A POSSIBLE ANALYTICAL EXPLANATION FOR THE MICROMETEORITE CONCENTRATION NEAR THE EARTH

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ABSTRACT

The Keplerian motion of a single particle in a dissipating medium, such as air, is analysed and a theory of micrometeorite capture and resulting orbital lifetime is developed. The equations of motion are derived for a planar, two-dimensional model, and all orbital variables are assumed to be perturbed slightly from their Keplerian values. The equations are then linearized and solved. Then a statistical model of the interplanetary micrometeorite flux is developed in which the distribution of velocities at infinity relative to the earth and masses of the particles are taken into account. The velocity distribution is taken to be $\varphi(V_\infty) = BV_\infty^4 e^{-\beta V_\infty^2}$. The distribution of the masses is taken as a constant number of particle flux M(m) at infinity. Finally, this statistical model is combined with the theory of capture and lifetime to furnish a possible explanation for the micrometeorite concentration near the earth.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont'd)

LIST OF SYMBOLS

A = frontal area of particle

a = semi-major axis

b = $\frac{2\epsilon}{1+\epsilon}$

C = angular momentum = $r^2\dot{\theta}$

C_D = drag coefficient

d = diameter of particle

D = drag force

E = specific total energy of orbit

g = local gravitational acceleration = $\frac{g_o R^2}{r^2}$

g_o = gravitational acceleration at sea level

H = scale height = $\mathcal{R}T/g$

K = drag parameter = $\frac{C_D A}{2m}\ \rho_p$

$K_n(p/2)$ = modified Bessel functions of the second kind with argument $p/2$

m = mass of particle

p = $\frac{r_p}{H} = \lambda r_p = \frac{\lambda C_o^2}{\mu(1+\epsilon)}$

P = pressure

q = $\epsilon/1+\epsilon$

r = radial distance measured from the center of the earth

R = radius of the earth

$\mathcal{R}$ = gas constant

t = time, epoch

T = temperature

u = reciprocal of the radial distance = $\frac{1}{r}$

LIST OF SYMBOLS (Cont'd)

V	=	local velocity of the particle
V_{∞}	=	relative velocity of the particle with respect to the earth
z	=	altitude about the earth surface
γ	=	dummy variable
δ	=	$\dfrac{\epsilon - 1}{1 + \epsilon}$
ϵ	=	eccentricity of the orbit
θ	=	polar angle measured as shown in figure 1
ξ	=	impact parameter
φ	=	angle between θ direction and the velocity vector
λ	=	reciprocal of scale height = $1/H$
μ	=	gravitational constant = $g_o R^2$
ρ_m	=	material density of particle
ρ	=	density of air
$\sigma(r)$	=	density ratio = ρ/ρ_p
$(\ )_o$	=	Keplerian values
$(\ )_p$	=	values at perigee
$(\ )_i$	=	initial values
$(\ )_f$	=	final values

I. INTRODUCTION

The main sources of information concerning the existence of interplanetary particles in the vicinity of the earth prior to the advent of rockets and satellites were: 1) scattering of sunlight by particles in the space, i.e., the zodiacal light measurements; 2) influx of meteors into the earth's atmosphere; 3) the accretion of fine particles on the surface of the earth; and 4) the sediments of particles of extra-terrestrial origin on the ocean beds. These interplanetary particles are called micrometeorites (sometimes dust particles) and are only a few microns in size. These micrometeorites consist of either stony or metallic material. The consensus of a majority of people regarding the origin of these particles is that they are of cometary origin.

After the coming of the space age, numerous impact measurements of micrometeorites in space from rockets and satellites have been obtained, and the measurements do confirm the existence of these tiny particles in the interplanetary space near the earth as well as far away from the earth. The numerous measurements from various rockets and space vehicles are very widely scattered, and there seems to be no proper correlation between them. However, a definite trend has been established by these measurements, and that is that the particle concentration in the neighborhood of the earth is very high and it falls off smoothly to insignificant numbers far away from the earth in interplanetary space. Whipple (1) has inferred as early as 1960 by studying the various impact measurements that the particle concentration near the earth is about 10^5 times that in interplanetary space. More recently, Alexander (2) has concluded from the Mariner II micrometeorite impact measurements that the concentration near the earth is 10^4 times that in space far away from the earth. From the most recent Mariner IV measurements, Alexander, et al. (3) state again that the new data from Mariner IV substantiate the conclusion derived from the Mariner II flux measurements.

Many physical mechanisms have been offered to explain this high concentration of interplanetary particles in the neighborhood of the earth. Whipple (4) has discussed a few mechanisms. One of them is the capture of particles from the matter ejected during impacts of large meteorites on the moon. He proposes that some of the particles thrown up have velocities larger than the lunar escape velocity. Consequently, these particles leave the moon and go into terrestrial orbits. Then he considers it possible for these orbits to converge towards the earth and thus enhance the population of particles near the earth. However, Whipple has not offered any quantitative basis for this hypothesis. He has also considered other mechanisms such as gravitational concentration, electrostatic explosion of frigid particles, etc. These do not appear to account for the observed high concentrations. Beard (5) had predicted earlier that the concentration near the earth is about 10^3 times that in outer space, based on a theoretical model of gravitational capture between sun-earth-particle system. This again does not account for the observed high concentrations. Hibbs (6) has concluded, by studying the data of measured impacts of micrometeorites from the satellite Explorer I, that the particles must be in closed orbits around the earth. Consequently, he has suggested that these particles were captured by the earth into closed orbits due to aerodynamic drag while passing through the upper regions of the earth's atmosphere.

In this paper the supposition that large observed particle concentrations come from closed orbits around the earth and that they are captured by the aerodynamic drag of the atmosphere is considered. Particles captured at certain heights from the surface of the earth will have very long orbital lifetimes and thereby enhance the population near the earth. Consequently, a theoretical model for this process is set up, and a theory of the atmospheric capture mechanism and the resulting orbital lifetime for these micrometeorites is developed. This theory is based on the perturbation of a Keplerian orbit due to the presence of aerodynamic drag. A plane, two-dimensional problem in the ecliptic plane of the earth is considered. Here, the atmosphere is assumed to be spherical and symmetric about the earth and non-rotating. This theory is then combined with a statistical model of the micrometeorite flux distribution in the interplanetary space to obtain a complete mechanism to account for this high concentration near the earth.

II. METHOD OF APPROACH

The problem consists of two parts. The first part deals with the analysis of the orbital motion of a single particle in a dissipating medium, such as air, to develop a theory for the capture mechanism and the resulting orbital lifetime. The second part deals with the construction of a statistical model of the micrometeorite flux distribution in the interplanetary space, which is then combined with the first part to obtain the complete picture.

In the absence of a dissipating medium, the trajectory of a particle moving under the influence of a central force field is a Keplerian conic section. The angular momentum and the total energy of such an orbit are constants. However, when a dissipating medium is present, the orbit of the particle is not quite Keplerian, and the angular momentum and the total energy are not constants any more. Hence the presence of a dissipating medium, such as air, perturbs the motion of the particle away from a Keplerian motion. The magnitude of this perturbation is very small. The first part of the problem investigates the effect of this perturbation on the orbital parameters based on a Keplerian trajectory. A two-dimensional planar problem around the earth is considered. The gravitational field of the earth is taken as that of a point mass at the earth's center, and the variations in the field due to the non-sphericity of the earth are neglected. Other minor perturbing forces due to the sun, moon, and other planets, electromagnetic effects, etc., are neglected.

The only major perturbing force considered in this paper is the aerodynamic resistance due to the atmosphere surrounding the earth. This resistance decreases very rapidly with increasing distance from the earth. Since the magnitude of the perturbing force is quite small, mathematical perturbation techniques can be used to analyze the problem. The rotations of the earth and the atmosphere are neglected. It is also assumed that the atmosphere is spherically symmetric about the earth.

The aerodynamic drag force, D, normally defined in terms of the dimensionless drag coefficient, C_D, is written as

$$D = \tfrac{1}{2} C_D A \rho V^2 \tag{2.1}$$

where A is the frontal area of the particle, ρ is the air density, and V is the velocity of the particle.

The Mach number dependence of C_D is neglected. As the region of interest lies above 50 miles altitude, the Newtonian approximation for the estimation of C_D is used. This approximation says that the mean free path of the air molecules is much larger than the characteristic length of the body and that C_D is constant with a value very close to 2. It is convenient to rewrite equation (2.1) in the form of drag per unit mass. Consequently,

$$\frac{D}{m} = \frac{C_D A}{2m} \rho_p \left(\frac{\rho}{\rho_p}\right) V^2 = K\sigma(r)V^2 \tag{2.2}$$

where m is the mass of the particle and ρ_p is some reference density (in this case, the density at perigee), $\sigma(r)$ is the density ratio, and $K \equiv (C_D A/2m)\rho_p$ is assumed to be a constant and has the dimensions of $(\text{length})^{-1}$.*

Under the above mentioned assumptions, the equations of motion of a particle are set up and the appropriate perturbations are evaluated.

Then a statistical model of the distribution of the micrometeorite flux in the interplanetary space is developed. Here it is assumed that these particles in space have different masses and velocities. These velocities are defined in terms of the relative velocities at infinity with respect to the earth (sometimes called the hyperbolic excess velocities). It is clear that these velocities and masses should be considered in estimating the total flux. Consequently, simple models of the statistical distribution of the velocities and masses are used. Finally, the theory of captures and the resulting lifetimes are combined with the statistical model to obtain the overall integrated flux.

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* If the charge effects on the particles are included, the basic phenomenon is affected only to the extent that the value of C_D will be different from 2. Since K is assumed to be constant, this change in C_D requires a corresponding change in the size and mass of the particle.

III. ANALYSIS OF THE MOTION OF A SINGLE PARTICLE IN A DISSIPATING MEDIUM

1. Derivation of the Equations of Motion

Consider a particle of mass m travelling along a Keplerian trajectory far away from the earth with an eccentricity $\epsilon > 1$, as shown in Figure 1. Then its trajectory with respect to the earth is a hyperbola whose asymptotes are given by $\pm\theta = \cos^{-1}\frac{1}{\epsilon}$. As mentioned before, we are considering a planar problem.

Let V be the velocity of the orbiting particle. Using polar coordinates r and θ, the velocity V is written as

$$\vec{V} = \frac{d\vec{r}}{dt} = \dot{r}\,\vec{i}_r + r\dot{\theta}\,\vec{i}_\theta \tag{3.1}$$

where the dots denote differentiation with respect to time and i's are unit vectors.

Then the acceleration of a particle with unit mass is given by

$$\frac{d\vec{V}}{dt} = \frac{d}{dt}(\dot{r}\,\vec{i}_r + r\dot{\theta}\,\vec{i}_\theta) = (\ddot{r} - r\dot{\theta}^2)\,\vec{i}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\vec{i}_\theta\,. \tag{3.2}$$

As the particle approaches closer to the earth, it begins to experience the air drag mentioned previously. Then the particle is acted upon by both the gravitational force and the drag force. From Figure 1, this force is written as

$$\vec{F} = (-mg + D\sin\varphi)\,\vec{i}_r - D\cos\varphi\,\vec{i}_\theta$$

where φ is the angle between the velocity vector and the θ direction. But from the geometry,

$$\sin\varphi = -\dot{r}/V$$

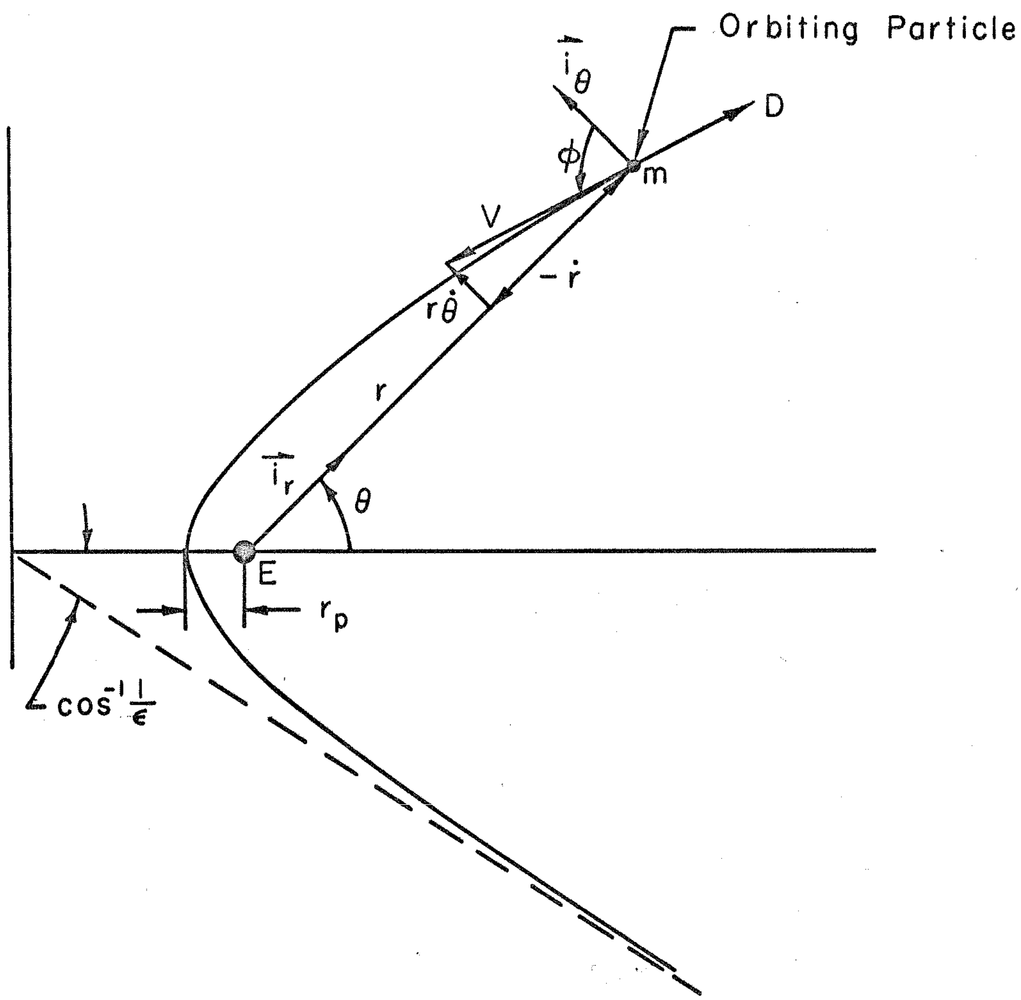

Figure 1. Notation showing the motion of the particle along a hyperbolic orbit with the center E as a focus

and

$$\cos\varphi = r\dot{\theta}/V \ .$$

Therefore, the force equation is rewritten as

$$\vec{F} = -\left(mg + \frac{D\dot{r}}{V}\right)\vec{i}_r - D\frac{r\dot{\theta}}{V}\vec{i}_\theta \ . \tag{3.3}$$

Newton's second law, for a particle with constant mass m, gives the equation of motion as

$$\frac{d\vec{V}}{dt} = \frac{\vec{F}}{m} \ .$$

This leads to the following equations of motion along r and θ directions, respectively:

$$\ddot{r} - r\dot{\theta}^2 = -g - \frac{D\dot{r}}{mV} \tag{3.4}$$

and

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{Dr\dot{\theta}}{mV} \ . \tag{3.5}$$

Also, the velocity is given by

$$V^2 = \dot{r}^2 + r^2\dot{\theta}^2 \ . \tag{3.6}$$

In the limiting case, when the dissipative drag force is zero, equations (3.4) and (3.5) will be shown to reduce to the Keplerian case. When the right hand side is zero, equation (3.5) can be written as

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \Rightarrow r^2\dot{\theta} = \text{constant} = C \ . \tag{3.7}$$

This shows the constancy of the angular momentum as defined by the Keplerian motion. Writing $r = 1/u$, we have

$$\dot{r} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta} = -C\frac{du}{d\theta}$$

by using (3.7). Similarly, the second derivative of r is written as

$$\ddot{r} = \dot{\theta} \frac{d}{d\theta} (\dot{r}) = -C^2 u^2 \frac{d^2 u}{d\theta^2} \quad . \tag{3.8}$$

Substituting (3.8) and (3.7) into (3.4) in the limiting case when $D = 0$, we get

$$\frac{d^2 u}{d\theta^2} + u = \frac{g_o R^2}{C^2} \tag{3.9}$$

where $g = (g_o R^2)/r^2 = g_o R^2 u^2$. This is the Kepler equation of motion which has the general solution of a conic section:

$$\frac{1}{u} = r = \frac{C^2}{g_o R^2} \; \frac{1}{1 - \epsilon \cos(\theta - \theta_o)} \quad . \tag{3.10}$$

Now the equations of motion given by equations (3.4) and (3.5) are not in a convenient form to analyze the motion. By a few manipulations, they can be put into a simpler form, as shown below. Differentiation of the expression for the particle velocity given by equation (3.6) yields

$$V\dot{V} = \dot{r}\,\ddot{r} + r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\,\ddot{\theta} \quad . \tag{3.11}$$

Multiplying equation (3.4) by $\dot{r}$ and rearranging gives

$$\dot{r}\,\ddot{r} = -\frac{D\dot{r}^2}{mV} - g\dot{r} + r\dot{r}\dot{\theta}^2 \quad . \tag{3.12}$$

Substituting this expression for $\dot{r}\,\ddot{r}$ in equation (3.11), we get

$$V\dot{V} = -\frac{D\dot{r}^2}{mV} - g\dot{r} + 2r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\,\ddot{\theta} \quad . \tag{3.13}$$

Again, multiplication of equation (3.5) by $r\dot{\theta}$ gives

$$2r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\,\ddot{\theta} = -\frac{Dr^2\dot{\theta}^2}{mV} \quad . \tag{3.14}$$

Finally, substituting equation (3.14) into (3.13), we obtain

$$V\dot{V} = - g\dot{r} - \frac{DV}{m} . \tag{3.15}$$

But $g = (g_o R^2)/r^2$. Therefore, equation (3.15) is rewritten as

$$\frac{d}{dt}\left(V^2 - \frac{2g_o R^2}{r} \right) = - 2K\sigma(r)V^3 \tag{3.16}$$

where equation (2.2) has been substituted for D/m.

Similarly, the elimination of the last term on the right hand side of equations (3.4) and (3.5) yields the other equation

$$\ddot{r} - r\dot{\theta}^2 - \frac{\dot{r}\ddot{\theta}}{\dot{\theta}} - 2\frac{\dot{r}^2}{r} = - \frac{g_o R^2}{r^2} . \tag{3.17}$$

Equations (3.16) and (3.17) are the governing differential equations for a particle in a dissipating medium with the forces acting along the velocity vector and perpendicular to it, respectively.

Now it is convenient to transform the independent variable t to θ. This changes the dynamic problem to an orbital problem. Therefore,

$$\frac{d}{dt} = \dot{\theta}\frac{d}{d\theta} . \tag{3.18}$$

Then

$$\dot{r} = - r^2\dot{\theta}\frac{d}{d\theta}\left(\frac{1}{r}\right) , \tag{3.19}$$

and

$$\ddot{r} = - \dot{\theta}^2 r^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) - \ddot{\theta}\, r^2 \frac{d}{d\theta}\left(\frac{1}{r}\right) + 2r^3\dot{\theta}^2\left[\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]^2 . \tag{3.20}$$

By using equations (3.18), (3.19), and (3.20), we obtain the governing orbital equations from equations (3.16) and (3.17) as

$$\frac{d}{d\theta}\left(V^2 - \frac{2g_o R^2}{r}\right) = - \frac{2K\sigma V^3}{\dot{\theta}} \tag{3.21}$$

and

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{g_o R^2}{r^4 \dot{\theta}^2} \quad . \tag{3.22}$$

Also, the velocity equation is rewritten as

$$V^2 = \dot{\theta}^2 \left[r^2 + \left(\frac{dr}{d\theta}\right)^2 \right] \quad . \tag{3.23}$$

Then equations (3.21), (3.22), and (3.23) are the basic equations relevant to this problem.

2. Linearization

As mentioned previously, the effect of atmospheric air drag is quite small and the trajectory of the particle deviates slightly from the Keplerian trajectory in vacuum. Therefore, it is reasonable to assume that the various orbital parameters are only slightly perturbed from their Keplerian values. As these effects are small, the squares of these effects can be neglected. Consequently, the various orbital quantities are written as

$$\begin{aligned} r &= r_o + \Delta r + O(\Delta r^2) + \dots \\ V &= V_o + \Delta V + \dots \\ \theta &= \theta_o + \Delta\theta + \dots \end{aligned} \tag{3.24}$$

where the subscript o is used to indicate the Keplerian values of the parameters and $\Delta(\ )$ refer to the first order perturbations.

With these assumptions, the equations of motion (3.21) and (3.22), together with (3.23), are linearized. Now the linearized differential operators are

$$\frac{d}{d\theta} = \frac{d\theta_o}{d\theta}\frac{d}{d\theta_o} = \frac{d\theta_o}{d(\theta_o + \Delta\theta)}\frac{d}{d\theta_o}$$

$$= \left(1 - \frac{d\Delta\theta}{d\theta_o}\right)\frac{d}{d\theta_o}$$

$$\frac{d^2}{d\theta^2} = \left(1 - \frac{d\Delta\theta}{d\theta_o}\right)\frac{d}{d\theta_o}\left\{\left(1 - \frac{d\Delta\theta}{d\theta_o}\right)\frac{d}{d\theta_o}\right\} \tag{3.25}$$

$$= \left(1 - 2\frac{d\Delta\theta}{d\theta_o}\right)\frac{d^2}{d\theta_o^2} - \left(1 - \frac{d\Delta\theta}{d\theta_o}\right)\frac{d^2\Delta\theta}{d\theta_o^2}\frac{d}{d\theta_o}$$

$$= \frac{d^2}{d\theta_o^2} - 2\frac{d\Delta\theta}{d\theta_o}\frac{d^2}{d\theta_o^2} - \frac{d^2\Delta\theta}{d\theta_o^2}\frac{d}{d\theta_o} \quad . \tag{3.26}$$

Now

$$\frac{1}{r} = \frac{1}{r_o + \Delta r} = \frac{1}{r_o}\left(1 - \frac{\Delta r}{r_o} + \ldots\right) = \frac{1}{r_o} - \frac{\Delta r}{r_o^2}$$

and

$$V^2 = (V_o + \Delta V)^2 = V_o^2 + 2V_o\Delta V \ .$$

For the Keplerian case we have, from equation (3.21),

$$\frac{d}{d\theta_o}\left(V_o^2 - 2\frac{g_oR^2}{r_o}\right) = 0 \quad . \tag{3.27}$$

Using these relations, the linearized form of equation (3.21) is given by

$$\frac{d}{d\theta_o}\left(V_o\Delta V + \frac{g_oR^2}{r_o^2}\ \Delta r\right) = -\frac{K\sigma_oV_o^3}{\dot{\theta}_o} \tag{3.28}$$

where $\sigma_o = \sigma(r_o)$.

The Keplerian form of equation (3.22) is

$$\frac{d^2}{d\theta_o^2}\left(\frac{1}{r_o}\right) + \frac{1}{r_o} = \frac{g_oR^2}{r_o^4\dot{\theta}_o^2} = \frac{\mu}{C_o^2} \tag{3.29}$$

where $\mu \equiv g_oR^2$ is the gravitational constant of the earth and C_o =

$r_o^2\dot{\theta}_o$ is the angular momentum constant. With these relations, the linearized form of (3.22) is

$$\frac{1}{r_o^2}\frac{d^2}{d\theta_o^2}\Delta r + 2\frac{d}{d\theta_o}\left(\frac{1}{r_o^2}\right)\frac{d\Delta r}{d\theta_o} + \left[\frac{d^2}{d\theta_o^2}\left(\frac{1}{r_o^2}\right) + \frac{1}{r_o^2} - \frac{4\mu}{C_o^2 r_o}\right]\Delta r$$

$$= \frac{2}{r_o}\frac{d\Delta\theta}{d\theta_o} - \frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\frac{d^2\Delta\theta}{d\theta_o^2}\ , \tag{3.30}$$

and the linearized form of (3.23) is

$$V_o\Delta V = \dot{\theta}_o^2\left[r_o\Delta r + \frac{dr_o}{d\theta_o}\cdot\frac{d\Delta r}{d\theta_o} + r_o^2\frac{d\Delta\theta}{d\theta_o}\right]. \tag{3.31}$$

At this stage, it is useful to obtain the linearized form of the specific total energy of the orbiting particle. The total energy per unit mass is written as

$$E = \tfrac{1}{2}V^2 - \frac{g_o R^2}{r} \tag{3.32}$$

Writing this in difference form, we obtain

$$\Delta E = V_o\Delta V + \frac{g_o R^2}{r_o^2}\Delta r \tag{3.33}$$

But this is the same expression given inside the brackets in equation (3.28). Consequently, we obtain a first order differential equation for ΔE from equation (3.28) as

$$\frac{d\Delta E}{d\theta_o} = -\frac{K\sigma_o V_o^3}{\dot{\theta}_o}\ . \tag{3.34}$$

Then the perturbed quantities Δr, $\Delta\theta$, ΔE, and ΔV can be determined by solving the linearized equations (3.28), (3.30), (3.31), and (3.34).

