THE EFFECTS OF VERTICAL GUSTS ON AN ELASTIC WING

Thesis

by

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ABSTRACT

The case of a wing flexible in bending and rigid in torsion entering a vertical gust is treated by consideration of the corresponding two-dimensional case. The bending stiffness of the wing is expressed as a spring constant, the aerodynamic forces acting are taken from the two-dimensional theory of airfoils in non-uniform motion, and the differential equation of motion of the wing is solved by operational methods.

Cases of sharp-edged and exponentially graded gusts are considered, and in each case the deflection is calculated in dimensionless form for three values of a dimensionless stiffness parameter. As a numerical example these results are put into dimensional form for a wing whose elastic properties are supposed to be typical of modern airplane wings.

The results obtained for the cases considered show only in one instance a maximum deflection greater than the steady-state value. Even in the sharp-gust case the deflection increases comparatively slowly following the entrance of the wing into the gust. The effects of grading the gust are to decrease slightly the rate of deflection and to reduce the amplitude of the oscillations produced by the gust.

ACKNOWLEDGMENT

This investigation was undertaken at the suggestion of Dr. W. R. Sears, as an application of the theory developed in von Kármán and Sears's paper: Airfoil Theory for Non-Uniform Motion, (Ref. 1). It is desired to express great appreciation for the very considerable assistance received from Dr. Sears during the progress of development of the basic methods and subsequent calculations made in this paper.

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I. INTRODUCTION

The problem of the behavior of an elastic wing during and subsequent to its entrance into a vertical gust has been considered by several authors (Refs. 2, 3, 4), who have made various assumptions regarding the properties of the wing and the nature of the aerodynamic forces. These forces have usually been calculated by neglecting entirely the "non-stationary flow" effects or by including them in certain terms and neglecting them in others. The results have often shown that wing deflections (and hence stresses) considerably greater than those corresponding to steady-state values are produced by certain combinations of gust gradient and wing elastic properties.

The present thesis is the first step in an attempt to determine whether this conclusion may be due in part to the neglection of non-stationary effects.

For this purpose the case of a wing elastic in bending but perfectly rigid in torsion is considered. It is treated approximately by assuming two-dimensional flow conditions at a typical section of the wing, and by representing the elastic properties of the wing by a vertical spring restraining the vertical motion of this section. The schematic setup is sketched in Fig. 1.

The aerodynamic forces are then calculated from the theory of two-dimensional thin airfoils in non-uniform motion, in order to take into account the "lag" in the build-up of the lift due to both the gust and the vertical motion of the wing.

II. THE EQUATION OF MOTION

The equation of motion for the wing section under consideration is

$$m\ddot{z} + kz = L \tag{1}$$

where m = mass of wing, including additional apparent mass of air, per unit span

z = upward deflection of the wing

k = spring constant representing bending stiffness

L = instantaneous total lift per unit span

and where dots indicate differentiation with respect to the time, t.

An expression has been developed in Ref. 1 for the lift on a rigid wing entering a vertical gust distribution w(s) where $s = \bar{U}t$ = distance travelled by the wing, in half-chords:

$$L_g(s) = \pi \rho U c \int_0^s w(\sigma)\, \Psi'(s - \sigma) d\sigma \tag{2}$$

where ρ = air density

U = velocity of flight (ft./sec.) = $\bar{U}c/2$

$\bar{U}$ = velocity of flight in half-chords/sec.

c = wing chord (ft.)

$\Psi(s)$ = a function calculated by v. Kármán and

Sears, giving the lift on a rigid airfoil entering a sharp-edged gust, s being measured from the instant the leading edge begins to enter the gust.

Since the wing is not rigid in the present case, there must be added to L_g the lift due to the vertical velocity of the wing. This can be shown to be

$$L_v(s) = \pi\rho Uc \left\{ \int_0^s \dot{z}(\sigma) \; \Phi'(s - \sigma)d\sigma \; - \dot{z}(s) \left[1 - \Phi(0)\right] \right\} \quad (3)$$

where $1-\Phi(s)$ is Wagner's function (see Ref. 1) which gives the lift on an airfoil following a sudden change of its angle of attack at s = 0.

It is convenient to use s as the independent variable in Eq. (1). Then, writing $L = L_g + L_v$, we have

$$m\overline{U}^2 z'' \; + \; kz \; = \; L_g + L_v \qquad (4)$$

where the primes represent differentiation with respect to s.

III. THE EQUATION OF MOTION IN OPERATIONAL FORM

The present problem is most conveniently solved by means of Heaviside's operational methods. The notation used here is that used by Pipes in Ref. 5, except that the independent variable will be denoted by s instead of t. If a function f(s) satisfies certain existence and continuity conditions on itself and its first derivatives, it may be represented by the complex integral:

$$f(s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{F(p)\ e^{ps}dp}{p} \qquad (5)$$

where $$F(p) = p \int_{-\infty}^{\infty} f(s)\ e^{-ps}dt$$

and a is a real number great enough so that all singularities of $F(p)/p$ occur for $R(p) < a$. The function $F(p)$ is called the direct Laplacian Transform or image of $f(s)$ and is written $F(p) \doteqdot f(s)$, after the notation of van der Pol. A few of the more useful theorems and transforms are given here without proof to show the method of transformation from the ordinary differential equation to the operational form.

