SPONTANEOUS BREAKDOWN OF CONFORMAL AND CHIRAL INVARIANCE

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ABSTRACT

We discuss the implications of a theory in which scale and chiral invariance are spontaneously broken, and the dilaton σ appears as a mixture of the two isoscalar members of the scalar nonet. From standard assumptions for the conformal properties of the axial-vector current, we predict the ratio of the $\sigma\pi\pi$ and σNN coupling constants, and test the hypothesis of f-dominance. At first, we neglect the effects of scale violation. Then, the calculated width of the dilaton, Γ_σ, appears to exceed the limit given by the Adler-Weisberger sum rule for $\pi\pi$ scattering, while f-dominance seems to work. Using the method of collinear dispersion relations, we estimate scale-breaking effects, which are found to be large. In the real world, our result for Γ_σ agrees with other expectations, both experimental and theoretical. However, the spin-2 gravitational radius of the pion is found to be double the prediction of f-dominance. This is consistent with experimental indications that f-dominance fails. We discuss meson-baryon scattering and its relation to parameters measuring the breaking of chiral symmetry in the energy density. Our interpretation of a recent result of Cheng and Dashen is that scale invariance is spontaneously broken, and chiral SU(2) x SU(2) is a much better summetry than SU(3). By requiring consistency with a sum rule of von Hippel and Kim, we find that data for πN scattering are not consistent with the dimension of the chiral-violating scalar being -1.

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I. HISTORY AND PHILOSOPHY OF BROKEN SCALE INVARIANCE

The investigation of scale and conformal transformations and the associated quantum number, dimension, began in 1910. However, only in the last two or three years has this subject received widespread attention. After noting the main historical developments, we discuss the modern theory of broken scale invariance, which forms an important extension of current algebra. The chapter concludes with some introductory remarks about our investigation and its relation to other aspects of the theory. Elementary technical remarks which complement the main text can be found in Appendix A.

I. 1. Rôle of the Stress-Energy Tensor

There is an elementary rule, known to all students of Physics, which states that all equations should balance dimensionally. Adopting the natural units $\hbar = 1$, $c = 1$, the only independent dimensional unit is that of length, L; it has the same dimensions as time and mass. The Schrödinger wave function $\Psi(x)$ has the dimensional character $L^{-3/2}$, since $\int_V d^3x \Psi^* \Psi$ is a probability. In the Schrödinger equation for a hydrogen atom,

$$-\left(\frac{\nabla^2}{2m} + \frac{e^2}{|\vec{x}|}\right) \Psi(t, \vec{x}) = i \frac{\partial}{\partial t} \Psi(t, \vec{x}) , \qquad (1.1)$$

both sides of the equation have the dimensional quality $L^{-5/2}$.

One must not confuse these elementary remarks with the more sophisticated meaning that we shall attach to the term, "dimension".

We shall investigate the hypothesis that the world of strongly interacting particles* is approximately invariant under scale transformations

$$x_\mu \to x'_\mu = \rho x_\mu \quad , \quad (\rho > 0), \tag{1.2}$$

and the associated special conformal transformations

$$x_\mu \to x'_\mu = (x_\mu - c_\mu x^2)/(1 - 2c\cdot x + c^2 x^2) \quad . \tag{1.3}$$

These transformations correspond to unitary transformations $U(\rho, c)$ on the vector space formed by solutions of the equations of motion under consideration. Dimension is a property of linear operators on this space.

Taking Eq. (1.1) as an example, linear operators on the vector space formed by the functions $\Psi(t, \vec{x})$ are constructed from $\vec{x}$, t, $\partial/\partial\vec{x}$, $\partial/\partial t$, and constants. The symbol for mass, m, actually represents the linear operator mI, where I is the unit operator. So, although m is a quantity of the type L^{-1} in length units, it is assigned dimension zero, according to our use of the term. The dimension of an operator is measured by its dependence on $\vec{x}$, t, $\partial/\partial\vec{x}$, $\partial/\partial t$. Thus, Eq. (1.1) is not invariant under the transformations $U(\rho, c)$, because $\nabla^2/2m$ has dimension -2, while $e^2/|\vec{x}|$ has dimension -1.

*Electromagnetic, weak and gravitational interactions are treated in the lowest order of perturbation theory.

The group of conformal transformations on Minkowski space consists of Poincaré transformations, (i.e., 3-rotations, boosts, and translations), and the transformations given by Eqs. (1.2) and (1.3). Conformal transformations were first considered shortly after Einstein formulated the Special Theory of Relativity in 1905. Ignoring gravity, relativity requires the equations for closed physical systems to retain their form under Poincaré transformations. It was gradually realized that the formulation of a relativistic theory of gravity would involve the consideration of all coordinate transformations. This prompted the discovery of Bateman[1] and Cunningham[1] that the largest group of space-time transformations which leaves Maxwell's equations invariant is the conformal group. This property results from the absence of dimensional constants in the theory--photons have zero mass and couple to the dimensionless quantity, charge.

The observation of Bateman and Cunningham means that Maxwell's equations have the same form in uniformly accelerated frames as in inertial frames[2]. By substituting $x_\mu = (t, 0, 0, 0)$ and $c_\mu = (0, 0, 0, -\frac{1}{2}a)$ in Eq. (1.3), we obtain $x'_3 = \frac{1}{2} a \tau'^2$ in terms of the proper time $\tau' = \sqrt{x'^2}$ in the primed frame. This corresponds to uniform acceleration a in the x'_3 - direction, i.e., hyperbolic motion $(x'_3 + a^{-1})^2 - x'^2_o = a^{-2}$ in x'-space.

The theory of broken conformal symmetry must not be confused with other extensions of these considerations, such as "Conformal Relativity"[3]. In conformal relativity, one imagines that masses transform like $\partial/\partial x$, so the equations of motion for free massive particles, such as the Dirac equation $(i\not\partial - m)\Psi = 0$, are invariant. Such transformations may be worth consideration in classical physics. However, in quantum mechanics, each value of the mass corresponds to a different vector space, so Conformal Relativity cannot be used as a symmetry theory for the vector space which describes the states of systems of elementary particles.

It should also be noted that the conformal transformations to be discussed here are not general-relativistic transformations. From the point of view required by "General Relativity", the effect of gravity would have to be included in all equations such that covariance under all coordinate transformations on general-relativistic space is achieved. The proper time $d\tau = \sqrt{g_{\mu\nu}(x)\, dx^{\mu} dx^{\nu}}$ is invariant under such transformations. These properties are not shared by the theory of broken conformal symmetry. The group of conformal transformations on Minkowski space is not a symmetry of the world, although it is a symmetry of theories of massless free particles such as the photon. In Eqs. (1.2) and (1.3), x_{μ} is a 4-vector in Minkowski space, so an element of proper time $d\tau$ is given by $\sqrt{dx_0^2 - d\vec{x}^2}$. In terms of small increments d_1x, d_2x at x which become d_1x', d_2x' at x', Eqs. (1.2) and (1.3) imply

$$d_1x' \cdot d_2x' = \rho^2 \, d_1x \cdot d_2x \ , \tag{1.4a}$$

$$d_1x' \cdot d_2x' = (1 - 2c\cdot x + c^2x^2)^{-2} \, d_1x \cdot d_2x \ , \tag{1.4b}$$

respectively. Thus $d\tau$ is not invariant under scale or special conformal transformations. On the other hand, angles are locally preserved, since $d_1x\cdot d_2x/(d_1x^2\ d_2x^2)^{\frac{1}{2}}$ is an invariant. This property accounts for the name "conformal".

For our purposes, gravitational effects are regarded as external, which means that gravitational fields are not operators on the vector space of particle states. The influence of gravity is measured by the local stress-energy tensor operator $\theta_{\mu\nu}(x)$, in first-order perturbation theory for the gravitational coupling. The coupling of an external field $\delta g_{\mu\nu}(x)$ to a system of elementary particles is given by the action

$$\delta A_{1,2} = \frac{1}{2}\int_{\sigma_1}^{\sigma_2} d^4x\ \theta^{\mu\nu}\ \delta g_{\mu\nu}\ , \tag{1.5}$$

where σ_1, σ_2 are space-like surfaces. This assumes that the linearized approximation of the relativistic theory of gravity remains valid at microscopic distances. Poincaré invariance is ensured by the conservation laws

$$\partial^\mu \theta_{\mu\nu} = 0 \quad , \tag{1.6}$$

$$\partial^\lambda \mathcal{M}_{\lambda\mu\nu} = \theta_{\mu\nu} - \theta_{\nu\mu} = 0 \quad , \tag{1.7}$$

where

$$\mathcal{M}_{\lambda\mu\nu} = x_\mu\,\theta_{\lambda\nu} - x_\nu\,\theta_{\lambda\mu} \tag{1.8}$$

is an angular momentum density, and the generators of Poincaré transformations are

$$P_\mu = \int d^3x\, \theta_{o\mu} , \tag{1.9}$$

$$M_{\mu\nu} = \int d^3x\, \mathcal{M}_{o\mu\nu} . \tag{1.10}$$

Elementary properties of matrix elements of $\theta_{\mu\nu}$, and the corresponding canonical Lagrangian formalism are discussed in Appendix A. Many textbooks on introductory field theory mention a non-fundamental quantity called the canonical energy-momentum tensor, $T_{\mu\nu} \neq T_{\nu\mu}$. When written in terms of $T_{\mu\nu}$, an angular momentum density contains a model-dependent term which is interpreted as a spin angular momentum density. Details of the relation[4] between $T_{\mu\nu}$ and $\theta_{\mu\nu}$ may also be found in Appendix A.

In 1921, not long after Noether[5] observed that a conservation law is implied by each invariance of the action integral, Bessel-Hagen[6] obtained the conservation laws which are required by conformal invariance of Maxwell's equations, (apart from Eqs. (1.6) and (1.7), of course). Invariance under conformal transformations involves "dilation" and "special conformal" currents:

$$\mathcal{D}_\mu(x) = x^\nu \theta_{\mu\nu} , \tag{1.11}$$

$$\mathcal{K}_{\mu\nu}(x) = (2x_\mu x^\lambda - \delta_\mu^\lambda x^2)\, \theta_{\lambda\nu} . \tag{1.12}$$

Bessel-Hagen's laws for Maxwell's theory are

$$\partial^{\mu} \mathscr{D}_{\mu} = 0 \quad , \tag{1.13}$$

$$\partial^{\nu} \mathscr{K}_{\mu\nu} = 0 \quad , \tag{1.14}$$

i.e., the trace $\theta_{\mu}^{\ \mu}$ of the stress-energy tensor for photons vanishes.

Until a few years ago, work on the conformal group was confined to extensions of the discoveries of Bateman, Cunningham, and Bessel-Hagen to other theories for free particles.(7) The only surprise was the observation of McLennan and Havas(7) that the usual canonical theory for massless spin-0 mesons cannot be formulated in exactly the same fashion. According to the textbooks,(8) the Lagrangian density $\frac{1}{2}(\partial\phi)^2$ leads to the expression $\partial_{\mu}\phi\ \partial_{\nu}\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2$ for the stress-energy tensor, which is not traceless. Following the "dictates" of canonical variational theory for the Lagrangian density, they included extra terms in $\mathscr{D}_{\mu}$ and $\mathscr{K}_{\mu\nu}$, so the direct connection between the vanishing of $\theta_{\mu}^{\ \mu}$ and scale invariance was lost. This approach was being followed as late as 1969.(9)

In 1962, Huggins(10) pointed out that the standard formula for $\theta_{\mu\nu}$ derived from a Lagrangian density arbitrarily disallows terms like $(\partial_{\mu}\partial_{\nu} - g_{\mu\nu}\ \partial^2)\phi^2$. The addition of such a term does not affect the expressions (1.9) and (1.10) for the Poincaré generators or the conservation equations (1.6) and (1.7). With a suitable choice of coefficient for the new term, an expression for $\theta_{\mu\nu}$ which satisfies Bessel-Hagen's conservation laws is obtained:

$$\theta_{\mu\nu} = \partial_{\mu}\phi\ \partial_{\nu}\phi - \frac{1}{2}\, g_{\mu\nu}(\partial\phi)^2 - \frac{1}{6}(\partial_{\mu}\partial_{\nu} - g_{\mu\nu}\ \partial^2)\phi^2 \quad . \tag{1.15}$$

Then we have $\theta_{\mu}^{\ \mu} = 0$ for the zero-mass spin-0 field ϕ. This "new, improved" stress-energy tensor was first proposed by Gürsey[11] in a different context. It was resurrected by Callan, Coleman and Jackiw (CCJ)[12] and Brown and Gell-Mann[13] because of the connection with conformal transformations. CCJ showed that, for renormalizable field theories, the new improved $\theta_{\mu\nu}$ has finite matrix elements when renormalized,* unlike the old $\theta_{\mu\nu}$.

The construction of the generators of dilations and special conformal transformations was initially tackled by Wess,[14] using the old $\theta_{\mu\nu}$. Given the new, improved $\theta_{\mu\nu}$ we can now write down the dilation operator

$$D(x_o) = \int d^3x\, x^{\mu}\, \theta_{o\mu} = \int d^3x\, \mathcal{D}_o \quad , \tag{1.16}$$

and the special conformal operators

$$K_{\mu}(x_o) = \int d^3x\, (2x_{\mu} x^{\lambda}\theta_{o\lambda} - x^2\theta_{o\mu}) = \int d^3x\, \mathcal{K}_{\mu o} \tag{1.17}$$

Eqs. (1.16) and (1.17) can be understood from another point of view.[15, 16] Instead of applying the infinitesimal transformations represented by $\delta\rho$ and δc_{μ}, we can impose the potentials

$$\delta g_{\mu\nu} = -2\delta\rho\; g_{\mu\nu}\,, \quad (\text{for } x_{\mu} \to x'_{\mu} = (1+\delta\rho)\, x_{\mu}) \;, \tag{1.18}$$

$$\delta g_{\mu\nu} = -4\delta c.x\; g_{\mu\nu}\,, (\text{for } x_{\mu} \to x'_{\mu} = x_{\mu} - \delta c_{\mu} x^2 + 2\,\delta c.x\, x_{\mu}) \;, \tag{1.19}$$

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* Strictly, the extra term should be written $(\partial_{\mu}\partial_{\nu} - g_{\mu\nu}\partial^2)(\phi + \epsilon)^2$, where ϵ is an infinite constant cancelling $\langle\phi\rangle_o$, which is also infinite. The finite constant $\epsilon' = \epsilon - \langle\phi\rangle_o$ is not determined by the proof of renormalizability.

with a corresponding change $\delta A_{1,2}$ in the action of the elementary particle system given by Eq. (1.5). We identify $\delta A_{1,2}$ with the change in action caused by the non-conservation of the generators G for transformations represented by the parameters $\delta\alpha$ according to the action principle[17]

$$\delta A_{1,2} = (G(1) - G(2))\,\delta\alpha \tag{1.20}$$

Therefore, taking $\delta\alpha = \delta\rho,\ \delta c_\mu$, we have

$$D(2)-D(1) = \int_{\sigma_1}^{\sigma_2} d^4x\ \theta_\mu^{\ \mu} \tag{1.21}$$

$$K_\mu(2)-K_\mu(1) = \int_{\sigma_1}^{\sigma_2} d^4x\ 2x_\mu\ \theta_\nu^{\ \nu} \tag{1.22}$$

(For example, the change in coordinates represented by $\delta c_\mu = (0,0,0,\frac{1}{2}\delta a)$ corresponds to uniform acceleration δa in the z-direction. Equivalently, one could suppose that a constant force is acting. The corresponding potential, $\frac{1}{2}\delta g_{oo} = -\delta az$, follows directly from Eq. (1.19)).

The integrands of Eqs. (1.21) and (1.22) may be written

$$\partial^\mu \mathscr{D}_\mu = \theta_\mu^{\ \mu} \quad , \qquad \partial^\nu \mathcal{K}_{\mu\nu} = 2x_\mu\ \theta_\nu^{\ \nu} \quad , \tag{1.23}$$

so, apart from additional conserved operators, Eqs. (1.21) and (1.22) imply Eqs. (1.16) and (1.17). The conserved operators are eliminated by requiring a consistent free field theory at $t = \pm\infty$.

In the limit of scale invariance, $\dot{D} = \int d^3x\ \theta_\mu^{\ \mu} = 0$, we can actually infer $\theta_\mu^{\ \mu} = 0$ because of the theorem (Appendix A) which states that $\int d^3x\ s(x) = 0$ implies $s(x) = 0$ if $s(x)$ is a local, spin-0 operator.[13] Therefore, scale invariance implies conformal invariance,

$\dot{K}_\mu = 0$.

The scale transformation (1.2) corresponds to a transformation

$$U(\rho) = \exp[i\, D(x_o) \log \rho] \tag{1.24}$$

on the vector space of particle states. The dimension ℓ of a (suitably chosen) field $\phi(x)$ is specified by

$$\phi(x) \to \phi'(x') = U(\rho)\, \phi(x)\, U(\rho)^{-1} = \rho^{-\ell} \phi(\rho x) \quad . \tag{1.25}$$

The infinitesimal form of Eq. (1.25) is

$$i[D(x_o), \phi(x)] = (-\ell + x\cdot\partial)\phi(x) \quad . \tag{1.26}$$

Because of the canonical commutation relations

$$[\phi(0,\vec{x}), \ \partial_o\phi(0,\vec{0})] = i\delta^3(\vec{x}) \quad , \tag{1.27a}$$

$$[\Psi(0,\vec{x}), \ \bar{\Psi}(0,\vec{0})]_+ = \gamma_o\delta^3(\vec{x}) \quad , \tag{1.27b}$$

free fermion fields Ψ have dimension $-3/2$, while free boson fields ϕ have dimension -1. To obtain the latter result, it is necessary to explicitly check that $\partial_o\phi$ has dimension -2, since the time dependence of $D(x_o)$ does not always permit the orders of operation of $D(x_o)$ and ∂_o to be interchanged. Further details are given in Appendix A.

In general, special conformal transformations with finite c_μ do not preserve the sign of $(x-y)^2$, where x and y are any two points in Minkowski space.[14] Therefore, care is needed when attempting to interpret the corresponding unitary transformations because of trouble with the causality condition

$$[A(x), B(y)]_- = 0 \quad , (\text{for } (x-y)^2 \text{ spacelike}), \tag{1.28}$$

where $A(x)$ and $B(y)$ are dynamical observables such as currents. However, infinitesimal transformations δc_μ do not display this unfortunate behavior, so causality places no restrictions on the use of equal-time commutation relations of the form(9)

$$i[K_\mu(x_o), \phi(x)] = \varkappa_\mu \phi - 2x^\alpha(\ell g_{\alpha\mu} + i\Sigma_{\alpha\mu})\phi + (2x_\mu x^\lambda - \delta_\mu^\lambda x^2)\partial_\lambda \phi \ , \tag{1.29}$$

where the nilpotent matrices $\varkappa_\mu$ characterize the conformal representation to which ϕ belongs, and $\Sigma_{\mu\nu}$ is the spin matrix of ϕ:

$$i[M_{\mu\nu}, \phi(x)] = (x_\mu\partial_\nu - x_\nu\partial_\lambda)\,\phi(x) - i\Sigma_{\mu\nu}\phi(x). \tag{1.30}$$

I. 2. Early Applications

The first attempt to connect the conformal group with experiments in particle physics was made by Kastrup. In a long series of papers,(18) he tried to apply the idea(14) that masses should be unimportant at high energies. In field-theoretic terms, the conformal-invariant kinetic energy terms are supposed to dominate the non-invariant mass terms in the Lagrangian. Then, treating the conformal group as an approximate degeneracy symmetry of the world, he deduced that high-energy amplitudes should be roughly "conformal invariant" in momentum space. Let us summarize the evolution of this point of view.

Evidently, belief in this reasoning leads to the expectation that high-energy amplitudes depend only on dimensionless functions of the momenta. For example, if $A(s,t)$ is the amplitude for the scattering of two spinless particles,*

$$A(s,t) = f(s/t) \quad , \qquad (s, t \text{ large},) \tag{1.31}$$

is obtained, implying that A is energy-independent at large scattering angles.

In order to derive Eq. (1.31), it is essential to assume that the limit in which all masses vanish is smoothly connected to the real world. The validity of this assumption will be questioned later. The zero-mass limit is the limit of scale invariance, $\langle \theta_\mu^{\ \mu} \rangle = 0$, if no scalar mesons are coupled to the vacuum via $\theta_{\mu\nu}$.** The amplitude $A_{\mu\nu} = \langle p_3, p_4 | \theta_{\mu,\nu} | p_1, p_2 \rangle$ can be expanded in powers of $k = p_3 + p_4 - p_1 - p_2$ according to Low's method for bremsstrahlung.(19) The $0(k^{-1})$ terms, $A_{\mu\nu}^{Born}$, are represented by Feynman graphs in which $\theta_{\mu\nu}$ hooks on to external lines. The non-singular term, $A_{\mu\nu}^{contact}$, given by

$$A_{\mu\nu} = A_{\mu\nu}^{Born} + A_{\mu\nu}^{contact} + 0(k^2) \quad , \tag{1.32}$$

is determined by the conservation laws (1.6) and (1.7), which may be written

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* For $p_1 + p_2 \to p_3 + p_4$, we define $s = (p_1 + p_2)^2$, $t = (p_3 - p_1)^2$.

** Discussion of this alternative begins in Section I. 3.

$$k^{\mu} A_{\mu\nu} = 0 \quad , \qquad A_{\mu\nu} = A_{\nu\mu} \quad , \qquad (1.33)$$

in momentum space. The condition of scale invariance, $A^{\mu}_{\mu} = 0$, requires

$$\left(s \frac{\partial}{\partial s} + t \frac{\partial}{\partial t} \right) A(s, t) = 0 \qquad (1.34)$$

which is equivalent to Eq. (1.31).

Kastrup further speculated that the large s behavior given by Regge pole theory at small t results from approximate conformal invariance. The t-dependence of amplitudes was assumed to be completely changed by the effects of finite masses, since Regge behavior is obviously not compatible with the t-channel dependence of Eq. (1.31).

The problem with this idea is summarized by the fact that Regge behavior was originally observed[20] in solutions of the Schrödinger equation, which severely breaks scale invariance. The situation is less clear in relativistic field theory. Because of renormalization, amplitudes contain logarithms as well as powers of the momenta and masses, so it is a question of whether the leading term at large s contains log s or not.

Now, it has been known for a long time[21] that there is a connection between renormalizability and the dimension of an interaction.* The Lagrangian density of a renormalizable field theory contains interaction terms with dimension $\ell \geq -4$. Examples are

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* The situation for nonpolynomial Lagrangians is reviewed in Ref. 22. Study of the corresponding amplitudes at high energy is just beginning.

$\lambda\varphi^3$ and $e A^{\mu} J_{\mu}$, which have dimensions -3 and -4 respectively.*
For superrenormalizable theories, (i.e., both the bare and physical coupling constants are finite), the dimension is greater than -4.

In early investigations of large-s behavior, classes of diagrams were summed by specifying the kernel of a Bethe-Salpeter equation. Superrenormalizable $\lambda\varphi^3$ theory was found to give Regge behavior, whereas similar examination of theories in which only mass terms break scale invariance gave $s^{\alpha} \log^{\gamma} s$ at large s,[23] so it seemed that the controlling factor is the manner in which scale invariance is broken. However, the present status of the subject is that some of the diagrams previously thrown away become unexpectedly important at large s when summed. In a rigorous study of perturbation theory for quantum electrodynamics, Cheng and Wu[24] have obtained logarithmic behavior at large s, (contra Kastrup's suggestion). In $\lambda\varphi^3$ theory, a power law in s is obtained only if λ is less than a critical value.[25] Since this power law is merely due to the Born term dominating at large s, we doubt that these attempts to connect Reggeism with broken scale invariance will prove profitable. Alternative schemes have been proposed, but the status of these suggestions is shrouded in controversy.[26]

Renormalization involves the introduction of a cutoff mass M which breaks scale invariance. Since the limit $M \to \infty$ is taken at some stage in the calculation, it is not surprising that scale invariance, unlike gauge invariance, cannot be preserved by this procedure.[27]

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* We refer to the unrenormalized dimension in this paragraph.

Barring the presence of arbitrary cutoffs, the logarithmic terms blow up in the limit of zero mass. Part of the infinity corresponds to the emission of an infinite number of massless particles accompanying each external particle which was well-defined in the massive theory. This phenomenon is called the "infrared problem". It causes single-particle states and the S-matirx to be ill-defined in this limit,* and destroys the belief that all high-energy amplitudes should be approximately conformal invariant in momentum space.[28] Broken conformal invariance cannot be treated as an approximate degeneracy symmetry for on-mass-shell amplitudes.

In particular, Eq. (1.31) must be abandoned. For example, infrared contributions dominate the radiative corrections to electromagnetic scattering processes at high energies and large momentum transfers.[29] It might be supposed that Eq. (1.31) could be saved by "going off-mass-shell". By avoiding particle states and considering vacuum-expectation values of T-products, the zero-mass limit can be discussed because there are no emitted particles which can initiate the infinite bremsstrahlung. However, when the zero-mass limit is applied to the vacuum-expectation value representing an off-mass-shell extension of $\langle p_3, p_4 | \theta_{\mu\nu} | p_1, p_2 \rangle_{massive}$, the analytic behavior p_i^2 about the on-mass-shell point $p_i^2 = 0$ is destroyed.[30] Once again, logarithmic terms appear, this time in

*There is one uninteresting exception, which is obtained by supposing that all interactions can be turned off before the masses vanish. Even then, the zero-mass limit need not be smooth--e.g., vector fields have mass singularities.

the form log (p_i^2/p_j^2). They invalidate the derivation of Eq. (1.31), because they do not scale.

In 1966, Kastrup[31] recognized that his earlier work could not provide a good description of high-energy processes because of the infrared problem. Therefore, he considered amplitudes in which soft mesons are emitted in addition to the two primary particles being scattered. In the CM frame, the differential cross section for the emission of N mesons was written

$$d\sigma^N/d\Omega = P_N(E,\theta)\, d\sigma^{TOT}/d\Omega \quad , \qquad (1.35)$$

at large energies and scattering angles E, θ of the primary particles. The probability distribution P_N of the soft mesons was assumed to be of the Poisson type:

$$P_N(E,\theta) = \overline{N}(E,\theta)^N\, e^{-\overline{N}(E,\theta)}/N! \quad , \qquad (1.36)$$

where $\overline{N}(E,\theta)$ is the mean multiplicity of the secondary mesons at (E,θ). By interpreting the total ("inclusive") cross section, $d\sigma^{TOT}/d\Omega$, as the "non-infrared" contribution to $d\sigma^N/d\Omega$ in Eq. (1.35), with all long-distance infrared effects incorporated in P_N, Kastrup speculated that it would mainly depend on short-distance interactions, which were supposed to be conformal invariant:

$$d\sigma^{TOT}/d\Omega = |f(\theta)|^2/E^2 \quad , \quad (E,\theta \text{ large}) \; . \qquad (1.37)$$

This intuition was guided by examples from electromagnetism,[29] with one important difference. Vacuum polarization causes the short-distance behavior of quantum electrodynamics to be modified by logarithmic terms. Thus Coulomb's Law becomes[32]

$$V(r) = \frac{q'q}{4\pi r}\left[1 - \frac{2\alpha}{3\pi}(\log mr + 5/6 + \log \nu)\right] + 0(\alpha^2) ,$$

$$(\nu = 1.781\ldots) , \qquad (1.38)$$

at distances r much smaller than the Compton wavelength of the electron. Therefore, logarithmic dependence is not necessarily due to infrared effects. For the purposes of Kastrup's argument, it must be supposed that strong interactions do not contaminate the leading r-singularity with logarithms.

Eq. (1.36) is a separate proposal which corresponds to independent emission of soft mesons. However, most of these soft mesons are pions, which carry isospin, so their emission is constrained by the requirement that the production of exotic states be damped. Therefore, it is not clear that P_N should be given by a Poisson distribution,[33] although experiments[34] are not inconsistent with this possibility provided that P_0 is treated as the probability for non-diffractive elastic scattering.

Scaling laws like Eq. (1.37) have become the subject of intensive investigation recently. The processes involved are typically of the form

$$A + B \to C + \text{anything} \quad , \tag{1.39}$$

where B is a hadron and all energy variables are large. The observation[(35)] of scaling laws for deep inelastic electroproduction (A = C = electron, B = proton in Eq. (1.39),) together with Bjorken's explanation[(36)] of the phenomenon, were followed by a large number of papers in which the scaling laws are "proved" in various models or from various assumptions and approximations. We refer to review articles[(37)] for comparisons of these methods, and restrict our attention to the connection with broken scale invariance.*

I.3. Modern Formulation of Broken Scale Invariance

A few years ago, Mack[(39)] and Wilson[(40)] replaced these qualitative observations with concrete proposals which provide a natural extension of current algebra, broken symmetry, and the corresponding set of low-energy theorems. These proposals, together with the work of Brown and Gell-Mann[(13)] on the manner in which conformal symmetry is violated, form the basis of our present-day understanding of the subject.

The central quantities in current algebra[(41)] are the octets of vector and axial-vector current densities $\mathfrak{F}_\mu^a$ and $\mathfrak{F}_{5\mu}^a$ $(a = 1 \ldots 8)$, which arise in electromagnetic and weak interactions of hadrons. In

*In view of the previous discussion, analyses in which the conformal group is treated as an approximate degeneracy symmetry in momentum space cannot be taken seriously. These papers are listed and criticized in the review articles by Carruthers (Ref. 38) and Wilson (Ref. 28).

first-order perturbation theory for the electromagnetic and weak coupling constants, hadronic amplitudes are proportional to matrix elements of the electromagnetic current density

$$J_\mu(x) = \mathfrak{F}^3_\mu + \frac{1}{\sqrt{3}} \mathfrak{F}^8_\mu \tag{1.40}$$

and the weak current density

$$J^W_\mu(x) = (\mathfrak{F}^{1+i2}_\mu + \mathfrak{F}^{1+i2}_{5\mu}) \cos\theta + (\mathfrak{F}^{4+i5}_\mu + \mathfrak{F}^{4+i5}_{5\mu}) \sin\theta \tag{1.41}$$

respectively, where $\theta \approx 15^\circ$ is the Cabibbo angle.[42]

The time-components of the current densities may be integrated to form the sixteen charges

$$\begin{aligned} F^a(x_o) &= \int d^3x \; \mathfrak{F}^a_o(x) \quad , \\ F^a_5(x_o) &= \int d^3x \; \mathfrak{F}^a_{50}(x) \quad . \end{aligned} \tag{1.42}$$

The lack of conservation of most of these charges is indicated by dependence on the time x_o. Only isospin $\vec{T} = (F^1, F^2, F^3)$ and hypercharge $Y = \frac{2}{\sqrt{3}} F^8$ are conserved by strong interactions. The group SU(3), which is used to classify particle states according to the "Eightfold Way",[43] is generated by the octet of charges, F^a.

The basic postulate of current algebra[41] requires that the SU(3) generators obey the SU(3) algebra even when they are not conserved, and further, that this property extend to an SU(3) x SU(3) algebra involving all sixteen charges:

$$\left[F^a(x_o), F^b(x_o)\right] = i f^{abc} F^c(x_o) ,$$
$$\left[F^a(x_o), F^b_5(x_o)\right] = i f^{abc} F^c_5(x_o) , \qquad (1.43)$$
$$\left[F^a_5(x_o), F^b_5(x_o)\right] = i f^{abc} F^c(x_o) .$$

The coefficients f^{abc} are the structure constants of SU(3). Applications of Eq. (1.43) have been reviewed by Adler and Dashen[44] and Renner.[45]

No such postulate is possible for the conformal group.[13] Equal-time commutation relations such as[46]

$$i\left[P_o, D(x_o)\right] = \dot{D}(x_o) - P_o , \qquad (1.44)$$

$$i\left[M_{io}, D(x_o)\right] = \int d^3x \, x_i \, \theta^\mu_\mu , \qquad (1.45)$$

are required by the known properties of $\theta_{\mu\nu}$ under Poincaré transformations, so the exact conformal algebra can be satisfied only in the limit of exact conformal symmetry, $\theta^\mu_\mu \to 0$. A complete account of the broken conformal algebra is given in Appendix B.

Equations such as (1.45) show that $D(x_o)$ and $K_\mu(x_o)$ do not have the Lorentz behavior indicated by indices such as μ because they are not conserved. The definitions (1.16) and (1.17) of these generators involve the plane of integration, x_o = constant. Taking $x_o = 0$ for simplicity, boosting corresponds to rotation of this plane about $x = 0$ towards the light cone, so the generator changes by an amount which can be determined by Gauss's theorem. In general, if $\phi(x)$ is a local operator with known behavior under Lorentz transformations, the operators

$$d_\phi(x) = i\,[D(x_o) - x\cdot P,\ \phi(x)] \quad , \tag{1.46}$$

$$k_\phi^\mu(x) = i\left[K^\mu(x_o) - 2x^\mu D(x_o) + 2x_\alpha M^{\alpha\mu} + 2x^\mu x\cdot P - x^2 P^\mu,\ \phi(x)\right], \tag{1.47}$$

are also local, but may have obscure properties under boost transformations. Similar considerations were involved in the formulation of the local generalizations

$$\left[F^a(x_o),\ \mathfrak{F}_\mu^b(x)\right] = i\, f^{abc}\, \mathfrak{F}_\mu^c(x) \quad , \text{ etc.}, \tag{1.48}$$

of Eq. (1.43). Corresponding to the fact that the non-conserved charges defined in Eq. (1.42) are not scalar or pseudoscalar operators, the assumption that the equal-time commutator in Eq. (1.48) is a vector or pseudovector implies the conditions[44, 47]

$$\left[\int d^3x\, x_i\, \partial^\mu \mathfrak{F}_\mu^a(0,\vec{x})\ ,\ \mathfrak{F}_\nu^b(0)\right] = 0 \ , \quad \text{etc.} \tag{1.49}$$

Similarly, if the gradient terms in

$$d_{\phi}(x) = -\ell\,\phi(x) + \text{grad. terms} \tag{1.50}$$

are supposed to be absent, the condition

$$\left[\int d^3x\, x_i\, \theta^{\mu}_{\mu}(0,\vec{x}) \;,\; \phi(0) \right] = 0 \tag{1.51}$$

is obtained. The example $\phi = \mathfrak{F}_{\nu}$ has appeared in the literature.[48]

The nature of the breakdown of a symmetry can be specified by giving the symmetry properties of the appropriate current divergences. In Gell-Mann's theory of broken chiral symmetry,[41] the energy density is written

$$\theta_{oo} = \overline{\theta}_{oo} - u_o - c\, u_8 \;, \tag{1.52}$$

where $\overline{\theta}_{oo}$ is invariant with respect to chiral SU(3) x SU(3), and u_o and u_8 belong to a set of scalar densities u_b and pseudoscalar densities v_b, (b = 0 ... 8), which form a $(3,\overline{3}) + (\overline{3},3)$ representation of SU(3) x SU(3):

$$\begin{aligned}
\left[F^a(x_o),\, u^b(x) \right] &= i\, f^{abc}\, u^c(x) \;, \\
\left[F^a(x_o),\, v^b(x) \right] &= i\, f^{abc}\, v^c(x) \;, \\
\left[F^a_5(x_o),\, u^b(x) \right] &= -i\, d^{abc}\, v^c(x) \;, \\
\left[F^a_5(x_o),\, v^b(x) \right] &= i\, d^{abc}\, u^c(x) \;,
\end{aligned} \tag{1.53}$$

where the Clebsch-Gordan coefficients d^{abc} are symmetric in $a = 1 \ldots 8$, $b,c = 0 \ldots 8$, with $d^{0bc} = \sqrt{2/3}\,\delta_{bc}$. The assumption that the equal-time commutators in Eq. (1.53) are spin-0 operators implies the same property for $[\theta_{oo}(x), F^a_{(5)}(x_o)]$, which can therefore be specified by integrating over d^3x and using the theorem of Appendix A for spin-0 operators:(13, 41)

$$\begin{aligned} i\left[\theta_{oo}(x),\ F^a(x_o)\right] &= \partial^\mu \mathfrak{F}^a_\mu(x) \quad , \\ i\left[\theta_{oo}(x),\ F^a_5(x_o)\right] &= \partial^\mu \mathfrak{F}^a_{5\mu}(x) \quad . \end{aligned} \tag{1.54}$$

The SU(3) x SU(3) properties of the current divergences follow by writing $\partial\mathfrak{F}$ as a linear combination of the u's or v's, using Eqs. (1.51), (1.52), and (1.53); e.g.

$$\partial^\mu \mathfrak{F}^a_{5\mu} = \frac{\sqrt{2} + c}{\sqrt{3}}\, v^a \quad , \quad (a = 1, 2, 3). \tag{1.55}$$

Thus, the deviation of c from $-\sqrt{2}$ measures SU(2) x SU(2) violation.(49, 50) Gell-Mann, Oakes, and Renner(49) obtained the value $c \simeq -1.25$.

The feature of this theory which influences our work is the absence of a term

$$S_{\mu\nu} = (\partial_\mu \partial_\nu - g_{\mu\nu}\,\partial^2)\, s(x) \tag{1.56}$$

in $\theta_{\mu\nu}$, where $s(x)$ is not invariant under SU(3) x SU(3) or gauge transformations. The existence of $S_{\mu\nu}$ would imply that Eq. (1.54) is invalid, and even our basic assumption

$$\left[F_5(0)\,,\ \dot{D}(0)\right] = 0 \tag{1.57}$$

would no longer be true; (i.e., $\mathfrak{F}_{5\lambda}$ would have mixed dimension). This point is discussed at length in Chapter II.

The breakdown of scale invariance has been similarly treated.(13,40) The energy density is decomposed into the scale-invariant term, $\bar{\bar{\theta}}_{oo}$, and scale-violating terms w_n:

$$\theta_{oo} = \bar{\bar{\theta}}_{oo} + \sum_n w_n \quad , \tag{1.58}$$

with $\dim \bar{\bar{\theta}}_{oo} = -4$, and $\dim w_n = \ell_n \neq -4$. Eqs. (1.44) and (1.58) imply

$$\dot{D}(x_o) = \int d^3x\ \theta^{\mu}_{\mu} = \sum_n \int d^3x\ (\ell_n + 4)\ w_n\ . \tag{1.59}$$

Therefore, the assumption that $\sum_n (\ell_n + 4)\ w_n$ is a scalar density is a necessary and sufficient condition for the theorem(13,16)

$$\theta_{\mu}^{\ \mu} = \sum_n (\ell_n + 4)\ w_n \tag{1.60}$$

to hold.

