APPLICATION OF THE TWO VARIABLE EXPANSION PROCEDURE TO THE COMMENSURABLE PLANAR RESTRICTED THREE-BODY PROBLEM

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ABSTRACT

The nearly commensurable case of the planar restricted three-body problem is treated by application of the two variable expansion procedure. The polar angle of the infinitesimal body, rather than the time, is taken as the independent variable. A set of four coupled first order differential equations, which govern the long-period behavior of the orbital elements, is obtained by imposing the requirement that the assumed form of the expansions must be self-consistent. The independent variable in these equations is the "slow variable". It is then found that the short-period perturbations of the motion of the infinitesimal body do not contain small divisors or secular terms.

Approximate solutions for the orbital elements are given, for two different cases. Both libratory and non-libratory solutions are found, depending upon the initial conditions. Numerical results are calculated from these solutions, and are compared to numerical computations recently reported in the literature.

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I. INTRODUCTION

The planar restricted three-body problem may be stated as follows: Two bodies move in circular orbits about their common center of mass, and are assumed to be point masses. A third body having infinitesimal mass moves in the orbital plane of the two large masses, under their combined gravitational attraction.

The above problem, although highly idealized, provides an approximate mathematical model of several actual problems which occur in celestial mechanics. One such problem is the motion of an asteroid (minor planet) about the sun. The mass of an asteroid is sufficiently small, in comparison to the masses of the sun and major planets, that the effect of the gravitational pull of the asteroid upon the motion of these larger bodies may be neglected.

The two largest planets in the solar system are Jupiter and Saturn, the mass of Saturn being approximately 0.299 that of Jupiter. (The next largest planet, Neptune, has a mass only 0.053 that of Jupiter.) The orbit of Jupiter lies much closer to the orbits of the asteroids than does the orbit of Saturn. Therefore, the perturbations of the motion of an asteroid caused by the gravitational attraction of Jupiter are much larger than those caused by any other single planet.

The orbit of Jupiter around the sun is nearly circular, its eccentricity being approximately 0.0482. The orbital inclinations of many of the asteroids, with respect to the sun-Jupiter plane, are only a few degrees. For the above reasons, a solution of the planar restricted three-body problem may be expected to provide an approximation to the motion of an asteroid around the sun.

The mass of Jupiter, although being large in comparison to the masses of the other planets, is only about 1/1047 that of the sun. This suggests the application of a perturbation procedure to obtain an approximate solution of the problem.

Another instance in which the planar restricted three-body problem may be used as an approximate model is the motion of an artificial earth satellite in the orbital plane of the earth-moon system. In this case the motion of the artificial satellite about the earth is perturbed by the gravitational attraction of the moon.

A serious difficulty occurs in the classical variation of constants solution of the problem, for those cases where the period of the infinitesimal body is commensurable with that of the perturbing body. This difficulty will be briefly described, following a discussion by Brouwer and Clemence.[1]

The equations of motion for the infinitesimal body are solved by the method of variation of constants. The first approximation yields a Keplerian orbit that may be described in terms of four orbital elements. The perturbations caused by the gravitational attraction of the body of mass μ' are taken into account in the next approximation, and a set of four first-order equations is obtained for the variation of the constants of integration; i.e. for the behavior of the orbital elements. For example, the equation for $\frac{da}{dt}$ is as follows:

$$\frac{da}{dt} = -\mu' \frac{2}{n_0 a_0} \sum_{j_1, j_2, j_3} j_1 C_{j_1 j_2 j_3} \sin\left[(j_1 n_0 + j_3 n')t + j_1 \epsilon_0 + j_2 \omega_0\right]$$

with the notation

a = semimajor axis of the orbit of the infinitesimal body
t = time
μ' = mass of the perturbing body
n = mean motion of the infinitesimal body
n' = mean motion of the perturbing body
ϵ = mean longitude of the infinitesimal body at epoch
ω = longitude of the pericenter of the infinitesimal body
$C_{j_1 j_2 j_3}$ = coefficients depending only on a and e (for the planar restricted three-body problem)
j_1, j_2, j_3 = integers which are summed over

and where a_o, n_o, e_o, ω_o, and ϵ_o are the corresponding unperturbed Keplerian values. The series on the r. h. s. of the above equation can be arranged in integral powers of the eccentricity e.

Equations similar in form to the above are obtained for $\frac{de}{dt}$, $\frac{d\omega}{dt}$, and $\frac{d\epsilon}{dt}$. These equations are integrated by neglecting the variation of the orbital elements of the infinitesimal body on the r. h. s., as is indicated by the use of a_o, n_o, ω_o, and ϵ_o instead of a, n, ω, and ϵ. The following result is obtained for the semimajor axis:

$$a = a_o + \delta a$$

where

$$\delta a = \mu' \frac{2}{n_o a_o} \sum_{j_1, j_2, j_3} \frac{j_1}{(j_1 n_o + j_3 n')} C_{j_1 j_2 j_3} \cos\left[(j_1 n_o + j_3 n')t + j_1 \epsilon_o + j_2 \omega_o\right]$$

The solutions for δe, $\delta\omega$, and $\delta\epsilon$ are similar in form to that for δa.

If the mean motions n_o and n' are approximately commensurable, there will exist a particular pair of integers $j_1 = J_1$ and $j_3 = J_3$ for which $(J_3 + J_1 \frac{n_o}{n'}) \cong 0$. The expressions for δa, δe, $\delta\omega$, and $\delta\epsilon$ will then contain terms which are divided by the small divisor $(J_3 + J_1 \frac{n_o}{n'})$.

For cases in which these small divisors occur, the above solution is not valid. This is because the orbital elements a, e, ω, and ϵ as given above undergo large oscillations having amplitude proportional to $(J_3 + J_1 \frac{n_0}{n'})^{-1}$, in violation of the approximation that was used in integrating the equations for $\frac{da}{dt}$, $\frac{de}{dt}$, $\frac{d\omega}{dt}$, and $\frac{d\epsilon}{dt}$. This is known as the "difficulty of small divisors".

The difficulty of small divisors also occurs in the variation of constants solution of the non-planar restricted three-body problem, as well as in the more general problem where the orbit of the perturbing body is taken as elliptic rather than circular. However, in order to investigate the basic features of the difficulty of small divisors, without becoming unnecessarily encumbered by algebraic detail, it is reasonable to consider the simplest problem where the difficulty occurs—the planar restricted three-body problem.

A qualitative method of treating the problem of small divisors has been given by Poincaré(2) for the case where the mean motions are in the ratio $\frac{J+1}{J}$, with J a positive integer. The time is taken as the independent variable, and all the short-period perturbations are neglected. Two approximate integrals of the long-period motion are obtained, because the Hamiltonian then contains neither the time nor the short-period angular variable. However, only the general form of the Hamiltonian is given, without specifying the expressions for those terms which are multiplied by the perturbing mass. Hence the time-dependence of the motion is not treated in a satisfactory manner.

Hagihara(3) later extended Poincaré's method to the case where the mean motions are in the ratio $\frac{J+K}{J}$, J and K being positive integers. Higher powers of the eccentricity are retained in the perturbing terms. However, in treating the time-dependence of the motion, several important perturbing terms have incorrectly been neglected, as the result of not having ordered the small quantities in a systematic manner.

Schubart(4) has published the results of extensive numerical computations for the nearly commensurable case of the restricted three-body problem. In his work, the short-period perturbations are removed by a numerical averaging process, and only the long-period effects are included in the orbital elements. These results provide considerable insight into the qualitative and quantitative features of the motion for a wide range of initial conditions.

The purpose of the work described in this thesis is to demonstrate how the two variable expansion procedure may be used to obtain a solution which is free of small divisors. This method establishes the proper time-like variable for the long-period motion, and clarifies the dependence of the amplitudes of the orbital elements on the small parameter in the problem. Both the short-period and long-period perturbations of the motion of the infinitesimal body can be determined.

II. EQUATIONS OF MOTION

The planar restricted three-body problem will be non-dimensionalized by choosing the units of mass, length, and time as follows: the unit of mass is chosen in such a way that the larger of the two massive bodies has mass $1-\mu$, and the smaller one has mass μ, where $0 < \mu \leq \frac{1}{2}$ for all cases; the unit of length is chosen such that the constant distance between the two massive bodies, as they revolve in their circular orbits, is equal to 1; the unit of time is chosen such that the constant angular velocity of the two large bodies about their common center of mass is equal to 1.

The center of mass will lie on the line joining the two large bodies, at a distance μ from the body of mass $1-\mu$. The center of mass is assumed to be moving at constant rectilinear velocity with respect to an inertial frame of reference.

Let the non-rotating X-Y coordinate system have its origin fixed at the center of mass. This frame of reference will be an inertial one. The line of centers will rotate about the mass center with unit angular velocity. Choose the angular orientation of the X-Y system in such a way that the positive X axis coincides with the position of mass μ at time $t = 0$. The line of centers then makes an angle t with the positive X axis.

The geometrical situation is shown in Figure 1.

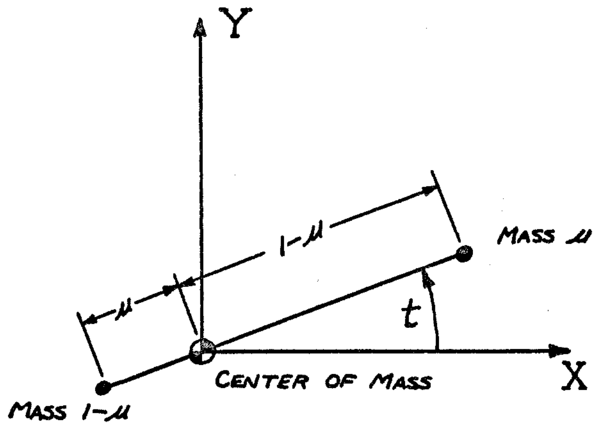

Figure 1. Barycentric Coordinate System

Let the $X^* - Y^*$ system be a non-rotating reference frame centered at the body of mass $1-\mu$. As seen from the inertial frame X-Y, the origin of coordinates of the $X^* - Y^*$ system will move at constant angular velocity in a circle of radius μ about the center of mass, and hence the $X^* - Y^*$ frame is not an inertial one. Let the $X^* - Y^*$ system have the same fixed angular orientation as does the X-Y system. The positive X^*-axis will then pass through the position of mass μ at $t = 0$. Therefore the line of centers will make an angle t with the positive X^*-axis.

The geometrical situation in the $X^* - Y^*$ system is shown in Figure 2.

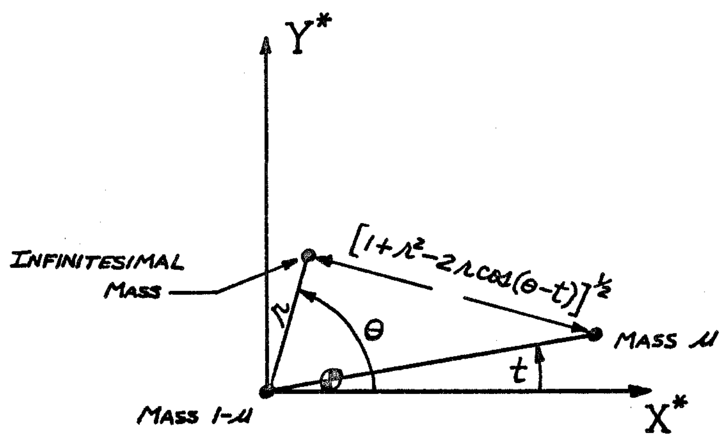

Figure 2. Heliocentric Coordinate System

Let r denote the distance of the infinitesimal body from the origin of the $X^* - Y^*$ system, and let θ denote the angle from the positive X^* axis to the radius vector of the infinitesimal body. The distance between the infinitesimal body and the body of mass μ is then equal to $\left[1+r^2-2r\cos(\theta-t)\right]^{1/2}$.

The equations of motion of the infinitesimal body may easily be derived in terms of r and θ, considered as functions of the time t. They are as follows (where $\dot{r} = \frac{dr}{dt}$, etc.):

$$r\ddot{\theta}+2\dot{r}\dot{\theta} = \mu\sin(\theta-t) - \mu\frac{\sin(\theta-t)}{\left[1+r^2-2r\cos(\theta-t)\right]^{3/2}} \tag{1}$$

$$\ddot{r}-r\dot{\theta}^2+\frac{1}{r^2} = -\mu\cos(\theta-t)+\mu\frac{1}{r^2}+\mu\frac{\left[-r+\cos(\theta-t)\right]}{\left[1+r^2-2r\cos(\theta-t)\right]^{3/2}} \tag{2}$$

In applying the two variable expansion procedure that will later be used to solve these equations, a different set of variables is more useful. The new form of the equations will make it easier to treat in a proper manner the terms which would otherwise produce small divisors.

Introduce the variable

$$s = \frac{1}{r} \tag{3}$$

Then transform to θ instead of t as the independent variable, so that $s = s(\theta)$, $t = t(\theta)$. This may be done by means of the following relations:

$$\dot{\theta} = \frac{1}{\left(\frac{dt}{d\theta}\right)} \qquad \dot{s} = \frac{\left(\frac{ds}{d\theta}\right)}{\left(\frac{dt}{d\theta}\right)}$$

$$\ddot{\theta} = \frac{-\left(\frac{d^2t}{d\theta^2}\right)}{\left(\frac{dt}{d\theta}\right)^3} \qquad \ddot{s} = \frac{-\left(\frac{ds}{d\theta}\right)\left(\frac{d^2t}{d\theta^2}\right)}{\left(\frac{dt}{d\theta}\right)^3} + \frac{\left(\frac{d^2s}{d\theta^2}\right)}{\left(\frac{dt}{d\theta}\right)^2}$$

The equations of motion for the planar restricted three-body problem then assume the following form:

(4) $$\frac{d}{d\theta}\left(s^2\frac{dt}{d\theta}\right) = \mu\left\{-s^3\left(\frac{dt}{d\theta}\right)^3\sin(\theta-t) + \left(s^2\frac{dt}{d\theta}\right)^3\frac{\sin(\theta-t)}{\left[1+s^2-2s\cos(\theta-t)\right]^{3/2}}\right\}$$

(5) $$\frac{d^2s}{d\theta^2} + s - \left(s^2\frac{dt}{d\theta}\right)^2 = \mu\left\{\begin{array}{l} s^2\left(\frac{dt}{d\theta}\right)^2\cos(\theta-t) - s\frac{ds}{d\theta}\left(\frac{dt}{d\theta}\right)^2\sin(\theta-t) \\ -\left(s^2\frac{dt}{d\theta}\right)^2 + \left(s^2\frac{dt}{d\theta}\right)^2\frac{\left[1-s\cos(\theta-t)+\frac{ds}{d\theta}\sin(\theta-t)\right]}{\left[1+s^2-2s\cos(\theta-t)\right]^{3/2}}\end{array}\right\}$$

It is seen that both the time $t(\theta, \mu)$ and the independent variable θ appear <u>explicitly</u> in the equations of motion, in the terms which involve $\sin(\theta-t)$ and $\cos(\theta-t)$. The problem is therefore non-autonomous.

Because of the manner in which the orientation of the X^*- Y^* axes was specified, the initial condition on t is as follows:

(6) $$t(\theta_0, \mu) = 0$$

where θ_0 is the initial angle between the radius vector to the infinitesimal body and the line of centers of the two massive bodies.

The terms which involve $\left[1+s^2-2s\cos(\theta-t)\right]^{-3/2}$ lead to the occurrence of small divisors. These terms represent the gravitational attraction of the body of mass μ upon the infinitesimal body. The term $-\mu\left(s^2\frac{dt}{d\theta}\right)^2$ on the r.h.s. of eq. (5) occurs as a result of having chosen $1-\mu$, instead of 1, for the mass of the larger body. The remaining terms on the r.h.s. of eqs. (4) and (5) are "apparent forces" which result from the fact that the X^* - Y^* system is not an inertial reference frame. These "apparent forces" do not lead to small divisors.

Eqs. (4) and (5) are an exact mathematical representation of the planar restricted three-body problem, valid for all values of $0<\mu\leq\frac{1}{2}$. These equations possess one exact integral, the well-known Jacobi integral:

$$\frac{1}{\left(s^2\frac{dt}{d\theta}\right)^2}\left[\frac{1}{2}\left(\frac{ds}{d\theta}\right)^2+\frac{1}{2}s^2-\left(s^2\frac{dt}{d\theta}\right)-s\left(s^2\frac{dt}{d\theta}\right)^2\right] \tag{7}$$

$$+\mu\left[s+\frac{1}{s}\cos(\theta-t)-\frac{s}{\sqrt{1+s^2-2s\cos(\theta-t)}}\right]=C$$

where C depends only on the initial conditions.

In the remainder of this work, it will be assumed that $0<\mu<<\frac{1}{2}$. The quantity μ may then be treated as a small parameter in the equations of motion.

III. METHOD OF AVOIDING SMALL DIVISORS

The occurrence of small divisors in the variation of constants treatment of the problem results from having neglected the variation of the mean motion, and the other orbital elements, while carrying out the integration of the perturbation equations. The small divisors are produced by the integration of terms whose period is very large compared to the orbital period of the infinitesimal body. This suggests the existence of a second time scale, the "slow-time" scale, over which important changes occur in the orbital elements.

The physical reason for the occurrence of the difficulty is the fact that the perturbing force is nearly resonant with the motion of the infinitesimal body. This near-resonance aspect of the motion will now be discussed briefly.

Assume that the infinitesimal body moves in an elliptical orbit about the larger mass $1-\mu$. This elliptical orbit will be perturbed by the gravitational force exerted by the mass μ. The distance between the infinitesimal body and the perturbing body will be approximately a periodic function of time, so that the perturbing force is also nearly periodic. If the orbital period of the infinitesimal body is approximately a rational fraction of the orbital period of the perturbing body, the perturbing force oscillates with a nearly resonant frequency. The improper mathematical treatment of this near-resonance leads to the occurrence of small divisors.

The problem at hand is to derive a set of equations which gives an adequate description of the behavior of the orbital elements, in the presence of the nearly-resonant perturbing forces.

1. Justification for Use of the Two Variable Expansion Procedure

The two variable expansion procedure has been discussed in the literature by Cole and Kevorkian,[5] and by Kevorkian.[6] It is a systematic method of constructing an expansion, of the solution of an ordinary differential equation containing a small parameter, which remains valid for large values of the independent variable. This method is especially useful in problems where a small perturbing force produces important effects which occur over a time scale that is large compared to the time scale of the main features of the motion.

In applying the two variable procedure, it is assumed that the exact solution may be represented by an expansion which depends explicitly upon two different time (or time-like) variables, a "fast time" variable and a "slow time" variable. The use of two different variables introduces an indeterminacy into the various terms of the expansion. This indeterminacy is removed by requiring that the assumed form of the expansion must be self-consistent.

When the two variable expansion procedure is applied to the planar restricted three-body problem, the orbital elements will exhibit only long-period effects. Short-period perturbations will be taken into account by the second term of the expansion. However, it is precisely in the long-period effects that the fundamental difficulty of the problem lies. Thus the use of the two variable expansion procedure leads directly to a study of the basic difficulties of the problem.

The variation of constants approach yields both short-period and long-period effects in the orbital elements. The short-period effects must be removed before the fundamental difficulty of the problem can be studied.

2. The Form of the Expansions

For $\mu \ll \frac{1}{2}$, the terms on the r.h.s. of eqs. (4) and (5) may be treated as small perturbations, provided that $[1+s^2-2s\cos(\theta-t)]^{3/2}$ does not become arbitrarily small. This implies that the infinitesimal body must not make a "close approach" to the body of mass μ. Close approaches cannot occur for orbits which lie entirely within the orbit of the perturbing body; i.e. for orbits having $s(\theta,\mu)>1$ for all θ. For cases where $s(\theta,\mu)\leq 1$ during part of the orbit, the perturbations will remain small only if $[1+s^2-2s\cos(\theta-t)]^{3/2}$ remains bounded away from zero.

Orbits for which $[1+s^2-2s\cos(\theta-t)]^{3/2}$ approaches 0 will not be considered in this work.

The solution of eqs. (4) and (5) will be sought by use of the two variable expansion procedure in the following form:

(8a) $$s(\theta,\mu) = s_0(\theta,\tilde{\theta},\mu) + \mu s_1(\theta,\tilde{\theta},\mu) + o(\mu)$$

(8b) $$t(\theta,\mu) = t_0(\theta,\tilde{\theta},\mu) + \mu t_1(\theta,\tilde{\theta},\mu) + o(\mu)$$

where the slow variable is

(9) $$\tilde{\theta} = \mu^{1/2}\theta$$

The essential features of the difficulty of small divisors occur in the terms of $O(\mu)$; i.e. in the solutions for $s_1(\theta, \tilde{\theta}, \mu)$ and $t_1(\theta, \tilde{\theta}, \mu)$. Hence, for the purpose of resolving the basic difficulty, the terms of higher order in μ may be neglected.

Derivatives are to be calculated by the rule

(10) $$\frac{d}{d\theta} = \frac{\partial}{\partial\theta} + \mu^{1/2}\frac{\partial}{\partial\tilde{\theta}}$$

The following expansions are obtained by applying this derivative rule to expansions (8a) and (8b):

(11a) $$\frac{ds}{d\theta} = \frac{\partial s_0}{\partial\theta} + \mu^{1/2}\frac{\partial s_0}{\partial\tilde{\theta}} + \mu\frac{\partial s_1}{\partial\theta} + o(\mu)$$

(11b) $$\frac{dt}{d\theta} = \frac{\partial t_0}{\partial\theta} + \mu^{1/2}\frac{\partial t_0}{\partial\tilde{\theta}} + \mu\frac{\partial t_1}{\partial\theta} + o(\mu)$$

(11c) $$s^2\frac{dt}{d\theta} = s_0^2\frac{\partial t_0}{\partial\theta} + \mu^{1/2}\left(s_0^2\frac{\partial t_0}{\partial\tilde{\theta}}\right) + \mu\left(s_0^2\frac{\partial t_1}{\partial\theta} + 2s_0 s_1\frac{\partial t_0}{\partial\theta}\right) + o(\mu)$$

Applying the derivative rule again,

(12a) $$\frac{d^2 s}{d\theta^2} = \frac{\partial^2 s_0}{\partial\theta^2} + \mu^{1/2}\left(2\frac{\partial^2 s_0}{\partial\theta\partial\tilde{\theta}}\right) + \mu\left(\frac{\partial^2 s_1}{\partial\theta^2} + \frac{\partial^2 s_0}{\partial\tilde{\theta}^2}\right) + o(\mu)$$

(12b) $$\frac{d}{d\theta}\left(s^2\frac{dt}{d\theta}\right) = \frac{\partial}{\partial\theta}\left(s_0^2\frac{\partial t_0}{\partial\theta}\right) + \mu^{1/2}\left[\frac{\partial}{\partial\tilde{\theta}}\left(s_0^2\frac{\partial t_0}{\partial\theta}\right) + \frac{\partial}{\partial\theta}\left(s_0^2\frac{\partial t_0}{\partial\tilde{\theta}}\right)\right]$$
$$+ \mu\left[\frac{\partial}{\partial\theta}\left(s_0^2\frac{\partial t_1}{\partial\theta} + 2s_0 s_1\frac{\partial t_0}{\partial\theta}\right) + \frac{\partial}{\partial\tilde{\theta}}\left(s_0^2\frac{\partial t_0}{\partial\tilde{\theta}}\right)\right] + o(\mu)$$

These expansions may be used to express the l.h.s. of eqs. (4) and (5), retaining all terms of $O(\mu^0)$, $O(\mu^{1/2})$, and $O(\mu)$.