Theorem I

If $g(p) \doteqdot h(s)$, then $\frac{d^n h(s)}{ds^n} \doteqdot p^n g(p) - \sum_{k=0}^{n-1} \frac{d^k h(0)}{ds^k}\ p^{n-k}$

Theorem III

If $g_1(p) \doteqdot h_1(s)$ and $g_2(p) \doteqdot h_2(s)$, then

$$\frac{g_1(p)g_2(p)}{p} \doteqdot \int_0^s h_1(u)h_2(s-u)du$$

Elementary Transforms

If $h(s) = 1(t) = \left.\begin{matrix} 1 \text{ for } s \geqq 0 \\ 0 \text{ for } s < 0 \end{matrix}\right\}$, then $1 \doteqdot h(s)$

If $h(s) = e^{-\alpha s}$, then $\frac{p}{p+\alpha} \doteqdot e^{-\alpha s}$

If $h(s) = (1 - e^{-\alpha s})$, then $\frac{\alpha}{p+\alpha} \doteqdot (1 - e^{-\alpha s})$

Each term of Eq. (4) can now be put into operational form, as follows:

(i) By use of Theorem I, assuming that $z(s) = 0$ for $s \leqq 0$, the first term can be written as

$$m\bar{U}^2 z''(s) \doteqdot m\bar{U}^2 p^2 h(p)$$

where $z(s) \doteqdot h(p)$.

(ii) By use of Theorems I and III, from Eq. (2), noting that $\Psi(0) = 0$, L_g becomes

$$L_g(s) \doteqdot \pi \rho U^2 \cdot (c^2/2)\ g(p)\,\psi(p)$$

where $w(s)/\bar{U} \doteqdot g(p)$ and $\Psi(s) \doteqdot \psi(p)$.

(iii) By use of Theorems I and III, from Eq. (3), noting that $\dot{z}(s) = \bar{U}z'(s) \doteqdot \bar{U}ph(p)$, L_v becomes

$$L_v(s) \doteqdot -\pi \rho \bar{U}^2 \cdot (c^2/2)\, ph(p) \cdot \left[1 - \varphi(p)\right]$$

where $\Phi(s) \doteqdot \varphi(p)$.

The equation of motion then becomes

$$\left\{m\bar{U}^2 p^2 + k + \pi \rho \bar{U}^2 \cdot (c^2/2)\ p\left[1 - \varphi(p)\right]\right\} \cdot h(p)$$

$$= \pi \rho \bar{U}^2 \cdot (c^2/2)\ g(p)\ \psi(p) \qquad (6)$$

or

$$z(s) \doteqdot h(p) = \frac{B\ \psi(p)}{p^2 + A + Bp\left[1 - \varphi(p)\right]}\ g(p) \qquad (7)$$

where $A = k/m\bar{U}^2$ and $B = \pi \rho c^2/2m$.

The parameter A may be called the "dimensionless stiffness", while B is twice the ratio of the additional apparent air mass to the total mass m, and may be considered as the reciprocal of the dimensionless mass parameter.

IV. APPROXIMATIONS TO THE LIFT FUNCTIONS

The solution for z(s) for various convenient gust functions $w(s) \doteqdot g(p)$ can now be carried out if $\psi(p)$ and $\varphi(p)$ are known. Sears (Ref. 6) gives these operators in terms of certain Bessel functions which are practically intractable in the present problem. However, an approximation to Wagner's function $1 - \Phi(s)$ has been suggested by Jones (Ref. 7), having a particularly simple image:

$$1 - \Phi(s) \doteqdot 1 - a_1 e^{-\lambda_1 s} - a_2 e^{-\lambda_2 s}$$

for which (8)

$$1 - \varphi(p) = 1 - \frac{a_1 p}{p + \lambda_1} - \frac{a_2 p}{p + \lambda_2}$$

where $a_1 = .165$, $a_2 = .335$, $\lambda_1 = .0455$, and $\lambda_2 = .300$.

