measurement apparatus and making a sharp, projective measurement.

Of course, this model, too, often does not match with the physical reality of a laboratory or its example quantum system under study. A real physical quantum system is an open system, unavoidably coupled to the environment surrounding it, making it difficult to define the boundary of the "system." Often, the best we can hope for is to shape the coupling of system to environment

so that we may retrieve information about the system without disrupting it in ways we do not want. (If we can do this, then we are justified in our model's separation between "system" and "environment.") A physical measurement process can almost always be modeled not as a direct measurement of the system, but instead a measurement of the environment interacting with the system of interest. Measurements also necessarily take time; the instantaneous measurement of introductory quantum mechanics is as much an approximation as its isolation.

We must also deal with the fact that our measurement may not tell us everything about our system, because of the limited way in which the environment interacts with the system, or because our measurement cannot distinguish between different states of the environment or, as a result, the system. As experimenters, our understanding of the state of our quantum system is shaped by the system's history and dynamics (the realm of the Schrödinger equation), and also by the fallible way we measure the system, noise which may be introduced through the environment or measurement, or our simple failure to measure all of the parts of the environment which carry information about the system. We must almost always think of our quantum system as being in a mixed state, acknowledging our lack of knowledge about the system.

Mixed states cannot be modeled as wavevectors ($|\psi\rangle$), and must instead be modeled using density matrices ($\sum_i c_i|\psi_i\rangle\langle\psi_i|$), which we propagate in time using the master equation. Density matrices reside in a much larger space than wavefunctions — they have many more degrees of freedom — which makes accurate simulations of dynamics a challenge for large systems. In particular, any system which is coupled with an electromagnetic field can be difficult to simulate because the field, modeled as a harmonic oscillator, is infinite dimensional (there's no top to the ladder of states). It is usually reasonable to define a cut-off energy, but this can still result in very large density matrices, especially if the system consists of tensor products of this large field space with other system components. Coupling with a two-level system quadruples the number of elements in the density matrix.

One way to tackle this challenge is by simulating only fully observed systems. That is, only systems in which the experimenter measures every output channel. Such systems can be simulated using a stochastic Schrödinger equation, and their states remain pure, and take the form of wavefunctions. The quantum simulations ("trajectories") in this thesis are all of this form. The statistical behavior of such systems is identical to partially observed systems over long times (or multiple samples of identical systems). Measurements should not be able to change the fundamental statistics of the underlying system.

Given a physical system, however, the experimenter often does not have the option of measuring every output channel. An atom may emit light into all 4π steradians, for example, which cannot all be covered in detectors. To build her best guess as to the system state, the experimenter is forced to use the master equation as a "filter" — an equation which propagates her current best guess about the

state of the system, with filter input from the output of the measurement process. The experimenter's best guess is a mixed state, and propagating the master equation will be computationally intense (and very difficult to do in real time). Should the experimenter desire to use this filter to decide on a control signal to drive the system to a particular state, she will be hard pressed to update her best guess in real time, and the control task will be very challenging. This thesis attempts to develop techniques to build simple models for quantum systems which capture the dynamics of interest (and potential utility) and which may also be propagated much more easily, potentially in real time, by a computer or purpose-built circuit. Such simple models might also provide insight into the system by elucidating the components which result in particular behavior.

The general technique I use to build simpler models is to find a linear or nonlinear submanifold of the full space of possible dynamics, in which the particular dynamics of a system are generally confined. I adapt various techniques for finding such subspaces developed for other applications to the case of quantum dynamics, and illustrate both their successes and failures.

1.1 Thesis overview

The remainder of this Introduction introduces the example physical system whose dynamics we will attempt to model with simple dynamical systems: a two-level atom in a high-finesse optical cavity, a situation known as "cavity quantum electrodynamics" (cavity QED). I review the dominant model for cavity QED, the Jaynes-Cummings model, give an abbreviated derivation of the Maxwell-Bloch equations, and a short summary of known interesting dynamics observed in these equations and in corresponding simulations of cavity QED. I close with a brief introduction to a few topics in stochastic calculus, to lay the groundwork for the remainder of the thesis.

Chapter 2 provides an overview of the technique of projecting filtering equations onto manifolds. I then turn to a particular manifold — a linear space of density matrices — and project the stochastic master equation for cavity QED onto this space, deriving nonlinear dynamical equations for the local coordinates.