IV. SOLUTIONS

1. Solution for Δr

The differential equation for the radial perturbation Δr is obtained from equations (3.28), (3.30), and (3.31). If the initial conditions are known, then equation (3.28) can be directly integrated. For hyperbolic orbits, i.e., $\epsilon > 1$, the initial point is given by $\theta_o = \theta_1 = \cos^{-1}\frac{1}{\epsilon}$. For all other cases, i.e., $\epsilon \leq 1$, the apogee point of the particle, for which $\theta_o = 0$, is chosen as the initial point. As the problem concerns the perturbation of the Keplerian orbit, the initial conditions for the perturbed quantities are matched to the Keplerian values at the initial point, $t = 0$. Hence the initial conditions are

$$\Delta r = \frac{d\Delta r}{d\theta_o} = \Delta\theta = \frac{d\Delta\theta}{d\theta_o} = 0 \quad \text{at } t = 0,$$

and at $t = 0$,

$$\begin{aligned} &\text{for } \epsilon > 1, \quad \theta_o = \theta_1 = \cos^{-1}\frac{1}{\epsilon}, \\ &\text{for } \epsilon \leq 1, \quad \theta_o = \theta_1 = 0. \end{aligned} \tag{4.1}$$

Substitution of equation (3.31) into (3.28) and an integration with the initial conditions given by (4.1) yields

$$\dot{\theta}_o^2 \left[r_o \Delta r + \frac{dr_o}{d\theta_o}\frac{d\Delta r}{d\theta_o} + r_o^2 \frac{d\Delta\theta}{d\theta_o} \right] + \frac{\mu}{r_o^2}\Delta r = -K\int_{\theta_1}^{\theta_o} \frac{\sigma_a V_a^3}{\dot{\theta}_a}\, d\theta .$$

Solving for $d\Delta\theta/d\theta_o$, this is rewritten as

$$\frac{d\Delta\theta}{d\theta_o} = \frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\frac{d\Delta r}{d\theta_o} - \left(\frac{1}{r_o} + \frac{\mu}{C_o^2}\right)\Delta r - \frac{Kr_o^2}{C_o^2}\int_{\theta_1}^{\theta_o} \frac{\sigma_a V_a^3}{\dot{\theta}_a}\, d\theta . \tag{4.2}$$

When this is differentiated once with respect to θ_o, it gives

$$\frac{d^2\Delta\theta}{d\theta_o^2} = \frac{d^2}{d\theta_o^2}\left(\frac{1}{r_o}\right)\frac{d\Delta r}{d\theta_o} + \frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\frac{d^2\Delta r}{d\theta_o^2} - \left(\frac{\mu}{C_o^2}+\frac{1}{r_o}\right)\frac{d\Delta r}{d\theta_o} - \frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\Delta r$$
$$- \frac{2Kr_o}{C_o^2}\frac{dr_o}{d\theta_o}\int_{\theta_1}^{\theta_o}\frac{\sigma_a V_a^3}{\dot{\theta}_a}d\theta - \frac{Kr_o^2}{C_o^2}\frac{\sigma_o V_o^3}{\dot{\theta}_o} \quad . \tag{4.3}$$

Substitution of equations (4.2) and (4.3) into (3.30) to eliminate the first and the second derivatives of $\Delta\theta$ yields the following equation for Δr:

$$\left[\frac{1}{r_o^2} + \left(\frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\right)^2\right]\frac{d^2\Delta r}{d\theta_o^2} + \left[2\frac{d}{d\theta_o}\left(\frac{1}{r_o^2}\right) + \frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\frac{d^2}{d\theta_o^2}\left(\frac{1}{r_o}\right)\right.$$
$$\left. - \frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\left(\frac{1}{r_o} + \frac{\mu}{C_o^2}\right) - \frac{2}{r_o}\frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\right]\frac{d\Delta r}{d\theta_o}$$
$$+\left[\frac{d^2}{d\theta_o^2}\left(\frac{1}{r_o^2}\right) + \frac{1}{r_o^2} - \frac{4\mu}{C_o^2 r_o} - \left(\frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\right)^2 + \frac{2}{r_o}\left(\frac{1}{r_o} + \frac{\mu}{C_o^2}\right)\right]\Delta r$$
$$= \left[-\frac{2Kr_o}{C_o^2} + \frac{2Kr_o}{C_o^2}\frac{dr_o}{d\theta_o}\frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\right]\int_{\theta_1}^{\theta_o}\frac{\sigma_a V_a^3}{\dot{\theta}_a}d\theta + \frac{Kr_o^2}{C_o^2}\frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\frac{\sigma_o V_o^3}{\dot{\theta}_o} \, . \tag{4.4}$$

By using the Keplerian equation (3.29), equation (4.4) is further simplified.

Now the Keplerian equation for the velocity is

$$V_o^2 = \dot{\theta}_o^2\left[r_o^2 + \left(\frac{dr_o}{d\theta_o}\right)^2\right] = \dot{\theta}_o^2\left[r_o^2 + r_o^4\left(\frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\right)^2\right] \tag{4.5}$$

or

$$\frac{1}{r_o^2} + \left[\frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\right]^2 = \frac{V_o^2}{C_o^2} \quad . \tag{4.6}$$

Finally, using equation (4.6) together with equation (3.29), the differential equation (4.4) for the radial perturbation Δr is reduced to the simple form

$$\frac{V_o^2}{C_o^2}\frac{d^2\Delta r}{d\theta_o^2}+\frac{V_o^2}{C_o^2}\Delta r = -\frac{2Kr_o^3}{C_o^4}V_o^2\int_{\theta_1}^{\theta_o}\frac{\sigma_a V_a^3}{\dot\theta_a}d\theta + \frac{Kr_o^2}{C_o^2}\frac{d}{d\theta_o}\left(\frac{1}{r_o}\right)\frac{\sigma_o V_o^3}{\dot\theta_o} .$$

It is found that the coefficient of the first derivative $d\Delta r/d\theta_o$ collapses to zero and both $d^2/(d\theta_o^2)\,\Delta r$ and Δr have the same coefficient V_o^2/C_o^2. Finally, we have

$$\frac{d^2\Delta r}{d\theta_o^2}+\Delta r = -K\left(\frac{V_o}{\dot\theta_o}\right)\frac{dr_o}{d\theta_o}\sigma_o - \frac{2Kr_o^3}{C_o^2}\int_{\theta_1}^{\theta_o}\frac{\sigma_a V_a^3}{\dot\theta_a}d\theta . \qquad (4.7)$$

Because of the simple form of equation (4.7), its solution, satisfying the initial conditions, is immediately written as:

$$\Delta r(\theta_o) = \int_{\theta_1}^{\theta_o}\sin(\theta_o-\theta)\,G(\theta)d\theta \qquad (4.8)$$

where $G(\theta)$, representing the forcing function, is equal to

$$G(\theta) = -K\left(\frac{V_o}{\dot\theta_o}\right)\frac{dr_o}{d\theta_o}\sigma_o - \frac{2Kr_o^3}{C_o^2}\int_{\theta_1}^{\theta}\frac{\sigma_a V_a^3}{\dot\theta_a}d\theta . \qquad (4.9)$$

However, $G(\theta)$ has to be expressed explicitly in terms of the orbital parameters before the integral can be evaluated.

Now, for a Keplerian trajectory, the position of the particle is given by the orbital equation

$$r_o = \frac{C_o^2}{\mu(1-\epsilon\cos\theta_o)} \tag{4.10}$$

where $\mu = g_o R^2$, $C_o = r_o^2 \dot{\theta}_o$, and ϵ is the eccentricity of the orbit.

From equation (4.10), $\dot{r}_o$ is obtained by differentiating it with respect to time. Differentiating (4.10) once, we get

$$\dot{r}_o = - \frac{C_o^2}{\mu(1-\epsilon\cos\theta_o)^2} \cdot \epsilon \sin\theta_o \dot{\theta}_o ,$$

but

$$\dot{\theta}_o = \frac{C_o}{r_o^2} = \frac{\mu^2}{C_o^3}(1-\epsilon\cos\theta_o)^2 . \tag{4.11}$$

Hence

$$\dot{r}_o = - \frac{\mu\epsilon}{C_o} \sin\theta_o . \tag{4.12}$$

From these two relations, the velocity of the orbiting particle is obtained in the form

$$V_o^2 = \dot{r}_o^2 + r_o^2\dot{\theta}_o^2 = \dot{r}_o^2 + C_o\dot{\theta}_o = \frac{\mu^2}{C_o^2}(1+\epsilon^2-2\epsilon\cos\theta_o). \tag{4.13}$$

Also,

$$\frac{dr_o}{d\theta_o} = - \frac{C_o^2 \epsilon \sin\theta_o}{\mu(1-\epsilon\cos\theta_o)^2} . \tag{4.14}$$

Now we will investigate the model for the air density. As stated previously, the atmosphere is assumed to be spherically symmetric and non-rotating. It is also assumed that the density does not vary with time and it depends only on the radial distance r from the center of the earth. As the scale height H is small compared to the perigee distance r_p, the flat earth hydrostatic equilibrium equation for the variation of the pressure with the altitude is written as*

$$\frac{dP}{dr} = -g\rho \tag{4.15}$$

where g is the local gravitational acceleration.

Assuming that the atmosphere follows the perfect gas law, $P = \rho \mathcal{R} T$, equation (4.15) is written as

$$\frac{dP}{dr} = -\frac{g}{\mathcal{R}T} P \ . \tag{4.16}$$

Since the region of discussion involved is confined to a few scale heights about the perigee distance, the variation in the gravitational acceleration is neglected and g is assumed to be constant in the above analysis. Equation (4.16) can now be integrated if it is assumed that the temperature distribution at the orbital altitudes is isothermal. Consequently, using the above assumptions, integration of equation (4.16) yields

$$\frac{P}{P_p} = \frac{\rho}{\rho_p} = \sigma(r_o) = e^{-\lambda(r-r_p)} \tag{4.17}$$

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* It will be shown that this is a very good approximation to the spherical earth pressure distribution.
The hydrostatic equilibrium equation for a spherical earth is

$$\frac{d}{dr}(4\pi r^2 P)dr = -g\rho 4\pi r^2 dr$$

or

$$\frac{dP}{dr} + \frac{2}{r} P = -g\rho \ ,$$

but $\frac{2}{r} P/g\rho \ll 1$ in general. Hence the hydrostatic equilibrium for a spherical earth can be approximated to equation (4.15):

$$\frac{dP}{dr} = -g\rho \ .$$

where $\lambda = g/\mathcal{R}T$ = constant = $1/H$. H is defined as the scale height. The subscript p denotes the reference condition at the perigee point r_p. At the perigee point $r_o = r_p$, $\theta_o = \pi$. Hence, from equation (4.10), r_p is given as

$$r_p = \frac{C_o^2}{\mu(1+\epsilon)} \quad . \tag{4.18}$$

Therefore, the density distribution given by equation (4.17) is written in the form

$$\sigma(r_o) = \sigma(\theta_o) = e^{\lambda r_p} \cdot e^{-\frac{\lambda C_o^2}{\mu}\left(\frac{1}{1-\epsilon\cos\theta_o}\right)} \quad . \tag{4.19}$$

However, r_o is not defined for values of θ_o between zero and $\cos^{-1}\frac{1}{\epsilon}$ for hyperbolic trajectories. Consequently, we set

$$\sigma(\theta_o) = 0 \qquad \text{for } 0 \leq \theta_o < \theta_1 = \cos^{-1}\frac{1}{\epsilon} \quad . \tag{4.20}$$

Substituting the relations (4.10), (4.11), (4.13), together with (4.14), in equation (4.9), the forcing function $G(\theta_o)$ is written as

$$G(\theta_o) = \frac{KC_o^4}{\mu^2}\,\epsilon\,\frac{(1+\epsilon^2-2\epsilon\cos\theta_o)^{\frac{1}{2}}}{(1-\epsilon\cos\theta_o)^4}\sin\theta_o \cdot \sigma(\theta_o)$$
$$- \frac{2KC_o^4}{\mu^2}\,\frac{1}{(1-\epsilon\cos\theta_o)^3}\int_{\theta_1}^{\theta_o}\frac{(1+\epsilon^2-2\epsilon\cos\theta)^{3/2}}{(1-\epsilon\cos\theta)^2}\,\sigma(\theta)d\theta \quad . \tag{4.21}$$

Hence the solution for Δr is

$$\Delta r(\theta_f) = \int_{\theta_1}^{\theta_f}\sin(\theta_f-\theta_o)\left\{\frac{KC_o^4\epsilon}{\mu^2}\,\frac{(1+\epsilon^2-2\epsilon\cos\theta_o)^{\frac{1}{2}}}{(1-\epsilon\cos\theta_o)^4}\sin\theta_o\cdot\sigma(\theta_o)\right.$$
$$\left. - \frac{2KC_o^4}{\mu^2}\,\frac{1}{(1-\epsilon\cos\theta_o)^3}\int_{\theta_1}^{\theta_o}\frac{(1+\epsilon^2-2\epsilon\cos\gamma)^{\frac{3}{2}}}{(1-\epsilon\cos\gamma)^2}\,\sigma(\gamma)d\gamma\right\}d\theta_o . \tag{4.22}$$

where θ_f is some final value of θ_o.

This representation for Δr is valid for all values of eccentricity starting from zero. However, the evaluation of the integral is not easy for all values of ϵ. Consequently, the integral is first evaluated for values of ϵ near 1 in a closed form. Then a long, cumbersome series solution valid for $0 \leq \epsilon < 1$ is also obtained.

Let us now consider the first case when $\epsilon \sim 1$. Now, $G(\theta_o)$ is written as

$$G(\theta_o) = G_1(\theta_o) - G_2(\theta_o) ,$$

where

$$G_1(\theta_o) \equiv \frac{KC_o^4\epsilon}{\mu^2} \frac{(1+\epsilon^2-2\epsilon\cos\theta_o)^{\frac{1}{2}}}{(1-\epsilon\cos\theta_o)^4} \sin\theta_o \sigma(\theta_o) \tag{4.23}$$

and

$$G_2(\theta_o) = \frac{2KC_o^4}{\mu^2} \frac{1}{(1-\epsilon\cos\theta_o)^3} \int_{\theta_1}^{\theta_o} \frac{(1+\epsilon^2-2\epsilon\cos\gamma)^{\frac{3}{2}}}{(1-\epsilon\cos\gamma)^2} \sigma(\gamma)d\gamma. \tag{4.24}$$

Then

$$\Delta r(\theta_f) = \int_{\theta_1}^{\theta_f} \sin(\theta_f-\theta_o)G_1(\theta_o)d\theta_o - \int_{\theta_1}^{\theta_f} \sin(\theta_f-\theta_o)G(\theta_o)d\theta_o. \tag{4.25}$$

This integral for the perturbation in the radial distance r will be evaluated for $\theta_f = \pi$. Therefore, equation (4.25) is written as

$$\Delta r(\pi, \epsilon) = \int_{\theta_1}^{\pi} \sin\theta_o G_1(\theta_o)d\theta_1 - \int_{\theta_1}^{\pi} \sin\theta_o G_2(\theta_o)d\theta_o$$

$$\equiv I_1 - I_2 \tag{4.26}$$

It is convenient to change the independent variable θ_o to a new variable s given by the following transformation

$$\frac{1}{s} = \tan^2 \theta_o/2 \,. \tag{4.27}$$

Then

$$d\theta_o = -\frac{ds}{s^{\frac{1}{2}}} \cdot \frac{1}{1+s} \quad .$$

Also, at $\theta_o = \theta_1$, $1/\epsilon = \cos\theta_1 = (s-1)/(1+s)$.

Solving for s,

$$s = \frac{1+\epsilon}{\epsilon - 1} = \frac{1}{\delta} \, ,$$

and at $\theta_o = \pi$, $s = 0$.

Now,

$$\sin\theta_o = \frac{2s^{\frac{1}{2}}}{1+s}$$

$$1-\epsilon\cos\theta_o = (1+\epsilon)\left(\frac{1-\delta s}{1+s}\right)$$

and

$$1+\epsilon^2 - 2\epsilon\cos\theta_o = (1+\epsilon)^2 \left(\frac{1+\delta^2 s}{1+s}\right) \, ,$$

and

$$\delta \equiv \frac{\epsilon - 1}{1+\epsilon} \quad . \tag{4.28}$$

Let us consider the integral defined by I_1. Substitution of the above relations and (4. 19) yields for I_1 the following expression:

$$I_1 = \frac{4KC_o^4}{\mu^2(1+\epsilon)^3} \, \epsilon \, e^{\lambda r_p} \int_0^{1/\delta} \frac{s^2(1+\delta^2 s)^{\frac{1}{2}}}{(1-\delta s)^4} (1+s)^{\frac{1}{2}} \, e^{-\lambda r_p\left(\frac{1+s}{1-\delta s}\right)} ds \,. \tag{4.29a}$$

Even in this form this integral is very complicated. Some simplification is necessary to evaluate it. Since we are interested in the solution for $\epsilon \sim 1$, it is recognized that δ, as defined by (4.28), goes to zero as $\epsilon \rightarrow 1$. Therefore, the integrand is expanded in terms of δ for $\delta << 1$. Terms of order δ^2 and higher are neglected. Thus we have

$$(1 + \delta^2 s)^{\frac{1}{2}} = 1 + \tfrac{1}{2}\,\delta^2 s + \ldots$$

$$(1-\delta s)^{-4} = 1 + 4\delta s + 10\delta^2 s^2 + \ldots$$

and

$$e^{-\lambda r_p\left(\frac{1+s}{1-\delta s}\right)} = e^{-\lambda r_p(1+s)}\,[1-\lambda r_p \delta s(1+s) + \ldots]\ .$$

Consequently, after expansion, equation (4.29a) reduces to the form

$$I_1 = \frac{4KC_o^4}{\mu^2(1+\epsilon)^3}\,\epsilon \int_0^{1/\delta} \sqrt{s(1+s)}\ [1+(4-p)\delta s-\delta p s^2+O(\delta^2)]e^{-ps}\,ds \qquad (4.29b)$$

where $p \equiv \lambda r_p = \dfrac{\lambda C_o^2}{\mu(1+\epsilon)}$.

It is pointed out here that the upper limit of the integral $1/\delta$ is very large for small δ and goes to infinity as $\delta \rightarrow 0$. Also, in this analysis, the exponent p is very large and positive, and hence the main contribution to the integral comes when $s \simeq 0$. Therefore, the upper limit in the above integral is taken to be ∞ in all the following integrals. This corresponds to the asymptotic expansion of the integral using Watson's lemma. With these arguments, a function T(p) is defined as

$$T(p) \equiv \int_0^\infty \sqrt{s(1+s)}\; e^{-ps}\, ds \; . \tag{4.30a}$$

Then the integral given by (4.29b) takes the form

$$I_1 = \frac{4KC_o^4}{\mu^2(1+\epsilon)^3}\, \epsilon \left[T - (4-p)\,\delta \frac{dT}{dp} - \delta p \frac{d^2T}{dp^2} \right] . \tag{4.30b}$$

Now it is observed that $T(p)$ is the Laplace transform of the function $f(s) = s^{\frac{1}{2}}(1+s)^{\frac{1}{2}}$. Hence, from the tables, $T(p)$ is given by

$$T(p) = \frac{1}{2p}\, e^{p/2}\, K_1(p/2)$$

where $K_1(p/2)$ is the modified Bessel function of the second kind.

Now

$$\frac{dT}{dp} = \frac{d}{dp}\left(\frac{1}{2p}\, e^{p/2} K_1(p/2)\right) = \left(\frac{1}{4} - \frac{1}{p}\right) e^{p/2}\, \frac{K_1(p/2)}{p} - \frac{1}{4p}\, e^{p/2} K_o(p/2)$$

and similarly,

$$\frac{d^2T}{dp^2} = \frac{e^{p/2}}{4p}\left[\left(\frac{3}{p} - 1\right) K_o(p/2) + \left(1 - \frac{4}{p} + \frac{12}{p^2}\right) K_1(p/2)\right] .$$

Combining and substituting these relations in the expression (4.30b) for I_1, we get

$$I_1 = \frac{KC_o^4 \epsilon}{\mu^2(1+\epsilon)^3}\, \frac{e^{p/2}}{p} \left\{ \delta K_o(p/2) + \left[2 - 4\delta\left(1 - \frac{1}{p}\right)\right] K_1(p/2) \right\} . \tag{4.31}$$

Let us now consider the integral defined by I_2. It is given by

$$I_2 = \frac{2KC_o^4}{\mu^2} \int_{\theta_1}^{\pi} \frac{\sin\theta_o\, d\theta_o}{(1-\epsilon\cos\theta_o)^3} \left[\int_{\theta_1}^{\theta_o} \frac{(1+\epsilon^2 - 2\epsilon\cos\gamma)^{3/2}}{(1-\epsilon\cos\gamma)^2}\, \sigma(\gamma)\, d\gamma \right] . \tag{4.32}$$

Integration of equation (4.32) by parts yields the following expression for I_2:

$$I_2 = -\frac{KC_o^4}{\mu^2\epsilon}\frac{e^{\lambda r_p}}{(1+\epsilon)^2}\int_{\theta_1}^{\pi}\frac{(1+\epsilon^2-2\epsilon\cos\theta_o)^{\frac{3}{2}}}{(1-\epsilon\cos\theta_o)^2}e^{-\frac{\lambda C_o^2}{\mu}\frac{1}{1-\epsilon\cos\theta_o}}d\theta_o$$

$$+\frac{KC_o^4}{\mu^2\epsilon}e^{\lambda r_p}\int_{\theta_1}^{\pi}\frac{(1+\epsilon^2-2\epsilon\cos\theta_o)^{\frac{3}{2}}}{(1-\epsilon\cos\theta_o)^4}e^{-\frac{\lambda C_o^2}{\mu}\frac{1}{1-\epsilon\cos\theta_o}}d\theta_o . \qquad (4.33)$$

Now if $F(p,\epsilon)$ is defined as

$$F(p,\epsilon) = \int_{\theta_1}^{\pi}\frac{(1+\epsilon^2-2\epsilon\cos\theta_o)^{\frac{3}{2}}}{(1-\epsilon\cos\theta_o)^2}e^{-\frac{p(1+\epsilon)}{1-\epsilon\cos\theta_o}}d\theta \qquad (4.34)$$

where $(\lambda C_o^2)/\mu = p(1+\epsilon)$, then the last integral on the right hand side of equation (4.33) can be expressed as the second derivative of F with respect to p. With this definition, the integral I_2 is written as

$$I_2 = -\frac{KC_o^4}{\mu^2\epsilon}\frac{e^p}{(1+\epsilon)^2}\left\{F - \frac{d^2F}{dp^2}\right\} . \qquad (4.35)$$

This integral is now evaluated in the same way as I_1. Using the transformation given in equation (4.27) for the new independent variable, s, the integral for F in equation (4.34) is written as

$$F = (1+\epsilon)\int_0^{1/\delta}\frac{ds}{\sqrt{s(1+s)}}\frac{(1+\delta^2 s)^{\frac{3}{2}}}{(1-\delta s)^2}e^{-p(\frac{1+s}{1-\delta s})} . \qquad (4.36)$$

For $\delta \ll 1$, this reduces to the following form after expansion of the integrand for small δ. Thus,

$$F = (1+\epsilon)e^{-p}\int_0^{1/\delta}\frac{ds}{\sqrt{s(1+s)}}\ (1 + (2-p)\delta s - \delta p s^2)e^{-ps} .$$

Using the same argument as before, we set the upper limit of the above integral equal to infinity. Again, a new function S is defined as

$$S \equiv \int_0^{\infty} \frac{1}{\sqrt{s(1+s)}} e^{-ps} ds . \qquad (4.36a)$$

Then F is written in the form

$$F = (1+\epsilon)e^{-p}\left\{S-(2-p)\,\delta \frac{dS}{dp} - \delta p \frac{d^2S}{dp^2}\right\} . \qquad (4.37)$$

Here, S(p) is again identified as the Laplace transform of the function $f(s) = 1/\sqrt{s(1+s)}$. From tables, this is evaluated as

$$S = e^{p/2} K_o(p/2)$$

where $K_o(p/2)$ is the modified Bessel function of the second kind.