This suggests the possibility of a similar approximation to the Kármán-Sears function $\Psi(s)$:

$$\Psi(s) \doteq 1 - b_1 e^{-\mu_1 s} - b_2 e^{-\mu_2 s}$$

for which (9)

$$\psi(p) = 1 - \frac{b_1 p}{p + \mu_1} - \frac{b_2 p}{p + \mu_2}$$

It is found by trial that the following numerical values provide the best approximation*:

$$b_1 = b_2 = 1/2, \quad \mu_1 = .130, \; \mu_2 = 1$$

Using the forms of φ and ψ given in Eqs. (8) and (9), Eq. (7) becomes

$$z(s) \doteqdot h(p) = B \frac{(p+\mu_1)(p+\mu_2) - b_1 p(p+\mu_2) - b_2 p(p+\mu_1)}{(p+\mu_1)(p+\mu_2)}$$

$$\cdot \frac{(p+\lambda_1)(p+\lambda_2)}{(p^2 + Bp + A)(p+\lambda_1)(p+\lambda_2) - Bp^2\left[a_1(p+\lambda_2) + a_2(p+\lambda_1)\right]}$$

$$\cdot g(p) \qquad \text{- - - - - - - - - -} \qquad (10)$$

If the final value of $w(s)$ is $w(\infty)$, the final steady-state deflection will be $z(\infty) = \pi \rho U w(\infty) c/k$ $= (B/A) \cdot w(\infty)/\bar{U}$. Hence Eq. (10) can be considered as an expression for the dimensionless deflection

$$z(s) \Big/ \left(\frac{B}{A} \frac{w(\infty)}{\bar{U}}\right) = z^*(s),$$ say, which depends only on the two dimensionless parameters A and B.

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* It will be seen that the approximate form of $\Psi(s)$ does not have a vertical tangent at $s = 0$ as does the exact Kármán-Sears function. This is not expected to have a great effect on the accuracy of the results obtained.

V. EVALUATION OF THE OPERATOR

The operator $h(p)$ in Eq. (10) can be evaluated according to the following reasoning: It can be shown (Ref. 5) that for operators of the type under consideration

$$z(s) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{ps}h(p)}{p}\,dp = \sum \left(\text{Residues of } \frac{e^{ps}h(p)}{p}\right) \qquad (11)$$

Hence, assuming that $g(p)$ is given in convenient form, the evaluation of $h(p)$ in Eq. (10) depends only on the determination of the zeros of the denominator. These occur when $p = -\mu_1, -\mu_2$, and the four roots of

$$p^4 + \alpha p^3 + \beta p^2 + \gamma p + \delta = 0$$

where $\alpha = \lambda_1 + \lambda_2 + B(1 - a_1 - a_2) = .3455 + B/2$

$$\beta = A + B\lambda_1(1 - a_2) + B\lambda_2(1 - a_1) + \lambda_1\lambda_2$$

$$= A + .2807B + .01365$$

$$\gamma = B\lambda_1\lambda_2 + A(\lambda_1 + \lambda_2) = .01365B + .3455A$$

$$\delta = A\lambda_1\lambda_2 = .01365A$$

VI. SCOPE OF THE PRESENT INVESTIGATION

The deflection, in dimensionless form, is calculated here for two gust profiles:

(i) sharp-edged gust: $w(s)/\bar{U} = 1(s) = \begin{cases} 0 & \text{for } s < 0 \\ 1 & \text{for } s \geqq 0 \end{cases}$

for which $g(p) = 1$

(ii) graded gust: $w(s)/\overline{U} = \begin{cases} 0 \text{ for } s < 0 \\ 1 - e^{-as} \text{ for } s \geqq 0 \end{cases}$

for which $g(p) = a/(p + a)$

The value taken here for a is 0.75, which corresponds to the gust profile drawn in Fig. 3. This is a gust which reaches 90% of its final strength about 3 half-chords from its edge.

For these two cases the calculations of $z^*(s)$ have been carried out for B = 2/7 and A = .3380, .0845, and .0375. The results are plotted in Figs. 2 and 3.

As a numerical example, the following properties have been assumed for the wing:

$c = 7.5$ ft.

$m = 0.7354$ slugs/ft. span

$k = 622.5$ lbs./ft./ft. span

These values (to which the value B = 2/7 corresponds) have been taken from a thesis by A. E. Lombard, Jr., and are supposed to represent typical values for modern airplane wing construction.† Using these numerical values the dimensionless curves of Figs. 2 and 3 have been replotted in Figs. 4 and 5.

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† Lombard expresses the same values as follows:

$$\frac{\text{(wing mass)}}{\text{(additional apparent air mass)}} = 6.0 \text{ at sea level}$$

$$\text{natural frequency in bending} = \sqrt{k/(\text{wing mass})} = 10\pi$$

The analytic expressions for the results are given, together with some of the detailed calculations, in Section VIII of this thesis.

VII. DISCUSSION AND CONCLUSIONS

In Figs. 2 and 3 it is seen that the effect of increasing the dimensionless stiffness is to increase the rate of the dimensionless deflection and to increase the tendency of the wing to oscillate. In the case $A = .3380$ and the sharp-edged gust, the oscillation carries the deflection beyond the steady-state asymptotic value.

The effect of even as slight a grading of the gust as employed here is seen to be great. The deflections increase more slowly and the amplitudes of the oscillations are diminished. It seems probable that further grading of the gust profile would cause the oscillations to disappear almost entirely, thus eliminating any possibility of extremely large deflections.