Eq. (1.60) is often called the "Virial Theorem" because it resembles the well-known theorem of classical mechanics.(51) The virial, $\lambda = q \cdot p$, where (q, p) are canonical momenta, is the classical analogue of the dilation operator $D(0) = \int d^3x\, x^i \theta_{0i}(0,\vec{x}) \sim -i\vec{x}\cdot\vec{\nabla}$. Denoting time-averaging by $\langle\ \rangle$, $\langle \dot{\lambda} \rangle$ usually vanishes, which implies that the Hamiltonian $H = H(q,p)$ satisfies $\langle p\cdot \frac{\partial}{\partial p} H\rangle = \langle q\cdot\frac{\partial}{\partial q} H\rangle$. For $H = p^2/2m + V(\vec{r})$, with $V = \sum_\ell V_\ell$, (V_ℓ = homogeneous function of degree ℓ in $\vec{r}$), the classical virial theorem is

$$\langle \vec{p}\cdot\dot{\vec{r}}\rangle = 2\langle T\rangle = \langle \vec{r}\cdot\vec{\nabla}V\rangle = \sum_\ell \ell\, \langle V_\ell\rangle \quad , \tag{1.61}$$

where T is the kinetic energy.

The majority of the many successful applications of current algebra depend on the hypothesis that the axial-vector current is partially conserved (PCAC).(52) According to this hypothesis, $\langle \partial^\mu \mathfrak{J}_{5\mu}\rangle$ satisfies an unsubtracted dispersion relation in the square of the momentum transfer, t ; near $t = 0$, the dispersion integral is dominated by the pole of the appropriate 0^- meson, (π, K, or η). For example the relevant matrix elements for nucleons are

$$\langle N(P + \tfrac{1}{2}k)\, |\mathfrak{J}_{5\lambda}^{a}|\, N(P - \tfrac{1}{2}k)\rangle$$

$$= \bar{u}(P + \tfrac{1}{2}k)\ \tfrac{1}{2}\tau^a \left[-i\,\gamma_\lambda\gamma_5 F_A(k^2) - i k_\lambda \gamma_5 F_p(k^2)\right] u(P - \tfrac{1}{2}k) \ , \tag{1.62}$$

and

$$\langle N(P + \tfrac{1}{2}k)\, |\partial^\lambda \mathfrak{J}_{5\lambda}^{a}|N(P - \tfrac{1}{2}k)\rangle = \bar{u}(P + \tfrac{1}{2}k)\ \tfrac{1}{2}\tau^a \gamma_5\, u(P - \tfrac{1}{2}k)\, D_N(k^2) \quad , \tag{1.63}$$

with $a = 1,2,3$ and

$$D_N(t) = 2M_N F_A(t) + t F_P(t)$$

$$= \frac{m_\pi^2 \, g_{\pi NN}}{f_\pi (m_\pi^2 - t)} + \frac{1}{\pi} \int_{(3m_\pi)^2}^{\infty} \frac{dt' \, \mathrm{Im}\, D(t')}{(t' - t)} \quad . \tag{1.64}$$

The pion decay constant, f_π, is defined by

$$\langle \pi^o(q) | \mathcal{F}^3_{5\lambda} | 0 \rangle = -i \, q_\lambda / 2f_\pi \quad , \tag{1.65}$$

$\sqrt{2}\, g_{\pi NN}$ is the $pn\pi^+$ coupling constant, and $F_A(t)$, $F_P(t)$ are the axial and induced pseudoscalar form factors of the nucleon. When the continuum integral is neglected at $t = 0$, Eq. (1.64) implies the Goldberger-Treiman relation(53)

$$f_\pi \simeq g_{\pi NN} / (2M_N \, g_A) \quad , \tag{1.66}$$

where $g_A = F_A(0) \simeq 1.24$ is the ratio of the axial-vector to vector coupling constants in neutron β-decay. Similar relations may be deduced from K and η PCAC. Adopting the no-subtraction hypothesis, the distance of the pole from $t = 0$, (i. e., $m^2(0^-, \underline{8})$), is a rough measure of the accuracy of these relations.

It is possible to regard the success of formulae like Eq. (1.66) as an indication of approximate chiral invariance of the Hamiltonian, but not of the vacuum, so that parity-doubled multiplets do not appear; i. e., chiral invariance is said to be "spontaneously broken".(54) For example, in the limit of chiral

SU(2) x SU(2) symmetry, $\partial^\mu \mathfrak{F}^a_{5\mu}$ vanishes for a = 1, 2, 3, and Eq. (1.64) becomes

$$F_P(t) = -2M_N F_A(t)/t \quad . \tag{1.67}$$

Either M_N vanishes, or there is a pole at t = 0 due to the presence of a massless pion. The latter alternative is much more attractive, since $m_\pi^2 \to 0$ closely approximates the real world, while parity-doubling would be expected in the other case. Then, by evaluating the residue of the pion pole in $F_P(t)$, we recover Eq. (1.66) as an exact relation. Since matrix elements of $\partial^\mu \mathfrak{F}_{5\mu}$ contain the factor $m_\pi^2/(m_\pi^2 - t)$, the limit $\partial^\mu \mathfrak{F}_{5\mu} \to 0$ is non-uniform in t ; e.g.,

$$\lim_{\partial^\mu \mathfrak{F}_{5\mu} \to 0} \lim_{t \to 0} D_N(t) = 2M_N g_A \quad . \tag{1.68}$$

The non-invariance of the vacuum arises through Eq. (1.65). When the vacuum is chirally transformed, soft mesons are added to form a new state which, in the limit of chiral symmetry, could also be called a vacuum. Thus there is an infinitely degenerate set of vacua in this limit. Only one of them corresponds to the unique vacuum state of the real world. It is distinguished by the requirement[55]

$$F^a|0\rangle = 0 \quad , (\approx 0 \text{ in the real world}). \tag{1.69}$$

Eq. (1.69) summarizes the observation that particle multiplets may be classified according to the SU(3) group generated by F^a, a=1. . . 8, and not the group $\overline{SU(3)}$ generated by* $\overline{F^a} = \exp(i\beta F_5^8)\, F^a \exp(-i\beta F_5^8)$, for example. The violation of physical SU(3) is not spontaneous.

An infinitesimal chiral transformation is accompanied by the emission or absorption of a finite number of soft mesons. Thus, if $\phi(x)$ is a local operator, and the limits $q_m \to 0$ are taken correctly,** the soft-meson formula (e.g., for pions)[44,45]

$$\lim_{\substack{q_m \to 0 \\ m=1\ldots n}} \langle B, \pi(q_1), \pi(q_2), \ldots\ldots, \pi(q_j)\, |\phi(0)|\, \pi(q_{j+1}), \ldots\ldots, \pi(q_n), A \rangle$$

$$= (-2if_\pi)^n \langle B\,|\, [F_5, [F_5, \ldots\ldots [F_5, \phi(0)]\ldots\ldots]]\,|\, A\rangle \tag{1.70}$$

corresponds to the α^n term in a power-series expansion of $e^{i\alpha F_5}\, \phi\, e^{-i\alpha F_5}$. In the real world, Eq. (1.70) becomes an approximate relation which can be obtained by pole-dominance methods. Sum rules may be obtained by supposing that the n-pion amplitude obeys an unsubtracted dispersion relation in a suitable variable.

From the previous discussion, it would appear that m_π^2 and m_K^2 indicate the magnitude of SU(2) x SU(2) and SU(3) x SU(3) violation, respectively. However, this conclusion is not

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* In the special case $\beta = \pi\sqrt{3}$, we have Kuo's transformation.[56] See Ref. 55 for comments on Kuo's work.

** For example, in πN scattering, the limit of zero pion energy should be applied to the forward amplitude.

automatic--it might be an accident that m_π is small. The test of this idea is the relative accuracy of the predictions of pion and kaon PCAC. This is a controversial point at present.(57, 58) The main issue is whether the data for $K_{\ell 3}$ decay(59) imply a large violation of the soft-pion prediction.(60) This involves contradictory experimental data and a theoretical extrapolation of order $m_K m_\pi$, not $0(m_\pi^2)$. In our view, the experimental data are not good enough to indicate the correct theoretical extrapolation to the soft-pion point. Since the other soft-pion theorems are in excellent agreement with experiment*, we assume that m_π^2 measures SU(2) x SU(2) violation.

Since SU(3) is a degeneracy symmetry, the variational principle(61)**

$$\frac{\partial}{\partial\lambda} \langle\langle \psi(\lambda) | \theta_{oo}(\lambda) | \psi(\lambda) \rangle\rangle = \langle\langle \psi(\lambda) | \partial\theta_{oo}(\lambda)/\partial\lambda | \psi(\lambda) \rangle\rangle \qquad (1.71)$$

may be applied to $\theta_{oo}(\lambda) = \overline{\theta}_{oo} - u_o - c\lambda u_8$ to obtain

$$\langle \pi | c u_8 | \pi \rangle = \tfrac{1}{2}(m_\eta^2 - m_\pi^2)(1 + 0(\epsilon)) \quad , \qquad (1.72)$$

where $\epsilon \simeq 0.2$ is a parameter indicated the inaccuracy ($\sim 20\%$) of SU(3)-symmetric results. Applying Eq. (1.70) with $n = 1$, the left-hand side of Eq. (1.72) may be evaluated, implying the result of

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* We disagree with the claim of Brandt and Preparata(57) that the failure of naive soft-pion theorems for electromagnetic reactions such as $\pi^0 \to 2\gamma$, $\eta \to 3\pi$ is connected with a large violation of SU(2) x SU(2) invariance. The contents of this paragraph are more fully discussed in Chapter III.

** The states $|\psi(\lambda)\rangle\rangle$ are normalized to one particle per unit volume. States $|\psi\rangle$ are normalized invariantly. See Appendix A.

Gell-Mann, Oakes and Renner:[(49)*]

$$c = -\sqrt{2} + 2\sqrt{2}\, m_\pi^2/m_\eta^2 + 0(m_\pi^4/m_\eta^4\,,\; m_\pi^2\,\epsilon/m_\eta^2) \simeq -1.25 \;. \tag{1.73}$$

The manner in which conformal invariance is realized may involve similar considerations. Mack[(39)] observed that the failure of the argument leading to Eq. (1.31) can be traced to the failure of $A_\mu^\mu(k=0) = \langle p_3, p_4 | \theta_\mu^\mu | p_1, p_2 \rangle$ to vanish in the limit of conformal invariance. Therefore, the limit $\theta_\mu^\mu \to 0$ is non-uniform in momentum transfer squared, the vacuum is not invariant under conformal transformations, and A_μ^μ has to be responsible for the soft-particle emission which forced Kastrup to adopt Eq. (1.37) instead of Eq. (1.31). In the zero-mass limit, this corresponds to infinite pair creation by the conformal generators when the vacuum is conformally transformed. In that case, it is difficult to construct a rule governing the soft-particle emission.

Mack proposed a completely different and much simpler mechanism for the soft-particle emission. He supposed that matrix elements of $\partial^\mu \mathcal{D}_\mu = \theta_\mu^\mu$ at low frequencies are, to a good approximation, given by a nearby pole in the θ_μ^μ channel which is due to a low-lying isoscalar S-wave $\pi\pi$ resonance which we call the dilaton, σ.

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*Equation (1.73) does not depend on the value of ξ, defined by $\langle \pi | u | \pi \rangle = \xi m_\pi^2$. PCAC gives $\xi = 0(1)$, but does not determine the actual value of ξ. We thank Professor K. Wilson for pointing this out.

This assumption requires $\langle B | \theta^{\mu}_{\mu} | A \rangle$ to be universally proportional to the amplitude for $A \rightarrow B + \sigma(\text{soft})$, i.e., dilatons couple to mass. The dilation current is said to be partially conserved (PCDC hypothesis).

The relation between PCDC and conformal symmetry is analogous to that between PCAC and chiral symmetry.[13] In the limit of conformal invariance, the dilaton becomes massless and, in general, matrix elements of $\theta_{\mu\nu}$ have a dilaton pole at zero momentum transfer. The presence of this pole allows heavy particles such as the baryons to remain massive in this limit. The action integral becomes conformal-invariant, but the presence of the massless Nambu-Goldstone boson σ is responsible for the non-invariance of the vacuum:

$$\langle \sigma(k) | \theta_{\mu\nu} | 0 \rangle = - \frac{1}{3} F_{\sigma} (k_{\mu} k_{\nu} - g_{\mu\nu} k^2) \quad . \tag{1.74}$$

Imposition of the condition $\theta^{\mu}_{\mu} \rightarrow 0$ results in exact relations for soft-dilaton amplitudes. The universal constant of proportionality, F_{σ}, which is given in the units of a mass, is analogous to the PCAC proportionality constant, $(2f_{\pi})^{-1}$. In order that the scale-invariant relations for soft-dilation amplitudes remain approximately true as scale invariance is broken, the low-mass dilaton state is assumed to dominate an unsubtracted dispersion relation for $\langle \theta^{\mu}_{\mu} \rangle$ at small values of the momentum transfer. That is the PCDC hypothesis. Conformal invariance is said to be spontaneously broken.

Some aspects of Mack's proposal (1967) were anticipated several years earlier. In 1960, Gell-Mann and Levy[52] noticed that, in the σ-model, "the σ-coupling is responsible for the nucleon mass." With the benefit of hindsight, we can now say that this occurs because the σ-model obeys the PCDC hypothesis in the special case $(2f_\pi)^{-1} = -F_\sigma$. In 1962, Gell-Mann[62] suggested that matrix elements of θ^μ_μ obey unsubtracted dispersion relations which can be dominated by scalar mesons. However, the connection with scale and conformal transformations was not mentioned.

The simplest scale-invariant calculation is the derivation of the σMM coupling constant, $G_{\sigma MM}$, where M is a spin-0 meson. The form-factor expansion

$$\langle M(P + \tfrac{1}{2}k) | \theta_{\mu\nu} | M(P - \tfrac{1}{2}k) | \rangle = 2P_\mu P_\nu H_1(k^2) + (k_\mu k_\nu - g_{\mu\nu}k^2) H_2(k^2) \tag{1.75}$$

is required by the conservation laws of Poincaré invariance, Eqs. (1.6) and (1.7). Eq. (1.9) implies $H_1(0) = 0$. In the limit of scale invariance, $\theta^\mu_\mu \to 0$, Eq. (1.75) becomes

$$H_2(k^2) = (2m_M^2 - \tfrac{1}{2}k^2)\, H_1(k^2)/3k^2 \quad . \tag{1.76}$$

Since a σ-pole is permitted to appear at $k^2 = 0$ in $H_2(k^2)$, m_M need not vanish. Use of Eq. (1.74) to evaluate the residue of this pole implies the scale-invariant result[39, 62, 63, 64]

$$F_\sigma G_{\sigma MM} = 2m_M^2 \quad . \tag{1.77}$$

Because the normalization of fermion states differs from that of boson states by a factor $(2 \cdot \text{mass})^{-1}$, the corresponding result for the dilaton-baryon coupling $\bar{u}'u\, g_{\sigma BB}$ is*

$$F_\sigma\, g_{\sigma BB} = M_B \tag{1.78}$$

We expect m_σ^2 to give a rough indication of inaccuracies in these scale-invariant results, just as $m^2(0-,\underline{8})$ measures the effects of the violation of chiral invariance. The estimation of corrections due to the lack of scale invariance of the real world is a major aim of this investigation.

There remains one aspect of the discussion in Section I. 2 which needs a quantitative formulation--Kastrup's idea that strongly interacting systems are approximately scale-invariant at short distances. The appropriate formalism was constructed by Wilson(40) two years ago. He considered operator-product expansions of the form

$$A(x + y/2)\, B(x - y/2) = \sum_n C_n(y)\, O_n(x) \quad , \tag{1.79}$$

where $A(x)$ and $B(x)$ are local operators with dimensions ℓ_A, ℓ_B. The sum $\sum_n$ is countably infinite, $\{O_n(x)\}$ is a set of local operators (including the identity), and $C_n(y)$ are c-number functions of the 4-vector y_μ. Wilson considered short-distance expansions only; i. e., he required all components of y to be small. Examples of

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* If needed, some elementary details which supplement this discussion can be found in Appendix A.

Eq. (1.79) arose[65] in studies of renormalization in perturbation theory, and in the problem of constructing a field theory for composite particles. Wilson postulated that Eq. (1.79) is a general property of local relativistic quantum mechanics. It is obviously true for free field theories, and is also valid for renormalized interacting fields $A(x)$, $B(x)$ to all orders in perturbation theory.[66] Therefore, in its most general form, Eq. (1.79) constitutes a very weak assumption.*

The strong assumption is that, to zeroth order in scale violation, only a limited number of operators $0_n(x)$ of dimensions ℓ_n appear, where the leading singularities of the corresponding $C_n(y)$'s are homogeneous functions of the form $y^{\ell_A + \ell_B - \ell_n}$; i.e., the leading singularities are determined by a scale-invariance argument. The exponent of y is simply obtained by inspection, or by commuting both sides of Eq. (1.79) with the dilation operator $D(x_o)$, with $x_o = y_o$, $\vec{x} \neq \vec{y}$. Logarithmic functions of y, which are due to scale-breaking effects, are assumed to be less singular, so the scale-invariant terms dominate at short distances y. This means that $D(x_o)$ is a slowly varying function of time.

This point of view is difficult for a field theorist to accept. In renormalizable field theories, (i.e., dim $\mathcal{L}_{int} = -4$ to first order in perturbation theory), the leading singularity of C_n is logarithmically more singular than the corresponding (scale-invariant) singularity for free fields, due to the effects of vacuum polarization. In particular, renormalized quantum electrodynamics is not scale-invariant at short distances; (see Eq. (1.38)). Non-renormalizable theories (dim $\mathcal{L}_{int} < -4$), necessarily treated in lowest order, increase the singularity by a factor y^{-1}. Only superrenormalizable field theories (dim $\mathcal{L}_{int} > -4$) are scale-invariant at short distances.[67] In conventional field theory,* the only known example is the unrealistic $\lambda\phi^3$ interaction.

However, the success of current algebra favors Wilson's ideas. In practice, Eq. (1.79) is applicable only when the fields A(x) and B(x) are the local operators from which current algebra is constructed: the vector and axial-vector currents $\mathfrak{F}^a_\mu(x)$ and $\mathfrak{F}^a_{5\mu}(x)$, the stress-energy tensor $\theta_{\mu\nu}$, and derivatives such as $\partial^\mu \mathfrak{F}^a_{5\mu}(x)$ and $\partial_\mu \mathfrak{F}_\nu - \partial_\nu \mathfrak{F}_\mu$. According to Wilson's hypotheses, the term containing currents in the operator-product expansion of the unequal-time commutator of two currents is given by

$$\tfrac{1}{2}\pi^2 \left[\mathfrak{F}^a_\mu(x),\ \mathfrak{F}^b_\nu(0)\right] = (\beta d^{abc} + \gamma \delta^{ab}\delta^{co})\, E^{(-2)}(x^2)\, \epsilon_{\mu\nu}{}^{\lambda\eta} x_\lambda \mathfrak{F}^c_{5\eta}(0)$$

$$+ f^{abc}\left\{\alpha(g_{\mu\nu}x^\eta - \delta^\eta_\nu x_\mu - \delta^\eta_\mu x_\nu)E^{(-2)}(x^2) + (1+4\alpha)x_\mu x_\nu x^\eta E^{(-3)}(x^2)\right\}\mathfrak{F}^c_\eta(0)$$

$$+ \ldots\ldots, \qquad (1.80)$$

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*This discussion is restricted to polynomial interactions in 4-dimensional Minkowski space. Further examples of superrenormalizable polynomial interactions exist if one takes the liberty of reducing the number of spatial dimensions. Of much greater interest is the existence of many 4-dimensional nonpolynomial Lagrangians which are superrenormalizable, and do not appear to have the diseases of the $\lambda\phi^3$ interaction. To our knowledge, no work has been done on the short-distance behavior of such theories. See Ref. 22.

with similar expressions for products involving an axial-vector current.[68] The c-number functions $E^{(-n)}$ contain powers of x^2, and vanish for spacelike x^2:

$$E^{(-n)}(x^2) = (-x^2 + i\epsilon x_o)^{-n} - (-x^2 - i\epsilon x_o)^{-n}$$

$$= -2\pi i\, \epsilon(x_o)\, \delta^{(n-1)}(x^2)/(n-1)! \quad , \quad (n = \text{integer} > 0). \qquad (1.81)$$

The constants α, β, γ are not determined by current algebra; the limit $x_o \to 0$ of Eq. (1.80) yields

$$\left[\mathcal{F}_o^a(0,\vec{x})\,,\, \mathcal{F}_o^b(0,\vec{0})\right] = i\, f^{abc}\, \mathcal{F}_o^c\, \delta^3(\vec{x})\,, \text{ etc.}, \qquad (1.82)$$

which is the completely local version of Eqs. (1.43) and (1.48). Thus operator-product expansions at short distances are generalizations of current algebra.

If logarithms were present in Eq. (1.80), they would destroy the equal-time commutators (1.82)--for small times x_o, the coefficient of $\delta^3(\vec{x})$ would contain the quantity $\log m x_o$, which diverges when the equal-time limit is taken. In order to obtain the singularity structure of Eq. (1.80), the scale-violating terms in the energy density must have dimension $\ell_n > -4$. Also, according to the basic hypothesis of current algebra, Eq. (1.82) does not depend on the magnitude of the violation of SU(3) x SU(3) symmetry. Therefore, the term u in θ_{oo} which breaks SU(3) x SU(3) invariace must also break scale invariance, and have a dimension $\ell_u > -4$.[40] Thus the decompositions (1.52) and (1.58) of θ_{oo} may be combined to give

$$\theta_{oo} = \bar{\bar{\theta}}_{oo} + \delta + u \quad , \tag{1.83}$$

where $\bar{\bar{\theta}}_{oo}$ is both scale- and chiral- invariant, and δ breaks scale invariance but not chiral invariance. Also, we note that if there are any scale-violating pieces in the currents, they have dimension $\rangle$-3.

Some understanding of the nature of Schwinger terms (S. T.)[69] can be gained if the hypothesis of scale invariance at short distances is accepted. For currents, we have the example

$$\left[\mathfrak{F}^a_\mu(x),\ \mathfrak{F}^b_\nu(0)\right] = (\partial_\mu \partial_\nu - g_{\mu\nu}\partial^2)\ E^{(-2)}(x^2)\ \delta^{ab}\ \xi\ I + \ldots. , \quad . \tag{1.84}$$

Assuming that the constant ξ does not vanish, there is a finite third-order S. T. $-i\pi^2 \xi \nabla_i \vec{\nabla}^2 \delta^3(\vec{x})$ in $[\mathfrak{F}_o(0,\vec{x}),\ \mathfrak{F}_i(0,\vec{0})]$. The term originally discovered by Schwinger is infinite:

$$1^{st}\text{-order S. T. in } [\mathfrak{F}_o(0,\vec{x}),\ \mathfrak{F}_i(0,\vec{0})] = -\lim_{x_o \to 0} \pi^2 i \xi \nabla_i \delta^3(\vec{x})/x_o^{\ 2}. \tag{1.85}$$

Contrary to the usual lore, there is a second-order S. T. in $[\mathfrak{F}_o(0,\vec{x}),\ \mathfrak{F}_o(0,\vec{0})]$, but practical calculations are not affected because it is a c-number. Commutators involving the stress-energy tensor may be similarly analyzed:

$$\begin{aligned}\left[\ \theta_{\mu\nu}(x),\ \theta_{\alpha\beta}(0)\right] = \Big\{ & 4\partial_\mu \partial_\nu \partial_\alpha \partial_\beta + 2(g_{\mu\nu}\partial_\alpha\partial_\beta + g_{\alpha\beta}\partial_\mu\partial_\nu - g_{\mu\nu}g_{\alpha\beta}\partial^2)\partial^2 \\ & -3(g_{\mu\alpha}\partial_\nu\partial_\beta + g_{\mu\beta}\partial_\nu\partial_\alpha + g_{\nu\alpha}\partial_\mu\partial_\beta + g_{\nu\beta}\partial_\alpha\partial_\mu - g_{\mu\alpha}\partial_{\nu\beta}\partial^2 - g_{\nu\alpha}g_{\mu\beta}\partial^2)\partial^2 \Big\} . \\ & E^{(-2)}(x^2)\ \zeta I + \ldots. \quad .\end{aligned} \tag{1.86}$$

If the constant ζ is not zero, a fifth-order S. T. $4\pi^2 i\zeta\nabla_i(\vec{\nabla}^2)^2\delta^3(\vec{x})$ appears in $[\theta_{oo}(0,\vec{x}),\ \theta_{oi}(0,\vec{0})]$.

A consequence of scale invariance at short distances is "scaling" of the total cross section $\sigma_{TOT}(q^2)$ for $e^+e^- \rightarrow$ anything at large, time-like values of q^2 (with q = sum of e^+, e^- momenta):

$$\sigma_{TOT}(q^2) = -16\pi^2\alpha^2/3q^4 \int d^4x\, e^{iq\cdot x} \langle 0\,|J^\mu(x)\, J_\mu(0)|\, 0\rangle$$

$$\sim -32\pi^4\alpha^2\,\xi/q^2 \sim -12\pi^3\,\xi\,\sigma(e^+e^-\rightarrow\mu^+\mu^-). \qquad (1.87)$$

The q^2-dependence of Eq. (1.87) was first obtained by Bjorken[70] from different considerations. The connection with scale invariance was noted by Wilson.[28] In the event that ξ vanishes, the dominating term has the q^2-dependence $(q^2)^{-5-\ell}$ at high energies, where ℓ is the minimum dimension of the scale-violating terms in the stress-energy tensor.[28]*

On the other hand, many extra assumptions are needed in order to obtain the scaling laws for deep inelastic electroproduction. The relevant matrix element is**

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* This analysis neglects two-photon exchange, which may involve sufficient dynamical enhancement to dominate the one-photon contribution, even though it is formally suppressed by a factor α^2. For example, suppose that the two-photon term scales, as if dominated by the c-number part of $J_\alpha(x)\, J_\beta(y)\, J_\nu(z)\, J_\delta(0)$ for $x\approx y\approx z\approx 0$; (in general, other values of x, y, z are as important). For $\xi = 0$, $\ell = -2$, the two photon term begins to dominate at energies greater than $\sim\sqrt{137}$ GeV, because the cross section is then given by

$$\sigma_{TOT}(q^2) = C_1\,\alpha^2(1\text{ GeV})^4/(q^2)^3 + C_2\,\alpha^4/q^2,\ (C_1, C_2 = \text{constants}).$$

** Momentum transfer q = initial minus final electron momenta.

$$W_{\alpha\beta}(q,p) = \frac{1}{2\pi} \int d^4x\, e^{iq\cdot x} \langle p | [J_\alpha(x), J_\beta(0)] | p \rangle$$

$$= (q_\alpha q_\beta / q^2 - g_{\alpha\beta}) W_1(q^2, \nu)$$

$$+ M^{-2}(p_\alpha - q_\alpha p\cdot q/q^2)(p_\beta - q_\beta p\cdot q/q^2) W_2(q^2, \nu) , \quad (1.88)$$

with $M\nu = q\cdot p$, and nucleon spins averaged. In terms of the ingoing and outgoing energies E, E' and scattering angle θ of the electron in the laboratory frame, the total inelastic scattering cross section from an unpolarized nucleon is[71]

$$\frac{d^2\sigma}{dE'd\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \left[2W_1(q^2, \nu)\sin^2(\theta/2) + W_2(q^2, \nu) \cos^2(\theta/2) \right] . \quad (1.89)$$

Brandt[72] and Ioffe[72] have shown that, in the deep inelastic limit considered by Bjorken[36] ($-q^2, \nu \to \infty$ with $\omega = -q^2/q\cdot p$ fixed), points near the light cone are emphasized. This follows from the LAB frame parametrization

$$p = (M, 0, 0, 0) \quad , \quad q = \nu(1, 0, 0, \sqrt{1 + M\omega/\nu}) . \quad (1.90)$$

The factor $e^{iq\cdot x}$ in Eq. (1.88) oscillates rapidly except for the region $|x_0 - x_3| \lesssim \nu^{-1}$, i.e., near the light cone $x^2 = 0$. Therefore, it is necessary to extend Wilson's hypotheses to the light cone[73] if a connection between the scaling laws and broken scale invariance is sought. In order to obtain Bjorken's result that W_1, νW_2 are functions of ω alone in the deep inelastic region, the operators $0_{\alpha_1\alpha_2\cdots\alpha_J}$ in the light-cone expansion of $[J_\mu(x), J_\nu(0)]$ must have dimension [68]

$$\ell(J) = -(J+2) \quad , \quad (J = 2,4,6, \ldots\ldots) \quad . \qquad (1.91)$$

Then the p-dependence of $W_{\mu\nu}(q,p)$, obtained from the spin-J part of*

$$\langle p | 0_{\alpha_1\alpha_2\cdots\alpha_J} | p \rangle = c_J p_{\alpha_1} p_{\alpha_2} \cdots\cdots p_{\alpha_J} + \text{traces} \quad , \qquad (1.92)$$

correctly matches the q-dependence given by $\int d^4x\, e^{iq\cdot x} C_{\mu\nu}^{\alpha_1\alpha_2\cdots\alpha_J}(x)$.

At present, it is not understood how these methods may be applied to inclusive processes (Eq. (1.39)) in which no leptons participate. In field theory, it is formally possible to write the total cross section as the matrix element of an operator product. However, in order to emphasize the region near the light cone $y^2 = 0$, this expression must be replaced by one in which the fields $A(x)$, $B(x)$ forming the operator product in Eq. (1.79) represent systems of

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* "Traces" stands for terms such as $g_{\alpha_1\alpha_2} p_{\alpha_3} \cdots p_{\alpha_J}$, $g_{\alpha_1\alpha_2} g_{\alpha_3\alpha_4} p_{\alpha_5} \cdots p_{\alpha_J}$, ..., which contribute $q^2 (p\cdot q)^{J-2}$, $(q^2)^2 (p\cdot q)^{J-4}$, ... relative to the contribution $(q\cdot p)^J$ of the term which scales, and are therefore negligible in the deep inelastic limit.

infinite mass.* In an approach which involves on-mass-shell amplitudes, $A(x)$ has to be the source of at least two strongly interacting particles. For $A(x)$ to be local, these particles X, Y must be strongly correlated so that their amplitudes $\mathcal{A}$ factorize,

$$\mathcal{A}(X + i \rightarrow Y + f) = a(X,Y) \langle f|A(0)|i\rangle, \; ((X - Y)^2 \rightarrow \infty), \quad (1.93)$$

as in the leptonic case. In strong interaction physics, the idea of factorization occurs mainly in Regge theory. Assuming that Reggeism works at large momentum transfers, $A(x)$ would be the source of a reggeon. The conceptual difficulties involved here should be compared with the derivations of scaling laws from the multiperipheral[(75)] or parton[(76)] models, where it makes little difference whether leptons are present or not.

I. 4. Comments on our Research and Related Work

We have investigated Mack's proposal that conformal invariance is spontaneously broken. Our method involves the simultaneous use of PCDC, PCAC and well-known techniques of current algebra. These notions are combined in a completely general fashion. Of course, our results are approximate in the real world, because we have to saturate dispersion integrals with

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*In Eqs. (1.87) and (1.88), $J^{\mu}(x) = A(x) = B(x)$ is the source of an infinite-mass virtual photon. In the ambitious scheme of Brandt and Orzalesi (Ref. 74) for strong interactions, the infinite-mass condition is satisfied by taking the formal field-theoretic expression an infinite distance off mass shell, (thereby ensuring that their assumption cannot be checked experimentally).

low-lying meson states. In this section, we give a superficial treatment of our assumptions and some of our results, postponing controversy and detailed discussion to other chapters. Most of our work has already appeared in published[15] or preprint[77,78] form.

From the formulae (1.23) for the divergences of the dilation and special conformal currents, we see that the corresponding Nambu-Goldstone bosons (dilatons) must be spin-0 isoscalar mesons, even though the special conformal generators carry a vector index. In order that SU(3) is realized as a degeneracy symmetry with no accompanying Nambu-Goldstone bosons, the dilaton states must become SU(3) singlets in the limit of scale invariance. We assume that there is just one dilaton, σ, but our formulae are easy to generalize if this turns out to be invalid. This assumption is motivated by the nonet structure of the meson spectrum. It corresponds to the existence of only two $(J^P, I^G) = (0^+, 0^+)$ mesons in the real world. They are mixtures of the SU(3) singlet state with dilation quality and the eighth member of an octet. We identify these mesons as the currently fashionable $\epsilon(700)$ and $\epsilon'(1060)$ resonances.* When the mixing of these particles is properly taken into account, $m_\sigma^2 = m_\epsilon^2$ measures the violation of scale and conformal invariance.

These mesons are also called $\eta_{0^+}(700)$ and $\eta_{0^+}(1060)$ or S(1060); see Ref. 79. We shall use the symbols σ, ϵ interchangeably. At no stage do we refer to $\sigma(410)$, which was observed in a few early experiments, but now appears to be extinct.

Originally, it was thought that this set of assumptions is inconsistent with the facts.[63,80] Taken at face value, Eq. (1.77) implies $G_{\sigma\pi\pi} = 0(m_\pi^2)$, so that, comparing it with Eq. (1.78) for nucleons, the dilaton width would be only a few MeV. This contradicts all expectations concerning the resonance structure of the $(0^+, 0^+)$ channel. Experiments[79,81] vaguely indicate $\Gamma_\epsilon \approx 400$ MeV for the width, a value which theory strongly supports via the Adler-Weisberger sum rule for $\pi\pi$ scattering,[82]

$$8 f_\pi^{\;2} = \frac{1}{\pi} \int_{2m_\pi}^{\infty} dW\, W\, (\sigma_{\pi^+\pi^-}(W) - \sigma_{\pi^+\pi^+}(W))/(W^2 - m_\pi^2) \;, \tag{1.94}$$

where W is the center-of-mass energy. Adler[82] pointed out that the ρ and f contributions to the integral are not large enough to satisfy the sum rule: there must be a large contribution from the $(0^+, 0^+)$ channel. It is known that $\epsilon'(1060)$ couples weakly to pions, so satisfaction of Eq. (1.94) depends on Γ_σ being of the order of several hundred MeV.

However, when the magnitude of scale violation is taken into account, one realizes that Eqs. (1.77) and (1.78) can be approximately true only if the square of the mass involved, (i.e., m_M^2 or M_B^2,) is large enough to swamp scale-violating effects, represented by m_σ^2. By assuming Eq. (1.57), we were able to prove that the next order in scale violation is given by[15]

$$F_\sigma G_{\sigma\pi\pi} \approx m_\sigma^2 \;. \tag{1.95}$$

The calculation assumes that SU(2) x SU(2) is a very good symmetry.[49,50]

This conclusion was independently arrived at by J. Ellis[83] from similar physical assumptions. A superficial comparison of his work with ours does not give the impression that any connection exists, because the formalisms are so different. In fact, the effective Lagrangian formalism of Ellis is designed to reproduce the results of the traditional methods of current algebra, which we used. Kleinert and Weisz,[84] who were unaware of our work, translated Ellis's work into a language which resembles ours.

It is instructive to look back at earlier attempts to calculate the width of the $\sigma(700)$, and to understand why initial attempts to combine chiral and conformal invariance gave either no result or the wrong result. It turns out that the σ-model is very misleading; over-reliance on it for gaining insight caused much confusion.

Before broken scale invariance became fashionable, the standard practice, (especially in hard-pion calculations,) was to assume that the matrix elements of the so-called "σ-terms"*

$$\Sigma^{ab} = (\sqrt{2/3}\, u_o + \sqrt{1/3}\, u_8)\, \delta^{ab} = i\, [F_5^a, v^b], \; (a,b=1,2,3), \tag{1.96}$$

are dominated by a σ-meson pole. Then one obtains $(2f_\pi)^{-1} G_{\sigma\pi\pi} = -m_\sigma^2$,

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*Note that $\{\sqrt{2/3}\, u_o + \sqrt{1/3}\, u_8, \vec{v}\}$ form a $(\frac{1}{2},\frac{1}{2})$ representation of SU(2) x SU(2). The unfortunate name "σ-terms" is a legacy of the σ-model, in which the "σ" field is given by $\Sigma_{ab} = \delta_{ab}$ "σ".

because the σ-term can be evaluated by applying a soft-pion theorem (Eq. 1.70) with n = 1). However, this is a "fake" derivation, because there is no way of telling which combination of u_o and u_8 should be dominated by the σ-meson, while the soft-pion theorem can be arbitrarily changed by varying the angle Ψ in the operator $\Sigma(\Psi)$ whose matrix elements are to be dominated:

$$\Sigma(\Psi) = (\sqrt{2/3}\, u_o + \sqrt{1/3}\, u_8) \sin\Psi + (\sqrt{1/3}\, u_o - \sqrt{2/3}\, u_8) \cos\Psi . \quad (1.97)$$

If $\Psi = \pi/2$ is assumed, (as indicated by the σ-model), the Adler-Weisberger rule (1.94) is completely saturated by the σ-pole alone, leaving no room for the ρ and f contributions. (This is hardly surprising since the ρ and f mesons do not appear in the σ-model).

The correct way to approach the problem is to write the Goldberger-Treiman relation for the β-decay of a dilaton into a pion[85,15]

$$(2f_\pi)^{-1}\, G_{\sigma\pi\pi} \simeq m_\sigma^2\, F_{\sigma\pi}(0) \quad , \qquad (1.98)$$

where $F_{\sigma\pi}(0)$ is the appropriate axial coupling constant, (i.e., the quantity analogous to g_A in Eq. (1.66)). The failure of PCDC for $\langle\pi|\theta^\mu_\mu|\pi\rangle$, (which gives Eq. (1.77) for pions,) indicates that in the real world, one cannot dominate $\langle\pi|\Sigma(\Psi)|\pi\rangle$ successfully by the σ. However, for the purposes of comparison with linear models,[63] for which the extrapolation from the σ-pole to zero momentum transfer involves no error, we give the result of the naive σ-dominance-plus-soft-pion-theorem calculation:

$$F_{\sigma\pi}(0) = -\sin\Psi . \tag{1.99}$$

Thus, if we had appealed to an SU(3) x SU(3) σ-model, where the "σ" field transforms like u_o, we would have obtained $F_{\sigma\pi}(0) = -\sqrt{2/3}$, instead of $F_{\sigma\pi}(0) = -1$ given by the ordinary SU(2) x SU(2) model.

We note that, within the accuracy of their saturation assumptions, the work of Gilman and Harari(86) is correct: they do not fix Ψ arbitrarily.* They require satisfaction of the sum rule (1.94), together with all other current-algebraic and superconvergent sum rules implied by Regge asymptotics for pion-baryon scattering. They find $\Psi \simeq \pi/4$ is needed in order to fit the data for the width of the ρ. Then the σ-pole contributes $\sin^2\Psi \simeq \frac{1}{2}$ of the right-hand side of Eq. (1.94).

