AN APPLICATION OF MATRIX METHODS TO WING THEORY

Thesis

by

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The numerical values given by Mr. Fischer have a considerable number of errors. These calculations have been corrected by Mr. Nagamatsu and the corrected calculations are presented in Report No. A55 of Northrop Aircraft, Inc. on file in the Aeronautics Library reprint file under Northrop. (6/13/44)

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ABSTRACT

The calculation of spanwise lift distribution of a wing by a new method proposed by Theodore von Kármán and W. R. Sears* depends on knowledge of certain "characteristic values" (eigenvalues) and "characteristic functions" (eigenfunctions) of the wing planform. These functions are solutions of a homogeneous boundary-value problem of the third kind.

In the present paper the eigenvalues and the eigenfunctions, in series form, are calculated for a class of planforms by the method of successive multiplication of matrices.

The class of planforms considered is that of trapezoidal wings with rounded tips. The eigenvalues and eigenfunctions are calculated for taper ratios 1:1, 2:1, 3:1, and 4:1; they are independent of aspect ratio. It is found that for intermediate taper ratios they can be determined with reasonable accuracy by graphical interpolation.

* To be published shortly.

AN APPLICATION OF MATRIX METHODS TO WING THEORY

INTRODUCTION

A new method for calculating spanwise lift distribution has been developed by Professors Theodore von Kármán and W. R. Sears, and is to appear shortly in a paper to be published by these authors. This method involves the solution of the boundary-value problem of the lifting line in terms of the characteristic values (eigenvalues) and characteristic functions (eigenfunctions) of the planform.

The eigenfunctions are solutions of the following boundary-value problem of the "third kind":

$$4f\varphi(r,\theta) + \lambda \frac{\partial \varphi(r,\theta)}{\partial r} = 0, \quad \text{for } r = 1,$$

$$\text{where} \quad f = \frac{\sin\theta}{cm/c_s m_s}.$$

$$\nabla^2\varphi = 0, \text{ for } r > 1 \qquad (1)$$

$$\varphi \rightarrow 0 \text{ as } r \rightarrow \infty$$

$$\varphi(1,-\theta) = -\varphi(1,\theta)$$

This system of equations has an infinite number of solutions of undetermined amplitude for certain values of λ , say $\lambda = \lambda_1$, λ_2 , ... , λ_n , called by Professor von Karman the eigenvalues of the planform. The solution $\varphi_n(r,\theta)$ corresponding to $\lambda = \lambda_n$ is called the n^{th} eigenfunction of the planform. Since the function f is independent of the aspect ratio, obviously the λ_n's and φ_n 's are also.

NOTATION

$f = \dfrac{\sin\theta}{cm/c_s m_s}$

θ = measure of spanwise position along wing, $\cos\theta = \dfrac{y}{b/2}$

b = wing span

y = spanwise coordinate, measured from root

$c = c(\theta)$ = wing chord distribution

m = slope of lift curve $\dfrac{dc_\ell}{d\alpha}$ of any section, per radian

$(\ )_s$ = root section

$(\ )_t$ = tip section

$T = \dfrac{c_s}{c_t}$ = taper ratio

$\phi(r,\theta)$ = velocity potential in the transformed Trefftz Plane

λ_n = eigenvalue of the planform

$\phi_n(r,\theta)$ = eigenfunction of the planform

$\phi_n(\theta), \phi(\theta)$ = shorthand for $\phi_n(1,\theta), \phi(1,\theta)$

r,θ = polar coordinates in the transformed Trefftz Plane

$C_m^{(n)}$ = Fourier coefficients of $\phi_n(\theta)$

$K_n = \sum_{m=1}^{\infty} m(C_m^{(n)})^2$

$M_n = \dfrac{\pi}{4}\ell_n K_n$

b_k = Fourier coefficient of the function $f(\theta)$

$[\ \ ]$ represents a square matrix

$\{\ \ \}$ represents a column matrix

STATEMENT OF THE PROBLEM BY MEANS OF FOURIER SERIES

The most general form of ϕ_n which satisfies equations (1) is

$$\phi_n(r,\theta) = \sum_{m=1}^{\infty} C_m^{(n)} \frac{\sin m\theta}{r^m} \tag{2}$$

Then

$$\frac{\partial \phi_n(r,\theta)}{\partial r} = -\sum_{m=1}^{\infty} C_m^{(n)} \frac{m \sin \theta}{r^{m+1}}$$

(Note: Hereafter, let $\phi_n(\theta)$ or $\phi(\theta)$ designate $\phi_n(1,\theta)$ or $\phi(1,\theta)$.)

Following I. Lotz*, let us write

$$f(\theta) = \sum_{k=0}^{\infty} b_k \cos k\theta \tag{3}$$

Then, substituting in equation (1), we get

$$4 \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} C_m^{(n)} b_k \sin m\theta \cos k\theta - \lambda_n \sum_{m=1}^{\infty} mC_m^{(n)} \sin m\theta = 0$$

Using the relation

$$\sin m\theta \cos k\theta = \frac{1}{2} \left[\sin(m+k)\theta + \sin(m-k)\theta\right]$$

we get

$$2 \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} C_m^{(n)} b_k \left[\sin(m+k)\theta + \sin(m-k)\theta\right] - \lambda_n \sum_{m=1}^{\infty} mC_m^{(n)} \sin m\theta = 0 \tag{4}$$

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* Reference 2.

By equating coefficients of sin θ, sin 2θ, ... , sin nθ, ... , we get the following infinite set of linear equations for the coefficients $C_i^{(n)}$:

$$\left.\begin{aligned}
&C_1^{(n)}\left[-\lambda_n+2(2b_0-b_2)\right] + 2\left[C_2^{(n)}(b_1-b_3) + C_3^{(n)}(b_2-b_4) + C_4^{(n)}(b_3-b_5) + \dots\right] = 0\\
&2C_1^{(n)}(b_1-b_3) + C_2^{(n)}\left[-2\lambda_n+2(2b_0-b_4)\right] + 2\left[C_3^{(n)}(b_1-b_5) + C_4^{(n)}(b_2-b_6) + \dots\right] = 0\\
&2\left[C_1^{(n)}(b_2-b_4) + C_2^{(n)}(b_1-b_5)\right] + C_3^{(n)}\left[-3\lambda_n+2(2b_0-b_6)\right] + 2\left[C_4^{(n)}(b_1-b_7) + \dots\right] = 0\\
&2\left[C_1^{(n)}(b_3-b_5) + C_2^{(n)}(b_2-b_6) + C_3^{(n)}(b_1-b_7)\right] + C_4^{(n)}\left[-4\lambda_n+2(2b_0-b_8)\right] + \dots = 0\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}\right\} (5$$

By dividing through equations (5) by 2, and letting

$$\frac{1}{2}\lambda_n = \ell_n$$

we get

$$\left.\begin{aligned}
&C_1^{(n)}(2b_0-b_2) + C_2^{(n)}(b_1-b_3) + C_3^{(n)}(b_2-b_4) + C_4^{(n)}(b_3-b_5) + \dots = C_1^{(n)}\ell_n\\
&C_1^{(n)}(b_1-b_3)/2 + C_2^{(n)}(2b_0-b_4)/2 + C_3^{(n)}(b_1-b_5)/2 + C_4^{(n)}(b_2-b_6)/2 + \dots = C_2^{(n)}\ell_n\\
&C_1^{(n)}(b_2-b_4)/3 + C_2^{(n)}(b_1-b_5)/3 + C_3^{(n)}(2b_0-b_6)/3 + C_4^{(n)}(b_1-b_7)/3 + \dots = C_3^{(n)}\ell_n\\
&C_1^{(n)}(b_3-b_5)/4 + C_2^{(n)}(b_2-b_6)/4 + C_3^{(n)}(b_1-b_7)/4 + C_4^{(n)}(2b_0-b_8)/4 + \dots = C_4^{(n)}\ell_n\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}\right\} (6$$

Equations (6) are identical to

$$\ell_n C_i^{(n)} = \sum_{j=1}^{\infty} k_{ij} C_j^{(n)} \tag{7}$$

where the matrix of k_{ij}'s is

$$\left[k_{ij}\right] = \begin{bmatrix} (2b_0-b_2), & (b_1-b_3), & (b_2-b_4), & (b_3-b_5), & \cdots \\ (b_1-b_3)/2, & (2b_0-b_4)/2, & (b_1-b_5)/2, & (b_2-b_6)/2, & \cdots \\ (b_2-b_4)/3, & (b_1-b_5)/3, & (2b_0-b_6)/3, & (b_1-b_7)/3, & \cdots \\ (b_3-b_5)/4, & (b_2-b_6)/4, & (b_1-b_7)/4, & (2b_0-b_8)/4, & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

For a surface such as an airplane wing planform, which is symmetrical with respect to the airplane center line, the expansion for $f(\theta)$, equation (3), contains only even values of k. Hence all b_n's with odd subscripts are zero for this case. We see that if we combine the odd rows of equation (6) into one set of equations and the even rows into another, the first set will contain only $C_i^{(n)}$'s with odd subscripts (i.e., $C_1^{(n)}$, $C_3^{(n)}$, ...), while the second will contain only those with even subscripts (i.e., $C_2^{(n)}$, $C_4^{(n)}$, ...).

These two sets are written as follows:

$$\left.\begin{array}{l}
c_1^{(n)}(2b_0-b_2) + c_3^{(n)}(b_2-b_4) + c_5^{(n)}(b_4-b_6) + \ldots = c_1^{(n)}\ell_n \\
c_1^{(n)}(b_2-b_4)/3 + c_3^{(n)}(2b_0-b_6)/3 + c_5^{(n)}(b_2-b_8) + \ldots = c_3^{(n)}\ell_n \\
c_1^{(n)}(b_4-b_6)/5 + c_3^{(n)}(b_2-b_8)/5 + c_5^{(n)}(2b_0-b_{10}) + \ldots = c_5^{(n)}\ell_n \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}\right\} \text{The "Odd Set"}$$

and,

$$\left.\begin{array}{l}
c_2^{(n)}(2b_0-b_4)/2 + c_4^{(n)}(b_2-b_6)/2 + c_6^{(n)}(b_4-b_8)/2 + \ldots = c_2^{(n)}\ell_n \\
c_2^{(n)}(b_2-b_6)/4 + c_4^{(n)}(2b_0-b_8)/4 + c_6^{(n)}(b_2-b_{10})/4 + \ldots = c_4^{(n)}\ell_n \\
c_2^{(n)}(b_4-b_8)/6 + c_4^{(n)}(b_2-b_{10})/6 + c_6^{(n)}(2b_0-b_{12})/6 + \ldots = c_6^{(n)}\ell_n \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}\right\} \text{The "Even Set"}$$

The odd $c_i^{(n)}$'s are the Fourier constants of the eigenfunctions that are symmetrical about the wing centerline; the even are those of the eigenfunctions that are antisymmetrical.

NUMERICAL CALCULATION BY THE MATRIX METHOD

These two sets of linear equations can each be solved for the ℓ_n's and the $C_i^{(n)}$'s by matrix methods, provided that the convergence is such that it is satisfactory to limit them to a convenient finite number of terms. For the purposes of this investigation five equations of five terms each are used in each set. This allows the solution for ten eigenvalues and ten eigenfunctions, each of the latter expressed as a Fourier series of five terms. We shall refer to the set with odd-numbered subscripts as the "odd set," and to the other as the "even set." While the following discussion of the method of solution applies equally to both sets, we shall only concern ourselves with the odd set for the sake of clarity.

Theory of the Matrix Method.