It is now necessary to discuss the manner in which the perturbing terms on the r.h.s. of the equations of motion may be expanded in powers of μ. Since only the terms of $O(\mu^0)$, $O(\mu^{1/2})$, and $O(\mu)$ are to be retained, it is sufficient to use the $O(\mu^{o})$ approximation to the quantities in braces on the r. h. s. of eqs. (4) and (5).

The terms which involve powers of $s, \frac{ds}{d\theta}$, and $\frac{dt}{d\theta}$ may be expanded as above. The only remaining terms are those which involve $\sin(\theta-t)$ and $\cos(\theta-t)$.

By the expansion for $t(\theta, \mu)$ we have

(13) $$(\theta-t) = \theta - t_0(\theta,\tilde{\theta},\mu) - \mu t_1(\theta,\tilde{\theta},\mu) + o(\mu)$$

The two variable expansion procedure will be used to make $t_1(\theta, \tilde{\theta}, \mu)$ a bounded function of θ. Therefore $\mu t_1(\theta, \tilde{\theta}, \mu)$ will remain a quantity of $O(\mu)$, and may be dropped from eq. (13), so that

(14) $$\begin{aligned} \sin(\theta-t) &= \sin[(\theta-t_0) - \mu t_1 + o(\mu)] \\ &= \sin(\theta-t_0) + O(\mu t_1) \\ &= \sin(\theta-t_0) + O(\mu) \end{aligned}$$

Similarly,

(15) $$\cos(\theta-t) = \cos(\theta-t_0) + O(\mu)$$

The following expansion is therefore valid for the terms on the r. h. s. of eq. (4):

(16) $$\mu\left\{-s^3\left(\frac{dt}{d\theta}\right)^3\sin(\theta-t) + \left(s^2\frac{dt}{d\theta}\right)^3\frac{\sin(\theta-t)}{[1+s^2-2s\cos(\theta-t)]^{3/2}}\right\}$$

$$= \mu\left\{-s_0^3\left(\frac{\partial t_0}{\partial\theta}\right)^3\sin(\theta-t_0) + \left(s_0^2\frac{\partial t_0}{\partial\theta}\right)^3\frac{\sin(\theta-t_0)}{[1+s_0^2-2s_0\cos(\theta-t_0)]^{3/2}}\right\} + O(\mu^2)$$

A similar expansion is valid for the terms on the r. h. s. of eq. (5).

Thus the perturbation terms of $O(\mu)$ involve only the quantities $s_o(\theta, \tilde{\theta}, \mu)$ and $t_o(\theta, \tilde{\theta}, \mu)$ and their derivatives. However, this approximation will be valid only if it can be shown that $t_1(\theta, \tilde{\theta}, \mu)$ and $s_1(\theta, \tilde{\theta}, \mu)$ are indeed bounded functions of θ.

3. Solution of the $O(\mu^o)$ Equations

The terms multiplied by μ^o in the equations of motion lead to the following equations:

$$\frac{\partial}{\partial\theta}\left(s_o^2 \frac{\partial t_o}{\partial\theta}\right) = 0 \tag{17}$$

$$\frac{\partial^2 s_o}{\partial\theta^2} + s_o = \left(s_o^2 \frac{\partial t_o}{\partial\theta}\right)^2 \tag{18}$$

These are the equations of Keplerian motion. That is, if the perturbing mass μ were equal to zero, the infinitesimal body would describe an unperturbed Keplerian orbit about the large mass.

In this work only direct orbits will be considered. That is, it will be assumed that both the infinitesimal body and the perturbing mass μ revolve about the large mass in a counterclockwise direction (see Figure 2).

Eqs. (17) and (18) will be solved, regarding θ and $\tilde{\theta}$ as being two entirely different variables. Eq. (17) has the solution

$$s_o^2 \frac{\partial t_o}{\partial\theta} = \frac{1}{\sqrt{a(1-e^2)}} \tag{19}$$

where

$a = a(\tilde{\theta}, \mu)$ = semimajor axis of the orbit of the infinitesimal body

$e = e(\tilde{\theta}, \mu)$ = eccentricity of the orbit of the infinitesimal body

Eq. (19) defines the angular momentum of the orbit. For retrograde orbits, eq. (19) would be replaced by $s_o^2 \frac{\partial t_o}{\partial \theta} = -a^{-1/2}(1-e^2)^{-1/2}$.

Only elliptical orbits $(0 \leq e < 1)$ will be considered here. Parabolic and hyperbolic orbits $(e \geq 1)$ do not produce the difficulty of small divisors, because the motion of the infinitesimal body is not periodic in these cases.

Eq. (18) becomes

$$\frac{\partial^2 s_o}{\partial \theta^2} + s_o = \frac{1}{a(1-e^2)} \tag{20}$$

The general solution of this equation is

$$s_o(\theta, \tilde{\theta}, \mu) = \frac{1}{a(1-e^2)} + A(\tilde{\theta}, \mu)\cos\theta + B(\tilde{\theta}, \mu)\sin\theta$$

where A and B are arbitrary functions. In terms of the Keplerian orbital elements, these functions are

$$A(\tilde{\theta}, \mu) = \frac{e\cos\omega}{a(1-e^2)} \quad ; \quad B(\tilde{\theta}, \mu) = \frac{e\sin\omega}{a(1-e^2)}$$

where

$\omega = \omega(\tilde{\theta}, \mu)$ = longitude of pericenter of the orbit of the infinitesimal body

Therefore,

$$s_o(\theta, \tilde{\theta}, \mu) = \frac{1 + e\cos(\theta - \omega)}{a(1-e^2)} \tag{21}$$

The quantity $t_o(\theta, \tilde{\theta}, \mu)$ may be obtained from the relation

(22) $$\frac{\partial t_o}{\partial \theta} = \frac{1}{\Delta_o^2}\left(\Delta_o^2 \frac{\partial t_o}{\partial \theta}\right) = \frac{a^{3/2}(1-e^2)^{3/2}}{[1+e\cos(\theta-\omega)]^2}$$

so that

(23) $$t_o(\theta,\tilde{\theta},\mu) = \int_{\theta_o}^{\theta} \frac{a^{3/2}(1-e^2)^{3/2}}{[1+e\cos(\theta-\omega)]^2}\, d\theta$$

where

$$\theta = \theta_o \quad at \; t_o = 0$$

Eq. (22) is still satisfied if an arbitrary function of $\tilde{\theta}$ is added to $t_o(\theta, \tilde{\theta}, \mu)$.

If one expands the integrand on the r. h. s. of eq. (23) in a Taylor series about $e = 0$, and then holds $\tilde{\theta}$ fixed while carrying out the integral w. r. t. θ, the following expression is obtained:

(24) $$t_o(\theta,\tilde{\theta},\mu) = T(\tilde{\theta},\mu) + a^{3/2}\theta + \left\{\begin{matrix}\text{short-period sinusoidal functions} \\ \text{of } \theta, \text{ multiplied by } e, e^2, e^3, \text{ etc.}\end{matrix}\right\}$$

where $T(\tilde{\theta}, \mu)$ is an arbitrary function which defines the position of the infinitesimal body in its orbit.

4. <u>Occurrence of Small Divisors in s_1 and t_1</u>

The unbounded part of $t_o(\theta, \tilde{\theta}, \mu)$ is entirely contained in the quantity $[T + a^{3/2}\theta]$. Therefore

(25) $$(\theta - t_o) = (1 - a^{3/2})\theta - T + \left\{\begin{matrix}\text{short-period sinusoidal functions} \\ \text{of } \theta, \text{ multiplied by } e, e^2, e^3, \text{ etc.}\end{matrix}\right\}$$

It follows that

$$(26)\quad \sin(\theta - t_0) = \sin\left[(1-a^{3/2})\theta - T\right] + \left\{\begin{matrix}\text{short-period sinusoidal functions} \\ \text{of } \theta, \text{ multiplied by } e, e^2, e^3, \text{ etc.}\end{matrix}\right\}$$

A similar expansion would be valid for $\cos(\theta - t_o)$.

Therefore, if eq. (24) were used for $t_o(\theta, \tilde{\theta}, \mu)$ it would be found that the equations for $s_1(\theta, \tilde{\theta}, \mu)$ and $t_1(\theta, \tilde{\theta}, \mu)$ would contain forcing functions which would involve $\sin[(1-a^{3/2})\theta - T]$ and $\cos[(1-a^{3/2})\theta - T]$. Since $\tilde{\theta}$ is held fixed during the integrations w.r.t. θ, the quantity $(1-a^{3/2})$ would appear as a constant frequency. In combination with other frequencies which are present in the perturbing terms, these terms would produce sinusoidal functions of θ having frequencies close to zero and others with frequencies close to 1, for certain values of $a^{3/2}$. Upon integration w.r.t. θ, these terms would produce small divisors in s_1 and t_1.

By expressing the perturbing terms as functions of θ and the orbital elements $a^{3/2}, e, \omega, T$, and then expanding in periodic series' to determine which frequencies occur, it may be shown that small divisors would occur in $s_1(\theta, \tilde{\theta}, \mu)$ and $t_1(\theta, \tilde{\theta}, \mu)$ for direct elliptical orbits in those cases where the semimajor axis has a value such that

$$a^{3/2} \cong \frac{n-m}{n}$$

where n and m are relatively prime positive integers, with $n > m$.

It may also be shown that the perturbing terms which are multiplied by the first power of the eccentricity would produce small divisors only for commensurabilities with $m = 1$; the perturbing terms multiplied by e^2 would produce small divisors for both the m=1 and m=2 cases; those multiplied by e^3 would produce small divisors for the m=1, m=2, and m=3 cases; etc. Correspondingly, one would expect the behavior of the orbital elements to be somewhat different for the various values of m.

For brevity, this analysis will not be carried out here. However, it should be mentioned that the occurrence of small divisors in the above form is equivalent to the corresponding difficulty encountered in the variation of constants treatment of the problem.

Although retrograde (clockwise) elliptical orbits will not be discussed here, small divisors would occur for certain cases where $a^{3/2}$ is the ratio of two positive integers. These small divisors could be avoided by a method similar to that which will be discussed in the next section.

5. Explicit Inclusion of Commensurability in the Expansions

As discussed above, small divisors would occur if the semimajor axis is such that $a^{3/2}(\tilde{\theta}, \mu)$ is near one of the values $\frac{n-m}{n}$. This suggests that the near-commensurability should be taken into account from the outset, and that the semimajor axis should be expanded in the form

(27) $$a^{3/2}(\tilde{\theta},\mu) = \frac{n-m}{n} + \mu^{1/2}\hat{a}^{3/2}(\tilde{\theta},\mu)$$

The corresponding derivative is

(28) $$\frac{da^{3/2}}{d\tilde{\theta}} = \mu^{1/2}\frac{d\hat{a}^{3/2}}{d\tilde{\theta}}$$

The expression for $t_o(\theta, \tilde{\theta}, \mu)$ must now be re-examined, taking into account expansion (27). The expression given in eq. (24) was obtained by holding the slow variable $\tilde{\theta}$ fixed while carrying out the integration w.r.t. θ. Such a procedure is valid for the terms which do not give rise to unbounded quantities proportional to θ. Therefore

$$(29) \quad t_o(\theta,\tilde{\theta},\mu) = T(\tilde{\theta},\mu) + \int_{\theta_o}^{\theta} a^{3/2} d\theta - 2a^{3/2} e \sin(\theta-\omega) + \frac{3}{4} a^{3/2} e^2 \sin 2(\theta-\omega)$$

$$+ \left\{ \begin{array}{l} \text{similar short-period sinusoidal functions} \\ \text{of } \theta, \text{ multiplied by } e^3, e^4, --- \end{array} \right\}$$

There is no non-uniform approximation to the unbounded part of $t_o(\theta, \tilde{\theta}, \mu)$ caused by dropping the terms multiplied by $e^3, e^4, ---$, since the integrals of all such terms w.r.t. θ are bounded.

Using eq. (27) for $a^{3/2}(\tilde{\theta}, \mu)$, one obtains

$$(30) \quad \int_{\theta_o}^{\theta} a^{3/2} d\theta = \int_{\theta_o}^{\theta} \left[\frac{n-m}{n} + \mu^{1/2} \hat{a}^{3/2}(\tilde{\theta},\mu) \right] d\theta = \frac{(n-m)}{n}(\theta-\theta_o) + \mu^{1/2} \int_{\theta_o}^{\theta} \hat{a}^{3/2}(\tilde{\theta},\mu) d\theta$$

If the integral on the r.h.s. of eq. (30) can be expressed as a function of $\tilde{\theta}$ alone, rather than as a function of both θ and $\tilde{\theta}$, it will be possible to distinguish between the unbounded behavior of $t_o(\theta, \tilde{\theta}, \mu)$ which is proportional to θ and the unboundedness which is proportional to $\tilde{\theta}$. This will make it possible to avoid the occurrence of small divisors in $s_1(\theta, \tilde{\theta}, \mu)$ and $t_1(\theta, \tilde{\theta}, \mu)$.

To accomplish this it is necessary to use the relation $\tilde{\theta} = \mu^{1/2}\theta$ when carrying out the integral on the r.h.s. of eq. (30). Therefore

(31) $$\mu^{1/2}\int_{\theta_0}^{\theta} \hat{a}^{3/2}(\tilde{\theta},\mu)\,d\theta = \int_{\mu^{1/2}\theta_0}^{\mu^{1/2}\theta} \hat{a}^{3/2}(\tilde{\theta},\mu)\,d(\mu^{1/2}\theta) = \int_{\tilde{\theta}_0}^{\tilde{\theta}} \hat{a}^{3/2}(\tilde{\theta},\mu)\,d\tilde{\theta}$$

Introduce the notation

(32) $$\tau(\tilde{\theta},\mu) = T(\tilde{\theta},\mu) - \frac{(n-m)}{n}\theta_0$$

Eq. (29) may now be written as follows:

(33) $$t_0(\theta,\tilde{\theta},\mu) = \frac{(n-m)}{n}\theta + \tau(\tilde{\theta},\mu) + \int_{\tilde{\theta}_0}^{\tilde{\theta}} \hat{a}^{3/2}\,d\tilde{\theta} - 2a^{3/2}e\sin(\theta-\omega)$$

$$+\frac{3}{4}a^{3/2}e^2\sin 2(\theta-\omega) + \begin{Bmatrix}\text{similar short-period sinusoidal functions}\\ \text{of } \theta, \text{ multiplied by } e^3, e^4, \text{ - - -}\end{Bmatrix}$$

For brevity, the following notation will be used, whenever it is convenient:

(34) $$\phi(\tilde{\theta},\mu) = \tau(\tilde{\theta},\mu) + \int_{\tilde{\theta}_0}^{\tilde{\theta}} \hat{a}^{3/2}\,d\tilde{\theta}$$

The corresponding derivative is

(35) $$\frac{d\phi}{d\tilde{\theta}} = \frac{d\tau}{d\tilde{\theta}} + \hat{a}^{3/2}$$

Eq. (33) then becomes

(36) $$t_0(\theta,\tilde{\theta},\mu) = \frac{(n-m)}{n}\theta + \phi(\tilde{\theta},\mu) - 2a^{3/2}e\sin(\theta-\omega) + \frac{3}{4}a^{3/2}e^2\sin 2(\theta-\omega)$$

$$+ \begin{Bmatrix}\text{similar short-period sinusoidal functions}\\ \text{of } \theta, \text{ multiplied by } e^3,\ e^4, \text{ - - - .}\end{Bmatrix}$$

This expression will be used for t_o from this point on.

The term $\frac{(n-m)}{n}\theta$ represents the unbounded behavior of t_o which is proportional to θ, and $\phi(\tilde{\theta}, \mu)$ represents a possible unboundedness of t_o on the $\tilde{\theta}$ scale. A geometrical interpretation of ϕ will be given later.

Having expressed t_o by eq. (36) it is necessary to express the derivatives $\frac{\partial t_o}{\partial \theta}$ and $\frac{\partial t_o}{\partial \tilde{\theta}}$ in a self-consistent manner. The former is given by

(37) $$\frac{\partial t_o}{\partial \theta} = \frac{1}{\Delta_o^2}\left(\Delta_o^2 \frac{\partial t_o}{\partial \theta}\right) = \frac{a^{3/2}(1-e^2)^{3/2}}{\left[1+e\cos(\theta-\omega)\right]^2}$$

By the derivative rule (10) we expect that

(38a) $$\frac{dt_o}{d\theta} = \frac{\partial t_o}{\partial \theta} + \mu^{1/2}\frac{\partial t_o}{\partial \tilde{\theta}}$$

Formally applying the derivative rule to eq. (33), it is found that

(38b) $$\frac{d}{d\theta}(t_o) = \frac{n-m}{n} + \mu^{1/2}\left(\frac{d\tau}{d\tilde{\theta}} + \hat{a}^{3/2}\right) + \left\{\frac{\partial}{\partial \theta}\text{derivatives of short-period terms}\right\}$$
$$+ \mu^{1/2}\left\{\frac{\partial}{\partial \tilde{\theta}}\text{ derivatives of short-period terms}\right\}$$

From eqs. (37), (38a), and (38b) it follows that

(39) $$\frac{\partial t_o}{\partial \tilde{\theta}} = \frac{d\tau}{d\tilde{\theta}} + \left\{\frac{\partial}{\partial \tilde{\theta}}\text{derivatives of short-period terms}\right\}$$
$$= \frac{d\phi}{d\tilde{\theta}} - \hat{a}^{3/2} + \left\{\frac{\partial}{\partial \tilde{\theta}}\text{derivatives of short-period terms}\right\}$$

The quantity $\tau(\tilde{\theta}, \mu)$ should be regarded as the fourth orbital element. The quantity $\phi(\tilde{\theta}, \mu)$ is completely defined in terms of τ and $\hat{a}^{3/2}$ by eq. (34).

6. Geometrical Significance of $\phi(\tilde{\theta}, \mu)$

Using the approximation

$$t(\theta,\mu) = t_0(\theta,\tilde{\theta},\mu) + O(\mu) \tag{40}$$

it follows that

$$\begin{aligned}(\theta - t) &= (\theta - t_0) + O(\mu) \\ &= \frac{m}{n}\theta - \phi(\tilde{\theta},\mu) + 2a^{3/2}e\sin(\theta-\omega) - \frac{3}{4}a^{3/2}e^2\sin 2(\theta-\omega) \\ &\quad + \begin{Bmatrix}\text{similar short-period sinusoidal functions} \\ \text{of } \theta, \text{ multiplied by } e^3, e^4, \text{ - - -}\end{Bmatrix} + O(\mu)\end{aligned} \tag{41}$$

The quantity $(\theta - t)$ represents the angle from the line of centers of the two large masses to the radius vector of the infinitesimal body. The geometrical situation is shown in Figure 3.

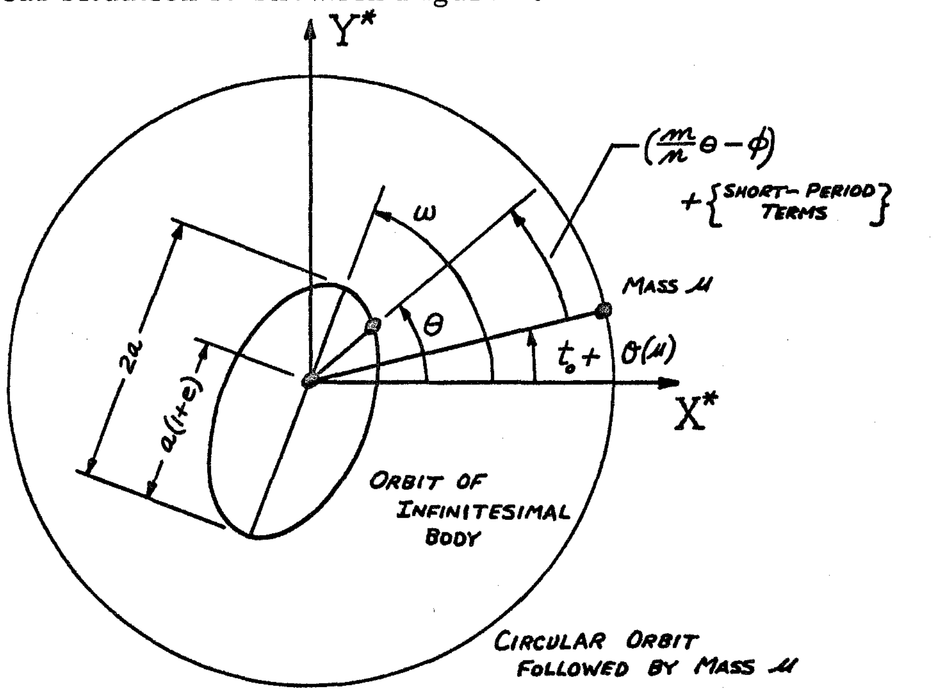

Figure 3. Geometry of the Orbit

The elements $a(\tilde{\theta}, \mu)$ and $e(\tilde{\theta}, \mu)$ specify the size and shape of the slowly-varying elliptical orbit. The longitude of pericenter $\omega(\tilde{\theta}, \mu)$ specifies its angular orientation. The quantity $\phi(\tilde{\theta}, \mu)$ specifies the position of the infinitesimal body in its orbit.

Consider the geometrical situation which occurs every nth time the infinitesimal body is at pericenter. Between two such occurrences, the infinitesimal body will have completed exactly n revolutions in its elliptical orbit, and the mass μ will have completed approximately (n-m) revolutions in its circular orbit. At each such instant,

$$\theta = \omega(\tilde{\theta}, \mu) + p \cdot 2n\pi \quad ; \text{ p a non-negative integer}$$

so that eq. (41) becomes

$$(\theta - t) = \left(\frac{m}{n}\omega - \phi\right) + pm \cdot 2\pi + O(\mu) \tag{42}$$

The simple form of eq. (42) results from the fact that each of the short-period terms in t_o vanishes when $\theta = \omega + p \cdot 2n\pi$. The geometrical situation when the infinitesimal body is at pericenter is shown in Figure 4.

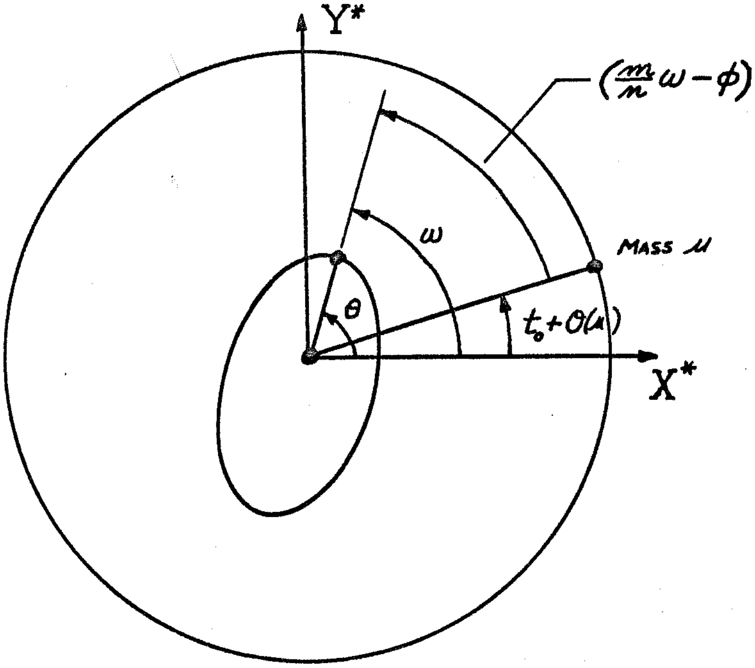

Figure 4. Geometrical Significance of $(\frac{m}{n}\omega - \phi)$

Thus the quantity $(\frac{m}{n}\omega - \phi)$ is equal to the angle between the pericenter of the infinitesimal body and the position of the mass μ, measured every nth time the infinitesimal body is at pericenter.