Now $$\frac{dS}{dp} = \tfrac{1}{2}e^{p/2}\{K_o(p/2)-K_1(p/2)\}$$

and $$\frac{d^2S}{dp^2} = \tfrac{1}{2}e^{p/2}\left\{K_o(p/2) - K_1(p/2) + \frac{K_1(p/2)}{p}\right\} .$$

Substitution of the above expressions for S, dS/dp, and d^2S/dp^2 in equation (4.37) yields the following expression for F:

$$F = \frac{(1+\epsilon)}{2} e^{-p/2}\left\{2(1-\delta)K_o(p/2)+\delta K_1(p/2)\right\} . \qquad (4.38)$$

Differentiation of equation (4.38) twice with respect to p yields the following relations

$$\frac{dF}{dp} = -\frac{(1+\epsilon)}{4} e^{-p/2}\left\{(2-\delta)[K_o(p/2)+K_1(p/2)] + \frac{2\delta K_1(p/2)}{p}\right\}$$

and

$$\frac{d^2F}{dp^2} = \frac{(1+\epsilon)}{4} e^{-p/2}\left\{(2-\delta)\,[K_o(p/2)+K_1(p/2)] + \frac{\delta K_o(p/2)}{p} + 2\left(1+\frac{2\delta}{p}\right)\frac{K_1(p/2)}{p}\right\} .$$

Substituting this value of the second derivative of F into equation (4.35), the integral I_2 is evaluated as

$$I_2 = \frac{-KC_o^4}{4\epsilon(1+\epsilon)\mu^2} e^{p/2} \left\{ (2-3\delta)[K_o(p/2)-K_1(p/2)] - \frac{\delta K_o(p/2)}{p} - 2(1+\frac{2\delta}{p}) \frac{K_1(p/2)}{p} \right\} \tag{4.39}$$

Then the solution for the perturbation Δr in the radial distance in moving from $\theta_o = \theta_1$ to $\theta_o = \pi$ is obtained by subtracting I_2 from I_1. Thus,

$$\Delta r(\pi, \epsilon) = I_1 - I_2$$

Substituting for I_1 and I_2 from equations (4.31) and (4.39) and rearranging the terms, the following solution for $\Delta r(\pi, \epsilon)$ is obtained when the eccentricity of the orbit ϵ is close to 1:

$$\Delta r(\pi, \epsilon) = \frac{-KC_o^4}{4\mu^2\epsilon(1+\epsilon)} e^{p/2} \left\{ (2-3\delta)[K_1(p/2)-K_o(p/2)] - \frac{\delta^2}{p}\frac{1}{(1+\epsilon)}\left[(1-3\epsilon)\left((K_o(p/2)+\frac{4}{p}K_1(p/2)\right) -2(1+5\epsilon)K_1(p/2)\right] \right\}$$

As all the computations have been carried to order δ, the above expression can be written as

$$\Delta r(\pi, \epsilon) = \frac{-KC_o^4 \cdot e^{p/2}}{4\mu^2\epsilon(1+\epsilon)} \left\{ (2-3\delta)[K_1(p/2)-K_o(p/2)] + O(\delta^2) \right\} \tag{4.40}$$

Note that when the initial trajectory of the particle is parabolic, i.e., $\epsilon = 1$ and $\delta = 0$, the solution for Δr reduces to

$$\Delta r(\pi, 1) = - \frac{KC_o^4}{4\mu^2} e^{p/2} [K_1(p/2) - K_o(p/2)] \tag{4.41}$$

Now a few numerical calculations will be carried out to estimate the radial perturbation Δr in order to inspect the validity of the linearization process.

Substitution for δ from equation (4.28) in equation (4.40) yields

$$\Delta r(\pi, \epsilon) = - (\frac{5-\epsilon}{4\epsilon}) Kr_p^2 e^{p/2} \left\{K_1(p/2) - K_o(p/2)\right\} \tag{4.42}$$

where equation (4.18) has been used to eliminate C_o and μ.

Now the argument p of the Bessel function given by

$$p = \lambda r_p$$

is very large for the whole range of perigee distances involved in this problem. Consequently, the asymptotic expansions of both $K_o(p/2)$ and $K_1(p/2)$ valid for arguments greater than about 3 will be used in the numerical calculations here. Now the asymptotic expansion of $K_n(p/2)$ for large p is

$$K_n(p/2) \simeq (\frac{\pi}{p})^{\frac{1}{2}} e^{-p/2} \left\{1 + \frac{4n^2-1}{1!\,4p} + \frac{(4n^2-1)(4n^2-9)}{2!\,(4p)^2} + \ldots\right\} \tag{4.43}$$

It is sufficient to keep terms up to order $1/p$. Thus

$$K_o(p/2) = (\pi/p)^{\frac{1}{2}} e^{-p/2} \left\{1 - \frac{1}{4p} + O(\frac{1}{p^2}) \text{ ---}\right\} \tag{4.44a}$$

and

$$K_1(p/2) = (\pi/p)^{\frac{1}{2}} e^{-p/2} \left\{1 + \frac{3}{4p} + O(\frac{1}{p^2}) \text{ ---}\right\} \tag{4.44b}$$

Substitution of these in equation (4.42) yields

$$\Delta r(\pi, \epsilon) = -\left(\frac{5-\epsilon}{4\epsilon}\right) \frac{Kr_p^2}{p} \left(\frac{\pi}{p}\right)^{\frac{1}{2}} \tag{4.45}$$

Here ϵ is taken to be 1. Then

$$\Delta r(\pi, 1) = -\frac{Kr_p^2}{p} (\pi/p)^{\frac{1}{2}} \tag{4.46}$$

The validity of the linearization approximation depends on the value of Δr. In the linearization process it has been assumed that $\Delta r << r$. As $\lambda \Delta r$ estimates the error in drag force, it is also required that $\lambda \Delta r$ should be small compared to 1.*

For computation purposes particles will be assumed to be spherical in shape and to have uniform material density ρ_m. Further, ρ_m is assumed to be equal to 1 gm/c.c.. Let the diameter of the particles be d microns. With these assumptions a few calculations of Δr for different values of diameter d and r_p are made. Properties of the atmosphere that are used here are taken from U. S. Standard Atmosphere, 1962 (7). The computed values of Δr and $\lambda \Delta r$ are shown in Table I.

From the table we can establish the validity of linearization. It is clear that the smaller particles have to be captured at higher altitudes than the bigger particles. The theory holds good for particles of 1 micron size in the regions of the atmosphere about 225 km above the earth and thereafter. For the largest particle

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* From equations (4.23) to (4.25) it is thus seen that the "small parameter" with respect to which the problem is linearized is $\lambda K C_o^4/\mu^2$ and the appropriate length used to make both λ and K non-dimensional is $C_o^2/\mu = r_{p_o}(1+\epsilon)$.

d (microns)	Altitude z (km)	r_p (km)	Scale ht $H = 1/\lambda$ (km)	Δr (km)	$\lambda \Delta r$	$\frac{\Delta r}{r_p}$
1	175	6553	38.497	-35.18	-0.9138	-0.00537
	200	6578	43.62	-20.607	-0.4724	-0.00313
	225	6603	47.876	-12.614	-0.2635	-0.00191
10	125	6503	14.019	-14.345	-1.0232	-0.0022
	150	6528	29.462	- 6.305	-0.2140	-0.00097
100	100	6478	6.362	-17.075	-2.6839	-0.00263
	125	6503	14.019	- 1.435	-0.1023	-0.00022
1000	75	6453	5.998	-135.96	-22.66	-0.0210
	100	6478	6.362	- 1.7075	-0.2684	-0.000263

TABLE I

Estimation of the range of the validity of the linearization process of 1000 micron size the theory is good for all regions above 100 km. Thus the linearized theory holds good for all particles in the region of the atmosphere above 200-225 kms from the earth's surface.

As mentioned earlier in this section a series solution for Δr valid for $0 \leqslant \epsilon < 1$ is obtained. The derivation of this solution is given in Appendix A.

2. <u>Solutions for ΔE</u>

Here two solutions, one valid for values of ϵ close to 1 and another asymptotic solution valid approximately over the whole range of ϵ from zero to one, are derived.

The appropriate differential equation governing the perturbation ΔE of the specific total energy of the particle is given by equation (3.34) as

$$\frac{d}{d\theta_o} \Delta E = -\frac{K\sigma_o v_o^3}{\dot{\theta}_o} \tag{3.34}$$

and the initial condition corresponding to the unperturbed Keplerian orbit is $\Delta E = 0$ at $t = 0$. At $t = 0$, $\theta_1 = 0$ for $\epsilon \leqslant 1$ and $\theta_1 = \cos^{-1}\frac{1}{\epsilon}$ for $\epsilon > 1$.

By integrating the above equation once and using the above initial condition, the solution for ΔE is obtained as

$$\Delta E(\theta_f) = -K \int_{\theta_1}^{\theta_f} \frac{\sigma_o v_o^3}{\dot{\theta}_o} \, d\theta_o \tag{4.47}$$

Substituting for $\dot{\theta}_o$, v_o and σ_o from the relations (4.11), (4.13) and (4.19), equation (4.47) is written in the form

$$\Delta E(\theta_f) = -K\mu e^p \int_{\theta_1}^{\theta_f} \frac{(1+\epsilon^2-2\epsilon\cos\theta_o)^{\frac{3}{2}}}{(1-\epsilon\cos\theta_o)^2} e^{-\frac{\lambda C_o^2/\mu}{1-\epsilon\cos\theta_o}} d\theta_o \qquad (4.48)$$

This expression for ΔE is valid for all values of eccentricity.

Equation (4.48) is evaluated first for $\theta_f = \pi$ and then the resulting solution is doubled to obtain the total loss ΔE_T in specific energy for the first pass around the earth. Thus,

$$\Delta E_T = -2K\mu e^p \int_{\theta_1}^{\pi} \frac{(1+\epsilon^2-2\epsilon\cos\theta_o)^{\frac{3}{2}}}{(1-\epsilon\cos\theta_o)^2} e^{-\frac{\lambda C_o^2/\mu}{1-\epsilon\cos\theta_o}} d\theta_o \qquad (4.49)$$

By using the transformation given in equation (4.27), the above equation is written in terms of the new variable s as

$$\Delta E_T = -2K\mu(1+\epsilon)\, e^p \int_{o}^{1/\delta} \frac{e^{-p(\frac{1+s}{1-\delta s})}}{\sqrt{(1+s)s}} \frac{(1+\delta^2 s)^{\frac{3}{2}}}{(1-\delta s)^2} ds \qquad (4.50)$$

For convenience this is written in the form

$$\Delta E_T = -2K\mu e^p F(p, \delta) \qquad (4.51)$$

where F is defined by equation (4.36).

The total energy loss ΔE_T is now evaluated for values of ϵ close to 1. When ϵ is near 1, $\delta = \epsilon - 1/1+\epsilon$ has a value close to zero, i.e., $\delta << 1$. Consequently, we expand the integrand near $\delta \simeq 0$ and integrate term by term. This has been carried out in part 1 of this section and the solution for F is given by equation (4.38). Consequently we can write the solution for ΔE_T for one complete pass valid when $|\epsilon - 1| \sim 0$ as

$$\Delta E_T = -(1+\epsilon)K\mu e^{p/2}\left\{2(1-\delta)K_o(p/2)+\delta K_1(p/2)\right\} \tag{4.52}$$

where $p = \lambda r_p$.

This solution for ΔE_T is correct up to order $O(\delta)$ and consequently gives a good estimation of the specific energy loss for one complete pass when the eccentricity of the orbit is close to 1.

In order to calculate the lifetime of a particle, a solution for the loss of the specific energy valid over the whole range of eccentricities from zero to one is required. As mentioned previously an asymptotic expansion of the integral for the energy loss is obtained for all $0 < \epsilon \leqslant 1$. In this case the energy loss is estimated for one complete orbit, that is, for $\theta_f = 2\pi$. It is found that $\Delta E(2\pi, \epsilon)$ is equal to $2\Delta E(\pi, \epsilon)$. Also, it is pointed out that $\theta_1 = 0$ for this case. This says that the upper limit of the integral in equation (4.50) is exactly equal to ∞. Let us now consider the integral given by (4.36) with the upper integral limit equal to ∞. Thus

$$F = (1+\epsilon)\int_o^\infty \frac{e^{-p\left(\frac{1+s}{1-\delta s}\right)}}{\sqrt{s(1+s)}} \frac{(1+\delta^2 s)^{3/2}}{(1-\delta s)^2}\, ds \tag{4.53}$$

The exponent p appearing in the integral is as before

$$p = \frac{\lambda C_o^2}{\mu(1+\epsilon)} = \lambda r_p$$

where λ is the reciprocal of the scale height H and r_p is the distance of the closest approach of the particle.

For the whole range of values of eccentricity considered, this exponent $p = \lambda r_p$ is very large and positive. Then the major contribution to the integral comes when the quantity $1+s/1-\delta s$ is close to zero. Consequently we can expand the integrand near $1+s/1-\delta s \simeq 0$ and integrate term by term to obtain the asymptotic expansion of the integral given in equation (4.53).

Now a new variable x is defined in the following manner. Let

$$1 + x \equiv \frac{1+s}{1-\delta s} \tag{4.54}$$

Then $dx = \dfrac{b}{(1-\delta s)^2}\,ds$ and $b \equiv 1 + \delta$. Now

at $s = 0;\ x = 0$

and

at $s = \infty;\ x = \dfrac{2\epsilon}{1-\epsilon}$

Solving equation (4.54) for s, s is obtained as

$$s = \frac{x}{b(1+\frac{\delta}{b}x)} \tag{4.55}$$

$$1 + s = 1 + \frac{x}{b(1+\frac{\delta}{b}x)} = \frac{1+x}{1+\frac{\delta}{b}x}$$

$$1 + \delta^2 s = 1 + \frac{\delta^2 x}{1+\delta+\delta x} = \frac{1+\delta x}{1+\frac{\delta}{b}x}$$

$$1 - \delta s = 1 - \frac{\delta x}{1+\delta+\delta x} = \frac{1}{1+\frac{\delta}{b}x}$$

Substituting these expressions in equation (4.53) for the function F we obtain

$$F = \frac{(1+\epsilon)}{b^{\frac{1}{2}}} \int_0^{\frac{2\epsilon}{1-\epsilon}} e^{-p(1+x)}\, \frac{dx}{x^{\frac{1}{2}}}\, \frac{(1+\delta x)^{3/2}}{(1+x)^{\frac{1}{2}}}\, \frac{1}{(1+\frac{\delta}{b}x)^{\frac{1}{2}}} \tag{4.56}$$

where $b = 1 + \delta = \dfrac{2\epsilon}{1+\epsilon}$; $\dfrac{\delta}{b} = \dfrac{1-\epsilon}{2\epsilon}$ and $\delta = \dfrac{\epsilon - 1}{1+\epsilon}$. Rewriting this in a more convenient form

$$F = \frac{(1+\epsilon)}{b^{\frac{1}{2}}} \, e^{-p} \int_{o}^{2\epsilon/1-\epsilon} e^{-px} \frac{dx}{x^{\frac{1}{2}}} \, g(x) \tag{4.57}$$

where $g(x) \equiv \dfrac{(1+\delta x)^{3/2}}{(1+x)^{\frac{1}{2}}} \cdot \dfrac{1}{(1+\frac{\delta}{b}x)^{\frac{1}{2}}}$ (4.58)

As pointed out previously, p is large and positive and the major contribution from the integral comes when x is near zero. Consequently an asymptotic evaluation of this integral is possible. This is done by expanding the expression for g(x) in (4.58) near x = 0 in a power series, and integrating term by term using Watson's Lemma. Near x = 0 the power series expansions of the following functions are

$$(1+\delta x)^{3/2} \simeq 1 + \frac{3}{2}\delta x + \frac{3}{8}\delta^2 x^2 - \frac{1}{16}\delta^3 x^3 + ---$$

$$(1+x)^{-\frac{1}{2}} \simeq 1 - \tfrac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + ---$$

$$(1+\frac{\delta}{b}x)^{-\frac{1}{2}} \simeq 1 - \frac{\delta}{2b}x + \frac{3}{8}\frac{\delta^2}{b^2}x^2 - \frac{5}{16}\frac{\delta^3}{b^3}x^3 + ---$$

and hence the expansion for g(x) near x = 0 is given by

$$g(x) \simeq 1 + a_1 x + a_2 x^2 + a_3 x^3 + O(x^4) \tag{4.59}$$

where $a_1 \equiv \dfrac{1-8\epsilon+3\epsilon^2}{4\epsilon(1+\epsilon)}$

$$a_2 \equiv \frac{1}{32\epsilon^2(1+\epsilon)^2}\left\{3-16\epsilon + 50\epsilon^2 + 16\epsilon^3 - 5\epsilon^4\right\} \tag{4.60}$$

$$a_3 \equiv \frac{1}{128\epsilon^3(1+\epsilon)^3}\left\{5-22\epsilon + 23\epsilon^2 - 36\epsilon^3 - 277\epsilon^4 - 22\epsilon^5 + 9\epsilon^6\right\}$$

Under this approximation the integral for the total energy loss ΔE for one complete orbit takes the form

$$\Delta E(2\pi,\epsilon) = \frac{-2(1+\epsilon)K\mu}{b^{\frac{1}{2}}} \int_{0}^{2\epsilon/1-\epsilon} e^{-px} \frac{dx}{x^{\frac{1}{2}}} \left\{1+a_1x+a_2x^2+a_3x^3+O(x^4)+ \text{---}\right\} \tag{4.61}$$

This is evaluated by using Watson's Lemma. Integrating term by term from zero to infinity, the asymptotic expansion of the integral for $\Delta E(2\pi,\epsilon)$ is obtained in terms of the powers of $1/p$ as

$$\Delta E\ (2\pi,\epsilon) = -2K\mu(1+\epsilon)\ \left(\frac{\pi}{pb}\right)^{\frac{1}{2}}\left\{1 + \frac{a_1}{2p} + \frac{3a_2}{4p^2} + \frac{15a_3}{8p^3} + O\left(\frac{1}{p^4}\right)\right\} \tag{4.62}$$

This is valid for all $0< \epsilon \leqslant 1$. However it is pointed out that this solution for the perturbation in specific total energy given by equation (4.62) is not good for values of ϵ very close to zero. This expansion breaks down when $\epsilon = 0$.

V. MECHANISM FOR THE CAPTURE OF A PARTICLE BY THE EARTH

Consider a particle to be orbiting around the sun. Let its velocity relative to the velocity of earth, which is also orbiting around the sun, be V_∞. V_∞ is also called the hyperbolic excess velocity or velocity at infinity with respect to earth. For simplicity, this particle is assumed to be travelling in the same direction as the earth. However, the mechanism holds for particles moving in both directions. When the particle begins to feel the gravitational pull of its big neighbour, the earth, its trajectory starts to get perturbed. Then it begins to move along a hyperbolic trajectory with respect to earth. If there is no dissipating medium near the planet of attraction it will merely be deflected by the planet and will move back into another trajectory around the sun. If there is a dissipating medium then the particle will lose some of its total energy whence it can either be captured by the earth or escape back to infinity depending on the energy loss. Thus in a dissipating medium around a planet a mechanism for capture can be established by studying the loss in the total energy of the particle during the first pass around the earth.

Let the initial energy of the incoming particle before encountering the dissipating medium be E_o per unit mass. Other perturbing forces due to sun, moon, etc., are neglected as they are small compared to the drag effect of the medium. The particle is travelling along a hyperbolic trajectory with the earth as one focus. Its eccentricity then is $\epsilon > 1$.

The total specific energy of an orbiting particle in general is given by

$$E_o = -\frac{2\mu}{a} \tag{5.1}$$

where a is the semi-major axis of the orbit. For a hyperbolic trajectory, according to Kepler's planetary laws, "a" is negative and hence the energy of the system is positive.

Now from the properties of a Keplerian orbit the relation (4.10) for the radial distance r, written as a function of C_o, μ, θ and ϵ can be expressed as a function of a, θ and ϵ. Thus

$$r_o = \frac{a(1-\epsilon^2)}{(1-\epsilon\cos\theta_o)} \tag{5.2}$$

when $\theta_o = \pi$, we obtain the relation between the perigee distance r_p, the semi-major axis a and the eccentricity ϵ as

$$r_p = a(1-\epsilon) \tag{5.3}$$

Since a is negative for a hyperbolic trajectory with $\epsilon > 1$, it is noted that r_p is always positive.

Eliminating a between equations (5.3) and (5.1), the specific total energy of the orbit is written in the form

$$E_o = -\frac{\mu}{2a} = \frac{\mu}{2r_p}(\epsilon - 1) \tag{5.4}$$

As the air density decreases rapidly with increasing altitude above the earth, particles with highly eccentric orbits are most affected by drag within a small section of the orbits when they are closest to the earth, i.e., most of the energy loss takes place near the perigee point. Because of this drag effect the total energy of the particle is reduced after the first pass around the perigee. Suppose this new total energy is still positive. Then its eccentricity is greater than 1. Hence the particle escapes back to infinity. If, on the other hand, the energy loss is such that the new total energy becomes negative, then Kepler's laws say the particle has a bounded orbit and hence is captured by the earth. Therefore the condition for capture of the particle is that its total energy after the first complete pass around the perigee must be negative.

Now let E be the total energy of the particle after the first pass around the earth. Then the criterion for capture to occur is

$$E = E_o + \Delta E_T \leqslant 0 \tag{5.5}$$

In order to evaluate this condition equations (4.52) or (4.62) can be used. However it is found that it is more convenient to use the result of equation (4.52). Thus recalling equation (4.52), we have

$$\Delta E_T = -(1+\epsilon)K\mu e^{p/2}\left\{2(1-\delta)K_o(p/2) + \delta K_1(p/2)\right\}$$

Substituting for δ from equation (4.28), the above expression for ΔE_T takes the form

$$\Delta E_T = -K\mu e^{p/2}\left\{4K_o(p/2) + (\epsilon - 1)\, K_1(p/2)\right\} \tag{5.6}$$

As pointed out previously, the argument p appearing in the Bessel functions K_o and K_1 is very large and hence the asymptotic expansions for $K_o(p/2)$ and $K_1(p/2)$ are used to simplify the expression for the total energy loss ΔE_T in equation (5.6). The asymptotic expansions of $K_o(p/2)$ and $K_1(p/2)$ are given by equations (4.44) in section 4. Only terms of $O(1/p)$ are used. Consequently, the substitution of equations (4.44) in equation (5.6) yields the following simple expression for ΔE_T.

$$\Delta E_T = -K\mu \left(\frac{\pi}{p}\right)^{\frac{1}{2}} \left[3 + \epsilon - \frac{1}{4p}(7-3\epsilon)\right] \tag{5.7}$$

Substituting equation (5.7) for ΔE_T and equation (5.4) for E_o in equation (5.5) the criterion for the capture of a particle is written as

$$\frac{\mu(\epsilon-1)}{2r_p} - K \left(\frac{\pi}{p}\right)^{\frac{1}{2}} \left[3 + \epsilon - \frac{1}{4p}(7-3\epsilon)\right] \leqslant 0 \tag{5.8}$$

with $p = \lambda r_p = r_p/H$

Finally, solving this for ϵ as a function of r_p and K, the criterion for the capture of a particle is obtained as

$$\epsilon \leqslant \frac{2+K\left(\frac{\pi}{p}\right)^{\frac{1}{2}}[12r_p-7H]}{2-K\left(\frac{\pi}{p}\right)^{\frac{1}{2}}[4r_p+3H]}$$

As all particles orbiting with $\epsilon < 1$ have bound orbits around the earth, the above condition is rewritten as

$$1 \leqslant \epsilon \leqslant \frac{2+K\left(\frac{\pi}{p}\right)^{\frac{1}{2}}[12r_p-7H]}{2-K\left(\frac{\pi}{p}\right)^{\frac{1}{2}}[4r_p+3H]} \tag{5.9}$$

This says that a particle having a mass m will be captured by the earth at a given perigee distance r_p if the value of its initial eccentricity ϵ satisfies the inequality given by equation (5.9).