When the curves are plotted in dimensional form for the numerical example considered (Figs. 4 and 5), it is seen that the rates of deflection are actually not much different for the various flying speeds and that what appeared to be the most dangerous case ($A = .3380$ in Fig. 2) actually corresponds to the lowest speed of flight and hence is not critical. In fact, the deflection curves for this wing are generally of a favorable character, and it seems that calculation of gust load factors by the usual methods should be quite conservative.

The accuracy of the numerical results for $z(s)$ presented here and in Section VIII is believed to be about $\pm 5\%$ or 6% of the asymptotic value $z(\infty)$. Hence the values for $z(s)$ near $s = 0$ are not dependable, and have been omitted from the plotted curves.

It is intended that further calculations shall be carried out in order to extend the method to more values of the parameters.

VIII. CALCULATIONS

1. Sharp Gust: $g(p) = 1, \quad B = 2/7 = .2856$

$$z^*(s) \doteq h^*(p) = A\,\frac{\left[(p+.130)(p+1) - p(2p+1.130)/2\right](p+.0455)(p+.300)}{(p+.130)(p+1)(p^4+\alpha p^3+\beta p^2+\gamma p+\delta)} \qquad --- (12)$$

1a. A = .3380:

$\alpha = .3455 + B/2 = .3455 + .1428 = .4883$

$\beta = A + .2807B + .01365 = .3380 + .0802 + .01365 = .4318$

$\gamma = .01365B + .3455A = .00390 + .11678 = .1207$

$\delta = .01365A = .0046$

The real roots are found by Horner's method, and are found to be

$$p_1 = -.0455, \qquad p_2 = -.278$$

(The first of these provides a factor which cancels the factor $(p + .0455)$ in the numerator above.) The other two roots are found from the remaining quadratic factor; they are

$$p_{3,4} = -.08165 \pm .600\, i$$

The residues are now calculated by the formula

$$R_i = \lim_{p \to p_i} \left\{ (p + p_i)\, \frac{h^*(p)e^{ps}}{p} \right\}$$

where $h^*(p)$ is the operator in Eq. (12). For example,

$$R_2 = \lim_{p \to -.278} \left\{ (p - .278)\, \frac{h^*(p)e^{ps}}{p} \right\}$$

$$= .338\,\frac{\left[(.130 - .278)(1 - .278) + .278(1.130 - .556)\right]\,(.300 - .278)e^{-.278s}}{(.130 - .278)(1 - .278)(-.278 + .08165 + .600i)(-.278 + .08165 - .600i)\cdot(-.278)}$$

The other residues are calculated in a similar manner. The residues at the two complex poles combine to give a sinusoidal term. The final result for this case is found to be

$$z^*(s) = 1 - .0171e^{-.278s} - .5383e^{-.130s} - .1371e^{-s}$$
$$-.4985e^{-.082s}\sin(34.4s + 38°)$$

1b. A = .0845:

$\alpha = .4827, \ \beta = .1784, \ \gamma = .0332, \ \delta = .00116$

$p_i = -.044, \ -.235, \ -.101 \pm .317i$

$$z^*(s) = 1 - .6539e^{-.1305} - .0422e^{-s} - .0472e^{-.044s} - .0065e^{-.235s}$$
$$-.5798e^{-.101s}\sin(18.16s + 25.3°)$$

1c. A = .0375:

$\alpha = .4886, \ \beta = .1315, \ \gamma = .0169, \ \delta = .00051$

$p_i = -.0424, \ -.1885, \ -.1285 \pm .2180i$

$$z^*(s) = 1 - .1109e^{-.0424s} + .2110e^{-.189s} - 1.1032e^{-.130s}$$
$$- .0200e^{-s} - .5723e^{-.129s}\sin(12.5s - 2.91°)$$

2. Graded Gust: $g(p) = 0.75/(p + 0.75)$, $B = 2/7 = .2856$

$$z^*(s) \doteqdot h^*(p) = \left[h^*(p) \text{ of Eq. (12)}\right] \cdot \frac{0.75}{p + 0.75} \qquad (13)$$

The results for these cases are obtained from those of 1a,b,c above by multiplying each residue by the factor $0.75/(p_i + 0.75)$ and adding a term corresponding to the residue at $p = -0.75$. The results are as follows:

2a. A = .3380:

$$z^*(s) = 1 - .0272e^{-.278s} - .6510e^{-.130s} + .4113e^{-s} - .7614e^{-.75s}$$
$$- .0730e^{-.082s}\sin(34.4s - 4.1°)$$

2b. A = .0845:

$$z^*(s) = 1 - .7918e^{-.130s} + .1265e^{-s} - .0501e^{-.044s} - .0095e^{-.235s}$$
$$-.2654e^{-.75s} - .2809e^{-.101s}\sin(18.16s - 0.8°)$$

2c. A = .0375:

$$z^*(s) = 1 - .1176e^{-.0424s} + .2822e^{-.189s} - 1.3353e^{-.130s}$$
$$+ .0600e^{-s} - .00018e^{-.75s} - .3284e^{-.129s}\sin(12.5s - 22.3°)$$