7. Dependence of the Orbital Elements on μ

The eccentricity is assumed to depend on $\tilde{\theta}$ and μ in the following manner:

(43) $$e(\tilde{\theta}, \mu) = e_o + \mu^{1/2}\hat{e}(\tilde{\theta}, \mu) \quad ; \; e_o \text{ a constant}$$

The corresponding derivative is

(44) $$\frac{de}{d\tilde{\theta}} = \mu^{1/2}\frac{d\hat{e}}{d\tilde{\theta}}$$

In certain cases it will be possible to use the approximation $e = e_o + O(\mu^{1/2})$. However, if e_o is sufficiently small, it is necessary to retain both terms on the r. h. s. of eq. (43).

The quantities ω and τ are both unbounded functions of $\tilde{\theta}$, in general. They will be assumed to depend on $\tilde{\theta}$ and μ in the following manner:

$$\omega(\tilde{\theta},\mu) = \omega_0 + \mu^{1/2}\hat{\omega}(\tilde{\theta},\mu) \quad ; \quad \omega_0 \text{ a constant} \tag{45}$$

$$\tau(\tilde{\theta},\mu) = \tau_0 + \mu^{1/2}\hat{\tau}(\tilde{\theta},\mu) \quad ; \quad \tau_0 \text{ a constant} \tag{46}$$

The corresponding derivatives are as follows:

$$\frac{d\omega}{d\tilde{\theta}} = \mu^{1/2}\frac{d\hat{\omega}}{d\tilde{\theta}} \tag{47}$$

$$\frac{d\tau}{d\tilde{\theta}} = \mu^{1/2}\frac{d\hat{\tau}}{d\tilde{\theta}} \tag{48}$$

It is not necessary to assume in advance that e_o, ω_o, and τ_o are constants. However, if one begins with eqs. (43), (45), and (46) it will be found that $\frac{de_o}{d\tilde{\theta}} = 0$, $\frac{d\omega_o}{d\tilde{\theta}} = 0$, $\frac{d\tau_o}{d\tilde{\theta}} = 0$. By assuming e_o, ω_o, and τ_o to be constants from the outset, these unnecessary calculations are avoided.

The quantities $\hat{\omega}(\tilde{\theta}, \mu)$ and $\hat{\tau}(\tilde{\theta}, \mu)$ will be unbounded functions of $\tilde{\theta}$ in general. Hence it is not correct to write $\omega = \omega_o + O(\mu^{1/2})$ or $\tau = \tau_o + O(\mu^{1/2})$. Both terms on the r. h. s. of eqs. (45) and (46) must be retained.

By substitution of the expansions (11) and (12) into eqs. (4) and (5), the following equations are obtained from the terms which are formally of $O(\mu^{1/2})$:

(49) $$\frac{\partial}{\partial\tilde{\theta}}\left(s_o^2\frac{\partial t_o}{\partial\theta}\right)+\frac{\partial}{\partial\theta}\left(s_o^2\frac{\partial t_o}{\partial\tilde{\theta}}\right)=0$$

(50) $$2\frac{\partial^2 s_o}{\partial\theta\partial\tilde{\theta}}-2\left(s_o^2\frac{\partial t_o}{\partial\theta}\right)\left(s_o^2\frac{\partial t_o}{\partial\tilde{\theta}}\right)=0$$

It will now be shown that because of the form of the expansions for $\frac{da^{3/2}}{d\tilde{\theta}}$, $\frac{de}{d\tilde{\theta}}$, $\frac{d\omega}{d\tilde{\theta}}$, and $\frac{d\tau}{d\tilde{\theta}}$, the terms which occur in eqs. (49) and (50) are actually of $O(\mu^{1/2})$, instead of $O(\mu^{0})$. By eq. (19),

(51) $$\frac{\partial}{\partial\tilde{\theta}}\left(s_o^2\frac{\partial t_o}{\partial\theta}\right)=\frac{d}{d\tilde{\theta}}\left[\frac{1}{a^{1/2}(1-e^2)^{1/2}}\right]$$
$$=\mu^{1/2}\left[\frac{-1}{3a^2(1-e^2)^{1/2}}\frac{d\hat{a}^{3/2}}{d\tilde{\theta}}+\frac{e}{a^{1/2}(1-e^2)^{3/2}}\frac{d\hat{e}}{d\tilde{\theta}}\right]$$

From eq. (21), it follows that

(52) $$\frac{\partial^2 s_o}{\partial\theta\partial\tilde{\theta}}=\cos\theta\frac{d}{d\tilde{\theta}}\left[\frac{e\sin\omega}{a(1-e^2)}\right]-\sin\theta\frac{d}{d\tilde{\theta}}\left[\frac{e\cos\omega}{a(1-e^2)}\right]$$
$$=\mu^{1/2}\left[\frac{e\cos\omega}{a(1-e^2)}\frac{d\hat{\omega}}{d\tilde{\theta}}-\frac{2}{3}\frac{e\sin\omega}{a^{5/2}(1-e^2)}\frac{d\hat{a}^{3/2}}{d\tilde{\theta}}+\frac{(1+e^2)\sin\omega}{a(1-e^2)^2}\frac{d\hat{e}}{d\tilde{\theta}}\right]\cos\theta$$
$$+\mu^{1/2}\left[\frac{e\sin\omega}{a(1-e^2)}\frac{d\hat{\omega}}{d\tilde{\theta}}+\frac{2}{3}\frac{e\cos\omega}{a^{5/2}(1-e^2)}\frac{d\hat{a}^{3/2}}{d\tilde{\theta}}-\frac{(1+e^2)\cos\omega}{a(1-e^2)^2}\frac{d\hat{e}}{d\tilde{\theta}}\right]\sin\theta$$

By carrying out the indicated derivatives in eq. (39), and then multiplying by $s_o^2 = a^{-2}(1-e^2)^{-2}[1+e\cos(\theta-\omega)]^2$, the following result is obtained:

(54) $$s_o^2\frac{\partial t_o}{\partial\tilde{\theta}}=\mu^{1/2}\left[\left(\frac{1}{a^2}+\frac{5e^2}{2a^2}\right)\frac{d\hat{\tau}}{d\tilde{\theta}}+\left(\frac{2e}{a^{1/2}}+O(e^3)\right)e\frac{d\hat{\omega}}{d\tilde{\theta}}\right]+\mu^{1/2}\left[\frac{-e}{2a^{1/2}}\frac{d\hat{e}}{d\tilde{\theta}}-\frac{5e^2}{4a^2}\frac{d\hat{a}^{3/2}}{d\tilde{\theta}}\right]\sin 2(\theta-\omega)$$
$$+\mu^{1/2}\left[-\left(\frac{2}{a^{1/2}}+\frac{3e^2}{a^{1/2}}\right)\frac{d\hat{e}}{d\tilde{\theta}}-\frac{2e}{a^2}\frac{d\hat{a}^{3/2}}{d\tilde{\theta}}\right]\sin(\theta-\omega)+\mu^{1/2}\left[\left(\frac{2}{a^{1/2}}+\frac{4e^2}{a^{1/2}}\right)e\frac{d\hat{\omega}}{d\tilde{\theta}}+\frac{2e}{a^2}\frac{d\hat{\tau}}{d\tilde{\theta}}\right]\cos(\theta-\omega)$$
$$+\mu^{1/2}\left[\frac{e^2}{2a^2}\frac{d\hat{\tau}}{d\tilde{\theta}}+\left(\frac{e}{2a^{1/2}}+O(e^3)\right)e\frac{d\hat{\omega}}{d\tilde{\theta}}\right]\cos 2(\theta-\omega)+\mu^{1/2}\left[\begin{array}{c}\text{similar terms multiplied}\\ \text{by } e^3, e^4, \ldots\end{array}\right]$$

<u>Note:</u> There is no equation (53).

By differentiation of eq. (54) w. r. t. θ, it follows that

$$\frac{\partial}{\partial \theta}\left(s_0^2 \frac{\partial t_0}{\partial \tilde{\theta}}\right) = O(\mu^{1/2}) \tag{55}$$

Thus, each term which occurs in eqs. (49) and (50) is actually of $O(\mu^{1/2})$, rather than $O(\mu^{o})$. These terms must therefore be included in the $O(\mu)$ equations. Hence there are no $O(\mu^{1/2})$ equations to solve.

8. The $O(\mu)$ Equations

By use of eqs. (11), (12), (16), (49), (50), and (55) it may be shown that the $O(\mu)$ terms of the equations of motion lead to the following equations:

$$\begin{aligned}\frac{\partial}{\partial \theta}\left[\left(s_0^2 \frac{\partial t_1}{\partial \theta} + 2 s_0 s_1 \frac{\partial t_0}{\partial \theta}\right) + \frac{1}{\mu^{1/2}}\left(s_0^2 \frac{\partial t_0}{\partial \tilde{\theta}}\right)\right] &= \frac{1}{3a^2(1-e^2)^{1/2}} \frac{d\hat{a}^{3/2}}{d\tilde{\theta}} - \frac{e}{a^{1/2}(1-e^2)^{3/2}} \frac{d\hat{e}}{d\tilde{\theta}} \\ &\quad - s_0^3 \left(\frac{\partial t_0}{\partial \theta}\right)^3 \sin(\theta - t_0) + \left(s_0^2 \frac{\partial t_0}{\partial \theta}\right)^3 \frac{\sin(\theta - t_0)}{\left[1 + s_0^2 - 2 s_0 \cos(\theta - t_0)\right]^{3/2}}\end{aligned} \tag{57}$$

$$\begin{aligned}\frac{\partial^2 s_1}{\partial \theta^2} + s_1 &= \left[\frac{-2e\cos\omega}{a(1-e^2)} \frac{d\hat{\omega}}{d\tilde{\theta}} + \frac{4}{3} \frac{e \sin\omega}{a^{5/2}(1-e^2)} \frac{d\hat{a}^{3/2}}{d\tilde{\theta}} - \frac{2(1+e^2)\sin\omega}{a(1-e^2)^2} \frac{d\hat{e}}{d\tilde{\theta}}\right] \cos\theta \\ &\quad + \left[\frac{-2e\sin\omega}{a(1-e^2)} \frac{d\hat{\omega}}{d\tilde{\theta}} - \frac{4}{3} \frac{e\cos\omega}{a^{5/2}(1-e^2)} \frac{d\hat{a}^{3/2}}{d\tilde{\theta}} + \frac{2(1+e^2)\cos\omega}{a(1-e^2)^2} \frac{d\hat{e}}{d\tilde{\theta}}\right] \sin\theta \\ &\quad + 2\left(s_0^2 \frac{\partial t_0}{\partial \theta}\right)\left[\left(s_0^2 \frac{\partial t_1}{\partial \theta} + 2 s_0 s_1 \frac{\partial t_0}{\partial \theta}\right) + \frac{1}{\mu^{1/2}}\left(s_0^2 \frac{\partial t_0}{\partial \tilde{\theta}}\right)\right] + s_0^2 \left(\frac{\partial t_0}{\partial \theta}\right)^2 \cos(\theta - t_0) - \left(s_0^2 \frac{\partial t_0}{\partial \theta}\right)^2 \\ &\quad - s_0 \frac{\partial s_0}{\partial \theta}\left(\frac{\partial t_0}{\partial \theta}\right)^2 \sin(\theta - t_0) + \left(s_0^2 \frac{\partial t_0}{\partial \theta}\right)^2 \frac{\left[1 - s_0 \cos(\theta - t_0) + \frac{\partial s_0}{\partial \theta} \sin(\theta - t_0)\right]}{\left[1 + s_0^2 - 2 s_0 \cos(\theta - t_0)\right]^{3/2}}\end{aligned} \tag{58}$$

The quantity $\frac{1}{\mu^{1/2}}\left(s_o^2 \frac{\partial t_o}{\partial \tilde{\theta}}\right)$ is of $O(\mu^{o})$, as may be seen from eq. (54).

The notation $\frac{1}{\mu^{1/2}}\left(s_o^2 \frac{\partial t_o}{\partial \tilde{\theta}}\right)$ is merely a convenient way of writing this term.

Note: There is no equation (56).

9. Series Expansion of the Perturbing Terms

In order to express the perturbing terms which involve $\sin(\theta - t_o)$ and $\cos(\theta - t_o)$ in a useful form, it is necessary to expand these quantities in powers of e. The amount of algebraic labor that is required increases very rapidly as higher powers of e are retained. For this reason, all terms multiplied by $e^3, e^4,$ --- will be neglected in the remainder of this work. For orbits with small eccentricities, this should yield a reasonable approximation. The approximation could be improved in a straightforward manner, merely by retaining the higher powers of e.

Using eq. (36) for t_o, the quantity $\sin(\theta - t_o)$ may be expanded in powers of e as follows:

$$
\begin{aligned}
(59) \qquad \sin(\theta - t_o) = {} & \sin\left(\tfrac{m}{n}\theta - \phi\right) + 2a^{3/2} e \sin(\theta - \omega)\cos\left(\tfrac{m}{n}\theta - \phi\right) \\
& - a^3 e^2 \sin\left(\tfrac{m}{n}\theta - \phi\right) + a^3 e^2 \cos 2(\theta - \omega) \sin\left(\tfrac{m}{n}\theta - \phi\right) \\
& - \tfrac{3}{4} a^{3/2} e^2 \sin 2(\theta - \omega) \cos\left(\tfrac{m}{n}\theta - \phi\right) + \left\{ \begin{array}{l} \text{similar sinusoidal functions} \\ \text{of } \theta, \text{ multiplied by } e^3, e^4, -- \end{array} \right\}
\end{aligned}
$$

The quantity $\cos(\theta - t_o)$ may be expanded in a similar form.

The perturbing terms on the r.h.s. of eqs. (57) and (58) may then be expanded in powers of e. For example,

$$
\begin{aligned}
(60) \qquad -s_o \frac{\partial s_o}{\partial \theta} \left(\frac{\partial t_o}{\partial \theta}\right)^2 \sin(\theta - t_o) = {} & a e \sin(\theta - \omega) \sin\left(\tfrac{m}{n}\theta - \phi\right) + a^{5/2} e^2 \cos\left(\tfrac{m}{n}\theta - \phi\right) \\
& - a^{5/2} e^2 \cos 2(\theta - \omega) \cos\left(\tfrac{m}{n}\theta - \phi\right) - \tfrac{3}{2} a e^2 \sin 2(\theta - \omega) \sin\left(\tfrac{m}{n}\theta - \phi\right) \\
& + \left\{ \begin{array}{l} \text{similar sinusoidal functions} \\ \text{of } \theta, \text{ multiplied by } e^3, e^4, -- \end{array} \right\}
\end{aligned}
$$

Similar expansions can be made for the terms $-s_0^3\left(\frac{\partial t_0}{\partial\theta}\right)^3 \sin(\theta-t_0)$ and $s_0^2\left(\frac{\partial t_0}{\partial\theta}\right)^2 \cos(\theta-t_0)$.

The expansions of $\left(s_0^2\,\frac{\partial t_0}{\partial\theta}\right)^3 \sin(\theta-t_0)\left[1+s_0^2-2s_0\cos(\theta-t_0)\right]^{-3/2}$ and $\left(s_0^2\,\frac{\partial t_0}{\partial\theta}\right)^2\left[1-s_0\cos(\theta-t_0)+\frac{\partial s_0}{\partial\theta}\sin(\theta-t_0)\right]\left[1+s_0^2-2s_0\cos(\theta-t_0)\right]^{-3/2}$ in powers of e are quite lengthy, and are therefore given in the appendix.

The r.h.s. of eqs. (57) and (58) have now been expressed as functions of θ and the orbital elements $a^{3/2}, e, \omega$, and ϕ. However, the integration of these equations cannot be carried out explicitly with the r.h.s. in its present form.

A convenient way to carry out the integration is to express the various periodic functions of θ in their Fourier series' expansions, and then to integrate these series' termwise. The use of Fourier series' identifies the various frequencies which occur in the perturbing terms, thereby making it possible to identify and remove the terms which would otherwise produce quantities proportional to θ in s_1 and t_1.

There are several ways in which the Fourier expansions could be carried out. The one that will be used here is convenient when one wishes to determine the numerical values of the Fourier coefficients. It is sufficient to use the following three Fourier series expansions:

(61a) $$\left[1+a^2-2a\cos\left(\tfrac{m}{n}\theta-\phi\right)\right]^{-3/2} = \tfrac{1}{2}A_0(a)+\sum_{k=1}^{\infty}A_k(a)\cos k\left(\tfrac{m}{n}\theta-\phi\right)$$

(61b) $$\left[1+a^2-2a\cos\left(\tfrac{m}{n}\theta-\phi\right)\right]^{-5/2} = \tfrac{1}{2}B_0(a)+\sum_{k=1}^{\infty}B_k(a)\cos k\left(\tfrac{m}{n}\theta-\phi\right)$$

(61c) $$\left[1+a^2-2a\cos\left(\tfrac{m}{n}\theta-\phi\right)\right]^{-7/2} = \tfrac{1}{2}C_0(a)+\sum_{k=1}^{\infty}C_k(a)\cos k\left(\tfrac{m}{n}\theta-\phi\right)$$

The Fourier coefficients are given by

$$A_k(a) = \frac{1}{\pi}\int_0^{2\pi}\left[1+a^2-2a\cos x\right]^{-3/2}\cos kx\,dx \tag{62a}$$

$$B_k(a) = \frac{1}{\pi}\int_0^{2\pi}\left[1+a^2-2a\cos x\right]^{-5/2}\cos kx\,dx \tag{62b}$$

$$C_k(a) = \frac{1}{\pi}\int_0^{2\pi}\left[1+a^2-2a\cos x\right]^{-7/2}\cos kx\,dx \tag{62c}$$

for $k = 0, 1, 2, ---$. The value of $a(\tilde{\theta}, \mu)$ is held fixed in carrying out these integrations with respect to x.

If all the perturbing terms multiplied by e^3 were retained, it would be necessary to express the quantity $\left[1+a^2-2a\cos(\frac{m}{n}\theta-\phi)\right]^{-9/2}$ in its Fourier expansion. In general, one additional Fourier expansion of the above type is required for each additional power of e that is retained in the perturbing terms.

The series representation of each perturbing term can be obtained from the above Fourier expansions, by termwise multiplication. For example,

$$\left[1+a^2-2a\cos(\tfrac{m}{n}\theta-\phi)\right]^{-3/2}\sin\theta\cos(\tfrac{m}{n}\theta-\phi) = \tfrac{1}{2}A_0\sin\theta\cos(\tfrac{m}{n}\theta-\phi)$$
$$+\frac{1}{4}\sum_{k=1}^{\infty}A_k\left\{\begin{array}{l}-\sin\left[\frac{(-n+mk+m)}{n}\theta-(k+1)\phi\right]-\sin\left[\frac{(-n+mk-m)}{n}\theta-(k-1)\phi\right]\\ +\sin\left[\frac{(n+mk+m)}{n}\theta-(k+1)\phi\right]+\sin\left[\frac{(n+mk-m)}{n}\theta-(k-1)\phi\right]\end{array}\right\} \tag{63}$$

Similar expansions can be made for each of the perturbing terms.

These Fourier coefficients may be expressed in terms of the hypergeometric function. For example,

$$A_k(a) = \frac{2a^k \Gamma(k+\frac{3}{2})}{\Gamma(\frac{3}{2})\Gamma(k+1)} F(\tfrac{3}{2}, k+\tfrac{3}{2}, k+1, a^2) \tag{64}$$

Similar expressions are valid for $B_k(a)$ and $C_k(a)$. They may also be expressed in terms of the complete elliptic integrals of the first and second kinds, $K(a)$ and $E(a)$, respectively. The recursion relations for the hypergeometric function may be used to prove certain relationships between the Fourier coefficients.

In order to obtain results related to the behavior of the orbital elements for a specific numerical value of μ, it is necessary to know the numerical values of the Fourier coefficients. These coefficients could be calculated directly from the definitions in eqs. (62a), (62b), and (62c), by numerical integration over the range $0 \leqslant x \leqslant 2\pi$.

However, these values may also be obtained from extensive tables published by Brown and Brouwer[7]. These tables give numerical values of $G_{3/2}^{(k)}(a)$, $G_{5/2}^{(k)}(a)$, and $G_{7/2}^{(k)}(a)$ for $0.0 \leqslant a \leqslant 0.845$, where

$$A_k(a) = a^k(1-a^2)^{-3/2} G_{3/2}^{(k)}(a)$$

$$B_k(a) = a^k(1-a^2)^{-5/2} G_{5/2}^{(k)}(a)$$

$$C_k(a) = a^k(1-a^2)^{-7/2} G_{7/2}^{(k)}(a)$$

for $k = 0, 1, 2, ---$. The quantities $G_{3/2}^{(k)}$, $G_{5/2}^{(k)}$, and $G_{7/2}^{(k)}$ are known as Laplace coefficients.

10. Removal of Resonant Perturbing Terms

The quantity $\left[\left(s_o^2 \frac{\partial t_1}{\partial \theta} + 2s_o s_1 \frac{\partial t_o}{\partial \theta} + \frac{1}{\mu^{1/2}}\left(s_o^2 \frac{\partial t_o}{\partial \tilde{\theta}}\right)\right]$ must be known explicitly in terms of θ before eq. (58) can be solved. Hence eq. (57) will be solved first. After expressing each of the perturbing terms as discussed above, eq. (57) can be written in the following form:

$$(65)\quad \frac{\partial}{\partial \theta}\left[\left(s_o^2 \frac{\partial t_1}{\partial \theta} + 2s_o s_1 \frac{\partial t_o}{\partial \theta}\right) + \frac{1}{\mu^{1/2}}\left(s_o^2 \frac{\partial t_o}{\partial \tilde{\theta}}\right)\right] = \frac{1}{3a^2(1-e^2)^{1/2}} \frac{d\hat{a}^{3/2}}{d\tilde{\theta}} - \frac{e}{a^{1/2}(1-e^2)^{3/2}} \frac{d\hat{e}}{d\tilde{\theta}}$$

$$+ h_1(\theta, a^{3/2}, e, \omega, \phi)$$

where the bounded function h_1 is composed of terms of the following types:

(a) several infinite series' which are multiplied by e^o, e, e^2, etc. and which contain sinusoidal functions of θ, whose frequencies are independent of $\tilde{\theta}$. These infinite series' result from the expansion of the term $\left(s_o^2 \frac{\partial t_o}{\partial \theta}\right)^3 \left[1 + s_o^2 - 2s_o \cos(\theta - t_o)\right]^{-3/2} \sin(\theta - t_o)$ in powers of e.

(b) sinusoidal functions of θ which result from the expansion of $-s_o^3 \left(\frac{\partial t_o}{\partial \theta}\right)^3 \sin(\theta - t_o)$ in powers of e.