Chapter 3 makes the work of Chapter 2 more concrete by describing a process, Proper Orthogonal Decomposition, for determining a linear density matrix space onto which to project the dynamics. I analyze cavity QED dynamics in phase and absorptive bistability regimes, and demonstrate some successes, and some failures, of this process for generating accurate filters.

Chapter 4 turns away from linear manifolds, and introduces nonlinear manifolds generated with an algorithm for Local Tangent Space Alignment. We turn again to phase and absorptive bistability regimes. In the phase bistability case, we are able to fit the manifold coordinates with simple combinations of expectation values, and derive a set of situation-specific "Maxwell-Bloch" equations, which I then compare with previously-derived equations for this system, and evaluate as a filter. The

absorptive bistability case is more complex, but I lay the groundwork for deriving similar systems of equations.

The concluding Chapter 5 draws the results together, examines the successes and failures of the examined model reduction techniques, and suggests directions for future work.

1.2 Model system: Cavity quantum electrodynamics

A single atom in a high-finesse optical cavity constitutes a canonical system in quantum optics. Experimental work on such systems (such as [3, 4, 5, 6, 7], among many others) has demonstrated strong coupling between the atom and the optical field, meaning that the presence or absence of a single photon drastically changes the environment for a single atom, and correspondingly, that the state of the single atom strongly affects the behavior of the field in the cavity. There are multiple parameter ranges in which strong coupling occurs. Strong coupling can be identified as a regime in which the ratio of the square of the atom-field coupling rate to the product of the cavity decay and atomic spontaneous emission rates is large. This ratio is called the "cooperativity." Two limits of interest are the "bad cavity" and "good cavity" limits: for a fixed value of the cooperativity, the cavity decay rate may be large compared with the spontaneous emission ("bad cavity"), or small ("good cavity"). We usually scale time by the atomic spontaneous emission rate, so it ends up dropping out; this means that a small cavity decay rate, for a fixed cooperativity, implies a small atom-field coupling rate (inside a higher-finesse cavity), and *vice versa*.

1.2.1 The Jaynes-Cummings model

A simple model for the atom-cavity system is the Jaynes-Cummings model [8]. In this model, the atom is approximated as a two-level system, there is only one harmonic mode of the field in the cavity. The two-level atom is equivalent to a single spin, and the operators which act on it are the Pauli matrices and their linear combinations: σ_- lowers the atom into its ground state, while $\sigma_+ = \sigma_-^\dagger$ excites the atom. The field is acted upon by a and $a^\dagger$, the familiar annihilation and creation operators for a simple harmonic oscillator. We operate in a rotating frame, at the frequency of the driving field. The Hamiltonian in this model takes the form

$$H = \Delta_c a^\dagger a + \Delta_a \sigma_+ \sigma_- + i g_0 \left(a^\dagger \sigma_- - a \sigma_+ \right) + i\mathcal{E} \left(a^\dagger - a \right), \tag{1.1}$$

where Δ_c is the detuning between the field and the cavity, Δ_a is the detuning between the field and the atomic transition, g_0 is the coupling rate between the atom and the cavity field, and $\mathcal{E}$ is the driving (classical) field strength. Stepping through the terms of the equation:

- $\Delta_c a^\dagger a$ accounts for the difference in energy between on- and off-resonant drive of the cavity by the external field.

- $\Delta_a \sigma_+ \sigma_-$ similarly accounts for the difference in energy between on- and off-resonant drive of the atom by the external field.

- $ig_0 \left(a^\dagger \sigma_- - a\sigma_+\right)$ is responsible for exchange of energy between the atom and the field at rate $2g_0$. The first term creates an excitation in the field, while driving the atom to the ground state; that is, the atom emits a photon into the field. The second is the inverse process, where the atom absorbs one photon out of the field, and becomes excited.

- $i\mathcal{E}\left(a^\dagger - a\right)$ is the external driving field, acting to excite the phase quadrature of the cavity mode.

A Hamiltonian like (1.1) would suffice if we were concerned only with the unobserved dynamics of a closed quantum system. However, for useful systems, the "closed" model will not suffice. Instead, we must extend the picture to include the system's interaction with its environment, and the behavior of an observer making measurements on the system. The observer may know the quantum state fully, or imperfectly, and so we model the dynamics of such an "open quantum system" with a master equation, which propagates the motion of a density matrix. The interaction of the system and its environment is governed by probabilities, so that the time at which the atom or field changes state (such as emitting a photon) is random. We therefore require a stochastic model for the propagation of the density matrix, which allows for the introduction of noise resulting from this inherently probabilistic behavior.