It is more convenient to use the relative velocity V_∞ of the particle instead of the eccentricity ϵ in studying the capture mechanism. Now the energy integral for a Keplerian motion gives the following relation

$$\tfrac{1}{2}V^2 - \frac{\mu}{r} = E = -\frac{\mu}{2a} \tag{5.10}$$

Since V_∞ is defined as the velocity of the particle at infinity with respect to the earth, V_∞ can be obtained by letting $r \rightarrow \infty$ in equation (5.10). Thus

$$\tfrac{1}{2}V^2\,(r = \infty) = -\frac{\mu}{2a} = \tfrac{1}{2}V_\infty^{\,2} \tag{5.11}$$

Then by using equation (5.4) the desired relation between V_∞ and ϵ for a given r_p is obtained from (5.11) as

$$V_\infty^{\,2} = \frac{\mu}{r_p}(\epsilon - 1) \tag{5.12}$$

But $(\mu/r_p)^{\frac{1}{2}}$ is defined as the circular velocity at the perigee altitude.

$$\frac{\mu}{r_p} = V_{C_p}^2 \tag{5.13}$$

Then solving for ϵ

$$\epsilon - 1 = \frac{V_\infty^{\,2}}{V_{C_p}^2} = \frac{V_\infty^{\,2} r_p}{\mu} \tag{5.14}$$

Now subtracting 1 from the expression in equation (5.9) we obtain

$$0 \leqslant \epsilon - 1 \leqslant \frac{2+K(\frac{\pi}{p})^{\frac{1}{2}}(12r_p - 7H)}{2-K(\frac{\pi}{p})^{\frac{1}{2}}(4r_p + 3H)} - 1$$

or

$$0 \leqslant \epsilon - 1 \leqslant \frac{4K(\frac{\pi}{p})^{\frac{1}{2}}(4r_p - H)}{2-K(\frac{\pi}{p})^{\frac{1}{2}}(4r_p + 3H)} \tag{5.15}$$

Finally substituting equation (5.14) in (5.15), the required capture criterion in terms of the velocity at infinity is written as

$$0 \leqslant V_\infty^2 \leqslant \frac{4\mu K(\frac{\pi}{p})^{\frac{1}{2}}(4r_p - H)}{r_p[2-K(\frac{\pi}{p})^{\frac{1}{2}}(4r_p + 3H)]} \tag{5.16}$$

This relation says that a particle having a given mass m will be captured by the earth at a given perigee distance r_p, if V_∞ of the particle satisfies the capture condition given by (5.16) for these values of r_p and m. Then the particle will have a bound elliptic orbit around the earth.

VI. LIFETIME CALCULATIONS

After the particle has been captured by the earth, it becomes a natural satellite moving along a very highly eccentric orbit. As mentioned previously, the particle suffers an energy loss due to aerodynamic drag as it passes through the perigee region of the orbit. This causes the particles to undergo a drop in height at its next apogee. Due to the exponential nature of the density distribution in the atmosphere, the particle suffers very little or zero drag near the apogee region and consequently there is negligible loss in height at the next perigee point. Thus it can be said that the particle suffers a continuous apogee loss while suffering zero or very little perigee loss, i.e., perigee location remains almost constant as it orbits around the earth.

This explanation holds good for all orbits that are not circular; however for near circular orbits this breaks down and the estimation of lifetime becomes inaccurate according to the theory. But by this time the lifetime is almost zero. Hence the error in the estimation of lifetime is negligible. Thus, the highly eccentric orbit of the particle decays into a near circular orbit after a certain number of orbits with negligible loss in r_p. Then the orbital lifetime is defined as the number N_o of orbits required to circularise a given eccentric orbit into an orbit with $\epsilon \simeq 0$. The orbital lifetime N_o is derived in the following way.

The perturbation ΔE per one revolution can be expressed in terms of the perturbation Δa of the semi-major axis a. The relation between a and E given in equation (5.1) is differentiated once and written in the difference form as

$$\Delta E = \frac{\mu}{2a^2} \cdot \Delta a \tag{6.1}$$

from which Δa is given by

$$\Delta a = \frac{2a^2}{\mu} \Delta E \tag{6.2}$$

Substituting for ΔE from equation (4.62) the change in Δa for one revolution is obtained from (6.2) as

$$\Delta a(2\pi, \epsilon) = -4Ka^2(1+\epsilon)\left[\frac{\pi(1+\epsilon)}{2p\epsilon}\right]^{\frac{1}{2}}\left\{1 + \frac{1-8\epsilon+3\epsilon^2}{8\epsilon(1+\epsilon)p} + O(\frac{1}{p^2})\right\} \tag{6.3}$$

This relation is valid over most of the range of ϵ except near $\epsilon \sim 0$. Now the value of Δa given in equation (6.3) is for one revolution, i.e., for $\Delta n = 1$. Consequently equation (6.3) can be written as

$$\frac{\Delta a}{\Delta n} = -\ 4Ka^2(1+\epsilon)(\frac{1+\epsilon}{2\epsilon})^{\frac{1}{2}}(\frac{\pi}{p})^{\frac{1}{2}}\left[1 + \frac{1-8\epsilon+3\epsilon^2}{8\epsilon(1+\epsilon)p} + O(\frac{1}{p^2})\right]$$

Now using total differentials for $\Delta a/\Delta n$ the above expression is rewritten as

$$\frac{da}{dn} = -4Ka^2(1+\epsilon)(\frac{1+\epsilon}{2\epsilon})^{\frac{1}{2}}(\frac{\pi}{p})^{\frac{1}{2}}\left(1 + \frac{1-8\epsilon+3\epsilon^2}{8\epsilon(1+\epsilon)p} + ---\right) \tag{6.4}$$

Therefore

$$dn = -\frac{1}{4a^2K}\frac{1}{(1+\epsilon)}(\frac{p}{\pi})^{\frac{1}{2}}(\frac{2\epsilon}{1+\epsilon})^{\frac{1}{2}}\left(\frac{da}{1 + \frac{1-8\epsilon+3\epsilon^2}{8\epsilon(1+\epsilon)p} + ---}\right)$$

Finally expanding the denominator for large p, this is written in the following form

$$dn = -\frac{1}{2K}\left(\frac{p}{2\pi}\right)^{\frac{1}{2}} \frac{\epsilon^{\frac{1}{2}}}{a^2(1+\epsilon)^{3/2}} \left(1 - \frac{1-8\epsilon+3\epsilon^2}{8\epsilon(1+\epsilon)p} + ---\right) da \tag{6.5}$$

We define the number of orbits a particle makes in dropping from a_i to a_f as N given by

$$N = -\int_{a_f}^{a_i} dn \tag{6.6}$$

where a_i and a_f are the initial and final values of a. Consequently integration of equation (6.5) yields

$$N = \frac{1}{2K}\left(\frac{p}{2\pi}\right)^{\frac{1}{2}} \int_{a_f}^{a_i} \frac{da}{a^2} \left(\frac{\epsilon^{1/3}}{1+\epsilon}\right)^{3/2} \left[1 - \frac{1-8\epsilon+3\epsilon^2}{8\epsilon(1+\epsilon)p}\right] \tag{6.7}$$

Here p and K appear as parameters. Before the orbital life is determined from the above integral, the integrand has to be expressed in terms of the semi-major axis a, i.e., the eccentricity ϵ has to be expressed explicitly in terms of a.

The relation between a and ϵ as given by (5.3) is

$$r_p = a(1-\epsilon)$$

Solving for ϵ

$$\epsilon = \frac{a-r_p}{a} \tag{6.8}$$

Hence

$$1 + \epsilon = \frac{2a-r_p}{a} \tag{6.9a}$$

and

$$1 - 8\epsilon + 3\epsilon^2 = \frac{1}{a^2}[3r_p^2 - 2a(2a-r_p)] \tag{6.9b}$$

Substituting these expressions in equation (6.7) the integral for N takes the form

$$N = \frac{1}{2K}\left(\frac{p}{2\pi}\right)^{\frac{1}{2}} \int_{a_f}^{a_i} \frac{da}{a} \frac{(a-r_p)^{\frac{1}{2}}}{(2a-r_p)^{3/2}} \left\{1 - \frac{1}{8p}\frac{3r_p^2-2a(2a-r_p)}{(a-r_p)(2a-r_p)}\right\} \tag{6.10a}$$

or

$$N = \frac{1}{2K}\left(\frac{p}{2\pi}\right)^{\frac{1}{2}} \int_{a_f}^{a_i} \frac{(a-r_p)^{\frac{1}{2}}}{(2a-r_p)^{3/2}} \frac{da}{a}$$

$$- \frac{1}{16Kp}\left(\frac{p}{2\pi}\right)^{\frac{1}{2}} \int_{a_f}^{a_i} \frac{3r_p^2-2a(2a-r_p)}{(a-r_p)^{\frac{1}{2}}(2a-r_p)^{5/2}} \frac{da}{a} \tag{6.10b}$$

This can be integrated if the following transformation is used. Let

$$2(a-r_p) \equiv r_p \tan^2\psi \tag{6.11}$$

where ψ is the new variable. r_p is assumed constant in the analysis as mentioned earlier. Thus

$$da = r_p \tan\psi \sec^2\psi d\psi$$

Using this transformation the integral for N in equation (6.10b) is transformed into the following form

$$N = \frac{1}{2K}\left(\frac{p}{\pi}\right)^{\frac{1}{2}} \frac{1}{r_p} \int \frac{\sin^2\psi\cos\psi d\psi}{2-\sin^2\psi}$$

$$- \frac{1}{8Kr_p} \cdot \frac{1}{p}\left(\frac{p}{\pi}\right)^{\frac{1}{2}} \int \frac{3-(1+\sec^2\psi)\sec^2\psi}{(1+\sec^2\psi)\sec^3\psi} d\psi \tag{6.12}$$

This expression is in a more convenient form for integration.

This is evaluated by using Dwight's integral tables, as

$$N = \frac{1}{2Kr_p}\left(\frac{p}{\pi}\right)^{\frac{1}{2}}\left[\sqrt{2}\tanh^{-1}\frac{\sin\psi}{\sqrt{2}} - \sin\psi\right]$$

$$- \frac{1}{8Kr_p}\frac{1}{p}\left(\frac{p}{\pi}\right)^{\frac{1}{2}}\left[\frac{3}{\sqrt{2}}\tanh^{-1}\frac{\sin\psi}{\sqrt{2}} - \sin^3\psi - \sin\psi\right] \qquad (6.13)$$

But $\sin\psi$ is given by (6.11) as

$$\sin\psi = \tan\psi\cos\psi = \frac{\tan\psi}{[1+\tan^2\psi]^{\frac{1}{2}}}$$

$$= \sqrt{2}\left(\frac{a-r_p}{2a-r_p}\right)^{\frac{1}{2}} \qquad (6.14)$$

Finally after substituting equation (6.14) in (6.13) the total number of orbits a particle makes in a given interval a_i to a_f is written as

$$N = \frac{1}{Kr_p}\left(\frac{p}{2\pi}\right)^{\frac{1}{2}}\Big\{\left(1 - \frac{3}{8p}\right)\left[\tanh^{-1}q_i - \tanh^{-1}q_f\right]$$

$$- \left(1 - \frac{1}{4p}\right)\left[q_i - q_f\right]$$

$$+ \frac{1}{2p}\left[q_i^3 - q_f^3\right]\Big\} \qquad (6.15)$$

where $q^2 \equiv \dfrac{a-r_p}{2a-r_p}$

By using equations (6.8) and (6.9a) q is written as

$$q^2 \equiv \frac{a-r_p}{2a-r_p} = \frac{\epsilon}{1+\epsilon} \qquad (6.16)$$

By substituting for q from (6.16) in (6.15), N is expressed explicitly as a function of ϵ. Thus

$$N(\epsilon_f) = \frac{1}{Kr_p} \left(\frac{p}{2\pi}\right)^{\frac{1}{2}} \left\{ \left(1 - \frac{3}{8p}\right) \left[\tanh^{-1} \left(\frac{\epsilon}{1+\epsilon}\right)_i^{\frac{1}{2}} - \tanh^{-1} \left(\frac{\epsilon}{1+\epsilon}\right)_f^{\frac{1}{2}} \right] \right.$$

$$- \left(1 - \frac{1}{4p}\right) \left[\left(\frac{\epsilon}{1+\epsilon}\right)_i^{\frac{1}{2}} - \left(\frac{\epsilon}{1+\epsilon}\right)_f^{\frac{1}{2}} \right]$$

$$\left. + \frac{1}{2p} \left[\left(\frac{\epsilon}{1+\epsilon}\right)_i^{3/2} - \left(\frac{\epsilon}{1+\epsilon}\right)_f^{3/2} \right] \right\} \tag{6.17}$$

where subscripts i and f refer to initial and final values.

This is the number of orbits a particle makes in decaying from an orbit of eccentricity ϵ_i to a final orbit of eccentricity ϵ_f. As mentioned previously, the orbital lifetime N_o is defined as the number of orbits the particle makes while its eccentricity decays to zero, i.e., $N_o = N(\epsilon_f = 0)$.

By setting $\epsilon_f = 0$ in equation (6.17) the orbital lifetime is obtained as

$$N_o = \frac{1}{Kr_p} \left(\frac{p}{2\pi}\right)^{\frac{1}{2}} \left\{ \left(1 - \frac{3}{8p}\right) \tanh^{-1} \left(\frac{\epsilon}{1+\epsilon}\right)_i^{\frac{1}{2}} \right.$$

$$\left. - \left(1 - \frac{1}{4p}\right) \left(\frac{\epsilon}{1+\epsilon}\right)_i^{\frac{1}{2}} + \frac{1}{2p} \left(\frac{1}{1+\epsilon}\right)_i^{3/2} \right\} \tag{6.18}$$

It has already been noted in part 1 of section IV that the usefulness of the solution for the energy perturbation ΔE is not good when ϵ has a value very close to zero. The same inaccuracy is carried over into the derivation of N_o. (Auxiliary calculations near $\epsilon \simeq 0$ showed the error in the estimation of N_o to be about 15%.) Since we are looking for a few orders of magnitude estimation of the flux concentration this error is considered to be insignificant.

Finally, when the particles are captured their eccentricities are very close to one. Therefore we will set $\epsilon_i = 1$ in the estimation of the total orbital lifetime as it is counted from the time the particles are captured. This only causes an insignificant error in N_o. Hence the total orbital lifetime is

$$N_o = \frac{1}{Kr_p}\left(\frac{p}{2\pi}\right)^{\frac{1}{2}}\left[0.1743 + \frac{0.09216}{4p}\right] = N_o(r_p, m) \qquad (6.19)$$

VII. CONSTRUCTION OF STATISTICAL MODEL OF MICROMETEORITE DISTRIBUTION

So far in this analysis, the motion of a single particle in a dissipating medium has been considered where the particle is assumed to be obeying Kepler's planetary laws at every instant. A theory for the capture mechanism and the resulting orbital lifetime has been developed from the solutions for the perturbation ΔE of the specific total energy of the particle.

Now a statistical model of the particle flux distribution in the interplanetary space is developed. In this model an observer located on a sphere of radius $r_{ob} > R$ counts the number of particles crossing this sphere from both sides.

Let us consider a swarm of particles orbiting around the sun at earth's distance from the sun. These particles are assumed to have a mass distribution as well as a velocity distribution. Since these particles are orbiting at the earth's distance, an observer on the earth will see particles moving all around him. With such a picture in mind, let us observe the behavior of a single particle orbiting around the sun. As long as this particle does not feel the gravitational attraction of the earth its orbit will be unperturbed. Perturbations due to other planets, etc., are neglected here. When the particle feels the gravitational pull of the earth, its orbit is perturbed and its motion becomes hyperbolic with respect to the earth. Let us attach a plane perpendicular to the direction of motion of this particle and position this plane at the place where the particle begins to feel the gravitational pull of the earth. It will be located at an infinite distance from the earth. Also it serves as a reference plane that divides the two body motion, i.e., sun-particle motion, from the three body motion, i.e., sun-particle-earth motion. Once the particle pierces this plane, it is dominated by the earth's gravitational pull.

It is clear from this description of the particle motion that there exists such planes for all the different particles orbiting at earth's distance from the sun. The implication here is that every particle has its own plane at infinite distance from the earth which separates it from the region in which the gravitational pull of the earth is present.

However for the construction of the present statistical model, these various planes are superimposed upon each other to obtain a single representative plane at an infinite distance from the earth. This superposition simplifies the construction of the model. Particles crossing this plane feel the presence of the earth, otherwise their orbits are unperturbed by the earth. This is shown in figure 2.

This plane is designated the "ξ-plane." Distance along this plane is measured in terms of ξ. After the particles pierce this plane they will move along hyperbolic trajectories with respect to earth.

Let the relative velocity of the particles with respect to earth be V_∞ for all the particles beyond the ξ-plane. We will now trace the path of a particle after it has pierced the ξ-plane at a given ξ. The particle comes under the gravitational pull of the earth and hence approaches the earth along a hyperbolic trajectory.

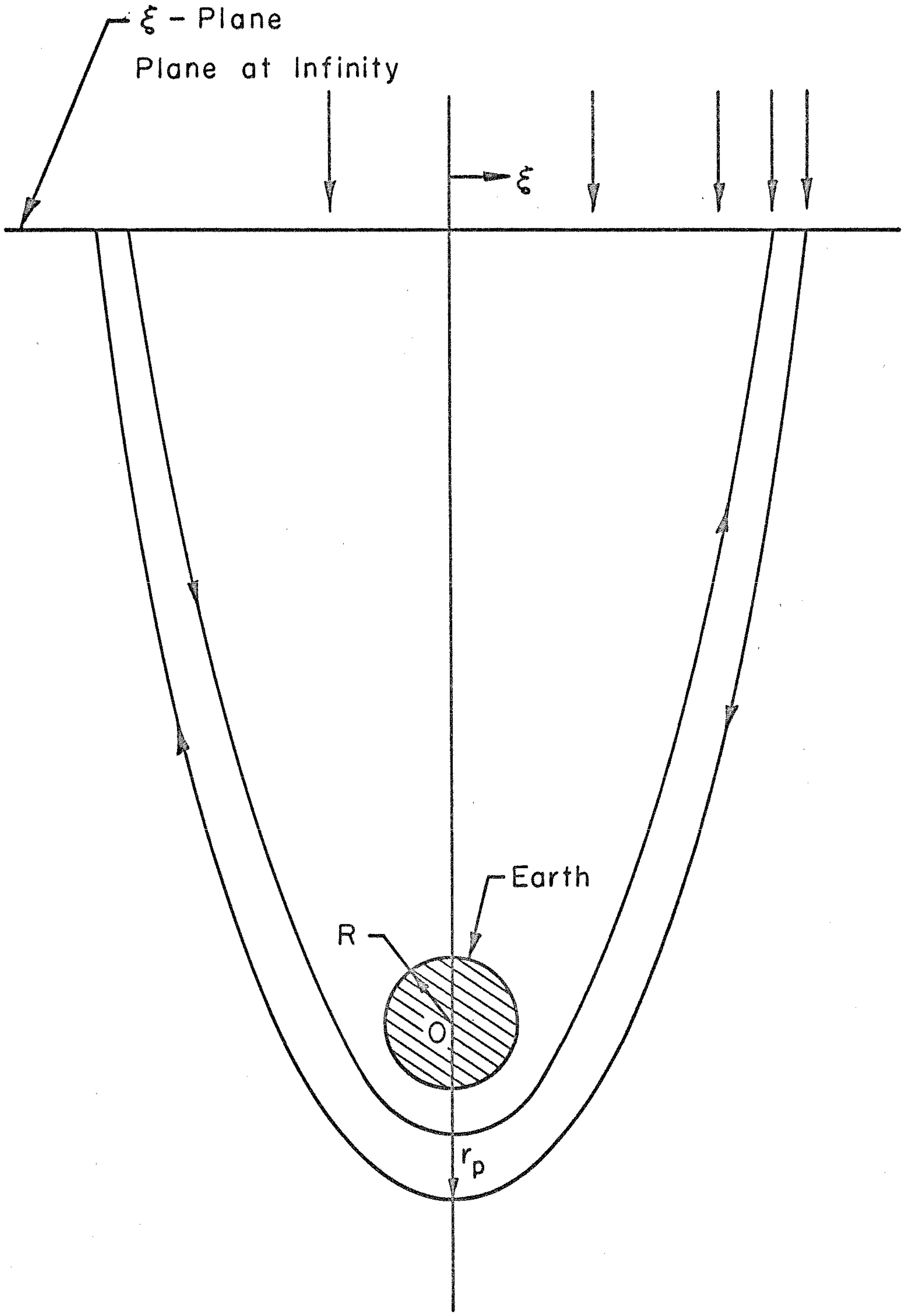

Figure 2. Diagram showing "ξ-plane"

The distance of closest approach r_p to the earth depends on the initial angular momentum and velocity V_∞. Now from the constancy of the angular momentum we have

$$\xi V_\infty = r_p V_p \tag{7.1}$$

where V_p is the velocity of the particle at perigee. And the energy integral gives the relation

$$V^2 = \frac{2\mu}{r} + V_\infty^2 \tag{7.2}$$

which for $r = r_p$ gives

$$V_p^2 = \frac{2\mu}{r_p} + V_\infty^2 \tag{7.3}$$

Eliminating V_p between (7.3) and (7.2), and solving for ξ gives

$$\xi = \left(1 + \frac{2\mu}{r_p V_\infty^2}\right)^{\frac{1}{2}} r_p \tag{7.4}$$

This distance ξ is called the impact parameter. Equation (7.4) says that all particles having impact parameters less than or equal to ξ for a given V_∞ will intersect the sphere of radius r_p.

Now let M(m) be the number of particles with mass between m and m + dm in the interplanetary space streaming through a unit area per unit time. Then M(m)dm represents the differential flux of micrometeorites with masses between m and m+dm.

From zodiacal light measurements and other observations a general model for the interplanetary particle flux distribution has been established by various investigators in this field. This is represented in the following form

$$M(m)dm = C_1 m^{-y}\, dm \tag{7.5}$$

where C_1 and y are constants. These constants have not been clearly established. Various authors have suggested different values for C_1 and y.

However, it has been established that M increases with decreasing mass of the particle. It is clear that the exponent y must have a value greater than 1 in order that the cumulative mass distribution is finite. This differential distribution function M(m) is independent of the radial distance r and also the impact parameter ξ. It is a measure of the constant flux distribution far away in the interplanetary space.