The essential point of the work of Ellis(83) and this author(15) was that Eq. (1.57) fixes $F_{\sigma\pi}(0)$ within the accuracy of the saturation assumptions involved:(15)

$$F_{\sigma}F_{\sigma\pi}(0)\, f_{\pi} \approx \tfrac{1}{2} \tag{1.100}$$

Eq. (1.100) implies Eq. (1.95) when combined with the Goldberger-Treiman relation (1.98). The derivation is considered in Chapter II. Eq. (1.100) becomes exact in the limit of scale invariance; in that case, it can be derived from the conservation equations $\partial^{\mu}\mathfrak{F}_{5\mu}=0$, $\theta^{\mu}_{\mu}=0$, without additional assumptions.(15)**

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* Our Ψ is the same as the Ψ of Gilman and Harari.

** See Appendix C and the discussion in Chapter II.

Ellis's analysis was preceded by a paper by Isham, Salam, and Strathdee (ISS)[87] in which the method of effective Lagrangians for both chiral and conformal symmetry is introduced. Although ISS made use of Eq. (1.57), they were misled by the σ-model into putting $\Psi = \pi/2$. Hence, they obtained $F_\sigma = -(2f_\pi)^{-1}$ and the connection with the $\sigma\pi\pi$ coupling was lost. Ellis provided the necessary generalization of their work.

Prior to our work, there were some attempts[88] to make use of equal-time commutators involving the dilation and special conformal generators. However, due to some invalid steps in the course of very complicated manipulations, the results obtained were not correct and looked unpromising.* Our point of view was very different, and the analysis much simpler.

In phenomenological tests of dispersion relations for backward πN scattering, it is necessary to include a large contribution due to the exchange of a scalar meson. Assuming that the $\sigma(700)$ is responsible, and taking its width to be about 400 MeV, the σNN coupling is found to be roughly the same as the πNN coupling:[89,90]

$$g_{\sigma NN}^{2}/4\pi \approx 12 \quad . \tag{1.101}$$

Making use of PCDC for $\langle N | \theta^{\mu}_{\mu} | N \rangle$, (Eq. (1.78) with B = nucleon,) and Eq. (1.95), all the unknown constants drop out, and we find[15,83]

*Kleinert and Weisz (Ref. 84) took the trouble to explicitly point out the errors.

$$g_{\sigma\pi\pi} / g_{\sigma NN} \approx m_\sigma^2 / 2m_\pi M_N , \qquad (1.102)$$

where $g_{\sigma\pi\pi} = (2m_\pi)^{-1} G_{\sigma\pi\pi}$ is a dimensionaless constant (like $g_{\sigma NN}$). The left-hand side of Eq. (1.102) is about 1, compared with $m_\sigma^2/2m_\pi M_N \simeq 2$.

We regarded this result as satisfactory, but Kleinert and Weisz,[84] holding to a more optimistic view of the accuracy of the saturation assumptions, decided that the dispersion relation for $\langle N | \theta_\mu^\mu | N \rangle$ needs a subtraction.* That would mean that PCDC is an invalid hypothesis and the whole endeavour has come to nought. To show that this is not necessarily the case, we extended the method of collinear dispersion relations[91] to the calculation of scale-violating effects. We were able to give an equally good argument for the result[77, 78]**

$$g_{\sigma\pi\pi}/g_{\sigma NN} \approx m_\sigma^2 (2m_\pi M_N)^{-1} (1 - m_\sigma^2/m_{A_1}^2) , \qquad (1.103)$$

which is in good agreement with experiment. The difference between Eqs. (1.103) and (1.102) is a measure of the uncertainty involved in making a prediction for $G_{\sigma\pi\pi}$ from the theory of broken scale invariance. Therefore, PCDC is consistent with our picture of the meson spectrum and estimates of the σNN coupling.

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* In particular, the Adler-Weisberger sum rule (1.94) is over-saturated when Eq. (1.102) is treated as an accurate formula for $G_{\sigma\pi\pi}$. Note that de Alwis (Ref. 92) also has an over-optimisitc view of its accuracy.

** See Chapter III.

We have also investigated some consequences of the equal-time commutator

$$[K_\mu(0), \mathfrak{F}_{5\lambda}(0)] = 0 \quad , \tag{1.104}$$

which is a slightly stronger condition than Eq. (1.57). From the matrix element of Eq. (1.104) between a one-pion state and the vacuum, we can estimate the slope of the spin-2 form factor of $\langle \pi | \theta_{\mu\nu} | \pi \rangle$.[77,78] This estimate may be compared with the result of assuming f-dominance. While f-dominance appears to work quite well in the limit of scale invariance, there is a large scale-violating effect which indicates that f-dominance may be a poor approximation in the real world.* Again, we use the method of collinear dispersion relations. Our conclusion is supported by a recent estimate[90] of the fNN coupling constants.

Because of its large coupling to mesons as well as baryons, the dilaton greatly influences the application of broken chiral symmetry to low-energy meson-baryon scattering. In particular, the recent discovery of Cheng and Dashen[108] that $-(\sqrt{2} + c)\Sigma(\pi/2)/\sqrt{3}$, the SU(2) x SU(2) violating part of θ_{00}, contributes 110 MeV to the nucleonic matrix element does not necessarily mean that SU(2) x SU(2) cannot be a much better symmetry than SU(3).[77] If scale invariance is broken to the same extent as SU(3) x SU(3)

*Using the hard-meson method, Raman (Ref. 93) has independently arrived at a similar conclusion. He starts from an assumption equivalent to Eq. (1.104), but his methods of calculation and approximation are different from ours. The connection with scale-violation is not noted.

symmetry, (i.e., "δ vanishes in the limit of chiral SU(3) x SU(3) invariance"), the dilaton pole enhances $\langle N|u_0|N\rangle$ relative to $\langle N|u_8|N\rangle$, with the result that we are able to retain most of the attractive scheme of Gell-Mann, Oakes, and Renner,[49] including Eq. (1.73).

Our most important result concerns ℓ_u, the dimension of u or of a current divergence $\partial^\mu \mathfrak{F}_{5\mu}$. Using collinear dispersion relations, Fubini and Furlan[91] and von Hippel and Kim[113] have related the threshold amplitude for meson-baryon scattering to the equal-time commutator $\langle N|\left[\ddot{F}_5 + m_M^2 F_5, \partial^\mu \mathfrak{F}_{5\mu}\right]|N\rangle$. Only the connected part of this matrix element is involved. Contrary to the general belief, we show that, unless ℓ_u is -1, the $\left[\ddot{F}_5, \partial^\mu \mathfrak{F}_{5\mu}\right]$ part of this equal-time commutator should have the same order of magnitude as the other part, which is the quantity estimated by Cheng and Dashen. Making use of the experimental fact that the isospin-symmetric πN scattering length is very small we observe that $\langle N|\left[\ddot{F}_5, \partial^\mu \mathfrak{F}_{5\mu}\right]|N\rangle$ (connected) does not vanish, and therefore, ℓ_u cannot be -1.* We also point out that a recent criticism of the von Hippel-Kim paper by Ellis[103] does not affect our analysis.

We are unaware of any conflict between experiment and our scheme for broken conformal and chiral invariance. Some other possibilities, both respectable and doubtful, are briefly discussed in Sections III.3 and III.4.

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*Fritzsch and Gell-Mann (Ref. 143) and Mandula, Schwimmer, Weyers, and Zweig (Ref. 144) note the possibility of directly measuring ℓ_u in deep inelastic neutrino scattering.

In private discussions, we have found many people reluctant to consider the spontaneously broken variety of scale invariance. We can distinguish three different attitudes:

(i) Dilaton theories do not coincide with the theoretical interests of many workers;

(ii) Some say that m_σ is too large to allow the successful application of PCDC;

(iii) There is a feeling that the existence of scaling laws for high-energy inclusive processes such as deep inelastic electroproduction is harder to explain when a dilaton is present.

We do not entirely disagree with attitude (ii); at any rate, neither (i) nor (ii) constitutes an argument against spontaneous breakdown. Let us conclude this chapter by explaining why we do not share the sentiments of (iii).[78]

As we have often emphasized, conformal symmetry must not be treated as a degeneracy symmetry in momentum space because of the infrared problem. Exclusive processes are not expected to obey equations like (1.31) at high energies. The observation of a scaling law for an inclusive process means that bremsstrahlung effects are not dominant--i.e., only short-distance behavior matters. The PCDC hypothesis of Mack constrains low-energy behavior: at <u>large</u> distances, certain potentials are supposed to have the form $\exp(-m_\sigma r)/r$. This does not obviously conflict with Wilson's hypothesis of scale invariance at short distances. In terms of the intuitive discussion accompanying Eq. (1.35), Mack is concerned with the factor $P_N(E, \theta)$, Wilson with $d\sigma^{TOT}/d\Omega$.

However, this discussion does not completely answer the doubts represented by (iii). If we are given the set of operators $0_N(x)$ with dimension ℓ_n in the operator-product expansion in Eq. (1.79), the assumption that the leading singularities of the $C_n(y)$ are given by $y^{\ell_A + \ell_B - \ell_n}$ is easily understood (if one forgets renormalized field theory)--it corresponds to requiring that $D(x_o)$ be a slowly varying function of the time x_o. PCDC would seem to be an excellent way to arrange this. The problem is understanding why the set of operators 0_n is constrained to obey rules like Eq. (1.91). In the language of effective Lagrangians,(83) one can construct factors $\exp(\ell\,\sigma/F_\sigma)$ from a dilaton field $\sigma(x)$ with the anomalous transformation law

$$i\,[D(x_o),\ \sigma(x)] = -F_\sigma + x\cdot\partial\sigma(x) \quad . \tag{1.105}$$

It is easy to derive the formula

$$i\,[D(x_o),\ \exp(\ell\,\sigma(x)/F_\sigma)] = (-\ell + x\cdot\partial)\ \exp(\ell\,\sigma(x)/F_\sigma) \ . \tag{1.106}$$

The exponentials are used to fix up the dilation behavior of various terms in the Lagrangian to make it scale-symmetric; e.g.,

$$-M\bar{\Psi}\Psi \rightarrow -M\bar{\Psi}\Psi\exp(-\sigma/F_\sigma) = (-M + M_\sigma/F_\sigma - \tfrac{1}{2}M\sigma^2/F_\sigma^2 + \dots)\,\bar{\Psi}\Psi\ ; \tag{1.107}$$

(immediately, one gets $g_{\sigma NN} = M_N/F_\sigma$). Dilaton theory appears to be too flexible to allow the development of a rule for determining dim 0_n, because multiplication by $\exp(\ell\sigma/F_\sigma)$ permits us to arbitrarily change the dimension ℓ_n of a given operator to $(\ell + \ell_n)$.

We think that this problem exists with or without dilatons. In the above example, it is caused by the availability of a c-number F_σ which has the same units as a mass. In a non-dilaton theory, the limit of scale invariance (zero-mass limit) is unstable because it is impossible to banish all mass-like c-numbers in the process of renormalization. One finds that the dimension of a field can be "anomalous"(40, 27, 94)--instead of taking the free field value, the dimension becomes a function of the strength of a coupling constant in the theory. Therefore, we are faced with just as much flexibility as in a dilaton theory.

Now, it might be argued that, since renormalized field theory predicts that the $C_n(y)$'s are contaminated with logarithms (except for unrealistic superrenormalizable theories), it should be disregarded and free field theory should serve as a guide. In that case, rules like (1.91) are easy to explain, but it makes no difference whether dilatons are present or not. If the dimension of 0_n is changed through multiplication by $\exp(\ell\sigma/F_\sigma)$, the same factor must appear on the other side of the equation; (otherwise, the expansion is not valid for matrix elements involving σ particles).

Therefore, we do not believe that the treatment of scaling laws via operator-product expansions is connected with the manner in which scale invariance is broken, unless the non-leading singularities are considered. Perhaps the magnitude of the non-leading singularities is controlled by the magnitude of scale violation, and is therefore a means of distinguishing these possibilities.* To understand the leading singularities, we have to appeal to rules which look artificial at present; e.g., if the $0_{\alpha_1\alpha_2 \cdots \alpha_J}$ (including $\theta_{\mu\nu}$) are supposed to belong to an infinite representation of the conformal group,**, Eq. (1.91) follows immediately.[68]

Our investigation forms part of a program to discover the nature of the terms which break scale and chiral invariance in the energy density. Finding experiments which distinguish the various possibilities presents a considerable challenge for theorists. These experiments need not be difficult to perform. For example, our discussion of ℓ_u depends on measuring low-energy πN phase shifts with reasonable accuracy. At present, our knowledge of these amplitudes is almost adequate.

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* This works only for e^+e^- annihilation, and then only if the two-photon term is small enough. In deep inelastic electroproduction, the traces discussed in the footnote to Eq. (1.92) are likely to dominate the non-leading singularities.

** That is, they satisfy an algebra.

II. LIMITS OF CONFORMAL AND CHIRAL INVARIANCE

We proceed to develop a consistent, general theory to describe the consequences of combining broken conformal and chiral symmetry. Here we concentrate on understanding the constraints implied by symmetry arguments; the use of dispersion-theoretic methods is postponed to the next chapter. Remarks about some controversial contributions to this field are included.

II. 1. Realization of Scale Invariance with One Dilaton

One of the characteristics of the meson spectrum which the quark model explains is the appearance of nonets. Each SU(3) octet is accompanied by a singlet, which can mix with the eight member of the octet via SU(3) violation. Experimental data are not good enough to demonstrate nonet structure for the scalar mesons. In particular, experiments do not even vaguely suggest the number of $(J^P, I^G) = (0^+, 0^+)$ resonances in the region of interest (below 1.5 or 2 GeV). Our interpretation of dilatons is that they are $^3P_0$ states of the quark model, so we expect that the nonet property holds for scalar mesons.*

There is fairly strong evidence for the presence of some members, $\delta(960)$ and $\epsilon'(1060)$ with quantum numbers $(0^+, 1^-)$ and

*The opposite viewpoint would involve treating dilatons as scalar gluons. The Lagrangian density would contain both quark and dilaton fields, following the idea that the rôle of dilatons as Nambu-Goldstone particles differentiates them from other mesons. However, to be consistent, the whole pseudoscalar octet would have to be treated in the same fashion.

$(0^+, 0^+)$, of a possible scalar nonet.[79] Evidence for the corresponding strange particles $\varkappa(\sim 1\ \mathrm{GeV})$ is obscure.[81] The ninth member could be the $\epsilon(700)$ meson, but its parameters depend on indirect and controversial analyses of data.[79,81] Assuming that an excited nonet of scalar mesons is too massive to concern us (if it exists), we are left with two $(0^+, 0^+)$ resonances, $\epsilon = \sigma$ and ϵ', which can give important contributions to unsubtracted dispersion relations for $\langle \theta^\mu_\mu \rangle$.

In the limit of scale invariance, SU(3) becomes an exact degeneracy symmetry, so the singlet and octet states are no longer mixed. The vacuum is SU(3) invariant, so from

$$\langle \sigma | \theta_{\mu\nu} | 0 \rangle = -\frac{1}{3} F_\sigma (k_\mu k_\nu - g_{\mu\nu} k^2) \quad , \tag{2.1}$$

we see that only the SU(3) singlet state has dilaton quality. For these reasons, we assume that there is only one dilaton. Since SU(3) x SU(3) also becomes an exact symmetry in the limit of scale invariance, there are nine massless Nambu-Goldstone bosons: the scalar singlet σ and the pseudoscalar octet π, K, η. All other hadrons, including η' and the scalar octet, are supposed to remain massive and degenerate in hypercharge within SU(3) multiplets.

It has been claimed[95] that theories in which conformal invariance is realized via a Nambu-Goldstone boson are self-contradictory. These analyses involve taking the limit of scale invariance. None of the "proofs" apply when scale invariance is broken, so the contradictions obtained merely reflect an incorrect limiting procedure. As pointed out by Callan and Carruthers,[96] the standard mistake is to assume that the Poincaré generators are given by Eqs. (1.9) and (1.10) in the scale-invariant limit. This procedure ignores the presence of the dilaton pole at zero momentum transfer in $\langle \theta_{\mu\nu} \rangle$; i.e., the integrals in Eqs. (1.9) and (1.10) are ill-defined in this limit. When scale invariance is broken, there is a potential $\exp(-m_\sigma r)/r$ due to σ-exchange. It contributes a term proportional to

$$J(m_\sigma) = \lim_{V\to\infty} \int_V d^3x\, \vec{\nabla}^2 \left(\exp(-m_\sigma r)/r\right) = \lim_{\text{rad.}\to\infty} \oint d\vec{S}\cdot\vec{\nabla}\left(\exp(-m_\sigma r)/r\right) \tag{2.2}$$

to the integral defining P_o in Eq. (1.9), with $J(m_\sigma) = 0$ for $m_\sigma \neq 0$, and $J(0) = -4\pi$. Hence, this definition of energy has a discontinuity at $m_\sigma = 0$. To understand the limit of scale invariance, the ambiguity associated with the dilaton pole must be removed.

It is convenient to write the PCDC hypothesis in the form*

$$\theta_{\mu\nu} = t_{\mu\nu} + \frac{\partial_\mu \partial_\nu - g_{\mu\nu}\partial^2}{3(m_\sigma^2 + \partial^2)}\, t_\alpha^\alpha , \tag{2.3}$$

where $\langle t_{\mu\nu} \rangle$ has no σ-pole. The trace of Eq. (2.3) is

$$\theta_\mu^\mu = \frac{m_\sigma^2}{m_\sigma^2 + \partial^2}\, t_\alpha^\alpha \quad . \tag{2.4}$$

*Eq. (2.3) was suggested by M. Gell-Mann (private communication), who has also obtained the corresponding expressions for PCAC and PCTC. See footnote 29 of Ref. 77.

The operator t^{α}_{α} does <u>not</u> vanish, irrespective of the magnitude of scale violation. Consider Eqs. (1.9) and (1.10) with $m_\sigma \neq 0$; substituting Eq. (2.3) and integrating by parts, we obtain[77]

$$P_\mu = \int d^3x \, t_{o\mu}(x) \quad , \tag{2.5}$$

$$M_{\mu\nu} = \int d^3x \, (x_\mu t_{o\nu} - x_\nu t_{o\mu}) \quad . \tag{2.6}$$

Now there is no problem classifying states according to their masses and spins in the scale-invariant limit--the offending σ-pole is no longer present, so Eqs. (2.5) and (2.6) provide convergent definitions of the Poincaré generators for $m_\sigma = 0$.

The expressions (1.16) and (1.17) cannot be altered in this manner, so the integrations over d^3x diverge in the limit $m_\sigma \to 0$. This formal difficulty also occurs when defining the axial charges F^a_5 in the limit of chiral symmetry. In practice, difficulties in carrying out the integration over infinite 3-space are harmless, since we always consider commutators $[D, Q]$ and sums of commutators $\exp i\alpha D \, Q \, \exp -i\alpha D$ for some operator Q. These expressions contribute the well-behaved terms $\delta^3(\vec{x})$, $\vec{\partial}\,\delta^3(\vec{x})$, ... to the integrand. Unlike the SU(3) and Poincaré generators, F^a_5, D and K_μ are not used to classify states.*

*The chiral classification of states by Gilman and Harari (Ref. 86) is performed at infinite momentum, where the pole term may be removed. Another harmless difficulty associated with the d^3x integration is involved in the definition of non-conserved charges. See Renner's book (Ref. 45) for a brief discussion and references.

Another way of discussing the effect of the σ-pole in defining D and K_μ is to state that, in the limit of conformal invariance, the pole term causes $\exp(i\alpha D)$ and $\exp(i\beta_\mu K^\mu)$ to generate a continuum of degenerate vacua. As θ^μ_μ vanishes, the unique physical vacuum becomes a member of this continuum. However, it can still be distinguished because Poincaré invariance is a degeneracy symmetry; i.e., the vacuum is invariant under Poincaré transformations:

$$P_\mu |0\rangle = 0 \quad , \quad M_{\mu\nu} |0\rangle = 0 \quad , \tag{2.7}$$

with P_μ and $M_{\mu\nu}$ given by Eqs. (2.5) and (2.6). Thus, the rôle of the Poincaré group as a subgroup of the conformal group is analogous to that of SU(3) as a subgroup of chiral SU(3) x SU(3). The Poincaré and SU(3) groups provide classifications of particle states because their generators annihilate the vacuum. However, the other chiral and conformal generators produce degenerate vacua, so the corresponding classification schemes for particle states need not appear.

The most interesting paradox connected with the limit $m_\sigma \rightarrow 0$ was found by Renner.* In the limit of scale invariance, Eq. (1.44) gives

$$i[P_o, D] = -P_o \quad , \tag{2.8}$$

implying

$$e^{i\alpha D} P_o e^{-i\alpha D} = e^{\alpha} P_o \quad . \tag{2.9}$$

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*Renner's observation was recorded by Ellis (Ref. 83).

If $|N\rangle$ is a nucleon state at rest, then defining $|\Psi\rangle = \exp(-i\alpha D)|N\rangle$,

$$P_o|\Psi\rangle = e^{\alpha} M_N |\Psi\rangle \tag{2.10}$$

is apparently the result. Since $|\Psi\rangle$ has the same quantum numbers as the nucleon, it should have an energy greater than or equal to M_N, so transformations with $\alpha < 0$ seem to be forbidden. This anomaly arises from an illegal interchange of limits in the equation[77]

$$\lim_{\theta^\mu_\mu \to 0} \lim_{k^2 \to 0} \langle N(p+k) | \theta^\mu_\mu | N(p)\rangle = M_N \; ; \tag{2.11}$$

i.e., the limit $\theta^\mu_\mu \to 0$ is non-uniform in the momentum transfer. The derivation of Eq. (2.10) assumes that the scale-invariant limit of Eq. (1.44) is uniform.[77,96]

In order to see whether the requirement

$$\langle N| e^{i\alpha D} P_o e^{-i\alpha D} |N\rangle \geq M_N \langle N|N\rangle \tag{2.12}$$

leads to any useful theorems, the effects of the breakdown of scale invariance must be included:

$$e^{i\alpha D(0)} \theta_{oo}(0,\vec{x}) \, e^{-i\alpha D(0)} = e^{(4+\vec{x}\cdot\vec{\nabla})\alpha} \theta_{oo}(0,\vec{x})$$

$$+ \sum_n \{ e^{(-\ell_n + \vec{x}\cdot\vec{\nabla})\alpha} - e^{(4+\vec{x}\cdot\vec{\nabla})\alpha} \} w_n(0,\vec{x}) \; , \tag{2.13a}$$

with $\ell = 0$ for w_n = c-number; the integrated form of this identity is[77]

$$e^{i\alpha D(0)} P_o e^{-i\alpha D(0)} = e^{\alpha} P_o + \sum_n (e^{-(\ell_n+3)\alpha} - e^{\alpha}) \int d^3x \, w_n(0,\vec{x}) \, . \tag{2.13b}$$

If the dimension ℓ of the scale-violating part w of θ_{oo} is unique, (apart from a c-number term), combination of Eqs. (2.12) and (2.13b) appears to yield

$$(\ell+3)\, e^{\alpha} + e^{-(\ell+3)\alpha} \geqslant \ell + 4 \quad ,$$

which would imply $\ell \geqslant -3$ or $\ell \leqslant -4$. Making use of the constraints on ℓ found by Wilson,[40] we would deduce $-3 \leqslant \ell \leqslant -1$, a result which is reasonable,* but not guaranteed by the above argument.

The flaw in the above reasoning arises from neglect of the contribution of the disconnected part, $\langle N|N\rangle \langle 0|e^{i\alpha D} P_o e^{-i\alpha D}|0\rangle$, to Eq. (2.12). This term dominates the inequality because of the infinity associated with the integration in $P_o' = \int d^3x\, \theta_{oo}$. The most striking demonstration of the effect of this term occurs when chiral transformations are investigated in the same manner:

$$\exp(i\alpha F_5^3)\, \theta_{oo}(0)\, \exp(-i\alpha F_5^3) = \theta_{oo} - \alpha \partial^{\mu} \mathfrak{F}_{5\mu}^{3} - \tfrac{1}{2}\alpha^2 \Sigma^{33} + 0(\alpha^3) \, , \quad (2.14a)$$

with $\Sigma^{ab} = [F_5^a, [F_5^b, \theta_{oo}]]$, or, if Σ^{ab} belongs to a $(\frac{1}{2}, \frac{1}{2})$ representation of SU(2) x SU(2), (as in Eq. (1.96)),

$$\exp(i\alpha F_5^3)\, \theta_{oo} \exp(-i\alpha F_5^3) = \theta_{oo} - \sin\alpha\, \partial^{\mu} \mathfrak{F}_{5\mu}^{3} + (\cos\alpha - 1)\Sigma^{33} \, . \quad (2.14b)$$

*Wilson (Ref. 97) notes that many vertex functions commonly encountered in current algebra diverge unless this condition holds.

Using a principle analogous to Eq. (2.12), we find

$$\langle N|\Sigma^{33}|N\rangle < 0 \quad . \tag{2.15}$$

At first sight, this result looks outrageous; in the free quark model

$$\mathscr{L} = :\bar{q}\, i\not\partial\, q: \; - :m_o\, \bar{q}(\lambda_0 + c\lambda_i)\, q: \quad , \tag{2.16}$$

$\langle q|:\Sigma^{33}:|q\rangle$ is the mass of an isodoublet quark. Closer inspection reveals that the mass term is normal-ordered, whereas the quantity which transforms as part of a $(3,\bar{3}) + (\bar{3},3)$ representation is $\bar{q}\,\lambda_a q$. Eq. (2.15) is correct because the left-hand side contains the infinite disconnected part $\langle N|N\rangle\langle 0|\Sigma^{33}|0\rangle$, which is negative.

Apart from the irrelevant case of the limit of scale invariance, the disconnected part does not vanish. Therefore, the only way of making use of a principle like Eq. (2.12) involves taking the vacuum expectation value of Eq. (2.13a). Note that we do not attempt to apply Eq. (2.13b), because the integrand loses its dependence on $\vec{x}$ when the vacuum expectation value is taken; thus the integration by parts which converts Eq. (2.13a) into Eq. (2.13b) gives an infinite answer, and produces surface terms which must not be neglected. From Eq. (2.13a), we have*

*A similar equation involving chiral transformations has been analyzed by Dashen (Ref. 55).

$$f(\alpha) = \langle 0 | e^{i\alpha D} \theta_{oo} e^{-i\alpha D} | 0 \rangle = \sum_n (e^{-\ell_n \alpha} - e^{4\alpha}) \langle 0 | w_n | 0 \rangle$$

$$= \sum_n e^{-\ell_n \alpha} \langle 0 | w_n | 0 \rangle + e^{4\alpha} \langle 0 | \bar{\bar{\theta}}_{oo} | 0 \rangle$$

$$\geq 0 \qquad (2.17)$$

The condition $d^2 f(0)/d\alpha^2 > 0$ ensures a local minimum at $\alpha = 0$:

$$d^2 f(0)/d\alpha^2 = \sum_n \ell_n(\ell_n + 4) \langle 0 | w_n | 0 \rangle$$

$$= -i \langle 0 | [D, \theta^\nu_\nu] | 0 \rangle = R(0) > 0 . \qquad (2.18a)$$

with

$$R(k^2) = i \int d^4x \, e^{ik \cdot x} \theta(x_o) \langle 0 | [\theta^\mu_\mu(x), \theta^\nu_\nu(0)] | 0 \rangle . \qquad (2.18b)$$

Eq. (2.18a) is consistent with the assumption that $R(k^2)$ satisfies an unsubtracted Lehmann-Källén representation:

$$R(k^2) = \frac{1}{\pi} \int \frac{dt \, \mathrm{Im}\, R(t)}{t - k^2} , \quad (\mathrm{Im}\, R(t) \geq 0). \qquad (2.18c)$$

This assumption is characteristic of a PCDC theory, in which matrix elements of θ^μ_μ emphasize low frequencies.

For large positive α, Eq. (2.17) implies

$$\langle 0 | \bar{\bar{\theta}}_{oo} | 0 \rangle \geq 0. \qquad (2.19)$$

If $\langle 0 | \bar{\bar{\theta}}_{oo} | 0 \rangle$ vanishes, the first non-vanishing term $\langle 0 | w_{Min.} | 0 \rangle$ for ℓ_n increasing must be positive. Similarly, looking at the case in which α becomes a large, negative number, either there is a positive c-number w_o, or the first non-vanishing term $\langle 0 | w_{Max.} | 0 \rangle$ as ℓ_n decreases is positive. We have no general method for imposing the constraint that $f(\alpha)$ have a global minimum $\alpha = 0$. However, all consequences of this condition may be obtained if the decomposition of θ_{oo} into scale-preserving and scale-violating pieces is not too complicated.

In particular, we consider Eq. (1.83) with the dimension ℓ_u of u assumed to be unique. Then δ cannot vanish[15]* because of the equations

$$\langle 0 | \theta^{\mu}_{\mu} | 0 \rangle = \sum_n (\ell_n + 4) \langle 0 | w_n | 0 \rangle = 0 \tag{2.20}$$

and $\langle 0 | u | 0 \rangle \neq 0$. Theories in which $\delta = 0$ is claimed actually assume that δ is a c-number δ_o given by

$$\delta_o \simeq 3m_\eta^2 (\ell_u + 4)/(64 f_\pi^2) \quad , \tag{2.21}$$

where we have substituted the result $\langle 0 | u | 0 \rangle \simeq -3m_\eta^2/(16 f_\pi^2)$ obtained by Gell-Mann, Oakes, and Renner.[49] For this model, Eq. (2.17) implies $-4 < \ell_u < 0$.

In another simple model, u and δ have unique dimensions ℓ_u, ℓ_δ. Then Eq. (2.17) is equivalent to

$$\ell_\delta > \ell_u \tag{2.22}$$

Eq. (2.22) does not contradict the SU(3) x SU(3) σ-model[63]

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*This point was also noticed by M. Gell-Mann (private communication).

$$\mathcal{L} = \tfrac{1}{2}\mathrm{Tr}\,\partial^{\mu}\mathcal{M}^{+}\partial_{\mu}\mathcal{M} + f_1(\mathrm{Tr}\,\mathcal{M}^{+}\mathcal{M})^2 + f_2\,\mathrm{Tr}\,\mathcal{M}^{+}\mathcal{M}\mathcal{M}^{+}\mathcal{M} \tag{2.23a}$$

$$+ g_1(\det\mathcal{M} + \det\mathcal{M}^{+}) + f_o\,\sigma_o + f_8\,\sigma_8 + c \quad ,$$

where $\mathcal{M} = \sum_{a=0}^{8} \lambda_a(\sigma_o + i\phi_a)$ is a 3x3 matrix of fields (σ_a, ϕ_a) which form a $(3,\bar{3}) + (\bar{3},3)$ representation of SU(3) x SU(3), because there is an extra c-number term

$$c = -bf_o - 2(2/3)^{3/2} g_1\, b^3 - 4f_1\, b^4 - \tfrac{4}{3} f_1\, b^4 \quad , \quad (b = \langle 0\,|\sigma_o|\,0\rangle) \; , \tag{2.23b}$$

which is usually ignored.

According to de Alwis and O'Donnell,[(64, 99)*] if the operator term w breaking scale invariance in θ_{oo} has unique dimension ℓ, then ℓ is necessarily -1. The assumption of de Alwis and O'Donnell, which they write in the ambiguous form "$\lim_{m_\sigma \to 0} \langle N'|\theta_{oo}|N\rangle = \langle N'|\lim_{m_\sigma\to 0}\theta_{oo}|N\rangle$", is easy to analyze when the correct relation

$$\lim_{\theta^{\mu}_{\mu}\to 0}\lim_{\vec{k}\to 0} \langle N(\vec{p}+\vec{k})|\,\bar{\bar{\theta}}_{oo}|N(\vec{p})\rangle = \lim_{\vec{k}\to 0}\lim_{\theta^{\mu}_{\mu}\to 0} \langle N(\vec{p}+\vec{k})|\,\bar{\bar{\theta}}_{oo}|N(\vec{p})\rangle \tag{2.24}$$

$$-(\ell+1)\, M_N/(\ell+4)$$

---

*Our analysis of this work was given in private correspondence with Drs. de Alwis and O'Donnell. See Ref. 98 for an alternative discussion with the same conclusion (for the case δ = c-number).

is considered. For degeneracy symmetries, it is permissible to interchange momentum transfer and symmetry limits of matrix elements. However, Eq. (2.11) demonstrates that spontaneously realized symmetries do not generally obey this rule. When it is remembered that a "smooth" definition of energy is obtained when the dilaton pole is first removed from $\theta_{\mu\nu}$, as in Eq. (2.5), we see that there is no reason preventing $\langle \bar{\bar{\theta}}_{oo} \rangle$ from developing a dilaton pole, in which case the limits in Eq. (2.24) cannot be interchanged. In the language of de Alwis and O'Donnell, there is no way of distinguishing their assumption from the alternative

$$" \lim_{m_\sigma \to 0} \langle N' | \theta_{oo} - \xi \theta^{\mu}_{\mu} | N \rangle = \langle N' | \lim_{m_\sigma \to 0} (\theta_{oo} - \xi \theta_{\mu}^{\mu}) | N \rangle " , \quad (2.25)$$

where ξ is, a priori, arbitrary. Their prescription gives $\xi = (\ell + 1)/(\ell + 4)$, which does not help to determine ℓ.

Although a large percentage of papers on broken scale invariance rely on formal arguments, the considerations of this section indicate that such reasoning has very limited application. In particular, "arguments" which can be formulated only in the limit of scale invariance should be ignored. However, if soft-dilaton theorems are being considered, there is no problem associated with the limit of scale invariance. By yielding exact theorems, this procedure often indicates the correct way to carry out the corresponding pole-dominance calculations.

II. 2. Width of a Single Dilaton

For the moment, we ignore the fact that the dilaton mixes with the eighth member of the scalar octet, and imagine that it is isolated. The effects of mixing will be considered in Section III. 1.

The only hadronic coupling constants of the dilaton which are presently measurable are $\bar{u}' u \, g_{\sigma NN}$ and $G_{\sigma\pi\pi} \, \delta^{ab}$ for protons and pions, (with a, b = isospin indices of the pions), where states are normalized invariantly. Applying PCDC to $\langle N | \theta_\mu{}^\mu | N \rangle$ and $\langle \pi | \theta_\mu{}^\mu | \pi \rangle$, and taking care to indicate the effects of scale violation, we have

$$F_\sigma \, g_{\sigma NN} = M_N + 0(m_\sigma^2/2 M_N) \quad , \tag{2.26}$$

$$F_\sigma \, G_{\sigma\pi\pi} = 2m_\pi^2 + 0(m_\sigma^2) \quad . \tag{2.27}$$

We investigate the possibility that, in Eq. (2.27), the scale-violating term is more important than the term $2m_\pi^2$.

Since pions are involved, it is natural to consider the application of approximate chiral SU(2) x SU(2) symmetry. Let us define the axial form factors for the β-decay of a dilaton into a pion:

$$\langle \sigma(k) | \mathcal{F}_{5\lambda}^{3}(0) | \pi^o(q) \rangle = -i \, (k+q)_\lambda \, F_{\sigma\pi}(t) + i \, (k-q)_\lambda G_{\sigma\pi}(t) \ , \tag{2.28}$$

'here $t = (q-k)^2$ is the momentum transfer squared. The ivergence of Eq. (2.28) is

$$D_{\sigma\pi}(t) = \langle \sigma(k) \,|\, \partial^\lambda \mathfrak{F}^3_{5\lambda}(0) \,|\, \pi^0(q) \rangle$$

$$= (m_\sigma^2 - m_\pi^2)\, F_{\sigma\pi}(t) - t\, G_{\sigma\pi}(t) \quad . \tag{2.29}$$

According to the PCAC hypothesis, $D_{\sigma\pi}(t)$ satisfies an unsubtracted dispersion relation,*

$$D_{\sigma\pi}(t) = \frac{m_\pi^2\, G_{\sigma\pi\pi}}{2f_\pi(m_\pi^2 - t)} + \frac{1}{\pi}\int_{9m_\pi^2}^{\infty} \frac{dt'\, \mathrm{Im}\, D_{\sigma\pi}(t')}{t' - t} \ , \tag{2.30}$$

which is dominated by the pion pole in the region $|t| \lesssim m_\pi^2$. At $t = 0$, we obtain the Goldberger-Treiman relation(85, 15)

$$(2f_\pi)^{-1}\, G_{\sigma\pi\pi} = m_\sigma^2\, F_{\sigma\pi}(0) + 0(m_\pi^2) \quad . \tag{2.31}$$

As noted in our introductory remarks in Section I. 4, the axial coupling constant, $F_{\sigma\pi}(0)$, is an SU(3) x SU(3) Clebsch-Gordan coefficient in linear models for chiral symmetry breaking.

Current algebra allows further progress. Placing the equal-time commutation relation

$$\left[F_5^{1+i2}(0),\ F_5^{1-i2}(0)\right] = 2\, F^3(0) \tag{2.32}$$

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* The pion decay constant f_π is defined in Eq. (1.65).

between π^+ states, the resulting formula

$$1 = (4p_o)^{-1} \sum_I (2\pi)^3 \delta^3(\vec{P}_I - \vec{p}) \left\{ |\langle I | \mathfrak{F}_{5o}^{1-i2}(0) | \pi^+(p) \rangle |^2 - |\langle I | \mathfrak{F}_{5o}^{1+i2}(0) | \pi^+(p) \rangle |^2 \right\} \tag{2.33}$$

becomes the Adler-Weisberger sum rule for $\pi\pi$ scattering, Eq. (1.94), when PCAC is applied to $\langle I | \partial^\lambda \mathfrak{F}_{5\lambda} | \pi(p) \rangle$ in the limit $p_z \to \infty$. For our purposes, it is more convenient to consider just the $p_z \to \infty$ step; we display the σ-contribution explicitly:

$$1 = |F_{\sigma\pi}(0)|^2 + (\epsilon', \rho, f, \ldots \text{ contributions})_{p_z \to \infty}. \tag{2.34}$$

The ϵ'-contribution is negligible, ($\Gamma_{\epsilon' \to \pi\pi} \approx 30$ MeV), and the ρ, f, ... resonances contribute about $\frac{1}{2}$: i.e., (82, 86)

$$|F_{\sigma\pi}(0)| \approx 1/\sqrt{2} \quad , \tag{2.35}$$

which corresponds to a σ-width of several hundred MeV. Obviously, the PCDC relation (2.27) is useless if we have correctly identified the dilaton as the resonance responsible for the satisfaction of Eq. (2.34). Note that we are not claiming that Eq. (2.27) is wrong.

Now we consider the constraints placed on $F_{\sigma\pi}(0)$ by the theory of broken scale invariance. At this point, we ignore the effects of symmetry violation, because the derivation of symmetric results is easier and more elegant. The calculation is performed with $m_\sigma = m(0^-, \underline{8}) = 0$ and $\theta_\mu^{\ \mu} = 0 = \partial^\lambda \mathcal{F}_{5\lambda}$; the corresponding analysis using pole dominance will be fully considered in Sections III. 1 and III. 2.