The Itô form of the stochastic master equation (SME) which governs the behavior of an atom-cavity system being observed with homodyne measurements of both the leaking cavity field and the atomic emission is

$$\begin{aligned} d\rho &= -i[H,\rho]dt + \kappa\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right)dt \\ &\quad +\gamma\left(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-\right)dt \\ &\quad +i\sqrt{2\kappa}\left(\rho a^\dagger - a\rho - \mathrm{Tr}[\rho\left(a^\dagger - a\right)]\rho\right)dW_1 \\ &\quad +i\sqrt{2\gamma}\left(\rho\sigma_+ - \sigma_-\rho - i\mathrm{Tr}[\rho\left(\sigma_+ - \sigma_-\right)]\rho\right)dW_2. \end{aligned} \tag{1.2}$$

Here dW_1 and dW_2 are uncorrelated Wiener processes corresponding to noise on the two different measurement processes, κ is the rate of field decay out the end of the cavity (usually toward a detector), and γ is the rate of decay of the atomic state (atomic spontaneous emission) measured by a second homodyne detection process. $\gamma_\perp$, not included above, is the atomic transverse relaxation rate, and we scale time so that this rate is 1. A free atom has a spontaneous emission rate $\gamma = 2$, and

a Fabry-Perot cavity of the sort we model does not significantly change the spontaneous emission characteristics of the atom. The cooperativity parameter mentioned above is defined as $C = \frac{g_0^2}{2\kappa\gamma_\perp} = \frac{g_0^2}{\kappa\gamma}$.

This stochastic master equation will maintain the purity of an initial state, because all the channels of information leaving the system are being measured. As a result, this equation is exactly equivalent to a stochastic Schrödinger equation (SSE), which is relatively easy to simulate due to its more benign linear scaling behavior with the number of field modes to be propagated. I will refer to the pure states which result from the propagation of such an equation as *quantum trajectories*, and they will be a prime source of our insight into system behavior from simulations throughout this thesis. My discussion of quantum trajectories here is grounded in the work of Mabuchi and Wiseman [9] and references therein, although for convenience I have created a somewhat narrower definition of quantum trajectories as pure state trajectories only. I calculate these trajectories throughout the thesis using the SSE integration built in to the Quantum Optics Toolbox for Matlab, written by Sze Tan [10].

The experimentalist is generally unable to measure the field leaking out the sides of the cavity, and has only the measurement record from the cavity field measurement. As a result, she must average over all possible measurement results for the missing atomic measurement, leaving a (likely) mixed state as her best state of knowledge about the system. The density matrix in this case evolves according to the following SME:

$$\begin{aligned} d\rho &= -i[H,\rho]dt + \kappa\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right)dt \\ &\quad +\gamma\left(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-\right)dt \\ &\quad +i\sqrt{2\kappa}\left(\rho a^\dagger - a\rho - \mathrm{Tr}[\rho\left(a^\dagger - a\right)]\rho\right)dW. \end{aligned} \tag{1.3}$$

It is this equation which we will project onto low-dimensional subspaces to produce reduced-order models (which we will then use as simple filters), and we will also use it to derive stochastic equations of motion for expectation values of system operators.

1.2.2 Filtering

Until now, we have simply discussed the stochastic master equation as a stochastic differential equation, without explicitly stating what we mean by the "state" ρ, or exactly how the noise on the observer's measurement record relates to the Wiener process W driving the noise (dW) terms in the equation. We could, if we chose, continue while only thinking of this equation as the abstract dynamics of some theoretical system. However, it is far more useful to think of the stochastic master equations, Eqns. (1.2) and particularly (1.3), as filtering equations. That is, that the state ρ which we are propagating is the experimenter's state of knowledge of the system. In this context, dW

represents the new information which the experimenter receives through her measurement, and is called the *innovation*. The innovation is defined as the difference between the measurement result and what the observer expected the result to be. In the case of a homodyne measurement of the phase quadrature of the cavity field, for example, the innovation is $measurement - expectation = dY - \frac{i}{2}\left\langle a^\dagger - a \right\rangle$. That this is a Wiener process reflects that the innovation is unbiased — the experimenter learns as much from a measurement result that is larger than her expectation as from a result that is smaller [11].