The odd set, then, comprises five linear equations of the form of equations (7). Let the right hand member of equations (7) be represented by N_i:

$$\sum_{j=1}^{9} k_{ij} C_j^{(n)} = N_i \tag{8}$$

Then equations (7) become

$$N_i = \ell_n C_i^{(n)} \tag{9}$$

The solution which is now going to be obtained* will be designated by $n = 1$. Begin by assuming an arbitrary set of values for the $C_i^{(1)}$'s, say ${}_0C_1^{(1)}$, ${}_0C_3^{(1)}$, ... , ${}_0C_9^{(1)}$, in which for convenience let ${}_0C_1^{(1)} = 1$, and substitute these into the five equations (8). This yields five numerical values, say ${}_1N_1$, ${}_1N_3$, ... , ${}_1N_9$. Now, had the assumed values ${}_0C_1^{(1)}$, ${}_0C_3^{(1)}$, ... , ${}_0C_9^{(1)}$ been an exact solution of equations (7), then, since ℓ_1 is the same value for all five equations, it follows from equation (9) that we would find

$$ {}_1N_1 : {}_1N_3 : \dots : {}_1N_9 = 1 : {}_0C_3^{(1)} : \dots : {}_0C_9^{(1)} \tag{10a} $$

and

$$ \frac{{}_1N_1}{1} = \frac{{}_1N_3}{{}_0C_3^{(1)}} = \dots = \frac{{}_1N_9}{{}_0C_9^{(1)}} = \ell_1 \tag{10b} $$

If the assumed values are not correct, we will not find these equations to be satisfied. We then assume as our second approximations, ${}_1C_1^{(1)}$, ${}_1C_3^{(1)}$, ... , ${}_1C_9^{(1)}$, the values $\frac{{}_1N_1}{{}_1N_1}$, $\frac{{}_1N_3}{{}_1N_1}$, ... , $\frac{{}_1N_9}{{}_1N_1}$, and repeat the calculation in equations (8) of five new values, say ${}_2N_1$, ${}_2N_3$, ... , ${}_2N_9$.

It can be shown that this process is convergent, so that $\frac{{}_\nu N_1}{{}_\nu N_1}$, $\frac{{}_\nu N_3}{{}_\nu N_1}$, ... , $\frac{{}_\nu N_9}{{}_\nu N_1}$ converge to definite values $C_1^{(1)}$, $C_3^{(1)}$, ... ,

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* The demonstration given here is essentially that of reference 3, pages 196-201.

$C_9^{(1)}$, and $\frac{{}_{\nu}N_1}{{}_{\nu-1}C_1^{(1)}}$, $\frac{{}_{\nu}N_3}{{}_{\nu-1}C_3^{(1)}}$, ... , $\frac{{}_{\nu}N_9}{{}_{\nu-1}C_9^{(1)}}$ converge to a fixed value ℓ_1. We have thus found values of five unknowns, namely, the four ratios $\frac{C_3^{(1)}}{C_1^{(1)}}$, $\frac{C_5^{(1)}}{C_1^{(1)}}$, $\frac{C_7^{(1)}}{C_1^{(1)}}$, $\frac{C_9^{(1)}}{C_1^{(1)}}$, and the number ℓ_1.

It is of interest to note that this discussion is directly applicable to the vibration of a mechanical system having five degrees of freedom, in which case $C_1^{(1)}$, $C_3^{(1)}$, ... , $C_9^{(1)}$ are the coefficients of the principal modes of vibration and $\ell_1 = \omega_1^2$, the square of a characteristic frequency. It can be shown that ω_1^2 is always the largest root of the frequency equation and that $\frac{N_1}{N_1}$, $\frac{N_3}{N_1}$, ... , $\frac{N_9}{N_1}$ are the coefficients of the principal mode corresponding to the highest frequency.

The actual calculation of the values of ℓ_1 and the $C^{(1)}$'s can be represented symbolically with the aid of matrix algebra. The matrix $\left[k_{ij}\right]$ of equation (7) is classified as a square matrix because it has an equal number of rows and columns. Similarly, the following are column matrices:

$$\left\{C_j^{(n)}\right\} = \left\{\begin{matrix} C_1^{(n)} \\ C_3^{(n)} \\ \cdot \\ \cdot \\ \cdot \\ C_9^{(n)} \end{matrix}\right\} \quad ; \quad \left\{N_i\right\} = \left\{\begin{matrix} N_1 \\ N_3 \\ \cdot \\ \cdot \\ \cdot \\ N_9 \end{matrix}\right\} .$$

Then equation (8) can be written in matrix notation as

$$\left[k_{ij} \right] \left\{ c_j^{(n)} \right\} = \left\{ N_i \right\} \tag{11}$$

where, by equation (9),

$$\left\{ N_i \right\} = \ell_n \left\{ c_i^{(n)} \right\} .$$

The operation indicated by the right hand side of equation (7) is equivalent to multiplying the elements of each row of the square matrix $\left[k_{ij} \right]$ by the corresponding element of the column matrix $\left\{ c_j^{(n)} \right\}$ and summing the products. This process is identical with that indicated by equation (11) and is defined as a multiplication of matrices.

Therefore the method of solution is as follows: We assume arbitrary values ${}_0\left\{ c_j^{(1)} \right\}$ which yield a first approximation ${}_1\left\{ N_1 \right\}$. The second approximation ${}_1\left\{ c_j^{(1)} \right\}$ is obtained by dividing each element of ${}_1\left\{ N_1 \right\}$ by the first element ${}_1N_1$. This process is then repeated until ${}_\nu\left\{ c_j^{(1)} \right\}$ no longer changes. An advantage of this method is that incidental numerical errors have no effect on the final result since ${}_\nu\left\{ c_j^{(n)} \right\}$ is arbitrary. By the same token, after one becomes a little experienced he can accelerate the process by observing the trend of the first few approximations and anticipating the approximate answer. In addition, one is not forced to choose ${}_0\left\{ c_j^{(n)} \right\}$ entirely arbitrarily, since the underlying theory gives an indication as to the order of magnitude of $\left\{c_j^{(n)}\right\}$ by the following reasoning.

The Case of Elliptic Planform.

For the case of an elliptic planform,

$$\frac{\mathcal{C}m}{\mathcal{C}_s m_s} = \sin\theta \; ; \quad \text{hence} \quad f = 1$$

$$\text{and} \quad b_0 = 1, \; b_k = 0 \text{ for } k = 2, 3, \ldots.$$

Equations (7) become in this case

$$\ell_n c_i^{(n)} = \frac{2}{i} c_i^{(n)}.$$

The condition for solubility of this set is

$$\begin{vmatrix} (\ell_n - \frac{2}{1}) & 0 & 0 & 0 & \cdots \\ 0 & (\ell_n - \frac{2}{2}) & 0 & 0 & \cdots \\ 0 & 0 & (\ell_n - \frac{2}{3}) & 0 & \cdots \\ 0 & 0 & 0 & (\ell_n - \frac{2}{4}) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 .$$

This yields the values $\ell_n = 2/n$. The equations above are then

$$\frac{2}{n} c_i^{(n)} = \frac{2}{i} c_i^{(n)} ,$$

for which the solutions are given by

$$C_n^{(n)} = 1,\ C_m^{(n)} = 0 \quad \text{for } m \neq n.$$

This furnishes the indication mentioned above as to approximate values to be assumed for ${}_0\left\{C_j^{(1)}\right\}$. For, since an elliptic wing has $C_1^{(1)} = 1$, $C_3^{(1)} = C_5^{(1)} = C_7^{(1)} = C_9^{(1)} = 0$, we would expect most practical wings to have $C_i^{(1)}$'s of the same order; hence the first approximation should be ${}_0\left\{C_j^{(1)}\right\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$.

Calculation of "Harmonics."

Having found the "fundamental solution," ℓ_1; $C_1^{(1)}$, $C_3^{(1)}$, $C_5^{(1)}$, $C_7^{(1)}$, $C_9^{(1)}$, corresponding to the largest ℓ_n, we can now proceed to solve for the "harmonics," namely,

$$\ell_3 ;\ C_1^{(3)},\ C_3^{(3)},\ \dots ,\ C_9^{(3)} \qquad \text{(3rd set)}$$

$$\ell_5 ;\ C_1^{(5)},\ C_3^{(5)},\ \dots ,\ C_9^{(5)} \qquad \text{(5th set)}$$

$$\ell_7 ;\ C_1^{(7)},\ C_3^{(7)},\ \dots ,\ C_9^{(7)} \qquad \text{(7th set)}$$

and

$$\ell_9 ;\ C_1^{(9)},\ C_3^{(9)},\ \dots ,\ C_9^{(9)} \qquad \text{(9th set)}$$

for which successive ℓ_n's diminish in size. Until this point we have only obtained the first set as a solution of the original five equations, and we cannot obtain by the matrix method any other set from these same five equations as they stand. However, we can eliminate the fundamental solution by use of a property possessed by the $C_i^{(n)}$'s known as "orthogonality." The orthogonality is obtained as follows: Since

$$\phi_i(\theta) = \sum_{n=1}^{\infty} C_n^{(i)} \sin n\theta$$

and by equations (1)

$$4f(\theta)\,\phi_j(\theta) = -\lambda_j \left(\frac{\partial \phi_j}{\partial r}\right)_{r=1}$$

$$= +\lambda_j \sum_{m=1}^{\infty} mC_m^{(j)} \sin m\theta \quad ,$$

we can write

$$8\int_0^{\pi} \phi_i\,\phi_j\, f d\theta = \lambda_j \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mC_m^{(j)} C_n^{(i)} \int_0^{\pi} \sin m\theta \sin n\theta \, d\theta$$

$$+ \lambda_i \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} mC_m^{(i)} C_n^{(j)} \int_0^{\pi} \sin m\theta \sin n\theta \, d\theta$$

$$= \frac{\pi}{2}(\lambda_j + \lambda_i) \sum_{m=1}^{\infty} mC_m^{(i)} C_m^{(j)}$$

or, again,

$$0 = \frac{\pi}{2}(\lambda_j - \lambda_i) \sum_{m=1}^{\infty} mC_m^{(i)} C_m^{(j)} \quad .$$

Hence $$\sum_{m=1}^{\infty} mC_m^{(i)}C_m^{(j)} = 0 \quad \text{for } i \neq j \tag{12}$$

If $i = j$,

$$\int_0^{\pi} (\phi_i)^2 f d\theta = \frac{\pi}{8}\lambda_i \sum_{m=1}^{\infty} m(C_m^{(i)})^2$$

$$= \frac{\pi}{8}\lambda_i K_i$$

$$= \frac{\pi}{4}\ell_i K_i$$

$$= M_i, \text{ say,}$$

where

$$\sum_{m=1}^{\infty} m(C_m^{(i)})^2 = K_i \tag{13}$$

We make use of the orthogonality property, equation (12), to eliminate the fundamental solution (equation (13) will be of use later). Writing equation (12) explicitly:

$$1C_1^{(1)}C_1^{(n)} + 3C_3^{(1)}C_3^{(n)} + \dots + 9C_9^{(1)}C_9^{(n)} = 0, \text{ for } n \neq 1$$

or,

$$C_1^{(n)} = -\frac{1}{C_1^{(1)}}\left(3C_3^{(1)}C_3^{(n)} + 5C_5^{(1)}C_5^{(n)} + 7C_7^{(1)}C_7^{(n)} + 9C_9^{(1)}C_9^{(n)}\right),$$

for $n \neq 1$,

and since $C_1^{(1)} = 1$, we get

$$C_1^{(n)} = -3C_3^{(1)}C_3^{(n)} - 5C_5^{(1)}C_5^{(n)} - 7C_7^{(1)}C_7^{(n)} - 9C_9^{(1)}C_9^{(n)}, \text{ for } n \neq 1 \tag{14}$$

Writing equation (14) for $n = 3$ and substituting for $C_1^{(3)}$ in the five equations (7), we are left with only four unknowns, $C_5^{(3)}/C_3^{(3)}$, $C_7^{(3)}/C_3^{(3)}$, $C_9^{(3)}/C_3^{(3)}$, and ℓ_3, so that there is one superfluous equation. Hence let us drop one equation, say the first one ($i = 1$). In the light of our mechanical analogy, the remaining four equations determine the oscillations of a mechanical system with four degrees of freedom whose frequencies and normal modes are identical with four frequencies and normal modes of the original five-degree system. Using again the matrix method used for the calculation of the first set, we find $C^{(3)}(= 1)$, $C_5^{(3)}$, $C_7^{(3)}$, $C_9^{(3)}$, and ℓ_3. Finally we calculate $C_1^{(3)}$ from equation (14), which completes the solution of the third set.