Now consider the following picture of the model shown in figure 3. The ξ-plane is divided into annular regions as shown. Particles are assumed to be streaming through these annular zones. The sizes of these zones are given by ξ_R, ξ_D, ξ_G and ξ. For a given V_∞, ξ_R is determined by the radius of the earth, ξ_D is calculated from the perigee radius corresponding to the capture of a particle of a given mass m and ξ_G is finally determined by the radius of the observer sphere. ξ corresponds to impact parameters of particles that are greater than ξ_G or ξ_D depending on the radius of the observer sphere. Particles with $\xi > \xi_G > \xi_D$ or $\xi > \xi_D > \xi_G$ are not seen by the observer and hence are not counted. Now ξ_R is determined by substituting $r_p = R$ in equation (7.4). Thus

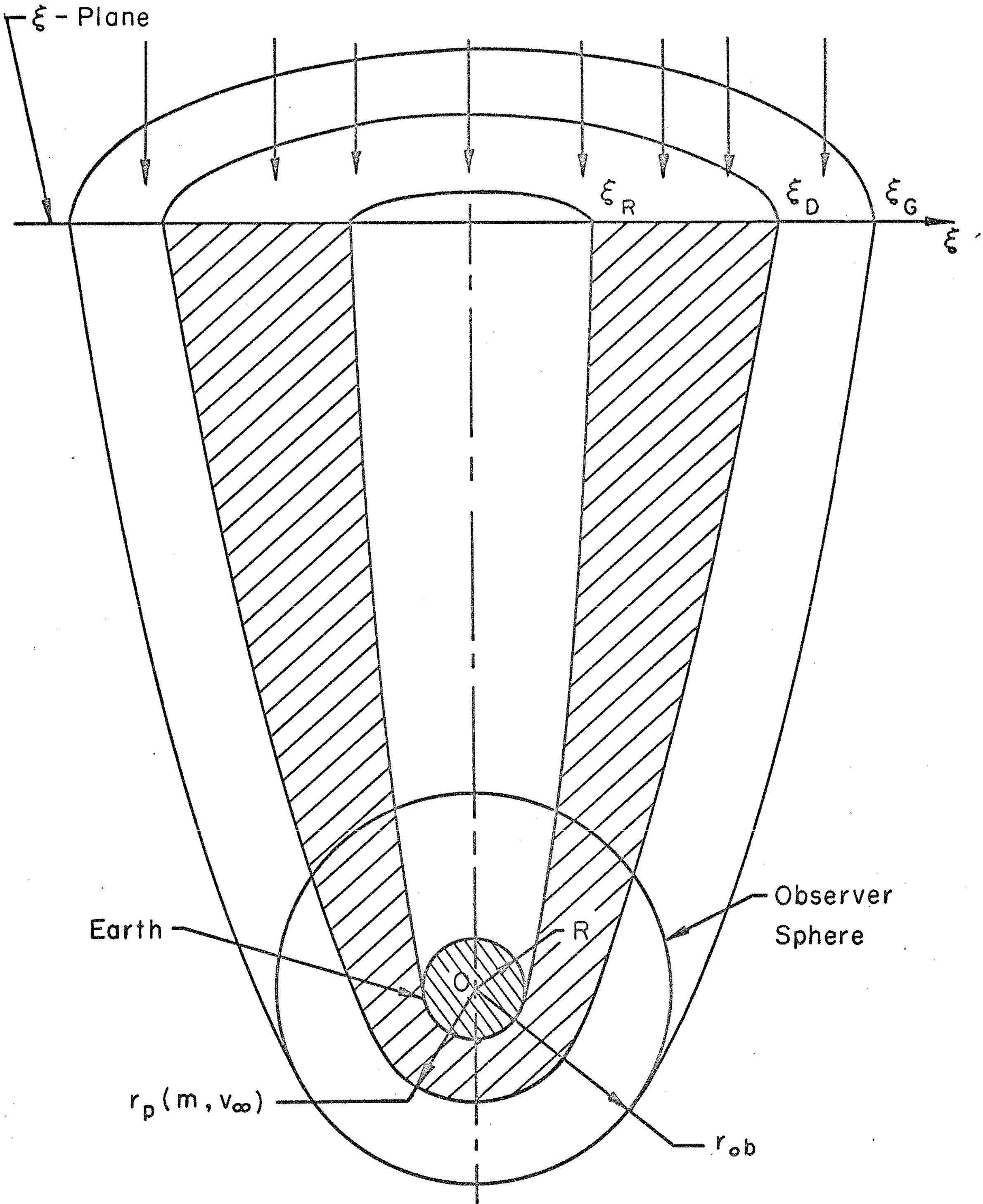

Figure 3. Diagram of counting zones for $\xi_G > \xi_D$ shown in cross section

$$\xi_R = R\left(1 + \frac{2\mu}{RV_\infty^2}\right)^{\frac{1}{2}} \tag{7.6}$$

ξ_R corresponds to that impact parameter where particles with $\xi \leq \xi_R$ impact directly on the earth.

Now an observer sitting on the sphere of radius r_{ob} counts the particle flux crossing the sphere as described below.

All particles having impact parameters $\xi \leq \xi_R$ collide with the earth directly and are destroyed. Hence the observer sees these particles only once as they pass by him to collide with the earth. Therefore the total flux he counts for particles with $\xi \leq \xi_R$ is given by

$$N'_R = \frac{\pi \xi_R^2}{4\pi r_{ob}^2} M(m) = \left[\frac{R^2}{4r_{ob}^2}\left(1 + \frac{2\mu}{RV_\infty^2}\right)\right] M(m) \tag{7.7}$$

Here r_{ob} is the radius of the observer sphere.

Thus the quantity in the square brackets represents the fraction of the particles that strike the earth for a given V_∞. We therefore define a flux multiplier $\frac{N'_R}{M} \equiv N_R(V_\infty)$ which represents the fraction he counts.

ξ_D is the impact parameter that corresponds to a given perigee distance r_p at which a particle of mass m and velocity V_∞ is captured. As explained before capture is the result of energy loss due to the presence of the aerodynamic drag. Hence ξ_D is determined from the criterion for capture given in section 5. This criterion given in equation (5.16) estimates the maximum perigee distance r_p at which an incoming particle of mass m and velocity

V_∞ is captured by the earth. Therefore ξ_D is calculated by choosing a given V_∞ and m and calculating the corresponding perigee distance $r_p(m, V_\infty)$ from equation (5.16) and then substituting this r_p and V_∞ in equation (7.4). Consequently, for a given sized particle ξ_D takes the value

$$\xi_D = r_p(m, V_\infty) \left[1 + \frac{2\mu}{r_p(m, V_\infty) V_\infty^2}\right]^{\frac{1}{2}} \tag{7.8}$$

All particles with this mass m and velocity V_∞ will be captured by the earth as natural satellites of the earth if their impact parameters have values between $\xi_R \leqslant \xi \leqslant \xi_D$.

Now these captured particles have certain orbital lifetimes N_o as derived in section 6. This number N_o varies inversely with the perigee distance and density at perigee as observed from equation (6.19). Since the density ρ_p increases exponentially with decreasing r_p, the lifetime N_o decreases with decreasing r_p. But r_p is a function of ξ as shown in equation (7.4) varying directly with ξ. Hence, for a particle with an impact parameter ξ close to ξ_R, N_o is very small and for a particle with ξ near ξ_D, N_o is large. Therefore it is observed that N_o is a function of ξ for a captured particle.

Consider one such captured particle. Its impact parameter is $\xi_R < \xi < \xi_D$. Then our observer sitting on the observer sphere of radius r_{ob} counts the particle twice per one orbit as it goes into the sphere on one side and comes out on the other side. But the particle makes $N_o(\xi)$ orbits. Therefore the total number of times that the observer counts this particle is $2N_o(\xi)$. Then the total flux for all particles is the integral of $N_o(\xi)$ over all ξ. Therefore the total flux designated N_D' for $\xi_R \leq \xi \leq \xi_D$ is

$$N_D' = \frac{2 \times 2\pi M}{4\pi r_{ob}^2} \int_{\xi_R}^{\xi_D} N_o(\xi)\, \xi d\xi \tag{7.9}$$

The limits of integration ξ_R and ξ_D are calculated from equations (7.6) and (7.8) for each V_∞. In terms of the flux multiplier, this is written as

$$N_D \equiv \frac{N_D'}{M} = \frac{1}{r_{ob}^2} \int_{\xi_R}^{\xi_D} N_o(\xi)\, \xi d\xi \tag{7.10}$$

Now ξ_G is directly related to r_{ob}. It can either be greater than ξ_D or less than ξ_D depending on the value of r_{ob} of the observer sphere. Thus we have two possible cases.

<u>Case 1:</u> $\xi_G > \xi_D$

In this case the observer counts some more particles as the observer radius r_{ob} is bigger than capture radius r_p. Particles with impact parameters $\xi_D \leq \xi \leq \xi_G$ are perturbed by the earth and hence pass around the earth along a hyperbolic trajectory. Then the observer sees these particles passing through the sphere on one side and going out to infinity on the other side. Thus he counts these particles twice. Therefore the total flux designated N_G' for all particles with $\xi_D \leq \xi \leq \xi_G$ is given by

$$N_G' = \frac{2 \times \pi M}{4\pi r_{ob}^2} (\xi_G^2 - \xi_D^2) \tag{7.11}$$

where

$$\xi_G = r_{ob}\left(1 + \frac{2\mu}{V_\infty^2 r_{ob}}\right)^{\frac{1}{2}} \tag{7.12}$$

In terms of the flux multiplier N_G, it is

$$N_G \equiv \frac{N_G'}{M} = \frac{1}{2r_{ob}^2}\,(\xi_G^2 - \xi_D^2) \tag{7.13}$$

All the other particles with $\xi > \xi_G$ are not counted in the analysis as the observer does not see them piercing the observer sphere. Consequently, the flux of all particles with a given V_∞ and m integrated over all ξ when $\xi_G > \xi_D$ is

$$N_1' = N_R' + N_D' + N_G'$$

or in terms of the flux multiplier N_1 this is

$$N_1 = \frac{N_1'}{M} = \frac{1}{4r_{ob}^2}\left\{\xi_R^2 + 4\int_{\xi_R}^{\xi_D} N_o(\xi)\xi d\xi + 2(\xi_G^2 - \xi_D^2)\right\} \tag{7.14}$$

<u>Case 2:</u> $\xi_G < \xi_D$

In this case the observer sphere radius r_{ob} is smaller than the drag capture radius $r_p(m, V_\infty)$. The observer will not see the captured particles lying in the annular zone for $\xi_G < \xi < \xi_D$ till their perigee distance reduces to a value smaller than the observer radius r_{ob}. This will occur only after the particles have achieved a near circular orbit. But in the present analysis it is assumed that the perigee position remains approximately constant. Consequently the observer will not count the particles that are captured outside of ξ_G. However, calculations have shown that this affects only the smaller sized particles. The implication here is that the basic model is unaltered.

Consequently Case 1, where $\xi_G > \xi_D$, is chosen for computation in order to include particles of the smaller 1 micron size in the overall picture.

Now all the fluxes given by equations (7.7), (7.9) and (7.11) are divided by $4\pi r_{ob}^2$. Consequently, they give the values of the flux averaged over the observer sphere. The flux multiplier given by equation (7.14) estimates the average fraction of particles, with a given initial velocity V_∞, crossing an arbitrarily oriented counter of unit area located on the observer sphere of radius r_{ob} in unit time.

Equation (7.14) representing the total spatial flux multiplier N_1 is a function of V_∞ and m. In order to obtain the total integrated flux distribution, N_1 has to be integrated over both V_∞ and m. However, in this paper only the velocity integration is performed, thereby giving a differential flux distribution of masses between m and m + dm.

VIII. COMPUTATION OF MICROMETEORITE DISTRIBUTIONS

The differential flux distribution is computed by integrating the statistical model developed in the last section first over the impact parameter ξ and then over the velocity V_∞.

1. Integration over the Impact Parameter ξ

Now from equation (7.14) we have for the flux multiplier

$$N_1 = \frac{1}{4r_{ob}^2} \left\{ \xi_R^2 + 2\left(\xi_G^2 - \xi_D^2\right) + 4\int_{\xi_R}^{\xi_D} N_o(\xi)\xi d\xi \right\} \qquad (8.1)$$

with

$$\xi_R^2 = R^2\left(1 + \frac{2\mu}{V_\infty^2 R}\right)$$

$$\xi_D^2 = r_p^2(V_\infty, m)\left(1 + \frac{2\mu}{V_\infty^2 r_p(m, V_\infty)}\right)$$

and

$$\xi_G^2 = r_{ob}^2\left(1 + \frac{2\mu}{V_\infty^2 r_{ob}}\right)$$

Both ξ_R and ξ_G are directly determined once V_∞ is fixed. But the estimation of ξ_D is a little complicated. First we have to determine the maximum capture perigee distance r_p at which a particle with a given V_∞ and m is captured. This is done by using the capture criterion given by relation (5.16). Then this r_p together with the velocity V_∞ is used to determine ξ_D. Once ξ_D is calculated for a given V_∞, then the first two terms on the right hand side of equation (8.1) are determined. However, the evaluation of the integral in the last term of the right hand side is still complicated.

Let us now define the integral in equation (8.1) as

$$T_1 \equiv 4 \int_{\xi_R}^{\xi_D} N_o(\xi)\, \xi d\xi \tag{8.2}$$

Now equation (6.19) is rewritten in the form

$$N_o(r_p, m) = \frac{0.1743}{A_1 \rho_p r_p} \left(\frac{r_p}{2\pi H}\right)^{\frac{1}{2}} \left[1 + \frac{0.1322H}{r_p}\right] \tag{8.3}$$

where $A_1 \equiv \frac{C_D A}{2m} = \frac{K}{\rho_p}$

Substitution for N_o from (8.3) in (8.2) yields

$$T_1 = \frac{0.6972}{A_1} \int_{\xi_R}^{\xi_D} \frac{\xi d\xi}{\rho_p r_p} \left(\frac{r_p}{2\pi H}\right)^{\frac{1}{2}} \left[1 + \frac{0.1322H}{r_p}\right] \tag{8.4}$$

In order to evaluate T_1, the integrand in (8.4) has to be expressed explicitly as a function of ξ. The relation between ξ and r_p is given by (7.4). Solving for r_p we get

$$r_p = \frac{\mu}{V_\infty^2} \left[\left(1 + \frac{V_\infty^4 \xi^2}{\mu^2}\right)^{\frac{1}{2}} - 1\right] = r_p(\xi) \tag{8.5}$$

Since the density ρ_p and the scale height H are implicit functions of r_p, they cannot be expressed as explicit functions of ξ. However both ρ_p and H are tabulated for a wide range of values of r_p. Consequently we have turned to numerical integration for the evaluation of the function T_1. Simpson's rule is used in the numerical integration.

It is observed that the computation becomes very involved if the integral in equation (8.4) is numerically integrated in terms of the impact parameter ξ. This is because the tabulated values of ρ_p and H are expressed in terms of r_p at some convenient intervals. Then for every value ξ, r_p as calculated from (8.5) can be different from those for which ρ_p and H are tabulated. This leads to interpolation. This is easily overcome by using r_p as the variable of integration instead of ξ. Thus rewriting equation (7.4)

$$\xi^2 = r_p^2 \left(1 + \frac{2\mu}{r_p V_\infty^2}\right) = r_p^2 + \frac{2\mu r_p}{V_\infty^2} \tag{8.6}$$

Differentiation of equation (8.6) yields the necessary relation between the differentials $d\xi$ and dr_p. Thus

$$2\xi d\xi = 2r_p\, dr_p + \frac{2\mu}{V_\infty^2}\, dr_p$$

Therefore

$$\xi d\xi = \left(1 + \frac{\mu}{V_\infty^2 r_p}\right) r_p\, dr_p \tag{8.7}$$

Limits of integration are easily determined. When

$$\xi = \xi_R \;;\; r_p = R$$

and (8.8)

$$\xi = \xi_D \;;\; r_p = r_p(m, V_\infty) \text{ at capture.}$$

Substitution of (8.7) and (8.8) in (8.4) yields the following integral for T_1

$$T_1 = \frac{2.7815 \times 10^{-1}}{A_1} \int_R^{r_{p_c}} \left(\frac{r_p}{H}\right)^{\frac{1}{2}} \left(1 + \frac{\mu}{r_p V_\infty^2}\right)\left(1 + \frac{0.1322H}{r_p}\right) \frac{dr_p}{\rho_p} \tag{8.9}$$

This is simplified further if r_p is replaced by the variable z given by the following relation

$$r_p = R + z \tag{8.10}$$

where R is the earth's radius.

Here z is the height above the earth's surface at which the perigee point of the particle is located.

When $r_p = R$; $z = 0$ and $r_p \equiv r_{p_c}(m, V_\infty)$ at capture; $z = z_c$.

Then

$$T_1 = \frac{2.7815 \times 10^{-1}}{A_1} \int_0^{z_c} \left(\frac{r_p}{H}\right)^{\frac{1}{2}} \left(1 + \frac{\mu}{r_p V_\infty^2}\right)\left(1 + \frac{0.1322H}{r_p}\right) \frac{dz}{r_p} \tag{8.11}$$

$$\equiv \frac{2.7815 \times 10^{-1}}{A_1} \int_0^{z_c} T_o(z)dz \tag{8.11a}$$

and hence the flux multiplier over all ξ is given by

$$N_1 = \frac{1}{4r_{ob}^2} \left\{R^2\left(1 + \frac{2\mu}{RV_\infty^2}\right) + 2\left[r_{ob}^2\left(1 + \frac{2\mu}{V_\infty^2 r_{ob}}\right) - r_{p_c}^2\left(1 + \frac{2\mu}{V_\infty^2 r_{p_c}}\right)\right] + \frac{2.7815 \times 10^{-1}}{A_1} \int_0^{z_c} \left(\frac{r_p}{H}\right)^{\frac{1}{2}} \left(1 + \frac{\mu}{r_p V_\infty^2}\right)\left(1 + \frac{0.1322H}{r_p}\right) \frac{dz}{\rho_p}\right\} \tag{8.12}$$

Equation (8.11) is first integrated numerically according to Simpson's Rule and then the flux multiplier is evaluated from (8.12). The Simpson's Rule is given by

$$\int_{x_o}^{x_n} f(x)dx = \frac{\Delta x}{3} (f_o + 4f_1 + 2f_2 + --- +2f_{n-2}+4f_{n-1}+f_n) \quad (8.13)$$

and $x_n = x_o + n\Delta x$ and n is even, with

$f_n = f(x_n)$; $\Delta x = x_1 - x_o = x_2 - x_1 = ---$

The integrand in equation (8.11) is a smooth function. Therefore n is chosen to be 10 for all integrations. The evaluation of N_1 is as follows.

First a particle of given size and mass is chosen. Then for this particle a capture perigee distance r_p is chosen. Using the capture criterion given in section 5 the initial V_∞ necessary for this particle to be captured at this r_p is computed. With these values of m, V_∞ and r_p the integral given in (8.11) is evaluated numerically. All values of ρ_p and H are taken from U. S. Standard Atmosphere 1962 (7). Then N_1 is computed from (8.12).

Again another r_p is chosen for the same particle. And a new V_∞ is computed from the capture criterion. Another value of N_1 is computed for this new V_∞ and so on. Thus evaluations of the flux distribution over a wide range of values of the velocity V_∞ from zero to about 25 km/sec are carried out.

Also this computation scheme has been used to calculate the flux for four different values of the radius of the observer sphere ranging from 7500 km to 25,000 km.

In all the computations the particles are assumed to be spherical in shape and their material density ρ_m to be 1 gm/c.c. as before.

A sample calculation of the flux multiplier N_1 for a particle, 10 microns in diameter, is shown below. Now

$$A_1 = \frac{C_D A}{2m} = \frac{C_D}{2} \cdot \frac{1.5}{\rho_m d}$$

for a spherical particle. Here ρ_m is the density of the particle and d is the diameter. The value of the drag coefficient C_D, as mentioned previously, is very nearly equal to 2. Consequently, it is taken as 2 in the computations. Then

$$A_1 = 1.5 \times 10^{-4} \frac{(\mathrm{km})^2}{\mathrm{kg}}$$

for our test particle.

Now the capture perigee is chosen to be $z_c = 700$ km. Hence $r_p = R + z_c = 6378 + 700 = 7078$ km. The radius of the earth is taken as $R \equiv 6378$ km. At this altitude $\rho_p = 1.537 \times 10^{-4}$ kgm / $(\mathrm{km})^3$ and $H = 97.435$ km. Now

$$K = A_1 \rho_p$$

Hence

$$K = 2.3055 \times 10^{-8}/\mathrm{km}$$

$$p = \frac{r_p}{H} = 72.6433 \text{ and } p^{\frac{1}{2}} = 8.5231.$$

Substituting all these values in (5.16), the velocity V_∞ necessary for the capture of the particle is calculated to be

$$V_\infty^2 (r_p, m) = 1.5234 \times 10^{-2}$$

or

$$V_{\infty}(r_p, m) = 0.1234 \text{ km/sec}$$

This says that a spherical particle 10 microns in diameter and 1 gm/c.c. density orbiting around the sun with relative velocity $V_{\infty} = 0.1234$ km/sec will be captured by the earth as a natural satellite of the earth when its perigee distance is less than or equal to 700 km above the earth surface.

Then with these values of K, V_{∞} and z_c, the flux of particles originating from the capture mechanism is calculated from the integral for T_1 given by equation (8.11).

The following Table II illustrates this integration. Using Simpson's Rule of integration,

$$\int_0^{700} T_o dz = \frac{70}{3} \times 7.9577 \times 10^8 = 1.85679 \times 10^{10}$$

Therefore

$$T_1 = \frac{2.7815 \times 10^{-1}}{K_1} \times \int_0^{700} T_o dz$$

$$= 3.4431 \times 10^{13}$$

And

$$\xi_R^2 = 33.3814 \times 10^{10}$$

$$\xi_D^2 = 37.0456 \times 10^{10}$$

The observer sphere radius is chosen to be $r_{ob} = 7500$ km for this test case.

z (km)	r_p (km)	ρ_p kgm/(km)^3	$H = 1/\lambda$ (km)	$\frac{r_p}{H} = \lambda r_p$ $= p$	$(\frac{r_p}{H})^{\frac{1}{2}}$ $= p^{\frac{1}{2}}$	T_o
0	6378	1.225×10^{9}	8.4345	756.180	27.4987	9.2131×10^{-5}
70	6448	8.7535×10^{4}	6.5733	980.9380	31.3199	1.4525
140	6518	3.3940	23.257	280.2597	16.7408	1.9812×10^{4}
210	6588	2.558×10^{-1}	45.315	145.3823	12.0574	1.8743×10^{5}
280	6658	5.315×10^{-2}	55.859	119.1929	10.9176	8.0837×10^{5}
350	6728	1.465×10^{-2}	65.028	103.4631	10.1717	2.7044×10^{6}
420	6798	4.816×10^{-3}	73.613	92.3478	9.6098	7.6934×10^{6}
490	6868	1.801×10^{-3}	81.324	84.4523	9.1898	1.9476×10^{7}
560	6938	7.464×10^{-4}	87.433	79.3522	8.9080	4.5097×10^{7}
630	7008	3.304×10^{-4}	92.786	75.5286	8.6907	9.8408×10^{7}
700	7078	1.537×10^{-4}	97.435	72.6433	8.5231	2.0543×10^{8}

TABLE II

Sample calculation for flux multiplier N_1

$$\xi_G^2 = 39.2658 \times 10^{10}$$

From (8.12) the total flux of particles with velocities $V_\infty = 0.1234$ km/sec is calculated to be

$$N_1^{'} = 1.5471 \times 10^5 \; M \; (10\mu)$$

or the flux multiplier

$$N_1 = \frac{N_1^{'}}{M} = 1.5471 \times 10^5 \tag{8.14}$$

Similar calculations for this 10 microns particle for various capture perigee distances are carried out. Calculations are also made for three other particle sizes with diameters equal to 1 micron, 100 microns and 1000 microns and the four different observer sphere radii mentioned previously. The results of these computations are plotted in figures 5-8 in terms of the flux multiplier N_1 and V_∞ on a log-log plot. These curves show that the flux multiplier decreases monotonically with increasing V_∞. It is observed that this number goes to infinity like V_∞^{-4} as $V_\infty \to 0$. This can be explained in the following manner.