In carrying out the integration of eq. (65) w. r. t. θ, the slow variable $\tilde{\theta}$ will be held fixed. Therefore any term which depends only on $\tilde{\theta}$ (i.e. which is independent of θ) would produce an unbounded term proportional to θ in the quantity $\left(s_o^2 \frac{\partial t_1}{\partial \theta} + 2s_o s_1 \frac{\partial t_o}{\partial \theta}\right)$. This would lead to the occurrence of similar unbounded terms in $s_1(\theta, \tilde{\theta}, \mu)$ and $t_1(\theta, \tilde{\theta}, \mu)$, contrary to the assumptions of the original two variable expansion.

Several terms which are independent of θ will occur in the infinite series'. These are the terms which produce small divisors in the variation of constants solution. For example, if the integers m and n have values such that there exists a non-negative integer k such that $\frac{n}{m} - 1 = k$, then the $(\frac{n}{m}-1)$th term of several of the infinite series' will contain the quantity

(66) $$\sin\left[\frac{(-n+km+m)}{m}\theta-(k+1)\phi\right] = -\sin\frac{n}{m}\phi$$

Each of the series' will contain one or more terms of the above type, depending upon the values of m and n. By a careful inspection of the series' which occur on the r.h.s. of eq. (65), the sum of all such terms may be determined.

From this point on, only the case $m = 1$ will be discussed in detail. This is the most important case for comparison of the results with the motion of asteroids.

In order that $\left(s_o^2 \frac{\partial t_1}{\partial\theta} + 2s_o s_1 \frac{\partial t_o}{\partial\theta}\right)$ will not contain a term proportional to θ, the sum of all terms on the r.h.s. of eq. (65) which are independent of θ must vanish. This requirement yields the following equation:

(67) $$\frac{-1}{3a^2(1-e^2)^{1/2}}\frac{d\hat{a}^{3/2}}{d\hat{\theta}} + \frac{e}{a^{1/2}(1-e^2)^{3/2}}\frac{d\hat{e}}{d\hat{\theta}} = -\frac{\alpha_n}{a^2}e\sin(\omega-n\phi)$$
$$+\frac{\beta_n}{a^2}e^2\sin 2(\omega-n\phi) + \left\{\begin{matrix}\text{similar terms multiplied}\\ \text{by } e^3, e^4, \text{ - - -}\end{matrix}\right\}$$

The quantities α_n and β_n are functions of $a^{3/2}$ only, and are defined in the appendix. They are the sum of several of the Fourier coefficients, each multiplied by some power of $a^{3/2}$.

For the case $m=2$, the r.h.s. of eq. (67) would not contain a term multiplied by e; the leading term would be multiplied by e^2. For $m=3$, the leading term would be multiplied by e^3, etc.

After the terms which are independent of θ have been removed by means of eq. (67), eq. (65) can be integrated with respect to θ, holding $\tilde{\theta}$ fixed. The result will be free of small divisors, but will not be written out explicitly here.

The expression for the integral of eq. (65) can then be substituted into eq. (58). The result will be as follows:

$$(68)\quad \frac{\partial^2 \Delta_1}{\partial\theta^2} + \Delta_1 = \left[\frac{-2e\cos\omega}{a(1-e^2)}\frac{d\hat{\omega}}{d\tilde{\theta}} + \frac{4}{3}\frac{e\sin\omega}{a^{5/2}(1-e^2)}\frac{d\hat{a}^{3/2}}{d\tilde{\theta}} - \frac{2(1+e^2)\sin\omega}{a(1-e^2)^2}\frac{d\hat{e}}{d\tilde{\theta}}\right]\cos\theta$$
$$+\left[\frac{-2e\sin\omega}{a(1-e^2)}\frac{d\hat{\omega}}{d\tilde{\theta}} - \frac{4}{3}\frac{e\cos\omega}{a^{5/2}(1-e^2)}\frac{d\hat{a}^{3/2}}{d\tilde{\theta}} + \frac{2(1+e^2)\cos\omega}{a(1-e^2)^2}\frac{d\hat{e}}{d\tilde{\theta}}\right]\sin\theta$$
$$-\frac{1}{a(1-e^2)} + h_2\left(\theta, a^{3/2}, e, \omega, \phi, \frac{d\hat{e}}{d\tilde{\theta}}, \frac{d\hat{\omega}}{d\tilde{\theta}}, \frac{d\hat{\tau}}{d\tilde{\theta}}\right)$$

where the bounded function h_2 contains terms of the following types:

(a) several infinite series' which are multiplied by e^0, e, e^2 etc., and which contain sinusoidal functions of θ whose frequencies are independent of $\tilde{\theta}$. These series' result from the expansion of the quantity $\left(s_o^2 \frac{\partial t_o}{\partial\theta}\right)^2\left[1-s_o\cos(\theta-t_o) + \frac{\partial s_o}{\partial\theta}\sin(\theta-t_o)\right]\left[1+s_o^2-2s_o\cos(\theta-t_o)\right]^{-3/2}$ in powers of e, and also from the corresponding terms in $\left[\left(s_o^2\frac{\partial t_1}{\partial\theta} + 2s_o s_1 \frac{\partial t_o}{\partial\theta}\right) + \frac{1}{\mu^{1/2}}\left(s_o^2\frac{\partial t_o}{\partial\tilde{\theta}}\right)\right]$.

(b) sinusoidal functions of θ which result from the expansion of the quantities $s_o^2\left(\frac{\partial t_o}{\partial\theta}\right)^2\cos(\theta-t_o)$ and $s_o\frac{\partial s_o}{\partial\theta}\left(\frac{\partial t_o}{\partial\theta}\right)^2$.

$\cdot\sin(\theta - t_o)$ in powers of e, and also from the corresponding terms contained in $\left[\left(s_o^2 \frac{\partial t_1}{\partial\theta} + 2s_o s_1 \frac{\partial t_o}{\partial\theta} + \frac{1}{\mu^{\frac{1}{2}}}\left(s_o^2 \frac{\partial t_o}{\partial\tilde{\theta}}\right)\right)\right]$.

If a term in $\sin\theta$ or $\cos\theta$ were to occur on the r.h.s. of eq. (68), the response to this term would contain the unbounded quantity $\theta\sin\theta$ or $\theta\cos\theta$. This would clearly be a resonance effect, and would violate the assumption that $\mu s_1(\theta, \tilde{\theta}, \mu)$ remains a small quantity of $O(\mu)$.

Several such terms in $\sin\theta$ and $\cos\theta$ are contained in the infinite series'. For example, if the integers m and n have values such that there exists a non-negative integer k for which $\frac{2n}{m} - 1 = k$, the $(\frac{2n}{m} - 1)$th term of several of the infinite series' will contain the quantity

$$(69) \qquad \sin\left[\frac{(-n+km+m)}{n}\theta - (k+1)\phi\right] = \cos\frac{2n}{m}\phi\sin\theta - \sin\frac{2n}{m}\phi\cos\theta$$

Each of the infinite series' will contain one or more such terms, provided that m and n have the necessary values. By a careful inspection of the r.h.s. of the equation for $\frac{\partial^2 s_1}{\partial\theta^2} + s_1$, the sum of all terms in $\sin\theta$ and $\cos\theta$ may be determined.

In order for $s_1(\theta, \tilde{\theta}, \mu)$ not to contain a term proportional to θ, the sum of the terms in $\sin\theta$ and $\cos\theta$ must vanish, for all values of $\tilde{\theta}$. This requires that the coefficients of $\sin\theta$ and $\cos\theta$ must vanish separately, for all values of $\tilde{\theta}$. This leads to the following equations, for the case $m = 1$:

(70) $$\frac{e\sin\omega}{a(1-e^2)}\frac{d\hat{\omega}}{d\tilde{\theta}}+\frac{2}{3}\frac{e\cos\omega}{a^{5/2}(1-e^2)}\frac{d\hat{a}^{3/2}}{d\tilde{\theta}}-\frac{(1+e^2)\cos\omega}{a(1-e^2)^2}\frac{d\hat{e}}{d\tilde{\theta}}=(\tfrac{1}{2}K_n)\sin n\phi+(\tfrac{1}{2}\rho)e\sin\omega$$

$$-(\tfrac{1}{2}\gamma_n)e\sin(\omega-2n\phi)+(\tfrac{1}{2}\delta_n)e^2\sin n\phi+(\tfrac{1}{4}\eta_n)e^2\sin(2\omega-n\phi)$$

$$+(\tfrac{1}{4}\xi_n)e^2\sin(2\omega-3n\phi)+\left\{\begin{matrix}\text{similar terms}\\ \text{multiplied by } e^3, e^4, ---\end{matrix}\right\}$$

(71) $$\frac{e\cos\omega}{a(1-e^2)}\frac{d\hat{\omega}}{d\tilde{\theta}}-\frac{2}{3}\frac{e\sin\omega}{a^{5/2}(1-e^2)}\frac{d\hat{a}^{3/2}}{d\tilde{\theta}}+\frac{(1+e^2)\sin\omega}{a(1-e^2)^2}\frac{d\hat{e}}{d\tilde{\theta}}=(\tfrac{1}{2}K_n)\cos n\phi+(\tfrac{1}{2}\rho)e\cos\omega$$

$$+(\tfrac{1}{2}\gamma_n)e\cos(\omega-2n\phi)+(\tfrac{1}{2}\delta_n)e^2\cos n\phi+(\tfrac{1}{4}\eta_n)e^2\cos(2\omega-n\phi)$$

$$-(\tfrac{1}{4}\xi_n)e^2\cos(2\omega-3n\phi)+\left\{\begin{matrix}\text{similar terms}\\ \text{multiplied by } e^3, e^4, ---\end{matrix}\right\}$$

The quantities $\kappa_n, \rho, \gamma_n, \delta_n, \eta_n$, and ξ_n depend only on $a^{3/2}$. They are defined in the appendix.

After the terms in $\sin\theta$ and $\cos\theta$ have been removed from eq. (68) by means of eqs. (70) and (71), the solution for s_1 will be as follows:

(72) $$s_1(\theta,\tilde{\theta},\mu)=\frac{-1}{a(1-e^2)}+h_3\left(\theta,a^{3/2},e,\omega,\phi,\frac{d\hat{e}}{d\tilde{\theta}},\frac{d\hat{\omega}}{d\tilde{\theta}},\frac{d\hat{\tau}}{d\tilde{\theta}}\right)$$

where the bounded function h_3 contains terms of the following types:

(a) several infinite series' which are multiplied by e^o, e, e^2, etc., and which contain sinusoidal functions of θ. These infinite series' do not contain any small divisors.

(b) sinusoidal functions of θ which are multiplied by e, $\sin\omega$, $\cos\omega$, $\sin n\phi$, $\cos n\phi$, $\frac{d\hat{e}}{d\tilde{\theta}}$, $\frac{d\hat{\omega}}{d\tilde{\theta}}$, and $\frac{d\hat{\tau}}{d\tilde{\theta}}$.

The derivatives $\frac{d\hat{e}}{d\tilde{\theta}}$, $\frac{d\hat{\omega}}{d\tilde{\theta}}$, and $\frac{d\hat{\tau}}{d\tilde{\theta}}$ may be eliminated from the equation for s_1 after the expressions for these derivatives have been found in terms of $a^{3/2}$, e, ω, and φ. The resulting expression for $s_1(\theta, \tilde{\theta}, \mu)$ will be free from small divisors.

The quantity $\frac{\partial t_1}{\partial \theta}$ may be expressed as follows:

$$
(73) \quad \frac{\partial t_1}{\partial \theta} = -\frac{2}{s_0^3}\left(s_0^2 \frac{\partial t_0}{\partial \theta}\right) s_1(\theta, \tilde{\theta}, \mu) + \frac{1}{s_0^2}\left\{\left[\text{integral of eq. (65)}\right] - \frac{1}{\mu^{1/2}}\left(s_0^2 \frac{\partial t_0}{\partial \tilde{\theta}}\right)\right\}
$$

$$
= \frac{-2a^{5/2}(1-e^2)^{5/2} s_1}{\left[1+e\cos(\theta-\omega)\right]^3} + \frac{a^2(1-e^2)^2}{\left[1+e\cos(\theta-\omega)\right]^2}\left\{\left[\text{integral of eq. (65)}\right] - \frac{1}{\mu^{1/2}}\left(s_0^2 \frac{\partial t_0}{\partial \tilde{\theta}}\right)\right\}
$$

When the expressions for s_1 and $\left\{\left[\text{integral of eq. (65)}\right] - \frac{1}{\mu^{\frac{1}{2}}}\left(s_o^2 \frac{\partial t_o}{\partial \tilde{\theta}}\right)\right\}$ are substituted into eq. (73), the following equation is obtained:

$$
(74) \quad \frac{\partial t_1}{\partial \theta} = \left[-4 - 12e^2 + O(e^3)\right]\frac{d\hat{\tau}}{d\tilde{\theta}} - \left[6a^{3/2}e + O(e^3)\right] e \frac{d\hat{\omega}}{d\tilde{\theta}}
$$

$$
+ h_4\left(\theta, a^{3/2}, e, \omega, \phi, \frac{d\hat{e}}{d\tilde{\theta}}, \frac{d\hat{\omega}}{d\tilde{\theta}}, \frac{d\hat{\tau}}{d\tilde{\theta}}\right)
$$

where the bounded function h_4 contains terms of the following types:

(a) several infinite series' which are multiplied by e^o, e, e^2, etc. and which contain sinusoidal functions of θ, whose frequencies are independent of $\tilde{\theta}$.

(b) sinusoidal functions of θ which are multiplied by e, sin ω, cos ω, sin nφ, $\frac{d\hat{e}}{d\tilde{\theta}}$, $\frac{d\hat{\omega}}{d\tilde{\theta}}$, and $\frac{d\hat{\tau}}{d\tilde{\theta}}$.

In carrying out the integration of eq. (74), the same considerations that were discussed in relation to the integration of eq. (65) will apply. The sum of all terms on the r.h.s. which are independent of θ must vanish for all $\tilde{\theta}$. This requirement yields the following equation:

$$(75)\quad \left[4+12e^2+O(e^3)\right]\frac{d\hat{\tau}}{d\tilde{\theta}} + \left[6a^{3/2}e+O(e^3)\right]e\frac{d\hat{\omega}}{d\tilde{\theta}} = \nu + \lambda_n e\cos(\omega - n\phi) + \sigma e^2$$

$$+\left(\tfrac{1}{2}\zeta_n\right)e^2\cos 2(\omega - n\phi) \quad + \quad \left\{\begin{array}{l}\text{similar terms} \\ \text{multiplied by } e^3, e^4, \text{---}\end{array}\right\}$$

The quantities ν, λ_n, σ, and ζ_n depend only on $a^{3/2}$, and are defined in the appendix.

After the terms which are independent of θ have been removed from eq. (74) by means of eq. (75), eq. (74) may be integrated w.r.t. θ, holding $\tilde{\theta}$ fixed. The result is of the following form:

$$(76)\qquad t_1(\theta,\tilde{\theta},\mu) = h_5\left(\theta, a^{3/2}, e, \omega, \phi, \frac{d\hat{e}}{d\tilde{\theta}}, \frac{d\hat{\omega}}{d\tilde{\theta}}, \frac{d\hat{\tau}}{d\tilde{\theta}}\right)$$

where the bounded function h_5 contains terms of the following types:

(a) several infinite series' which are multiplied by e^0, e, e^2, etc. and which contain sinusoidal functions of θ, whose frequencies are independent of $\tilde{\theta}$. These series' are free from small divisors.

(b) sinusoidal functions of θ which are multiplied by e, $\sin\omega$, $\cos\omega$, $\sin n\phi$, $\cos n\phi$, $\frac{d\hat{e}}{d\tilde{\theta}}$, $\frac{d\hat{\omega}}{d\tilde{\theta}}$, and $\frac{d\hat{\tau}}{d\tilde{\theta}}$.

The derivatives $\frac{d\hat{e}}{d\tilde{\theta}}$, $\frac{d\hat{\omega}}{d\tilde{\theta}}$, and $\frac{d\hat{\tau}}{d\tilde{\theta}}$ may be eliminated from the expression for t_1 by use of eqs. (67), (70), (71), and (75).

Thus the assumed form of the two variable expansions given in eqs. (8a) and (8b) has been shown to yield a self-consistent approximation to the solution of eqs. (4) and (5), provided that the orbital elements satisfy the four first-order differential equations (67), (70), (71), and (75). The perturbation terms $\mu s_1(\theta, \tilde{\theta}, \mu)$ and $\mu t_1(\theta, \tilde{\theta}, \mu)$, as given in eqs. (72) and (76), will be free from small divisors and will remain small quantities of $O(\mu)$.

If the perturbing terms of $O(\mu^2)$ were taken into account, the r.h.s. of eqs. (67), (70), (71), and (75) would also contain $O(\mu)$ terms involving $a, e, \omega,$ and ϕ. The short-period perturbations would be accounted for by terms $\mu^2 s_2(\theta, \tilde{\theta}, \mu)$ and $\mu^2 t_2(\theta, \tilde{\theta}, \mu)$, similar in nature to s_1 and t_1 .

Therefore an approximate solution for the motion of the infinitesimal body, which remains valid for large values of θ, has been obtained for the case of nearly commensurable mean motions. The difficulty of small divisors has been avoided in this solution by requiring that the orbital elements must satisfy a set of four first order differential equations.

IV. BEHAVIOR OF THE ORBITAL ELEMENTS

In section III it was shown that the difficulty of small divisors can be avoided by requiring that the orbital elements of the infinitesimal body must satisfy a set of four coupled first-order equations, having the independent variable $\tilde{\theta} = \mu^{1/2}\theta$ rather than θ. In this section, some approximate solutions of these equations will be given.

1. Equations for the Orbital Elements

Eq. (67) gives one relation between $\frac{d\hat{a}^{3/2}}{d\tilde{\theta}}$ and $\frac{d\hat{e}}{d\tilde{\theta}}$. A second relation may be obtained by multiplication of eq. (70) by $-a(1-e^2)\cos\omega$ and multiplication of eq. (71) by $a(1-e^2)\sin\omega$, followed by addition of the results:

$$(77)\quad -\frac{2e}{3a^{3/2}}\frac{d\hat{a}^{3/2}}{d\tilde{\theta}} + \frac{(1+e^2)}{(1-e^2)}\frac{d\hat{e}}{d\tilde{\theta}} = (\tfrac{1}{2}aK_n)\sin(\omega-n\phi) + (\tfrac{1}{2}a\gamma_n)e\sin 2(\omega-n\phi)$$

$$+(\tfrac{1}{2}a\delta_n - \tfrac{1}{4}a\eta_n - \tfrac{1}{2}aK_n)e^2\sin(\omega-n\phi) - (\tfrac{1}{4}a\zeta_n)e^2\sin 3(\omega-n\phi) + O(e^3)$$

Multiplication of eq. (67) by $-2a^{1/2}e(1-e^2)^{1/2}$, followed by addition of the result to eq. (77) yields

$$(78)\quad \frac{d\hat{e}}{d\tilde{\theta}} = (\tfrac{1}{2}aK_n)\sin(\omega-n\phi) + (\tfrac{1}{2}a\gamma_n)e\sin 2(\omega-n\phi) - (\tfrac{1}{4}a\zeta_n)e^2\sin 3(\omega-n\phi)$$

$$+(\tfrac{1}{2}a\delta_n - \tfrac{1}{4}a\eta_n + \tfrac{2}{a^{3/2}}\alpha_n - \tfrac{1}{2}aK_n)e^2\sin(\omega-n\phi) + O(e^3)$$

From eq. (67) it then follows that

$$(79)\quad \frac{d\hat{a}^{3/2}}{d\tilde{\theta}} = (3\alpha_n + \tfrac{3}{2}a^{5/2}K_n)e\sin(\omega-n\phi) + (-3\beta_n + \tfrac{3}{2}a^{5/2}\gamma_n)e^2\sin 2(\omega-n\phi)$$

$$+ O(e^3)$$

Similarly, multiplication of eq. (70) by $a(1-e^2)\sin\omega$ and eq. (71) by $a(1-e^2)\cos\omega$, followed by addition of the results, yields the following:

$$(80)\quad e\frac{d\hat{\omega}}{d\tilde{\theta}} = \left(\tfrac{1}{2}aK_n\right)\cos(\omega-n\phi) + \left(\tfrac{1}{2}a\rho\right)e + \left(\tfrac{1}{2}a\gamma_n\right)e\cos 2(\omega-n\phi)$$
$$+\left(\tfrac{1}{2}a\delta_n+\tfrac{1}{4}a\eta_n-\tfrac{1}{2}aK_n\right)e^2\cos(\omega-n\phi) - \left(\tfrac{1}{4}a\xi_n\right)e^2\cos 3(\omega-n\phi) + O(e^3)$$

Eq. (75) then yields the following equation, after dropping all terms in e^3, e^4, etc:

$$(81)\quad \frac{d\hat{\tau}}{d\tilde{\theta}} = \tfrac{1}{4}\nu + \left(\tfrac{1}{4}\lambda_n - \tfrac{3}{4}a^{5/2}K_n\right)e\cos(\omega-n\phi) + \left(\tfrac{1}{8}\varsigma_n - \tfrac{3}{4}a^{5/2}\gamma_n\right)e^2\cos 2(\omega-n\phi)$$
$$+\left(\tfrac{1}{4}\sigma - \tfrac{3}{4}\nu - \tfrac{3}{4}a^{5/2}\rho\right)e^2 + O(e^3)$$

Since the angular quantity $(\omega-n\phi)$ occurs frequently in the above equations, its behavior as a function of $\tilde{\theta}$ will be of considerable importance. Using the expressions for $\frac{d\omega}{d\tilde{\theta}}$ and $\frac{d\phi}{d\tilde{\theta}}$ defined previously, one obtains

$$\frac{d(\omega-n\phi)}{d\tilde{\theta}} = -n\hat{a}^{3/2} + \mu^{1/2}\left(\frac{d\hat{\omega}}{d\tilde{\theta}} - n\frac{d\hat{\tau}}{d\tilde{\theta}}\right)$$

Using eqs. (80) and (81) this becomes

$$(82)\quad \frac{d(\omega-n\phi)}{d\tilde{\theta}} = -n\hat{a}^{3/2} + \mu^{1/2}\left[\left(\tfrac{1}{2}aK_n\right)\frac{\cos(\omega-n\phi)}{e}\right]$$
$$+\mu^{1/2}\left\{\begin{array}{l}\left(\tfrac{1}{2}a\rho-\tfrac{1}{4}n\nu\right) - n\left(\tfrac{1}{4}\sigma-\tfrac{3}{4}\nu-\tfrac{3}{4}a^{5/2}\rho\right)e^2 \\ +\left[\tfrac{1}{2}a\delta_n+\tfrac{1}{4}a\eta_n-\tfrac{n}{4}\lambda_n+\left(\tfrac{3}{4}na^{5/2}-\tfrac{1}{2}a\right)K_n\right]e\cos(\omega-n\phi) \\ +\left[\tfrac{1}{2}a\gamma_n+n\left(\tfrac{3}{4}a^{5/2}\gamma_n-\tfrac{1}{8}\varsigma_n\right)e^2\right]\cos 2(\omega-n\phi) - \left(\tfrac{1}{4}a\xi_n\right)e\cos 3(\omega-n\phi) \\ +O(e^2)\text{ terms from }\frac{d\hat{\omega}}{d\tilde{\theta}} + O(e^3)\end{array}\right\}$$

The second term on the r.h.s. of eq. (82) is written separately from the other terms of $O(\mu^{1/2})$, because if $e(\tilde{\theta}, \mu)$ is small of $O(\mu^{1/2})$ this term will become $O(\mu^{\circ})$.