According to Wilson's theory of broken scale invariance,[40] the breakdown of chiral invariance is also a scale-violating effect. Therefore, in the limit of scale invariance, we have

$$[F_5, \theta_{\mu\nu}] = 0 \quad , \tag{2.36}$$

implying

$$i\,[D, \mathcal{F}_{5\lambda}(x)] = (3 + x\cdot\partial)\,\mathcal{F}_{5\lambda}(x) \quad . \tag{2.37}$$

It might be thought that Eq. (2.37) should contain 3-gradient terms of the form $\partial^\alpha \mathcal{J}_{5\alpha\lambda}$ in order to be generally correct, where $\mathcal{J}_{5\alpha\lambda}$ is an antisymmetric pseudotensor with dimension -2, (possibly the axial counterpart of the tensor current obeying PCTC. However, examination of the operator product $[\theta_{\mu\nu}(x,)\ \mathcal{F}_{5\lambda}(0)]$ reveals that the presence of such a term is inconsistent with the known expression for $[M_{\mu\nu}, \mathcal{F}_{5\lambda}(0)]$; (see the next section).

An important matrix element for our work is

$$\mathcal{M}_{\mu\nu} = \langle \pi(P + \tfrac{1}{2}k)\,|\,\theta_{\mu\nu}(0)\,|\,\pi(P - \tfrac{1}{2}k)\rangle \quad . \tag{2.38}$$

We choose the form factor expansion[15]

$$\mathcal{M}_{\mu\nu} = (2P_\mu P_\nu - (k_\mu k_\nu - g_{\mu\nu}k^2)/6)\,F_1(k^2) + (k_\mu k_\nu - g_{\mu\nu}k^2)F_2(k^2) \ , \tag{2.39}$$

(where Eq. (1.9) requires $F_1(0) = 1$), because the dispersion theory of $F_1(t)$ and $F_2(t)$ is simplified when the mass of the pion is neglected; intermediate states with (J^P, I^G) equal to $(2^+, 0^+)$ and $(0^+, 0^+)$ contribute to $\mathrm{Im}\, F_1(t)$ and $\mathrm{Im}\, F_2(t)$ respectively. Applying the condition of scale invariance to the trace of Eq. (2.39),

$$\mathcal{M}_\mu^\mu = 2m_\pi^2 F_1(k^2) - 3k^2 F_2(k^2) \quad , \tag{2.40}$$

we find[15]

$$F_2(k^2) = 0 \quad . \tag{2.41}$$

Thus, the effects of scale violation are responsible for the presence of the induced scalar form factor, $F_2(t)$, in Eq. (2.39).

From Eqs. (2.29) and (2.31), the formulae[15]

$$G_{\sigma\pi\pi} = 0 \quad , \tag{2.42}$$

$$G_{\sigma\pi}(t) = 0 \quad , \tag{2.43}$$

are also valid in the limit of scale invariance.

The spirit of the following analysis[15] resembles that of the original derivations of chiral-symmetric results by Nambu and his collaborators.[54] We begin by defining the time-ordered product

$$A_{\mu\lambda}(k,q) = i \int d^4x\, e^{ik\cdot x} \langle 0 | T(\mathcal{D}_\mu(x) \mathcal{F}_{5\lambda}^3(0)) | \pi^0(q) \rangle \quad , \tag{2.44}$$

which contains the dilation current $\mathcal{D}_\mu(x)$, defined in Eq. (1.11). Since we are working in the limit of scale invariance, (i.e., $\partial^\mu \mathcal{D}_\mu = 0$), the divergence of Eq. (2.44) takes the very simple form

$$-k^\mu A_{\mu\lambda}(k,q) = \langle 0 | [D, \mathcal{F}^3_{5\lambda}(0)] | \pi^0(q)\rangle + 0(k) . \tag{2.45}$$

Substituting Eqs. (1.65) and (2.37), the right-hand side may be written

$$\langle 0 | [D, \mathcal{F}^3_{5\lambda}(0)] | \pi^0(q)\rangle = 3 \ q_\lambda \ (2f_\pi)^{-1} . \tag{2.46}$$

In order to evaluate the left-hand side of Eq. (2.45), we need consider only those contributions to $A_{\mu\lambda}(k,q)$ which are singular in k_α. The appropriate diagrams, which are displayed in Fig. 1, represent the amplitude

$$\begin{aligned} P_{\alpha\nu\lambda}(k,q) = & -\tfrac{i}{3}(k+q)_\lambda F_\sigma F_{\sigma\pi}((k-q)^2)\,(k_\alpha k_\nu - g_{\alpha\nu}k^2)/k^2 \\ & -\frac{i F_1(k^2)}{2f_\pi}(k-q)_\lambda \Big[2(q-\tfrac{1}{2}k)_\alpha (q-\tfrac{1}{2}k)_\nu \\ & - (k_\alpha k_\nu - g_{\alpha\nu}k^2)/6\Big] / (2q\cdot k - k^2) . \end{aligned} \tag{2.47}$$

In writing Eq. (2.47), we have made use of Eqs. (2.28) and (2.39) for the vertices shown in Fig. 1, subject to the constraints (2.41), (2.42) and (2.43). The formula (1.11) for the dilation current implies

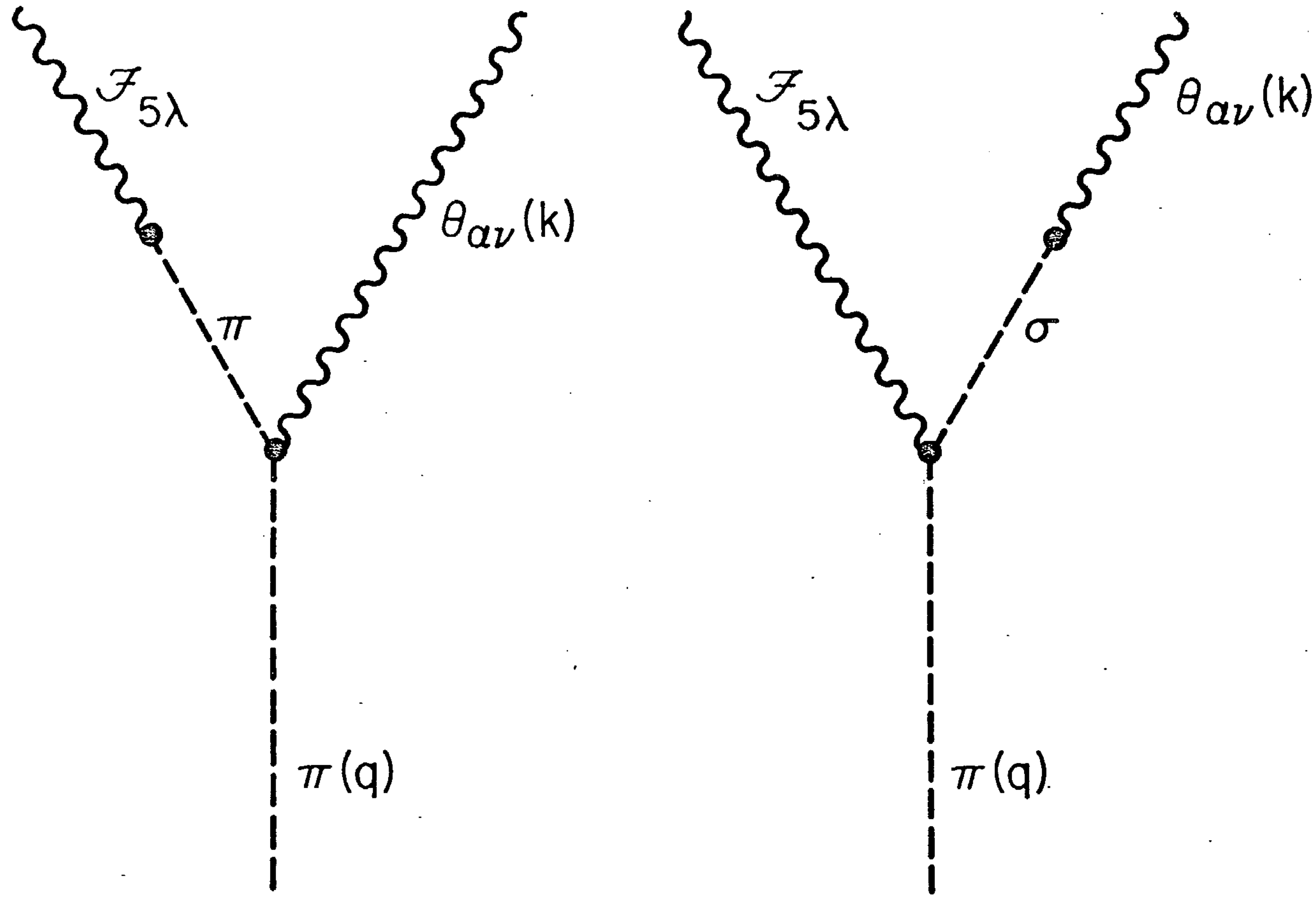
$\mathcal{F}_{5\lambda}$
π
$\theta_{\alpha\nu}(k)$
$\pi(q)$
$\mathcal{F}_{5\lambda}$
σ
$\theta_{\alpha\nu}(k)$
$\pi(q)$

Fig. 1

$$- k^{\mu} A_{\mu\lambda}(k,q) = i\, k^{\mu} \frac{\partial}{\partial k_{\nu}} P_{\mu\nu\lambda}(k,q) + 0(k)$$

$$= i \frac{\partial}{\partial k_{\nu}} (k^{\mu} P_{\mu\nu\lambda}) - i P^{\mu}_{\mu\lambda} + 0(k), \qquad (2.48)$$

where the last step was included because it saves much labor when Eqs. (2.47) and (2.48) are combined:

$$- k^{\mu} A_{\mu\lambda}(k,q) = (F_{\sigma} F_{\sigma\pi}(0) + f_{\pi}^{-1})\, q_{\lambda} + 0(k) \quad . \qquad (2.49)$$

Combining Eqs. (2.45), (2.46) and (2.49), we find[15]

$$F_{\sigma} F_{\sigma\pi}(0)\, f_{\pi} = \tfrac{1}{2} \quad . \qquad (2.50)$$

In Appendix C, we give methods of deriving this result where only the conservation equations $\theta^{\mu}_{\mu} = 0$, $\partial^{\nu} \mathfrak{F}_{5\nu} = 0$ are used, so the validity of Eq. (2.37) in the limit of scale invariance is confirmed.

We shall fully treat the question of the effects of scale breaking in Chapter III. However, at this point, we have some reasons for supposing that Eq. (2.50) is roughly correct in the real world. Theories of spontaneous violation of a symmetry assume that the fundamental decay constant, (F_{σ} for scale invariance, f_{π} for chiral SU(2) x SU(2) invariance), is practically unaffected by the magnitude of that symmetry violation. The main cause of uncertainty is the dependence of $F_{\sigma\pi}(0)$ on the breakdown of scale invariance. Essentially, $F_{\sigma\pi}(0)$ is an SU(3) x SU(3) Clebsch-Gordan coefficient, so it may be slightly changed by mixing of the dilaton with $\epsilon'(1060)$ meson. Also, there is considerable ambiguity about the point t_o at which $F_{\sigma\pi}(t_o)$ should be evaluated, because t_o can be $0(m_\sigma^2)$. Here, a typical uncertainty involves the factor $(1 - m_\sigma^2/m_{A_1}^{\ 2})$. Therefore, Eq. (2.50) should be applicable to the real world if one is prepared to tolerate a discrepancy involving factors of 1.5 or 2.[15] This is not worse than the inaccuracies observed for many chiral SU(3) x SU(3)-symmetric results.[49,100] In fact, SU(3) x SU(3) and scale breaking may be of the same magnitude: m_σ^2 is supposed to be not much larger than m_η^2.

We can further investigate the accuracy of Eq. (2.50) by considering Eq. (2.39) in the limit of chiral SU(2) x SU(2) invariance, but with scale invariance broken. In order to proceed, we must assume the validity of Eq. (1.57), or Eq. (2.37) for $\lambda = 0$. As we have remarked in Section I.3, Eq. (1.57) is valid in the usual theory of broken chiral symmetry. Note that if it existed, the scalar operator $s(x)$ defined in Eq. (1.56) would necessarily vanish in the limit of scale invariance, have a dimension greater than -2, and, to affect our argument, belong to the $(3,\bar{3}) + (\bar{3},3)$ representation of SU(3) x SU(3), (for example), rather than $(1,8) + (8,1)$. To phrase our assumption another way, the energy density, not just the Hamiltonian and other Poincaré generators, is supposed to be chiral invariant in the limit of chiral invariance; then $\dot{F}_5 = 0$ gives

$$[F_5, \theta_{\mu\nu}] = 0 \qquad (2.51)$$

As we shall demonstrate immediately, Eq. (2.51) implies the constraint[15]

$$F_2(0) = -1/3 \quad . \qquad (2.52)$$

A brief derivation of Eq. (2.52) follows from the formula

$$\partial^\lambda R_{\lambda\mu\nu}{}^*(x) = 0 \quad , \qquad (2.53)$$

where

$$R_{\lambda\mu\nu}{}^*(x) = [\mathcal{J}_{5\lambda}(x), \theta_{\mu\nu}(0)]_{\text{ret.}} \qquad (2.54)$$

is a retarded commutator made covariant by careful treatment of the singularity at $x = 0$.* Eq. (2.51) has been assumed in writing Eq. (2.53). Then

$$q^\lambda \int d^4x \, e^{iq\cdot x} \langle \pi(q') \,|\, R_{\lambda\mu\nu}{}^*(x) \,|\, 0 \rangle = 0 \qquad (2.55)$$

implies

$$\lim_{q\to 0} \langle \pi(q') \,|\, \theta_{\mu\nu}(0) \,|\, \pi(q) \rangle = 0 \quad , \qquad (2.56)$$

which may be combined with Eq. (2.39) to yield Eq. (2.52).

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*We presented this derivation in Ref. 77. Previously (Ref. 15), we dealt with the less singular quantity $R^{\mu}_{\lambda\mu}(x)$, and the corresponding derivation may be found in Appendix C. The covariantization of quantities like $R_{\lambda\mu\nu}$ is considered in Ref. 101. As long as the current is conserved, the question of covariance turns out to be irrelevant in the derivation of the soft- meson theorem, (here Eq. (2.56)). Following tradition, we add stars to symbols to denote that they have been made covariant.

In the limit of SU(2) x SU(2) invariance, the induced scalar form factor $F_2(t)$ satisfies an unsubtracted dispersion relation, because Eq. (2.40) shows that it is damped even more strongly than $\mathcal{M}_\mu^\mu$ at large t, and matrix elements of $\langle \theta_\mu^\mu \rangle$ satisfy unsubtracted dispersion relations (PCDC hypothesis). While σ-dominance of $\mathcal{M}_\mu^\mu$ at small t does not appear to be a good approximation, the corresponding assumption for $F_2(t)$ has a good chance of working, since $F_2(0)$ is proportional to the gradient of $\mathcal{M}_\mu^\mu$ at $t = 0$:(15,77)

$$F_2(t) \simeq -\frac{1}{3} m_\sigma^2/(m_\sigma^2 - t) \quad , \tag{2.57}$$

with $|t| \lesssim m_\sigma^2$. Note the consistency of Eqs. (2.41), (2.52) and (2.57). The property that the limits $m_\sigma^2 \to 0$, $t \to 0$ cannot be interchanged is characteristic of theories involving a Nambu-Goldstone boson. We can identify the residue of the σ-pole in Eq. (2.57) with coupling constants of the dilaton:(15,83)

$$F_\sigma G_{\sigma\pi\pi} \approx m_\sigma^2 \quad . \tag{2.58}$$

When Eqs. (2.31) and (2.58) are combined, we obtain Eq. (2.50) as an approximate relation.

If we had neglected the $0(m_\sigma^2)$ term in Eq. (2.27), Eq. (2.50) would have allowed us to write

$$(2f_\pi)^{-1} G_{\sigma\pi\pi} = 2m_\pi^2 G_{\sigma\pi}(0) \quad , \qquad \text{(naive)} ;$$

unless the dilation appears as an extremely narrow resonance, i.e., $|F_{\sigma\pi}(0)| << 1$, this relation violently contradicts the Goldberger-Treiman relation (2.31).

However, we may substitute the original Goldberger-Treiman relation (1.66) and Eq. (2.26) in Eq. (2.50) to obtain[15]

$$g_{\sigma NN}/g_{\pi NN} \approx F_{\sigma\pi}(0)/g_A \quad . \tag{2.59}$$

Thus $|F_{\sigma\pi}(0)| << 1$ would require the dilaton to couple weakly to nucleons. This is not possible if the dilaton is assumed to be the scalar particle exchanged in NN and πN scattering. Therefore, we conclude that Eq. (2.27) is correct but useless--in practice, Eq. (2.58), (which is consistent with Eq. (2.27)), should be used.

Combining Eqs. (2.26) and (2.58), we derive the relation[15, 83]

$$g_{\sigma\pi\pi}/g_{\sigma NN} \approx m_\sigma^2/(2m_\pi M_N) \quad , \tag{2.60}$$

with $g_{\sigma\pi\pi} = G_{\sigma\pi\pi}/2m_\pi$. Thus σ-exchange in πN and NN scattering can be compared with or predicted from the mass and width of the dilaton. The order of magnitude of these exchange forces indicates that the dilaton has a width of several hundred MeV. This agrees with the discussion based on the Adler-Weisberger sum rule for $\pi\pi$ scattering; (see Eq. (2.35)). The positive sign of $g_{\sigma\pi\pi}/g_{\sigma NN}$ as given by Eq. (2.60) agrees with the sign obtained from the scalar-exchange contribution to πN scattering.

Following the prejudices established early in this chapter, we try fitting data for the $\epsilon(700)$ meson to our formula. If the width of the ϵ is taken to be 400 MeV, i.e.,

$$g_{\epsilon\pi\pi}{}^2/4\pi \simeq 11 \quad , \tag{2.61}$$

Eq. (1.101) for $g_{\epsilon NN}$ implies

$$g_{\epsilon\pi\pi}/g_{\epsilon NN} \simeq 1 \quad , \tag{2.62}$$

whereas the right-hand side of Eq. (2.60) is

$$m_\epsilon{}^2/(2m_\pi \bar{M}_N) \simeq 2 \quad . \tag{2.63}$$

In view of the uncertainty in the PCDC predictions which lead to Eq. (2.60), the discrepancy between Eqs. (2.62) and (2.63) is not significant.[15]

Eq. (2.58) may be related to the work of Chang and Freund.[102] If the dilaton is treated as a $^3P_0$ state in the quark model, generalized W-spin symmetry predicts[102]

$$G_{\sigma\rho\rho} = G_{\sigma\pi\pi} \quad , \tag{2.64}$$

where $\sigma\rho\rho$ coupling is $-g_{\alpha\beta}\, G_{\sigma\rho\rho} + k_\alpha k_\beta \bar{G}_{\sigma\rho\rho}$ for a dilaton with momentum k_μ. PCDC applied to $\langle \rho | \theta^\mu_\mu | \rho \rangle$ yields

$$F_\sigma G_{\sigma\rho\rho} = 2m_\rho^2 + 0(m_\sigma{}^2) \quad . \tag{2.65}$$

Chang and Freund showed that the disastrous conclusion $m_\rho^2 = m_\pi^2$ follows from Eqs. (2.27), (2.64) and (2.65), if one ignores all $0(m_\sigma^2)$ terms; (that motivated their use of a "tenth" piece in θ_μ^μ). Using Eqs. (2.58) and (2.65), we find(15)

$$G_{\sigma\rho\rho} \approx (2m_\rho^{\ 2}/m_\sigma^2) \ G_{\sigma\pi\pi} \quad . \tag{2.66}$$

The mass of the dilaton is not supposed to be much less than m_ρ, so the effects of scale violation in Eq. (2.65) are probably serious. Therefore, the deviation of Eq. (2.66) from the W-spin result (2.64) is not significant.

The success of our calculation depends on the assumption that SU(2) x SU(2) is a much better symmetry than scale invariance. The same cannot be said for SU(3) x SU(3). The Goldberger-Treiman relation for the σKK coupling is

$$(2f_K)^{-1} G_{\sigma KK} = m_\sigma^2 \ F_{\sigma K}(0) + 0(m_K^{\ 2}) \ . \tag{2.67}$$

Evidently, we cannot claim that the $0(m_K^{\ 2})$ term is unimportant, especially if we are considering a theory in which scale invariance is automatically realized in the limit of SU(3) x SU(3) symmetry, (i.e., $\delta \to 0$ when $u \to 0$). Also, mixing becomes important.

It is amusing to compare our method of deriving $G_{\sigma\pi\pi}$ with that of Ellis,(83) which is phrased entirely in the language of effective non-linear Lagrangians. Ellis begins with the observation that an octet of pseudoscalar meson fields M^a obeying the non-linear chiral transformation law $[F_5^a, M^b] = i f^{ab}(M)$ necessarily has dimension zero if Eq. (1.57) is valid. By constructing chirally covariant derivatives $D_\mu M^a = \partial_\mu M^a + 0(M^2 \partial M)$, (where the exact form of $0(M^2 \partial M)$ does not concern us here), one can form chiral-invariant kinetic energy terms $T = \frac{1}{2} D_\mu M^a D^\mu M^a$. The dimension of T is -2, so an extra chiral-invariant factor* $\exp(-2\sigma/F_\sigma)$ must be included to give a scale-invariant result. Proceeding in this manner, Ellis obtains the mesonic Lagrangian(103, 104, 98)

$$\mathcal{L} = \frac{1}{2} D_\mu M^a D^\mu M^a \exp(-2\sigma/F_\sigma) + \frac{1}{2} \partial_\mu \sigma \, \partial^\mu \sigma \exp(-2\sigma/F_\sigma)$$

$$- \frac{1}{4} \ell_u \langle 0|U|0\rangle \exp(-4\sigma/F_\sigma) + \frac{1}{4}(\ell_u + 4) \langle 0|U|0\rangle \quad (2.68)$$

$$- \; U \exp(\ell_u \sigma/F_\sigma) \quad .$$

The first three terms are scale-invariant, where the third term depends on the details of the symmetry violation given by the last two terms. In terms of the decomposition (1.83) of θ_{oo}, the fourth and fifth terms are $-\delta$ and $-u$ respectively. By assuming that δ is a c-number, Ellis fixes the coefficients of the third and fourth terms, since $\mathcal{L}$ must not contain an overall constant or a term linear in σ. Note that, a priori, any value of ℓ_u is

*See Eqs. (1.105-7).

is allowable. The factor U has the series expansion

$$U = -U_o - c\, U_8 = \langle 0 | U | 0 \rangle + \tfrac{1}{2} m_\pi^2 \pi^2 + \tfrac{1}{2} m_K^2 K^2 + \tfrac{1}{2} m_\eta^2 \eta^2 + 0(M^4) \, , \tag{2.69}$$

(i. e., dim U = 0).

To calculate the $\sigma\pi\pi$ coupling constant, one finds the coefficient of $\frac{1}{2}\sigma\pi^2$ in $\mathscr{L}$, replacing $i\partial_\mu$ by the appropriate momentum (as in the rules for Feynman diagrams):

$$\begin{aligned} \mathscr{L} &\to \tfrac{1}{2}\partial_\mu \pi \partial^\mu \pi (-2\sigma/F_\sigma) - \tfrac{1}{2} m_\pi^2 \pi^2 (\ell_u \sigma/F_\sigma) \\ &\to (2\, P_{\pi_1} \cdot P_{\pi_2} - \ell_u m_\pi^2)\, F_\sigma^{-1} (\tfrac{1}{2}\sigma\pi^2) \, , \end{aligned} \tag{2.70}$$

which implies

$$F_\sigma G_{\sigma\pi\pi} = m_\sigma^2 - (\ell_u + 2)\, m_\pi^2 \, . \tag{2.71}$$

The term depending on m_π^2 is obviously negligible.* To emphasize the lack of dependence of our calculation on ℓ_u, we have always omitted it.[15, 77] Otherwise, Eqs. (2.71) and (2.58) are identical.

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* Some workers (Refs. 84 and 92) have taken the trouble to calculate this term using formalism which resembles ours. Note that the extra term is important if the σKK coupling is being considered; but then mixing is also important.

Whereas we were led to discover the m_σ^2 term by considering the relative magnitudes of symmetry violation, Ellis gets it from the "momentum dependence of the $\sigma\pi\pi$ vertex," (in the jargon of users of effective Lagrangians).

To conclude this section, we comment further on Eqs. (2.41) and (2.52), which, at first sight, contradict each other. As Eq. (2.57) demonstrates, this behavior of $F_2(t)$ at $t = 0$ is required in a consistent theory of spontaneous breakdown of scale and chiral invariance. However, we regard the difference between Eqs. (2.41) and (2.52) as significant only for dilaton theories. In the scale-invariant limit of a theory with no dilatons, all masses vanish, so chiral invariance is no longer realized in the Nambu-Goldstone manner and has no connection with soft-meson amplitudes. Therefore, this difference is significant only if the zero-mass limit of Eq. (2.52) is supposed to be smooth, in spite of the infrared problem. This limit might be smooth for amplitudes which are vacuum-expectation values of an operator product expansion near the light cone, (i.e., a long way off-mass-shell). However, Eq. (2.52) is a low-energy result arising from the behavior of the operator product $R_{\lambda\mu\nu}{}^*(x)$ at large x^2, so we expect that the difference between Eqs. (2.41) and (2.52) is generated by infrared effects if no dilatons are present.

A contrary point of view has been expressed by Jackiw.[105] He introduces the term $S_{\mu\nu}$ of Eq. (1.56) in order to make the chiral-invariant prediction for $F_2(0)$ agree with the result of allowing $\theta_\mu{}^\mu$ to vanish, (even though he admits that the zero-mass limit might be singular). In addition, this attitude requires $\ell_u = -1$. It is possible that these results are right. We can only prove that $S_{\mu\nu}$ vanishes in the limit of scale invariance. Provided that $S_{\mu\nu}$ has dimension greater than -4, it does not affect the leading singularities of operator product expansions. It contributes a scale-violating effect to deep inelastic neutrino scattering, but this would be swamped by the trace terms mentioned in the footnote to Eq. (1.92). To have a hope of directly measuring its effects, one would have to perform the experiment $\nu\bar{\nu} \rightarrow$ anything hadronic. The current commutation relations impose constraints on $S_{\mu\nu}$, but these are not strong enough to allow any definite conclusions. Jackiw presents an example, the σ-model with a modified $\theta_{\mu\nu}$, in which his conclusions are valid.* While these conclusions are plausible, the reasons given for accepting them are not.[77]**

It is also possible that $S_{\mu\nu}$ exists in a theory with dilatons. However, it would be smaller than the term proposed by Jackiw, because the magnitude of scale violation is much smaller; (e.g., Eq. (2.52) would become $F_2(0) = -\frac{1}{3}(1 + m_\sigma^2/s_o)$, where $|s_o| \gtrsim 1\,\text{GeV}^2$ remains finite for $m_\sigma \rightarrow 0$). If it can be shown that the dimension of

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*This model does not work in a dilaton theory because it must be supposed that $1/2f_\pi$ vanishes in the scale-invariant limit; otherwise $S_{\mu\nu}$ does not vanish in that limit.

**Jackiw's work has also been criticized in Ref. 98, but only within the context of dilaton theory. Jackiw prefers not to entertain the idea that scale invariance could be spontaneously broken.

scalar operators (like s(x) in Eq. (1.56)) is not greater than -2, then the presence of $S_{\mu\nu}$ is not allowed in a theory of broken scale invariance. For the moment, we can only assume this.

II. 3. Conformal Invariance and Tensor Meson Dominance

We examine the short-distance behavior of the commutator of $\theta_{\mu\nu}$ and $\mathfrak{F}^a_{5\lambda}$. The main uncertainty in this procedure involves identifying the "licensed operators" 0_n which appear in the expansion (1.79). We restrict our attention to operators with dimension greater than or equal to -4, since the corresponding $C_n(x)$ are then sufficiently singular to be of interest. Applying the principles laid down by Wilson,[40] we obtain*

$$96\pi^2 \left[\theta_{\mu\nu}(x), \mathfrak{F}^a_{5\lambda}(0)\right] = -6\partial_{\mu\nu\lambda\alpha} \log x^2 \mathfrak{F}_5^{a\alpha}(0)$$

$$+ (\partial_{\mu\nu\lambda\alpha} x_\beta + 3\,\partial_{\mu\nu\lambda\beta} x_\alpha - 3\,\partial_{\mu\nu\alpha\beta} x_\lambda) \log x^2 \partial^\alpha \mathfrak{F}_5^{a\beta}(0) + \ldots \tag{2.72}$$

$$\ldots - 4(\ell_u + 3)(\partial_\mu \partial_\nu - g_{\mu\nu}\partial^2)\, x_\lambda\, E^{(-1)}(x^2)\, \partial^\alpha \mathfrak{F}^a_{5\alpha}(0)$$

$$+ r(\partial_\mu \partial_\nu - g_{\mu\nu}\partial^2)\left\{x_\lambda\, E^{(-2)}(x^2)\, \partial^\alpha \mathfrak{F}^a_{5\alpha}(x)\right\} + \ldots ,$$

where $\partial_{\mu\nu\alpha\beta}$ is a differential operator which is conserved and

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* The symbol + ... - denotes separation of the scale-invariant and scale-violating contributions to the expansion.

traceless in the indices μ, ν ;

$$\partial_{\mu\nu\alpha\beta}=6\partial_\mu\partial_\nu\partial_\alpha\partial_\beta - 3(\partial_\mu\partial_\alpha g_{\nu\beta} + \partial_\nu\partial_\alpha g_{\mu\beta} + \partial_\mu\partial_\beta g_{\nu\alpha} + \partial_\nu\partial_\beta g_{\mu\alpha})\partial^2$$

$$+ 3(g_{\mu\alpha}g_{\nu\beta} + g_{\nu\alpha}g_{\mu\beta})\partial^4 - 2(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)(\partial_\alpha\partial_\beta - g_{\alpha\beta}\partial^2). \tag{2.73}$$

Of the constant coefficients appearing in Eq. (2.72), only r is not determined by the hypotheses*--zero trace and divergence for the leading singularities, together with the equal-time commutation relations (ETCR) required by Poincaré invariance. Further, these ETCR forbid scale-invariant contributions of the form

$$c_1(\epsilon_{\mu\lambda\beta\gamma}\partial_\nu + \epsilon_{\nu\lambda\beta\gamma}\partial_\mu)\partial^\gamma \, E^{(-1)}(x^2)\,\mathfrak{F}^{\alpha\beta}(0) + c_2\,\partial_{\mu\nu\lambda\alpha}\log x^2 \partial_\beta \mathcal{J}_5^{\alpha\beta}(0)$$

$$+ c_3\partial_{\mu\nu\lambda\alpha}x_\beta E^{(-1)}(x^2)\,\mathcal{J}_5^{\alpha\beta}(0) + c_4(2\partial_{\mu\nu\lambda\alpha}x_\beta - \partial_{\mu\nu\alpha\beta}x_\alpha)\log x^2\,\theta_5^{a\alpha\beta}(0) \tag{2.74}$$

which would otherwise be allowed; ($\theta^a_{5\mu\nu}(x)$ is the axial part of a (1,8) + (8,1) tensor $(\theta^a_{\mu\nu}, \theta^a_{5\mu\nu})_{a=1\ldots8}$, where $\theta^a_{\mu\nu}$ is the tensor octet whose presence is indicated by the electroproduction data[106]). The singular functions $E^{(-n)}(x^2)$ were defined by Eq. (1.81), while "$\log x^2$" obeys $\partial_\mu \log x^2 = -x_\mu E^{(-1)}(x^2)$; (we might have

*However, r must vanish for $\ell_u \leqslant -2$.

written $i\pi\theta(x^2)$ instead of "log x^2").

From this operator product expansion, we wish to determine whether the standard assumption(9, 68, 107),

$$[K_\mu(0), \mathcal{F}_{5\lambda}(0)] = 0 \tag{2.75}$$

should be believed. Since the K_μ are second moments of $\theta_{\mu\nu}$, the only terms in the expansion which can contribute involve a singularity $(x)^{-5}$, i.e., the last term of Eq. (2.72), which corresponds to Jackiw's term $S_{\mu\nu}$ appearing in $\theta_{\mu\nu}$--see Eq. (1.56). However, we assume the usual theory of broken chiral symmetry ($r = 0$), so Eq. (2.75) is valid, and there are no Schwinger terms in $[\theta_{\mu\nu}(0,\vec{x}), \mathcal{F}_{5\mu}(0)]$ more singular than $\vec{\partial}\delta^3(\vec{x})$.

Starting with Eq. (2.75), we may perform a calculation(77) which is analogous to our derivation of Eq. (2.50). Again, we work in the limit of scale invariance, and defer to the next chapter discussion of a more realistic but less elegant approach involving dispersion relations.

The appropriate time-ordered product is

$$M_{\mu\nu\lambda}(k,q) = i\int d^4x\, e^{ik\cdot x} \langle 0 |\, T(\mathcal{K}_{\mu\nu}(x)\, \mathcal{F}^3_{5\lambda}(0))\, | \pi^0(q) \rangle, \tag{2.76}$$

which obeys the identity

$$k^\nu M_{\mu\nu\lambda}(k,q) = 0(k) \tag{2.77}$$

because of Eq. (2.75). Again, we must calculate the contributions to $M_{\mu\nu\lambda}$ which are singular in k. From the definition (1.12) of the conformal current $\mathcal{K}_{\mu\nu}(x)$, it is easy to see that this involves second derivatives of the pole terms given by Eq. (2.47):

$$k^{\nu} M_{\mu\nu\lambda}(k,q) = -k^{\nu}\left[2\frac{\partial}{\partial k^{\mu}}\frac{\partial}{\partial k_{\alpha}} - \delta^{\alpha}_{\mu}\frac{\partial}{\partial k^{\beta}}\frac{\partial}{\partial k_{\beta}}\right] P_{\alpha\nu\lambda}(k,q) + 0(k) . \tag{2.78}$$

As in our previous calculation, the simplest way of evaluating an expression such as Eq. (2.78) is to take the k^{ν} factor inside the derivatives:

$$k^{\nu} M_{\mu\nu\lambda}(k,q) = -\left[2\frac{\partial}{\partial k^{\mu}}\frac{\partial}{\partial k_{\alpha}} - \delta^{\alpha}_{\mu}\frac{\partial}{\partial k^{\beta}}\frac{\partial}{\partial k_{\beta}}\right] k^{\nu} P_{\alpha\nu\lambda}(k,q) \tag{2.79}$$

$$+ 2\frac{\partial}{\partial k^{\mu}} P^{\nu}_{\nu\lambda}(k,q) + 0(k) .$$

(The symmetry of $P_{\alpha\nu\lambda}$ in (α, ν) considerably reduces the length of this expression). Combining Eqs. (2.47), (2.77) and (2.79), we have

$$2i\, g_{\mu\lambda}(F_{\sigma}F_{\sigma\pi}(0) - (2f_{\pi})^{-1}) = 4i\, q_{\mu}q_{\lambda}(F_{\sigma}F_{\sigma\pi}'(0) - (2f_{\pi})^{-1} F_{1}'(0)), \tag{2.80}$$

which implies Eq. (2.50) together with a new relation[77]

$$F_{\sigma\pi}'(0)/F_{\sigma\pi}(0) = F_{1}'(0) . \tag{2.81}$$

The primes denote differentiation with respect to the momentum transfer squared.

If poles due to the $A_1(1070)$ meson and the spin-2 SU(3) singlet, which corresponds to a mixture of the f(1260) and f'(1515) states, are supposed to dominate $F_{\sigma\pi}(t)$ and $F_1(t)$ respectively, Eq. (2.81) implies

$$m^2(2^+) \approx m^2(A_1) \quad . \tag{2.82}$$

When the magnitude of SU(3) mass splitting is considered, the agreement of this result with the observed meson spectrum is reasonable. However, it turns out that Eq. (2.81) is strongly affected by scale-violating effects, and the results of A_1 and f dominance are not consistent. These considerations form the subject of the next chapter.

III. DISPERSION THEORY AND ESTIMATES OF SYMMETRY VIOLATION

Soft-meson theorems are exact in the limit in which the current which couples the meson to the vacuum is conserved. However, in the real world, these results become approximate; one has to saturate unsubtracted dispersion relations for the channel characterized by the quantum numbers of the meson involved. Typically, only the meson pole itself is taken into account, because the evaluation of contributions of many-body intermediate states seems to be hopeless in the absence of a complete theory of strong interactions. In a few fortunate instances, one can directly substitute experimental cross-sections or phase shifts to obtain a relatively accurate estimate of the effects of symmetry violation. When trying to understand the nature of symmetry-violating terms in θ_{oo}, the accurate saturation of such dispersion integrals in very important.

It is evident that the application of conformal-invariant results to amplitudes in the real world requires a great deal of care:

(i) Conformal symmetry is badly broken, with no candidate for the dilaton below 0.5 GeV. We assume that the mass of the dilaton is significantly less than 1 GeV, ($500 \lesssim m_\sigma \lesssim 800$ MeV).

(ii) This symmetry violation may be accompanied by mixing of the dilaton with other $(J^P, I^G) = (0^+, 0^+)$ mesons.

Unfortunately, in attempting to obtain corrections for Eqs. (2.50) and (2.81), we cannot improve on the approximation of keeping only the meson pole. We try to make up for this deficiency by saturating two different dispersion integrals for the same quantity. By proceeding in this manner, we can give rough estimates of the uncertainties in our results.

To conclude this chapter, predictions of the theory of broken chiral symmetry are re-examined. This is necessary, because the presence of a dilaton can affect low-energy theorems for pseudoscalar meson amplitudes if more than one such meson is present. In particular, we show that, in the recent calculation of Cheng and Dashen,[108] the soft-pion theorem which they assume is still valid in a dilaton theory. This leads to an alternative interpretation of their result.

III.1. Effects of Mixing

A phenomenological analysis of mixing in the nonet picture has been carried out by Carruthers.[89] From his results, we find[77]

$$F_{\sigma} \approx 102 \text{ MeV} \qquad , \qquad F_{\epsilon'} \approx 68 \text{ MeV} \quad . \qquad (3.1)$$

These values correspond to a small value of the coupling of the octet state to the vacuum via $\theta_{\mu\nu}$ when $|\sigma = \epsilon\rangle$ and $|\epsilon'\rangle$ are (roughly) ideally mixed. This feature of Carruthers' analysis is ensured by his use of SU(3) symmetry to relate coupling constants; i.e., it is (correctly) supposed that SU(3) is realized as a degeneracy symmetry.* Eq. (3.1) is very approximate because PCDC for baryons has been used.