To solve for the $C^{(5)}$'s and ℓ_5, we again make use of the orthogonality property. Writing

$$C_1^{(3)}C_1^{(5)} + 3C_3^{(3)}C_3^{(5)} + \dots + 9C_9^{(3)}C_9^{(5)} = 0 \tag{15}$$

and in addition writing equation (14) for $n = 5$:

$$C_1^{(5)} = -3C_3^{(1)}C_3^{(5)} - 5C_5^{(1)}C_5^{(5)} - 7C_7^{(1)}C_7^{(5)} - 9C_9^{(1)}C_9^{(5)} \tag{16}$$

i.e., $$C_1^{(5)} = C_1^{(5)}\left(C_3^{(5)}, C_5^{(5)}, C_7^{(5)}, C_9^{(5)}\right),$$

we can substitute $C_1^{(5)}$ from equation (16) into equation (15) and solve for $C_3^{(5)}$:

$$C_3^{(5)} = C_3^{(5)}\left(C_5^{(5)}, C_7^{(5)}, C_9^{(5)}\right) \tag{17}$$

We now utilize equation (17) to reduce the set of four equations and four unknowns to three, dropping the first equation. Employing the matrix method to solve this matrix of three rows and columns, we get $C_5^{(5)}(=1)$, $C_7^{(5)}$, $C_9^{(5)}$, and $\mathcal{L}_5$. Finally, we solve for $C_3^{(5)}$ from equation (17) and for $C_1^{(5)}$ from equation (16).

Similarly, for the seventh set,

$$C_1^{(5)}C_1^{(7)} + 3C_3^{(5)}C_3^{(7)} + \ldots + 9C_9^{(5)}C_9^{(7)} = 0 \tag{18}$$

and writing equation (14) for $n = 7$,

$$C_1^{(7)} = -3C_3^{(1)}C_3^{(7)} - 5C_5^{(1)}C_5^{(7)} - 7C_7^{(1)}C_7^{(7)} - 9C_9^{(1)}C_9^{(7)} \tag{19}$$

i.e., $$C_1^{(7)} = C_1^{(7)}\left(C_3^{(7)}, C_5^{(7)}, C_7^{(7)}, C_9^{(7)}\right) ,$$

and equation (17) for $n = 7$, giving

$$C_3^{(7)} = C_3^{(7)}\left(C_5^{(7)}, C_7^{(7)}, C_9^{(7)}\right) \tag{20}$$

we can substitute equations (19) and (20) into (18) and get

$$C_5^{(7)} = C_5^{(7)}\left(C_7^{(7)}, C_9^{(7)}\right) \tag{21}$$

and use (21) to reduce the former matrix to one of two rows and columns, which we shall represent as

$$\begin{bmatrix} a & d \\ c & b \end{bmatrix} .$$

We are now left with a matrix equation which is equivalent to the equations

$$\left.\begin{aligned} (a-\ell)C_7^{(7)} + \quad dC_9^{(7)} &= 0 \\ cC_7^{(7)} + (b-\ell)C_9^{(7)} &= 0 \end{aligned}\right\} \tag{22}$$

It would be a simple matter to solve for ℓ_7 and ℓ_9 since the "compatibility condition," that is, the condition that a solution of equations (22) exists, is that the determinant

$$\begin{vmatrix} (a-\ell), & d \\ c, & (b-\ell) \end{vmatrix} = 0 \tag{23}$$

which is equivalent to the quadratic equation

$$(a-\ell)(b-\ell) - cd = 0 .$$

This equation gives two roots: ℓ_7 and ℓ_9 $(\ell_7 > \ell_9)$. Substituting ℓ_7 into either of equations (22), we get the ratio $C_9^{(7)}/C_7^{(7)}$. But since $C_7^{(7)} = 1$, we now know $C_7^{(7)}$ and $C_9^{(7)}$, and can obtain $C_5^{(7)}$, $C_3^{(7)}$, and $C_1^{(7)}$ from equations (21), (20), and (19), respectively. Repeating for ℓ_9, writing equations (21), (20), and (19) for $n = 9$, we get numerical values of $C_1^{(9)}$, $C_3^{(9)}$, $C_5^{(9)}$, $C_7^{(9)}$, and $C_9^{(9)}$.

In actual practice it was found that solving equation (23) involves taking the square root of the difference of two nearly equal numbers which introduced considerable inaccuracies. So instead of the procedure described above for solving equation (22), the following alternative method was found to be more accurate. By expanding (23) we get

$$\ell^2 - \ell(a + b) + ab - cd = 0 = (\ell - \ell_7)(\ell - \ell_9)$$

$$= \ell^2 - \ell(\ell_7 + \ell_9) + \ell_7\ell_9 .$$

Equating coefficients we get

$$a + b = \ell_7 + \ell_9 \qquad (24)$$

$$\text{and} \qquad ab - cd = \ell_7\ell_9$$

It should be noted that (24) is a general result for a square matrix with any number of rows, viz., the sum of the diagonals equals the sum of the ℓ's. We shall soon make use of (24), but first let us solve the two-row square matrix by the matrix method for $C_7^{(7)}(= 1)$, $C_9^{(7)}$, and ℓ_7. Then we obtain $C_5^{(7)}$, $C_3^{(7)}$, and $C_1^{(7)}$ from equations (21), (20), and (19), respectively. Now we can use (24):

$$\ell_7 + \ell_9 = a + b$$

to solve for ℓ_9. Substituting for ℓ_9 into either of equations (22), we get $C_7^{(9)}/C_9^{(9)}$, where $C_9^{(9)} = 1$. Then, writing equations (21), (20), and (19) for $n = 9$, we solve for $C_5^{(9)}$, $C_3^{(9)}$, and $C_1^{(9)}$. This completes the solution for all five odd sets comprising five ℓ_n's and twenty-five $C_m^{(n)}$'s.

Numerical Checks.

There are two numerical checks which can be made for each set. The first is that every set of $C_m^{(n)}$'s, when substituted into the original five-row square matrix should satisfy the matrix equation, equation (11). This check should be applied immediately after finding each successive set. The second check is that the sum of the diagonals of the original matrix, designated

$$a + b + c + d + e \quad ,$$

where these diagonals occupy the positions as shown

$$\begin{bmatrix} a & & & & \\ & b & & & \\ & & c & & \\ & & & d & \\ & & & & e \end{bmatrix} ,$$

must equal the sum:

$$\ell_1 + \ell_3 + \ell_5 + \ell_7 + \ell_9 \quad .$$

The "Even" Set.

Identically the same procedure as explained heretofore is to be followed for solving the even set of five equations. This will yield ℓ_2, ℓ_4, ℓ_6, ℓ_8, ℓ_{10}, and twenty-five $C_m^{(n)}$'s: $C_m^{(2)}$, $C_m^{(4)}$, $C_m^{(6)}$, $C_m^{(8)}$, $C_m^{(10)}$.

Additional Calculations.

φ_n can be calculated from equation (2):

$$\phi_n(\theta) = \sum_{m=1}^{\infty} C_m^{(n)} \sin m\theta$$

For example,

$$\phi_7(\theta) = C_1^{(7)} \sin \theta + C_3^{(7)} \sin 3\theta + C_5^{(7)} \sin 5\theta + C_7^{(7)} \sin 7\theta + C_9^{(7)} \sin 9\theta$$

Hence ϕ_n can be evaluated numerically for any θ. We are now also able to calculate K_n from equation (13):

$$K_n = \sum_{m=1}^{\infty} m(C_m^{(n)})^2 \qquad (13)$$

The following quantities have now been numerically determined:

	Ten	ℓ_n's	(odd and even sets),
	fifty	$C_m^{(n)}$'s	(" " " "),
	ten	ϕ_n's	(" " " "),
and	ten	K_n's	(" " " ").

RANGE OF THE PRESENT CALCULATIONS

Since these quantities are all independent of aspect ratio, they can be calculated for a wide class of planforms by varying the single parameter of taper ratio. Calculations have been carried out for trapezoidal wings* with rounded tips for four taper ratios, c_s/c_t, namely 1:1, 2:1, 3:1, and 4:1. Actually it was unnecessary, in the present calculation, to consider the rounding of the tips when the taper ratio was 2:1 or greater. The chord distribution $cm/c_s m_s$ of the tip of the rectangular wing was arbitrarily chosen so that rounding began at

$$\frac{y}{b/2} = \cos\theta = \frac{5}{6}$$

and the shape was semi-circular for an aspect ratio of 6. For any other aspect ratio the shape given by the chord function can be shown to be an ellipse.

Values of ℓ_n, $C_m^{(n)}$, and $1/K_n$ are tabulated and plotted in the present paper as a function of taper ratio, c_s/c_t, and odd and even $\varphi_n(\theta)$ are tabulated as a function of θ for each taper ratio; a sample of an odd φ is plotted.

---

* i.e., a linear distribution of $cm/c_s m_s$ as a function of the span-wise coordinate y.

DISCUSSION OF RESULTS

The cases of 4:1 and 2:1 taper ratio were computed before those of 1:1 and 3:1. The first two were computed with one less significant figure than the last two. It is believed that the number of significant figures is three and four for the two groups, respectively. The reason for adding an additional figure was that in checking some sets of $C_m^{(n)}$'s in the original matrix for T = 2:1 the check was poor. While this may have been due partly to taking the square root of a difference of two nearly equal large numbers (mentioned in the discussion above), it was felt advisable to carry one more significant figure. Also the alternative procedure was introduced eliminating the square root operation as such. The accuracy from then on was very satisfactory in checking each set of $C_m^{(n)}$'s in the original matrix and it is recommended that any further calculations for other taper ratios be made following the latter procedure and carrying the same number of significant figures.

With regard to the curves, there are a few points which look doubtful, such as the value of ℓ_6 for T = 4 in Fig. 8. Lack of time has prevented investigating possible sources of error. Also, in one instance, for T = 2, there was found a slight disagreement in $\ell_2 + \ell_4 + \ell_6 + \ell_8 + \ell_{10}$ as compared with the sum of the matrix diagonals, the value being 2.1570 and 2.1613, respectively; in all other comparisons these two sums differed only in the last place, if at all.

The reciprocal of K_i was plotted, since this form is to be used in calculating spanwise characteristics. The coefficients $C_m^{(n)}$ are cross plotted against T, the curves for each value of n being grouped on one page. The scales are somewhat consistent with the accuracy to which the various $C_m^{(n)}$'s must be known for further calculations. Where the scale prohibits, $C_m^{(n)}$ for m = n, which always equals 1.0, is omitted. The curves for any one value of n are not necessarily supposed to comprise a single family. Instead, it is observed that there exists a resemblance between such curves as $C_3^{(1)}$, $C_5^{(3)}$, $C_7^{(5)}$, and $C_9^{(7)}$ and again between $C_5^{(1)}$, $C_7^{(3)}$, and $C_9^{(5)}$, and similarly.