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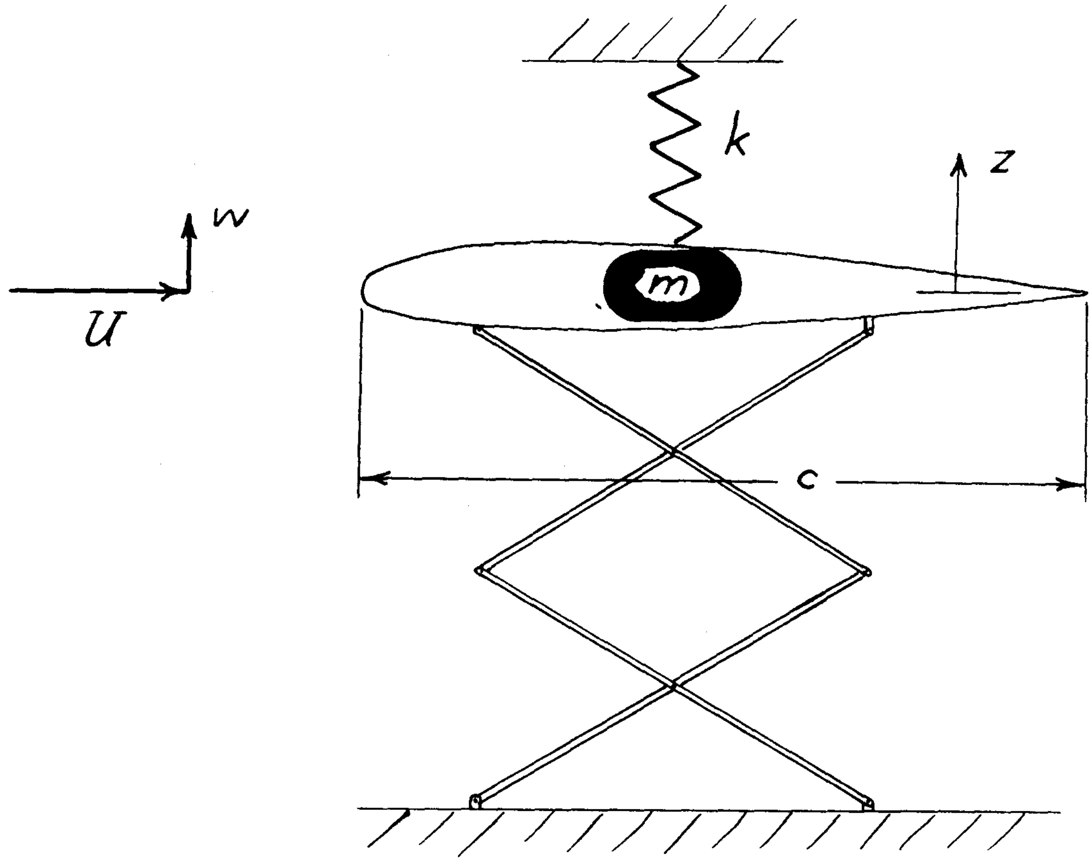