A good illustration of the effect of mixing is provided by PCDC for the matrix element $\langle \phi | \theta_\mu^\mu | \phi \rangle$:[77]

$$F_\sigma G_{\sigma\phi\phi} + F_{\epsilon'} G_{\epsilon'\phi\phi} = 2m_\phi^2 + 0(m_\sigma^2) \quad . \tag{3.2}$$

If the scalar nonet is taken to be a set of $^3P_0$ states in the quark model, the apparent suppression of the $\epsilon' \to \pi\pi$ mode relative to $\epsilon' \to K\bar{K}$ suggests that ϵ' contains only strange quarks. Ellis[83] has pointed out that this standard picture implies

$$G_{\sigma\phi\phi} = 0 \quad , \tag{3.3}$$

because the corresponding quark diagram is disconnected. Thus Eq. (3.2) shows that the ϵ' pole provides an essential contribution to the unsubtracted dispersion relation for $\langle \phi | \theta_\mu^\mu | \phi \rangle$. As the limit of conformal invariance is approached, the ϵ' state loses its dilaton quality to the α state, and Eq. (3.2) becomes

$$F_\sigma G_{\sigma\phi\phi} = 2m_\phi^2 \quad , \tag{3.4}$$

*A recent claim (Ref. 109) that Carruthers' assumptions combined with Ward identities for the meson system give a large value for the octet coupling $\langle 8 | \theta^{\mu\nu} | 0 \rangle$ relative to the singlet coupling is incorrect; (add their Eqs. (63) and (64) to obtain a contradiction). Also, their claim $\ell_u = -2$ is a direct result of assuming Eq. (30) -- no Ward identities are needed. Unfortunately, Eq. (30) is not general valid, and is certainly not implied by SU(3) symmetry.

in agreement with Eq. (1.77).

Another obvious example[77] is the sum rule[83,110]

$$- i \langle 0 | [D, \theta_\mu{}^\mu] | 0 \rangle = i \int d^4x \, \theta(x_0) \langle 0 | [\theta_\nu{}^\nu(x), \theta_\mu^\mu(0)] | 0 \rangle$$

$$\approx m_\sigma^2 \, F_\sigma^2 + m_{\epsilon'}{}^2 F_{\epsilon'}^2 \quad . \tag{3.5}$$

Numerically, we observe

$$m_\sigma^2 F_\sigma^2 \simeq m_{\epsilon'}{}^2 \, F_{\epsilon'}{}^2 \tag{3.6}$$

so the contribution of the ϵ' pole to the sum rule is significant.

Eq. (3.5) tests the idea that δ is a c-number. In that case, we have

$$m_\sigma^2 F_\sigma^2 + m_{\epsilon'}^2 F_{\epsilon'}^2 \approx \ell_u(\ell_u + 4) \langle 0 | u | 0 \rangle \quad . \tag{3.7}$$

According to Gell-Mann, Oakes and Renner,[49] this vacuum expectation value is given by the approximate formula

$$\langle 0 | u | 0 \rangle \simeq - 3m_\eta^2/(16 \, f_\pi{}^2) \; , \tag{3.8}$$

which gives the numerical result[77]

$$- \ell_u(\ell_u + 4) \approx 5 \quad . \tag{3.9}$$

We think that the uncertainty in Eq. (3.9) is sufficiently large to allow $-3 \lesssim \ell_u \lesssim -1$.

We observe one more example in which the ϵ' contribution is important: the standard soft-dilaton theorems

$$F_\sigma \langle\sigma|u_o|0\rangle + F_{\epsilon'}\langle\epsilon'|u_o|0\rangle \approx \ell_u \langle 0|u_o|0\rangle \quad , \tag{3.10}$$

$$F_\sigma \langle\sigma|u_8|0\rangle + F_{\epsilon'}\langle\epsilon'|u_8|0\rangle \approx \ell_u \langle 0|u_8|0\rangle \approx 0 \ . \tag{3.11}$$

In some cases, it is expected that the ϵ' contribution is negligible compared with that of the dilaton. Consider the usual method(15, 84) of estimating $G_{\sigma\pi\pi}$; (i.e., the derivation of Eq. (2.58) by first obtaining Eq. (2.52)). Neglecting the mass of the pion, we combine an unsubtracted dispersion relation for $F_2(t)$, Eq. (2.52) and the decomposition

$$\mathrm{Im}\, F_2(t) = -\frac{\pi}{3}\,\delta(t-m_\sigma^2)F_\sigma G_{\sigma\pi\pi} - \frac{\pi}{3}\,\delta(t-m_{\epsilon'}^2)F_{\epsilon'}\, G_{\epsilon'\pi\pi} + f(t) \ , \tag{3.12}$$

to obtain(77)

$$1 = F_\sigma\, G_{\sigma\pi\pi}/m_\sigma^2 + F_{\epsilon'}\, G_{\epsilon'\pi\pi}/m_{\epsilon'}^2 - \frac{3}{\pi}\int dt\; f(t)/t \ . \tag{3.13}$$

The contribution of the ϵ' term is small; applying Eq. (2.31), we record this observation in the form

$$F_{\epsilon'}\, F_{\epsilon'\pi}(0) << F_\sigma\, F_{\sigma\pi}(0) \quad , \tag{3.14}$$

where the numerical difference involves a factor of about 10. If the continuum integral in Eq. (3.13) is also insignificant, Eq. (2.58) is obtained, and the prediction for the width of the dilaton is $\Gamma_{\sigma\to\pi\pi} \approx$ 1200 MeV. An optimistic view of the accuracy of this result might lead to the conclusion that it violates the Adler-Weisberger sum rule for $\pi\pi$ scattering (Eq. (1.94)). However, the next section is devoted to estimating $G_{\sigma\pi\pi}$ according to the method of collinear dispersion relations,[91] and there we conclude that our theory of broken scale invariance is not in conflict with the $\pi\pi$ sum rule. In the forthcoming calculation, it requires considerable effort, (recorded in Appendix D), to show that the ϵ' term may be neglected.

III. 2. Collinear Dispersion Relations and Violation of Conformal Invariance

In this section, we apply collinear dispersion relations to obtain symmetry-breaking corrections to Eqs. (2.50) and (2.81). In current algebra, this is one of a number of available procedures for obtaining the consequences of a given equal-time commutator. We pause briefly to examine the status of the method of collinear dispersion relations relative to the $P_z \to \infty$ and low-energy approaches.

The standard example from current algebra is[111]

$$\langle N_2\lambda_2\vec{P}_2|\left[F^a(\vec{q}_2), F^b(\vec{q}_1)\right]|N_1\lambda_1\vec{P}_1\rangle = i f^{abc}\langle N_2\lambda_2\vec{P}_2|F^c(\vec{q}_1+\vec{q}_2)|N_1\lambda_1\vec{P}_1\rangle \,, \tag{3.15}$$

where $|N\lambda\vec{P}\rangle$ denotes a hadronic state with momentum $\vec{P}$, helicity λ, and spin, mass and internal quantum numbers N, and $F^a(\vec{q})$ is given by $\int d^3x \exp(i\vec{q}.\vec{x})\,\mathcal{F}^a_o(0,\vec{x})$. To evaluate the left-hand side of Eq. (3.15), a complete set of states $|I\rangle$ is introduced via the identity $\sum_I |I\rangle\langle I| = 1$, the result being a class of sum rules for each value of $\vec{P} = \frac{1}{2}(\vec{P}_1+\vec{P}_2)$

Following the suggestion of Fubini and Furlan,[112] take the limit $P_z \to \infty$ and interchange it with $\sum_I$. This procedure is valid for "good" operators such as $F^a(\vec{q})$. One advantage of considering such a limit is that only connected matrix elements* $\langle I|\mathcal{F}_o|N\lambda\rangle_{conn.}$ with fixed $q^2 = (P_I-P)^2$ remain, so saturation of the sum rule is simpler and, for $|q^2| \lesssim m^2(0^-, \underline{8})$, PCAC may be applied. In terms of dispersion theory for the amplitude

$$(N_1\lambda_1P_1) + \mathcal{F}_\mu^b(q_1) \rightarrow (N_2\lambda_2P_2) + \mathcal{F}_\nu^a(q_2) \quad , \tag{3.16}$$

(with $P = \frac{1}{2}(P_1+P_2)$, $Q = \frac{1}{2}(q_1+q_2)$), the latter case corresponds to dispersing in two paths:

(i) Associated with the sum over states $\sum_I$, there is the path $\nu = P.Q$ varying, with $t = (P_2-P_1)^2 = (q_1-q_2)^2$, $q_1^2 \leq 0$, and $q_2^2 \leq 0$ all constant.

(ii) Associated with PCAC, there is a path q_i^2 varying (for i = 1 or 2 or both), with ν and t held fixed.

Disadvantages of the $P_z = \infty$ method are:

(1) For good-bad commutators, Z diagrams may not vanish for $P_z \rightarrow \infty$,** and the method fails for bad-bad commutators.

(2) The method obviously does not apply if one of the states is the vacuum.

A much smaller set of sum rules results from combining low-energy theorems with the appropriate unsubtracted dispersion relations. All of the $q^2 = 0$, $P_z = \infty$ rules may be expressed in this form. In addition, (1) and (2) do not apply. So far, we have been restricted to this type of calculation.

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* That is, meson-creation and Z diagrams vanish.

** Details are given in Ref. 44.

In 1968, Fubini and Furlan[91] introduced collinear dispersion relations, which involve a single dispersion path combining some of the features of paths (i) and (ii). In terms of Eq. (3.15), the complete set of intermediate states is introduced at $\vec{P} = 0$ instead of at infinite momentum, so meson-creation and Z diagrams must be evaluated as well as direct-channel diagrams. This additional complication was accepted in order that the following dispersion path could be used:

(iii) The variable is x, defined by $q_i = xP + k_i$, $(i = 1, 2)$, with $k_1 . P = 0 = k_2 . P$, t, k_1^2 and k_2^2 kept constant. In the Breit frame, $\vec{P} = 0$, this prescription becomes $q_i = (xP_o, \vec{q}_i)$, so we obtain variable-q_i^2, fixed-$\vec{q}_i$ sum rules. Meson amplitudes appear as the residues of poles in meson-creation diagrams.

The advantages of this method are:

(3) The lack of subtractions in each dispersion relation* is guaranteed by the existence of a Bjorken limit. We are now able to check this by looking at the short-distance behavior of the corresponding operator-product expansion; (i.e., the theory of broken scale invariance provides an extension of the PCAC hypothesis to the Fourier transforms of retarded commutators containing $\partial^\mu \mathcal{F}_{5\mu}$).

(4) There are no anomalous thresholds on the first Riemann sheet.

(5) The set of contributing states $|I\rangle$ is very restricted because of antular momentum and parity selection rules, particularly for the case $\vec{k}_i = 0$.

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* Alternatively, the need for a subtraction may be indicated by scale invariance at short distances.

(6) The method does not depend on whether an operator is "good" or "bad". For this reason, von Hippel and Kim[113] used collinear dispersion relations to relate meson-baryon scattering at threshold to $\langle B | [F_5^a(0), \partial^\mu \mathfrak{F}^b_{5\mu}(0)] | B \rangle$.

(7) The method works if one of the states is the vacuum.

Disadvantages of collinear dispersion relations are:

(8) The distance of the meson pole from the "soft-meson point" is often considerably larger than $m^2(0^-, \underline{8})$, which is the corresponding distance for the $P_z \to \infty$ and low-energy approaches.

(9) Apart from simple poles, the meson-creation and Z diagrams are potentially important but difficult to estimate; in practice, they are "thrown away". However, if one of the states is the vacuum, all diagrams are meson-creation diagrams. Because of (5), the error involved in neglecting the cut diagrams is about the same as in derivations of Goldberger-Treiman relations for the currents involved. The calculations in this section are of the latter type.

(10) There is a difficulty in evaluating the direct-channel cut contributions, because q_i^2 becomes large. The prescription given by Fubini and Furlan, and followed by von Hippel and Kim, is to replace them with on-shell S-wave phase shifts, (i.e., $q_i^2 \to m_i^2$). This has no hope of working unless the cut contribution is small, and is dominated by terms in the region $q_i^2 = 0(m_i^2)$. (Then the approximation is very good because it can be regarded as the application of PCAC to PCAC corrections). Unfortunately, there is no prescription which indicates when this is the case--a small value for the on-shell

estimate does not guarantee success, (although a large value would ensure failure). This problem does not arise if one state is the vacuum.

We conclude that the $P_z \to \infty$ method should be applied when both the initial and final states contain a single particle, while collinear dispersion relations are more reliable if one state is the vacuum. Low-energy theorems are more limited in scope, but may be applied in both cases.

We wish to evaluate

$$\langle 0 | [D(0), \mathfrak{F}_{5\lambda}^{3}(0)] | \pi^{0}(q) \rangle = 3\, q_{\lambda}/2f_{\pi} \quad , \tag{3.17}$$

$$\langle 0 | [K_{\mu}(0), \mathfrak{F}_{5\lambda}^{3}(0)] | \pi^{0}(q) \rangle = 0 \quad , \tag{3.18}$$

in a dispersion-theoretic manner. Since D and K_μ are moments of the bad operator $\theta_{\mu\nu}$, and one of the states is the vacuum, (1) and (2) forbid the $P_z \to \infty$ method. As an alternative to the low-energy approach already considered, we proceed to use collinear dispersion relations. All of the advantages (3-7) hold; of the disadvantages, (10) does not apply, (8) is not serious, and (9) involves errors similar to those encountered in PCDC and σ-dominance of $F_2(t)$.

The plan of the calculation is basically simple, but the computational details are fairly complicated. Here, we describe how to arrive at the results; the full derivation is given in Appendix E.

We consider the amplitude for $\mathfrak{F}_{5\lambda}$ to interact with $\theta_{\mu}^{\ \mu}$ and form a pion:

$$T_\lambda(k,q) = i \int d^4x \; e^{ik\cdot x}\theta(x_o)\langle 0|[\theta_\mu{}^\mu(x), \mathcal{F}^3_{5\lambda}(0)]|\pi^o(q)\rangle$$

$$= i\, q_\lambda X_1 + i\, k_\lambda X_2 \; . \qquad (3.19)$$

Since T_λ is a retarded commutator, it may be written as a dispersion relation along some path of integration in the s-t plane, where s and t are defined by

$$s = k^2 \quad , \quad t = (q-k)^2 \quad . \qquad (3.20)$$

The prescription for a collinear dispersion relation is

$$k = (zm_\pi, \vec{0}) \qquad , \qquad q = (m_\pi, \vec{0}) \quad , \qquad (3.21)$$

where z is the variable of integration in the dispersion relation. This condition specifies the path of integration to be the parabola

$$4m_\pi^2 \, s = (t - s - m_\pi^2)^2 \quad , (t \geq 0) \; , \qquad (3.22)$$

in the s-t plane. The behavior of T_λ at large z is determined by the leading singularity of the operator product $[\theta_\mu{}^\mu(x), \mathcal{F}^3_{5\lambda}(0)]$ at short distances x_μ, (i.e., the scale-violating part of Eq. (2.72)). This singularity goes like* $(x)^{-3}$, so, by dimensional analysis, it contributes a term

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*This turns out to be the case even if r does not vanish.

$$\int d^4x \; e^{im_\pi z x_o} (x)^{-3} \sim z^{-1} \tag{3.23}$$

to T_λ. This agrees with a formula proposed by Bjorken in a paper[70] which implicitly contains many features of the theory of broken scale invariance:

$$T_\lambda = - C_\lambda / m_\pi z + 0(z^{-2}) , \tag{3.24}$$

where C_λ is the equal-time commutator

$$C_\lambda = \int d^4x \; e^{im_\pi z x_o} \; \delta(x_o) \langle 0 | [\theta_\mu{}^\mu(x), \mathfrak{F}_{5\lambda}^3(0)] | \pi^o(q) \rangle . \tag{3.25}$$

Eq. (3.24) implies an unsubtracted dispersion relation for $X_1(z)$,

$$X_1(z) = \frac{1}{\pi} \int dz' \; \text{Im}\, X_1(z')/(z' - z) , \tag{3.26}$$

and a superconvergence-like relation for $X_2(z)$,

$$0 = \int dz \; \text{Im}\, X_2(z) . \tag{3.27}$$

Using methods analogous to those developed in Chapter II, the retarded commutator T_λ may be related to the equal-time commutators (3.17) and (3.18), implying the low-energy theorems[77]

$$2f_\pi X_1(x) = z^{-1} - \tfrac{1}{2} + 0(z) , \tag{3.28}$$

$$2f_\pi dX_1(z)/dz = - z^{-2} - \tfrac{3}{2} F_2(0) + 3m_\pi^2 F_1'(0) + \tfrac{1}{4} + 0(z) , \tag{3.29}$$

respectively. When combined with Eq. (3.26), Eqs. (3.28) and (3.29) become sum rules. The derivation appears in Appendix E.

The situation is illustrated in Fig. 3. The solid, curved line represents the dispersion path given by Eq. (3.22). Poles in the amplitude T_λ are indicated by dotted lines. The Bjorken limit is approached when s and t both become large and positive along the parabola. The collinear dispersion path crosses the pion-pole line $t = m_\pi^2$ at $(0, m_\pi^2)$ and $(4m_\pi^2, m_\pi^2)$, where z takes the values 0, 2 respectively. These points are denoted by (a) and (b) in Fig. 3, and correspond to the dispersion diagrams (a) and (b) of Fig. 2.* The low-energy theorems (3.28) and (3.29) constrain the behavior of $T_\lambda(z)$ in the vicinity of point (a). The singular terms z^{-1} and $-z^{-2}$ of Eqs. (3.28) and (3.29) are present because the pion-pole line passes through point (a). The dilaton-pole line $s = m_\sigma^2$ intersects the dispersion path at the points (c) $= (m_\sigma^2, (m_\sigma - m_\pi)^2)$ and (d) $= (m_\sigma^2, (m_\sigma + m_\pi)^2)$, which correspond to diagrams (c) and (d) of Fig. 2; z takes the values m_σ/m_π, $-m_\sigma/m_\pi$ at these points.

The amplitude for $\theta_\mu^{\ \mu}$ to interact with two pions is projected anywhere along the pion-pole line $t = m_\pi^2$ except in the region $0 < s < 4m_\pi^2$, which lies outside the physical regions for the process. Since the dispersion path (3.22) passes through the edges of these physical regions, the residues of the pole diagrams (a) and (b) are

*Here we need only the trace $g^{\alpha\nu}\theta_{\alpha\nu}$; in the limit of scale invariance, Fig. 2 becomes Fig. 1.

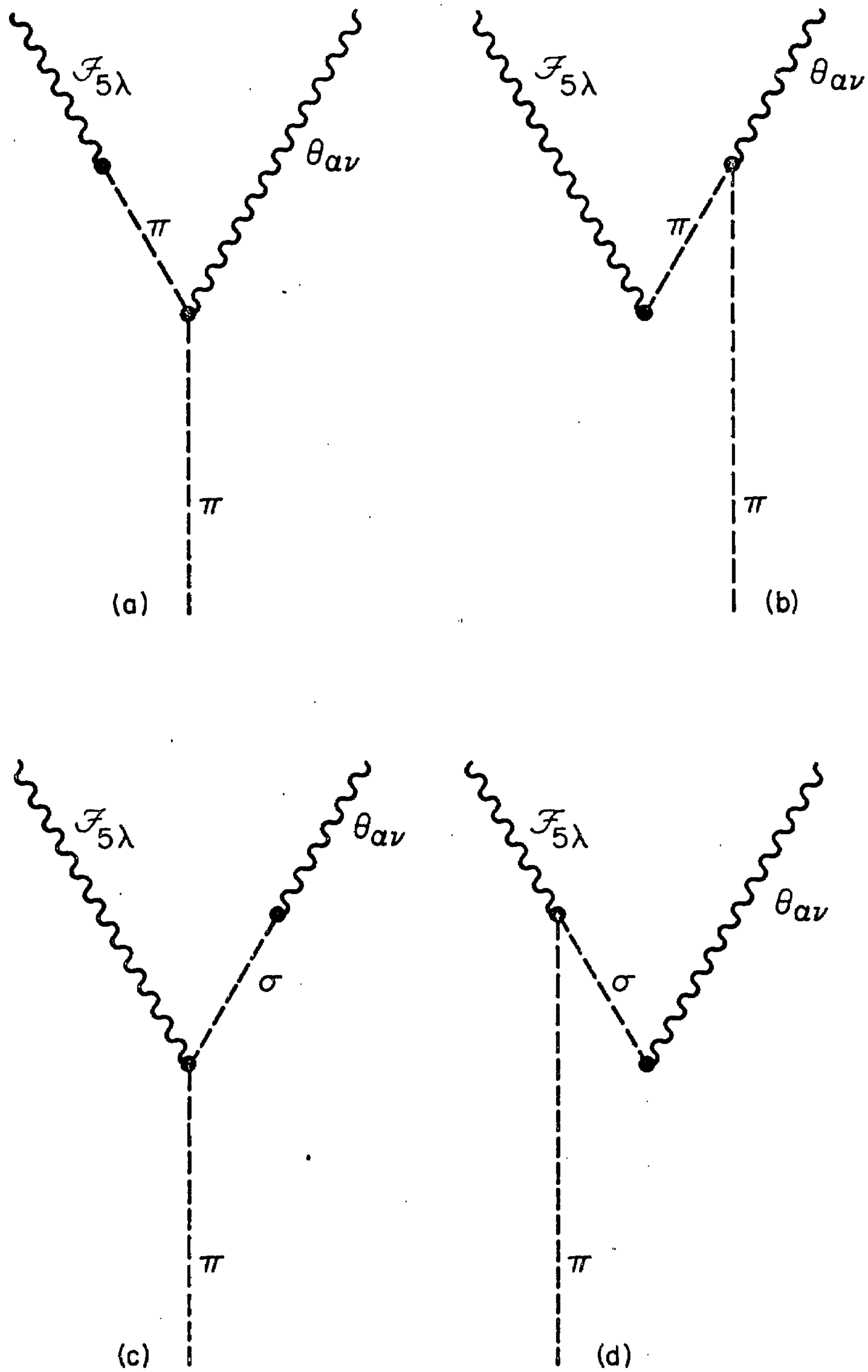
$\mathcal{F}_{5\lambda}$
$\theta_{a\nu}$
π
π
(a)
$\mathcal{F}_{5\lambda}$
$\theta_{a\nu}$
π
π
(b)
$\mathcal{F}_{5\lambda}$
$\theta_{a\nu}$
σ
π
(c)
$\mathcal{F}_{5\lambda}$
$\theta_{a\nu}$
σ
π
(d)

Fig. 2

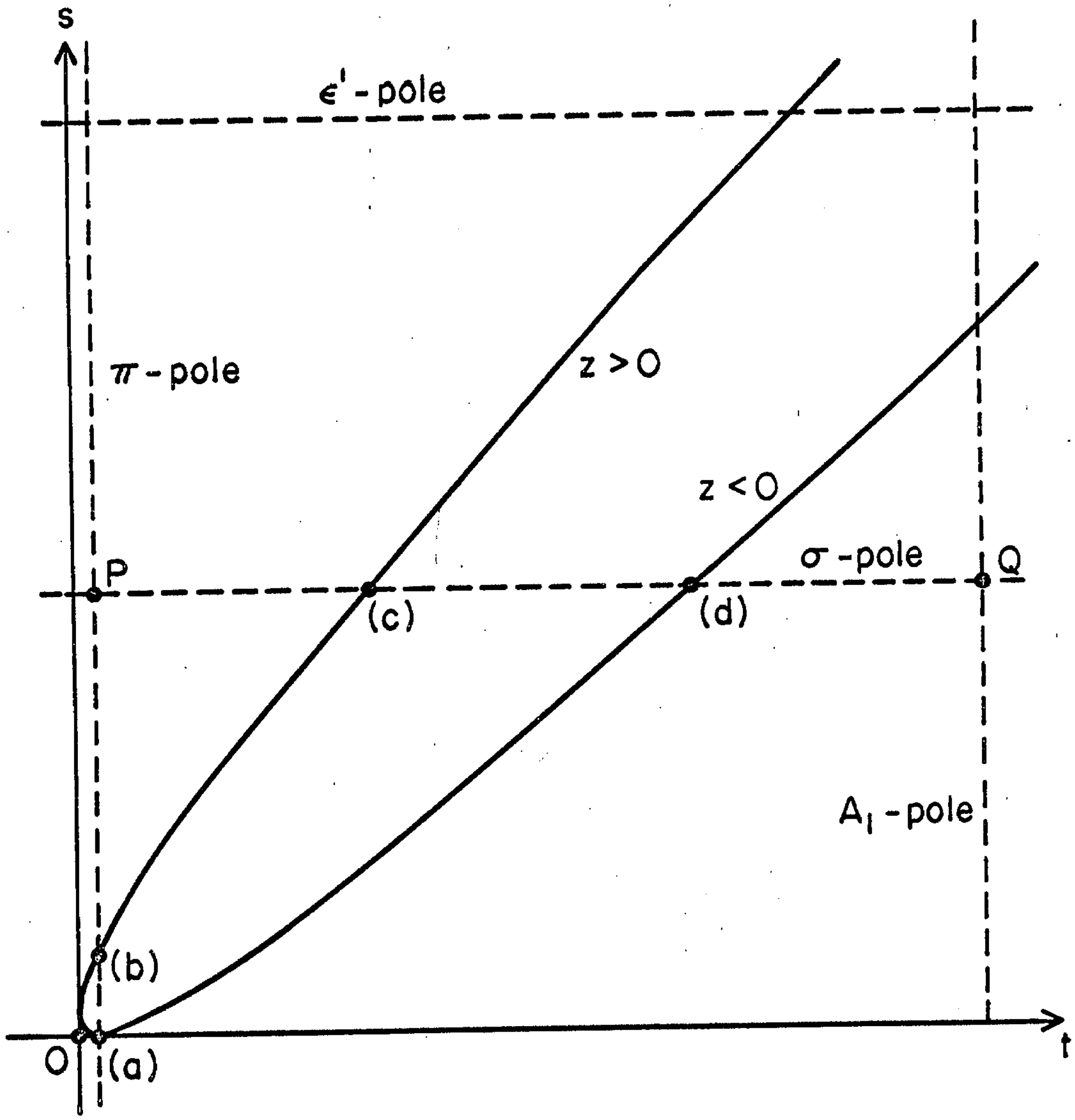

Fig. 3

proportional to $\langle\pi(\vec{0})|\theta_\mu^\mu|\pi(\vec{0})\rangle$ and $\langle 0|\theta_\mu{}^\mu|\pi(\vec{0}), \pi(\vec{0})\rangle$ respectively. Similarly, contributions to our dispersion integrals from the dilaton pole at $z = \pm m_\sigma/m_\pi$ contain the form factors $F_{\sigma\pi}(t)$, $G_{\sigma\pi}(t)$ of Eq. (2.28) evaluated at the edges of the physical regions for the processes $\sigma \to \pi + \ell + \bar{\nu}_\ell$ and $\sigma + \pi \to \ell + \bar{\nu}_\ell$, i.e., at $t = (m_\sigma \pm m_\pi)^2$. These observations are a direct consequence of the condition (3.21) defining the collinear dispersion path.

The amplitudes $\langle 0|\theta_\mu{}^\mu|\pi, A_1\rangle$ and $\langle A_1|\theta_\mu{}^\mu|\pi\rangle$ appear at points on the A_1-pole line above the $z > 0$ and below the $z < 0$ branches of the dispersion path. However, since the spin of the A_1 is not zero, the pion cannot be emitted in an S-wave. Therefore, these amplitudes vanish at threshold, i.e., on the curve given by Eq. (3.22). Reflecting advantage (5) of using a collinear dispersion relation, only spin-0 states contribute to our sum rules.

It is convenient to separate out the contributions of the pion and dilaton poles in the sum rules:

$$x_i(z) = x_i(s,t) = \mathrm{Im}\,X_i - \mathrm{Im}(\pi\text{-pole} + \sigma\text{-pole})_i \quad , \quad (i = 1,2) \; ; \tag{3.30}$$

then, after performing the analysis described in Appendix E, we can rewrite Eqs. (3.27), (3.28) and (3.29) as the exact sum rules[77]

$$(2f_\pi)^{-1}\left(F_1(4m_\pi^2) - 1 - 6F_2(4m_\pi^2)\right) = \frac{m_\sigma F_\sigma}{2m_\pi}\Big[F_{\sigma\pi}(+) - G_{\sigma\pi}(+) - F_{\sigma\pi}(-) + G_{\sigma\pi}(-)\Big] + \frac{1}{\pi}\int dz\, x_2(z) \; , \tag{3.31}$$

$$(2f_\pi)^{-1}\left(\tfrac{1}{2}+\tfrac{1}{2}F_1(4m_\pi^2)-3\,F_2(4m_\pi^2)\right)=\tfrac{1}{2}F_\sigma\Big[F_{\sigma\pi}(+)+F_{\sigma\pi}(-)$$

$$+\,G_{\sigma\pi}(+)+G_{\sigma\pi}(-)\Big]-\frac{1}{\pi}\int dz\; x_1(z)/z\;, \tag{3.32}$$

$$(2f_\pi)^{-1}\left(\tfrac{1}{4}-\tfrac{1}{4}\,F_1(4m_\pi^{\;2})+3m_\pi^{\;2}\,F_1'(0)+\tfrac{3}{2}F_2(4m_\pi^2)-\tfrac{3}{2}F_2(0)\right)$$

$$=\frac{m_\pi F_\sigma}{2m_\sigma}\Big[F_{\sigma\pi}(+)+G_{\sigma\pi}(+)-F_{\sigma\pi}(-)-G_{\sigma\pi}(-)\Big]+\frac{1}{\pi}\int dz\; x_1(z)/z^2\;, \tag{3.33}$$

where we have established the notation

$$F_{\sigma\pi}(\pm)=F_{\sigma\pi}((m_\sigma\pm m_\pi)^2)\;,\quad F_{\epsilon'\pi}(\pm)=F_{\epsilon'\pi}((m_{\epsilon'}\pm m_\pi)^2)\;, \tag{3.34}$$

and so on. At first sight, these sum rules look complicated, but further inspection allows the derivation of simple, approximate results.

The reader may have noticed that the contributions from the ϵ' pole have been absorbed in the definition of the $x_i(z)$, yet it is not obvious that this term is small. Isolating the ϵ'-pole terms, we find that the axial form factors for the β-decay of ϵ' into a pion are evaluated at points $t=(m_{\epsilon'}\pm m_\pi)^2$ close to the A_1 pole at $t=m_{A_1}^{\;2}$. For this reason, the demonstration in Appendices D and E that the ϵ' term may be neglected is essential. It is an unusual feature of our calculation that the inclusion of mixing is non-trivial.

The first simplification of Eqs. (3.31), (3.32) and (3.33) involves the good approximation of keeping only the terms of lowest order in m_π^2. In order to understand the magnitude of the continuum integrals and their dependence on m_π^2, an integration variable such as s or t should be used instead of z. For example, the continuum integral of Eq. (3.31) may be written

$$\int dz\, x_2(z) = \left(\int_{z>0} + \int_{z<0} \right) dz\, x_2(m_\pi^2 z^2,\, m_\pi^2 (1-z)^2)$$

$$= \int_0^\infty \frac{ds}{2m_\pi\sqrt{s}} \left[x_2(s,\, s+m_\pi^2 - 2m_\pi\sqrt{s}) - x_2(s,\, s+m_\pi^2 + 2m_\pi\sqrt{s}) \right]$$

$$= -2\int_0^\infty ds\, \frac{\partial}{\partial t} x_2(s,s) + 0(m_\pi^2) \quad . \tag{3.35}$$

A similar analysis of the other continuum integrals yields

$$\int dz\, x_1(z)/z = \int_0^\infty ds\, x_1(s,s)/s + 0(m_\pi^2) \quad , \tag{3.36}$$

$$\int dz\, x_1(z)/z^2 = -2m_\pi^2 \int_0^\infty ds\, \frac{\partial}{\partial t}\, x_1(s,s)/s + 0(m_\pi^4) \quad . \tag{3.37}$$

Nothing unexpected happens in Eqs. (3.35) and (3.36), but Eq. (3.37) shows that there is a hidden factor m_π^2 in Eq. (3.33), (corresponding to the factor $q_\mu q_\lambda$ in Eq. (2.80)), which should be removed before symmetry limits are considered or continuum integrals are neglected.

At this point, we pause to compare our sum rules with the results obtained in Chapter II. According to Eqs. (2.41) and (2.43),

$F_2(t)$ and $G_{\sigma\pi}(t)$ vanish in the limit of scale invariance. Since the $x_i(z)$ are proportional to θ^μ_μ and contain no σ-poles, they also vanish. Thus, the superconvergence sum rule (3.31) reduces to the trivial equation $0 = 0$; we expected this because it was not derived from an equal-time commutation relation. Eq. (3.32) reduces to Eq. (2.50)--both of these equations come from Eq. (2.37). Finally, when the artificial factor m_π^2 is removed, Eq. (3.33) becomes

$$(2f_\pi)^{-1} \; F_1'(0) = F_\sigma \, F_{\sigma\pi}'(0) \quad , \tag{3.38}$$

which, when combined with Eq. (2.50), yields Eq. (2.81) as expected.

We proceed with the approximation of neglecting terms of higher order in m_π^2. From Eq. (2.29) for $\langle \sigma | \partial^\mu \mathfrak{F}_{5\mu} | \pi \rangle$,

$$m_\sigma^2 \, F_{\sigma\pi}(t) \; - \; t \, G_{\sigma\pi}(t) = 0(m_\pi^2/m_\sigma^2) \tag{3.39}$$

is valid in the neighborhood of $t = m_\sigma^2$. In particular, Eq. (3.39) implies[77]

$$F_{\sigma\pi}(m_\sigma^2) = G_{\sigma\pi}(m_\sigma^2) \quad + \quad 0(m_\pi^2/m_\sigma^2) \quad , \tag{3.40}$$

$$m_\sigma^2(F_{\sigma\pi}'(m_\sigma^2) - G_{\sigma\pi}'(m_\sigma^2)) = G_{\sigma\pi}(m_\sigma^2) \quad + \quad 0(m_\pi^2/m_\sigma^2) \quad , \tag{3.41}$$

$$m_\sigma^2(F_{\sigma\pi}''(m_\sigma^2) - G_{\sigma\pi}''(m_\sigma^2)) = 2\,G_{\sigma\pi}'(m_\sigma^2) \quad + \quad 0(m_\pi^2/m_\sigma^4) \quad , \tag{3.42}$$

and so on. Eqs. (3.35), (3.36), (3.37), (3.40) and (3.41) allow the sum rules to be rewritten as follows:[77]

$$-3F_2(0) = 2f_\pi F_\sigma F_{\sigma\pi}(m_\sigma^2) - \frac{2f_\pi}{\pi}\int ds\,\frac{\partial}{\partial t}\,x_2(s,s) + 0(m_\pi^2/m_\sigma^2)\,, \tag{3.43}$$

$$\frac{1}{2}(1 - 3F_2(0)) = 2f_\pi F_\sigma F_{\sigma\pi}(m_\sigma^2) - \frac{f_\pi}{\pi}\int_0^\infty ds\;x_1(s,s)/s + 0(m_\pi^2/m_\sigma^2)\,, \tag{3.44}$$

$$F_1'(0) + 3F_2'(0) = 2f_\pi F_\sigma\left[2F_{\sigma\pi}'(m_\sigma^2) - F_{\sigma\pi}(m_\sigma^2)/m_\sigma^2\right] - \frac{2f_\pi}{\pi}\int_0^\infty ds\,\frac{\partial}{\partial t}\,x_1(s,s)/s + 0(m_\pi^2/m_\sigma^4). \tag{3.45}$$

For the purposes of comparison, we offer similar expressions which are equivalent to Eqs. (2.52) and (3.13):

$$F_2(0) = -1/3 + 0(m_\pi^2)/m_\sigma^2)\,, \tag{3.46}$$

$$1 = 2f_\pi F_\sigma F_{\sigma\pi}(0) - \frac{3}{\pi}\int_0^\infty ds'\;\bar{f}(s')/s' + 0\,(m_\pi^2/m_\sigma^2)\,, \tag{3.47}$$

where $\bar{f}(t)$ is given by the last two terms of Eq. (3.12). Eq. (3.47) corresponds to the dispersion path $t = 0$, $s \geq 0$ in Fig. 3.

When Eq. (3.46) is substituted in Eq. (3.44), we find[77]

$$1 = 2f_\pi F_\sigma F_{\sigma\pi}(m_\sigma^2) - \frac{f_\pi}{\pi}\int_0^\infty dx\;x_1(s,s)/s + 0(m_\pi^2/m_\sigma^2)\,. \tag{3.48}$$

(Apart from the continuum term, Eq. (3.43) happens to give the same answer).* Comparing the continuum terms of Eqs. (3.47) and (3.48), we observe that both involve semi-infinite paths of integration with apparently similar rates of convergence and similar sets of contributing diagrams. So at this level, we are unable to determine that one continuum integral is significantly smaller than the other.

However, numerical agreement of the results of ignoring the continuum integrals of both Eqs. (3.47) and (3.48) would be surprising. According to the usual estimates[86, 114] of the $A_1\sigma\pi$ coupling, the A_1 pole should cause considerable variation in $F_{\sigma\pi}(t)$ between $t = 0$ and $t = m_\sigma^2$. In fact, Carruthers[89] has shown that these estimates agree with the result of assuming A_1-pole dominance of $F_{\sigma\pi}(t)$, in which case we have

$$F_{\sigma\pi}(0)/F_{\sigma\pi}(m_\sigma^2) \simeq 1 - m_\sigma^{\,2}/m_{A_1}^{\,2} \quad . \tag{3.49}$$

The corresponding dispersion relations appear in Appendix D. In Fig. 3, the relevant dispersion path is $s = m_\sigma^{\,2}$, $t > 0$. The variation of $F_{\sigma\pi}(t)$ is mainly controlled by the magnitude of the double pole in T_λ at $Q = (m_\sigma^{\,2}, m_{A_1}^{\,2})$. The quantity of interest,

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* If the term $S^{\mu\nu}$ of Eq. (1.56) were present, the left-hand sides of Eqs. (3.47) and (3.48) would be modified by the same scale-violating factor. Then the accidental agreement of the pole-dominance approximation for Eqs. (3.43) and (3.44) would be lost. The continuum integral of Eq. (3.44) is likely to converge more rapidly than that of Eq. (3.43). That is why our conclusions are drawn from Eq. (3.44), while Eq. (3.43) is treated as a check on the consistency of our approximations.