If the perturbing terms of $O(\mu^2)$ from eqs. (4) and (5) had been retained, equations (78)-(82) would contain additional terms of higher order in μ on the r.h.s. These additional terms would involve $a^{3/2}$, e, ω, and ϕ.

Having obtained the equations for the behavior of the orbital elements, it is useful to distinguish between those terms which occur on the r.h.s. of eqs. (78)-(82) because of the nearly commensurable periods, and those which would also occur in the non-commensurable case. Each term which contains a sinusoidal function of $(\omega - n\phi)$ is solely the result of the commensurability. In the non-commensurable case these terms would not occur. The terms which involve the coefficients ρ, ν, and σ are not the result of the commensurability, and would therefore occur in the non-commensurable case as well.

Thus, if $\tilde{\theta} = \mu^{1/2}\theta$ were used as the slow variable for the non-commensurable case of the planar restricted three-body problem, it would be found that

$$\frac{da^{3/2}}{d\tilde{\theta}} = O(\mu^{3/2})$$

$$\frac{de}{d\tilde{\theta}} = O(\mu^{3/2})$$

$$\frac{d\omega}{d\tilde{\theta}} = \mu^{1/2}\left[\tfrac{1}{2}a\rho + O(e^2)\right] + O(\mu^{3/2})$$

$$\frac{d\tau}{d\tilde{\theta}} = \mu^{1/2}\left[\tfrac{1}{4}\nu + \left(\tfrac{1}{4}\sigma - \tfrac{3}{4}a^{5/2}\rho - \tfrac{3}{4}\nu\right)e^2 + O(e^3)\right] + O(\mu^{3/2})$$

This implies that $\tilde{\tilde{\theta}} = \mu\theta$ is the correct slow variable for the non-commensurable case.

A heuristic explanation of why the angle $(\omega - n\phi)$ will tend to oscillate about the value $0°$ will now be given, for the case $m=1$. This explanation is based on the crude approximation that the total effect, produced by the mass μ on the motion of the infinitesimal body during one complete orbit, will be qualitatively the same as the effect exerted near the point of closest approach to the perturbing body.

For the case $m=1$, the point of closest approach occurs once during every n revolutions of the infinitesimal body in its orbit. If $(\omega - n\phi) \cong 0°$, the point of closest approach occurs every nth revolution at approximately the time of pericenter passage.

Let $\theta = \theta_1$ designate an instant when the infinitesimal body is at pericenter, so that $\theta_1 = \omega(\tilde{\theta}_1, \mu) \equiv \omega_1$. (See Figure 4.) Let $\phi(\tilde{\theta}_1, \mu) \equiv \phi_1$ designate the value of ϕ at this same instant. Assume that $(\omega_1 - n\phi_1) = 0°$. After n additional complete revolutions in its orbit, the infinitesimal body will again be at pericenter, so that $\theta_2 = 2n\pi + \omega(\tilde{\theta}_2, \mu) \equiv 2n\pi + \omega_2$. However, ω_2 will differ slightly from ω_1, so that the infinitesimal body will have made slightly more or less than n complete revolutions about the large mass, measured in the non-rotating $X^* - Y^*$ system. Also, $\phi(\tilde{\theta}_2, \mu) \equiv \phi_2$ will differ slightly from ϕ_1. Since $a^{3/2} \cong \frac{n-1}{n}$, the mass μ will have made approximately $(n-1)$ complete revolutions about the large mass $(1-\mu)$.

If at the end of the above interval, the angle $(\frac{\omega_2}{n} - \phi_2)$ is small but $> 0°$, the infinitesimal body will be slightly displaced counterclockwise from the mass μ. The perturbing force at the point of closest approach will then act in a clockwise direction. This force will tend to decrease the counterclockwise angular velocity of the infinitesimal body. Since the mass μ moves at constant angular velocity, it will begin to "catch up" with the infinitesimal body during the next such interval $\theta_2 \leq \theta \leq \theta_3 = 4n\pi + \omega(\tilde{\theta}_3, \mu)$. Therefore, by the instant when $\theta = \theta_3$ the angle $(\frac{\omega}{n} - \phi)$ will have decreased somewhat, so that $(\frac{\omega_3}{n} - \phi_3) < (\frac{\omega_2}{n} - \phi_2)$.

Thus if $(\frac{\omega}{n} - \phi)$ is small but $> 0°$ a "restoring force" comes into play near the point of pericenter passage, and this restoring force tends to decrease the value of $(\frac{\omega}{n} - \phi)$. This situation will recur in the same qualitative manner at the end of each n revolutions, so long as $(\frac{\omega}{n} - \phi)$ is small and $> 0°$. Finally $(\frac{\omega}{n} - \phi)$ will become $< 0°$, and the restoring force will change sign. That is, when $(\frac{\omega}{n} - \phi)$ is small and $< 0°$ the restoring force will tend to increase the angle $(\frac{\omega}{n} - \phi)$ toward the value $0°$.

From the definition of $\phi(\tilde{\theta}, \mu)$ it follows that a change in $\frac{d\phi}{d\tilde{\theta}}$ requires a change in $\hat{a}^{3/2}(\tilde{\theta}, \mu)$. Hence oscillations of $(\omega - n\phi)$ about $0°$ will be accompanied by oscillations of $a^{3/2}(\tilde{\theta}, \mu)$ about some fixed value close to $\frac{n-1}{n}$.

2. Use of the Jacobi Integral

The Jacobi integral (7) will now be expressed in terms of the two variable expansions. Using eqs. (11) it may be shown that

$$(83)\quad \frac{1}{\left(s^2\frac{dt}{d\theta}\right)^2}\left[\frac{1}{2}\left(\frac{ds}{d\theta}\right)^2+\frac{1}{2}s^2-\left(s^2\frac{dt}{d\theta}\right)-s\left(s^2\frac{dt}{d\theta}\right)^2\right]=\left\{\frac{1}{2\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)^2}\left[\left(\frac{\partial s_0}{\partial\theta}\right)^2+s_0^2\right]-\frac{1}{\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)}-s_0\right\}$$

$$+\mu^{1/2}\frac{1}{\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)^2}\left\{\frac{\partial s_0}{\partial\theta}\frac{\partial s_0}{\partial\tilde\theta}+s_0^2\frac{\partial t_0}{\partial\tilde\theta}-\frac{\left(s_0^2\frac{\partial t_0}{\partial\tilde\theta}\right)}{\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)}\left[\left(\frac{\partial s_0}{\partial\theta}\right)^2+s_0^2\right]\right\}$$

$$+\mu\frac{1}{\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)^2}\left\{\begin{aligned}&\frac{\partial s_0}{\partial\theta}\frac{\partial s_1}{\partial\theta}+s_0s_1-\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)^2 s_1-2\frac{\left(s_0^2\frac{\partial t_0}{\partial\tilde\theta}\right)}{\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)}\frac{\partial s_0}{\partial\theta}\frac{\partial s_0}{\partial\tilde\theta}+\frac{3}{2}\frac{\left(s_0^2\frac{\partial t_0}{\partial\tilde\theta}\right)^2}{\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)^2}\left[\left(\frac{\partial s_0}{\partial\theta}\right)^2+s_0^2\right]\\&+\frac{\left[s_0^2\frac{\partial t_0}{\partial\theta}-\left(\frac{\partial s_0}{\partial\theta}\right)^2-s_0^2\right]}{\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)}\left(s_0^2\frac{\partial t_1}{\partial\theta}+2s_0s_1\frac{\partial t_0}{\partial\theta}\right)+\frac{1}{2}\left(\frac{\partial s_0}{\partial\tilde\theta}\right)^2-\frac{\left(s_0^2\frac{\partial t_0}{\partial\tilde\theta}\right)^2}{\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)}\end{aligned}\right\}$$

$$+\mu^{3/2}\left[\text{terms in }\frac{\partial s_0}{\partial\tilde\theta}\text{ and }\frac{\partial t_0}{\partial\tilde\theta}\right]\quad+\;O(\mu^2)$$

and that

$$(84)\quad \mu\left\{s+\frac{1}{s}\cos(\theta-t)-\frac{s}{\left[1+s^2-2s\cos(\theta-t)\right]^{1/2}}\right\}$$

$$=\mu\left\{s_0+\frac{1}{s_0}\cos(\theta-t_0)-\frac{s_0}{\left[1+s_0^2-2s_0\cos(\theta-t_0)\right]^{1/2}}\right\}+O(\mu^2)$$

The terms on the r.h.s. of eq. (83) which appear to be of $O(\mu^{1/2})$ are actually of $O(\mu)$, since $\frac{\partial s_0}{\partial\tilde\theta}$ and $\frac{\partial t_0}{\partial\tilde\theta}$ are of $O(\mu^{1/2})$. For this same reason, several terms involving $\frac{\partial s_0}{\partial\tilde\theta}$ which appear to be of $O(\mu)$ or of $O(\mu^{3/2})$ are actually of $O(\mu^2)$. It may be shown that

$$(85)\quad \frac{1}{2\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)^2}\left[\left(\frac{\partial s_0}{\partial\theta}\right)^2+s_0^2\right]-\frac{1}{\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)}-s_0=-\frac{1}{2a}-a^{1/2}(1-e^2)^{1/2}$$

The Jacobi integral may therefore be written as follows, in terms of the two variable expansions being used:

$$(86) \quad \left[\frac{-1}{2a} - a^{1/2}(1-e^2)^{1/2}\right] + \mu^{1/2}\frac{1}{\left(s_0^2\frac{\partial t_0}{\partial \theta}\right)^2}\left\{\frac{\partial s_0}{\partial \theta}\frac{\partial s_0}{\partial \tilde{\theta}} + \left(s_0^2\frac{\partial t_0}{\partial \tilde{\theta}}\right) - \frac{\left(s_0^2\frac{\partial t_0}{\partial \tilde{\theta}}\right)}{\left(s_0^2\frac{\partial t_0}{\partial \theta}\right)}\left[\left(\frac{\partial s_0}{\partial \theta}\right)^2 + s_0^2\right]\right\}$$

$$+\mu\frac{1}{\left(s_0^2\frac{\partial t_0}{\partial \theta}\right)^2}\left\{\begin{aligned}&\frac{\partial s_0}{\partial \theta}\frac{\partial s_1}{\partial \theta} + s_0 s_1 - \left(s_0^2\frac{\partial t_0}{\partial \theta}\right)^2 s_1 + \frac{\left[\left(s_0^2\frac{\partial t_0}{\partial \theta}\right) - \left(\frac{\partial s_0}{\partial \theta}\right)^2 - s_0^2\right]}{\left(s_0^2\frac{\partial t_0}{\partial \theta}\right)}\left(s_0^2\frac{\partial t_1}{\partial \theta} + 2s_0 s_1\frac{\partial t_0}{\partial \theta}\right)\\ &+\left(s_0^2\frac{\partial t_0}{\partial \theta}\right)^2 s_0 + \frac{1}{s_0}\left(s_0^2\frac{\partial t_0}{\partial \theta}\right)^2\cos(\theta - t_0) - \frac{s_0\left(s_0^2\frac{\partial t_0}{\partial \theta}\right)^2}{\left[1+s_0^2-2s_0\cos(\theta-t_0)\right]^{1/2}}\end{aligned}\right\}$$

$$+O(\mu^2) = C$$

where C is a constant which depends only on the initial conditions. It must be remembered that the terms in eq. (86) which involve $\frac{\partial s_0}{\partial \tilde{\theta}}$ and $\frac{\partial t_0}{\partial \tilde{\theta}}$ are of $O(\mu)$, rather than $O(\mu^{1/2})$.

By formal differentiation w.r.t. θ, followed by use of eqs. (57) and (58) to eliminate terms in $s_1, t_1, \frac{\partial s_1}{\partial \theta}, \frac{\partial t_1}{\partial \theta}$, etc., it may be shown that

$$(87) \quad \frac{\partial}{\partial \theta}\frac{1}{\left(s_0^2\frac{\partial t_0}{\partial \theta}\right)^2}\left\{\begin{aligned}&\frac{1}{\mu^{1/2}}\frac{\partial s_0}{\partial \theta}\frac{\partial s_0}{\partial \tilde{\theta}} + \frac{1}{\mu^{1/2}}\left(s_0^2\frac{\partial t_0}{\partial \tilde{\theta}}\right) - \frac{1}{\mu^{1/2}}\frac{\left(s_0^2\frac{\partial t_0}{\partial \tilde{\theta}}\right)}{\left(s_0^2\frac{\partial t_0}{\partial \theta}\right)}\left[\left(\frac{\partial s_0}{\partial \theta}\right)^2 + s_0^2\right] + \left(s_0^2\frac{\partial t_0}{\partial \theta}\right)^2 s_0\\ &+\frac{\partial s_0}{\partial \theta}\frac{\partial s_1}{\partial \theta} + s_0 s_1 - \left(s_0^2\frac{\partial t_0}{\partial \theta}\right)^2 s_1 + \frac{\left[s_0^2\frac{\partial t_0}{\partial \theta} - \left(\frac{\partial s_0}{\partial \theta}\right)^2 - s_0^2\right]}{\left(s_0^2\frac{\partial t_0}{\partial \theta}\right)}\left(s_0^2\frac{\partial t_1}{\partial \theta} + 2s_0 s_1\frac{\partial t_0}{\partial \theta}\right)\\ &+\frac{1}{s_0}\left(s_0^2\frac{\partial t_0}{\partial \theta}\right)^2\cos(\theta - t_0) - \frac{s_0\left(s_0^2\frac{\partial t_0}{\partial \theta}\right)^2}{\left[1+s_0^2-2s_0\cos(\theta-t_0)\right]^{1/2}}\end{aligned}\right\}$$

$$= \frac{1}{\mu^{1/2}}\frac{d}{d\tilde{\theta}}\left[\frac{1}{2a} + a^{1/2}(1-e^2)^{1/2}\right] + O(\mu)$$

The r.h.s. of eq. (87) is independent of θ, and depends at most on $\tilde{\theta}$. However, the quantity in braces on the l.h.s. does not contain any terms which are proportional to θ. Therefore the $\frac{\partial}{\partial \theta}$ derivative of this quantity cannot produce a term which is independent of θ. This implies that the r.h.s. must vanish; i.e. that

(88) $$\frac{d}{d\theta}\left[\frac{1}{2a}+a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}\right] = 0 + O(\mu^{\frac{3}{2}})$$

so that

(89) $$\frac{1}{2a}+a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}} = \text{constant} + O(\mu^{\frac{3}{2}})$$

Eq. (89) represents one of the four general integrals necessary to determine the behavior of the orbital elements. It is valid for all values of the integers $n > m > 0$.

By use of expansion (27) it may be shown that

$$\frac{1}{2a} = \frac{3n^{\frac{2}{3}}}{2(n-m)^{\frac{2}{3}}} - \frac{n\,a^{\frac{1}{2}}}{(n-m)} + O(\mu)$$

Eq. (89) then becomes

(90) $$a^{\frac{1}{2}}\left[\frac{n}{(n-m)} - (1-e^2)^{\frac{1}{2}}\right] = \text{constant} + O(\mu)$$

Eq. (90) may be used to express $a(\tilde{\theta}, \mu)$ in terms of $e^2(\tilde{\theta}, \mu)$ and the initial conditions.

3. Approximate Solution for $e(\tilde{\theta},\mu) \cong e_0$

Approximate solutions of eqs. (78)-(82) will now be investigated by neglecting the variation of e on the r.h.s. That is, the approximation

(91) $$e(\tilde{\theta},\mu) = e_0 + O(\mu^{\frac{1}{2}})\,; \qquad e_0 \text{ constant}$$

will be used on the r.h.s. of the equations. This approximation is not valid for extremely small e_0, since the variable part of e is then not negligible.

The coefficients $(\frac{1}{2} a \kappa_n)$, $(\frac{1}{2} a \gamma_n)$, etc., may be expanded in powers of $\mu^{1/2} \hat{a}^{3/2}$ about $a = (\frac{n-1}{n})^{2/3}$. Eqs. (78)-(82) then become, respectively,

$$(92) \quad \frac{d\hat{e}}{d\tilde{\theta}} = \begin{bmatrix} (\frac{1}{2} a K_n) + (a \gamma_n) e_0 \cos(\omega - n\phi) - (a \zeta_n) e_0^2 \cos^2(\omega - n\phi) \\ + (\frac{1}{2} a \delta_n - \frac{1}{4} a \eta_n + \frac{1}{4} a \zeta_n + \frac{2\alpha_n}{a^{3/2}} - \frac{1}{2} a K_n) e_0^2 \end{bmatrix} \sin(\omega - n\phi) + O(\mu^{1/2})$$

$$(93) \quad \frac{d\hat{a}^{3/2}}{d\tilde{\theta}} = \left[(3\alpha_n + \tfrac{3}{2} a^{5/2} K_n) e_0 + (-6\beta_n + 3a^{5/2} \gamma_n) e_0^2 \cos(\omega - n\phi) \right] \sin(\omega - n\phi) + O(\mu^{1/2})$$

$$(94) \quad \frac{d\hat{\omega}}{d\tilde{\theta}} = (\tfrac{1}{2} a \rho - \tfrac{1}{2} a \gamma_n) + \left[\frac{(\frac{1}{2} a K_n)}{e_0} + (\tfrac{1}{2} a \delta_n + \tfrac{1}{4} a \eta_n - \tfrac{1}{2} a K_n + \tfrac{3}{4} a \zeta_n) e_0 \right] \cos(\omega - n\phi)$$
$$+ (a \gamma_n) \cos^2(\omega - n\phi) - (a \zeta_n) e_0 \cos^3(\omega - n\phi) + O(e_0^2) + O(\mu^{1/2})$$

$$(95) \quad \frac{d\hat{\tau}}{d\tilde{\theta}} = \tfrac{1}{4} \nu + (\tfrac{1}{4} \lambda_n - \tfrac{3}{4} a^{5/2} K_n) e_0 \cos(\omega - n\phi) + (\tfrac{1}{4} S_n - \tfrac{3}{2} a^{5/2} \gamma_n) e_0^2 \cos^2(\omega - n\phi)$$
$$+ (\tfrac{1}{4} \sigma - \tfrac{3}{4} a^{5/2} \rho - \tfrac{3}{4} \nu - \tfrac{1}{8} S_n + \tfrac{3}{4} a^{5/2} \gamma_n) e_0^2 + O(\mu^{1/2})$$

$$(96) \quad \frac{d(\omega - n\phi)}{d\tilde{\theta}} = -n\hat{a}^{3/2} + \mu^{1/2} \left\{ \begin{array}{l} (\frac{1}{2} a \rho - \frac{1}{4} n \nu) + \frac{(\frac{1}{2} a K_n)}{e_0} \cos(\omega - n\phi) \\ + \left[\frac{1}{2} a \delta_n + \frac{a}{4} \eta_n - \frac{n}{4} \lambda_n + (\frac{3}{4} n a^{5/2} - \frac{a}{2}) K_n \right] e_0 \cos(\omega - n\phi) \\ + (\frac{1}{2} a \gamma_n) \cos 2(\omega - n\phi) - (\frac{1}{4} a \zeta_n) e_0 \cos 3(\omega - n\phi) + O(e_0^2) \end{array} \right\}$$
$$+ O(\mu)$$

The coefficients $(\frac{1}{2} a \kappa_n)$, $(a \gamma_n)$, etc., on the r.h.s. of eqs. (92)-96) depend only on n.

Since the angle $(\omega - n\phi)$ will be unbounded in many cases, it is necessary to include those terms of $O(\mu^{1/2})$ on the r.h.s. of eq. (96) which would contribute to a possible secular behavior of $(\omega - n\phi)$. Let nb designate the constant part of the $O(\mu^{1/2})$ terms in eq. (96). Then

(97) $$\frac{d(\omega-n\phi)}{d\tilde{\theta}} = -n\hat{a}^{3/2} + \mu^{1/2} n b + O(\mu^{1/2})$$

so that

(98) $$\frac{d\cos(\omega-n\phi)}{d\tilde{\theta}} = n(\hat{a}^{3/2} - \mu^{1/2} b)\sin(\omega-n\phi) \quad + O(\mu^{1/2})$$

Division of eq. (98) by eq. (93) yields the integral

(99) $$(\hat{a}^{3/2} - \mu^{1/2} b)^2 = S_1 + S_2 e_0 \cos(\omega-n\phi) + S_3 e_0^2 \cos^2(\omega-n\phi)$$

with

$$S_1 = \left[\hat{a}^{3/2}(\tilde{\theta}_0,\mu) - \mu^{1/2} b\right]^2 - S_2 e_0 \cos(\omega_0 - n\phi_0) - S_3 e_0^2 \cos^2(\omega_0 - n\phi_0)$$

$$S_2 = \frac{2}{n}\left(3\alpha_n + \frac{3}{2} a^{5/2} K_n\right)$$

$$S_3 = \frac{2}{n}\left(-3\beta_n + \frac{3}{2} a^{5/2} \gamma_n\right)$$

The value of $\cos(\omega\text{-}n\phi)$ must remain such that $(\hat{a}^{3/2} - \mu^{1/2} b)^2 \geq 0$.

By computing the numerical values of β_n and γ_n at $a = (\frac{n-1}{n})^{2/3}$, it may be shown that $S_3 < 0$ for the cases $n = 2, 3, 4$. It should be noted that the r.h.s. of eq. (99) would contain terms multiplied by e_0^3, e_0^4, --- if the corresponding terms in e^3, e^4, --- had been retained in eq. (79).

The behavior of $\cos(\omega\text{-}n\phi)$ as a function of $\tilde{\theta}$ will now be determined. From eq. (98),

$$\left[\frac{d\cos(\omega-n\phi)}{d\tilde{\theta}}\right]^2 = n^2(\hat{a}^{3/2} - \mu^{1/2} b)^2 \sin^2(\omega-n\phi)$$

Using eq. (99) and writing ξ in place of $\cos(\omega\text{-}n\phi)$, this becomes

(100) $$\left(\frac{d\xi}{d\tilde{\theta}}\right)^2 = n^2(1-\xi^2)(S_3 e_0^2 \xi^2 + S_2 e_0 \xi + S_1)$$

The r.h.s. may be factored as follows:

(101) $$m^2(1-\xi^2)(S_3 e_0^2 \xi^2 + S_2 e_0 \xi + S_1) = m^2 S_3 e_0^2 (1-\xi)(1+\xi)(\xi - P_1)(\xi - P_2)$$

where

(102) $$P_1 = \frac{-S_2 + \sqrt{S_2^2 - 4S_1S_3}}{2S_3 e_0} \quad ; \quad P_2 = \frac{-S_2 - \sqrt{S_2^2 - 4S_1S_3}}{2S_3 e_0}$$

Depending upon the initial conditions, P_1 and P_2 will be either both real-valued, or complex conjugates. Since $S_3 < 0$ it follows that $P_2 \geq P_1$ when P_1 and P_2 are real-valued. Also, $P_1 \leq \xi \leq P_2$ in order that $(\hat{a}^{3/2} - \mu^{1/2} b)$ be ≥ 0.

If the roots $P_1, P_2, 1, -1$, are all distinct, the value of ξ will not cross any of them, because this would make $\left(\frac{d\xi}{d\tilde{\theta}}\right)^2 < 0$. Thus $\cos(\omega - n\phi)$ will oscillate between two fixed limits. If P_1 and P_2 are complex conjugates, ξ will oscillate between the values 1 and -1, corresponding to a monotonic increase or decrease of $(\omega - n\phi)$.