1.3 The Maxwell-Bloch equations

In my efforts to derive simple classical (or semi-classical) models to approximate quantum systems, the canonical relevant example of such a simplification serves as a constant point of comparison. The Maxwell-Bloch equations are a set of five, coupled, deterministic differential equations for the expectation values of the five simplest operators in this system: a, $a^\dagger$, σ_-, σ_+, and σ_z. The first four will be familiar from the Hamiltonian; the last, σ_z, corresponds to the population difference between the atomic excited and ground states, and allows the system of equations to close without stretching any approximations to the breaking point. These operators may be combined to create the Hermitian operators

$$\begin{aligned} x &= \frac{1}{2}\left(a + a^\dagger\right) \\ y &= \frac{i}{2}\left(a^\dagger - a\right) \\ \sigma_x &= \sigma_+ + \sigma_- \\ \sigma_y &= i\left(\sigma_- - \sigma_+\right). \end{aligned} \tag{1.4}$$

These Hermitian operators correspond to possible observables of the atom-cavity system. x is the cavity field's amplitude quadrature; y is its phase. Using the spin analogy with a two-level atom, the Pauli matrix σ_x is the atomic component along the x axis, and similarly for σ_y and σ_z.

In the density matrix formalism, the expectation value of an operator $\mathcal{O}$ is given by $\mathrm{Tr}\left(\mathcal{O}\rho\right)$. Therefore, the equation of motion of that expectation value is $d\left\langle\mathcal{O}\right\rangle = \mathrm{Tr}\left(\mathcal{O}\left(\mathrm{d}\rho\right)\right)$. If we use the definition of $d\rho$ given by Eqn. (1.3) to calculate the equations of motion of the five operators above,

we derive this set of equations:

$$\begin{aligned}
d\langle a\rangle &= -\kappa(1+i\Theta)\langle a\rangle dt + g_0\langle\sigma_-\rangle dt + \mathcal{E}dt + i\sqrt{2\kappa}\left(\left\langle a(a^\dagger - a)\right\rangle - \left\langle a^\dagger - a\right\rangle\langle a\rangle\right)dW \\
d\left\langle a^\dagger\right\rangle &= -\kappa(1-i\Theta)\left\langle a^\dagger\right\rangle dt + g_0\langle\sigma_+\rangle dt + \mathcal{E}dt + i\sqrt{2\kappa}\left(\left\langle a^\dagger(a^\dagger - a)\right\rangle - \left\langle a^\dagger - a\right\rangle\left\langle a^\dagger\right\rangle\right)dW \\
d\langle\sigma_-\rangle &= -\gamma(1+i\Delta)\langle\sigma_-\rangle dt + g_0\langle a\sigma_z\rangle dt + i\sqrt{2\kappa}\left(\left\langle\sigma_-(a^\dagger - a)\right\rangle - \left\langle a^\dagger - a\right\rangle\langle\sigma_-\rangle\right)dW \\
d\langle\sigma_+\rangle &= -\gamma(1-i\Delta)\langle\sigma_+\rangle dt + g_0\left\langle\sigma_z a^\dagger\right\rangle dt + i\sqrt{2\kappa}\left(\left\langle\sigma_+(a^\dagger - a)\right\rangle - \left\langle a^\dagger - a\right\rangle\langle\sigma_+\rangle\right)dW \\
d\langle\sigma_z\rangle &= -2\gamma(1+\langle\sigma_z\rangle)dt - 2g_0\left\langle a^\dagger\sigma_- + \sigma_+ a\right\rangle dt \\
&\quad + i\sqrt{2\kappa}\left(\left\langle\sigma_z(a^\dagger - a)\right\rangle - \left\langle a^\dagger - a\right\rangle\langle\sigma_z\rangle\right)dW. \qquad (1.5)
\end{aligned}$$