FIGURE 1

DIAGRAM SHOWING SCHEMATICALLY THE CASE CONSIDERED

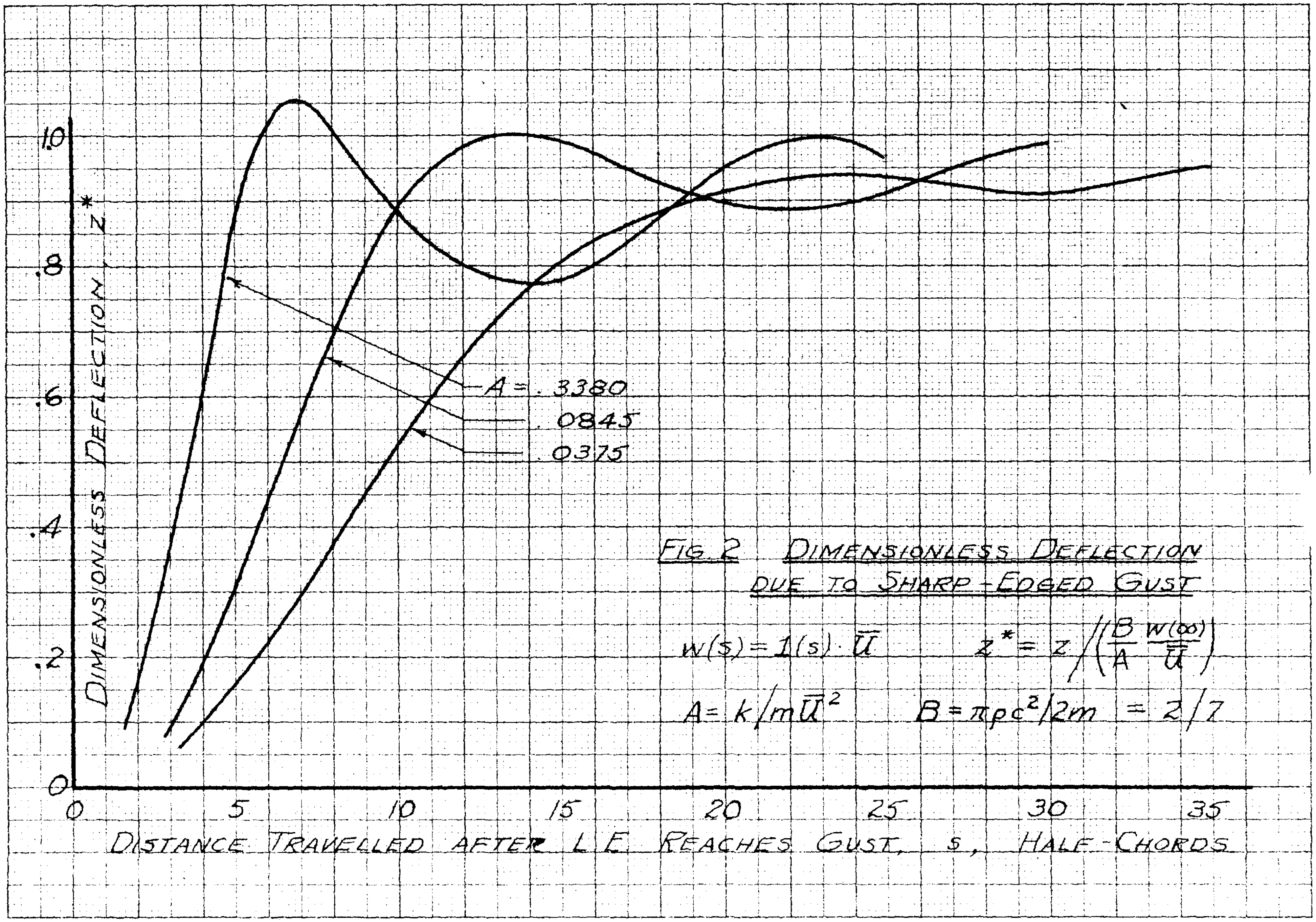


FIG. 2 DIMENSIONLESS DEFLECTION DUE TO SHARP-EDGED GUST

$w(s) = 1(s) \cdot \overline{U}$ $\quad z^* = z \Big/ \left(\frac{B}{A} \frac{w(\infty)}{\overline{U}}\right)$