To exhibit the explicit dependence of $\cos(\omega - n\phi)$ on $\tilde{\theta}$ for a typical case where $(\omega - n\phi)$ oscillates between two fixed limits, assume that the initial conditions $\hat{a}^{3/2}(\tilde{\theta}_0, \mu)$, e_0, and $(\omega_0 - n\phi_0)$ are such that $-1 < P_1 < \xi_0 < 1 < P_2$. Then $\cos(\omega - n\phi)$ will oscillate between the two values P_1 and 1. From eq. (97) it is seen that $\frac{d(\omega - n\phi)}{d\tilde{\theta}} = 0$ only when $(\hat{a}^{3/2} - \mu^{1/2} b) = 0$; that is, only when $\cos(\omega - n\phi) = P_1$. Therefore $(\omega - n\phi)$ will oscillate about $0°$ between the limits $\pm |\cos^{-1} P_1|$. This type of motion is known as libration. Keeping in mind that $S_3 < 0$, eq. (100) becomes

(103) $$\left(\frac{d\xi}{d\tilde{\theta}}\right)^2 = -n^2 S_3 e_0^2 (1-\xi)(1+\xi)(\xi-P_1)(P_2-\xi) \geq 0 \quad ; \quad \xi(\tilde{\theta}_0,\mu) = \xi_0$$

The solution of eq. (103) may be expressed in terms of elliptic functions.

For the sake of definiteness, assume that $\left[\hat{a}^{3/2}(\tilde{\theta}_0,\mu) - \mu^{1/2} b\right] < 0$ and $\cos^{-1} P_1 < (\omega_0 - n\phi_0) < 0°$.

Then $\frac{d(\omega-n\phi)}{d\tilde{\theta}}\Big|_{\tilde{\theta}=\tilde{\theta}_0} > 0$ and $\frac{d\xi}{d\tilde{\theta}}\Big|_{\tilde{\theta}=\tilde{\theta}_0} > 0$. After replacing ξ by $\cos(\omega - n\phi)$, the solution of eq. (103) becomes

(104) $$\cos(\omega-n\phi) = \frac{P_1 + \frac{1}{2}(1-P_1)\,sn^2\left[\frac{n e_0}{\sqrt{2}}(-S_3)^{1/2}(P_2-P_1)^{1/2}\right](\tilde{\theta}-\tilde{\theta}_1+\frac{1}{4}T)}{1 - \frac{1}{2}(1-P_1)\,sn^2\left[\frac{n e_0}{\sqrt{2}}(-S_3)^{1/2}(P_2-P_1)^{1/2}\right](\tilde{\theta}-\tilde{\theta}_1+\frac{1}{4}T)} \quad ; \tilde{\theta} > \tilde{\theta}_0$$

where

$$\tilde{\theta}_1 = \tilde{\theta}_0 + \frac{1}{n e_0 (-S_3)^{1/2}} \int_{\cos(\omega_0 - n\phi_0)}^{1} \frac{d\xi}{\sqrt{(\xi-1)(\xi+1)(\xi-P_1)(\xi-P_2)}}$$

and

(105) $$T = \frac{4}{n e_0 (-S_3)^{1/2}} \int_{P_1}^{1} \frac{d\xi}{\sqrt{(\xi-1)(\xi+1)(\xi-P_1)(\xi-P_2)}}$$

$$= \frac{4\sqrt{2}}{n e_0 (-S_3)^{1/2}(P_2-P_1)^{1/2}}\, K\left[\frac{(1-P_1)^{1/2}(1+P_2)^{1/2}}{\sqrt{2}\,(P_2-P_1)^{1/2}}\right]$$

The function sn is the Jacobian elliptic function, T is the oscillation period of the angle $(\omega - n\phi)$, and $K(k)$ is the complete elliptic integral of the first kind. When $\tilde{\theta} = \tilde{\theta}_0 + T$ the motion will begin to repeat itself. The quantity $(\hat{a}^{3/2} - \mu^{1/2} b)$ will oscillate between the values $\pm(S_1 + S_2 e_0 + S_3 e_0^2)^{1/2}$, attaining its maximum and minimum values at $(\omega - n\phi) = 0°$. Hence $a^{3/2}$ oscillates about the value $\frac{(n-1)}{n} + \mu b$ with the amplitude $\mu^{1/2}(S_1 + S_2 e_0 + S_3 e_0^2)^{1/2}$.

Since $\cos(\omega - n\phi)$ is periodic in $\tilde{\theta}$ with period $\frac{1}{2}T$, the constant b is given by

$$(106) \quad nb = \frac{2}{T} \int_{\tilde{\theta}_0}^{\tilde{\theta}_0 + \frac{1}{2}T} \left\{ \begin{aligned} &\left(\tfrac{1}{2}a\rho - \tfrac{1}{4}n\nu\right) + \frac{\left(\frac{1}{2}aK_n\right)}{e_0}\cos(\omega - n\phi) \\ &+ \left[\tfrac{1}{2}a\delta_n + \tfrac{1}{4}a\eta_n - \tfrac{n}{4}\lambda_n + \left(\tfrac{3}{4}na^{5/2} - \tfrac{1}{2}a\right)K_n\right] e_0 \cos(\omega - n\phi) \\ &+ \left(\tfrac{1}{2}a\gamma_n\right)\cos 2(\omega - n\phi) - \left(\tfrac{1}{4}a\zeta_n\right) e_0 \cos 3(\omega - n\phi) + O(e_0^2) \end{aligned} \right\} d\tilde{\theta}$$

The terms in $\cos(\omega - n\phi)$, $\cos 2(\omega - n\phi)$, and $\cos 3(\omega - n\phi)$ may be expressed as functions of $\tilde{\theta}$ by use of eq. (104). Since P_1, P_2, and T are independent of b correct to $O(\mu^{1/2})$, the value of b may be calculated from eq. (106), to within terms of $O(\mu^{1/2})$.

An analysis similar to the above may always be used to determine $\cos(\omega - n\phi)$ as an explicit function of $\tilde{\theta}$, when the roots P_1 and P_2 are real-valued. After this has been done, the expressions for $\hat{e}(\tilde{\theta}, \mu)$, $\hat{\omega}(\tilde{\theta}, \mu)$, and $\hat{\gamma}(\tilde{\theta}, \mu)$ can be obtained by integration of the known r.h.s. of eqs. (92), (94), and (95) w.r.t. $\tilde{\theta}$.

However, $\hat{a}^{3/2}$ can be obtained directly from eqs. (99) and (104). Care must be taken to choose the proper sign for $(\hat{a}^{3/2} - \mu^{1/2} b)$ when taking the square root of eq. (99). After $\hat{a}^{3/2}$ has been determined, the expression for $\hat{e}$ may be obtained by use of the integral (89). Using expansions (27) and (43) for $a^{3/2}$ and e, it may be shown that

$$\frac{1}{2a} + a^{1/2}(1-e^2)^{1/2} = \frac{n^{2/3}}{2(n-m)^{2/3}} + \frac{(n-m)^{1/3}}{n^{1/3}}(1-e_0^2)^{1/2} + \mu^{1/2}\left[\frac{-(n-m)^{1/3} e_0}{n^{1/3}(1-e_0^2)^{1/2}}\right]\hat{e}$$

$$+ \mu^{1/2}\left[\frac{n^{2/3}(1-e_0^2)^{1/2}}{3(n-m)^{2/3}} - \frac{n^{5/3}}{3(n-m)^{5/3}}\right]\hat{a}^{3/2} + O(\mu)$$

By eq. (89), this implies that

$$\hat{e} + \frac{m(1-e_0^2)^{1/2}}{3(n-m)e_0}\left[\frac{m}{(n-m)} - (1-e_0^2)^{1/2}\right]\hat{a}^{3/2} = \text{constant} + O(\mu^{1/2})$$

Specializing to the case m=1, and evaluating the constant of integration, this becomes

$$(107) \quad \hat{e} = \hat{e}(\tilde{\theta}_0,\mu) - \frac{m(1-e_0^2)^{1/2}}{3(n-m)e_0}\left[\frac{m}{(n-m)} - (1-e_0^2)^{1/2}\right]\left[\hat{a}^{3/2} - \hat{a}^{3/2}(\tilde{\theta}_0,\mu)\right] + O(\mu^{1/2})$$

Thus $\hat{e}$ remains bounded if $e_0 \neq 0$. The value of the eccentricity at any instant is then given by

$$e(\tilde{\theta},\mu) = e_0 + \mu^{1/2}\hat{e}(\tilde{\theta},\mu)$$

From eq. (104) it is seen that if the initial conditions are such that $P_1 = 1$, $\cos(\omega - n\phi)$ will have the constant value +1. This corresponds to the condition $(\omega - n\phi)$ = constant = 0°. From eq. (102), the condition $P_1 = 1$ implies that

$$S_3 e_0^2 + S_2 e_0 + S_1 = 0$$

Evaluating S_1 for the initial condition $(\omega - n\phi_0) = 0°$, this requires

$$\hat{a}^{3/2}(\tilde{\theta}_0,\mu) = \mu^{1/2} b$$

If P_1 is slightly less than 1, $(\omega - n\phi)$ will undergo infinitesimal oscillations with the period T, in accordance with eq. (104). The value of T is given by eq. (105) with $P_1 = 1$. Values of T calculated from eq. (105) are given below for several values of e_0, using the value $\mu = \frac{1}{1048}$, for the case n=2. It should be remembered that the present approximation (91) is not valid for $e_0 \to 0$.

e_0	Period
.065	612 years
.085	579
.105	576
.125	594
.145	644

The difference between the numerical values calculated from eq. (105) and those given by Schubart[4] (see Figure 5) is largely due to the neglect of terms in e_0^3, e_0^4, etc. from eq. (99). The agreement could be improved by inclusion of the higher powers of e_0 in the calculations, although a great deal of additional algebraic labor would be required even to determine the coefficient of e_0^3. Since the magnitudes of the numerical values of $\alpha_n, \beta_n, \gamma_n, \kappa_n$, etc. increase with n, the influence of the higher powers of e_0 is relatively greater for larger n.

Consider now the case in which P_1 and P_2 are complex conjugates. This implies that $(\hat{a}^{3/2} - \mu^{1/2} b)$ does not vanish, and is therefore of constant sign. The angle $(\omega - n\phi)$ will be a monotonic function of $\tilde{\theta}$, decreasing if $(\hat{a}^{3/2} - \mu^{1/2} b) > 0$ and increasing if $(\hat{a}^{3/2} - \mu^{1/2} b) < 0$. Eq. (100) becomes

$$(108) \qquad \left(\frac{d\xi}{d\tilde{\theta}}\right)^2 = -n^2 S_3 e_0^2 (1-\xi)(1+\xi)(\xi - P_1)(\xi - \bar{P}_1) \quad ; \quad \xi(\tilde{\theta}_0, \mu) = \xi_0$$

For the sake of definiteness assume that $\left[\hat{a}^{3/2}(\tilde{\theta}_0, \mu) - \mu^{1/2} b\right] < 0$ and that $\sin(\omega_0 - n\phi_0) > 0$. Then $\left.\frac{d(\omega - n\phi)}{d\tilde{\theta}}\right|_{\tilde{\theta} = \tilde{\theta}_0} > 0$ and $\left.\frac{d\xi}{d\tilde{\theta}}\right|_{\tilde{\theta} = \tilde{\theta}_0} < 0$. After replacing ξ by $\cos(\omega - n\phi)$ the solution of eq. (108) is

$$(109) \qquad \cos(\omega - n\phi) = \frac{(B-A) - (A+B)\,\mathrm{cn}\left[n e_0 (-S_3)^{1/2} (AB)^{1/2} (\tilde{\theta} - \tilde{\theta}_2)\right]}{(A+B) + (A-B)\,\mathrm{cn}\left[n e_0 (-S_3)^{1/2} (AB)^{1/2} (\tilde{\theta} - \tilde{\theta}_2)\right]} \quad ; \quad \tilde{\theta} > \tilde{\theta}_0$$

where

$$\tilde{\theta}_2 = \tilde{\theta}_0 + \frac{1}{n e_0 (-S_3)^{1/2}} \int_{-1}^{\xi_0} \frac{d\xi}{\sqrt{(1-\xi)(1+\xi)(\xi - P_1)(\xi - \bar{P}_1)}}$$

and

$$(110) \qquad T = \frac{2}{n e_0 (-S_3)^{1/2}} \int_{-1}^{1} \frac{d\xi}{\sqrt{(1-\xi)(1+\xi)(\xi - P_1)(\xi - \bar{P}_1)}}$$

and

$$A^2 = \left[1 - \tfrac{1}{2}(P_1 + \bar{P}_1)\right]^2 - \tfrac{1}{4}(P_1 - \bar{P}_1)^2$$

$$B^2 = \left[1 + \tfrac{1}{2}(P_1 + \bar{P}_1)\right]^2 - \tfrac{1}{4}(P_1 - \bar{P}_1)^2$$

The function cn is the Jacobian elliptic function, and T is the period of $(\omega - n\phi)$.

The constant b is easy to evaluate for this case. Since $(\omega - n\phi)$ varies from $(\omega_0 - n\phi_0)$ to $(\omega_0 - n\phi_0) + 2\pi$ during one period, the contribution to b from each of the terms in $\cos(\omega - n\phi)$, $\cos 2(\omega - n\phi)$, and $\cos 3(\omega - n\phi)$ vanishes. Therefore

(111) $$b = \tfrac{1}{2} a\rho - \tfrac{1}{4} n\nu + O(e_0^2)$$

The quantity $\hat{a}^{3/2}$ may be obtained as an explicit function of $\tilde{\theta}$ from eqs. (99) and (109), choosing the "-" sign when taking the square root of eq. (99). $(\hat{a}^{3/2} - \mu^{1/2} b)^2$ will oscillate between the values

$$\left[(\hat{a}^{3/2} - \mu^{1/2} b)^2\right]_{MAX} = \left[\hat{a}^{3/2}(\tilde{\theta}_0, \mu) - \mu^{1/2} b\right]^2 + S_2 e_0 \left[1 - \cos(\omega_0 - n\phi_0)\right]$$

$$+ S_3 e_0^2 \left[1 - \cos^2(\omega_0 - n\phi_0)\right] \qquad \text{at } (\omega - n\phi) = 0^\circ$$

and

$$\left[(\hat{a}^{3/2} - \mu^{1/2} b)^2\right]_{MIN} = \left[\hat{a}^{3/2}(\tilde{\theta}_0, \mu) - \mu^{1/2} b\right]^2 + S_2 e_0 \left[-1 - \cos(\omega_0 - n\phi_0)\right]$$

$$+ S_3 e_0^2 \left[1 - \cos^2(\omega_0 - n\phi_0)\right] \qquad \text{at } (\omega - n\phi) = 180^\circ$$

with the period T. The amplitude of $(\hat{a}^{3/2} - \mu^{1/2} b)^2$ is therefore

$$\left[\left(\hat{a}^{3/2}-\mu^{1/2}b\right)^2\right]_{MAX}-\left[\left(\hat{a}^{3/2}-\mu^{1/2}b\right)^2\right]_{MIN}=2S_2e_0$$

After $\hat{a}^{3/2}$ has been obtained as an explicit function of $\tilde{\theta}$, $\hat{e}(\tilde{\theta},\mu)$ can be obtained from eq. (107). Also, $\hat{\omega}(\tilde{\theta},\mu)$ and $\hat{\tau}(\tilde{\theta},\mu)$ can be expressed as the integrals of the r.h.s. of eqs. (94) and (95) w.r.t. $\tilde{\theta}$.

4. Approximate Solution for $e(\tilde{\theta},\mu)=\mu^{1/2}\hat{e}(\tilde{\theta},\mu)$

The solutions discussed in the previous section are not valid when $e_0=0$, because the variable part of e is then not negligible. Instead, eq. (43) becomes

(112) $$e(\tilde{\theta},\mu)=\mu^{1/2}\hat{e}(\tilde{\theta},\mu)$$

Eqs. (78)-(82) become, respectively,

(113) $$\frac{d\hat{e}}{d\tilde{\theta}}=\left(\tfrac{1}{2}aK_n\right)\sin(\omega-n\phi)+O(\mu^{1/2})$$

(114) $$\frac{d\hat{a}^{3/2}}{d\tilde{\theta}}=\mu^{1/2}\left(3\alpha_n+\tfrac{3}{2}a^{5/2}K_n\right)\hat{e}\sin(\omega-n\phi)+O(\mu)$$

(115) $$\frac{d\hat{\omega}}{d\tilde{\theta}}=\frac{1}{\mu^{1/2}}\left(\tfrac{1}{2}aK_n\right)\frac{\cos(\omega-n\phi)}{\hat{e}}+\left(\tfrac{1}{2}a\rho\right)+\left(\tfrac{1}{2}a\delta_n\right)\cos 2(\omega-n\phi)+\frac{\ell_1\hat{a}^{3/2}\cos(\omega-n\phi)}{\hat{e}}+O(\mu^{1/2})$$

(116) $$\frac{d\hat{\tau}}{d\tilde{\theta}}=\tfrac{1}{4}\nu+O(\mu^{1/2})$$

(117) $$\frac{d(\omega-n\phi)}{d\tilde{\theta}}=-n\hat{a}^{3/2}+\frac{\left(\tfrac{1}{2}aK_n\right)\cos(\omega-n\phi)}{\hat{e}}+\mu^{1/2}\left[\begin{array}{l}\tfrac{1}{2}a\rho-\tfrac{1}{4}n\nu+\left(\tfrac{1}{2}a\delta_n\right)\cos 2(\omega-n\phi)\\ +\dfrac{\ell_1\hat{a}^{3/2}\cos(\omega-n\phi)}{\hat{e}}\end{array}\right]+O(\mu)$$

where

$$\ell_1=\left.\frac{d\left(\tfrac{1}{2}aK_n\right)}{da^{3/2}}\right|_{a^{3/2}=\frac{n-1}{n}}$$

As in the previous section, the coefficients $(\tfrac{1}{2}a\kappa_n)$, $(3\alpha_n+\tfrac{3}{2}a^{5/2}\kappa_n)$, etc. have been expanded in powers of $\mu^{1/2}\hat{a}^{3/2}$ about $a=(\frac{n-1}{n})^{2/3}$, and therefore depend only on n.

Eqs. (113) and (114) have the integral

$$\hat{a}^{3/2} = \hat{a}^{3/2}(\tilde{e}_0, \mu) + \mu^{1/2} \frac{(3\alpha_m + \frac{3}{2} a^{5/2} K_m)}{(a K_m)} \left[\hat{e}^2 - \hat{e}^2(\tilde{e}_0, \mu) \right] \tag{118}$$

Hence the oscillations in $\hat{a}^{3/2}$ are $O(\mu^{1/2})$ for this case.

Since $(\omega - n\phi)$ will be unbounded in many cases, it is necessary to include those terms of $O(\mu^{1/2})$ on the r.h.s. of eq. (117) that would contribute to a possible secular behavior of $(\omega - n\phi)$. The remaining terms of $O(\mu^{1/2})$ will only produce bounded terms in $(\omega - n\phi)$. Let

$$m c = \left\{ \begin{array}{l} \frac{1}{2} a p - \frac{1}{4} m \nu - m \frac{(3\alpha_m + \frac{3}{2} a^{5/2} K_m)}{(a K_m)} \left[\hat{e}^2 - \hat{e}^2(\tilde{e}_0, \mu) \right] \\ + (\frac{1}{2} a \gamma_m) \cos 2(\omega - m\phi) + \ell_1 \hat{a}^{3/2}(\tilde{e}_0, \mu) \frac{\cos(\omega - m\phi)}{\hat{e}} \end{array} \right\}_{\text{CONSTANT PART}} \tag{119}$$

Eq. (117) then becomes

$$\frac{d(\omega - m\phi)}{d\tilde{\theta}} = \left[-m \hat{a}^{3/2}(\tilde{e}_0, \mu) + \mu^{1/2} m c \right] + (\tfrac{1}{2} a K_m) \frac{\cos(\omega - m\phi)}{\hat{e}} + O(\mu^{1/2}) \tag{120}$$

Eqs. (113) and (120) have the integral

$$\hat{e} \cos(\omega - m\phi) - R_1 \hat{e}^2 = R_2 \tag{121}$$

with

$$R_1 = \frac{m}{(a K_m)} \left[\hat{a}^{3/2}(\tilde{e}_0, \mu) - \mu^{1/2} c \right]$$

$$R_2 = \hat{e}(\tilde{e}_0, \mu) \cos(\omega_0 - m\phi_0) - R_1 \hat{e}^2(\tilde{e}_0, \mu)$$

Since $\cos^2(\omega - n\phi)$ must be ≤ 1, the value of $\hat{e}$ must always satisfy the condition

$$R_1^2 \hat{e}^4 + (2 R_1 R_2 - 1) \hat{e}^2 + R_2^2 \leq 0$$

Eq. (113) may now be written as

$$\left(\frac{d\hat{e}^2}{d\tilde{\theta}}\right)^2 = (aK_m)^2\left[-R_1^2\hat{e}^4 + (1-2R_1R_2)\hat{e}^2 - R_2^2\right] \geq 0 \tag{122}$$

The quantity on the r. h. s. may be factored as follows:

$$-R_1^2\hat{e}^4 + (1-2R_1R_2)\hat{e}^2 - R_2^2 = -R_1^2(\hat{e}^2 - Q_1)(\hat{e}^2 - Q_2) \geq 0$$

where

$$Q_1 = \frac{(1-2R_1R_2) + \sqrt{1-4R_1R_2}}{2R_1^2} \quad ; \quad Q_2 = \frac{(1-2R_1R_2) - \sqrt{1-4R_1R_2}}{2R_1^2} \tag{123}$$

Therefore

$$\left(\frac{d\hat{e}^2}{d\tilde{\theta}}\right)^2 = (aK_m)^2 R_1^2 (Q_1 - \hat{e}^2)(\hat{e}^2 - Q_2) \geq 0 \tag{124}$$

If Q_1 and Q_2 were complex conjugates, $\frac{d\hat{e}^2}{d\tilde{\theta}}$ would never vanish, and $\hat{e}^2$ would be an unbounded function of $\tilde{\theta}$. The present approximation (112) would not be valid, as $e(\tilde{\theta}, \mu)$ would become large. Therefore the approximate equations (113)-(117) will be valid only if Q_1, Q_2 are real-valued; i.e. only if

$$1 - 4R_1R_2 \geq 0 \tag{125}$$

For real-valued Q_1 and Q_2, it is seen that $Q_1 \geqslant Q_2$.

In order that $\left(\frac{d\hat{e}^2}{d\tilde{\theta}}\right)^2 \geqslant 0$ throughout the motion, it is necessary that

$$Q_2 \leq \hat{e}^2 \leq Q_1$$

Since the sign of $\frac{d\hat{e}^2}{d\tilde{\theta}}$ can change only when $\hat{e}^2 = Q_1$ or $\hat{e}^2 = Q_2$, the value of $\hat{e}^2$ will oscillate between these two limits. Correspondingly, the value of $\cos(\omega - n\phi)$ will oscillate between two limits. Using inequality (125), it may be shown that $Q_2 \geqslant 0$. In fact Q_2 vanishes, and $\hat{e}$ consequently passes through 0, only for the special case

$$\hat{e}(\tilde{\theta}_0, \mu) = \frac{\cos(\omega_0 - n\phi_0)}{R_1} = \frac{(aK_n)\cos(\omega_0 - n\phi_0)}{n\left[\hat{a}^{3/2}(\tilde{\theta}_0, \mu) - \mu^{1/2} c\right]}$$

For this case the maximum value of $\hat{e}^2$ is

$$Q_1 = \frac{1}{R_1^2} = \frac{(aK_n)^2}{n^2\left[\hat{a}^{3/2}(\tilde{\theta}_0, \mu) - \mu^{1/2} c\right]^2}$$

It is seen from eq. (123) that $Q_1 \to \infty$ when $\hat{a}^{3/2}(\tilde{\theta}_0, \mu) \to \mu^{1/2} c$. Therefore, $\hat{e}$ will remain of order unity only if $\hat{a}^{3/2}(\tilde{\theta}_0, \mu)$ is $O(\mu^{\circ})$. This is in agreement with the well-known fact that there are no periodic orbits of the first kind at commensurabilities with $m=1$.