This system of equations, however, is not closed: it includes the expectation values of products of operators. If we make the *ad hoc* approximation that these operator products can simply be factored ($\langle AB\rangle \rightarrow \langle A\rangle\langle B\rangle$), we are effectively neglecting correlations between observables (especially the correlations between the atom and the field), of the sort we might expect to see in quantum-limited behavior. However, this dramatically simplifies the system of equations. In particular, it eliminates all of the stochastic terms:

$$\begin{aligned}
d\langle a\rangle &= -\kappa(1+i\Theta)\langle a\rangle dt + g_0\langle\sigma_-\rangle dt + \mathcal{E}dt \\
d\left\langle a^\dagger\right\rangle &= -\kappa(1-i\Theta)\left\langle a^\dagger\right\rangle dt + g_0\langle\sigma_+\rangle dt + \mathcal{E}dt \\
d\langle\sigma_-\rangle &= -\gamma(1+i\Delta)\langle\sigma_-\rangle dt + g_0\langle a\rangle\langle\sigma_z\rangle dt \\
d\langle\sigma_+\rangle &= -\gamma(1-i\Delta)\langle\sigma_+\rangle dt + g_0\langle\sigma_z\rangle\left\langle a^\dagger\right\rangle dt \\
d\langle\sigma_z\rangle &= -2\gamma(1+\langle\sigma_z\rangle)dt - 2g_0\left(\left\langle a^\dagger\right\rangle\langle\sigma_-\rangle + \langle\sigma_+\rangle\langle a\rangle\right)dt. \qquad (1.6)
\end{aligned}$$

This resulting set of equations is known as the *Maxwell-Bloch equations*, and has its origins in a separate, semi-classical derivation [12]. The Maxwell-Bloch equations are derived by extending the optical Bloch equations (which characterize the excitation and coherence of a two-level atomic medium) to include interaction with a resonant (or near-resonant) coherent optical field, for example in a laser.

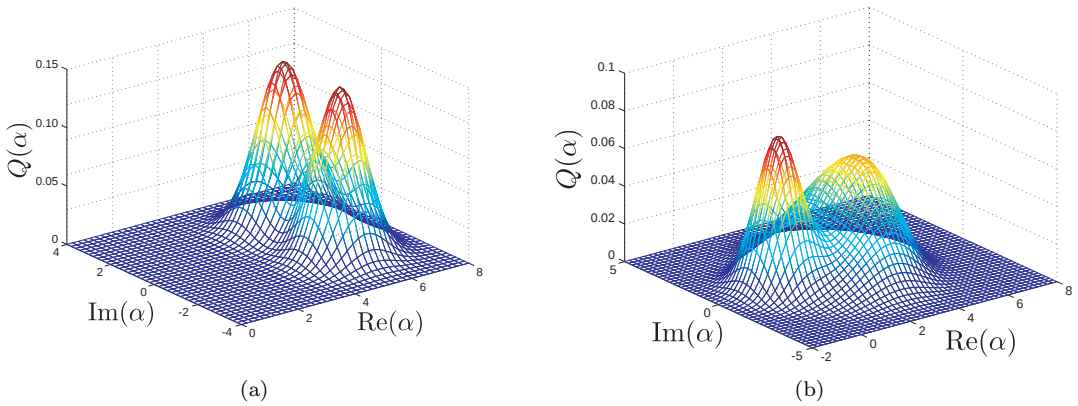
Now that this is simply a closed set of equations for deterministic, classical observables, the full machinery of classical dynamical systems analysis can be brought to bear on understanding the different regimes of dynamics modeled by these equations. The form of the correspondence between the calculated dynamics of the Maxwell-Bloch equations and the behavior of quantum systems with equivalent parameters remains an open question, and provides a significant portion of the motivation for the research described in this thesis.

1.4 Bistability and other interesting dynamics

The quantum dynamical system defined by the master equations (1.2) and (1.3) is known to exhibit several interesting behaviors, some of which correspond with the behavior of the Maxwell-Bloch equations (1.6). Of particular interest for this thesis are two parameter regimes in which the dynamics of both quantum and classical systems are "bistable." In classical, deterministic dynamics, a bistable system is one with two stable (or asymptotically stable) equilibria, and the system settles towards one or the other depending on initial conditions [13]. For the stochastic and quantum cases in this research, I have adapted this term, and assigned it a more phenomenological definition: a system is bistable if its dynamics show it to have two zones of phase space in which it is relatively stable. Noise may drive the system from one stable zone to the other (and back). I will commonly refer to the two zones as states, although they may not correspond to individual quantum states. The zones may roughly correspond to the stable points of a bistable deterministic system, or they may not. A more rigorous definition is likely possible using the techniques and language developed in stochastic dynamical systems theory (for background and foundations, see [14], [15] and [16]), but I have chosen this phenomenological definition for simplicity.