$A = k / m\overline{U}^2$ $\quad B = \pi \rho c^2 / 2m = 2/7$

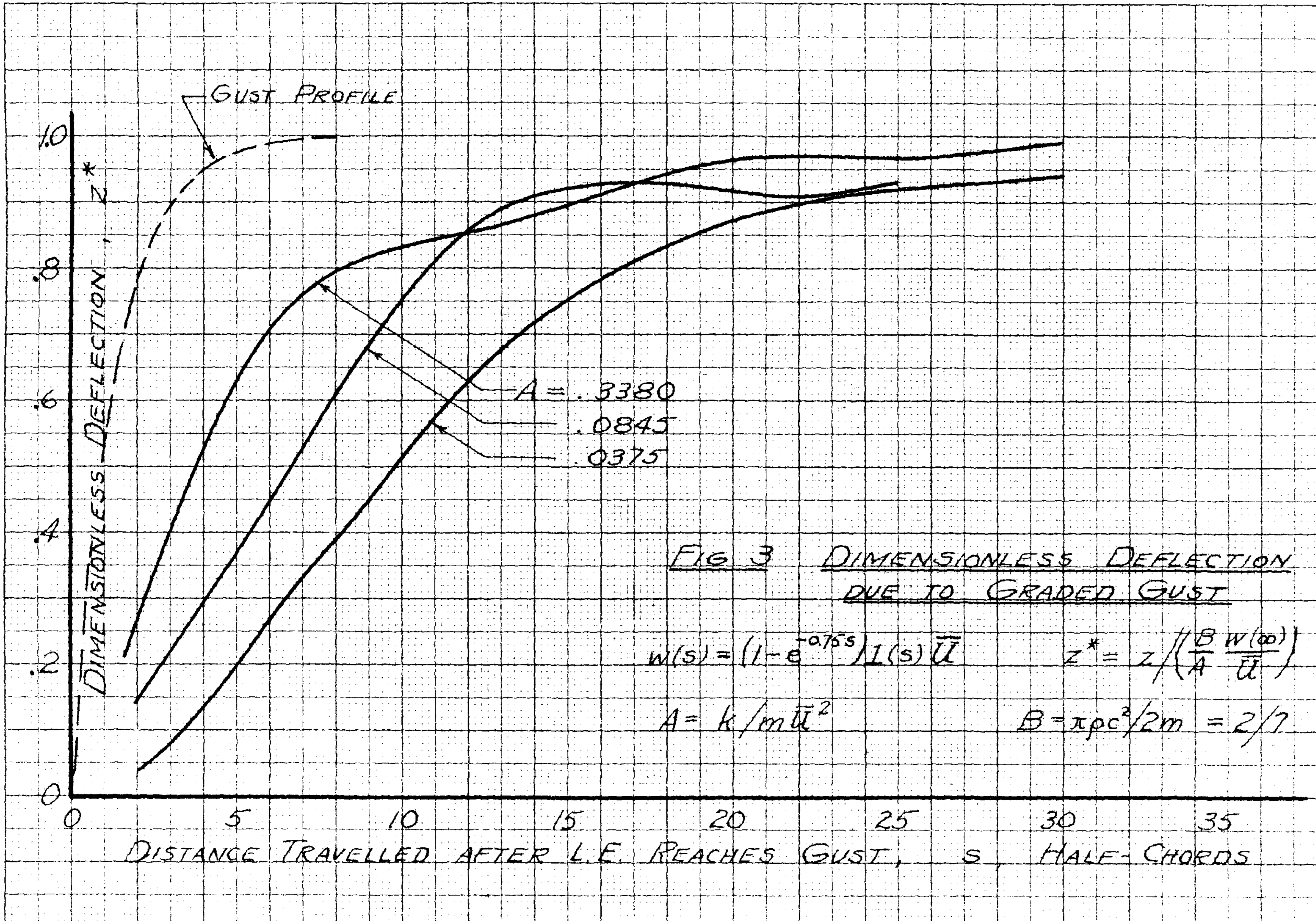
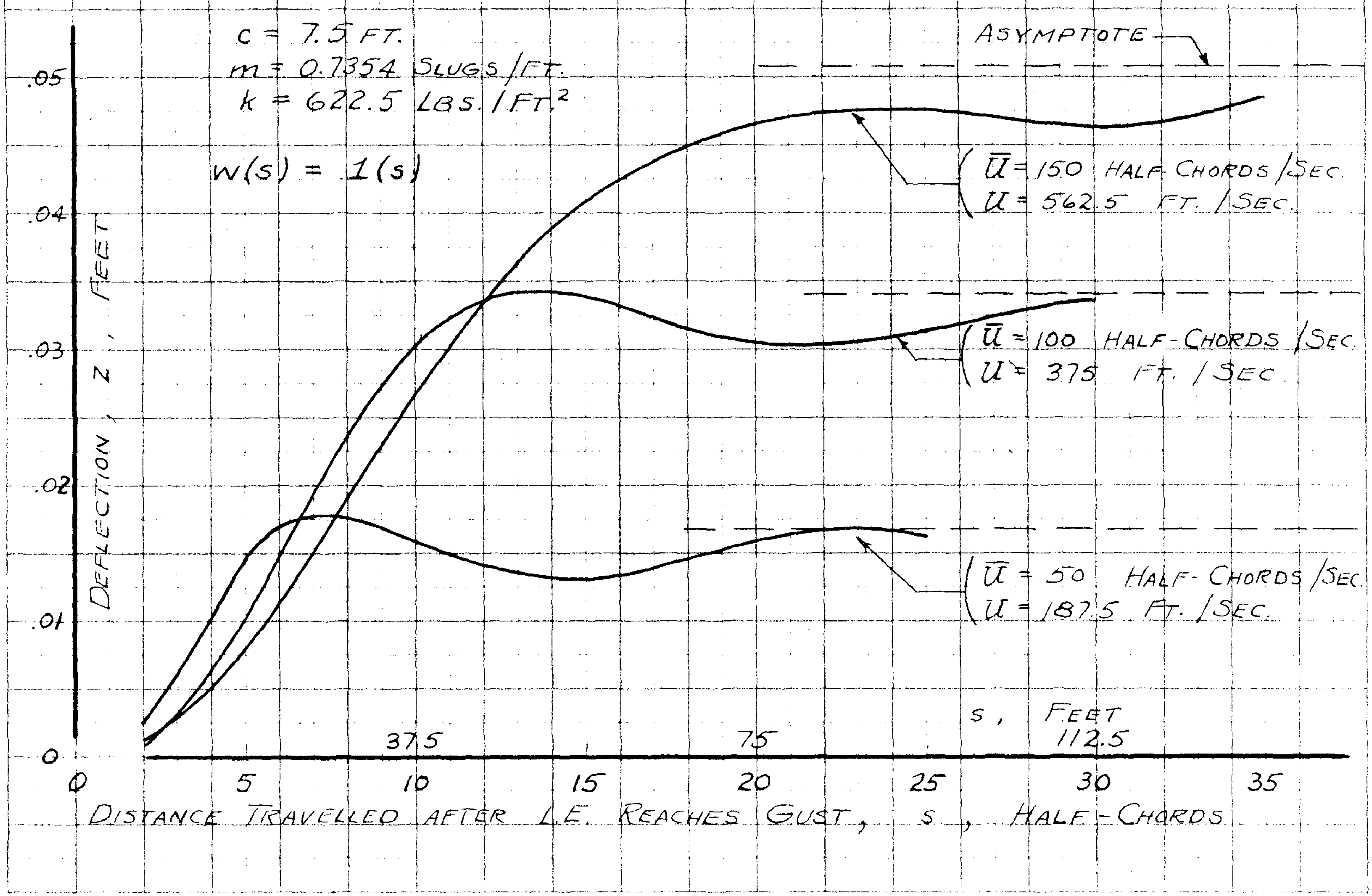