$G_{\sigma\pi\pi}$, is proportional to the residue of the double pole at $P = (m_\sigma^2, m_\pi^2)$. Whe continuum integrals are neglected, Eq. (3.47) yields the usual formula

$$F_\sigma \; G_{\sigma\pi\pi} \approx m_\sigma^2 \quad , \tag{3.50}$$

whereas Eqs. (3.48) and (3.49) require

$$F_\sigma \, G_{\sigma\pi\pi} \approx m_\sigma^{\;2} \,(1 - m_\sigma^2/m_{A_1}^{\;2}) \quad . \tag{3.51}$$

From the point of view of broken scale and chiral invariance, Eqs. (3.50) and (3.51) cannot be distinguished, because only terms $0(m_\sigma^{\;2})$ are determined by symmetry arguments. Numerically, the discrepancy between Eqs. (3.50) and (3.51) amounts to a factor of almost 2.

Of course, this numerical difference could be removed by suitably weakening the assumptions--for example, a tenth scalar meson could be introduced. We do not believe that such a procedure is called for at present. By saturating two different dispersion integrals, two estimates of $G_{\sigma\pi\pi}$ have resulted; the difference between these estimates is a measure of the uncertainty involved in predicting $G_{\sigma\pi\pi}$ by arguments based on broken scale invariance. We observe that, when the restriction (2.35) from the Adler-Weisberger sum rule for $\pi\pi$ scattering is combined with phenomenological estimates* of dilaton-baryon couplings, Eq. (3.51)

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*See Eq. (1.101) and Refs. 89 and 90.

is strongly favored. Having observed the numerical failure of PCDC for $\langle\pi|\theta_\mu^{\ \mu}|\pi\rangle$, it is less surprising that σ-pole dominance of $F_2(t)$ is such a crude approximation. However, we have no solid arguments which allow us to explain why Eq. (3.51) works much better than Eq. (3.50).

Since our analysis works so well for Eq. (3.44), we try the same approximations for Eq. (3.45), which involves the same path of integration. Ignoring the continuum integral, we find[77]

$$F_1'(0) \simeq -3F_2'(0) + 2f_\pi F_\sigma F_{\sigma\pi}(m_\sigma^2)\left[2F_{\sigma\pi}'(m_\sigma^2)/F_{\sigma\pi}(m_\sigma^2) - m_\sigma^{-2}\right]. \tag{3.52}$$

To estimate terms on the right-hand side of Eq. (3.52), we follow the prescription which led to Eq. (3.51):

$$F_\sigma F_{\sigma\pi}(m_\sigma^2)\ f_\pi \simeq \tfrac{1}{2}\quad , \tag{3.53}$$

$$F_{\sigma\pi}'(m_\sigma^2)/F_{\sigma\pi}(m_\sigma^2) \simeq (m_{A_1}^{\ 2} - m_\sigma^2)^{-1}\quad , \tag{3.54}$$

$$-3F_2'(0) \simeq F_\sigma G_{\sigma\pi\pi}/m_\sigma^4 \simeq m_\sigma^{-2} - m_{A_1}^{-2} \tag{3.55}$$

Eq. (3.54) is implied by A_1 dominance of $F_{\sigma\pi}(t)$. In Eq. (3.55), we first estimate the slope of $F_2(t)$ using σ-pole dominance, and then apply Eq. (3.51). Eq. (3.52) becomes[77]

$$F_1'(0) \approx \frac{m_{A_1}^2 + m_\sigma^2}{m_{A_1}^2 (m_{A_1}^2 - m_\sigma^2)} \tag{3.56}$$

Notice the strong dependence of $F_1'(0)$ on the magnitude of the violation of scale invariance. Scale-breaking effects are responsible for changing the scale-invariant estimate,* $F_1'(0) \simeq m_{A_1}^{-2}$, by a factor of almost 3. This factor accounts for most of the discrepancy between Eq. (3.56) and the prediction $F_1'(0) \simeq m_f^{-2}$ of f-dominance. Engels and Höhler[90] have estimated the fNN coupling constants from backward dispersion relations for πN scattering, obtaining an answer which is three times the value predicted using f-dominance.[115] The quoted error (~10 %) seems a bit optimistic, but their work does encourage the suspicion that f mesons do not couple universally. Then Eq. (3.56) is a reasonable result.

In the preceding analysis, our strongest assumption is that $F_{\sigma\pi}(t)$ is dominated by the A_1 pole in the region $|t| < m_{A_1}^2$. To obtain an indirect test of this hypothesis,[89] we note that it implies

$$F_{\sigma\pi}(0) \simeq g_{A_1} \; g_{A_1\sigma\pi} / 2m_{A_1}^2 \quad , \tag{3.57}$$

where $F_{\sigma\pi}(0)$ is given by the Goldberger-Treiman relation (2.31),

*See Eq. (2.81)

$-\frac{i}{2}\,\varepsilon\cdot(P_\sigma + P_\pi)\, g_{A_1\sigma\pi}$ is the $A_1\sigma\pi$ coupling,* ϵ_μ is the polarization of the A_1 meson, and

$$\langle A_1 | \mathcal{F}_{5\mu} | 0 \rangle = \epsilon_\mu g_{A_1} \tag{3.58}$$

would determine the rate for $A_1 \rightarrow \bar{\nu} + \ell$ if it were measurable.

The standard method for estimating g_{A_1} makes use of Weinberg's first and second sum rules[116] for SU(2) x SU(2) symmetry. From the theory of broken scale invariance, Wilson[40] has shown that these relations converge in the limit of SU(2) x SU(2) symmetry. Therefore, we accept the usual saturation approximation, which implies the formulae[116]

$$g_\rho^{\,2}/m_\rho^{\,2} \simeq g_{A_1}^{\,2}/m_{A_1}^{\,2} + (2f_\pi)^{-2} \quad , \tag{3.59}$$

$$g_\rho^{\,2} \simeq g_{A_1}^{\,2} \quad , \tag{3.60}$$

where

$$\langle \rho^o | \mathcal{F}^3_\mu | 0 \rangle = \epsilon_\mu\, g_\rho \tag{3.61}$$

is proportional to the amplitude for $\rho^o \rightarrow \ell^+ + \ell^-$. Within the theoretical and experimental uncertainties, data[79,117] for this

*The signs given for the momenta correspond to $\pi A_1 \rightarrow \sigma$.

process agree with the result of eliminating g_{A_1} from Eqs. (3.59) and (3.60):

$$g_\rho \simeq (2f_\pi)^{-1} (m_\rho^{-2} - m_{A_1}^{-2})^{-\frac{1}{2}} \quad . \tag{3.62}$$

In addition, Eq. (3.62) is consistent with the KSFR relation,[118] ρ dominance of the electromagnetic form factor for pions, and measurements of the width of the ρ meson.[79]* Therefore, we are confident that the right-hand side of Eq. (3.62) provides a good estimate for g_{A_1}, so Eq. (3.57) becomes

$$g_{A_1\sigma\pi}/G_{\sigma\pi\pi} \simeq \frac{2m_{A_1}^2}{m_\sigma^2} (m_\rho^{-2} - m_{A_1}^{-2})^{\frac{1}{2}} \quad , \tag{3.63}$$

which agrees with a set of formulae given by Gilman and Harari.[86]

Unfortunately, Eq. (3.63) is rather difficult to test experimentally. In order to avoid $A_1 \rightarrow \rho\pi$ decay, it is necessary to find a neutral peak for $A_1^0 \rightarrow \pi^+\pi^-\pi^0$, determine what proportion of it is due to threshold enhancement,[120] and isolate the $\sigma(700)$ peak in the $\pi^+\pi^-$ pair. The prediction $\Gamma_{A_1 \rightarrow \sigma\pi} = 30 - 60$ MeV for $\Gamma_{\sigma\rightarrow\pi\pi} = 300 - 600$ MeV is consistent with present data.

Our assumption of A_1 dominance is not invalidated by a remark of Ellis.[104] In effective Lagrangian models for the $A_1\sigma\pi$

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*Theoretical reviews appear in Refs. 45 and 119.

coupling constant, the pion must couple via a derivative so that an Adler consistency condition is satisfied; (i. e., the amplitude for $A_1 \to \sigma + \partial^\mu \mathfrak{F}_{5\mu}$ must vanish when the momentum of $\partial^\mu \mathfrak{F}_{5\mu}$ vanishes). Ellis notes that the coupling

$$\bar{\mathscr{L}}_{A_1 \sigma\pi} = g_{A_1\sigma\pi}\, \sigma \vec{A}_1{}^{\mu} \cdot \partial_\mu \vec{\pi} \tag{3.64}$$

is forbidden in conformally invariant models: the dilaton field σ always appears in the combination $\exp(-\sigma/F_\sigma)$, so Eq. (3.64) would imply the existence of $g_{A_1\sigma\pi}\, F_\sigma\, \vec{A}_1{}^{\mu} \cdot \partial_\mu \vec{\pi}$, which is not allowed. On the other hand, it is obvious from Eq. (3.57) that, according to our assumptions, $g_{A_1\sigma\pi}$ does not vanish, even in the limit of scale invariance.

Let us rephrase the argument in terms of low-energy theorems. Defining

$$\langle A_1 | \partial^\mu \mathfrak{F}_{5\mu} | \sigma \rangle = i\, \epsilon \cdot P_\sigma\, D_{A_1\sigma}(t) \tag{3.65}$$

with $t = (P_{A_1} - P_\sigma)^2$, PCAC gives

$$D_{A_1\sigma}(0) \simeq (2f_\pi)^{-1} \quad g_{A_1\sigma\pi} \quad . \tag{3.66}$$

The soft-dilaton theorems which could possibly be relevant are*

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*These expressions are derived using techniques similar to those developed in Chapter II; e.g., see the derivation of Eq. (2.50). Since $A_1 \to \sigma\pi$ is a P-wave decay, collinear dispersion relations may be written only for Eq. (3.68); i.e., $\partial^\mu \mathfrak{F}_{5\mu}$ and $\theta_\alpha{}^\alpha$ cannot couple to the spin-1 state $|A_1\rangle$ when all 3-momenta vanish. The resulting change in Eq. (3.72) is insignificant.

$$\langle A_1 | \partial^\mu \mathfrak{F}_{5\mu} | \sigma \rangle_{soft} \approx - iF_\sigma^{-1} \langle A_1 | [D(0), \partial^\mu \mathfrak{F}_{5\mu}(0)] | 0 \rangle$$

$$= \frac{\ell_u}{F_\sigma} \langle A_1 | \partial^\mu \mathfrak{F}_{5\mu} | 0 \rangle = 0 \quad , \tag{3.67}$$

$$\langle A_1 | \mathfrak{F}_{5\lambda} | \sigma \rangle_{soft} \approx \lim_{k \to 0} \frac{k^\mu}{F_\sigma} \int d^4x \, e^{ik \cdot x} \theta(x_o) \langle A_1 | [\mathcal{D}_\mu(x), \mathfrak{F}_{5\lambda}(0)] | 0 \rangle$$

$$- i \; F_\sigma^{-1} \langle A_1 | [D(0), \mathfrak{F}_{5\lambda}(0)] | 0 \rangle \tag{3.68}$$

$$= 0 \quad .$$

All the soft-dilaton amplitudes vanish, in agreement with the argument given by Ellis.

In order to continue the discussion, we need the form-factor expansion

$$\langle A_1 | \mathfrak{F}_{5\lambda} | \sigma \rangle = \epsilon \cdot P_\sigma [(P_\sigma + P_{A_1})_\lambda F_{A_1\sigma}(t) + (P_\sigma - P_{A_1})_\lambda G_{A_1\sigma}(t)]$$

$$- \epsilon_\lambda H_{A_1\sigma}(t) \quad . \tag{3.69}$$

The definitions (3.65) and (3.69) may be combined to yield

$$D_{A_1\sigma}(t) = (m_{A_1}^2 - m_\sigma^2) F_{A_1\sigma}(t) - t\, G_{A_1\sigma}(t) + H_{A_1\sigma}(t) \; . \tag{3.70}$$

Soft-dilaton amplitudes are obtained by setting the momentum associated with θ_μ^μ equal to zero, so Eqs. (3.67) and (3.68) should be interpreted as follows:

$$D_{A_1\sigma}(m_{A_1}^2) \approx 0 \quad , \tag{3.71}$$

$$H_{A_1\sigma}(m_{A_1}^2) \approx 0 \quad . \tag{3.72}$$

Comparing Eqs. (3.66), (3.71), and (3.72), we see that Eq. (3.72) is not useful, and the other theorems cannot be combined unless some way is found to extrapolate from $t = 0$ to $t = m_{A_1}^2$. Therefore, our analysis from the point of view of low-energy theorems indicates that there is no theorem for $g_{A_1\sigma\pi}$ unless extraneous assumptions are introduced.

Evidentally, the trouble with the Lagrangian (3.64) is that it does not allow for any t-dependence in $D_{A_1\sigma}(t)$ besides that due to the pion pole. If that were true, then Eqs. (3.66) and (3.71) would imply that $g_{A_1\sigma\pi}$ vanishes. In general, we have no right to expect pion pole dominance of $D_{A_1}(t)$ to work at $t = m_{A_1}^2$, so it must be possible to write an acceptable Lagrangian for the $A_1\sigma\pi$ coupling using Ellis's model. To obtain dependence of $D_{A_1\sigma}$ (no pion pole) on t, the correct expression must involve two more derivatives than are present in Eq. (3.64). In order to circumvent Ellis's argument, at least one of these extra derivatives must act on the dilaton field. In addition, the extrapolation is $0(m_{A_1}^2)$, not $0(m_\sigma^2)$, so the momentum of the A_1 must be involved. With the help of these clues, we arrive at the allowed coupling

$$\mathcal{L}_{A_1\pi\to n\sigma} = (F_\sigma\, g_{A_1\sigma\pi}/m_{A_1}^{\,2})\,(\partial^\alpha \vec{A}_1^{\,\mu} - \partial^\mu \vec{A}_1^{\,\alpha})\cdot \partial_\mu \vec{\pi}\,\partial_\alpha \exp(\sigma/F_\sigma)\,, \tag{3.73}$$

which vanishes for $n=0$, and is consistent with Eq. (3.57) for $n=1$.

The application of the method of collinear dispersion relations to related equal-time commutators is straightforward. Examples appear in Appendix F; these involve the dimension of the scale-violating terms in θ_{00}. Unfortunately, the resulting sum rules are are hard to saturate, or involve a coupling constant like(77)*

$$F_\sigma G_{\sigma\sigma\sigma} \simeq (1-\ell)\; m_\sigma^{\,2} \quad , \tag{3.74}$$

for a unique scale-breaking dimension ℓ.

So far, we have no indication that our scheme combining PCDC and PCAC is at variance with the facts. Now we examine the effect of our assumptions on the treatment of theorems for broken chiral symmetry.

III. 3. Magnitude of Breakdown of Chiral Symmetry

Basic to our approach to the calculation of soft-meson amplitudes has been the idea that the violation of chiral SU(2) x SU(2) symmetry is much smaller than the breakdown of conformal invariance. For example, we have assumed that, in the real world, the induced scalar form factor $F_2(t)$ is better approximated by Eq. (2.52) than Eq. (2.41). This implies that $G_{\sigma\pi\pi}$ is $0(m_\sigma^{\,2})$

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*Of course, PCDC for $\langle\sigma|\theta^\mu_\mu|\sigma\rangle$ does not fix ℓ in Eq. (3.74).

rather than $0(m_\sigma^4)$ or $0(m_\pi^2)$.

However, the position of chiral SU(3) x SU(3) in the hierarchy of symmetries is less clear. The standard SU(3) x SU(3) calculation includes only poles due to the octet of pseudoscalar mesons. On the other hand, 0^{++} poles are important in the event that δ is a c-number,* or, more generally, that the magnitude of scale violation is of the same order as the magnitude of SU(3) x SU(3) breaking, (i.e., "$\delta \to 0$ as $u \to 0$"). Our analysis is based on the latter model. Compared with theories in which there are no dilatons, our scheme has the following features:

(i) Theorems connected with chiral SU(2) x SU(2) or physical SU(3) are not altered if the relevant extrapolations in momentum squared are $0(m_\pi^2)$.

(ii) The interpretation of results obtained by comparing such theorems with experimental information may be radically different.

(iii) Any soft-meson theorem which does not belong to class (i), and which involves two pseudoscalar mesons which can form a $(J^P, I^G) = (0^+, 0^+)$ state at some stage of the extrapolation procedure, may be drastically altered.

To illustrate the effect of the dilaton pole, let us consider the limits of chiral and scale invariance. In general, chiral calculations performed in the limit of scale invariance differ from the usual analyses with $\theta_\mu^{\ \mu} \neq 0$. In a scale-invariant theory, extra insertions arise from diagrams in which the axial-vector current

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*For example, see Eq. (3.7).

can hook on to an external pion and turn it into a dilaton (and vice versa).* A good example is the threshold amplitude $T(0)$ for forward $\pi^+ p$ scattering, where we show only the dependence on t, the square of the momentum transfer. According to the Adler consistency condition,[121] $T(0)$ vanishes in the limit $m_\pi \to 0$ $\theta_\mu{}^\mu \neq 0$. In the limit of scale invariance, the extra insertion changes this result; instead, one obtains the formula (not to be applied to the real world)

$$T(0) = -2f_\pi F_{\sigma\pi}(0)\ g_{\sigma NN} = -\ g_{\sigma NN}{}^2/M_N \quad , \qquad (3.75)$$

where the last equality follows from Eqs. (1.78) and (2.50). That there is no contradiction can be seen by explicitly displaying the σ-pole in the non-scale-invariant $\pi^+ p$ scattering amplitude:

$$T(t) = \frac{m_\sigma{}^2}{m_\sigma{}^2 - t} \ \frac{g_{\sigma NN}{}^2}{M_N} + \overline{T}(t) \quad , \qquad (3.76)$$

with $\overline{T}(0) = -\ g_{\sigma NN}{}^2/M_N$ for both $\theta_\mu{}^\mu \neq 0$ and $\theta_\mu{}^\mu = 0$. The limits $m_\sigma{}^2 \to 0$, $t \to 0$ are not interchangeable in Eq. (3.76). Since the Adler consistency condition is in excellent agreement with experiment, chiral $SU(2) \times SU(2)$ symmetry provides a much better description of the real world than does conformal symmetry.

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* Consideration of such insertions was essential in the derivation of Eqs. (2.50) and (2.81); also, see Appendix C.

Because of the larger extrapolation associated with PCAC for kaons, it is difficult to check the Adler consistency condition for Kp scattering. Examination of the tables given by von Hippel and Kim[113] shows the importance of the extrapolation procedure in attempting to compare low-energy theorems with experiment. The poor agreement of the soft-kaon theorems with data for the scattering lengths encourages the idea that poles due to scalar mesons strongly affect the extrapolation. Theories of broken scale invariance apply here because we have shown that the relevant coupling constants are large; i.e., the presence of a dilaton pole can change the application of chiral SU(3) x SU(3) to such amplitudes.

Evidently, this viewpoint requires that SU(2) x SU(2) be regarded as a much better symmetry than SU(3) x SU(3), i.e., Eq. (1.52) holds with $c \simeq -1.25$, as proposed by Gell-Mann, Oakes, and Renner.[49] This has been challenged by Gaillard[122] and Brandt and Preparata,[123] who prefer $-c \ll \sqrt{2}$, a result based mainly on their analyses of $K_{\ell 3}$ decay. The relevant quantities are*

$$\langle \pi^{o}(q) | \mathfrak{F}_{\mu}^{4-i5} | K^{+}(k) \rangle = \frac{1}{\sqrt{2}} \left[(k+q)_{\mu} f_{+}(t) + (k-q)_{\mu} f_{-}(t) \right], \tag{3.77}$$

$$\begin{aligned}(m_K^2 - m_\pi^2) f_o(t) &= i \langle \pi^o | \partial^\mu \mathfrak{F}_\mu^{4-i5} | K^+ \rangle \\ &= (m_K^2 - m_\pi^2) f_+(t) + t f_-(t),\end{aligned} \tag{3.78}$$

*See the exhaustive review of Gaillard and Chounet (Ref. 124).

where $f_+(t)$, $f_o(t)$ are spin 1^-, 0^+ form factors respectively. Often, experimental results are obtained with the help of the parameters

$$\lambda_+ = m_\pi^2 f'_+(0)/f_+(0), \quad \lambda_o = m_\pi^2 f'_o(0)/f_o(0), \quad \xi(0) = f_-(0)/f_+(0) \, , \tag{3.79}$$

in terms of which the form factors may be parametrized linearly.

The situation is illustrated in Fig. 4, where the scalar form factor $f_o(t)$ is plotted in the region $0 \lesssim t \lesssim m_K^2$. At $t = 0$ (point A), Eq. (3.78) implies

$$f_o(0) = f_+(0) = 1 + 0(\varepsilon^2) \, , \tag{3.80}$$

where ε denotes SU(3) breaking (as in Eq. (1.72)), and the last equality follows from the Ademollo-Gatto theorem.[125] The soft-pion result, ("Callan-Treiman relation"),[60]

$$f_o(m_K^2) = f_\pi/f_K + 0(m_\pi^2/m_K^2) \quad , \tag{3.81}$$

involves the point $t = m_K^2$, which lies well outside the physical decay region

$$m_\ell^2 \leqslant t \leqslant (m_K - m_\pi)^2 \quad . \tag{3.82}$$

Eq. (3.81) is given by the point CT, where the error bar stands for $0(m_\pi^2/m_K^2)$ in a theory with $c \approx \sqrt{2}$. Gaillard and Brandt and

Preparata claim $0(m_\pi^2/m_K^2) \approx 1$, corresponding to $c \approx 0$; then their sum rule for $f_o(m_K^2)$ gives the point GBP. Using collinear dispersion relations, Banerjee[126] has obtained the sum rule

$$(1 + m_\pi/m_K)\ f_o((m_K - m_\pi)^2) \simeq \frac{f_\pi}{f_K}\left(1 + \frac{m_K}{m_\pi}\ \frac{\sqrt{2} + c}{\sqrt{2} - \frac{1}{2}c}\right)$$

$$+\ \varkappa\text{-meson pole contribution} \quad , \qquad (3.83)$$

which is extremely sensitive to changes in c. All estimates for the $\varkappa$-meson pole contribution lie within the limits specified by the error bars which we have attached to the points $(c \approx 0)$ and $(c \approx -1.25)$ in Fig. 4.

In practice, these theorems have to be extrapolated back to the point A, because the results of experiments are customarily quoted as values for $\xi(0)$ and λ_+. (This circumstance is forced by the limited amount of available data per experiment). The formula

$$\xi(0) = \frac{m_K^2 - m_\pi^2}{m_\pi^2}\ (\lambda_o - \lambda_+) \qquad (3.84)$$

connects these results with theoretical predictions for λ_o given by the slope of the extrapolation curve at $t = 0$. Four of these curves are displayed in Fig. 4.

If the confused experimental situation is supposed to favor

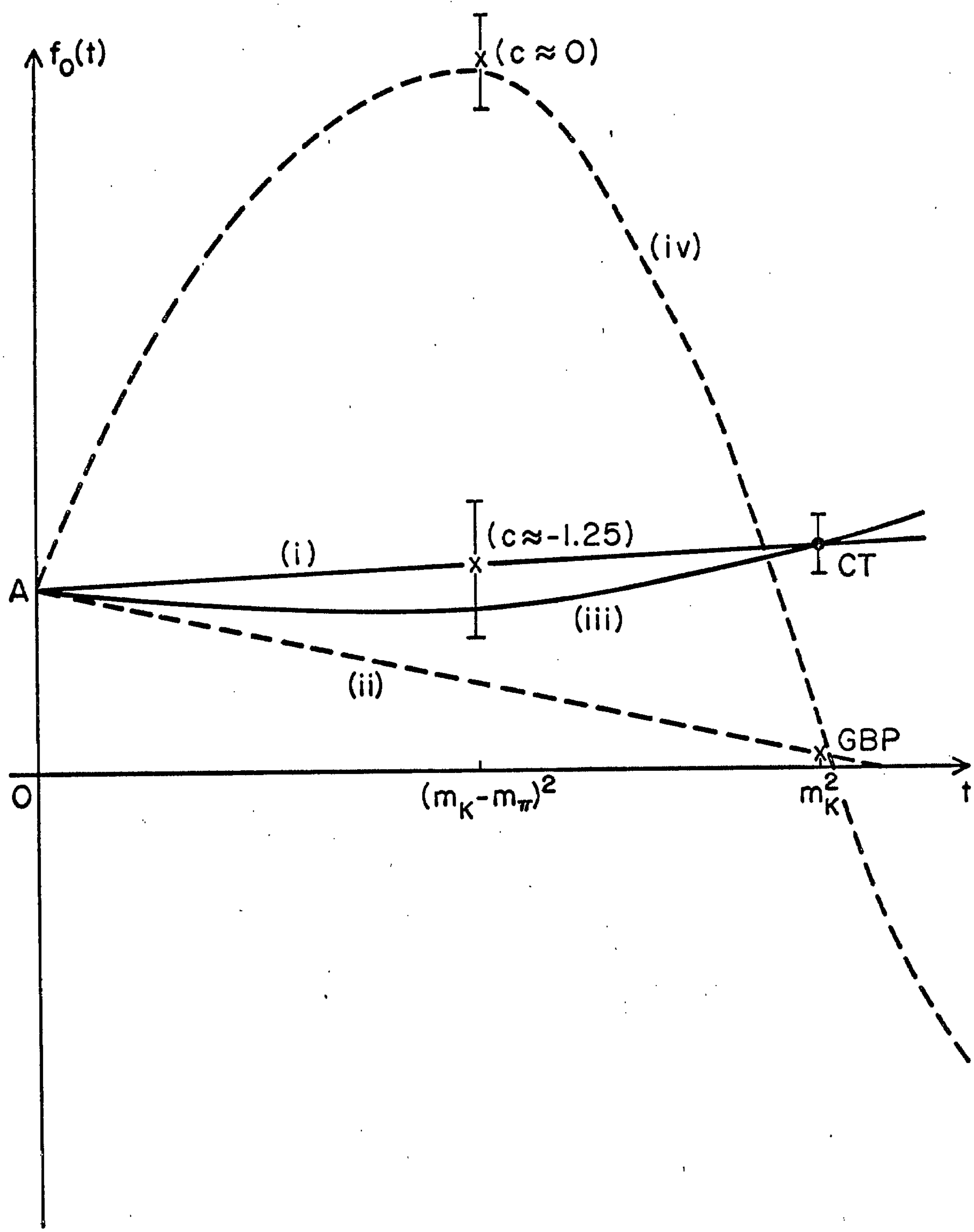

Fig. 4

$$\xi(0) \approx -1 \quad , \quad \lambda_+ \simeq \lambda_+(K^*) = .024 \quad , \qquad (3.85)$$

(where the latter value* is implied by K*(890) dominance of $f_+(t)$), only the curves (ii) (Gaillard-Brandt-Preparata) and (iii) (Banerjee) are allowed. However, from the point of view of dispersion theory for $f_o(t)$, the dip in (iii) is very mysterious, and, according to Eq. (3.83), the point GBP should be extrapolated back to A along curve (iv), not (ii). Therefore, we do not favor these alternatives; curve (iv) definitely contradicts the experiment.

Two recent experiments,[127] with more data than in previous measurements, indicate

$$\xi(0) \simeq -1 \quad , \quad \lambda_+ \simeq .06 - .08 \quad . \qquad (3.86)$$

Curve (i) fits these values. In addition, it is slowly varying, as required by dispersion theory, and passes through the CT point, as expected in a theory with $c \simeq -\sqrt{2}$. Of course, vector dominance fails.

We doubt that data from $K_{\ell 3}$ decays can be reliably interpreted until all parameterizations are avoided, and the form factors $f_+(t)$, $f_o(t)$ are plotted as functions of t. However, we conclude that the Gaillard-Brandt-Preparata scheme does not work.

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*Practically all theories for λ_+, (e.g., Weinberg's sum rules), predict this result, but these arguments are not compelling. Weinberg's sum rules for SU(3) x SU(3) are not necessarily valid--see Ref. 40.

If the term u breaking SU(3) x SU(3) in the energy density transforms as* $(3,\bar{3}) + (\bar{3},3)$, SU(2) x SU(2) is a much better symmetry than SU(3), (and vice versa). Otherwise, representations such as (8, 8) must be considered.

Recently, Cheng and Dashen[108] obtained the result

$$\sigma_{NN} = \langle N | - \frac{\sqrt{2} + c}{3} (\sqrt{2}\, u_o + u_8') | N \rangle \approx 110 \text{ MeV} \tag{3.87}$$

by using πN phase shifts, a fixed-t dispersion relation, and the low-energy theorem

$$T(0,0,m_\pi^2, m_\pi^2) = 4f_\pi^2 \sigma_{NN} + \text{"}0(m_\pi^4)\text{"} \tag{3.88}$$

for the amplitude

$$T(\nu, \nu_B, q^2, q'^2) = (m_\pi^2 - q^2)(m_\pi^2 - q'^2)(2f_\pi/m_\pi^2)^2\, i \int d^4x\, e^{iq'\cdot x}$$

$$\theta(x_o) \langle N(p') | [\partial^\mu \mathfrak{F}_{5\mu}^3(x), \partial^\nu \mathfrak{F}_{5\nu}^3(0)] | N(p) \rangle , \tag{3.89}$$

with $q = p' + q' - p$, $\nu = (p + p') \cdot (q + q')/4M_N$, and $\nu_B = -q\cdot q'/2M_N$. Since 110 MeV is not much smaller than energies associated with SU(3) breaking, they conclude that SU(2) x SU(2) and SU(3) violations are comparable in magnitude, which is contrary to expectations that

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*A term transforming as (1,8) + (8,1) could also be present, (Ref. 49). It has a negligible effect on soft-pion results, but could affect soft-kaon calculations; see Ref. 128.

c is near $-\sqrt{2}$. In a theory which does not contain dilatons, their conclusion appears to be unavoidable.

However, a different interpretation is available in a dilaton theory.(77) As a subgroup of SU(3) x SU(3), the physical SU(3) group is distinguished from other SU(3) groups in that its elements leave the vacuum invariant. Then physical SU(3) is not spontaneously broken and perturbation theory in the SU(3) violating parameter makes sense:

$$\langle N | c\, u_8 | n \rangle \simeq \frac{1}{2} M_\Sigma + \frac{1}{2} M_\Lambda - M_N = 215 \text{ MeV} . \quad (3.90)$$

This means that the dilaton state $|\sigma\rangle$ must be invariant under physical SU(3) transformations in the limit of scale invariance. As scale invariance is broken, the dilaton qualtiy is distributed between the $|\sigma\rangle$ and $|\varepsilon'\rangle$ states. Poles in $\langle u_8 \rangle$ due to the existence of $|\sigma\rangle$ and $|\varepsilon'\rangle$ arise from the non-dilaton, or octet, quality of these states. On the other hand, matrix elements of u_o have σ and ε' poles due to the dilaton quality in $|\sigma\rangle$ and $|\varepsilon'\rangle$. Therefore the magnitude of $\langle N | u_o | N \rangle$ is $0(m_\eta^{\;2} M_N / m_\sigma^{\;2})$, much larger than $\langle n | u_i | N \rangle$. In general, we expect

$$\langle \Psi | u_o | \Psi \rangle >> \langle \Psi | u_8 | \Psi \rangle \quad (3.91)$$

for all one-particle rest states $|\Psi\rangle$ except $|\sigma\rangle$ and $|0^-, 8\rangle$.

Because of these observations, there is no reason to abandon either the $(3,\bar{3}) + (\bar{3},3)$ form of the chiral SU(3) x SU(3) violating term in the energy density or the value -1.25 for c, if there is a dilaton. In order to apply Eq. (3.87) in a dilaton theory, we should first check the validity of neglecting the terms "$0(m_\pi^4)$" in Eq. (3.88), where the next order in m_π^2 is given by

$$"0(m_\pi^4)" = m_\pi^4 \frac{\partial^2}{\partial q^2 \, \partial q'^2} T(0,0,0,0) + "0(m_\pi^6)" . \qquad (3.92)$$

The pion poles at $q^2 = q'^2 = m_\pi^2$ do not contribute to Eq. (3.92). Parametrizing the failure of PCAC in terms of a "heavy pion", π^*, the contribution of the dilaton pole is

$$(2f_\pi/m_\sigma)^2 \, g_{\sigma NN} \, i \int d^4 x \, e^{iq' \cdot x} \langle \sigma | T(\partial^\mu \mathfrak{F}_{5\mu}(x), \partial^\nu \mathfrak{F}_{5\nu}(0)) | 0 \rangle \quad \text{(no } \pi \text{ poles)}$$

$$= 0 \left[M_N (2f_\pi)^4 \langle 0 | \partial^\mu \mathfrak{F}_{5\mu} | \pi^* \rangle^2 / (m_\sigma m_{\pi^*})^2 \right] , \qquad (3.93)$$

with $\langle 0 | \partial^\mu \mathfrak{F}_{5\mu} | \pi^* \rangle = 0(m_\pi^2/2f_\pi)$, so the correction terms have magnitude $0(4f_\pi^2 M_N m_\pi^4/(m_\sigma m_\pi{*})^2)$. Neglect of such terms appears to be a satisfactory approximation; a corollary is that the corresponding terms for KN scattering should not be thrown away: $m_K^4/m_\sigma^2 = 0(m_K^2)$. Therefore we use Eqs. (3.87) and (3.90) to obtain $\langle N | u_o | N \rangle = -1280$ MeV and

$$\langle N | u | N \rangle = \frac{3}{\sqrt{2}\,(\sqrt{2} + c)} \sigma_{NN} + \left(\frac{1}{c\sqrt{2}} - 1\right) \langle N | c u_8 | N \rangle$$

$$\simeq 1060 \text{ MeV} \quad . \qquad (3.94)$$

Within the 20% accuracy of Eq. (3.94), we have

$$\langle N|u|N\rangle \simeq M_N \quad , \quad \langle N|\bar{\bar{\theta}}_{oo}|N\rangle \simeq -\langle N|\delta|N\rangle \quad , \tag{3.95}$$

so the formula

$$\langle N|(\ell_\delta + 4)\,\delta + (\ell_u + 4)\, u|N\rangle = \langle N|\theta^\mu_\mu|N\rangle = M_N \tag{3.96}$$

suggests $\delta =$ c-number and $\ell_u = -3$; however, we are unable to exclude the possibility $-\langle N|\delta|N\rangle = 0(M_N)$.

More recently, Höhler, et. al.[129] have given another estimate of σ_{NN}, obtaining

$$\sigma_{NN} \simeq 60 \text{ MeV} \quad . \tag{3.97}$$

In a theory with $c = -1.25$, we find

$$\langle N|u|N\rangle \simeq 430 \quad \text{MeV} \quad , \tag{3.98}$$

so ℓ_u would be -2 in a model with $\delta =$ c-number. Höhler, et. al., note that there are large uncertainties in the real parts of the πN scattering amplitudes which they need in their calculation. However, they are unable to make their evaluation consistent with the Cheng-Dashen result. We have no explanation for this.

The estimates of σ_{NN} given by Cheng and Dashen and Höhler, et. al., supersede the work of von Hippel and Kim,[113] who were first to make a serious attempt to find $\langle N|u|N\rangle$.

They attempted to saturate collinear dispersion relations for SU(3) generalizations of the amplitude T defined in Eq. (3.89).* In terms of the function

$$\overline{T}(\nu) = T(\nu, -\nu^2/2M_B, \nu^2, \nu^2) \quad , \tag{3.99}$$

the forward amplitude for meson-baryon scattering at threshold is $\overline{T}(m_M)$, while the σ-commutator is given by the exact relation

$$\overline{T}(0) = -4f_\pi^{\,2} \langle B | \left[F_5, [F_5, \theta_{oo}]\right] | B \rangle \ . \tag{3.100}$$

These two quantities are connected by a collinear dispersion integral:(91,113)

$$\overline{T}(m_M) = \overline{T}(0) - (2f_M/m_M)^2\, i\langle B | [\ddot{F}_5(0), \partial^\mu \mathfrak{F}_{5\mu}(0)] | B\rangle_{\text{Conn.}} + \frac{2m_M^{\,2}}{\pi} \int_{-\infty}^{\infty} \frac{d\nu \ \text{Im}\ \overline{T}(\nu)}{\nu(\nu^2 - m_M^2)} \ . \tag{3.101}$$

In order to make the continuum integral converge, it is necessary to make a subtraction at $\nu = \infty$; this accounts for the appearance of the equal-time commutator

$$C = \langle B | [\ddot{F}_5(0) \ , \ \partial^\mu \mathfrak{F}_{5\mu}(0)] | B \rangle_{\text{Conn.}} \quad , \tag{3.102}$$

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*The SU(3) indices of the current divergences are symmetrized.

where "Conn." indicates that only the connected part should be considered. Von Hippel and Kim obtained $\langle N | u | N \rangle \simeq 0$ and $\sigma_{NN} \simeq 22$ MeV, in disagreement with the more recent results (3.97) and especially (3.87).

The von Hippel-Kim result has always been regarded as doubtful because of the difficulty involved in accurately saturating the continuum integral. Their analysis involves the assumption

$$\mathrm{Im}\ \overline{T}(\nu) \approx \mathrm{Im}\ A_o(|\nu|) \quad , \tag{3.103}$$

where $A_o(\nu)$ is the S-wave meson-baryon scattering amplitude in the s-channel. As noted in Section III. 2, this procedure neglects meson-creation and Z diagrams (apart from obvious poles), and replaces the S-wave direct-channel cut contribution to

$$\partial^\mu \mathfrak{F}_{5\mu}(q) + B(p) \rightarrow \partial^\mu \mathfrak{F}_{5\mu}(q) + B(p) \tag{3.104}$$

by the S-wave s-channel cut contribution for $MB \rightarrow MB$. The rationale for the latter step is that the only important cut terms are those with $|q^2| = \nu^2 = 0(m_M^2)$. However, this method is clearly inferior to those of Cheng and Dashen and Höhler, et. al., which depend only on the accuracy of the data, (apart from terms like "$0(m_\pi^4)$" in Eq. (3.88)).

There is another aspect of the von Hippel-Kim analysis which we question. The standard practice in this type of analysis is to ignore the equal-time commutator C. Thus, Fubini and Furlan[91] argue that C vanishes because the pion field $\partial^\mu \mathfrak{F}_{5\mu}$ should obey canonical commutation relations. In a theory of broken scale invariance, this happens only if $\partial^\mu \mathfrak{F}_{5\mu}$ has dimension $\ell_u = -1$. This assumption is completely arbitrary. The fact that $\partial^\mu \mathfrak{F}_{5\mu}$ has a pion pole has nothing to do with its dimension, since $\partial^\mu \mathfrak{F}_{5\mu} \exp(\ell\sigma/F_\sigma)$ couples pions to the vacuum with exactly the same strength; in fact, unless dilaton amplitudes are being investigated, these two pion fields are equally good candidates for smoothly extrapolating a pion amplitude of "off-mass-shell".