It seems probable from the character of the curves plotted herein that interpolation for taper ratios intermediate between those considered may be carried out with a fair expectation of accuracy. It is hoped that the work may be extended to slightly higher tapers (say 6:1) in the near future, and that the accuracy of the graphical interpolations drawn in Figures 2 to 6 and 9 to 13, inclusive, may be checked by calculations for some intermediate taper ratios. In the future it is suggested that better fairing of curves of $C_m^{(n)}$ vs. T may be found by use of some interpolation formula, or by fitting a curve such as a cubic to the four computed points. Lack of time prohibited investigations in this direction in the present work.

CONCLUSION

Numerical solutions by matrix methods have been found for a certain homogeneous boundary value problem of the third kind which occurs in the theory of three-dimensional airfoils. This constitutes a part of a theory of larger scope which is to be published shortly by Professors Theodore von Karman and W. R. Sears on the calculation of spanwise lift distribution.

REFERENCES

1. Theodore von Karman and W. R. Sears, A paper to be published shortly on a new method of calculating spanwise lift distribution.

2. I. Lotz, Berechnung der Auftriebsverteilung beliebig geformeter Flügel. Z.F.M., vol. 22, no. 7, April 14, 1931, S. 139-195.

3. Theodore von Karman and Maurice A. Biot, Mathematical Methods in Engineering, published by McGraw-Hill Book Co.

4. ANC-1 (1), Spanwise Air-Load Distribution.

RESULTS TABULATED

$c_m^{(n)}$ vs. T, Odd Set *

T	1	2	3	4
$c_3^{(1)}$	- .07769	-.0255	+.00988	+.0398
$c_5^{(1)}$	- .01807	-.0202	-.02891	-.0252
$c_7^{(1)}$	+ .01418	+.0005	+.00339	-.0029
$c_9^{(1)}$	- .00072	-.0018	-.00379	-.0028
$c_1^{(3)}$	+ .18198	+.0542	-.04168	-.1244
$c_5^{(3)}$	- .45508	-.2248	-.09933	-.0217
$c_7^{(3)}$	+ .10202	-.0588	-.09788	-.1008
$c_9^{(3)}$	+ .02428	+.0093	+.00004	-.0128
$c_1^{(5)}$	+ .40302	+.1253	+.14494	+.1183
$c_3^{(5)}$	+ .92264	+.3055	+.09621	+.0049
$c_7^{(5)}$	- .94996	-.4972	-.31252	-.1222
$c_9^{(5)}$	+ .51499	-.0495	-.12269	-.1825
$c_1^{(7)}$	+ .15560	+.0650	-.01037	-.0086
$c_3^{(7)}$	+ .64937	+.3661	+.27633	+.2230
$c_5^{(7)}$	+1.16137	+.5467	+.29002	+.0532
$c_9^{(7)}$	- .21948	-.9052	-.59709	-.3577
$c_1^{(9)}$	- .11764	+.0999	+.08235	+.0654
$c_3^{(9)}$	- .27929	+.3180	+.22744	+.1511
$c_5^{(9)}$	- .13294	+.6180	+.48541	+.3961
$c_7^{(9)}$	+ .47367	+.8470	+.64020	+.4066

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* Note: $c_n^{(n)} = 1$.

ℓ_n and $\frac{1}{K_n}$ vs. T, Odd Set

T	1	2	3	4
ℓ_1	1.73675	2.1464	2.45936	2.5739
ℓ_3	.44480	.6348	.75184	.8390
ℓ_5	.28055	.3695	.44787	.4920
ℓ_7	.18047	.2603	.31220	.3353
ℓ_9	.13141	.1437	.19522	.2353
$1/K_1$	.97928	.99602	.98984	.99226
$1/K_3$	.24115	.30482	.32070	.26802
$1/K_5$	.06090	.14188	.17042	.18457
$1/K_7$	.06466	.06144	.09209	.12026
$1/K_9$	.09169	.06156	.07659	.09079

$C_m^{(n)}$ vs. T, Even Set *

T	1	2	3	4
$C_4^{(2)}$	-.28483	-.1307	-.08318	-.0139
$C_6^{(2)}$	.02383	-.0339	-.04049	-.0613
$C_8^{(2)}$	.03223	-.0001	.00052	-.0097
$C_{10}^{(2)}$	-.01295	-.0018	-.00882	-.0033
$C_2^{(4)}$	.60327	.2237	.14099	-.0642
$C_6^{(4)}$	-.57590	-.3712	-.21920	-.4812
$C_8^{(4)}$	.12523	-.0531	-.12417	-.0975
$C_{10}^{(4)}$	.13242	.0163	.02261	.0188
$C_2^{(6)}$	.48326	.2213	.15367	.1939
$C_4^{(6)}$	.72676	.4584	.23208	.6974
$C_8^{(6)}$	-.80072	-.6964	-.38306	-.1878
$C_{10}^{(6)}$	.57927	-.0071	-.16350	-.1278
$C_2^{(8)}$	.17478	.2057	.05684	.0470
$C_4^{(8)}$	.70158	.5397	.36814	.2477
$C_6^{(8)}$	1.03162	.7348	.24938	.0441
$C_{10}^{(8)}$	-.34299	-1.1623	-.73948	-.4130
$C_2^{(10)}$	-.26515	.1978	.16777	.0832
$C_4^{(10)}$	-.44225	.4389	.31534	.2148
$C_6^{(10)}$	-.03942	.7244	.59911	.2286
$C_8^{(10)}$	.62668	.9277	.75184	.4828

* Note: $C_n^{(n)} = 1$.

ℓ_n and $\frac{1}{K_n}$ vs. T, Even Set

T	1	2	3	4
ℓ_2	.76547	1.0313	1.26378	1.3921
ℓ_4	.33098	.4730	.58019	.6450
ℓ_6	.21819	.3038	.36600	.5862
ℓ_8	.16872	.2262	.27372	.2948
ℓ_{10}	.10461	.1227	.16801	.2148
$1/K_2$	.42773	.48188	.49060	.49397
$1/K_4$	.14248	.20193	.22439	.18256
$1/K_6$	.05860	.09243	.12981	.11812
$1/K_8$	.05684	.03846	.06949	.10033
$1/K_{10}$	.07105	.04789	.05838	.08080

Summary of $\phi_n(\theta)$ vs. θ

(Note: For $\theta = 0$, all $\phi = 0$)

	θ^o =	10^o	20^o	30^o	40^o	50^o	60^o	70^o	80^o	90^o
T = 1	ϕ_1	.13360	.26603	.40690	.56774	.74095	.88393	.99327	1.03561	1.04472
	ϕ_3	.30317	.54566	.78816	1.03815	1.07360	.64004	-.19607	-1.01421	-1.35084
	ϕ_5	.91963	1.31101	1.58414	1.65159	.51033	-1.33965	-1.49896	.56557	1.94533
	ϕ_7	2.40051	2.40209	.58838	-.95728	-.36402	-.00500	.16632	-.00459	-.11292
	ϕ_9	1.18320	-.10854	-1.64142	-.73849	.81288	.42345	-.58498	-.12143	.55504
	ϕ_2	.19854	.39839	.60264	.83771	1.07397	1.15178	.93735	.50124	0
	ϕ_4	.60413	.87139	1.16530	1.43945	.91645	-.34980	-1.18387	-.92813	0
	ϕ_6	1.28035	1.42039	1.23964	.74551	-.78099	-.14720	.53667	1.92313	0
	ϕ_8	2.72671	2.15595	-.40410	-1.34459	-.53887	.70682	.09011	-.82041	0
	ϕ_{10}	1.19285	-.76779	-2.02131	-.13826	.96990	-.16993	-.32537	.52711	0
T = 2	ϕ_1	.1441	.3003	.4659	.6271	.7703	.8839	.9582	.9937	1.0030
	ϕ_3	.2913	.6253	.9348	1.0356	.7722	.1907	-.4644	-.9370	-1.1025
	ϕ_5	.4239	.9727	1.1663	.4927	-.6540	-1.1881	-.5402	.3316	1.2675
	ϕ_7	.6477	1.5204	1.0772	-.8130	-1.3597	.4489	1.4542	-.2436	-1.6596
	ϕ_9	2.4457	1.4627	-.7465	-.7059	.5077	.2848	-.5236	-.0694	.5529
	ϕ_2	.2267	.4847	.7545	.9684	1.0576	.9807	.7427	.3949	0
	ϕ_4	-.0593	.3624	1.0916	1.3493	.5971	-.7324	-1.5708	-1.2403	0
	ϕ_6	.5435	1.2237	1.1977	-.0485	-1.2573	-.8025	.7973	1.3266	0
	ϕ_8	.8937	2.0395	.7860	-1.6390	-.7226	1.5832	.2925	-1.7697	0
	ϕ_{10}	2.8754	1.1613	-1.1180	-.2359	.6565	-.2714	-.3377	.4840	0

Summary of $\varphi_n(\theta)$ vs. θ (cont.)

	θ° =	10°	20°	30°	40°	50°	60°	70°	80°	90°
T = 3	φ_1	.15589	.32427	.47951	.65791	.79373	.89398	.94616	.95650	.95403
	φ_3	.32473	.69101	.97839	.96957	.57845	-.03483	-.59694	-.93743	-1.04309
	φ_5	.42292	.91680	.94763	.14226	-.84898	-1.01112	-.20231	.80910	1.23856
	φ_7	.70114	1.16416	.01324	-.85136	-.91309	.60586	1.16480	-.40509	-1.59377
	φ_9	2.10144	1.11467	-.80877	-.54659	.60946	.20536	-.63027	-.02279	.70012
	φ_2	.24529	.52902	.80116	.98541	1.04297	.94612	.69250	.35121	0
	φ_4	.95028	.64471	.59139	1.12345	.27981	-1.35387	-1.23995	-.09076	0
	φ_6	.52949	1.11824	.80738	-.49417	1.14537	-.25804	.92316	.98561	0
	φ_8	.72680	1.20994	.14242	-1.15222	-.11842	1.23680	-.19914	-1.71428	0
	φ_{10}	2.50412	.85235	-1.13872	-.08624	.66462	-.34270	-.28301	.61790	0
T = 4	φ_1	.1688	.3498	.5315	.6888	.8073	.8853	.9397	.9427	.9351
	φ_3	.3543	.7373	.9901	.8927	.4298	-.1762	-.6775	-.9679	-1.0581
	φ_5	.4917	.9509	.8077	-.1415	-1.0079	-.8694	.0239	.7969	1.0531
	φ_7	.7328	.8854	.1030	-.8154	-.4765	.8125	.9949	-.5094	-1.5361
	φ_9	1.7725	.8048	-.8214	-.3630	.6829	.0657	-.7714	.0490	.9038
	φ_2	.2672	.5738	.8653	1.0372	1.0344	.8725	.6078	.3042	0
	φ_4	.1266	.4871	.8785	.7703	-.0391	-1.0223	-1.4159	-.9670	0
	φ_6	1.0698	1.6569	1.0451	-.3979	-1.1163	-.4879	.4117	.5430	0
	φ_8	.7916	.7955	-.2531	-.8155	.3007	1.0499	-.3763	-1.4964	0
	φ_{10}	1.8249	.2861	-1.0260	.2899	.7635	-.5618	-.4671	.5977	0