Bistability is useful in the engineering of practical devices, in particular for switching and binary memory. It is intimately related to hysteresis, in which a system prepared through two (or more) different time-varying processes settles into different stable states for the same set of system parameters. Static memory for computing makes use of the bistable behavior of magnetic domains — prepared with strong fields in one direction, they hold that state until actively switched. Most useful bistable systems are relatively noise-free, which contributes to their utility. As performance demands increase, however, devices must become smaller and faster, pushing them into the limit where their behavior is affected by thermal noise, and potentially quantum fluctuations. Current engineered devices are many orders of magnitude more energetic than the relevant quantum limits, but novel technologies for communication and computing (such as those developed for quantum computation and key distribution) may eventually approach it. Stabilizing the two "stable" states of a quantum bistable system in real time would require a model of the underlying quantum dynamics which can be computed alongside the system itself. This thesis is, in part, an attempt to examine potential tools which can be used to make such models, and evaluate their accuracy and utility.

In this thesis, I examine two distinct types of bistability in the cavity QED system: phase bistability and absorptive bistability. In the phase bistable regime, the field mode maintains a fixed amplitude, but switches between two states with opposite phase, with the switching events corresponding to atomic spontaneous emission events [9]. In the absorptive bistable regime, the field switches between a low-amplitude state, very near the vacuum state, and a high-amplitude state. For the parameter regime examined here, the upper state (or, perhaps more accurately,



(a) (b)

Figure 1.1: Q functions for the field modes of two bistable Cavity QED parameter regimes. (a) Phase bistability. (b) Absorptive Bistability.

zone) is much broader than a minimum-uncertainty state, and has significant variation in its phase. The appearance of the system from the observer's standpoint, then, is a system whose absorption changes suddenly between two values, thus "absorptive bistability." The relative ease of measuring field amplitude makes absorptive bistability particularly inviting for use as an optical switch or bit of memory. For both types of bistability, if one does a partial trace over the atomic portion of the density matrix, the Q function of the resulting field state takes the form of a bimodal distribution (Figure 1.1).

Phase bistability has been the subject of numerous publications over the course of the past twenty years. First noted by Alsing and Carmichael [17] and Kilin and Krinitskaya [18], it was further investigated by Mabuchi and Wiseman [9], who simulated the noisy switching behavior in a quantum trajectory formulation using the full stochastic master equation with homodyne detection to measure the phase quadrature directly, and no measurement of the spontaneous emission (following the example of Carmichael and collaborators [19], who modeled the system with homodyne detection of the cavity field and direct photo-detection of the atomic decay). Phase bistability occurs for resonant conditions (driving field, cavity, and atom all resonant), with a large driving field $\mathcal{E}$ relative to the atom-cavity coupling g_0, and g_0 large compared to the atomic spontaneous emission rate. In the limit of no atomic spontaneous emission ($\gamma = 0$), the Maxwell-Bloch equations have two stable points [17]

$$\begin{aligned} \langle a \rangle_{\pm} &= \frac{\mathcal{E} + g_0 \langle \sigma_- \rangle_{\pm}}{\kappa} \\ \langle \sigma_- \rangle_{\pm} &= \frac{-g_0}{4\mathcal{E}} \mp i \left[\frac{1}{4} - \left(\frac{g_0}{4\mathcal{E}} \right)^2 \right]^{1/2} \\ \langle \sigma_z \rangle &= 0, \end{aligned} \tag{1.7}$$

(with appropriate conjugate terms). Extending this insight into the case of quantum states, we see

that for $\mathcal{E} \gg g_0$, the two stable points become orthogonal quantum states. They now correspond to two coherent states of the field, each paired with an atomic state in a superposition of the excited and ground states, but with opposite phases. When we let $\gamma \neq 0$, but still small, the resulting spontaneous emission events correspond to the system switching between the two stable states [19].

Van Handel and Mabuchi [11] examined the phase bistable state from the perspective of projection filtering. They defined a three-dimensional, nonlinear manifold: one dimension measures the relative populations of two gaussian field states (paired with the corresponding atomic state appropriate for their phase), and the other two dimensions reflect the positions of the two gaussians. After an excellent, clear derivation of the filtering equation and rules for projecting it, they project the master equation onto this nonlinear manifold and derive a nonlinear set of stochastic differential equations to use as a simple filter. This filter behaves almost identically to the optimal filter, despite its simplicity, which implies that the underlying dynamics are fundamentally quite simple. If we take the two gaussian field modes to be fixed, the resulting 1-dimensional system is identical to the filter for a stationary Markovian jump process (the Wonham filter). Our analysis of the geometry of this system in Chapter 4 focuses on the geometry of the switching behavior; we should expect that the dynamics we derive will require the inclusion of the state position variables to characterize the transitions.