At various stages in their analysis, von Hippel and Kim[113] mention the fact that C does not vanish in the quark model. However their argument that C is negligible is circular. In the quark model, C involves an overall constant of proportionality which contains the quark mass m_q so their estimate of C depends on one's interpretation of the symbol m_q. Von Hippel and Kim assume

$$3\, m_q = \langle N | u_o | N \rangle \quad , \tag{3.105}$$

as indicated by a non-relativistic quark model for the SU(3) x SU(3) - violating term in θ_{oo}. Unfortunately, they also make use of their main result $\langle N | u_o | N \rangle \simeq -215$ MeV, which was computed <u>assuming</u> C = 0.

When Eq. (3.101) was first formulated,[91] it was accompanied by a statement that the usual soft-pion formula does not work if C is not small. This remark is <u>not valid</u> if the soft-pion formula is written correctly, as in Eq. (3.88). Ignoring the effects of σ-poles for the moment, we can analyze Eq. (3.101) in the following manner: the continuum integral is "$0(m_M^4)$", while $\overline{T}(m_M)$ and $\overline{T}(0)$ are "$0(m_M^2)$". Now, it is essential to distinguish the threshold amplitude

$$\overline{T}(m_M) = T(m_M, -m_M^2/2M_B, m_M^2, m_M^2) \tag{3.106}$$

from the amplitude $T(0,0,m_M^2,m_M^2)$ which Cheng and Dashen estimate. In fact, Eqs. (3.88) and (3.101) imply

$$\begin{aligned} -i\,(2f_M/m_M)^2\, C &= T(m_M, -m_M^2/2M_B, m_M^2, m_M^2) \\ &\quad + T(0,0, m_M^2, m_M^2) + \text{"}0(m_M^4)\text{"} \\ &= \text{"}0(m_M^2)\text{"} \quad . \end{aligned} \tag{3.107}$$

This is the <u>same</u> order of magnitude as the σ-commutator contribution $\overline{T}(0)$ to Eq. (3.101).

These observations may be formulated in a more general fashion by applying Wilson's theory of approximate scale invariance at short distances.[40] Let us examine $[v(x), v(0)]$ as ℓ_u varies. We wish to examine connected matrix elements of the equal-time commutator $[\dot{v}, v]$, so only q-numbers involving a singularity x^{-s} $(s \geq 2)$, are relevant. For $\ell_u = -1$, no q-numbers are available.

For the other integer values of ℓ_u, we obtain the following short-distance expansions:

$$[v(x), v(0)] = m_M^2 c_1 E^{(-1)}(x^2) u(0) + \ldots , (\ell_u = -2), \quad (3.108)$$

$$[v(x), v(0)] = (m_M^4/M_o^2)[c_2 \partial_\mu E^{(-1)}(x^2) \mathfrak{J}^\mu(0) + c_3 \partial_\mu \partial_\nu \log x^2 \theta^{\mu\nu}(0) + \ldots \ldots + c_4 E^{(-1)}(x^2) u(0) + \ldots],$$

$$(\ell_u = -3), \quad (3.109)$$

where the dependence on internal quantum numbers has been suppressed. The dimensionless numbers c_i do not depend on the magnitude of the violation of scale invariance, and have values $0(1)$ in models. In (3.109), masses are normalized relative to $M_o = 1$ BeV, and the term with coefficient c_4 is a scale-violating contribution. The diagonal matrix element C defined by Eq. (3.102) receives contributions from all terms in Eqs. (3.108) or (3.109) except the term proportional to c_2.

For $\ell_u = -2$, we have no reason to suppose that c_1 vanishes, and so we obtain

$$[\partial_o v(x), v(0)]_{x_o=0} = -4\pi^2 i\, m_M^2 \delta^3(\vec{x})\, u(0)\, c_1 , \quad (3.110)$$

which cannot be ignored relative to the contribution from the σ-term. For example, in a model containing canonical $0^+, 0^-$ fields, σ, ϕ, (i.e., dim σ = dim ϕ = -1), formulae such as

$$4\pi^2 [\phi(x), \phi(0)] = E^{(-1)}(x^{-2}) \quad I , \tag{3.111}$$

or the corresponding canonical commutation relations, imply

$$c_1 = (8\pi^2)^{-1} \quad , \tag{3.112}$$

where the scalar and pseudoscalar densities are defined by

$$u = \tfrac{1}{2} m_M^2 (\sigma^2 + \phi^2) \quad ,$$

$$v = \tfrac{1}{2} m_M^2 \sigma \phi \quad . \tag{3.113}$$

Apart from a difference in normalization factors, the model of Ellis[83] gives the same result.

The situation for $\ell_u = -3$ is more complicated, because there are two terms which may contribute to C:

$$[\partial_o v(x), v(0)]_{x_o=0} = (4\pi^2 i\, m_M^4/M_o^2)[c_2 \mathfrak{F}^i(0)\, \partial_i \delta^3(\vec{x}) + 2c_3 \theta_{oo}(0)\, \delta^3(\vec{x})$$

$$- c_4\, u(0)\, \delta^3(x)] \; ; \tag{3.114}$$

(the first term vanishes when integrated over 3-space). For example,

$$c_3 = 3c_4 \tag{3.115}$$

is obtained in the free quark model. Part of the "$0(m_M^4)$" term to be proportional to θ_{ii}, which does not contribute to C because of the self-stress theorem.* In the free quark model, the operator product of Eq. (3.109) has the form

$$4\pi^2 [\bar{q}(x)\gamma_5 q(x), \bar{q}(0)\gamma_5 q(0)] = \bar{q}(x)\{i\not\partial + m)E^{(-1)}(x^2)\} q(0)$$

$$+ \bar{q}(0)\{(-i\not\partial + m) E^{(-1)}(x^2)\} q(x). \quad (3.116)$$

Applying the Taylor series expansion $q(x) = q(0) + x\cdot\partial q(0) + \ldots$, we observe that $i\not\partial$ and m give rise to θ_{oo} and u, respectively, in Eq. (3.114). Actually, von Hippel and Kim incorrectly replace $i\dot{v}$ by $\left[v, \int d^3x\,(u_o + cu_8)\right]$, so they obtain $0(m_M^{\;6})$ terms, which turn out to be very small because of the use of Eq. (3.105).

Entirely different structure is observed in a boson model in which the pseudoscalar density is given by

$$v = \tfrac{1}{2} m_M^{\;2} \;\phi\; \exp(2\sigma/F_\sigma) \;, \quad (3.117)$$

for example. Then the right-hand side of Eq. (3.114) contains the quartic terms $\phi^2\exp(2\sigma/F_\sigma)$, $\exp(4\sigma/F_\sigma)$; in Ellis' model, the latter term gives the same situation as for the case $\ell_u = -2$. With or without dilatons, C cannot be neglected for $\ell_u = -3$, either.

*See Appendix A. To be consistent, Eq. (3.105) should read $3m_q = M_N$.

We conclude that the sum rules investigated by von Hippel and Kim do not specify the value of $\langle N|u|N\rangle$, unless a separate investigation demonstrates that ℓ_u has the canonical value -1; i.e., the equal-time commutator being evaluated is

$$E = \langle B|[\ddot{F}_5(0) + m_M^2 F_5(0)\,,\, \partial^\mu \mathfrak{F}_{5\lambda}(0)]|B\rangle_{\text{Conn.}} \quad . \tag{3.118}$$

This observation does not greatly affect their comparison of theory with experiment for the meson-baryon scattering lengths.

In dilaton theory, "$0(m_M^2)$", "$0(m_M^4)$", ... become "$0(m_M^2/m_\sigma^2)$", "$0(m_M^4/m_\sigma^2)$", ..., because of the presence of dilaton poles. Ellis[(103)*] has suggested that the approximation of von Hippel and Kim, Eq. (3.103), is not valid because of rapid variation due to the σ-pole. He presents a Lagrangian model which displays this variation, and he is able to fix an unknown constant such that the agreement between theory and experiment for scattering lengths obtained by von Hippel and Kim is not upset. However, the analysis mixes contributions from σ-poles with phenomenological estimates of cut contributions. Since one cannot tell how much the σ-pole contributes to the latter, some further comment seems necessary.

Eqs. (3.100), (3.101) and (3.118) imply the exact relation

$$(2f_M)^{-2}\, \overline{T}(m_M) = -i\, E/m_M^2 + I_{\text{Cont.}} \tag{3.119}$$

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* Ellis does not alter the von Hippel-Kim assumption that C is negligible.

for the threshold amplitude $\overline{T}(m_M)$, with*

$$I_{Cont.} = \frac{2m_M^2}{\pi(2f_M)^2} \int_{-\infty}^{\infty} \frac{d\nu \ \mathrm{Im} \ \overline{T}(\nu)}{\nu(\nu^2 - m_M^2)} = 0(m_M^4/m_\sigma^2 M_o) \ . \qquad (3.120)$$

In general, the other two terms in Eq. (3.119) are $0(M_o m_M^2/m_\sigma^2)$. Taking Ellis' point of view, the von Hippel-Kim method fails if $0(M_o\, m_M^2/m_\sigma^2)$ terms are introduced by the approximation (3.103). In theory, this is quite easy to arrange: the term

$$P(\nu, \nu_B, q^2, q'^2) = \frac{\xi}{m_\sigma^2 - \xi} \ \frac{m_\sigma^2 - 2m_M^2 + 4M_B \nu_B}{m_\sigma^2} \ Q \qquad (3.121)$$

(with $\xi = \xi(\nu_B, q^2, q'^2) = (q' - q)^2$, and Q an analytic function), vanishes on the collinear dispersion path:

$$P(\nu, -\nu^2/2M_B, \nu^2, \nu^2) = 0 \qquad (3.122)$$

Thus theoretical manipulation generates an arbitrary amount of rescattering integral with magnitude $0(M_o\, m_M^2/m_\sigma^2)$, depending on which function $\mathrm{Im}\, Q(\nu, \nu_B, m_M^2, m_M^2)$ one cares to choose. For example, we could have Q proportional to $T(\nu, \nu_B, m_M^2, m_M^2)$, in which case, the rescattering integral would be a linear combination of S- and P-wave MB amplitudes. Experimentally, the size of the

*Compare Eq. (3.93).

rescattering corrections found by von Hippel and Kim roughly indicates the contamination from $0(M_o m_M^2/m_\sigma^2)$ terms. A good example is the rescattering integral for the s-channel isoscalar $\bar{K}N$ amplitude, which contributes about 3 GeV to Eq. (3.119).

Evidently, the indicated procedure is to make an estimate of the magnitude of $I_{Cont.}$, treat it as an uncertainty, and then attempt to isolate the $0(M_o m_M^2/m_\sigma^2)$ terms, which should dominate. We restrict our attention to πN scattering amplitudes, because we are confident that $I_{Cont.}$ may be safely ignored. The justification is the same as that for the neglect of "$0(m_\pi^4)$" terms in Eq. (3.88), (the theorem used by Cheng and Dashen). If $a_0^{(t)}$ is the t-channel isospin-0 scattering length, Eq. (3.119) becomes

$$- i \ E/m_\pi^2 = (2f_\pi)^{-2} (1 + m_\pi/M_N) \, 4\pi \, a_0^{(t)} + \text{"}0(m_\pi^4)\text{"}. \tag{3.123}$$

All estimates of $a_0^{(t)}$ indicate that it is very small, although not everyone* agrees with the limits of uncertainty in the standard value

$$a_0^{(t)} = \frac{1}{3} a_{1/2}^{(s)} + \frac{2}{3} a_{3/2}^{(s)} = - (0.012 \pm 0.004)/m_\pi \tag{3.124}$$

quoted in the review by Moorhouse.[130] Eq. (3.124) implies

$$i \ E/m^2 \simeq 11 \quad \text{MeV} \quad , \tag{3.125}$$

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* See the discussion by Höhler, et. al., (ref. 129).

which is an order of magnitude smaller than the value for σ_{NN} obtained by Cheng and Dashen; (see Eq. (3.87)). From the definition of E, Eq. (3.118), we see that $[\partial_0 v(0,\vec{x}), v(0,\vec{0})]$ does not vanish; in fact, its contribution to Eq. (3.101) is as large as that of the σ-term. Therefore, assuming that the dimension ℓ_u of u is unique, our previous analysis implies

$$\ell_u \leqslant -2 \quad . \tag{3.126}$$

Eq. (3.126) assumes that there are no extra scalars of dimension -1 in the operator product expansion $[v(x), v(0)]$; otherwise, we would have $\ell_u \leqslant -3/2$. The important conclusion is that, if ℓ_u is unique, it cannot equal -1.

III.4. Concluding Remarks

Our final result, Eq. (3.126), provides a partial justification for assuming that the dilation operator D(0) commutes with the axial charge $F_5(0)$; (equivalently, $S_{\mu\nu}$, defined by Eq. (1.56), is not present in $\theta_{\mu\nu}$, or, the constant r vanishes in Eq. (2.72)). At various stages in the analysis, we have indicated that Eq. (3.126) is the condition needed. In turn, this statement assumes that (u_a, v_a) are the only spin-0 terms which break chiral invariance in an operator product expansion. This follows the suggestion of Wilson that only a limited set of "licensed" operators $0_n(x)$ is present in Eq. (1.79).

It is unfortunate that experiments on scalar mesons are not more definitive. Combined with the theoretical uncertainties of our calculation, there is a lot of room for error. However, we have been able to show that the dilaton has a large width, but not necessarily so large that the Adler-Weisberger sum rule for $\pi\pi$ scattering is oversaturated. Our result for the spin-2 gravitational radius of the pion is less positive, since we can argue that it is reasonable as long as f-dominance does not work. The most striking feature of these results, (Eqs. (3.50), (3.51), and especially (3.56)), is the strong dependence on the magnitude of scale violation. One must be very careful to look for these deviations from the standard soft-meson result when the symmetry is so badly broken.

Having observed that a dilaton couples strongly to both mesons and baryons, it is natural to consider the effect of t-channel dilaton poles on theorems for meson-baryon scattering at low energies. Of particular interest are possible interpretations of the recent result of Cheng and Dashen[108]; (see Eq. (3.87)):

(i) The breaking of SU(2) x SU(2) is almost as large as SU(3) violation, and the SU(3) x SU(3) breaking term u in θ_{oo} contains representations other than $(3,\bar{3}) + (\bar{3},3)$ and $(1,8) + (8,1)$; e.g., Cheng and Dashen adopt this proposal with u transforming like (8,8). This is a respectable possibility which is hard to discount.

(ii) SU(3) is a much better symmetry than SU(2) x SU(2), and u belongs to $(3,\bar{3}) + (\bar{3},3)$; (i.e., $|c| \ll \sqrt{2}$).[122,123] This scheme has trouble explaining $K_{\ell 3}$ decay; (see Fig. 4). We do not regard it as a viable theory.

(iii) The SU(2) x SU(2) and SU(3) x SU(3) breaking terms in θ_{oo} transform under different representations of SU(3) x SU(3). This model is usually dismissed as an ugly theory; purely experimental tests are very difficult to find.

(iv) SU(2) x SU(2) is a much better symmetry than SU(3), $u = -u_o - cu_8$ belongs to $(3,\bar{3}) + (\bar{3},3)$ with $c \simeq -1.25$, and there is a dilaton associated with approximate scale invariance. The magnitudes of scale and chiral SU(3) x SU(3) breaking are comparable. Thus, one imagines that δ vanishes in the limit of chiral invariance; (δ = c-number is a special case favored by Ellis[83,104]). This is the theory which we discuss in Section III.3.

Except for scheme (ii), consistency of the sum rule (3.101) for the threshold amplitude, the experimental value of the isospin-symmetric scattering length, and the Cheng-Dashen result requires $\ell_u \neq -1$.

Let us conclude by briefly mentioning some questions which have been raised about having $c \simeq -1.25$ and u belonging to $(3,\bar{3}) + (\bar{3},3)$.

Recently, Dashen[55] showed that the amplitude for $K_S^o \to 2\pi$ is $0(\epsilon^2)$ in a conventional $(3,\bar{3}) + (\bar{3},3)$ model, where ϵ is a parameter indicating the magnitude of SU(3) x SU(3) violation. This is not necessarily a difficulty because there is an overall normalization factor which must be estimated by other means. The dilaton scheme (iv) provides another way out. Essentially, the result is that the amplitude is dominated by the σ-pole. This justifies one of the assumptions of Dutta-Roy and Lapidus,[131] who also assume that the $K_1^o - K_2^o$ mass difference may be calculated in this fashion. Experimentally, these assumptions work quite well.

The most awkward problem in current algebra is the treatment of second-order electromagnetic processes in which the photons are not emitted. Whenever the effective electrodynamic Hamiltonian is supposed to cummute with F_5^3, the wrong answer is obtained. Brandt and Preparata[(123)] take this as evidence that SU(2) x SU(2) is a very bad symmetry. However, there are at least two other ways of proceeding:

(a) There may be a term u_3 in θ_{oo} which is not electromagneitc in origin. This scheme is attractive in that it provides a means of calculating the Cabibbo angle.[(132)]

(b) The formal expression is divergent, and therefore needs a subtraction, the $I = 1$ tadpole. This solution is advocated by Wilson.[(40)]

(Dilaton theory does not affect this type of calculation). With the theoretical freedom provided by (a) and (b), there is no problem in taking SU(2) x SU(2) to be a very good symmetry.

Although the non-vanishing rate for $\pi^o \rightarrow 2\gamma$ was once a problem, it is now reasonably well understood. (Again, Brandt and Preparata claim that their point of view is supported). Adler[(133)] and Bell and Jackiw[(134)] observed that the usual manipulations break down in spinor electrodynamics, and other models involving fermions. Because of the model-dependence of these investigations, there was some confusion as to their validity. The problem was put into perspective by Wilson,[40] who demonstrated that the soft-$\pi^o \to 2\gamma$ amplitude is proportional to the function $C_\eta(x, z)$ given by

$$\mathcal{J}^a_\mu(x)\, \mathcal{J}^b_\nu(0)\, \mathcal{J}^c_{5\lambda}(z) = d^{abc}\, \epsilon_{\mu\nu\lambda\eta}\, C^\eta(x, z)\, I + \ldots \quad , \qquad (3.127)$$

so there is no reason for $c \simeq -1.25$ to be invalid in a theory of broken scale invariance.

We can carry out a similar calculation for $\sigma \to 2\gamma$, ("anomalous PCDC"). The usual argument gives a vanishing rate, because

$$-\mathscr{L} = \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} \qquad (3.128)$$

is already scale-invariant, so the factor $\exp(\sigma/F_\sigma)$ cannot be added. In fact, repeating the argument of Wilson for the dilation current, the soft-$\sigma \to 2\gamma$ amplitude is proportional to

$$\mathcal{J}^a_\mu(x)\, \mathcal{J}^b_\nu(0)\, \theta_{\lambda\eta}(z) = \delta^{ab}\, C_{\mu\nu\lambda\eta}(x, z)\, I + \ldots \quad . \qquad (3.129)$$

Thus, the symmetry-breaking scheme outlined in this thesis appears to be a viable theory. However, some alternatives are difficult to discount.

APPENDIX A

ELEMENTARY TECHNICAL REMARKS

One-particle states are normalized invariantly,

$$\begin{aligned} \langle p'|p\rangle &= (2\pi)^3\, 2p_o\, \delta^3(\vec{p}\,'-\vec{p})\ , \quad \text{(bosons)}\ , \\ \langle P'|P\rangle &= (2\pi)^3\, P_o/M\, \delta^3(\vec{P}\,'-\vec{P}), \quad \text{(fermions)}\ , \end{aligned} \tag{A.1}$$

or occasionally to one particle per unit volume:

$$\langle\langle p'|p\rangle\rangle = (2\pi)^3\, \delta^3\, (\vec{p}\,'-\vec{p})\ , \quad \text{(bosons or fermions)}\ . \tag{A.2}$$

We have suppressed labels denoting spin or internal quantum numbers. We follow Sections 1 and 2 of "Conventions and Notation" in the book by Adler and Dashen,(44) except for our $\gamma_5 = \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$.

A useful theorem(13) states that $\int d^3x\, s(x) = 0$ implies $s(x) = 0$ if $s(x)$ is a local, spin-0 operator. To prove it, consider

$$s(K) = \int d^4x\, e^{iK.x}\, s(x) \qquad ; \tag{A.3}$$

the hypothesis becomes

$$s(K_o, \vec{0}) = 0 \qquad , \tag{A.4}$$

so Lorentz invariance implies $s(K) = 0$ for $K^2 > 0$. If K^2 is spacelike, $|\psi\rangle = s(K)|0\rangle$ must vanish -- otherwise K_μ would be the momentum of the state $|\psi\rangle$. Since $\langle\psi_2|\, s(K)\, |\psi_1\rangle$ is an analytic continuation of $\langle\psi_2, \overline{\psi}_1|s(K)|0\rangle$, $s(K)|0\rangle = 0$ implies(135) $s(K) = 0$, i.e., $s(x) = 0$.

Let us briefly summarize canonical field theory for the stress-energy tensor and scale and conformal transformations. One starts from a Lagrangian density $\mathscr{L}$ which is a function of fields ψ and a finite number of field derivatives:

$$\mathscr{L} = \mathscr{L}(\psi, \partial\psi, \partial^2\psi, \ldots, \partial^r\psi) \quad . \tag{A. 5}$$

Usually, only dependence on ψ and $\partial\psi$ is considered, but, as emphasized by Huggins,[10] this restriction is entirely artificial. What matters is the form of the equations of motion, which are given by, e.g.,

$$\frac{\delta\mathscr{L}}{\delta\psi} - \partial^\mu \frac{\delta\mathscr{L}}{\delta\,\partial^\mu\psi} + \partial^\mu\partial^\nu \frac{\delta\mathscr{L}}{\delta\,\partial^\mu\partial^\nu\psi} = 0 \tag{A. 6}$$

for $r = 2$.

The construction of the generators G_a of a transformation with parameters α_a proceeds via the Action Principle, Eq. (1.20), with

$$A_{1,2} = \int_{\sigma_1}^{\sigma_2} d^4x\, \mathscr{L}(x) \tag{A. 7}$$

An expression of the form

$$G_a(\sigma) = \int_\sigma d\sigma^\mu\, \mathscr{G}_{\mu a}(x) \tag{A. 8}$$

results, where (e.g., for $r = 2$)

$$\mathscr{G}_{\mu a}(x) = T_{\mu\nu}\frac{\delta x^\nu}{\delta\alpha^a} - \left[\frac{\delta\mathscr{L}}{\delta\,\partial^\mu\psi} - \partial_\nu \frac{\delta\mathscr{L}}{\delta\,\partial^\mu\partial_\nu\psi}\right]\frac{\delta\psi}{\partial\alpha^a}$$

$$+ \frac{\delta\mathscr{L}}{\delta\,\partial^\mu\partial_\nu\psi}\left[\partial_\nu\left(\frac{\delta x_\lambda}{\delta\alpha^a}\right)\partial^\lambda\psi - \partial_\nu\frac{\delta\psi}{\delta\alpha^a}\right] \tag{A. 9}$$

is a corresponding current, and

$$T_{\mu\nu}(x) = \left[\frac{\delta\mathcal{L}}{\delta\partial^{\mu}\psi} - 2\partial_{\lambda}\frac{\delta\mathcal{L}}{\delta\,\partial^{\mu}\partial_{\lambda}\psi}\right]\partial_{\nu}\psi + \partial_{\lambda}\left[\frac{\delta\mathcal{L}}{\delta\partial^{\mu}\partial_{\lambda}\psi}\partial_{\nu}\psi\right] - g_{\mu\nu}\mathcal{L} \qquad \text{(A. 10)}$$

is called the canonical energy-momentum tensor.

This prescription uniquely fixes G_i (within a unitary transformation induced by $\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \partial_{\mu}V^{\mu}$), but Eq. (A. 9) is not the only possible expression for the current $\mathcal{G}_{\mu a}$. This is important if the current has a physical interpretation as the source of electromagnetic, weak, or gravitational interactions. For example, the charge operator Q is obtained from the gauge transformation

$$\frac{\delta\psi}{\delta\alpha} = i\,q\,\psi \quad , \quad \frac{\delta x^{\nu}}{\delta\alpha} = 0 \quad , \qquad \text{(A. 11)}$$

where q is the charge annihilated by ψ. However, the most general expression for the electromagnetic current is

$$J_{\mu}(x) = \mathcal{G}_{\mu} + \partial^{\nu}\mathcal{J}_{\mu\nu} \quad , \qquad \text{(A. 12)}$$

where $\mathcal{J}_{\mu\nu}$ is an arbitrary, antisymmetric tensor.

Now consider the group of Poincaré transformations. Translations are characterized by

$$\frac{\delta\psi}{\delta\alpha^{\nu}} = 0 \quad , \quad \frac{\delta x^{\lambda}}{\delta\alpha^{\nu}} = \delta_{\nu}^{\ \lambda} \quad , \qquad \text{(A. 13)}$$

so the canonical current is $T_{\mu\nu}$ and the 4-momentum is

$$P_{\nu} = \int d^3x\, T_{o\nu} \quad . \qquad \text{(A. 14)}$$

Lorentz transformations are given by ($\alpha^{\mu\nu} = -\alpha^{\nu\mu}$)

$$\frac{\delta\psi}{\delta\alpha^{\mu\nu}} = +\frac{i}{2}\,\Sigma_{\mu\nu}\,\psi \quad , \quad \frac{\delta x^{\lambda}}{\delta\alpha^{\mu\nu}} = \tfrac{1}{2}(\delta_{\nu}^{\ \lambda}\,x_{\mu} - \delta_{\mu}^{\ \lambda}\,x_{\nu}) \quad , \qquad \text{(A. 15)}$$

where $\Sigma_{\mu\nu}$ is the spin matrix of ψ; (e.g., $\Sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ for spin-$\frac{1}{2}$ fields). From Eq. (A.9), the canonical angular momentum density takes the form

$$\mathcal{M}'_{\lambda\mu\nu} = \mathcal{L}_{\lambda\mu\nu} + \mathcal{S}_{\lambda\mu\nu} \tag{A.16}$$

where

$$\mathcal{L}_{\lambda\mu\nu} = x_\mu T_{\lambda\nu} - x_\nu T_{\lambda\mu} \tag{A.17}$$

is interpreted as the orbital angular momentum density, and the remainder $\mathcal{S}_{\lambda\mu\nu}$ is taken to be spin angular momentum density.

Since gravity couples to energy, the gravitational current $\theta_{\mu\nu}$ must also be a suitable energy-momentum density from which P_μ may be formed; (see Eq. (1.9)). Therefore, it should be possible to construct a candidate for the symmetric tensor $\theta_{\mu\nu}$ from $T_{\mu\nu}$ ($\neq T_{\nu\mu}$). According to the prescription given by Belinfante,[4] the appropriate construction is

$$\theta^{\text{Bel.}}_{\mu\nu} = T_{\mu\nu} - \partial^\lambda f_{\lambda\mu\nu} = \theta^{\text{Bel.}}_{\nu\mu} , \tag{A.18}$$

with

$$f_{\lambda\mu\nu} = \tfrac{1}{2}(\mathcal{S}_{\lambda\nu\mu} + \mathcal{S}_{\mu\lambda\nu} + \mathcal{S}_{\nu\lambda\mu}) = -f_{\mu\lambda\nu} . \tag{A.19}$$

Then

$$\mathcal{M}^{\text{Bel.}}_{\lambda\mu\nu} = x_\mu \theta^{\text{Bel.}}_{\lambda\nu} - x_\nu \theta^{\text{Bel.}}_{\lambda\mu} = \mathcal{M}'_{\lambda\mu\nu} - \partial^\eta(x_\mu f_{\eta\lambda\nu} - x_\nu f_{\eta\lambda\mu}) \tag{A.20}$$

differs from $\mathcal{M}'_{\lambda\mu\nu}$ by a term which vanishes upon integration over $d\sigma^\lambda$, and therefore is an acceptable total angular momentum density, (even though the expression for $\mathcal{M}^{\text{Bel.}}_{\lambda\mu\nu}$ resembles the classical formula $\vec{x} \times \vec{p}$ for orbital angular momentum):

$$M_{\mu\nu} = \int d\sigma^{\lambda}\, \mathcal{M}'_{\lambda\mu\nu} = \int d\sigma^{\lambda}\, \mathcal{M}^{\text{Bel.}}_{\lambda\mu\nu} \quad . \tag{A. 21}$$

Eq. (A. 18) becomes a candidate for the gravitational current, but, as Huggins noted,[10] this choice is not unique. In fact, the Belinfante prescription gives an extra term proportional to $(\partial_\mu \partial_\nu - g_{\mu\nu}\, \partial^2)\phi^2$ if $c\, \partial^2(\phi^2)$ is added to the Lagrangian, but the equations of motion are independent of the constant c. Similarly, different prescriptions can produce different $\theta_{\mu\nu}$'s from the same $\mathcal{L}$. Note that Eq. (A. 9) can be replaced by

$$\mathcal{J}^{\text{Bel.}}_{\mu a}(x) = \theta^{\text{Bel.}}_{\mu\nu} \frac{\delta x^\nu}{\partial\alpha^a} - f_{\lambda\mu\nu}\, \partial^\lambda \frac{\delta x^\nu}{\delta\alpha^a} - \left[\frac{\delta\mathcal{L}}{\delta\, \partial^\mu \psi} - \partial_\nu \frac{\delta\mathcal{L}}{\delta\, \partial^\mu \partial_\nu \psi}\right] \frac{\delta\psi}{\partial\alpha^a}$$
$$+ \frac{\delta\mathcal{L}}{\delta\, \partial^\mu \partial_\nu \psi}\left[\partial_\nu\left(\frac{\delta x_\lambda}{\delta\alpha^a}\right)\partial^\lambda\psi - \partial_\nu \frac{\delta\psi}{\delta\alpha^a}\right] \quad . \tag{A. 22}$$

To treat scale and conformal transformations, it is necessary to decide how the fields ψ transform. In the standard approach, the fields transform as irreducible representations of the conformal group; (see Mack and Salam[9] for the details). For scale transformations, the answer, Eq. (1. 25), is obvious. In the notation of this appendix, we have

$$\frac{\delta x^\mu}{\delta\rho} = x^\mu \quad , \qquad \frac{\delta\psi}{\delta\rho} = \ell\,\psi \quad . \tag{A. 23}$$

If the time-dependence of D, the generator of scale transformations, is ignored, then the values $\ell = -3/2,\ -1$ for fermions and bosons are obviously required by the canonical commutation relations (1.27). When the time-dependence is included, we have

$$i\,[D(x_o),\ \partial_o\,\phi(x)] = i\,\partial_o[D(x_o),\ \phi(x)] - i[\dot{D}(x_o),\ \phi(x)]$$

$$= (-\ell + 1 + x.\partial)\,\partial_o\phi - i[\dot{D}(x_o), \phi(x)]\ . \qquad (A.24)$$

Evidently, the condition for a consistent theory is

$$i[\dot{D}(x_o), \phi(x)] = 0 \qquad , \qquad (A.25)$$

i.e., the divergence of the dilation current should contain no derivatives. For this reason, Eq. (1.23) is valid only for the new, improved tensor defined in Eq. (1.15). For conformal transformations, we have

$$\frac{\delta x^\nu}{\delta c^\lambda} = 2\,x_\lambda\,x^\nu - \delta_\lambda^\nu\,x^2 \quad , \quad \frac{\delta\psi}{\partial c^\lambda} = 2\,x^\eta(\ell\,g_{\eta\lambda} + i\,\Sigma_{\eta\lambda}) - \kappa_\lambda\psi \quad , \qquad (A.26)$$

where κ_μ is a nilpotent matrix, (which vanishes for the low-spin fields considered here). Eq. (A.22) may be used to construct the generators D and K_μ and verify equations such as (A.25). The resulting expressions look complicated, but, for all interesting theories, they reduce to Eqs. (1.16) and (1.17); (for r = 1, the most general treatment has been given by CCJ[(12)]).

The various $\theta_{\mu\nu}$'s give matrix elements which have certain features in common. Typically, they differ only in terms like $k_\mu k_\nu - g_{\mu\nu}\,k^2$, where k_μ is the momentum transfer. However, the simplest and most general treatment of $\langle\theta_{\mu\nu}\rangle$ ignores canonical field theory, and involves the direct use of Eqs. (1.9) and (1.10). Thus, Eqs. (1.9) and (A.2) imply

$$2p_o\langle\langle p', s' | \int d^3x\, \theta_{o\mu}(0, \vec{x}) | p, s\rangle\rangle = (2p_o)(2\pi)^3\, p_\mu \delta^3(\vec{p}\,' - \vec{p})\delta_{s,s'} \quad , \tag{A. 27}$$

where s denotes the spin component of a single-particle state, so the diagonal matrix element

$$2p_o\langle\langle p, s | \theta_{\mu\nu}(0) | p, s\rangle\rangle \quad = \quad 2p_\mu p_\nu \tag{A. 28}$$

is completely specified. Eq. (A. 28) implies the self-stress theorem: the diagonal matrix element of θ_{ii} for a single particle at rest vanishes.

When the momentum transfer k_μ does not vanish, only terms 0(1) or 0(k) are specified. Eq. (1. 75) is the general expression for spin-0 mesons, obtained by expanding in the available momenta P_μ, k_μ to form a symmetric second-rank tensor obeying $k^\mu \langle\theta_{\mu\nu}\rangle = 0$. No extra constraints are implied by Eq. (1. 10). For spin-$\frac{1}{2}$ mesons, the same principles yield

$$\langle N(P + \tfrac{1}{2}k, s')| \theta_{\mu\nu}(0) | N(P - \tfrac{1}{2}k, s)\rangle = \bar{u}(P + \tfrac{1}{2}k, s')[\tfrac{1}{2}(\gamma_\mu P_\nu + \gamma_\nu P_\mu)G_1(k^2)$$
$$+ (P_\mu P_\nu / M)\, G_2(k^2) + (k_\mu k_\nu - g_{\mu\nu}k^2)G_3(k^2)]u(P - \tfrac{1}{2}k, s) \tag{A. 29}$$

Eq. (A. 28) implies

$$G_1(0) \; + \; G_2(0) \; = \; 1 \quad . \tag{A. 30}$$

Until very recently,[136] it was not realized[137] that Eq. (1. 10) provides a further constraint:

$$G_2(0) \quad = \quad 0 \tag{A. 31}$$

To demonstrate the truth of Eq. (A. 31), it is necessary to consider the boost operator

$$M_{io} = \int d^3x \; x_i \; \theta_{oo}(0, \vec{x}) \tag{A. 32}$$

From Eq. (A. 29), we obtain

$$\frac{\partial}{\partial k^i} \langle p+k, s' | \theta_{oo}(0) | p, s \rangle_{\vec{k}=0} = p_o \, \bar{u}(p, s') \left[\gamma_o \, G_1(0) + \frac{p_o}{M} G_2(0) \right] \gamma_i \, u(p, s) \, , \tag{A. 33}$$

and Eq. (A. 32) implies

$$\frac{\partial}{\partial k^i} \langle p + k, s' | \theta_{oo}(0) | p, s \rangle_{\vec{k}=0} = - i \int \frac{d^3k'}{(2\pi)^3} \langle p + k', s | M_{io} | p, s \rangle \, . \tag{A. 34}$$

Since the right-hand side of Eq. (A. 34) is the same for all theories, the value of $G_2(0)$ must be universal. Therefore, we may use free field theory to evaluate it; Eq. (A. 31) is the result.

Equations such as (1. 75) and (A. 29) are needed when expanding $\langle \psi_f | \theta_{\mu\nu} | \psi_i \rangle$ in powers of the momentum transfer for multiparticle states $| \psi_f \rangle$, $| \psi_i \rangle$. Examples of this type of calculation appear in Section I. 2 and Appendix C.

APPENDIX B: ALGEBRA OF THE CONFORMAL GENERATORS

The properties of $\theta_{\mu\nu}$ under Poincaré transformations are given by

$$i[P_\alpha, \theta_{\mu\nu}(x)] = \partial_\alpha \theta_{\mu\nu}(x) \ , \tag{B.1}$$

$$i[M_{\alpha\beta}, \theta_{\mu\nu}(x)] = (x_\alpha\partial_\beta - x_\beta\partial_\alpha)\theta_{\mu\nu} + g_{\alpha\mu}\theta_{\beta\nu} + g_{\alpha\nu}\theta_{\beta\mu} - g_{\beta\nu}\theta_{\alpha\mu} - g_{\beta\mu}\theta_{\alpha\nu} . \tag{B.2}$$

The Poincaré algebra

$$[P_\alpha, P_\mu] = 0 \ ,$$

$$i[P_\alpha, M_{\mu\nu}] = g_{\alpha\nu}P_\mu - g_{\alpha\mu}P_\nu \ ,$$

$$i[M_{\alpha\beta}, M_{\mu\nu}] = g_{\alpha\mu}M_{\beta\nu} - g_{\beta\mu}M_{\alpha\nu} + g_{\beta\nu}M_{\alpha\mu} - g_{\alpha\nu}M_{\beta\mu} \ , \tag{B.3}$$

and part of its extension to the conformal algebra

$$i[P_\alpha, D(x_o)] = -P_\alpha + g_{o\alpha}\int d^3x\theta_\lambda^\lambda \ ,$$

$$i[M_{\alpha\beta}, D(x_o)] = \int d^3x(x_\alpha g_{o\beta} - x_\beta g_{o\alpha})\theta_\lambda^\lambda \ ,$$

$$i[P_\alpha, K_\mu(x_o)] = 2[M_{\alpha\mu} - g_{\alpha\mu}D(x_o) + g_{o\alpha}\int d^3x x_\mu \theta_\lambda^\lambda] \ ,$$

$$i[M_{\alpha\beta}, K_\mu(x_o)] = g_{\alpha\mu}K_\beta(x_o) - g_{\beta\mu}K_\alpha(x_o) + \int d^3x\, 2x_\mu(x_\alpha g_{o\beta} - x_\beta g_{o\alpha})\theta_\lambda^\lambda \ , \tag{B.4}$$

are merely special cases of Eqs. (B.1) and (B.2).