From the capture mechanism in section 5, the relation between V_∞ and r_p for a given particle is written in the form

$$V_\infty^2 \leq \frac{4\mu A_1 \rho_p \left(\frac{\pi H}{r_p}\right)^{\frac{1}{2}} (4r_p - H)}{r_p \left[2 - A_1 \rho_p \left(\frac{\pi H}{r_p}\right)^{\frac{1}{2}} (4r_p + 3H)\right]} \tag{8.15}$$

with $\quad K = A_1 \rho_p = \dfrac{C_D A}{2m} \rho_p$

For large r_p this is written as

$$V_\infty^2 \leq \frac{4\mu A_1 \rho_p \left(\frac{\pi H}{r_p}\right)^{\frac{1}{2}} 4\, r_p}{r_p \left[2 - A_1 \rho_p \left(\frac{\pi H}{r_p}\right)^{\frac{1}{2}} \cdot 4 r_p\right]}$$

as $r_p >> H$.

As r_p grows very large $\rho_p \to 0$ exponentially. Consequently as $\rho_p \to 0$ we have

$$V_\infty^2 \leq 8\mu A_1 \rho_p \left(\frac{\pi H}{r_p}\right)^{\frac{1}{2}} \tag{8.16}$$

Thus the density ρ_p behaves like V_∞^2 as $\rho_p \to 0$.

Let us now examine the function T_1 defined by equation (8.11) as $V_\infty \to 0$. For very large r_p this can be written as

$$T_1 \approx \left(\frac{2.7815 \times 10^{-1}}{A_1}\right) \int_0^{z_c} \left(1 + \frac{\mu}{r_p V_\infty^2}\right) \left(\frac{r_p}{H}\right)^{\frac{1}{2}} \frac{dz}{\rho_p}$$

Then as $V_\infty \to 0$ this reduces to

$$T_1 \sim \int_0^{z_c} \frac{1}{r_p V_\infty^2} \left(\frac{r_p}{H}\right)^{\frac{1}{2}} \frac{dz}{\rho_p} \tag{8.17}$$

Substitution for the density from (8.16) in (8.17) yields the following expression of T_1 as $V_\infty \to 0$, i.e.,

$$T_1 \sim \int_0^{z_c} \frac{1}{r_p V_\infty^2} \frac{dz}{V_\infty^2}$$

$$\simeq \frac{1}{V_\infty^4} \int \frac{dz}{r_p}$$

Hence

$$T_1 \simeq \frac{1}{V_\infty^4} \ \ell n \frac{r_p}{R} \qquad \text{as } V_\infty \to 0$$

Therefore the singularity in the flux concentration arises from the fact that density ρ_p behaves like V_∞^2 as $\rho_p \to 0$. This in turn gives rise to the four zeros in V_∞ observed in the figures.

2. Integration over Velocity Distribution

It is observed that the interplanetary particles have a distribution of velocities V_∞. Consequently the computation of the total flux of particles must take this velocity distribution into consideration. Hence we define the total flux over the velocities as N_F'

$$N_F' = M \int_0^\infty N_1' \ \varphi(V_\infty) \ dV_\infty \tag{8.18}$$

or

$$N_F = \frac{N_F'}{M} = \int_0^\infty N_1 \ \varphi(V_\infty) \ dV_\infty \tag{8.19}$$

where N_F is the flux multiplier and $\varphi(V_\infty)$ represents the velocity distribution in the interplanetary space. Henceforth N_F will be studied in all the analysis.

Now the form of the distribution function $\varphi(V_\infty)$ is not known. However a simple model for the distribution function, based on physical explanations consistent with the observed behaviour of particles, is assumed.

Let us now consider a particle orbiting around the sun along some Keplerian trajectory in the same plane as the earth's orbital plane. This is shown in figure 4.

Now from Kepler's planetary laws objects with the same energy and orbital elements cannot exist as separate orbiting particles in the solar system. For only such objects the relative velocity V_∞ between them is zero. Then this says that, at the earth's distance from the sun, particles with relative velocity $V_\infty = 0$ with respect to the earth cannot exist in the solar system. All such particles will have been captured by the earth. Consequently it is concluded that the velocity distribution defined by $\varphi(V_\infty)$ must go to zero as V_∞ goes to zero. Hence

$$\varphi(V_\infty = 0) = \varphi(0) = 0 \tag{8.20}$$

It is also known from the observed meteor velocities that meteors with very large V_∞ are very few in the solar system. Thus it is expected that $\varphi(V_\infty)$ will go to zero as $V_\infty \to \infty$. Consequently it is concluded that

$$\varphi(V_\infty \to \infty) = 0 \tag{8.21}$$

Using these above-mentioned arguments a simple model for the velocity distribution function is written in the following form:

$$\varphi(V_\infty) = BV_\infty^\alpha \, e^{-\beta V_\infty^2} \tag{8.22}$$

where α and β are positive constants.

This model of the distribution function behaves like the tail end of a Gaussian distribution function as $V_\infty \to \infty$ and goes to zero as some power α of V_∞ as $V_\infty \to 0$. It is a simple function with two

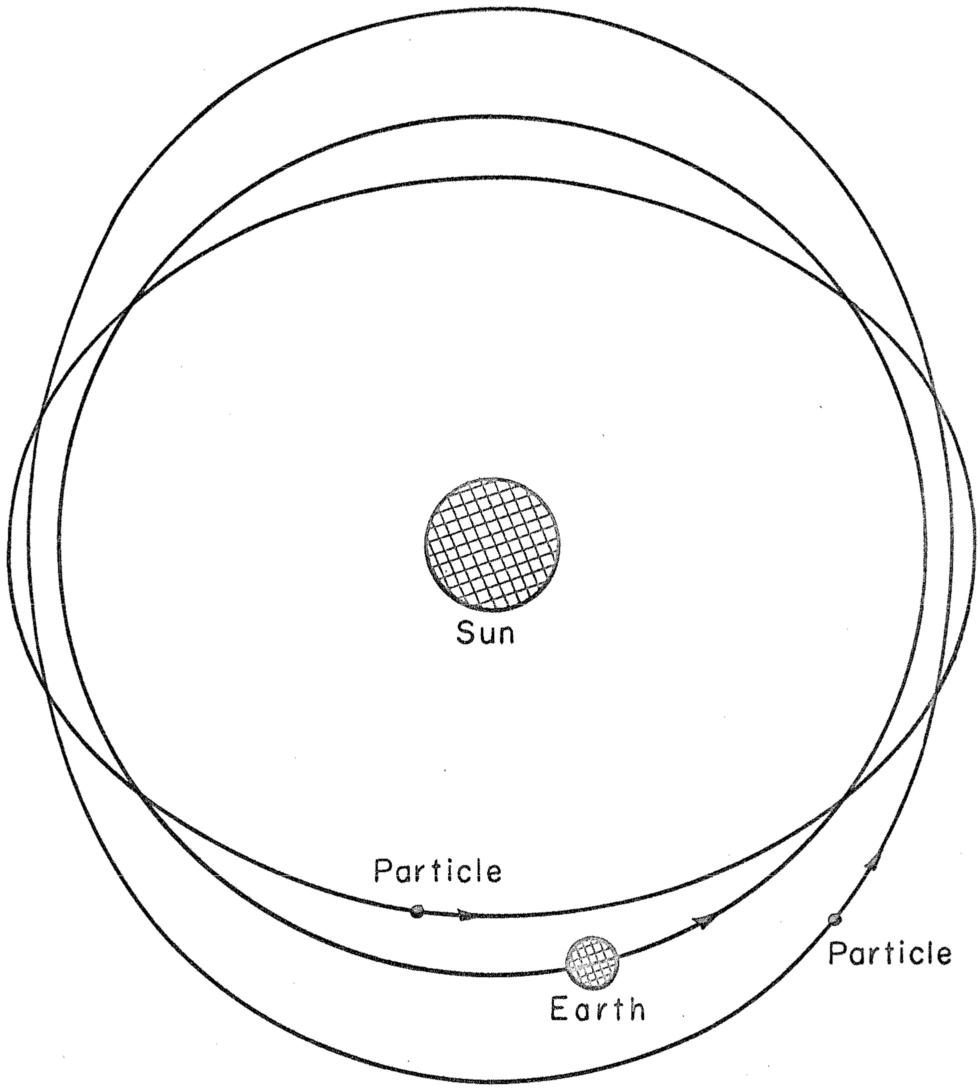

Figure 4. Schematic showing the orbits of the earth and particles around the sun

parameters α and β. The constants α and β are evaluated in the following way.

In part 1 of this section, it has been pointed out that the flux multiplier $N_1(V_\infty, m)$ has a singularity of the type V_∞^{-4} when V_∞ goes to zero. But the observational data from various measurements give a finite flux concentration in the vicinity of the earth. Consequently we require that the value of the exponent α in equation (8.22) be greater than 3 in order that the integral (8.19) be finite.

Qualitative argument shows that an appropriate value for α is 4. Now the orbit of a particle in space is described completely by the following six elements a, ϵ, i, t, ω and Ω. These elements are the semi-major axis, the eccentricity, the inclination of the orbital plane, the epoch that fixes the position of the particle in its orbit relative to some time scale, the orientation of the orbit in its plane and the angle of the ascending node measured from the vernal equinox respectively. However ω and Ω correspond to the orientation of the orbit in its plane and the orientation of the plane. If now $a = a_E$ while the other elements have the same values as those of the earth, then the relative velocity $V_\infty = 0$. Again if $\epsilon = \epsilon_E$ with the other elements the same V_∞ will again be zero. Similarly V_∞ will be equal to zero when i or t of the particle is equal to i_E or t_E with the other elements the same. When $\epsilon = \epsilon_E$ with all other elements being the same the concept of ω vanishes and similarly for $i = i_E$ the concept of Ω vanishes. Hence ω and Ω do not contribute to zeros of V_∞ when the other elements are the same. Thus there are four zeros corresponding to $V_\infty = 0$ whenever a, ϵ, i and t agree with those of the earth. From this simple argument, an appropriate value for α seems to be 4. Due to the lack of more detailed knowledge about the distribution function at this time α is chosen to equal 4. Thus

$$\varphi(V_\infty) = B\, V_\infty^4\, e^{-\beta V_\infty^2} \tag{8.23}$$

In this form, $\varphi(V_\infty)$ is a simple function with a single parameter β. Differentiating φ with respect to V_∞ and setting the derivative equal to zero, gives the value of $V_{\infty_{max}}$ for which φ has its maximum. This gives $V_{\infty_{max}}$ as

$$V_{\infty_{max}}^2 = \frac{2}{\beta} \tag{8.24}$$

Opik (8), from the study of the minimum size of the spherules collected from the sea beds, has remarked that the geocentric velocities of the micrometeorites are only slightly larger than the terrestrial escape velocity. In fact he concludes that the smaller particles have velocities between 11.1 and 12.2 km/sec which gives an upper limit for V_∞ as 5.1 km/sec. This implies that these small particles in the solar system have near circular orbits with their heliocentric velocities differing very little from that of the earth. This is not unexpected since the Poynting-Robertson effect acts on these small particles with eccentric orbits and circularises their orbits. The smaller the particle the more severe the effect. However there is a limit to the size which can exist in the solar system. This limit is obtained by equating the gravitational pull of the sun to the radiation pressure. This is given by Beard (9) as

$$d^* = \frac{0.600}{\rho_m} \text{ microns}$$

with ρ_m in gms/c.c..

For our particles this limit is about 0.6 microns. It is clear from this that particles of about 1 micron size have a smaller population as they are being blown away from the sun. Thus it is concluded from the above arguments that β can have a value greater than or equal to 1/3. Therefore

$$\beta \geqq 1/3 \tag{8.25}$$

for which $V_{\infty_{max}}$ at peak of $\varphi(V_\infty) \simeq 2.5$ km/sec. $\beta = 1/3$ corresponds to the smallest value that is consistent with the conclusions reached by Opik and others in this field. Most of the computations are made with $\beta = 1/3$. However, computations for $\beta > 1/3$ will be discussed later. Now the constant B in equation (8.23) is determined by the normalization

$$\int_0^\infty \varphi(V_\infty)\, dV_\infty = 1$$

or

$$B \int_0^\infty V_\infty^4 \, e^{-\beta V_\infty^2} \, dV_\infty = 1$$

The evaluation of this integral yields

$$B = \frac{8\beta^{\frac{5}{2}}}{3\sqrt{\pi}} \tag{8.26}$$

Finally the distribution function $\varphi(V_\infty)$ is completely determined as

$$\varphi(V_\infty) = \frac{8\beta^{\frac{5}{2}}}{3\sqrt{\pi}} V_\infty^4 \, e^{-\beta V_\infty^2} \tag{8.27}$$

This representation for $\varphi(V_\infty)$ is used in the integration of equation (8.19) to evaluate the flux multiplier over all the velocities. Again numerical integration using Simpson's Rule is carried out for different particle sizes and observer radii. These results are plotted on a log-log scale in figure 9 as a function of particle size. This number N_F represents the differential flux multiplier measured by an arbitrarily oriented stationary counter of unit area in unit time located on the observer sphere.

The flux multiplier due to a purely gravitational concentration can easily be calculated from equation (7.14) by letting the mass $m \to \infty$. When $m \to \infty, \xi_D \to \xi_R$. Thus

$$N_{1G} = \frac{1}{4r_{ob}^2} \left\{ 2\xi_G^2 - \xi_R^2 \right\} \tag{8.28}$$

Substitution for ξ_G and ξ_R from (7.12) and (7.6) in (8.28) yields

$$N_{1G}(V_\infty, m) = \frac{1}{2} - \frac{1}{4}\left(\frac{R}{r_{ob}}\right)^2 + \left(2 - \frac{R}{r_{ob}}\right) \frac{\mu}{2r_{ob}V_\infty^2} \tag{8.29}$$

Integrating N_{1G} over the velocity distribution function yields

$$N_{FG} = \int_0^\infty N_{1G}\varphi(V_\infty)\, dV_\infty$$

$$= \frac{1}{2} - \frac{1}{4}\left(\frac{R}{r_{ob}}\right)^2 + \frac{\beta}{3} \frac{\mu}{r_{ob}} \left(2 - \frac{R}{r_{ob}}\right) \tag{8.30}$$

For $\beta = 1/3$, this takes the value

$$N_{FG}(\beta = 1/3) = \frac{1}{2} - \frac{1}{4}\left(\frac{R}{r_{ob}}\right)^2 + \frac{\mu}{9r_{ob}}\left(2 - \frac{R}{r_{ob}}\right) \tag{8.31}$$

This is the asymptotic value of the flux multiplier N_F as the mass of the particle goes to infinity. This is shown in figure 9.

All the above-mentioned calculations can be extended for larger particles. However, a calculation of the total flux multiplier N_F for a 1 cm diameter particle shows that for larger particles N_F approaches N_{FG} very rapidly.

IX. RESULTS AND DISCUSSIONS

In this work, a theory of atmospheric capture of micrometeorites and the resulting orbital lifetimes has been developed. At the same time a statistical model of the micrometeorite flux in the interplanetary space has been derived. Then this has been combined with the capture theory to obtain a consistent picture of the flux distribution in the neighborhood of the earth.

The results of the computations are plotted in figure 9. In this plot the ordinate represents the differential flux multiplier and the abscissa represents the mass of the particle in terms of the diameter. This can be done without any difficulty as the particles have been assumed to be spherical with constant mass density.

It is observed in figure 9 that the computed values of the flux multiplier lie on a straight line in this range of particle sizes. Consequently, the functional relation between the flux multiplier and the size of the particle can be expressed in the form

$$N_F = \frac{N'}{M} = B_1 d^k \tag{9.1}$$

where d is in centimeters.

Now four of the curves shown in figure 9 correspond to the four different values of the observer radius, namely 7500 km, 10000 km, 15000 km and 25000 km. These curves show that the flux multiplier decreases as the observer radius increases. This confirms the reduction of the observed particle concentration as the observer recedes from the earth.

Constants B_1 and k are easily determined from the figure. They are given in Table III below.

TABLE III

r_{ob}(km)	β	B_1	k
7.5×10^3	1/3	7.701	-0.04615
1.0×10^4	1/3	6.64	-0.03273
1.5×10^4	1/3	5.225	-0.01734
2.5×10^4	1/3	3.556	-0.01155
7.5×10^3	2	99.04	-0.1044

The exponent k in equation (9.1) is found to be negative. Consequently the flux multiplier increases with decreasing particle size or mass. This implies that smaller sized particles are more affected by this mechanism than the larger ones. But it is noted that the magnitude of k is quite small and it cannot possibly account for the 4 orders of magnitude difference between the concentration near the vicinity of the earth and that in the interplanetary space. It is felt that the reason for this small value of the flux multiplier is due to the very small value of 1/3 chosen for the parameter β. This smallest value has been chosen so that it is consistent with the conclusions reached by Opik and others. But β could have larger values. Then a larger value of β implies that the orbiting particles with this value of β have more circular orbits than previously assumed. Such an assumption of large β is not very inconsistent. Consequently, a value of $\beta = 2$ has been used in the computation of the flux multiplier. This value of 2 for β corresponds to $V_\infty = 1$ km/sec at which $\varphi(V_\infty)$ has a maximum. The flux multiplier corresponding to $\beta = 2$ has been calculated and is plotted in figure 9. This new calculation shows that the value of the flux multiplier is increased by about 20 times for the smaller particles over that for $\beta = 1/3$. This leads to the conclusion that the flux multiplier is quite sensitive to the value of β. Exponent k for this case is evaluated from the curve and is found to be increased by about 3 times. Alexander (2, 3) infers that smaller particles have orbits that are more circular than the larger ones. This is not very unexpected as the Poynting-Robertson effect is more severe for smaller particles. This says that the smaller particles have more circular orbits. Then the implication here is that smaller particles have smaller relative velocities than the larger ones which in turn implies that β is larger for smaller particles. This leads to a conclusion that the value of β may depend on the mass of the particle. Then it is not too unreasonable for β to be as large as 10, which corresponds to $V_{\infty_{max}} \simeq 0.5$ km/sec at the peak of velocity distribution function. A value of 10 for β increases the flux multiplier by about 3 orders of magnitude which then brings it near the numbers that Alexander predicted from the Mariner II measurements. Thus the analysis offers an explanation for the micrometeorite concentrations near the earth, when the parameter β has a value of about 10 that is consistent with the above-mentioned discussions. However, it must be pointed out that the velocity distribution has been assumed by using consistent physical arguments.

Because of the present day technological advances in space exploration, the velocity distribution in the interplanetary space can be measured. This could be done by sending space probes containing microphone detectors through the interplanetary space. These detectors should be of large dimensions and be capable of detecting different sizes of masses and energies. Then these measurements could be used to determine the velocity distribution empirically and compare with the assumed velocity distribution mentioned above and also provide some data regarding the nature and dependence of the parameter β on the mass of the particle.