Fig 3 Dimensionless Deflection due to Graded Gust

$w(s) = (1 - e^{-0.75s}) 1(s) \bar{U}$ $z^* = z / \left(\frac{B}{A} \frac{w(\infty)}{\bar{U}} \right)$

$A = k / m \bar{U}^2$ $B = \pi \rho c^2 / 2m = 2/7$

FIG. 4 DEFLECTION DUE TO SHARP-EDGED GUST



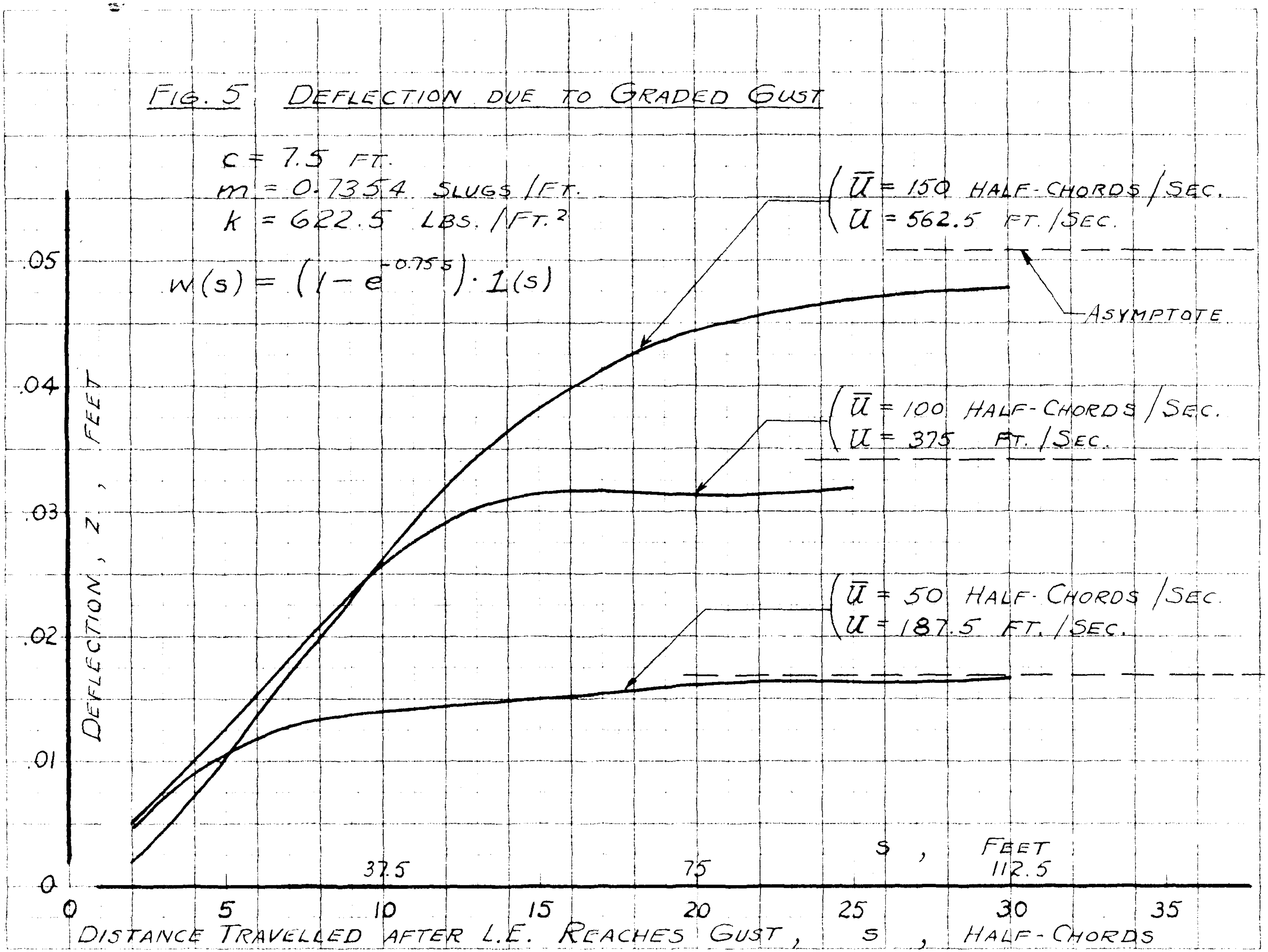


FIG. 5 DEFLECTION DUE TO GRADED GUST