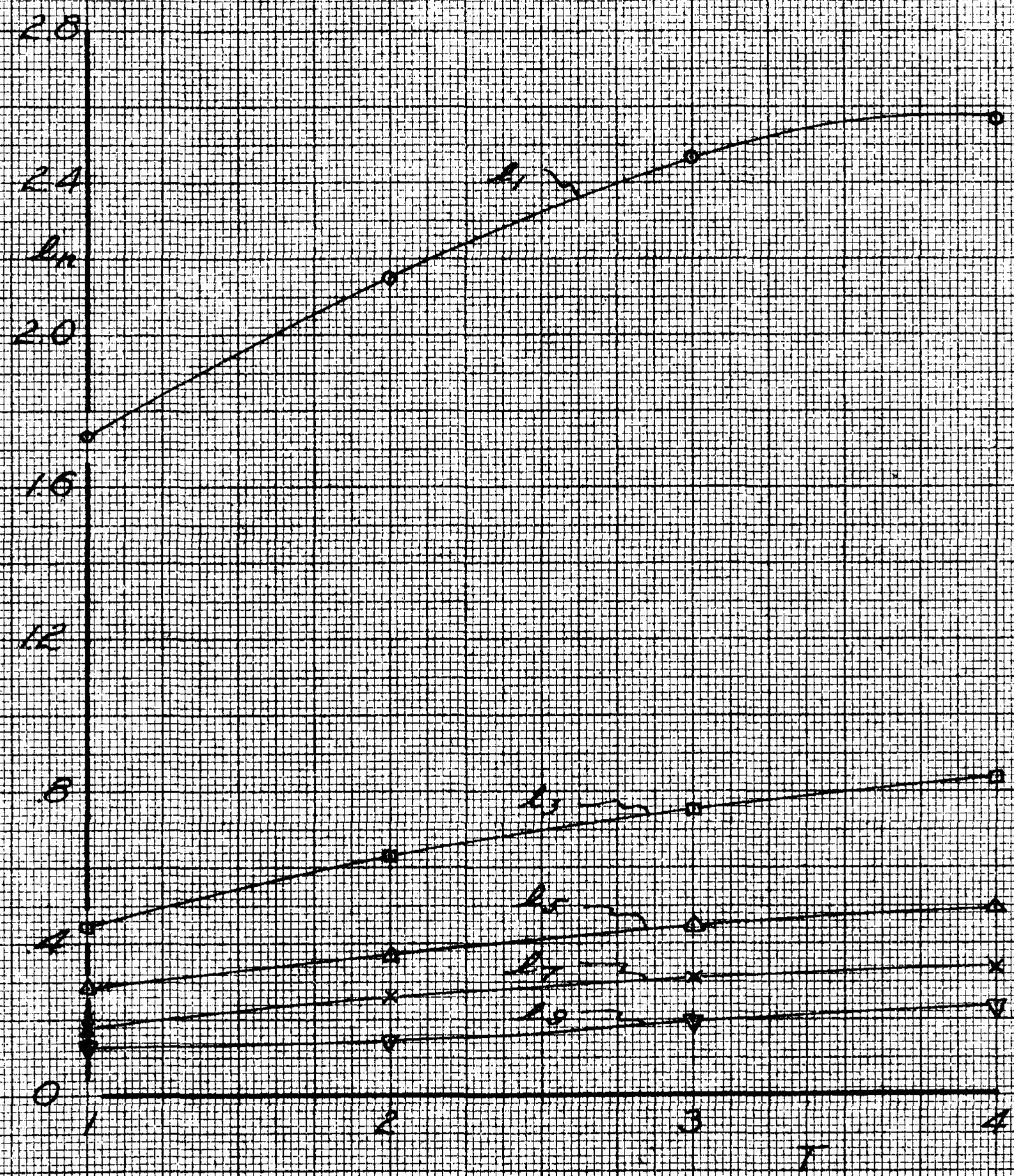

FIG. 1

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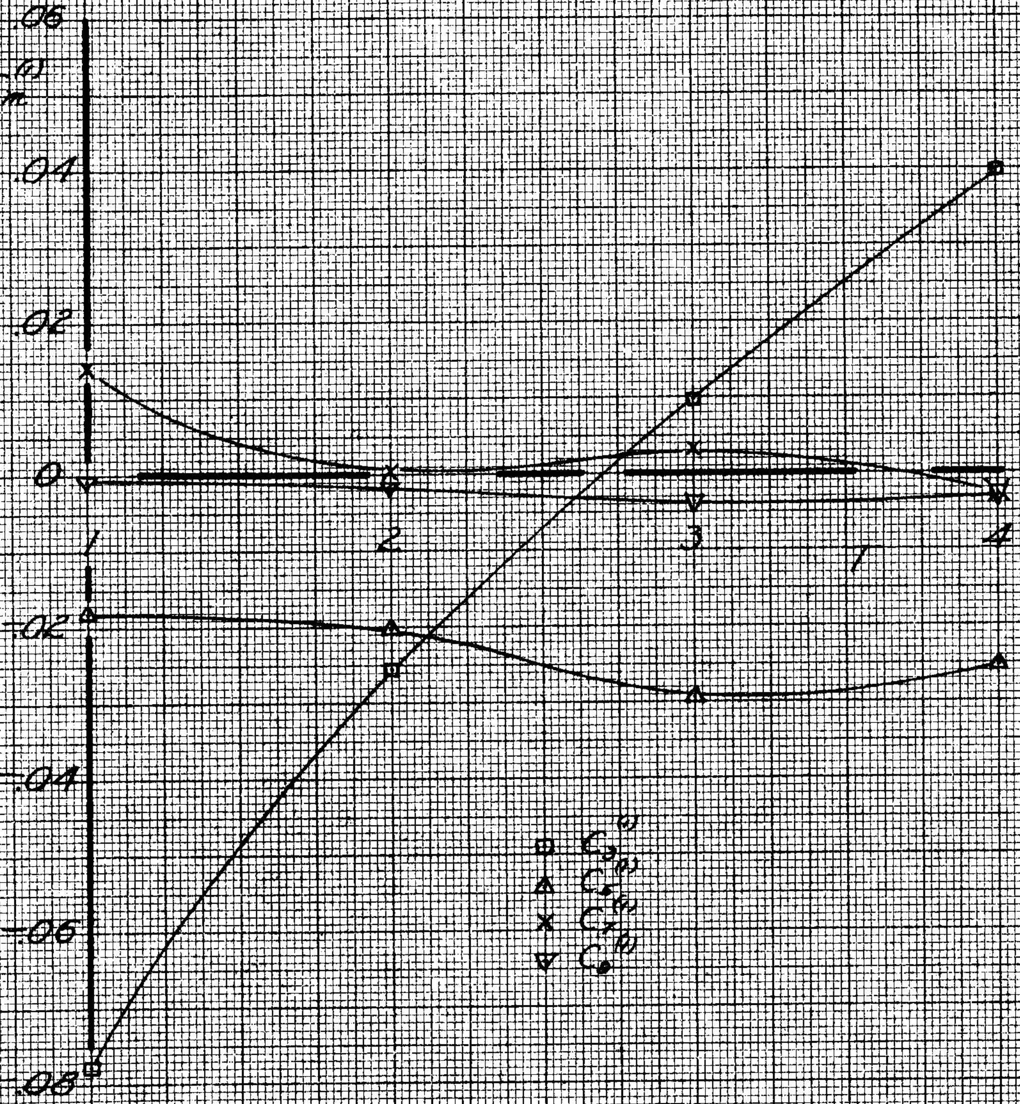