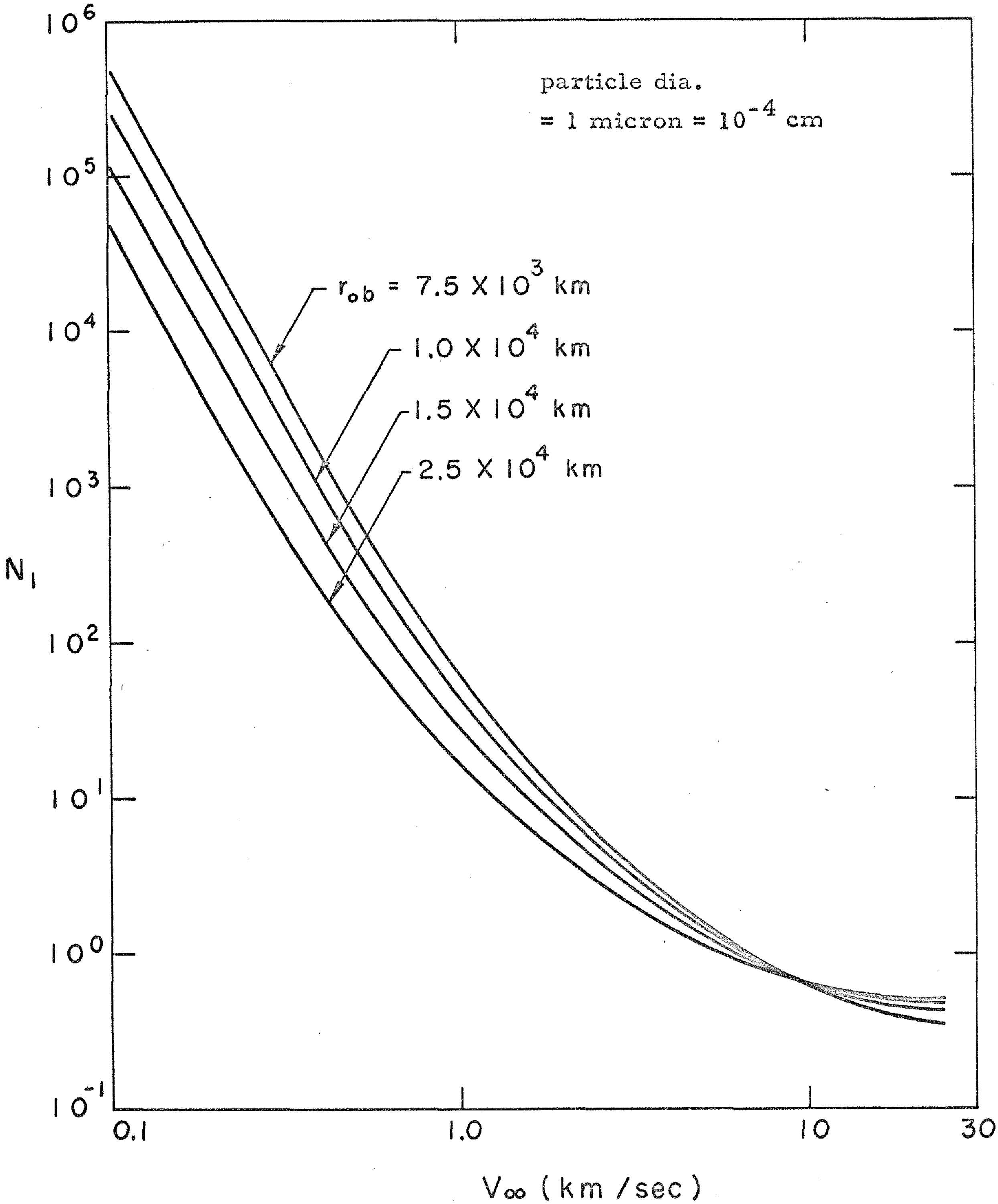

Figure 5. Flux Multiplier N_1 vs. Relative Velocity V_∞

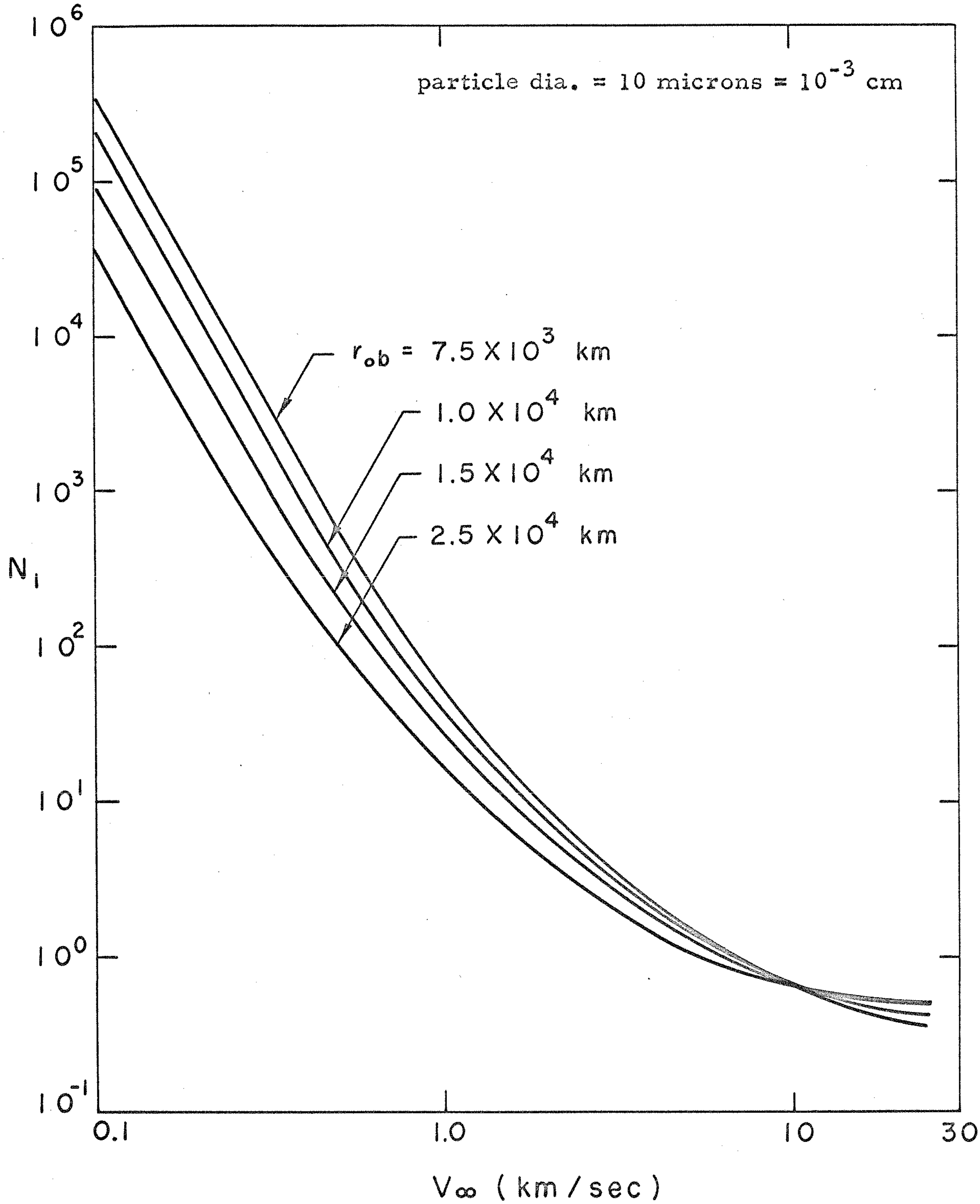

Figure 6. Flux Multiplier N_1 vs. Relative Velocity V_∞

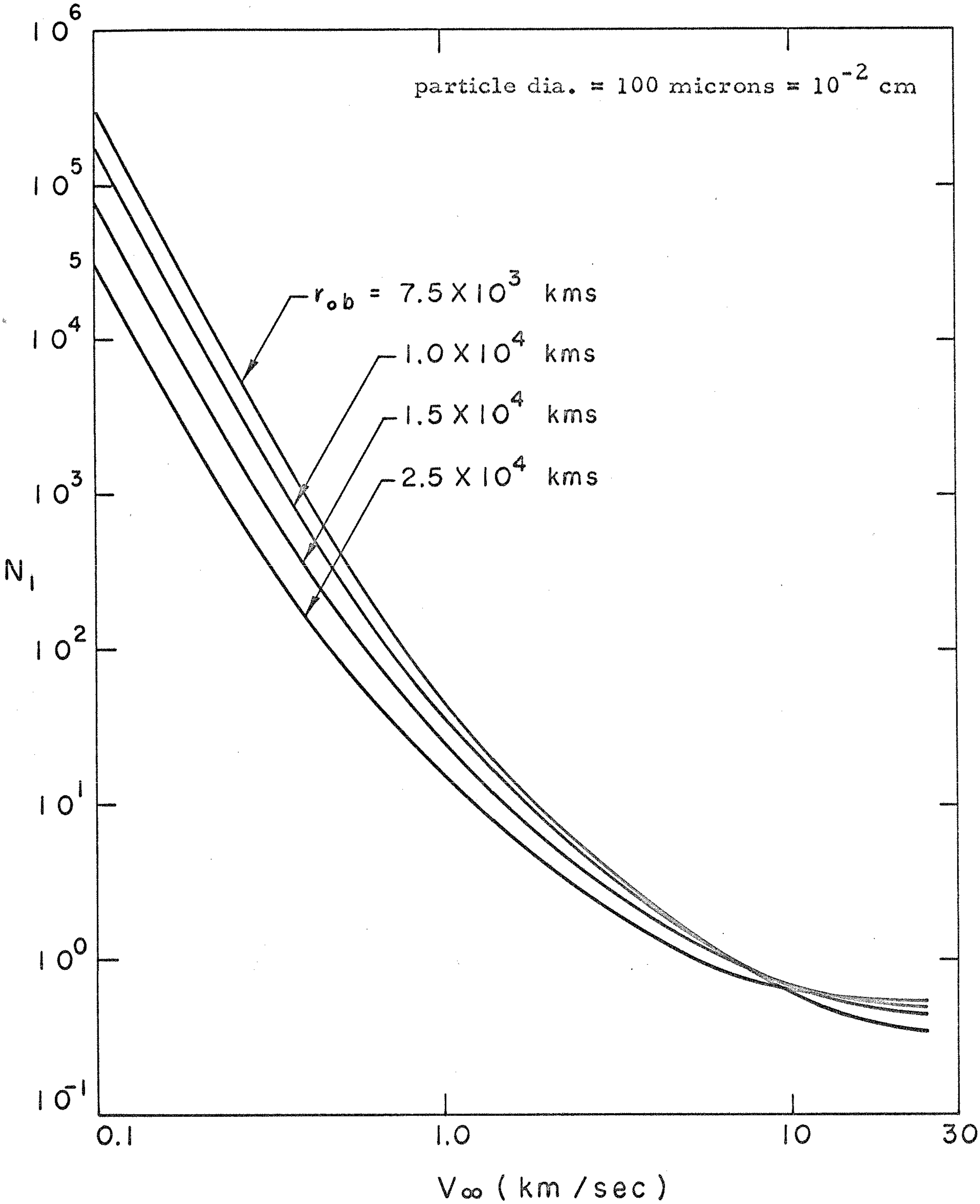

Figure 7. Flux Multiplier N_1 vs. Relative Velocity V_∞

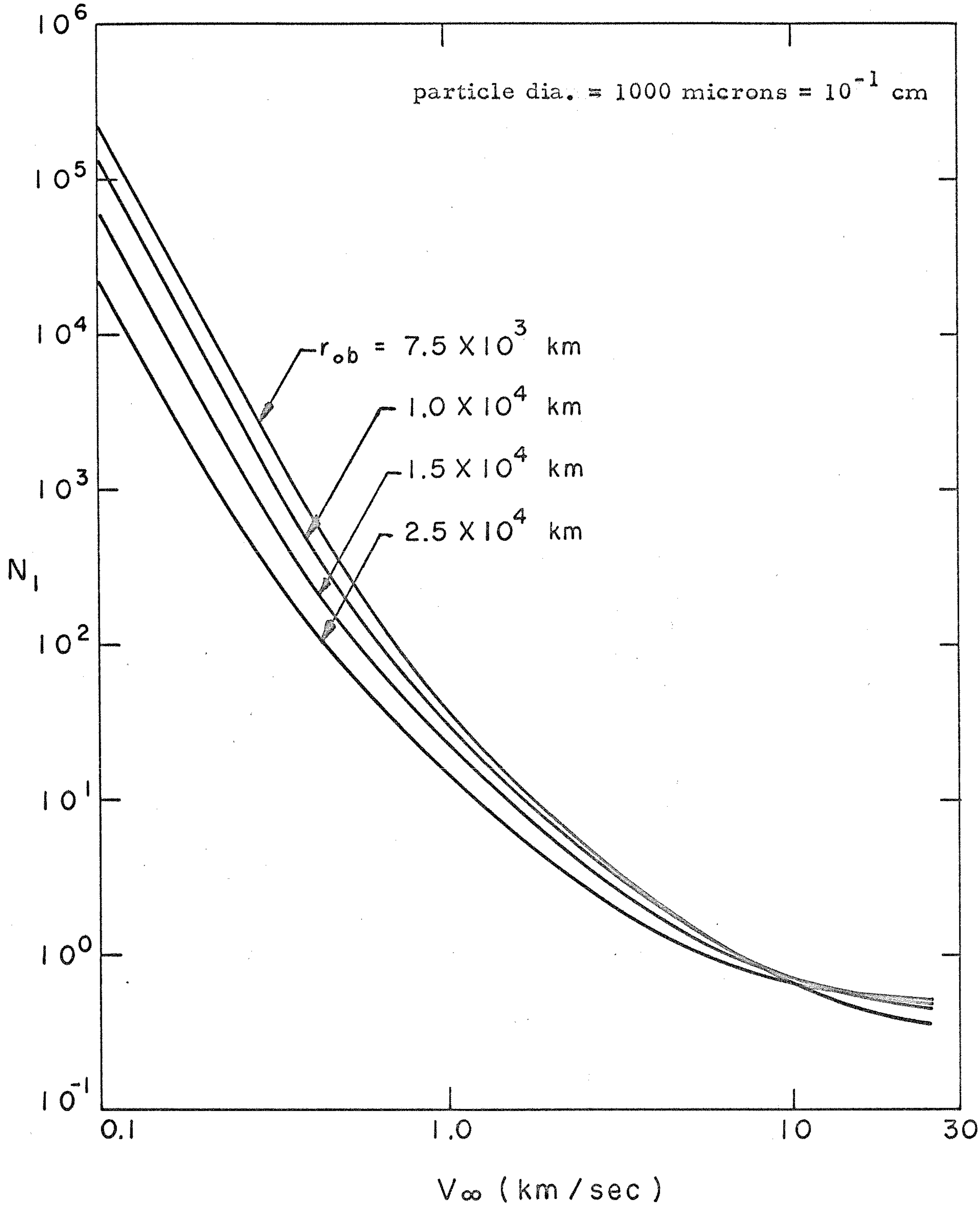

Figure 8. Flux Multiplier N_1 vs. Relative Velocity V_∞

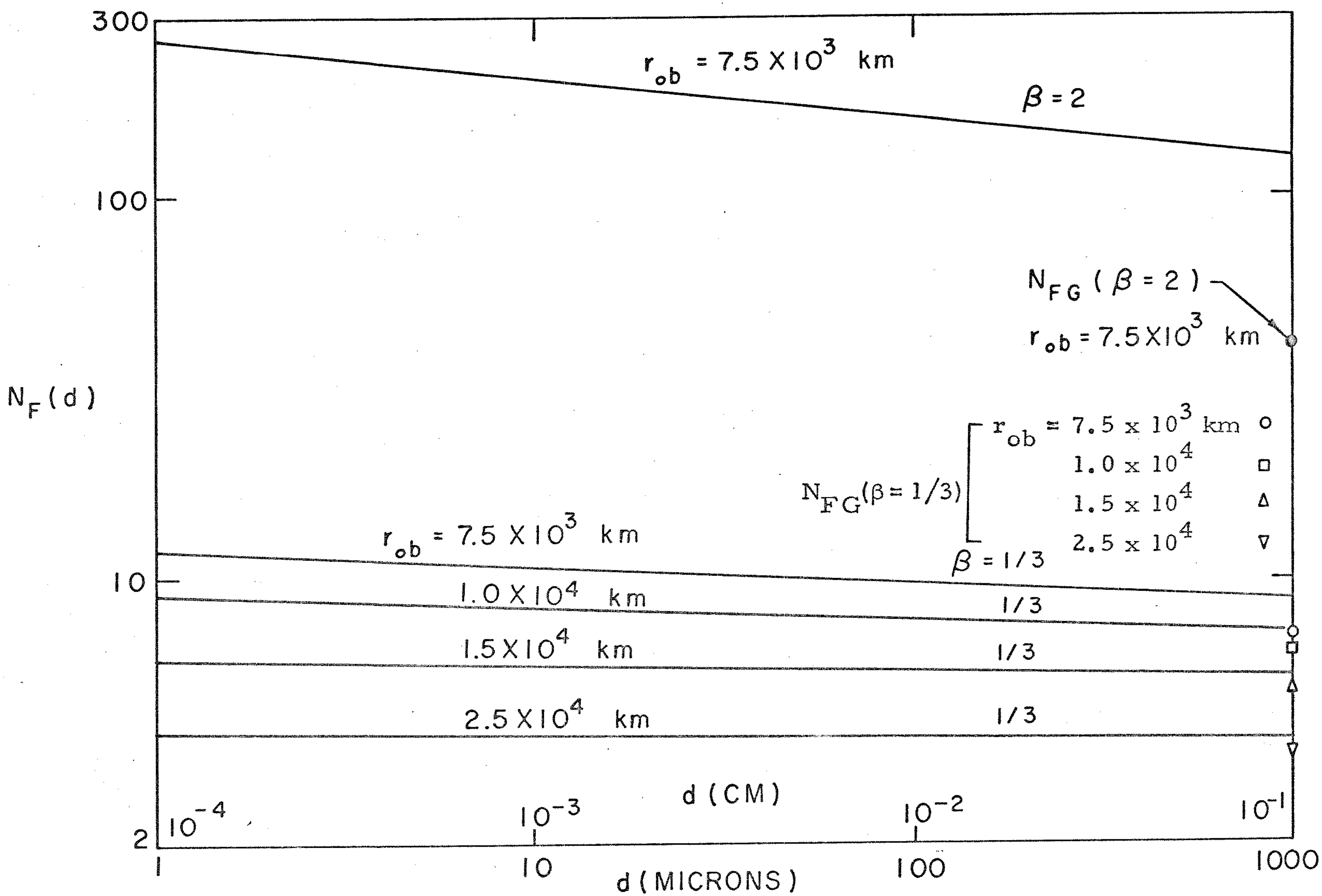

Figure 9. Flux multiplier N_F vs. particle diameter d for various r_{ob}

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APPENDIX A

DERIVATION OF SERIES SOLUTION FOR Δr FOR $0 \leqslant \epsilon < 1$

It is found more convenient to use the eccentric anomaly E instead of the true anomaly θ in the derivation of the series solution for Δr. The relation between E and θ valid for all values of $0 \leqslant \epsilon < 1$ is shown in figure A.

From the geometry of the figure

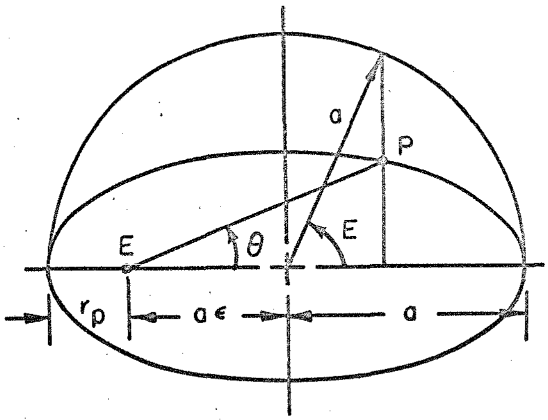

Figure A

$$r \cos \theta = a(\epsilon + \cos E) \tag{A.1a}$$

and

$$r \sin \theta = a(1-\epsilon^2)^{\frac{1}{2}} \sin E \tag{A.1b}$$

where $(1-\epsilon^2)^{\frac{1}{2}}$ is the scaling factor between a circle and an ellipse.

Squaring and adding these two relations, we get

$$r = a(1 + \epsilon \cos E) \tag{A.2}$$

Now the relation between θ and E is obtained by eliminating r from equations (A.1) and (A.2). Thus

$$\sin\theta = \frac{(1-\epsilon^2)^{\frac{1}{2}} \sin E}{1+\epsilon\cos E}$$

$$\cos\theta = \frac{\epsilon + \cos E}{1+\epsilon\cos E}$$

$$1-\epsilon\cos\theta = \frac{1-\epsilon^2}{1+\epsilon\cos E}$$

$$1+\epsilon^2-2\epsilon\cos\theta = (1-\epsilon^2)\left(\frac{1-\epsilon\cos E}{1+\epsilon\cos E}\right)$$

and

$$d\theta = \frac{(1-\epsilon^2)^{\frac{1}{2}}}{1+\epsilon\cos E}\, dE$$

Using these relations between θ and E in equation (4.22), the solution for the perturbation Δr for one orbit, valid for all values of eccentricity $0 \leqslant \epsilon < 1$, takes the form

$$\Delta r(2\pi,\epsilon) = -Ka^2 \left\{ \epsilon \int_0^{2\pi} (1-\epsilon\cos E)^{\frac{1}{2}} (1+\epsilon\cos E)^{\frac{1}{2}} \sin^2 E\, \sigma\, dE \right.$$
$$\left. -2\int_0^{2\pi} (1+\epsilon\cos E)\sin E\, dE \int_0^{E} (1-\epsilon\cos a)\left(\frac{1-\epsilon\cos a}{1+\epsilon\cos a}\right)^{\frac{1}{2}} \sigma\, da \right\} \tag{A.3}$$

After integration by parts, equation (A.3) takes the form

$$\Delta r(2\pi,\epsilon) = -Ka^2 \left\{ \epsilon \int_0^{2\pi} (1+\epsilon\cos E)\left(\frac{1-\epsilon\cos E}{1+\epsilon\cos E}\right)^{\frac{1}{2}} \sin^2 E\, \sigma\, dE \right.$$
$$+ \frac{1}{\epsilon}(1+\epsilon)^2 \int_0^{2\pi} (1-\epsilon\cos E)\left(\frac{1-\epsilon\cos E}{1+\epsilon\cos E}\right)^{\frac{1}{2}} \sigma\, dE$$
$$\left. - \frac{1}{\epsilon}\int_0^{2\pi} (1+\epsilon\cos E)^2 (1-\epsilon\cos E)\left(\frac{1-\epsilon\cos E}{1+\epsilon\cos E}\right)^{\frac{1}{2}} \sigma\, dE \right\} \tag{A.4}$$

Then combining all these three integrals, we get

$$\Delta r(2\pi,\epsilon) = \frac{-Ka^2}{\epsilon}\left\{ \int_0^{2\pi} \left(\frac{1-\epsilon\cos E}{1+\epsilon\cos E}\right)^{\frac{1}{2}} \sigma \times \right.$$

$$\left[\epsilon^2 \sin^2 E(1+\epsilon\cos E)+(1-\epsilon\cos E)(1+\epsilon)^2\right.$$

$$\left.\left. -(1-\epsilon\cos E)(1+\epsilon\cos E)^2\right] dE \right\}$$

Rearranging

$$\Delta r(2\pi,\epsilon) = -2(1+\epsilon)Ka^2 \int_0^{2\pi} (1-\cos E)\left(\frac{1-\epsilon\cos E}{1+\epsilon\cos E}\right)^{\frac{1}{2}} \sigma(E) dE \qquad (A.5)$$

The density distribution is given by

$$\sigma(E) = e^{\lambda(r_p - r)}$$

$$= e^{\lambda[a(1-\epsilon)-a(1+\epsilon\cos E)]}$$

$$= e^{-\lambda a\epsilon(1+\cos E)}$$

Consequently

$$\sigma(E) = e^{-\zeta(1+\cos E)} \qquad (A.6)$$

where $\zeta \equiv \lambda a \epsilon$

Therefore

$$\Delta r(2\pi,\epsilon) = -2(1+\epsilon)Ka^2 e^{-\zeta} \int_0^{2\pi} \frac{[1+\epsilon\cos^2 E-(1+\epsilon)\cos E]}{(1-\epsilon^2\cos^2 E)^{\frac{1}{2}}} e^{-\zeta\cos E} dE \qquad (A.7)$$

where the integrand in equation (A.5) is multiplied and divided by $(1-\epsilon\cos E)^{\frac{1}{2}}$. Now it is convenient to have the upper limit of integration of equation (A.7) as $\pi/2$ instead of 2π. Hence rewriting the above equation between the limits zero and $\pi/2$, we have

$$\Delta r = -4(1+\epsilon)Ka^2 e^{-\zeta}\left\{\int_0^{\pi/2} \frac{1+\epsilon\sin^2 E}{\sqrt{1-\epsilon^2\sin^2 E}}\left(e^{\zeta\sin E}+e^{-\zeta\sin E}\right)dE + (1+\epsilon)\int_0^{\pi/2}\frac{\sin E}{\sqrt{1-\epsilon^2\sin^2 E}}\left(e^{\zeta\sin E}-e^{-\zeta\sin E}\right)dE\right\} \tag{A.8}$$

The exponential function $e^{\pm\zeta\sin E}$ can be expanded in terms of the modified Bessel's function of the first kind (the details of this expansion are derived in Appendix B.1). Hence the square brackets containing the exponential terms in equation (A.8) are written in the form

$$e^{\zeta\sin E}+e^{-\zeta\sin E} = 2I_o(\zeta) + 4\sum_{n=1}^{\infty}(-1)^n I_{2n}(\zeta)\cdot\cos 2nE$$

and

$$e^{\zeta\sin E}-e^{-\zeta\sin E} = -4\sum_{n=1}^{\infty}(-1)^n I_{2n-1}(\zeta)\sin(2n-1)E \tag{A.9}$$

After the substitution of (A.9) in (A.8), the solution for Δr takes the form

$\Delta r(2\pi, \epsilon)$

$$= -8(1+\epsilon)Ka^2 e^{-\zeta} \int_0^{\pi/2} \frac{(1+\epsilon\sin^2 E)}{(1-\epsilon^2\sin^2 E)^{\frac{1}{2}}} dE \left[I_o(\zeta) + \sum_{n=1}^{\infty} (-1)^n I_{2n}(\zeta)\cos 2nE \right]$$

$$+16(1+\epsilon)^2 Ka^2 e^{-\zeta} \int_0^{\pi/2} \frac{\sin E}{(1-\epsilon^2\sin^2 E)^{\frac{1}{2}}} \sum_{n=1}^{\infty} (-1)^n I_{2n-1}(\zeta)\sin(2n-1)E\, dE$$

(A.10)

Now these integrals are evaluated from the table of integrals given on page 115 of reference 10. The results are derived in terms of an infinite series of the complete elliptic functions of the first kind $K_n(\epsilon)$ of order n. $K_n(\epsilon)$ is defined as

$$K_n(\epsilon) \equiv \int_0^{\pi/2} \frac{\cos 2nE}{(1-\epsilon^2\sin^2 E)} dE$$

$$= (-1)^n \sum_{\nu=n}^{\infty} \frac{[\Gamma(\nu+\frac{1}{2})]^2}{(\nu-n)!\,(\nu+n)!} \epsilon^{2\nu}, \quad 0 \leqslant \epsilon < 1 \qquad \text{(A.11)}$$

$$n = 0, 1, 2, \ldots$$

With this definition of $K_n(\epsilon)$, the solution for $\Delta r(2\pi, \epsilon)$ given in equation (A.10) is written in the form

$$\begin{aligned} \Delta r(2\pi, \epsilon) \equiv \Delta r_a & \\ = -4Ka^2 \Big\{ & [(2+\epsilon)K_o(\epsilon) - \epsilon K_1(\epsilon)] I_o(\zeta) \\ & + 2\sum_1^{\infty} (-1)^n K_n(\epsilon) [(2+\epsilon)I_{2n}(\zeta) + (1+\epsilon)I_{2n-1}(\zeta)] \\ & - \sum_1^{\infty} (-1)^n K_{n-1}(\epsilon) [\epsilon I_{2n}(\zeta) + 2(1+\epsilon)I_{2n-1}(\zeta)] \\ & - \epsilon \sum_1^{\infty} (-1)^n K_{n+1}(\epsilon) I_{2n}(\zeta) \Big\} \cdot (1+\epsilon)e^{-\zeta} \end{aligned} \qquad \text{(A.12)}$$

Note that the solution (A.12) for Δr is the perturbation Δr_a in the apogee distance for one complete orbit. The perturbation Δr_p of the perigee distance can be similarly obtained by evaluating the integrals from π to 3π. Then from the known solutions for Δr_a and Δr_p other perturbations such as Δa, $\Delta\epsilon$ can be determined.

The solution for Δr_a given by equation (A.12) holds good for all values of ϵ between zero and one. However the actual evaluation becomes very cumbersome for large values of ϵ as $K_n(\epsilon)$ converges very slowly. At $\epsilon = 1$, $K_n = \infty$. It is felt that a solution for $K_n(\epsilon)$ near ϵ equal to 1 would reduce the computational work for large values of ϵ. Consequently solutions of $K_n(\epsilon)$ near $\epsilon \sim 1$ are determined.