By calculation of the numerical values of the Fourier coefficients A_n, A_{n+1}, and A_{n-1}, it may be shown that $\kappa_n < 0$ for $n=2, 3, 4$. This is presumably also true for $n > 4$.

The solution of eq. (124) is

$$\hat{e}^2 = \tfrac{1}{2}(Q_1 + Q_2) + \tfrac{1}{2}(Q_1 - Q_2)\cos\left[(aK_n)R_1\tilde{\theta} + C\right] \tag{126}$$

with

$$C = -(aK_n)R_1\tilde{\theta}_0 + \cos^{-1}\left\{\frac{2}{(Q_1 - Q_2)}\left[\hat{e}^2(\tilde{\theta}_0, \mu) - \tfrac{1}{2}(Q_1 + Q_2)\right]\right\}$$

The value of $\cos^{-1}\{\ \}$ should be chosen such that

$$\left.\frac{d\hat{e}^2}{d\tilde{\theta}}\right|_{\tilde{\theta}=\tilde{\theta}_0} < 0 \qquad \text{if } \sin(\omega_0 - n\phi_0) > 0$$

$$\left.\frac{d\hat{e}^2}{d\tilde{\theta}}\right|_{\tilde{\theta}=\tilde{\theta}_0} > 0 \qquad \text{if } \sin(\omega_0 - n\phi_0) < 0$$

The expression for $\hat{a}^{3/2}$ then follows from eq. (118). The period of the oscillations of $\hat{e}$ is

(127)
$$T = \frac{2\pi}{(aK_m)R_1} = \frac{2\pi}{m\left[\hat{a}^{3/2}(\tilde{\theta}_0,\mu) - \mu^{1/2}c\right]}$$

Since $\hat{e} \geq 0$, it follows from eq. (121) that

(128)
$$\cos(\omega - m\phi) = R_1\left\{\tfrac{1}{2}(Q_1+Q_2) + \tfrac{1}{2}(Q_1-Q_2)\cos\left[(aK_m)R_1\tilde{\theta}+C\right]\right\}^{1/2} + R_2\left\{\tfrac{1}{2}(Q_1+Q_2) + \tfrac{1}{2}(Q_1-Q_2)\cos\left[(aK_m)R_1\tilde{\theta}+C\right]\right\}^{-1/2}$$

Since $\hat{e}$ and $\cos(\omega - n\phi)$ are periodic functions of $\tilde{\theta}$ with period T,

(129)
$$mc = \frac{1}{T}\int_{\tilde{\theta}_0}^{\tilde{\theta}_0+T}\left\{\begin{array}{l}\frac{1}{2}a\rho - \frac{1}{4}m\nu + (\frac{1}{2}a\chi_m)\cos 2(\omega-m\phi) + \ell_1\hat{a}^{3/2}(\tilde{\theta}_0,\mu)\frac{\cos(\omega-m\phi)}{\hat{e}} \\ -m\frac{(3\alpha_m + \frac{3}{2}a^{5/2}K_m)}{(aK_m)}\left[\hat{e}^2 - \hat{e}^2(\tilde{\theta}_0,\mu)\right]\end{array}\right\}d\tilde{\theta}$$

The constants R_1, R_2, Q_1, Q_2, and T are independent of c, correct to $O(\mu^{1/2})$. After expressing the integrand in terms of $\tilde{\theta}$ by use of eqs. (126) and (128), c may be evaluated from this integral, correct to $O(\mu^{1/2})$.

Having determined $\hat{e}$ and $\cos(\omega - n\phi)$ as functions of $\tilde{\theta}$, the expressions for $\hat{\omega}(\tilde{\theta}, \mu)$ and $\hat{\tau}(\tilde{\theta}, \mu)$ may be obtained by integration of the r.h.s. of eqs. (115) and (116).

The angle $(\omega - n\phi)$ will attain a maximum or minimum only if its derivative vanishes. By eq. (120) this can only occur when

$$\hat{e} = \frac{(\frac{1}{2}aK_m)\cos(\omega-m\phi)}{m\left[\hat{a}^{3/2}(\tilde{\theta}_0,\mu) - \mu^{1/2}c\right]} = \frac{\cos(\omega-m\phi)}{2R_1} \quad ; \ \hat{e} \geq 0$$

which corresponds to

$$\cos^2(\omega - n\phi) = 4R_1R_2 \; ; \quad \hat{e}^2 = \frac{R_2}{R_1} \tag{130}$$

If $\frac{R_2}{R_1}$ lies outside the range $Q_2 \leqslant \hat{e}^2 \leqslant Q_1$, the derivative $\frac{d(\omega-n\phi)}{d\tilde{\theta}}$ will never vanish, so that $(\omega-n\phi)$ will increase or decrease monotonically. However, for cases in which $(\omega-n\phi)$ oscillates between two fixed limits, eq. (130) may be used to determine the amplitude of oscillation, as a function of the initial conditions.

It follows from eqs. (126) and (128) that $\hat{e}$ and $(\omega-n\phi)$ are constant, corresponding to infinitesimal oscillations, if $Q_1 = Q_2 = \frac{1}{4R_1^2}$. This corresponds to

$$\hat{e}(\tilde{\theta},\mu) = \hat{e}(\tilde{\theta}_0,\mu) = \begin{cases} \dfrac{1}{2R_1} = \dfrac{aK_n}{2n\left[\hat{a}^{3/2}(\tilde{\theta}_0,\mu) - \mu^{1/2}c\right]} & ; R_1 > 0 \\[2ex] \dfrac{-1}{2R_1} = \dfrac{-aK_n}{2n\left[\hat{a}^{3/2}(\tilde{\theta}_0,\mu) - \mu^{1/2}c\right]} & ; R_1 < 0 \end{cases}$$

The choice of the sign follows from the fact that $\hat{e}(\tilde{\theta}_0,\mu) \geqslant 0$. Since $K_n < 0$, eq. (121) yields

$$\cos(\omega - n\phi) = \cos(\omega_0 - n\phi_0) = \begin{cases} +1 & ; \hat{a}^{3/2}(\tilde{\theta}_0,\mu) < \mu^{1/2}c \\ -1 & ; \hat{a}^{3/2}(\tilde{\theta}_0,\mu) > \mu^{1/2}c \end{cases}$$

It follows that infinitesimal oscillations with $e = \mu^{1/2}\hat{e}$ are possible only about $(\omega-n\phi) = 0°$ when $\hat{a}^{3/2}(\tilde{\theta}_0,\mu) < \mu^{1/2}c$ and only about $(\omega-n\phi) = 180°$ when $\hat{a}^{3/2}(\tilde{\theta}_0,\mu) > \mu^{1/2}c$.

In the derivation of eqs. (67), (70), (71), and (75) it was assumed that the angular rate $\frac{d\omega}{d\tilde{\theta}}$ of the pericenter angle $\omega(\tilde{\theta},\mu)$ is small in comparison to the angular rate $\frac{d\theta}{d\theta} = 1$ of the infinitesimal body. However, this assumption would be increasingly violated in the case of infinitesimal oscillations of $(\omega - n\phi)$ with $e = \mu^{1/2}\hat{e}$ if it were attempted to make calculations for $\hat{e}(\tilde{\theta}_0, \mu)$ very small. Accordingly the accuracy of the periods calculated from eq. (127) is not expected to be very good for extremely small values of $\hat{e}(\tilde{\theta}_0, \mu)$.

5. Comparison of Results with Calculations by Schubart

Extensive numerical calculations have been carried out by Schubart[4] for the nearly commensurable case of the restricted three-body problem. The following variables (written in terms of the present notation) are used in his work:

$$U = \frac{n}{m} a^{1/2}\left[1 - \frac{(n-m)}{n}(1-e^2)^{1/2}\right]$$

$$S = a^{1/2}\left[1 - (1-e^2)^{1/2}\right]$$

$$\lambda = \frac{m}{n}\theta - \phi$$

$$\sigma = -\omega + \frac{n}{m}\phi$$

The disturbing function or Hamiltonian is

$$F = \frac{1}{2a} + (1+m_1)^{1/2} a^{1/2}(1-e^2)^{1/2} + m_1\left\{-\frac{1}{\Delta}\cos(\theta - t) + \frac{\Delta}{\left[1+\Delta^2 - 2\Delta\cos(\theta-t)\right]^{1/2}}\right\}$$

where m_1 is the mass of the perturbing body and $(1+m_1)^{1/2}$ is its mean motion.

The short-period terms involving λ are "smoothed out" from F by a numerical averaging process, and the resulting quantity is denoted by $\bar{F}$. Only long-period terms are retained in $\bar{F}$. The following two integrals are then valid for the long-period effects:

$$U = constant$$
$$\bar{F} = constant$$

The first of these is equivalent to eq. (90), to within terms of $O(\mu)$.

Following a suggestion by Poincaré[2], the variables

$$x = (2S)^{1/2}\cos\sigma$$
$$y = (2S)^{1/2}\sin\sigma$$

are introduced and the results of the calculations are graphically presented in the form of curves $\bar{F}(x, y, U)$ = constant, drawn in the x-y plane for a fixed value of U. These curves bring out the nature of the behavior of $(\omega - \frac{n}{m}\phi)$ and the eccentricity e, for a wide range of initial conditions, and are therefore useful in obtaining an intuitive understanding of the motion. Since the numerical averaging process used to convert F to $\bar{F}$ was carried out on an electronic computer, without the necessity of expanding F in powers of e, the calculations are valid for orbits of all eccentricities $0 \leq e < 1$.

Although they clarify the qualitative nature of the motion, the curves $\bar{F}$ = constant do not provide information about its time dependence. However, reference[4] gives the period of infinitesimal librations of $(\omega - n\phi)$ about $0°$ as a function of the eccentricity, calculated by means of a numerical variational theory. This was carried out for the cases $n = 2$, $m = 1$ and $n = 3$, $m = 1$, using the numerical value $m_1 \cong {}^1/1047$ to correspond to the sun-Jupiter-asteroid problem. These values are plotted in Figure 5. The period of infinitesimal

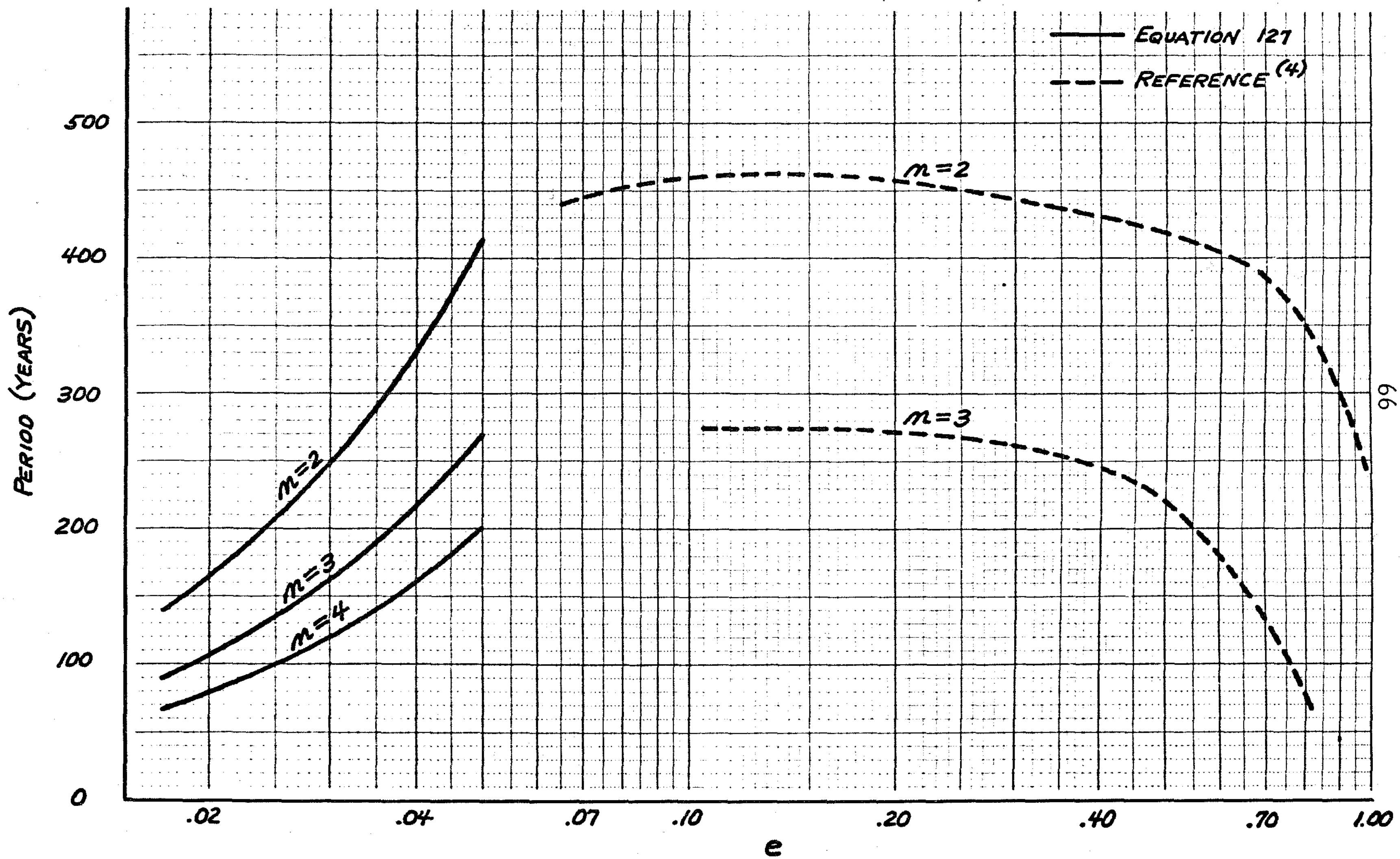

Figure 5. Period of Infinitesimal Librations

librations for very small eccentricities is not given in reference[4].

However, this can be calculated from eq. (127). The results are shown in Figure 5. The accuracy of the results decreases for large values of $\hat{e}(\tilde{\theta}_0, \mu)$ because the approximation (112) becomes unrealistic. Schubart's results for $n = 2$ indicate that the period decreases for very small e, and the values calculated from eq. (127) clearly show this. Although the curves for $n = 3$ do not fit together as well as do those for $n = 2$, a marked decrease in period is indicated for small eccentricities.

In order to exhibit a typical case of finite-amplitude librations of $(\omega - n\phi)$ about $0°$, the libration amplitude has been calculated from eq. (102) for the case $n = 2$ and using $\mu = \frac{1}{1048}$. In eq. (102) the value of e_0 is taken equal to $e_{initial} = e(\tilde{\theta}_0, \mu)$. To facilitate comparison of the present results with those given in reference[4], the libration is assumed to start from the initial condition $(\omega_0 - n\phi_0) = 0°$, and the initial condition $\hat{a}^{3/2}(\tilde{\theta}_0, \mu)$ has been adjusted for each value of $e_{initial}$ in such a way that the relation $U = a^{1/2}[2-(1-e^2)^{1/2}] = .8000$ is maintained.

The comparison of results is shown in Figure 6, and the agreement is good for large amplitudes of $(\omega - 2\phi)$. For the larger values of $e_{initial}$, the neglect of the higher powers of e_0 causes eq. (102) to yield amplitudes which are somewhat too small. The agreement could be improved by retention of all terms in e^3 throughout the calculations.

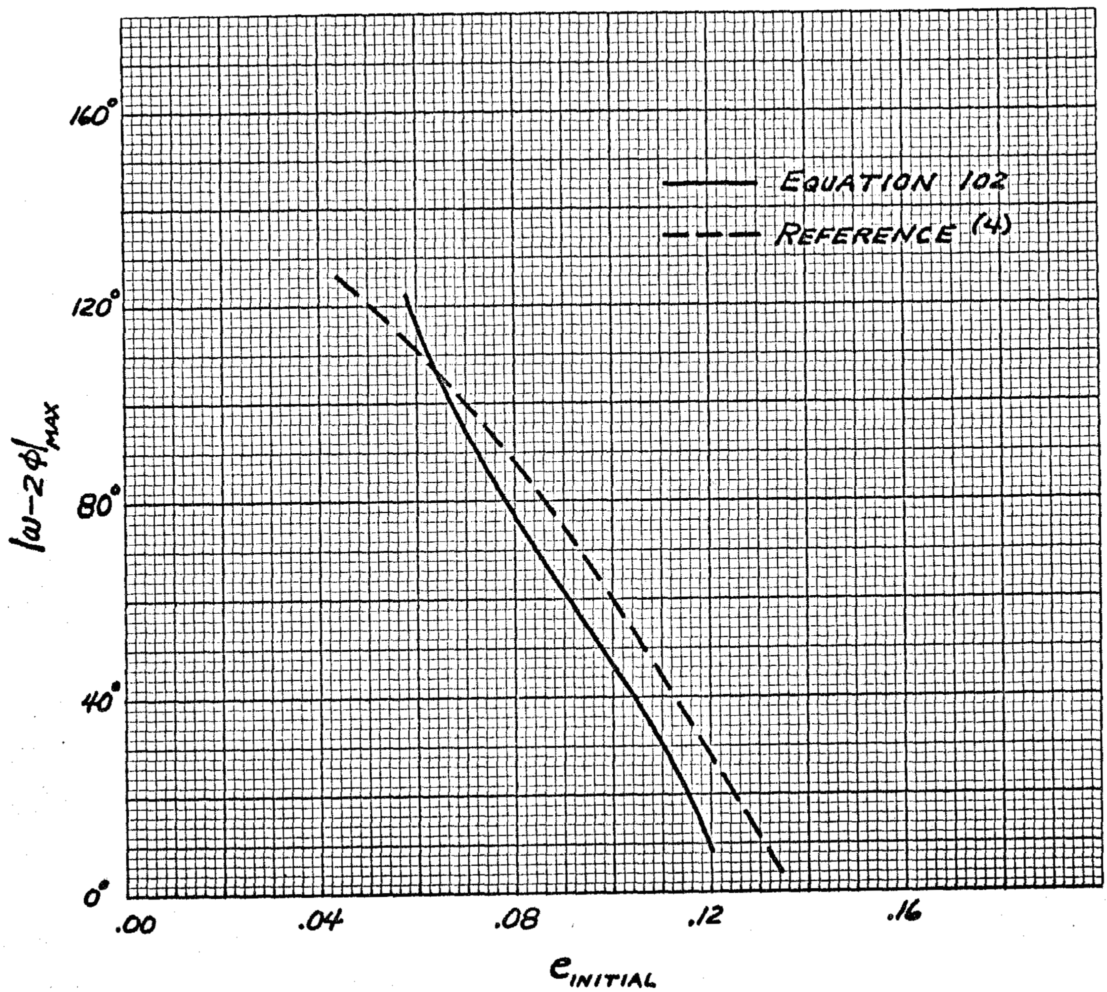

Figure 6. Libration Amplitude of $(\omega - 2\phi)$.
$n = 2$; $a^{1/2}[2-(1-e^2)^{1/2}] = .8000$

APPENDIX 1. Numerical Values of Commensurabilities

	$\frac{(n-1)}{n}$	$\frac{(n-2)}{n}$	$\frac{(n-3)}{n}$	$\frac{(n-4)}{n}$
n = 2	.500			
n = 3	.667	.333		
n = 4	.750	.500	.250	
n = 5	.800	.600	.400	.200
n = 6	.833	.667	.500	.333
n = 7	.857	.714	.571	.429
n = 8	.875	.750	.625	.500
n = 9	.889	.778	.667	.556
n = 10	.900	.800	.700	.600
n = ∞	1.000	1.000	1.000	1.000

APPENDIX 2. Expansions in Powers of e

$$\frac{\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)^3\sin(\theta-t_0)}{\left[1+s_0^2-2s_0\cos(\theta-t_0)\right]^{3/2}} = \frac{a^{3/2}\sin\left(\frac{m}{n}\theta-\phi\right)}{\left[1+a^2-2a\cos\left(\frac{m}{n}\theta-\phi\right)\right]^{3/2}}$$

$$+e\left\{\begin{aligned}&2a^3\frac{\sin(\theta-\omega)\cos\left(\frac{m}{n}\theta-\phi\right)}{\left[1+a^2-2a\cos\left(\frac{m}{n}\theta-\phi\right)\right]^{3/2}}-3a^4\frac{\sin(\theta-\omega)}{[\qquad]^{5/2}}-3a^{3/2}\frac{\cos(\theta-\omega)\sin\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{5/2}}\\&+\frac{3}{2}a^{5/2}\frac{\cos(\theta-\omega)\sin2\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{5/2}}+3a^4\frac{\sin(\theta-\omega)\cos2\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{5/2}}\end{aligned}\right\}$$

$$+e^2\left\{\begin{aligned}&\frac{a^{3/2}\left(\frac{3}{2}-a^3\right)\sin\left(\frac{m}{n}\theta-\phi\right)}{\left[1+a^2-2a\cos\left(\frac{m}{n}\theta-\phi\right)\right]^{3/2}}+a^{9/2}\frac{\cos2(\theta-\omega)\sin\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{3/2}}-\frac{3}{4}a^3\frac{\sin2(\theta-\omega)\cos\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{3/2}}\\&+\frac{9}{8}a^4\frac{\sin2(\theta-\omega)}{[\qquad]^{5/2}}-\frac{15}{4}a^{3/2}\frac{\sin\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{5/2}}-\frac{3}{4}a^{3/2}\frac{\cos2(\theta-\omega)\sin\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{5/2}}\\&-3a^3\frac{\sin2(\theta-\omega)\cos\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{5/2}}+\frac{3}{2}a^{5/2}(1-3a^3)\frac{\sin2\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{5/2}}\\&+\frac{9}{2}a^{11/2}\frac{\cos2(\theta-\omega)\sin2\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{5/2}}+\frac{15}{8}a^4\frac{\sin2(\theta-\omega)\cos2\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{5/2}}\\&+\frac{15}{2}a^4\frac{\sin2(\theta-\omega)}{[\qquad]^{7/2}}-\frac{15}{4}a^5\frac{\sin2(\theta-\omega)\cos\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{7/2}}-\frac{15}{4}a^{5/2}\frac{\sin2\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{7/2}}\\&+\frac{15}{16}a^{3/2}(4+a^2-12a^5)\frac{\cos2(\theta-\omega)\sin\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{7/2}}+\frac{15}{16}a^{3/2}(4+a^2+12a^5)\frac{\sin\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{7/2}}\\&-\frac{15}{4}a^{5/2}\frac{\cos2(\theta-\omega)\sin2\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{7/2}}-\frac{15}{2}a^4\frac{\sin2(\theta-\omega)\cos2\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{7/2}}\\&+\frac{15}{16}a^{7/2}(1-4a^3)\frac{\sin3\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{7/2}}+\frac{15}{16}a^{7/2}(1+4a^3)\frac{\cos2(\theta-\omega)\sin3\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{7/2}}\\&+\frac{15}{4}a^5\frac{\sin2(\theta-\omega)\cos3\left(\frac{m}{n}\theta-\phi\right)}{[\qquad]^{7/2}}\end{aligned}\right\}$$