FIG. 2

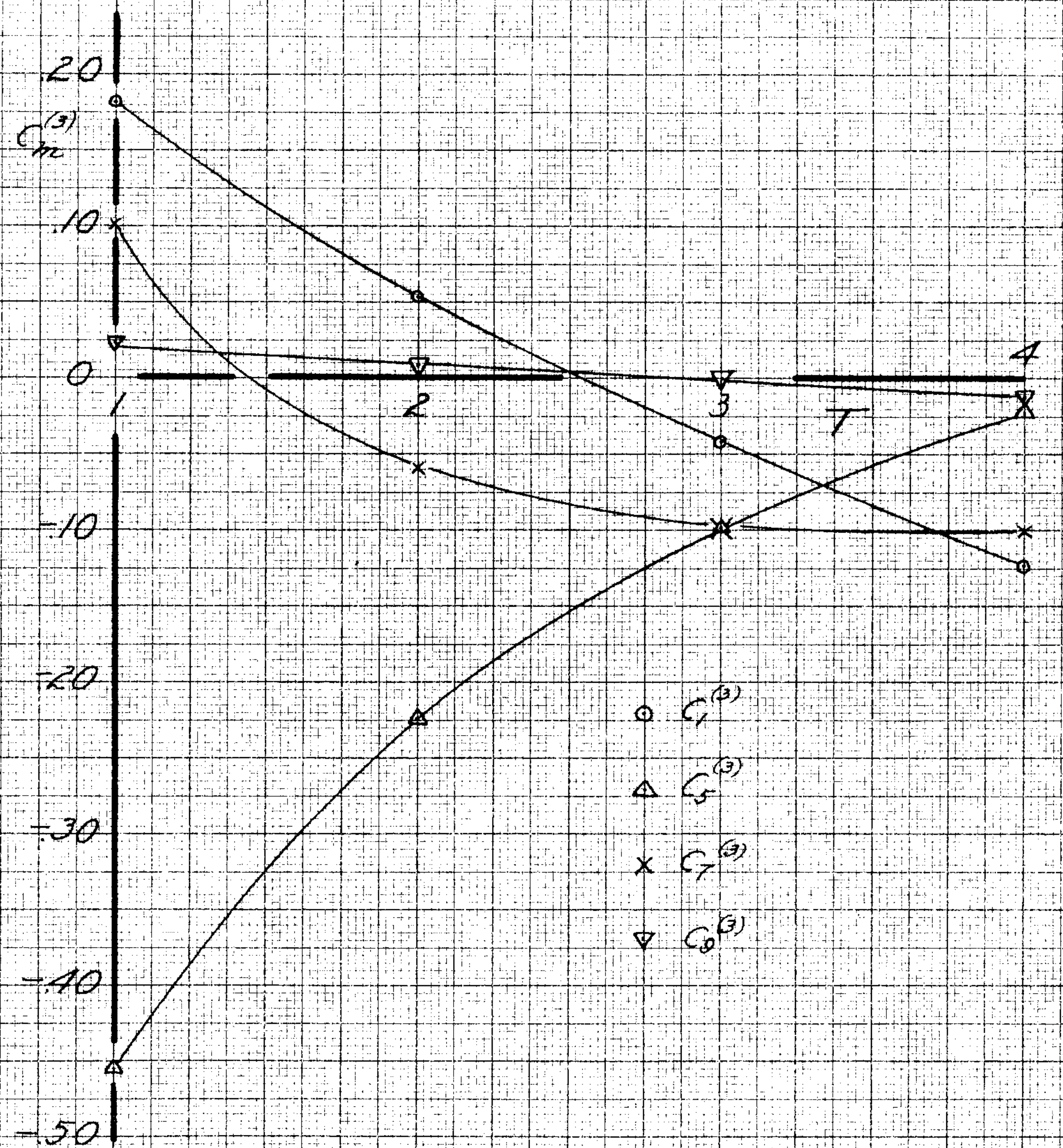

FIG. 3

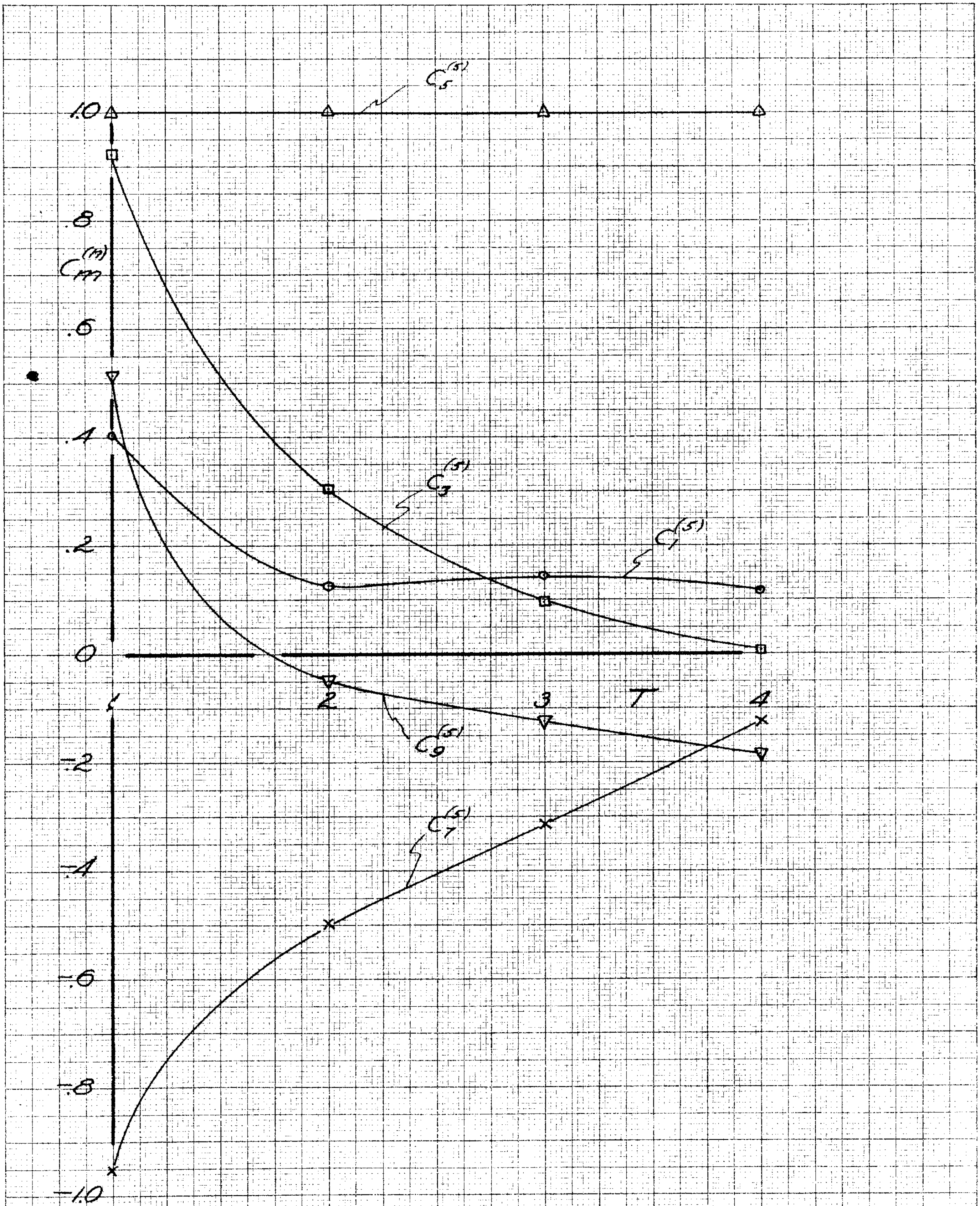

FIG. 4

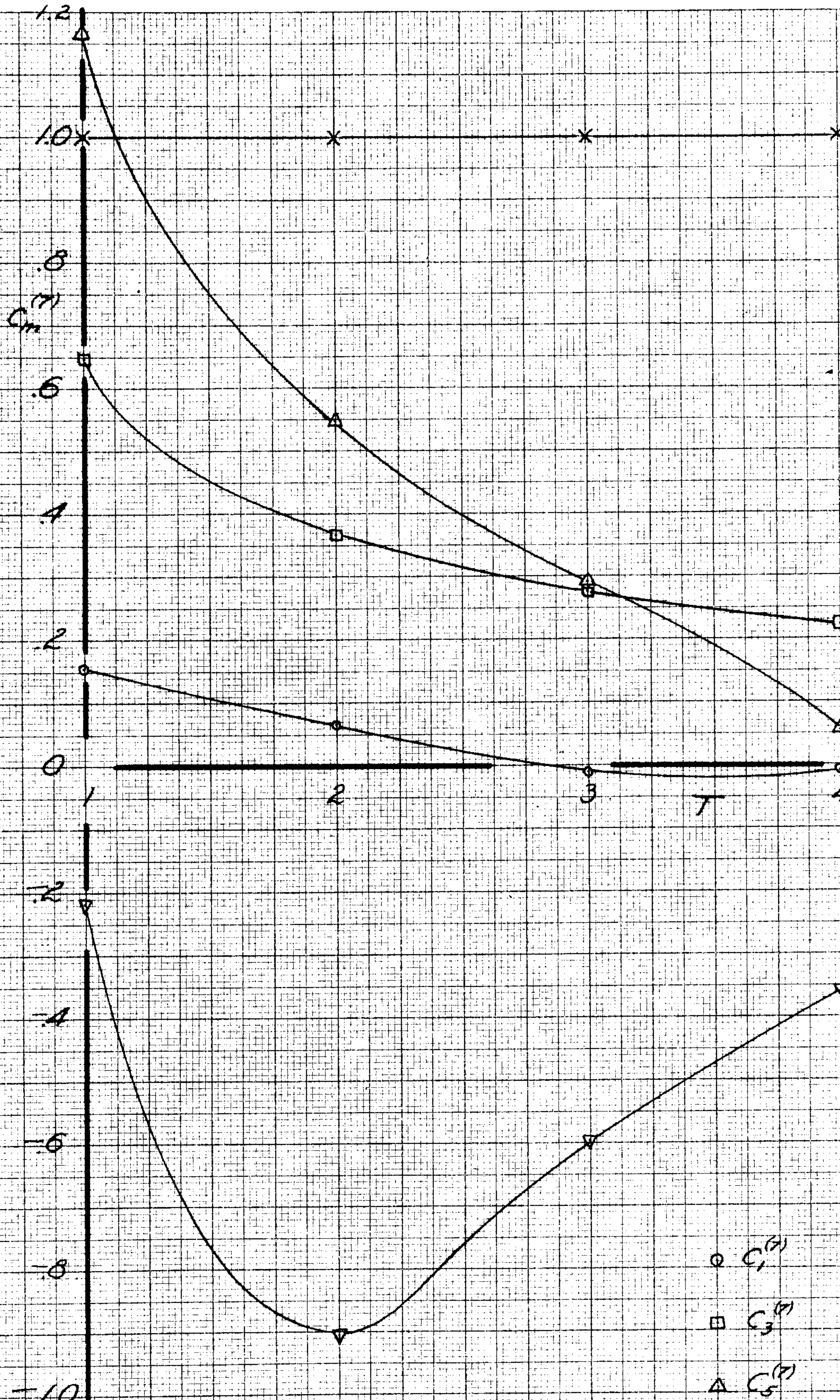

FIG. 5

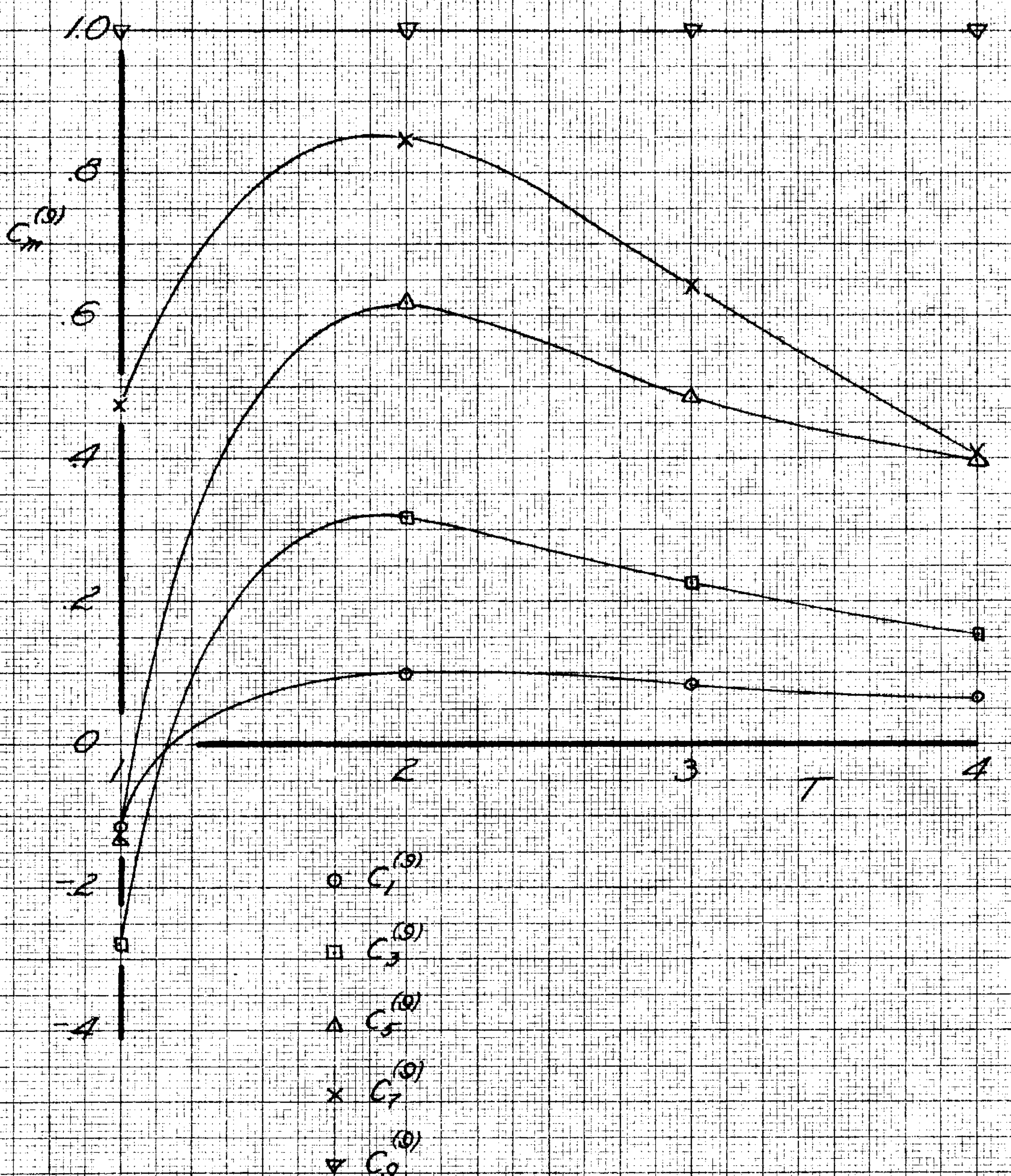

FIG. 6

SAMPLE EIGEN FUNCTION

φ_n VS θ
FOR $T=2$
(ODD SET)