The Maxwell-Bloch equations may be examined as classical dynamical systems in order to search for regimes with interesting behavior which may correspond to novel behavior in the related quantum system. Gang, Ning, and Haken ([20] and [21]; see [21] for a summary of earlier related, but limited, work by others) undertook a search for these regimes, and Armen and Mabuchi [22] extended this search and analyzed the behavior of quantum systems in several regimes. They examined the absorptive bistability regime (which I use as an example system to evaluate model reduction techniques in this thesis), as well as behavior near both super- and sub-critical Hopf bifurcations, which lead to classical systems exhibiting limit cycles surrounding stable or unstable fixed points. Absorptive bistability may be predicted by semi-classical analysis of a saturable absorber in an optical cavity [23]. Savage and Carmichael [24] were the first to examine the single-atom case using the model described here, and discuss how quantum fluctuations would affect the system.

Absorptive bistability occurs for a range of parameters, generally with quite low driving field amplitude. Similar behavior with a non-resonant driving field is referred to as "dispersive bistability." However, it is simplest to take all the detunings to be zero, and my calculations do so. The semiclassical system behavior then depends on the driving field strength and the dimensionless cooperativity. With $C = 10$, for example, the Maxwell-Bloch equations have two stable, and one unstable, solutions over a range of driving field strength; see Figure 1.2. The two stable states for a given driving field exhibit vastly different cavity field amplitude. Notably, the states are defined by the absolute value of their field amplitude, meaning that we do not gain much insight into the

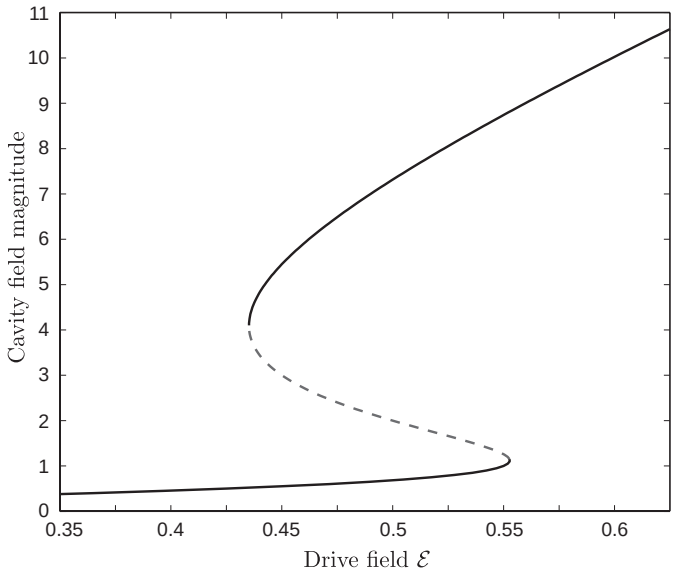


Figure 1.2: Semiclassical intracavity steady state field magnitude as a function of drive field $\mathcal{E}$, from [22]. The dashed portion indicates an unstable equilibrium. The parameter values are: $g_0 = \sqrt{2}$, $\gamma = 2$, $\kappa = 0.1$, $\Theta = 0$, and $\Delta = 0$ (with $\gamma_\perp = 1$ setting the scaling of time). The point used to examine absorptive bistability in this thesis has $\mathcal{E} = 0.56$.

relative quadrature (phase vs. amplitude) behavior. As [22] demonstrates, absorptive bistability parameters correspond to a quantum system which exhibits a bimodal Q function for the cavity field (see Figure 1.1b), and whose quantum trajectory simulations exhibit stochastic switching behavior.