$+ \left\{\text{similar terms in } e^3, e^4, \text{ - - -}\right\}$

$$\left(s_0^2\frac{\partial t_0}{\partial\theta}\right)^2 \frac{\left[1-s_0\cos(\theta-t_0)+\frac{\partial s_0}{\partial\theta}\sin(\theta-t_0)\right]}{\left[1+s_0^2-2s_0\cos(\theta-t_0)\right]^{3/2}} = \frac{a^2-a\cos(\frac{m}{n}\theta-\phi)}{\left[1+a^2-2a\cos(\frac{m}{n}\theta-\phi)\right]^{3/2}}$$

$$+e\left\{\begin{aligned}
&a(2a^{3/2}-1)\frac{\sin(\theta-\omega)\sin(\frac{m}{n}\theta-\phi)}{\left[1+a^2-2a\cos(\frac{m}{n}\theta-\phi)\right]^{3/2}} - a\frac{\cos(\theta-\omega)\cos(\frac{m}{n}\theta-\phi)}{[\qquad]^{3/2}}\\
&-6a^{9/2}\frac{\sin(\theta-\omega)\sin(\frac{m}{n}\theta-\phi)}{[\qquad]^{5/2}} + 3a(1+a^2)\frac{\cos(\theta-\omega)\cos(\frac{m}{n}\theta-\phi)}{[\qquad]^{5/2}}\\
&-\frac{9}{2}a^2\frac{\cos(\theta-\omega)}{[\qquad]^{5/2}} + 3a^{7/2}\frac{\sin(\theta-\omega)\sin 2(\frac{m}{n}\theta-\phi)}{[\qquad]^{5/2}} - \frac{3}{2}a^2\frac{\cos(\theta-\omega)\cos 2(\frac{m}{n}\theta-\phi)}{[\qquad]^{5/2}}
\end{aligned}\right\}$$

$$+e^2\left\{\begin{aligned}
&\frac{a^2(1-a^{1/2})}{\left[1+a^2-2a\cos(\frac{m}{n}\theta-\phi)\right]^{3/2}} + a(a^3-2)\frac{\cos(\frac{m}{n}\theta-\phi)}{[\qquad]^{3/2}} + \frac{\frac{3}{2}a^2(a^3-4)}{[\qquad]^{5/2}}\\
&+a^{5/2}(1-a^{3/2})\frac{\cos 2(\theta-\omega)\cos(\frac{m}{n}\theta-\phi)}{[\qquad]^{3/2}} + \frac{1}{4}a^{5/2}\frac{\sin 2(\theta-\omega)\sin(\frac{m}{n}\theta-\phi)}{[\qquad]^{3/2}}\\
&-\frac{3}{2}a^2(1+a^{3/2}-a^3)\frac{\cos 2(\theta-\omega)}{[\qquad]^{5/2}} + \frac{3}{4}a(7+4a^2-4a^5)\frac{\cos(\frac{m}{n}\theta-\phi)}{[\qquad]^{5/2}}\\
&+\frac{3}{4}a(2-4a^{3/2}-a^{7/2})\frac{\sin 2(\theta-\omega)\sin(\frac{m}{n}\theta-\phi)}{[\qquad]^{5/2}} + \frac{3}{4}a(3+4a^5)\frac{\cos 2(\theta-\omega)\cos(\frac{m}{n}\theta-\phi)}{[\qquad]^{5/2}}\\
&+\frac{3}{4}a^2(2a^3-3)\frac{\cos 2(\frac{m}{n}\theta-\phi)}{[\qquad]^{5/2}} + \frac{3}{8}a^2(9a^{3/2}-2)\frac{\sin 2(\theta-\omega)\sin 2(\frac{m}{n}\theta-\phi)}{[\qquad]^{5/2}}\\
&-\frac{3}{4}a^2(1+6a^3-2a^{3/2})\frac{\cos 2(\theta-\omega)\cos 2(\frac{m}{n}\theta-\phi)}{[\qquad]^{5/2}} + \frac{\frac{15}{8}a^2(4+a^2+4a^5)}{[\qquad]^{7/2}}\\
&+\frac{15}{8}a^2(4+a^2-4a^5)\frac{\cos 2(\theta-\omega)}{[\qquad]^{7/2}} - \frac{15}{16}a(4+11a^2+4a^5)\frac{\cos(\frac{m}{n}\theta-\phi)}{[\qquad]^{7/2}}\\
&+\frac{75}{4}a^{9/2}\frac{\sin 2(\theta-\omega)\sin(\frac{m}{n}\theta-\phi)}{[\qquad]^{7/2}} - \frac{15}{16}a(4+11a^2-4a^5)\frac{\cos 2(\theta-\omega)\cos(\frac{m}{n}\theta-\phi)}{[\qquad]^{7/2}}\\
&+\frac{15}{8}a^2(2+a^2-4a^5)\frac{\cos 2(\frac{m}{n}\theta-\phi)}{[\qquad]^{7/2}} - \frac{15}{2}a^{7/2}(1+a^2)\frac{\sin 2(\theta-\omega)\sin 2(\frac{m}{n}\theta-\phi)}{[\qquad]^{7/2}}\\
&+\frac{15}{8}a^2(2+a^2+4a^5)\frac{\cos 2(\theta-\omega)\cos 2(\frac{m}{n}\theta-\phi)}{[\qquad]^{7/2}} + \frac{15}{16}a^3(4a^3-1)\frac{\cos 3(\frac{m}{n}\theta-\phi)}{[\qquad]^{7/2}}\\
&+\frac{15}{4}a^{9/2}\frac{\sin 2(\theta-\omega)\sin 3(\frac{m}{n}\theta-\phi)}{[\qquad]^{7/2}} - \frac{15}{16}a^3(1+4a^3)\frac{\cos 2(\theta-\omega)\cos 3(\frac{m}{n}\theta-\phi)}{[\qquad]^{7/2}}
\end{aligned}\right\}$$

$+\left\{\text{similar terms in } e^3, e^4, \text{ - - -}\right\}$

APPENDIX 3. Expressions for Coefficients

$$\alpha_n(a) = \tfrac{1}{2}a^5 A_{n+1} + \tfrac{1}{2}a^5 A_{n-1} - \tfrac{3}{2}a^6 B_n - \tfrac{3}{4}a^{7/2} B_{n+1} + \tfrac{3}{4}a^{7/2} B_{n-1} + \left(\tfrac{3}{8}a^{9/2} + \tfrac{3}{4}a^6\right) B_{n+2}$$
$$+ \left(-\tfrac{3}{8}a^{9/2} + \tfrac{3}{4}a^6\right) B_{n-2}$$

$$\beta_n(a) = \left(\tfrac{3}{16}a^5 - \tfrac{1}{4}a^{13/2}\right) A_{2n+1} + \left(\tfrac{3}{16}a^5 + \tfrac{1}{4}a^{13/2}\right) A_{2n-1} - \tfrac{9}{16}a^6 B_{2n} + \left(\tfrac{3}{16}a^{7/2} + \tfrac{3}{4}a^5\right) B_{2n+1}$$
$$+ \left(-\tfrac{3}{16}a^{7/2} + \tfrac{3}{4}a^5\right) B_{2n-1} + \left(-\tfrac{15}{32}a^6 - \tfrac{9}{8}a^{15/2}\right) B_{2n+2} + \left(-\tfrac{15}{32}a^6 + \tfrac{9}{8}a^{15/2}\right) B_{2n-2}$$
$$- \tfrac{15}{4}a^6 C_{2n} + \left(-\tfrac{15}{16}a^{7/2} - \tfrac{15}{64}a^{11/2} + \tfrac{15}{16}a^7 + \tfrac{45}{16}a^{17/2}\right) C_{2n+1} + \left(\tfrac{15}{16}a^{9/2} + \tfrac{15}{8}a^6\right) C_{2n+2}$$
$$+ \left(\tfrac{15}{16}a^{7/2} + \tfrac{15}{64}a^{11/2} + \tfrac{15}{16}a^7 - \tfrac{45}{16}a^{17/2}\right) C_{2n-1} + \left(-\tfrac{15}{16}a^{9/2} + \tfrac{15}{8}a^6\right) C_{2n-2}$$
$$+ \left(-\tfrac{15}{64}a^{11/2} - \tfrac{15}{16}a^7 - \tfrac{15}{16}a^{17/2}\right) C_{2n+3} + \left(\tfrac{15}{64}a^{11/2} - \tfrac{15}{16}a^7 + \tfrac{15}{16}a^{17/2}\right) C_{2n-3}$$

$$K_n(a) = a^2 A_n + \tfrac{1}{2}a A_{n+1} - \tfrac{3}{2}a A_{n-1}$$

$$\gamma_n(a) = \tfrac{1}{2}a^{5/2} A_{2n+1} + \left(\tfrac{3}{2}a^{5/2} - \tfrac{1}{2}a\right) A_{2n-1} + \left(-3a^{7/2} - \tfrac{9}{4}a^2\right) B_{2n} + \left(-\tfrac{3}{4}a + \tfrac{3}{4}a^3 + \tfrac{3}{2}a^{9/2}\right) B_{2n+1}$$
$$+ \left(\tfrac{9}{4}a + \tfrac{3}{4}a^3 - \tfrac{3}{2}a^{9/2}\right) B_{2n-1} + \left(\tfrac{3}{4}a^{7/2} + \tfrac{3}{8}a^2\right) B_{2n+2} + \left(-\tfrac{9}{8}a^2 + \tfrac{9}{4}a^{7/2}\right) B_{2n-2}$$

$$\rho(a) = \left(-\tfrac{1}{2}a - 2a^{5/2}\right) A_1 + \left(-\tfrac{9}{4}a^2 + 3a^{7/2}\right) B_0 + \left(\tfrac{3}{2}a + \tfrac{3}{2}a^3\right) B_1 + \left(-\tfrac{3}{4}a^2 - 3a^{7/2}\right) B_2$$

$$\delta_n(a) = \left(a^2 - a^{5/2}\right) A_n + \left(a - \tfrac{1}{2}a^4\right) A_{n+1} + \left(-3a + \tfrac{3}{2}a^4\right) A_{n-1} + \left(-6a^2 + \tfrac{3}{2}a^5\right) B_n$$
$$+ \left(-\tfrac{9}{8}a + \tfrac{3}{2}a^3 - \tfrac{3}{2}a^6\right) B_{n+1} + \left(\tfrac{51}{8}a + \tfrac{3}{2}a^3 - \tfrac{3}{2}a^6\right) B_{n-1} + \left(\tfrac{3}{8}a^2 - \tfrac{15}{4}a^5\right) B_{n+2}$$
$$+ \left(-\tfrac{21}{8}a^2 + \tfrac{21}{4}a^5\right) B_{n-2} + \left(\tfrac{15}{2}a^2 + \tfrac{15}{8}a^4 + \tfrac{15}{2}a^7\right) C_n + \left(\tfrac{15}{8}a + \tfrac{75}{8}a^6 - \tfrac{135}{32}a^3\right) C_{n+1}$$
$$+ \left(-\tfrac{45}{8}a - \tfrac{195}{32}a^3 - \tfrac{105}{8}a^6\right) C_{n-1} + \left(-\tfrac{15}{8}a^2 + \tfrac{15}{16}a^4 - \tfrac{15}{4}a^7\right) C_{n+2}$$
$$+ \left(\tfrac{45}{8}a^2 + \tfrac{15}{16}a^4 - \tfrac{15}{4}a^7\right) C_{n-2} + \left(\tfrac{15}{32}a^3 - \tfrac{15}{8}a^6\right) C_{n+3} + \left(-\tfrac{45}{32}a^3 + \tfrac{45}{8}a^6\right) C_{n-3}$$
$$+ \left(-\tfrac{46}{32}a^3 + \tfrac{45}{8}a^6\right) C_{n+3}$$

$$\nu(a) = 2a^{3/2} - a^{9/2}A_0 + a^{7/2}A_1$$

$$\begin{aligned}
\eta_n(a) = {} & \left(\tfrac{9}{8}a^{5/2} - \tfrac{3}{2}a^4\right)A_{n+1} + \left(\tfrac{11}{8}a^{5/2} + \tfrac{1}{2}a^4\right)A_{n-1} + \left(-\tfrac{3}{2}a^2 - \tfrac{15}{4}a^{7/2} + \tfrac{3}{2}a^5\right)B_n \\
& + \left(\tfrac{9}{8}a + \tfrac{9}{2}a^{5/2} + \tfrac{3}{8}a^{9/2} + \tfrac{3}{2}a^6\right)B_{n+1} + \left(\tfrac{9}{8}a + \tfrac{3}{2}a^{5/2} - \tfrac{3}{8}a^{9/2} + \tfrac{3}{2}a^6\right)B_{n-1} \\
& + \left(-\tfrac{45}{16}a^{7/2} - \tfrac{27}{4}a^5\right)B_{n+2} + \left(-\tfrac{3}{4}a^2 + \tfrac{9}{16}a^{7/2} + \tfrac{9}{4}a^5\right)B_{n-2} + \left(\tfrac{15}{2}a^2 - 15a^{7/2} + \tfrac{15}{8}a^4 - \tfrac{15}{2}a^7\right)C_n \\
& + \left(-\tfrac{45}{8}a - \tfrac{195}{32}a^3 - \tfrac{45}{8}a^{9/2} + \tfrac{105}{8}a^6\right)C_{n+1} + \left(\tfrac{15}{8}a - \tfrac{135}{32}a^3 + \tfrac{105}{8}a^{9/2} - \tfrac{75}{8}a^6\right)C_{n-1} \\
& + \left(\tfrac{45}{8}a^2 + \tfrac{45}{4}a^{7/2} + \tfrac{15}{16}a^4 + \tfrac{15}{4}a^{11/2} + \tfrac{15}{4}a^7\right)C_{n+2} + \left(-\tfrac{15}{8}a^2 + \tfrac{15}{4}a^{7/2} + \tfrac{15}{16}a^4 - \tfrac{15}{4}a^{11/2} + \tfrac{15}{4}a^7\right)C_{n-2} \\
& + \left(-\tfrac{45}{32}a^3 - \tfrac{45}{8}a^{9/2} - \tfrac{45}{8}a^6\right)C_{n+3} + \left(\tfrac{15}{32}a^3 - \tfrac{15}{8}a^{9/2} + \tfrac{15}{8}a^6\right)C_{n-3} + \left(\tfrac{15}{32}a^3 - \tfrac{15}{8}a^{9/2} + \tfrac{15}{8}a^6\right)C_{-n+3}
\end{aligned}$$

$$\begin{aligned}
\xi_n(a) = {} & \left(\tfrac{3}{8}a^{5/2} - \tfrac{1}{2}a^4\right)A_{3n+1} + \left(\tfrac{1}{8}a^{5/2} + \tfrac{3}{2}a^4\right)A_{3n-1} + \left(\tfrac{3}{2}a^2 - \tfrac{3}{4}a^{7/2} - \tfrac{3}{2}a^5\right)B_{3n} \\
& + \left(\tfrac{3}{8}a + \tfrac{3}{2}a^{5/2} - \tfrac{3}{8}a^{9/2} - \tfrac{3}{2}a^6\right)B_{3n+1} + \left(-\tfrac{21}{8}a + \tfrac{9}{2}a^{5/2} + \tfrac{3}{8}a^{9/2} - \tfrac{3}{2}a^6\right)B_{3n-1} \\
& + \left(-\tfrac{15}{16}a^{7/2} - \tfrac{9}{4}a^5\right)B_{3n+2} + \left(\tfrac{3}{4}a^2 - \tfrac{69}{16}a^{7/2} + \tfrac{27}{4}a^5\right)B_{3n-2} + \left(-\tfrac{15}{2}a^2 - 15a^{7/2} - \tfrac{15}{8}a^4 + \tfrac{15}{2}a^7\right)C_{3n} \\
& + \left(-\tfrac{15}{8}a + \tfrac{135}{32}a^3 + \tfrac{105}{8}a^{9/2} + \tfrac{75}{8}a^6\right)C_{3n+1} + \left(\tfrac{45}{8}a + \tfrac{195}{32}a^3 - \tfrac{45}{8}a^{9/2} - \tfrac{105}{8}a^6\right)C_{3n-1} \\
& + \left(\tfrac{15}{8}a^2 + \tfrac{15}{4}a^{7/2} - \tfrac{15}{16}a^4 - \tfrac{15}{4}a^{11/2} - \tfrac{15}{4}a^7\right)C_{3n+2} + \left(-\tfrac{15}{32}a^3 - \tfrac{15}{8}a^{9/2} - \tfrac{15}{8}a^6\right)C_{3n+3} \\
& + \left(-\tfrac{45}{8}a^2 + \tfrac{45}{4}a^{7/2} - \tfrac{15}{16}a^4 + \tfrac{15}{4}a^{11/2} - \tfrac{15}{4}a^7\right)C_{3n-2} + \left(\tfrac{45}{32}a^3 - \tfrac{45}{8}a^{9/2} + \tfrac{45}{8}a^6\right)C_{3n-3}
\end{aligned}$$

$$\begin{aligned}
\lambda_n(a) = {} & \left(-\tfrac{1}{2}a^{7/2} + a^5\right)A_{n+1} + \left(\tfrac{3}{2}a^{7/2} - a^5\right)A_{n-1} + \tfrac{9}{2}a^{9/2}B_n + \left(-\tfrac{3}{2}a^{7/2} - \tfrac{3}{2}a^{11/2} - 3a^7\right)B_{n+1} \\
& + \left(-\tfrac{3}{2}a^{7/2} - \tfrac{3}{2}a^{11/2} + 3a^7\right)B_{n-1} + \left(\tfrac{3}{4}a^{9/2} + \tfrac{3}{2}a^6\right)B_{n+2} + \left(\tfrac{3}{4}a^{9/2} - \tfrac{3}{2}a^6\right)B_{n-2}
\end{aligned}$$

$$\begin{aligned}
\sigma(a) = {} & 3a^{3/2} + \left(-\tfrac{3}{2}a^{9/2} + a^5\right)A_0 + \left(\tfrac{5}{2}a^{7/2} + a^5 - a^{13/2}\right)A_1 + \left(6a^{9/2} - \tfrac{3}{2}a^6 - \tfrac{3}{2}a^{15/2}\right)B_0 \\
& + \left(-\tfrac{21}{4}a^{7/2} - 3a^{11/2} + 3a^{17/2}\right)B_1 + \left(\tfrac{9}{4}a^{9/2} + \tfrac{3}{2}a^6 - \tfrac{3}{2}a^{15/2}\right)B_2 + \left(-\tfrac{15}{2}a^{9/2} - \tfrac{15}{8}a^{13/2} - \tfrac{15}{8}a^{19/2}\right)C_0 \\
& + \left(\tfrac{15}{4}a^{7/2} + \tfrac{165}{16}a^{11/2} + \tfrac{15}{4}a^{17/2}\right)C_1 + \left(-\tfrac{15}{4}a^{9/2} - \tfrac{15}{8}a^{13/2} + \tfrac{15}{2}a^{19/2}\right)C_2 + \left(\tfrac{15}{16}a^{11/2} - \tfrac{15}{4}a^{17/2}\right)C_3
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{J}_n(a) = {} & 2a^{9/2}A_{2n} + \left(\tfrac{3}{8}a^{7/2} - \tfrac{7}{4}a^{5} + a^{13/2}\right)A_{2n+1} + \left(-\tfrac{19}{8}a^{7/2} - \tfrac{9}{4}a^{5} + a^{13/2}\right)A_{2n-1} + \left(3a^{9/2} + 6a^{6} - 3a^{15/2}\right)B_{2n} \\
& + \left(\tfrac{3}{4}a^{7/2} - 3a^{5} - \tfrac{3}{4}a^{7} - 3a^{17/2}\right)B_{2n+1} + \left(-\tfrac{21}{4}a^{7/2} + 3a^{5} + \tfrac{3}{4}a^{7} - 3a^{17/2}\right)B_{2n-1} \\
& + \left(-\tfrac{3}{4}a^{9/2} + \tfrac{3}{8}a^{6} + \tfrac{9}{2}a^{15/2}\right)B_{2n+2} + \left(\tfrac{9}{4}a^{9/2} - \tfrac{51}{8}a^{6} + \tfrac{9}{2}a^{15/2}\right)B_{2n-2} + \left(-15a^{9/2} - \tfrac{15}{4}a^{13/2} + 15a^{19/2}\right)C_{2n} \\
& + \left(\tfrac{15}{4}a^{7/2} + \tfrac{165}{16}a^{11/2} + \tfrac{75}{4}a^{7} - \tfrac{15}{4}a^{17/2}\right)C_{2n+1} + \left(\tfrac{15}{4}a^{7/2} + \tfrac{165}{16}a^{11/2} - \tfrac{75}{4}a^{7} - \tfrac{15}{4}a^{17/2}\right)C_{2n-1} \\
& + \left(-\tfrac{15}{4}a^{9/2} - \tfrac{15}{2}a^{6} - \tfrac{15}{8}a^{13/2} - \tfrac{15}{2}a^{8} - \tfrac{15}{2}a^{19/2}\right)C_{2n+2} + \left(\tfrac{15}{16}a^{11/2} + \tfrac{15}{4}a^{7} + \tfrac{15}{4}a^{17/2}\right)C_{2n+3} \\
& + \left(-\tfrac{15}{4}a^{9/2} + \tfrac{15}{2}a^{6} - \tfrac{15}{8}a^{13/2} + \tfrac{15}{2}a^{8} - \tfrac{15}{2}a^{19/2}\right)C_{2n-2} + \left(\tfrac{15}{16}a^{11/2} - \tfrac{15}{4}a^{7} + \tfrac{15}{4}a^{17/2}\right)C_{2n-3}
\end{aligned}
$$

Note: The coefficients C_{n-3}, C_{-n+3}, etc., are to be included only when $(n-3) \geq 0$, $(-n+3) \geq 0$, etc. The summation on k in the Fourier series' does not include $k < 0$.

NOTATION

The symbols which appear most often in the text are listed below:

μ	mass of the perturbing body divided by total mass of the system
t	time
r, θ	polar coordinates of the infinitesimal body; see Figure 2.
s	$\frac{1}{r}$
s_0, s_1	leading terms of the two variable expansion for s
t_0, t_1	" " " " " " " " t
$\tilde{\theta}$	the slow variable $(\tilde{\theta} = \mu^{1/2}\theta)$
$\theta_0, \tilde{\theta}_0$	initial values of θ and $\tilde{\theta}$ at $t=0$
a	semimajor axis of the orbit of the infinitesimal body
e	eccentricity " " " " " " "
ω	longitude of pericenter " " " " "
τ	quantity which defines the position of the infinitesimal body in its orbit
n, m	positive integers which specify the particular commensurability being considered
$e_0, \omega_0, \tau_0, \hat{a}^{3/2}, \hat{e}, \hat{\omega}, \hat{\tau}$	various terms in the expansions of the orbital elements
ϕ	a slowly-varying angular quantity which defines the position of the infinitesimal body in its orbit; see eq. (34).
$A_k(a), B_k(a), C_k(a)$	Fourier coefficients used to expand the periodic perturbing terms
k	summation index
$\alpha_n, \beta_n, \gamma_n, \kappa_n, \rho, \delta_n, \eta_n, \xi_n, \nu, \lambda_n, \sigma, \zeta_n$	various combinations of the Fourier coefficients
r.h.s.; l.h.s.	right-hand side; left-hand side

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