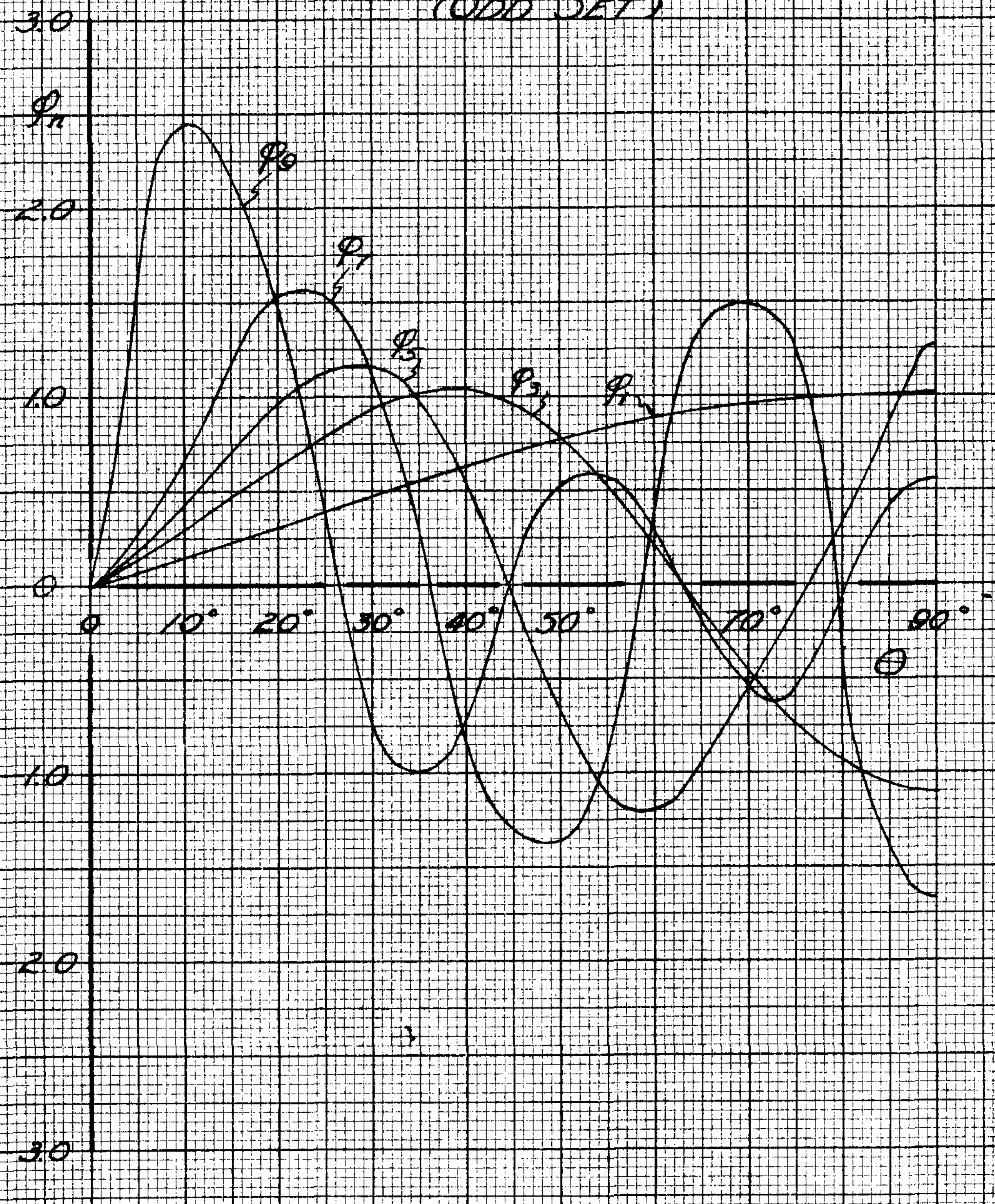

FIG. 7

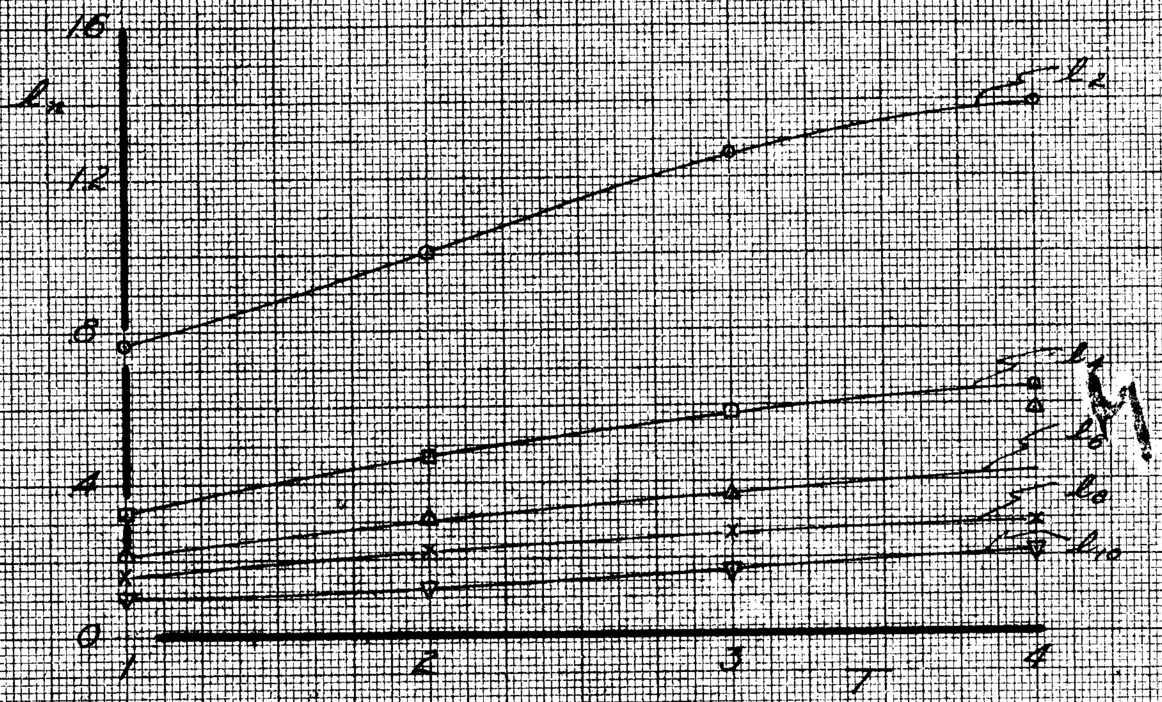

FIG. 8

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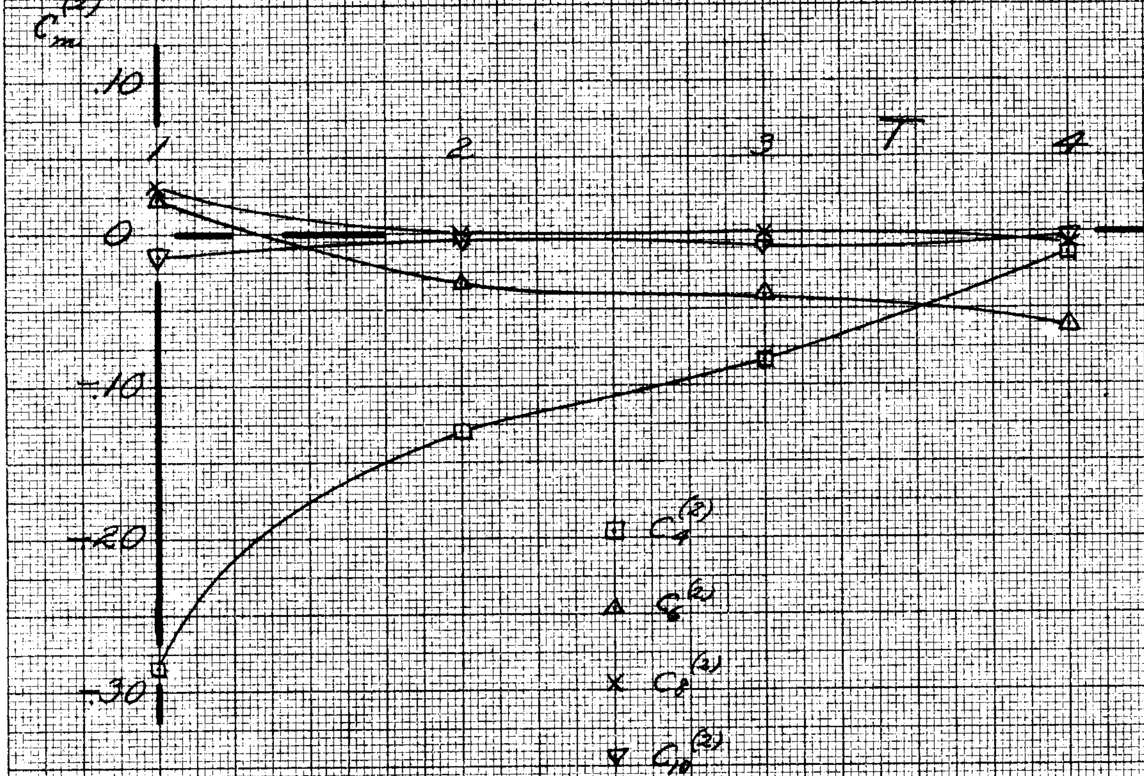