In contrast to the phase bistable regime, however, these two "states" may not be not superficially identical. For the parameter regime of interest in this research, the lower state's cavity field is very similar to a slightly displaced vacuum, while the upper state is much broader than a minimum-uncertainty coherent state, and may be better characterized as a semi-stable zone rather than a particular stable state. This shape is not universal. In the regime studied in [24], the two states are more similar in shape: the higher-amplitude portion of the Q function is only somewhat broader than the lower, and it lacks the spread in the phase quadrature that is pronounced with our parameters. For a fixed cooperativity, the shape of the Q function depends on the cavity decay rate, with more sharply peaked states in the good cavity case. Our trajectory simulations show a phase quadrature that stays very near zero in the lower state, and fluctuates dramatically in the upper state, corresponding to a pure state (as required by the quantum trajectory simulation) moving around significantly within the broad upper region defining the upper state. Making sense of this behavior, and what makes the upper state appear stable while not being a single state, motivates the examination of absorptive bistability throughout this thesis. While I find no global answer, I hope that the insights gained through attempts at model reduction will enable future work to better

understand this system.

1.5 An aside on stochastics

I will now give a brief overview of some results from stochastic calculus, to lay the groundwork for their use later in the thesis. These notes skim only the very surface of the field; I recommend Gardiner [25] and van Kampen [26] for more information. In stochastic calculus, we commonly have a Wiener process (usually denoted W), which is a continuous-time stochastic process which can be thought of as the integral of white noise. Rigorously, the differential form of writing stochastic equations of motion is nothing but a shorthand for the integral form. In stochastic differential equations, however, we write a stochastic increment dW, which is like a stochastic dt. We generally choose to scale it so that the mean value of dW^2 is dt, or "$dW \sim \sqrt{dt}$."

In classical, deterministic analysis, the integral can be defined as the sum of intervals, and in the limit of infinitesimal intervals, it does not matter whether we have chosen at each interval to use the function value at the start, middle, or end of the interval. In defining the stochastic integral, however, it does matter. We are forced to make a choice regarding which side of the infinitesimal interval to choose when summing, as they produce different results. The most common choices are those made by Itô and Stratonovich. Itô chose the start of each interval, while Stratonovich chose the midpoint. A given stochastic differential equation, therefore, must be accompanied by information about which kind of SDE it is. I follow the standard practice of writing $\circ dW$ for Stratonovich, and simply dW for Itô. Almost every equation in this thesis is in Itô form. There are many integrators designed for stochastic systems (see [27] for a summary), but I have restricted my work to a simple Itô-Euler integrator. (This is simply an Euler integrator with a stochastic term added.) When numerically integrating an SDE using this method, you must use the Itô form of the equation, because you are implicitly choosing the value at the start of each time interval for your integration. Other integrators use the Stratonovich form, or (like the Milstein integrator) add different correction terms.

Let us assume we have an Itô stochastic differential equation

$$dR(x,t) = A(x,t)dt + B(x,t)dW. \tag{1.8}$$

In order to transform this to the equivalent Stratonovich SDE, we must *subtract* the "Itô correction term"

$$\frac{1}{2}\left(\mathbf{D}B(x,t)\right)B(x,t)dt \tag{1.9}$$

where $\mathbf{D}$ indicates the derivative. (If Eqn. (1.8) is a linear equation such as $d\mathbf{v} = A\mathbf{v}dt + B\mathbf{v}dW$, the derivative is simply the matrix B, so you may commonly see the correction term written with

B^2.) Similarly, if we have a Stratonovich SDE

$$dR(x,t) = A(x,t)dt + B(x,t) \circ dW, \tag{1.10}$$

we *add* the Itô correction term (1.9) to transform it into Itô form.

The other critical difference between Itô and Stratonovich forms, for the purposes of this thesis, is how they respond to a transformation of coordinates. Projecting equations of motion, as I will do in Chapters 2 and 3, is such a transformation of coordinates. Under a transformation $\bar{x} = \phi(x)$, the coordinates of (1.8) transform into

$$\begin{aligned} \bar{A}(\bar{x}) &= A(x)\frac{d\phi}{dx} + \frac{1}{2}\left(B(x,t)\right)^2 \frac{d^2\phi}{dx^2} \\ \bar{B}(\bar{x}) &= B(x)\frac{d\phi}{dx}. \end{aligned} \tag{1.11}$$

In contrast, the coordinates in the Stratonovich form transform like vectors (with no second-order derivatives):

$$\begin{aligned} \bar{A}(\bar{x}) &= A(x)\frac{d\phi}{dx} \\ \bar{B}(\bar{x}) &= B(x)\frac{d\phi}{dx}. \end{aligned} \tag{1.12}$$

Projection is a geometric process, so we need to be able to treat the stochastic increments in our equations of motion as vectors. Therefore, when we project equations of motion they will be Stratonovich equations [11].