Now $K_n(\epsilon)$ satisfies the following differential equation as a function of ϵ. (The details of this derivation are given in Appendix B.2.) It is given by

$$\epsilon^2 (1-\epsilon^2)K_n'' + \epsilon(1-3\epsilon^2)K_n' - (\epsilon^2+4n^2)K_n = 0 \qquad \text{(A.13)}$$

where the primes denote differentiation with respect to ϵ. As we are interested in the solution of K_n for ϵ close to 1, a new independent variable η is introduced with the relation

$$\eta^2 = 1 - \epsilon^2 \qquad \text{(A.14)}$$

Then this gives

$$\frac{d}{d\epsilon} = -\frac{\epsilon}{\eta}\frac{d}{d\eta}$$

and

$$\frac{d^2}{d\epsilon^2} = -\frac{1}{\eta^3}\frac{d}{d\eta} + \frac{\epsilon^2}{\eta^2}\frac{d^2}{d\eta^2}$$

Hence in the new variable η, equation (A.13) takes the form

$$\eta(1-\eta^2)\frac{d^2K_n}{d\eta^2} + (1-3\eta^2)\frac{dK_n}{d\eta} - \eta\,(1+\frac{4n^2}{1-\eta^2})\,K_n = 0 \qquad \text{(A.15)}$$

Now by defining a new function $W_n(\eta)$ related to $K_n(\epsilon)$ by

$$K_n(\epsilon) = (1-\eta^2)^n\, W_n(\eta) \qquad \text{(A.16)}$$

equation (A.15) is further simplified. Then the function W_n satisfies the following differential equation

$$\eta(1-\eta^2)W_\eta'' + [1-(3+4n)\eta^2]\,W_n' - (1+2n^2)W_n = 0 \qquad \text{(A.17)}$$

Further simplification is made by using a new independent variable z given by the relation

$$z = \eta^2 \qquad \text{(A.18)}$$

This simplification is such that the reduced differential equation becomes a hypergeometric equation for which the general solution can be immediately written down. This hypergeometric equation is

$$z(1-z)W_n'' + (1-2(1+n)z)W_n' - (\frac{1+2n}{2})^2\, W_n = 0 \qquad \text{(A.19)}$$

One of the solutions is given by $F(\alpha, b, \gamma; z)$ where

$$\left.\begin{array}{l}\gamma = 1\\ \alpha+\beta+1 = 2(n+1)\\ \alpha\beta = (\frac{1+2n}{2})^2\end{array}\right\} \Rightarrow \alpha = \beta = \frac{1+2n}{2} \qquad \text{(A.20)}$$

Therefore

$$W_{1n} = F\left(\frac{1+2n}{2}, \frac{1+2n}{2}, 1; z\right) = 1 + \sum_{1}^{\infty} \frac{[(2s+2n-1)!]^2}{2^{2s}(s!)^2} z^s \tag{A.21}$$

and the second solution is given by

$$W_{2n} = W_{1n} \log z + \sum_{1}^{\infty} a_s z^s \tag{A.22}$$

where

$$a_s = \frac{[(2s+2n-1)!]^2}{2^{2s}(s!)^2} \left(2 \sum_{\ell=1}^{s} \frac{(1-2n)}{\ell(2\ell+2n-1)}\right) .$$

Then the required solution for our problem is given by

$$\begin{aligned} W_n &= A_n W_{1n} + B_n W_{2n} \\ &= A_n W_{1n} + B_n \left(W_{1n} \log z + \sum_{1}^{\infty} a_s z^s\right) . \end{aligned}$$

Transforming this solution back to η coordinates we get

$$W_n(\eta) = (A_n + B_n \log\eta) W_{1n}(\eta^2) + B_n \sum_{1}^{\infty} b_s \eta^{2s} \tag{A.23}$$

where $b_s = a_s/2$.

Then the solution for $K_n(\epsilon)$, when ϵ is close to 1, is

$$\begin{aligned} K_n(\epsilon) = (1-\eta^2)^n \Bigg\{ &(A_n + B_n \log\eta) F\left(\frac{1+2n}{2}, \frac{1+2n}{2}, 1; \eta^2\right) \\ &+ B_n \sum_{1}^{\infty} \frac{(2s+2n-1)!^2}{2^{2s}(s!)^2} \left(\sum_{\ell=1}^{s} \frac{1-2n}{\ell(2\ell+2n-1)}\right) \eta^{2s} \Bigg\} \end{aligned} \tag{A.24}$$

Now the constants A_n and B_n are evaluated by computing the following integral

$$K_n = \int_0^{\pi/2} \frac{\cos 2n\theta}{\sqrt{1-\epsilon^2 \sin^2\theta}} \, d\theta$$

as ϵ approaches 1. (The details are given in Appendix B.3.) Therefore we have

$$A_n = (-1)^n \log 4 - 2\psi(n;\nu)$$

$$\text{where } \psi(n;\nu) = \sum_{\nu=0}^{n-1} \frac{(-1)^n}{2n-2\nu-1} \text{ and } \psi(0;\nu) = 0 \qquad \text{(A.25)}$$

and

$$B_n = (-1)^{n+1}$$

Substitution of equation (A.24) together with (A.25) in the solution (A.12) for Δr_a reduces the computational work for large values of the eccentricity near 1.

APPENDIX B

1. Expansion of the Exponential Functions $e^{\pm\zeta\sin\theta}$ in a Series of a Modified Bessel Functions

The exponential function $e^{\frac{i\zeta}{2}(z - \frac{1}{z})}$ can be expanded in a series of Bessel functions (see, for example, Watson (17)) as

$$e^{\frac{i\zeta}{2}(z - \frac{1}{z})} = \sum_{-\infty}^{\infty} z^n J_n(i\zeta) \tag{B.1}$$

Now put $z = e^{i\varphi} \Longrightarrow \frac{1}{z} = e^{-i\varphi}$. Using the definition of modified Bessel function $I_n(\zeta) = i^{-n}J_n(i\zeta)$, the above series expansion is written in the form

$$\begin{aligned}
e^{\frac{i\zeta}{2}(z - \frac{1}{z})} &= e^{-\zeta\sin\varphi} \\
&= \sum_{-\infty}^{\infty} e^{in\varphi}(i)^n I_n(\zeta) \\
&= I_o(\zeta) + \sum_{1}^{\infty} e^{in\varphi}(i)^n I_n(\zeta) \\
&\qquad + \sum_{1}^{\infty} e^{-in\varphi}(i)^{-n} I_n(\zeta) \\
&= I_o(\zeta) + \sum_{1}^{\infty} (i)^n I_n(\zeta)\,[e^{in\varphi} + (-1)^n e^{-in\varphi}]
\end{aligned}$$

Therefore

$$\begin{aligned}
e^{-\zeta\sin\varphi} &= I_o(\zeta) - 2\sin\varphi\, I_1(\zeta) - 2\cos 2\varphi I_2(\zeta) \\
&\quad + 2\sin 3\varphi I_3(\zeta) + 2\cos 4\varphi I_4(\zeta) - 2\sin 5\varphi I_5(\zeta) \\
&\quad - 2\cos 6\varphi\, I_6(\zeta) + --- \\
&= I_o(\zeta) + 2\sum_{1}^{\infty} (-1)^n \sin(2n-1)\varphi\, I_{2n-1}(\zeta) \\
&\quad + 2\sum_{1}^{\infty} (-1)^n \cos 2n\varphi\, I_{2n}(\zeta)
\end{aligned} \tag{B.2}$$

Now let $z = -e^{i\varphi} \Rightarrow \frac{1}{z} = -e^{-i\varphi}$. With this relation for z, equation (B.1) yields

$$e^{\frac{i\zeta}{2}(z-\frac{1}{z})} = e^{\zeta \sin\varphi}$$

$$= \sum_{-\infty}^{\infty} (-1)^n e^{in\varphi} (i)^n I_n(\zeta)$$

$$= I_o(\zeta) + \sum_{1}^{\infty} (-i)^n e^{in\varphi} I_n(\zeta) + \sum_{1}^{\infty} (-i)^n e^{-in\varphi} I_n(\zeta)$$

$$= I_o(\zeta) + \sum_{1}^{\infty} (-i)^n I_n(\zeta) \quad e^{in\varphi} + (-1)^n e^{-in\varphi}$$

Therefore

$$e^{\zeta\sin\varphi} = I_o(\zeta) + 2 \sin\varphi \, I_1(\zeta) - 2 \cos 2\varphi \, I_2(\zeta)$$

$$- 2 \sin 3\varphi \, I_3(\zeta) + 2 \cos 4\varphi \, I_4(\zeta) + 2 \sin 5\varphi \, I_5(\zeta)$$

$$- 2 \cos 6\varphi \, I_6(\zeta) + ---$$

$$= I_o(\zeta) - 2 \sum_{1}^{\infty} (-1)^n \sin(2n-1)\varphi \, I_{2n-1}(\zeta)$$

$$+ 2 \sum_{1}^{\infty} (-1)^n \cos 2n\varphi \, I_{2n}(\zeta) \qquad \text{(B.3)}$$

From these two expressions for $e^{\pm\zeta\sin\varphi}$ the following relations

$$e^{\zeta\sin\varphi} + e^{-\zeta\sin\varphi} \quad \text{and} \quad e^{\zeta\sin\varphi} - e^{-\zeta\sin\varphi}$$

are evaluated. They are

$$e^{\zeta\sin\varphi} + e^{-\zeta\sin\varphi} = 2I_o(\zeta) + 4 \sum_{1}^{\infty} (-1)^n \cos 2n\varphi \, I_{2n}(\zeta)$$

and $\qquad$ (B.4)

$$e^{\zeta\sin\varphi} - e^{-\zeta\sin\varphi} = -4 \sum_{1}^{\infty} (-1)^n \sin(2n-1)\varphi \, I_{2n-1}(\zeta)$$

These are the relations used in equations (A.9).

2. Derivation of the Differential Equation Satisfied by $K_n(\epsilon)$

Consider the following functions

$$F_n(\epsilon,\varphi) = \int_0^{\varphi} \frac{\cos 2n\varphi}{\Delta}\, d\varphi$$

and

$$E_n(\epsilon,\varphi) = \int_0^{\varphi} \Delta \cdot \cos 2n\varphi\, d\varphi$$

where $\Delta = (1-\epsilon^2 \sin^2\varphi)^{\frac{1}{2}}$

Note that when $\varphi = \pi/2$; $F_n(\epsilon,\pi/2) \equiv K_n(\epsilon)$

Now

$$\frac{dF_n}{d\epsilon} = \int \epsilon \frac{\sin^2\varphi}{\Delta^3} \cos 2n\varphi\, d\varphi \qquad \text{(B.5)}$$

But $\quad \Delta^2 = 1 - \epsilon^2 \sin^2\varphi$

Solving for $\sin^2\varphi$

$$\sin^2\varphi = \frac{1-\Delta^2}{\epsilon^2} \qquad \text{(B.5a)}$$

Substituting this in (B.5), $dF_n/d\epsilon$ is written as

$$\frac{dF_n}{d\epsilon} = \frac{1}{\epsilon}\int \frac{\cos 2n\varphi}{\Delta^3}\, d\varphi - \frac{1}{\epsilon}\int \frac{\cos 2n\varphi}{\Delta}\, d\varphi$$

$$= \frac{1}{\epsilon}\int \frac{\cos 2n\varphi}{\Delta^3}\, d\varphi - \frac{F_n}{\epsilon} \qquad \text{(B.6)}$$

Let us now consider the derivative of the following function

$$\epsilon^2 \frac{d}{d\varphi}\left(\frac{\cos 2n\varphi \sin\varphi \cos\varphi}{\Delta}\right)$$

$$= \frac{-2n\epsilon^2 \sin 2n\varphi \sin\varphi \cos\varphi}{\Delta}$$

$$+ \frac{\epsilon^2 \cos 2n\varphi}{\Delta}(\cos^2\varphi - \sin^2\varphi) + \frac{\epsilon^4 \cos 2n\varphi \sin^2\varphi \cos^2\varphi}{\Delta^3}$$

$$= \frac{-2n\epsilon^2 \sin^2 n\varphi \sin\varphi \cos\varphi}{\Delta} + \frac{\cos 2n\varphi(\epsilon^2 - 2\epsilon^2 \sin^2\varphi + \epsilon^4 \sin^4\varphi)}{\Delta^3}$$

$$= \frac{-2n\epsilon^2 \sin 2n\varphi \sin\varphi \cos\varphi}{\Delta} + \frac{\cos 2n\varphi}{\Delta^3}[1 - 2\epsilon^2 \sin^2\varphi + \epsilon^4 \sin^4\varphi - 1 + \epsilon^2]$$

$$= \frac{-2n\epsilon^2 \sin 2n\varphi \sin\varphi \cos\varphi}{\Delta} + \frac{\cos 2n\varphi}{\Delta^3}(\Delta^4 - \eta^2)$$

where $\eta^2 = 1 - \epsilon^2$

Therefore

$$\epsilon^2 \frac{d}{d\varphi}\left(\frac{\cos 2n\varphi \sin\varphi \cos\varphi}{\Delta}\right)$$

$$= \Delta \cos 2n\varphi - \frac{\eta^2}{\Delta^3}\cos 2n\varphi \quad \frac{-2n\epsilon^2 \sin 2n\varphi \sin\varphi \cos\varphi}{\Delta} \qquad \text{(B.7a)}$$

Integrating (B.7a), we get

$$\epsilon^2 \frac{\cos 2n\varphi \sin\varphi \cos\varphi}{\Delta} = E_n - \eta^2 \int \frac{\cos 2n\varphi}{\Delta^3}\, d\varphi$$

$$+ 2n \sin 2n\varphi \cdot \Delta - 4n^2 E_n$$

And solving for the integral on the right hand side

$$\eta^2 \int \frac{\cos 2n\varphi}{\Delta^3}\, d\varphi = (1 - 4n^2)E_n + 2n \sin 2n\varphi\, \Delta$$

$$- \frac{\epsilon^2 \cos 2n\varphi \sin\varphi \cos\varphi}{\Delta} \qquad \text{(B.7b)}$$

Substituting (B.7b) in (B.6), $dF_n/d\epsilon$ is written in the form

$$\frac{dF_n}{d\epsilon} = \frac{1}{\epsilon\eta^2}(1 - 4n^2)E_n + \frac{2n}{\epsilon\eta^2}\sin 2n\varphi \cdot \Delta - \frac{\epsilon}{\eta^2}\frac{\cos 2n\varphi \cos\varphi \sin\varphi}{\Delta} - \frac{F_n}{\epsilon}$$

or

$$\frac{dF_n}{d\epsilon} = \frac{1}{\epsilon\eta^2}\left[(1-4n^2)E_n - \eta^2 F_n\right] + \frac{2n}{\epsilon\eta^2}\,\Delta\cdot\sin 2n\varphi - \frac{\epsilon}{\eta^2}\,\frac{\cos 2n\varphi\cos\varphi\sin\varphi}{\Delta}$$

Solving this for E_n, we have

$$(1-4n^2)E_n = \eta^2\left(\epsilon\frac{dF_n}{d\epsilon} + F_n\right) - 2n\Delta\sin 2n\varphi + \epsilon^2\frac{\cos 2n\varphi\sin\varphi\cos\varphi}{\Delta} \qquad \text{(B.8a)}$$

Now let us evaluate the derivative $dE_n/d\epsilon$.

$$\frac{dE_n}{d\epsilon} = -\int \frac{\epsilon\sin^2\varphi\cos 2n\varphi d\varphi}{\Delta}$$

$$= -\frac{1}{\epsilon}\int (1-\Delta^2)\frac{\cos 2n\varphi}{\Delta}\,d\varphi$$

where equation (B.5a) is used. This is written as

$$\frac{dE_n}{d\epsilon} = \frac{1}{\epsilon}(E_n - F_n)$$

Solving for F_n,

$$F_n = E_n - \epsilon\frac{dE_n}{d\epsilon} \qquad \text{(B.8b)}$$

As we are interested in the evaluation of $K_n(\epsilon)$ we set $\varphi = \pi/2$. Consequently

$$(1-4n^2)E_n = \eta^2\left(\epsilon\frac{dK_n}{d\epsilon} + K_n\right) \qquad \text{(B.9a)}$$

and

$$K_n = E_n - \epsilon\frac{dE_n}{d\epsilon} \qquad \text{(B.9b)}$$

Differentiation of equation (B.9a) once yields

$$(1-4n^2)\frac{dE_n}{d\epsilon} = -2\epsilon\left(\epsilon\frac{dK_n}{d\epsilon}+K_n\right)+\eta^2\left(2\frac{dK_n}{d\epsilon}+\epsilon\frac{d^2K_n}{d\epsilon^2}\right)$$

or

$$(1-4n^2)\frac{dE_n}{d\epsilon} = \eta^2\epsilon\frac{d^2K_n}{d\epsilon^2}+2(1-2\epsilon^2)\frac{dK_n}{d\epsilon}-2\epsilon K_n \qquad \text{(B.10)}$$

Eliminating both E_n and $dE_n/d\epsilon$ in equation (B.9b) by substituting (B.9a) and (B.10), the governing differential equation for K_n is obtained:

$$\epsilon^2(1-\epsilon^2)\frac{d^2K_n}{d\epsilon^2}+\epsilon(1-3\epsilon^2)\frac{dK_n}{d\epsilon}-(\epsilon^2+4n^2)K_n = 0$$

where (B.11)

$$K_n(\epsilon) = \int_0^{\pi/2}\frac{\cos 2n\varphi}{(1-\epsilon^2\sin^2\varphi)^{\frac{1}{2}}}\,d\varphi$$

This is the equation given in (A.13).

3. <u>Evaluation of the Constants A_n and B_n in the Solution of $K_n(\epsilon)$</u>

Let us consider the behavior of $K_n(\epsilon)$ near $\epsilon \sim 1$.

$$K_n(\epsilon) = \int_0^{\pi/2}\frac{\cos 2n\varphi}{(1-\epsilon^2\sin^2\varphi)^{\frac{1}{2}}}\,d\varphi$$

Let $\epsilon^2 = 1-\eta^2$ and assume that η is very small; with this value for ϵ^2, K_n takes the form

$$K_n = \int_0^{\pi/2}\frac{\cos 2n\varphi}{\sqrt{\cos^2\varphi+\eta^2\sin^2\varphi}}\,d\varphi \qquad \text{(B.12)}$$

This integral blows up as $\eta \to 0$. Therefore this integral will be split into two parts as shown

$$K_n = \int_0^{\frac{\pi}{2}-\alpha} (\quad)\, d\varphi + \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} (\quad)\, d\varphi$$

where α is very small

$$\equiv L_1 + L_2 \tag{B.13}$$

Since η is very near zero, L_1 is written as

$$L_1 = \int_0^{\frac{\pi}{2}-\alpha} \frac{\cos 2n\varphi}{\cos\varphi}\, d\varphi$$

Now put $\varphi = \frac{\pi}{2} - \theta$ in L_2. Then L_2 takes the form

$$L_2 = \int_0^{\alpha} \frac{\cos 2n(\frac{\pi}{2} - \varphi)}{\sqrt{\sin^2\theta + \eta^2\cos^2\theta}}$$

But $\cos 2n(\frac{\pi}{2} - \theta) = \cos n\pi \cos 2n\theta$

$$\simeq (-1)^n \quad \text{as} \quad \theta \to 0$$

$$\sin^2\theta \doteq \theta^2$$

$$\eta^2\cos^2\theta = \eta^2 - \eta^2\sin^2\theta = \eta^2 - \eta^2\theta^2$$

and

$$\sin^2\theta + \eta^2\cos^2\theta = \epsilon^2\theta^2 + \eta^2$$

Therefore

$$L_2 = \int_0^{\alpha} \frac{(-1)^n}{\sqrt{\eta^2+\epsilon^2\theta^2}}\, d\theta$$

$$= \frac{(-1)^n}{\epsilon} \log\left(\epsilon\,\theta + \sqrt{\eta^2+\epsilon^2\theta^2}\right)\Bigg|_0^{\alpha}$$

$$= \frac{(-1)^n}{\epsilon} \log \frac{\alpha\epsilon+\sqrt{\eta^2+\alpha^2\epsilon^2}}{\eta}$$

as $\eta \to 0$ this reduces to

$$\lim_{\substack{\eta\to 0\\ \epsilon\to 1}} L_2 = (-1)^n \log\frac{2\alpha}{\eta} \tag{B.14}$$

Now let us integrate L_1

$$L_1 = \int_0^{\frac{\pi}{2}-\alpha} \frac{\cos 2n\varphi}{\cos\varphi}\, d\varphi$$

$$= 2\sum_{\nu=0}^{n-1} (-1)^{\nu}\, \frac{\sin(2n-2\nu-1)(\frac{\pi}{2}-\alpha)}{(2n-2\nu-1)} + (-1)^n \log\frac{1+\sin(\frac{\pi}{2}-\alpha)}{\cos(\frac{\pi}{2}-\alpha)}$$

But $\quad \sin(2n-2\nu-1)\,(\frac{\pi}{2}-\alpha)$

$$= \sin\left\{\left[(n-\nu)\pi - \frac{\pi}{2}\right] - (2n-2\nu-1)\alpha\right\}$$

$$= \sin\left[2(n-\nu)-1\right]\frac{\pi}{2}\cos(2n-2\nu-1)\alpha$$

$$\quad -\cos(2n-2\nu-1)\frac{\pi}{2}\sin(2n-2\nu-1)\alpha$$

$$= \cos(2n-2\nu-1)\alpha\left[\sin(n-\nu)\pi\cos\frac{\pi}{2} - \cos(n-\nu)\pi\sin\frac{\pi}{2}\right]$$

$$\quad -\sin(2n-2\nu-1)\alpha\left[\cos(n-\nu)\pi\cos\frac{\pi}{2} + \sin(n-\nu)\pi\sin\frac{\pi}{2}\right]$$

$$= -\cos(n-\nu)\pi\cos(2n-2\nu-1)\alpha$$

$$= -\cos n\pi\cos\nu\pi\cos(2n-2\nu-1)\alpha$$

$$\simeq -(-1)^{n+\nu} \quad \text{as} \quad \alpha \to 0$$

Also

$$\frac{1+\sin(\frac{\pi}{2}-a)}{\cos(\frac{\pi}{2}-a)} \simeq \frac{1+\cos a}{\sin a}$$

$$\simeq \frac{2}{a} \text{ as } \quad a \to 0$$

Consequently,

$$L_1 = +\sum_{\nu=0}^{n-1} \frac{(-1)^{n+1}}{(2n-2\nu-1)} + (-1)^n \log \frac{2}{a}$$

Therefore

$$L_1+L_2 = \sum_{\nu=0}^{n-1} \frac{(-1)^{n+1}}{(2n-2\nu-1)} + (-1)^n \log \frac{2}{a} + (-1)^n \log \frac{2a}{\eta}$$

$$= (-1)^n \log \frac{4}{\eta} + 2\sum_{\nu=0}^{n-1} \frac{(-1)^{n+1}}{(2n-2\nu-1)} \qquad \text{(B.15)}$$

These are the values reached by $K_n(\epsilon)$ as $\epsilon \to 1$; that is, the constants A_n and B_n in $A_n + B_n \log\eta$ must have these values as $\epsilon \to 1$, $\eta \to 0$. Hence

$$A_n + B_n \log\eta = (-1)^n \log \frac{4}{\eta} + 2\sum_{\nu=0}^{n-1} \frac{(-1)^{n+1}}{(2n-2\nu-1)}$$

Therefore

$$A_n = (-1)^n \log 4 + 2\sum_{\nu=0}^{n-1} \frac{(-1)^{n+1}}{2n-2\nu-1}$$

and

$$B_n = -(-1)^n = (-1)^{1+n}$$

Let

$$\psi(n;\nu) = \sum_{\nu=0}^{n-1} \frac{(-1)^n}{2n-2\nu-1}$$

with $\psi(0;\nu) = 0$

Then

$$\left.\begin{aligned} A_n &= (-1)^n \log 4 - 2\psi(n;\nu) \\ \text{and} \quad & \\ B_n &= (-1)^{1+n} \end{aligned}\right\} \qquad (B.16)$$

These are the values of A_n and B_n given in (A.25) which are required for the solution of $K_n(\epsilon)$ in (A.24) for values of ϵ near 1.