In order to obtain the rest of the conformal algebra, it is necessary to consider local versions of Eqs. (B.1) and (B.2):[138,139]

$$i[\theta^{oo}(x), \theta^{oo}(y)]\delta(x_o - y_o) = (\theta^{oi}(x) + \theta^{oi}(y))\partial_i^x\delta(x-y) - \partial_i^x\partial_j^x\partial_k^y\partial_\ell^y\tau_1^{ij,k\ell} \ ,$$

$$i[\theta^{oo}(x), \theta^{oi}(y)]\delta(x_o - y_o) = (\theta^{ij}(x) - \theta^{oo}(y)g^{ij})\partial_j^x\delta(x-y)$$
$$- \partial_k^x\partial_\ell^x\partial_j^y[\tau_2^{k\ell,ij} - \tfrac{1}{2}(\partial_o^x + \partial_o^y)\tau_1^{k\ell,ij}] \ ,$$

$$i[\theta^{oo}(x), \theta^{ij}(y)]\delta(x_o-y_o) = (\partial_o\theta^{ij}(x)-\theta^{oi}(y)\partial^j_x-\theta^{oj}(y)\partial^i_x)\delta(x-y)$$
$$+\partial^x_k\partial^y_\ell[\tau_3^{k\ell,ij}+(\partial^x_o+\partial^y_o)\tau_2^{k\ell,ij}-\tfrac{1}{2}(\partial^x_o+\partial^y_o)^2\tau_1^{k\ell,ij}] \quad ,$$

$$i[\theta^{oi}(x), \theta^{oj}(y)]\delta(x_o-y_o) = -(\theta^{oj}(x)\partial^i_x+\theta^{oi}(y)\partial^j_x)\delta(x-y)-\partial^x_k\partial^y_\ell\tau_3^{ik,j\ell} \quad ,$$

$$i[\theta^{oi}(x), \theta^{k\ell}(y)]\delta(x_o-y_o) = (-\theta^{k\ell}(x)g^{ij}+\theta^{jk}(y)g^{i\ell}+\theta^{j\ell}(y)g^{ik})\partial^x_j\delta(x-y)$$
$$-\partial^x_j[\tau_4^{ij,k\ell}-\tfrac{1}{2}(\partial^x_o+\partial^y_o)\tau_3^{ij,k\ell}] \, , \qquad \text{(B. 5)}$$

with the latin indices i, j, ... denoting spatial components 1, 2, 3. The conservation laws (1. 6) and (1. 7) have been taken into account. The "Schwinger terms" $\tau_p^{ij,k\ell}(x,y)$ are local in time and bilocal in $\vec{x}$ and $\vec{y}$. Boulware and Deser(139) obtained the result $\langle 0|\tau_{2,4}|0\rangle \neq 0$, which is analogous to Schwinger's result $\langle 0|[J_o(0,\vec{x}), J_i(0)]|0\rangle \neq 0$.(69) There the resemblance ends, because the τ_p are q-numbers in simple models. For example, in the free quark model, we have

$$\iint d^3x d^3y \tau_3^{ij,k\ell}(x,y) = \tfrac{1}{2}\int d^3x \, q^+(g^{jk}\sigma^{i\ell}+g^{i\ell}\sigma^{jk}+g^{j\ell}\sigma^{ik}+g^{ik}\sigma^{j\ell})q\,\delta(x_o-y_o) \, . \qquad \text{(B. 6)}$$

The symmetry properties

$$\tau_p^{ij,k\ell}(x,y) = (-1)^p\tau_p^{k\ell,ij}(y,x) \quad ,$$
$$\tau_p^{ij,k\ell}(x,y) = \tau_p^{ji,k\ell}(x,y) \quad , \qquad \text{(B. 7)}$$

are generally valid.

The remaining equal-time commutators of the conformal algebra depend on the model-dependent terms τ_p:

$$i[D(x_o), K^o(x_o)] = -K^o(x_o)+\int d^3x\, x^2\theta^\lambda_\lambda+2\iint d^3x d^3y[\tau_2^{kk,ii}-\tfrac{1}{2}(\partial^x_o+\partial^y_o)\tau_1^{kk,ii}],$$

$$i[D(x_o), K^i(x_o)] = -K^i(x_o)+2\iint d^3x d^3y x_k \tau_3^{ik,jj} \quad ,$$

$$i[K^i(x_o), K^j(x_o)] = -4\iint d^3x d^3y (\delta^i_k x_m + g_{km}x^i - \delta^i_m x_k)$$
$$(\delta^j_\ell y_n + g_{\ell n} y^j - \delta^j_n y_\ell)\tau_3^{mk,n\ell}$$

$$i[K^o(x_o), K^i(x_o)] = 2\int d^3x x^2 x^i \theta^\lambda_\lambda$$
$$-4\iint d^3x d^3y (\delta^i_j y_m + g_{jm} y^i - \delta^i_m y_j)(x_o \tau_3^{kk,mj} - \tau_2^{kk,mj} + \tfrac{1}{2}(\partial^x_o + \partial^y_o)\tau_1^{kk,mj}) \ . \quad \text{(B. 8)}$$

Note the explicit dependence on θ^λ_λ in Eqs. (B. 7) and (B. 8).

The τ-dependent terms of Eq. (B. 8) usually vanish in simple models, (e. g. the quark model). For example, the virial theorem, Eq. (1. 60), and the assumption $\dim \theta_{oi} = -4$, imply

$$i[D(x_o), K_\mu(x_o)] = -K_\mu(x_o) + g_{o\mu}\int d^3x x^2 \theta^\lambda_\lambda \ . \quad \text{(B. 9)}$$

The presence of q-number τ-dependent terms in $[K_\mu, K_\nu]$ would indicate the presence of an operator of dimension -1 in the short-distance expansion of $\theta_{\mu\nu}(x)\theta_{\alpha\beta}(0)$. We know of no model with this property, so the relation

$$[K_\mu(x_o), K_\nu(x_o)] = 2\int d^3x x^2 (x_\mu g_{o\nu} - x_\nu g_{o\mu})\theta^\lambda_\lambda + \text{c-number} \quad \text{(B. 10)}$$

is probably correct.

The apparent absence of τ-dependence in the broken conformal algebra is not characteristic of algebras of coordinate generators. For example, consider the group of SL(3, R) transformations,

$$x_i \to x'_i = a_{ij}x_j \ , \quad a_{ij} \text{ real} \ , \quad \det(a_{ij}) = 1 \ . \quad \text{(B. 11)}$$

The corresponding set of eight generators is formed by the total angular momentum $J_k = \frac{1}{2}\epsilon_{ijk}M_{ij}$ and the generators of skew transformations,

$$Q_{ij}(x_o) = \int d^3x(x_i\theta_{oj}+x_j\theta_{oi} - \frac{2}{3}\delta_{ij}x_k\theta_{ok}) \quad . \tag{B. 12}$$

Inspection of the formula

$$\dot{Q}_{ij}(x_o) = 2\int d^3x(\theta_{ij} - \frac{1}{3}\delta_{ij}\theta_{kk}) \tag{B. 13}$$

leads to the conclusion that consideration of the limit of SL(3, R) symmetry is meaningless. Therefore, it is not surprising if the equal-time commutation relations for the SL(3, R) generators do not possess such a symmetry limit. Poincaré invariance determines

$$i[M_{ij}, M_{k\ell}] = \delta_{jk}M_{i\ell} - \delta_{ik}M_{j\ell} + \delta_{i\ell}M_{jk} - \delta_{j\ell}M_{ik} \quad ,$$

$$i[M_{ij}, Q_{k\ell}] = \delta_{jk}Q_{i\ell} - \delta_{ik}Q_{j\ell} - \delta_{i\ell}Q_{jk} + \delta_{j\ell}Q_{ik} \quad , \tag{B. 14}$$

but $[\overleftrightarrow{Q}, \overleftrightarrow{Q}]$ depends on $\tau_3^{ij,k\ell}$, and is therefore model-dependent. Referring to Eq. (B. 6), we see that the quark model possesses a special property: τ_3 may be written in terms of the quark spin

$$S_k = -\frac{1}{4}\epsilon_{ijk}\int d^3x\, q^+\sigma_{ij}q \quad . \tag{B. 15}$$

Equation (B. 15) leads to the observation of Dothan, Gell-Mann and Ne'eman[140] that $[\overleftrightarrow{Q}, \overleftrightarrow{Q}]$ is an orbital angular momentum $\vec{L} = \vec{J} - \vec{S}$ in the quark model (instead of a total angular momentum as implied by naive SL(3, R) invariance):

$$i[Q_{ij}, Q_{k\ell}] = \delta_{jk}L_{i\ell} + \delta_{ik}L_{j\ell} + \delta_{i\ell}L_{jk} + \delta_{j\ell}L_{ik} \quad , \tag{B. 16}$$

with $L_{ij} = \epsilon_{ijk}L_k$.

Equation (B. 16) may serve as an abstract definition of orbital angular momentum. Assuming that the 3-momentum density θ_{oi} has dimension -4, the SL(3, R) generators, and hence $\vec{L}$, commute with the dilation operator D. The transformation exp(-iαD) also conserves parity, isospin, G-parity, baryon number and hypercharge, but does not commute with an operator which alters the radial wave function of a state; e.g. such an operator may have eigenvalues N, the "total quantum number" in the quark model of Feynman, Kislinger, and Ravndal[141]. Thus, $\exp(-i\alpha D)$ would generate a tower of resonances with the same $(J^P, I^G, B, Y, \vec{L}, \ldots)$ and different $(N, \ldots)$. An example of one of these towers may be: nucleon $P_{11}(940)$, Roper resonance $P_{11}(1470)$, Roper recurrence $P_{11}(1780), \ldots$.

APPENDIX C

SOME LOW-ENERGY THEOREMS

This appendix contains alternative derivations of Eqs. (2.50) and (2.52). We mentioned these derivations in Ref. 15, but gave no details.

We begin with another derivation of $F_2(0) = -1/3$, i.e., Eq. (2.52). Scale invariance is broken ($m_\sigma \neq 0$), but we ignore the violation of chiral SU(2) × SU(2) symmetry. Using the method developed by Low[19] for bremsstrahlung, we expand the amplitude

$$P_{\mu\nu} = \langle p(p_2) | \theta_{\mu\nu}(0) | n(p_1), \pi(q) \rangle \tag{C.1}$$

in powers of $k = p_1 + q - p_2$. The $0(k^{-1})$ terms, $P_{\mu\nu}^{\rm Born}$, are represented by Feynman graphs in which $\theta_{\mu\nu}$ hooks on to external lines:

$$\begin{aligned} P_{\mu\nu}^{\rm Born} = \; & \bar{u}_2 \Big[\tfrac{1}{2}(\gamma_\mu (p_2+\tfrac{1}{2}k)_\nu + \gamma_\nu (p_2+\tfrac{1}{2}k)_\mu)\, G_1(k^2) \\ & + M^{-1}(p_2+\tfrac{1}{2}k)_\mu (p_2+\tfrac{1}{2}k)_\nu\, G_2(k^2) + (k_\mu k_\nu - g_{\mu\nu}k^2)\, G_3(k^2) \Big] . \\ & \frac{\not{p}_2 + \not{k} + M}{-2p_2\cdot k - k^2} \sqrt{2}\, g_{NN\pi}\, \gamma_5\, u_1 \\ & + \text{ u-channel nucleon Born term} \\ & + \bar{u}_2 \gamma_5 u_1 \sqrt{2}\, g_{NN\pi} (2q.k - k^2)^{-1} \Big[(2(q-\tfrac{1}{2}k)_\mu (q-\tfrac{1}{2}k)_\nu - (k_\mu k_\nu - g_{\mu\nu}k^2)/6) . \\ & F_1(k^2) + (k_\mu k_\nu - g_{\mu\nu}k^2) F_2(k^2) \Big] . \end{aligned} \tag{C.2}$$

The first term in Eq. (C.2) is the s-channel nucleon Born term. The u-channel term may be obtained from the s-channel term by writing the γ-matrices in reverse order and substituting $p_2 \to p_1$, $k \to -k$. The third term is generated by $\theta_{\mu\nu}$ hooking on to the pion to produce a pion pole.

The non-singular $0((k)^0)$ and $0(k)$ terms are represented by $P_{\mu\nu}^{\text{Contact}}$, with

$$P_{\mu\nu} = P_{\mu\nu}^{\text{Born}} + P_{\mu\nu}^{\text{Contact}} + 0(k^2) \quad . \tag{C.3}$$

Using the conservation laws (1.6) and (1.7), we find

$$P_{\mu\nu}^{\text{Contact}} = \sqrt{2}\, g_{NN\pi}\, \bar{u}_2\, \gamma_5\, u_1 \Big[\tfrac{1}{2} g_{\mu\nu} + (q_\mu k_\nu + q_\nu k_\mu - g_{\mu\nu} q.k) \frac{d}{dt}(G_1(t)+G_2(t) - F_1(t))_{t=0}\Big] . \tag{C.4}$$

To investigate the soft-pion limit, we discard all 0(q) terms:

$$P_{\mu\nu} = \text{s- and u-channel nucleon Born terms} - (\tfrac{1}{3} + F_2(0))\bar{u}_2 \gamma_5 u_1 \sqrt{2}\, g_{NN\pi}(k_\mu k_\nu - g_{\mu\nu} k^2)/k^2 + 0(k^2, q). \tag{C.5}$$

If we suppose that $\left[F_5, \theta_\mu^\mu\right] = 0$ is valid in the limit of chiral SU(2) x SU(2) invariance, the standard soft-pion argument for terms $0(q^{-1})$ and $0((q)^0)$ in $P_{\mu\nu}$ produces only the nucleon Born terms; hence, we obtain $F_2(0) = -1/3$, i.e., Eq. (2.52).

Now suppose that both scale and chiral transformations are symmetries of the world. If we define

$$\bar{P}_{\mu\nu} = \langle p(p_2) | \theta_{\mu\nu}(0) | n(p_1), \pi(q) \rangle \tag{C.6}$$

in this limit, the dilaton pole produces an extra term in the low-k expansion:

$$\overline{P}_{\mu\nu} = P_{\mu\nu}^{Born} + P_{\mu\nu}^{Contact} + F_\sigma \mathcal{M} (k_\mu k_\nu - g_{\mu\nu} k^2)/(3k^2) + 0(k^2) \quad , \tag{C. 7}$$

where

$$\mathcal{M} = - \overline{u}(p_2)\left[C(p_1 \cdot q, p_2 \cdot q) + \not{q}\, D(p_1 \cdot q, p_2 \cdot q)\right]\gamma_5\, u(p_1) \tag{C. 8}$$

is the amplitude for the process $\pi^+(q) + n(p_1) \rightarrow \sigma(k) + p(p_2)$. Since we are working in the limit of scale invariance, $\overline{P}_\mu^{\ \mu}$ vanishes, implying

$$C(0, 0) \quad = \quad 0 \quad . \tag{C. 9}$$

Eq. (C. 9) is analogous to Adler's consistency condition[121] for πN scattering.

We may obtain another consistency condition for C(0, 0) using conservation of the axial current. Thus, the amplitude

$$Q_\mu \quad = \quad \langle p(p_2), \sigma(k) |\, \mathfrak{F}_{5\mu} | n(p_1) \rangle \tag{C. 10}$$

obeys

$$q^\mu Q_\mu \quad = \quad 0 \quad . \tag{C. 11}$$

In the soft-pion limit, $q \rightarrow 0$, the only contributions to the left-hand side of Eq. (C. 11) come from $0(q^{-1})$ terms in Q^μ:

$$Q_\mu = \overline{u}_2 \left[g_{\sigma NN} \frac{\not{p}_1 + \not{q} + M}{-2p_1 \cdot q - q^2} (- i\, \gamma_\mu \gamma_5\, g_A) \right] u_1 + \text{u-channel Born term}$$
$$- i\sqrt{2}\; g_{NN\pi}\, \overline{u}_2\, \gamma_5\, u_1\, (2q \cdot k - q^2)^{-1} (k+q)_\mu\, F_{\sigma\pi}(0) + \frac{i q_\mu \mathcal{M}}{2 f_\pi q^2} + 0((q)^0) \quad . \tag{C. 12}$$

The first two terms arise from nucleon poles. The third term represents the contribution of a pion pole caused by $\mathfrak{F}_{5\mu}$ hooking on to the external dilaton line and turning it into a pion line. This term is singular in q because the pion and dilaton are degenerate in the limit of scale invariance. The fourth term is constructed by allowing $\mathfrak{F}_{5\mu}$ to turn into a pion, thereby producing another pion pole. We have made use of Eqs. (2.28) and (2.34).

Combining Eqs. (C.11) and (C.12), we find[15]

$$C(0,0) = \sqrt{2}\; g_{\pi NN} \left(g_{\sigma NN}/M - 2f_\pi F_{\sigma\pi}(0)\right) \quad . \tag{C.13}$$

Consistency of the consistency conditions (C.9) and (C.13) requires

$$2f_\pi F_{\sigma\pi}(0) = g_{\sigma NN}/M \tag{C.14}$$

which yields $F_\sigma F_{\sigma\pi}(0)\, f_\pi = \frac{1}{2}$ when Eq. (1.78) is applied.

APPENDIX D: COLLINEAR DISPERSION RELATIONS AND THE PROBLEM OF MIXING

The calculation in Section III. 2 is not complete without a demonstration that the ϵ' pole may be neglected. We have also treated this problem in ref. 77.

We apply the method of collinear dispersion relations to evaluate the equal-time commutator

$$\langle\sigma|[F_5{}^3, \mathfrak{F}_{5\nu}{}^3]|0\rangle = 0 \quad , \qquad \text{(D. 1)}$$

where σ here refers to any $(0^+, 0^+)$ meson. The details will be suppressed because our analysis is analogous to the treatment of $K_{\ell 3}$ decay given by Ademollo, Denardo, and Furlan[142].

According to Eq. (D. 1) and a standard Ward identity, the retarded commutator

$$R_\nu(k, q) = (2f_\pi/m_\pi{}^2)i\int d^4x e^{-iq\cdot x}\theta(x_o)\langle\sigma(k)|[\partial^\mu\mathfrak{F}_{5\mu}{}^3(x), \mathfrak{F}_{5\nu}{}^3(0)]|0\rangle$$

$$= i k_\nu V_1 + i q_\nu V_2 \quad , \qquad \text{(D. 2)}$$

satisfies the constraint

$$R_\nu(k, 0) = 0 \quad . \qquad \text{(D. 3)}$$

The usual analysis of the large q_o behavior of R_ν leads to the unsubtracted dispersion relation

$$0 = \int dy \, \mathrm{Im}\, V_1(y)/y \quad , \qquad \text{(D. 4)}$$

together with the superconvergence relation

$$0 = \int dy \, \mathrm{Im}\, V_2(y) \quad , \qquad \text{(D. 5)}$$

in the collinear frame $k = (m_\sigma, \vec{0})$, $q = (y m_\sigma, \vec{0})$.

Separation of the pion poles at $y = \pm m_\pi/m_\sigma$, $1 \pm m_\pi/m_\sigma$

from the imaginary part of R_ν yields

$$\mathrm{Im}\, V_1(y) = \pi\varepsilon(y-1)\delta((y-1)^2 - m_\pi^{\,2}/m_\sigma^{\,2}) D_{\sigma\pi}(y^2 m_\sigma^{\,2})/(m_\pi m_\sigma)^2$$
$$+ \pi\varepsilon(y)\delta(y^2 - m_\pi^{\,2}/m_\sigma^{\,2})[-F_{\sigma\pi}((1-y)^2 m_\sigma^{\,2})$$
$$+ G_{\sigma\pi}((1-y)^2 m_\sigma^{\,2})]/m_\sigma^{\,2} + v_1(y) \ , \qquad (D.6)$$

$$\mathrm{Im}\, V_2(y) = -\pi\varepsilon(y-1)\delta((y-1)^2 - m_\pi^{\,2}/m_\sigma^{\,2}) D_{\sigma\pi}(y^2 m_\sigma^{\,2})/(m_\pi m_\sigma)^2$$
$$-\pi\varepsilon(y)\delta(y^2 - m_\pi^{\,2}/m_\sigma^{\,2})[F_{\sigma\pi}((1-y)^2 m_\sigma^{\,2})$$
$$+ G_{\sigma\pi}((1-y)^2 m_\sigma^{\,2})]/m_\sigma^{\,2} + v_2(y) \ , \qquad (D.7)$$

so Eqs. (D. 4) and (D. 5) become

$$m_\sigma(G_{\sigma\pi}(+) - G_{\sigma\pi}(-)) = (m_\sigma - 2m_\pi)F_{\sigma\pi}(+) - (m_\sigma + 2m_\pi)F_{\sigma\pi}(-)$$
$$+ \frac{2m_\pi^{\,3}}{\pi}\int dy\, v_1(y)/y \ , \qquad (D.8)$$

$$(D_{\sigma\pi}(+) - D_{\sigma\pi}(-))/m_\pi^{\,2} = F_{\sigma\pi}(+) + G_{\sigma\pi}(+) - F_{\sigma\pi}(-) - G_{\sigma\pi}(-)$$
$$+ \frac{2m_\pi m_\sigma}{\pi}\int dy\, v_2(y) \ , \qquad (D.9)$$

where Eq. (D. 8) has been simplified by using Eq. (2.29), the definition of $D_{\sigma\pi}(t)$. In Eq. (D. 8), examination of the leading power in m_π gives a result consistent with Eqs. (3.40) and (3.41).

In the collinear frame, only 0^- intermediate states contribute to Im R_ν. The coupling of the $A_1(1070)$ meson to the axial current, $\langle A_1|\mathfrak{F}_{5\lambda}(0)|0\rangle = \varepsilon_\lambda g_{A_1}$, is implicitly contained in the form factors

$$F_{\sigma\pi}(t) = F_{\sigma\pi}(0) - \frac{t\, g_{A_1} g_{A_1\sigma\pi}}{2m_{A_1}^{\,2}(m_{A_1}^{\,2} - t)} + \frac{t}{\pi}\int \frac{dt'\, \mathrm{Im}\,\overline{F}_{\sigma\pi}(t')}{t'(t'-t)} \ , \qquad (D.10)$$

$$G_{\sigma\pi}(t) = -G_{\sigma\pi\pi}(2f_\pi)^{-1}(m_\pi^2 - t)^{-1} - \frac{g_{A_1} g_{A_1\sigma\pi}(m_\sigma^2 - m_\pi^2)}{2m_{A_1}^2(m_{A_1}^2 - t)}$$

$$+ \frac{1}{\pi}\int dt' \frac{\mathrm{Im}\,\overline{G}_{\sigma\pi}(t')}{t' - t} \quad , \tag{D. 11}$$

where $-\frac{i}{2}(\sigma+\pi)_\lambda g_{A_1\sigma\pi}$ is the $A_1\sigma\pi$ coupling. The inequality needed in Section III. 2 is

$$F_{\epsilon'\pi}(\pm) \;,\quad G_{\epsilon'\pi}(\pm) << F_{\sigma\pi}(m_\sigma^2) \;,\quad G_{\sigma\pi}(m_\sigma^2) \;; \tag{D. 12}$$

to make it plausible, we must show that the effect of the $A_1\epsilon'\pi$ coupling is negligible.

When ϵ' is the $(0^+, 0^+)$ meson involved in Eqs. (D. 8) and (D. 9), the points at which $F_{\epsilon'\pi}(t)$, $G_{\epsilon'\pi}(t)$, and $D_{\epsilon'\pi}(t)$ are evaluated straddle the point $t = m_{A_1}^2$. The near degeneracy of $\epsilon'(1060)$ and $A_1(1070)$ gives

$$(F_{\epsilon'\pi}(+) + F_{\epsilon'\pi}(-))_{A_1\,\mathrm{pole}} \simeq 0 \;, \tag{D. 13}$$

$$(G_{\epsilon'\pi}(+) + G_{\epsilon'\pi}(-))_{A_1\,\mathrm{pole}} \simeq 0 \;, \tag{D. 14}$$

whereas $D_{\epsilon'\pi}(t)$ does not have a pole near $t = m_{A_1}^2$, so Eq. (D. 9) becomes

$$2(F_{\epsilon'\pi}(-) + G_{\epsilon'\pi}(-))_{A_1\,\mathrm{pole}} \simeq (D_{\epsilon'\pi}(+) - D_{\epsilon'\pi}(-))/m_\pi^2$$

$$+ \text{continuum integrals} \quad . \tag{D. 15}$$

From PCAC, $D_{\epsilon'\pi}(t)$ and $D_{\sigma\pi}(t)$ satisfy unsubtracted dispersion relations; while we do not expect that the pion pole at $t = m_\pi^2$ dominates dispersion integrals for $D_{\epsilon'\pi}$ and $D_{\sigma\pi}$ at $t = m_{\epsilon'}^2$ or m_σ^2, the respective π-pole terms should indicate the correct orders of magnitude, i. e.,

$$D_{\epsilon'\pi}(\pm) \ll D_{\sigma\pi}(0) \quad , \quad D_{\sigma\pi}(m_\sigma^2) \quad . \tag{D. 16}$$

Therefore, $g_{A_1 \epsilon'\pi}$ is very small and the A_1-pole terms in $F_{\epsilon'\pi}(t)$ and $G_{\epsilon'\pi}(t)$ do not affect the validity of Eq. (D. 12).

APPENDIX E: DERIVATION OF COLLINEAR SUM RULES

This appendix contains some technical details associated with the collinear dispersion relations discussed in Section III.2. The analysis begins with the retarded commutators

$$T_\lambda(k,q) = i\int d^4x\, e^{ik\cdot x}\theta(x_o)\langle 0|[\theta^\mu_\mu(x), \mathfrak{F}_{5\lambda}(0)]|\pi(q)\rangle \quad , \tag{E. 1}$$

$$D_{\mu\lambda}(k,q) = i\int d^4x\, e^{ik\cdot x}\theta(x_o)\langle 0|[\mathcal{D}_\mu(x), \mathfrak{F}_{5\lambda}(0)]|\pi(q)\rangle \quad , \tag{E. 2}$$

$$K_{\mu\nu\lambda}(k,q) = i\int d^4x\, e^{ik\cdot x}\theta(x_o)\langle 0|[\mathcal{K}_{\mu\nu}(x), \mathfrak{F}_{5\lambda}(0)]|\pi(q)\rangle \quad , \tag{E. 3}$$

which are related to the equal-time commutators in Eqs. (3.17) and (3.18) by the identities

$$-ik^\mu D_{\mu\lambda}(k,q) = i\langle 0|[D(0), \mathfrak{F}_{5\lambda}(0)]|\pi(q)\rangle + T_\lambda(k,q) + 0(k) \;, \tag{E. 4}$$

$$-ik^\nu K_{\mu\nu\lambda}(k,q) = i\langle 0|[K_\mu(0), \mathfrak{F}_{5\lambda}(0)]|\pi(q)\rangle - 2i\,\partial T_\lambda/\partial k^\mu + 0(k) \,. \tag{E. 5}$$

The singularities in k of $D_{\mu\lambda}$ and $K_{\mu\nu\lambda}$ are given by derivatives of the pion-pole term

$$\begin{aligned}\overline{P}_{\alpha\nu\lambda}(k,q) = -i(k-q)_\lambda(2f_\pi)^{-1}[(2(q-\tfrac{1}{2}k)_\alpha(q-\tfrac{1}{2}k)_\nu - (k_\alpha k_\nu - g_{\alpha\nu}k^2)/6)F_1(k^2) \\ +(k_\alpha k_\nu - g_{\alpha\nu}k^2)F_2(k^2)]/(2q\cdot k - k^2)\end{aligned} \tag{E. 6}$$

with respect to k, so we obtain

$$-ik^\mu D_{\mu\lambda}(k,q) = -k^\mu \frac{\partial}{\partial k_\nu}\overline{P}_{\mu\nu\lambda} + 0(k) \;, \tag{E. 7}$$

$$-ik^\nu K_{\mu\nu\lambda}(k,q) = ik^\nu\left(2\frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k_\alpha} - \delta^\alpha_\mu \frac{\partial}{\partial k^\beta}\frac{\partial}{\partial k_\beta}\right)\overline{P}_{\alpha\nu\lambda} + 0(k) \quad . \tag{E. 8}$$

The substitution of Eq. (E.6) in Eqs. (E.7) and (E.8) is simplified if the identities

$$k^\mu \frac{\partial}{\partial k^\nu} = \frac{\partial}{\partial k^\nu}k^\mu - \delta^\mu_\nu \quad , \tag{E. 9}$$

$$k^\nu\left(2\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k_\alpha}-g^{\mu\alpha}\frac{\partial}{\partial k^\beta}\frac{\partial}{\partial k_\beta}\right)=\left(2\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k_\alpha}-g^{\mu\alpha}\frac{\partial}{\partial k^\beta}\frac{\partial}{\partial k_\beta}\right)k^\nu$$

$$-2g^{\alpha\nu}\frac{\partial}{\partial k_\mu}+2\left(g^{\mu\alpha}\frac{\partial}{\partial k_\nu}-g^{\mu\nu}\frac{\partial}{\partial k_\alpha}\right)\ , \qquad \text{(E. 10)}$$

are first applied (compare this procedure with the scale-invariant version, given in Eqs. (2.48) and (2.79)). The results are

$$-ik^\mu D_{\mu\lambda} = i(q-k)_\lambda 2m_\pi^2(2f_\pi)^{-1}/(2q\cdot k-k^2)+iq_\lambda/f_\pi + 0(k)\ , \qquad \text{(E. 11)}$$

$$-ik^\nu K_{\mu\nu\lambda} = 2\frac{\partial}{\partial k^\mu}\left\{\frac{(2m_\pi^2F_1(k^2)-3k^2F_2(k^2))(q-k)_\lambda}{(2q\cdot k-k^2)2f_\pi}\right\}$$

$$+(2f_\pi)^{-1}(-2g_{\mu\lambda}+4q_\mu q_\lambda F_1'(0))\ . \qquad \text{(E. 12)}$$

Then Eqs. (3.17), (3.18), (E.4), (E.5), (E.11), and (E.12) imply the low-energy theorems (3.28) and (3.29).

Now we turn to the problem of constructing sum rules corresponding to these low-energy theorems, together with the evaluation of the dispersion integrands.

The imaginary part of T_λ is given by the formula

$$\mathrm{Im}\ T_\lambda = iq_\lambda\,\mathrm{Im}\,X_1+ik_\lambda\,\mathrm{Im}\,X_2 = \tfrac{1}{2}(2\pi)^4\sum_I\left[\langle 0|\theta^\mu_\mu|I\rangle\langle I|\mathfrak{F}_{5\lambda}|\pi(q)\rangle\delta^4(k-P_I)\right.$$

$$\left.-\langle 0|\mathfrak{F}_{5\lambda}|I\rangle\langle I|\theta^\mu_\mu|\pi(q)\rangle\delta^4(q-k-P_I)\right]\ . \qquad \text{(E. 13)}$$

Let us isolate the contribution of the pion pole at $z = 0$ in Eq. (E.13):

$$\mathrm{Im}\ X_1(z) = -\pi\delta(z)(2f_\pi)^{-1}+\mathrm{Im}\ \overline{X}_1(z)\ , \qquad \text{(E. 14)}$$

$$\mathrm{Im}\ X_2(z) = \pi\delta(z)(2f_\pi)^{-1}+\mathrm{Im}\ \overline{X}_2(z)\ . \qquad \text{(E. 15)}$$

Then Eqs. (3.26) and (3.27) may be written

$$X_1(z) = (2f_\pi z)^{-1} + \frac{1}{\pi}\int dz'\ \mathrm{Im}\ \overline{X}_1(z')/(z'-z)\ , \qquad \text{(E. 16)}$$

$$0 = (2f_\pi)^{-1} + \frac{1}{\pi}\int dz\ \mathrm{Im}\ \overline{X}_2(z)\ . \qquad \text{(E. 17)}$$

Then, it is easy to combine Eqs. (E. 16), (3.28), and (3.29), and obtain the sum rules

$$\frac{2f_\pi}{\pi}\int dz\ \mathrm{Im}\ \overline{X}_1(z)/z = -\frac{1}{2}\ , \qquad \text{(E. 18)}$$

$$\frac{2f_\pi}{\pi}\int dz\ \mathrm{Im}\ \overline{X}_1(z)/z^2 = -\frac{3}{2}F_2(0) + 3m_\pi^2 F_1'(0) + \frac{1}{4}\ . \qquad \text{(E. 19)}$$

In the collinear frame, only $0^\pm$ states contribute to the sum over a complete set of states $|I\rangle$ in Eq. (E. 13). We expect that the pion poles at $z = 0, 2$ and σ-poles at $z = \pm m_\sigma/m_\pi$ will dominate, so these contributions are explicitly displayed:

$$\mathrm{Im}\,\overline{X}_1(z) = \pi\delta(z-2)\left[F_1(4m_\pi^2) - 6\,F_2(4m_\pi^2)\right]/2f_\pi$$

$$-\pi\,\varepsilon(z)\,\delta(z^2 - m_\sigma^2/m_\pi^2)\ F_\sigma\left[F_{\sigma\pi}((1-z)^2\,m_\pi^2)\right.$$

$$\left.+ G_{\sigma\pi}((1-z)^2\,m_\pi^2)\right] m_\sigma^2/m_\pi^2 + x_1(z)\ , \qquad \text{(E.20)}$$

$$\mathrm{Im}\,\overline{X}_2(z) = -\pi\delta(z-2)\left[F_1(4m_\pi^2) - 6\,F_2(4m_\pi^2)\right]/2f_\pi$$

$$+\pi\,\varepsilon(z)\,\delta(z^2 - m_\sigma^2/m_\pi^2)\,F_\sigma\left[-F_{\sigma\pi}((1-z)^2\,m_\pi^2)\right.$$

$$\left.+ G_{\sigma\pi}((1-z)^2\,m_\pi^2)\right] m_\sigma^2/m_\pi^2 + x_2(z)\ ; \qquad \text{(E. 21)}$$

(the $z = 0$ contribution appears in Eqs. (E.14) and (E.15). Summation over σ is understood if more than one scalar meson contributes. Diagrams which correspond to the pole terms are displayed in Fig. 2.

The sum rules given in the main text, Eqs. (3.31), (3.32), and (3.33), may be obtained from Eqs. (E.17), (E.18), and (E.19), respectively, by substituting Eqs. (E.20) and (E.21). Notice the similarity between the σ-pole terms of Eqs. (3.33) and (D.9)--in that case, the ϵ' problem is easily handled. (Note that there is no summation over σ in Appendix D; that is why we can isolate the ϵ' contribution and justify its neglect).

APPENDIX F: SUM RULES FOR DIMENSION

In order to obtain soft-meson theorems which explicitly involve dimension, commutators such as $[D, \theta_\mu^\mu]$ or $[F_5, \theta_\mu^\mu]$ must be considered. The cleanest example is Eq. (3.5), but, even in that case, the conclusions are not strong. Therefore, we give just a bare outline of the techniques.

Equation (3.74) for $G_{\sigma\sigma\sigma}$ may be derived as follows. Assuming that the dimension of θ_μ^μ, ℓ, is unique (apart from a c-number term), we have

$$\langle 0|[D, \theta_\mu^\mu]|\sigma\rangle = i\ell m_\sigma^2 F_\sigma \quad . \tag{F.1}$$

The corresponding retarded commutator is

$$S(z) = i\int d^4x \exp(ik\cdot x)\theta(x_o)\langle 0|[\theta_\mu^\mu(x), \theta_\nu^\nu(0)]|\sigma(p)\rangle \quad , \tag{F.2}$$

restricted to the collinear frame $k = (zm_\sigma, \vec{0})$, $p = (m_\sigma, \vec{0})$. Isolating the dilaton poles, we find

$$\mathrm{Im}\, S(z) = (\pi F_\sigma/2)\Big[\langle\sigma|\theta_\mu^\mu|\sigma\rangle(\delta(z-1)-\delta(z))+\langle 0|\theta_\mu^\mu|\sigma,\sigma\rangle(\delta(z-2)-\delta(z+1)\Big]+s(z), \tag{F.3}$$

where the σ-particles are understood to be at rest.

With the usual justification for an unsubtracted dispersion relation (valid for $\ell > -7/2$), together with the low-energy theorem

$$S(z) = m_\sigma^2 F_\sigma/z + (\ell + \tfrac{3}{2})m_\sigma^2 F_\sigma + 0(z) \quad , \tag{F.4}$$

obtained from Eq. (F.1), we find

$$(\ell + \tfrac{1}{2})m_\sigma^2 F_\sigma = \frac{3}{4} F_\sigma \langle 0|\theta_\mu^\mu|\sigma\sigma\rangle + \frac{1}{\pi}\int \frac{dz\, s(z)}{z} \quad . \tag{F.5}$$

Mixing effects may be small if σ and ϵ' are ideally mixed (or nearly so), because then the $\sigma\sigma\epsilon'$ and $\sigma\epsilon\epsilon'$ graphs are disconnected. To obtain the result implied by symmetry considerations alone, the continuum integral is ignored, and $\langle 0|\theta_\mu^\mu|\sigma,\sigma\rangle$ is extrapolated back to $\langle\sigma|\theta_\mu^\mu|\sigma\rangle$ via the σ-pole:

$$\langle 0|\theta_\mu^\mu|\sigma,\sigma\rangle \approx \langle\sigma|\theta_\mu^\mu|\sigma\rangle - 4m_\sigma^2 F_\sigma G_{\sigma\sigma\sigma}/3m_\sigma^2 \quad . \tag{F.6}$$

Combining Eqs. (F.5) and (F.6), we obtain[77]

$$F_\sigma G_{\sigma\sigma\sigma} \approx (1-\ell)m_\sigma^2 \quad . \tag{F.7}$$

It is amusing to derive this result using Ellis' Lagrangian model, Eq. (2.68). The appropriate term from the Lagrangian is

$$\mathcal{L}_{\sigma\sigma\sigma} = -\sigma(\partial\sigma)^2/F_\sigma - \langle 0|U|0\rangle(\ell^3-16\ell)(\sigma/F_\sigma)^3 \quad , \tag{F.8}$$

which simplifies to

$$F_\sigma\mathcal{L}_{\sigma\sigma\sigma} = -\sigma(\partial\sigma)^2+(4-\ell)m_\sigma^2\,\sigma^3/6 \rightarrow G_{\sigma\sigma\sigma}\,\sigma^3/3! \quad , \tag{F.8}$$

because of the mass formula

$$m_\sigma^2 = \ell(\ell+4)\langle 0|U|0\rangle/F_\sigma^2 \quad . \tag{F.10}$$

Equation (F.7) follows immediately.

The equal time commutator

$$[F_5(0), \theta_\mu^\mu(0)] = i(\ell_u+4)\partial^\mu\mathcal{F}_{5\mu} \tag{F.11}$$

looks promising until it is realized that the right-hand side is proportional to the PCAC corrections. One can see this explicitly by applying collinear dispersion relations to the vacuum-to-pion matrix element of Eq. (F.11). The result is essentially Eq. (2.71).

A more indirect method involves attempts to evaluate

$$Q(x) = [v(x), v(0)] \tag{F.12}$$

at equal times (i.e., $x_o = 0$). The discussion following Eq. (3.102) indicates that ℓ_u is the most important factor determining $Q(0,\vec{x})$.

A sum rule for $Q(0, \vec{x})$ is obtained by considering the Bjorken limit of the retarded commutator

$$R = i\int d^4x\, e^{iq\cdot x}\, \theta(x_o)\langle B|Q(x)|A\rangle \qquad (F.13)$$

in the collinear frame. The simplest example appears to be $|A\rangle = |K\rangle$, $|B\rangle = |0\rangle$, i.e., $K_{\ell 3}$ decay. However, practical difficulties are immediately evident, e.g. the K-pole contribution is obtained in the combination

$$f_o((m_K + m_\pi)^2) - f_o((m_K - m_\pi)^2) \quad , \qquad (F.14)$$

where $f_o(t)$ is the scalar form factor given by Eq. (3.78).

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