FIG. 9

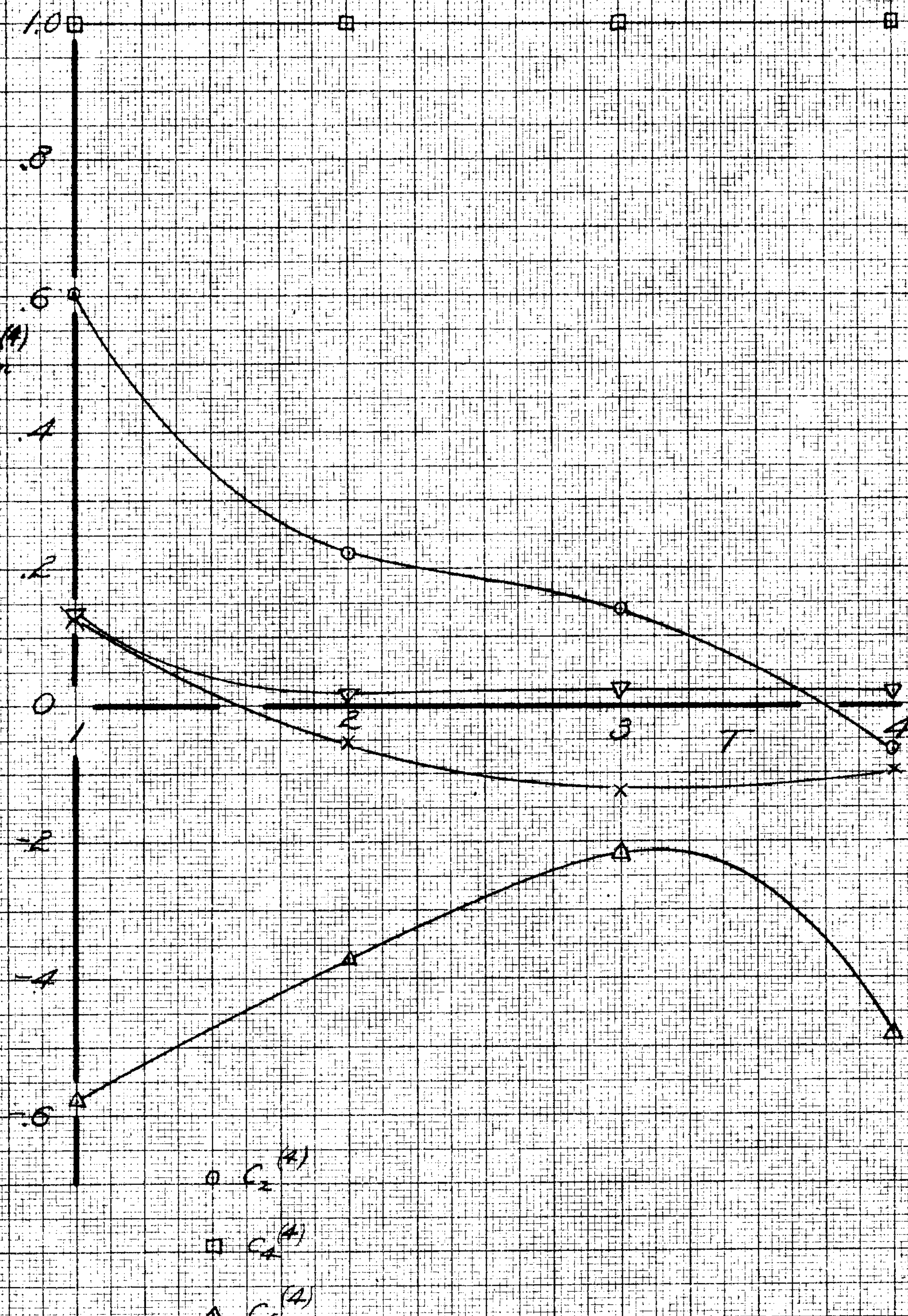

FIG. 10

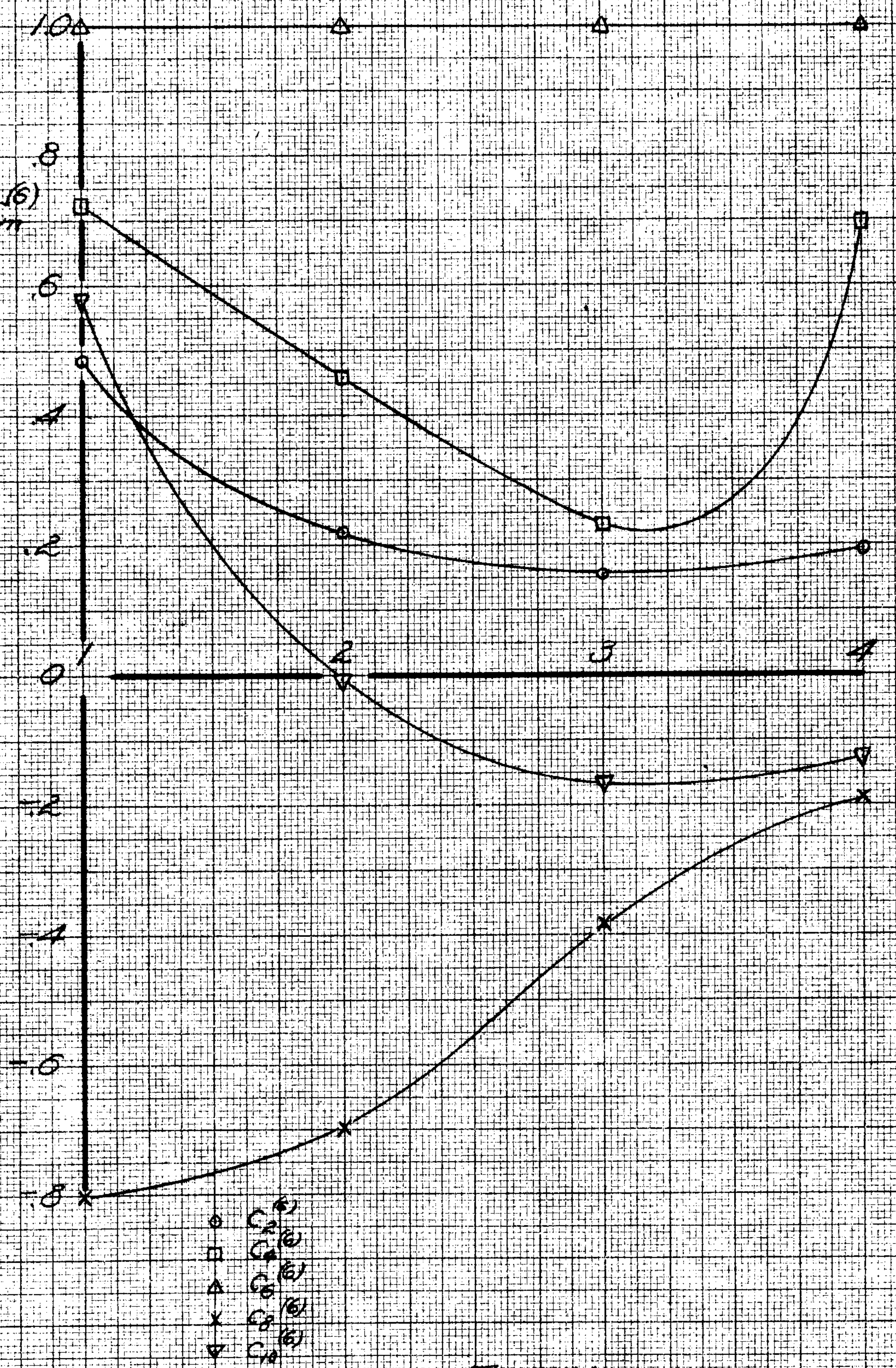

FIG. 11

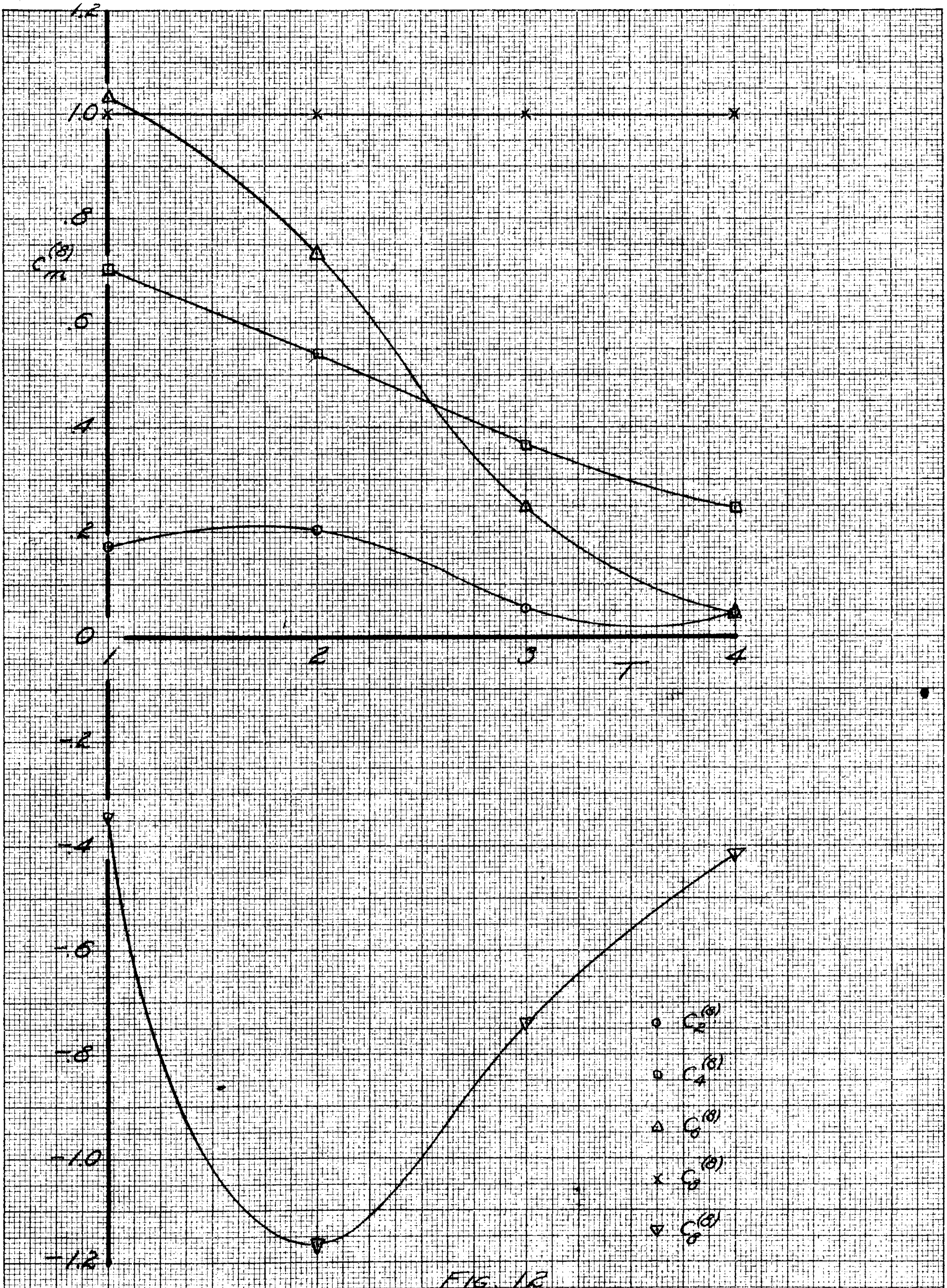

FIG. 12
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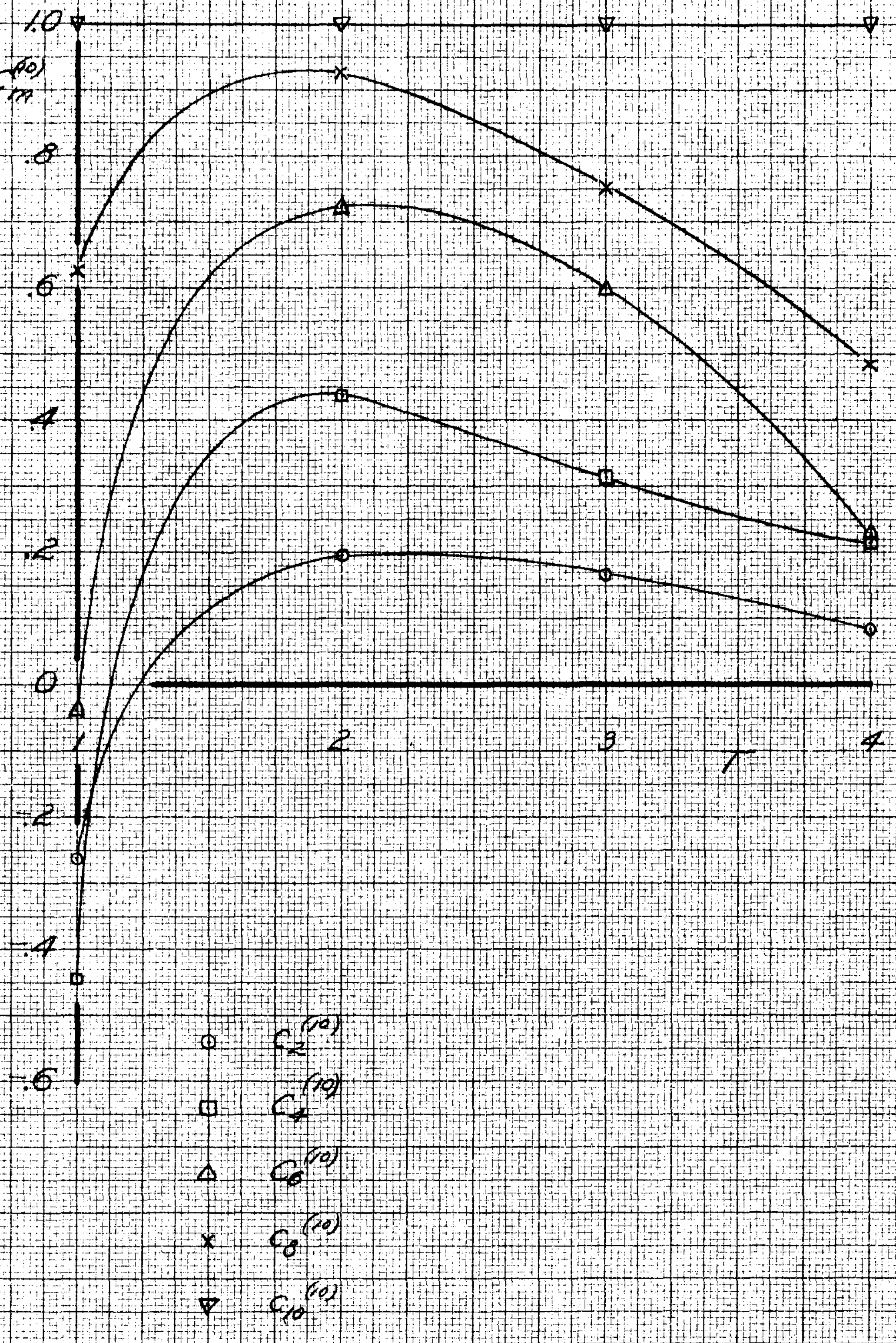

FIG. 13

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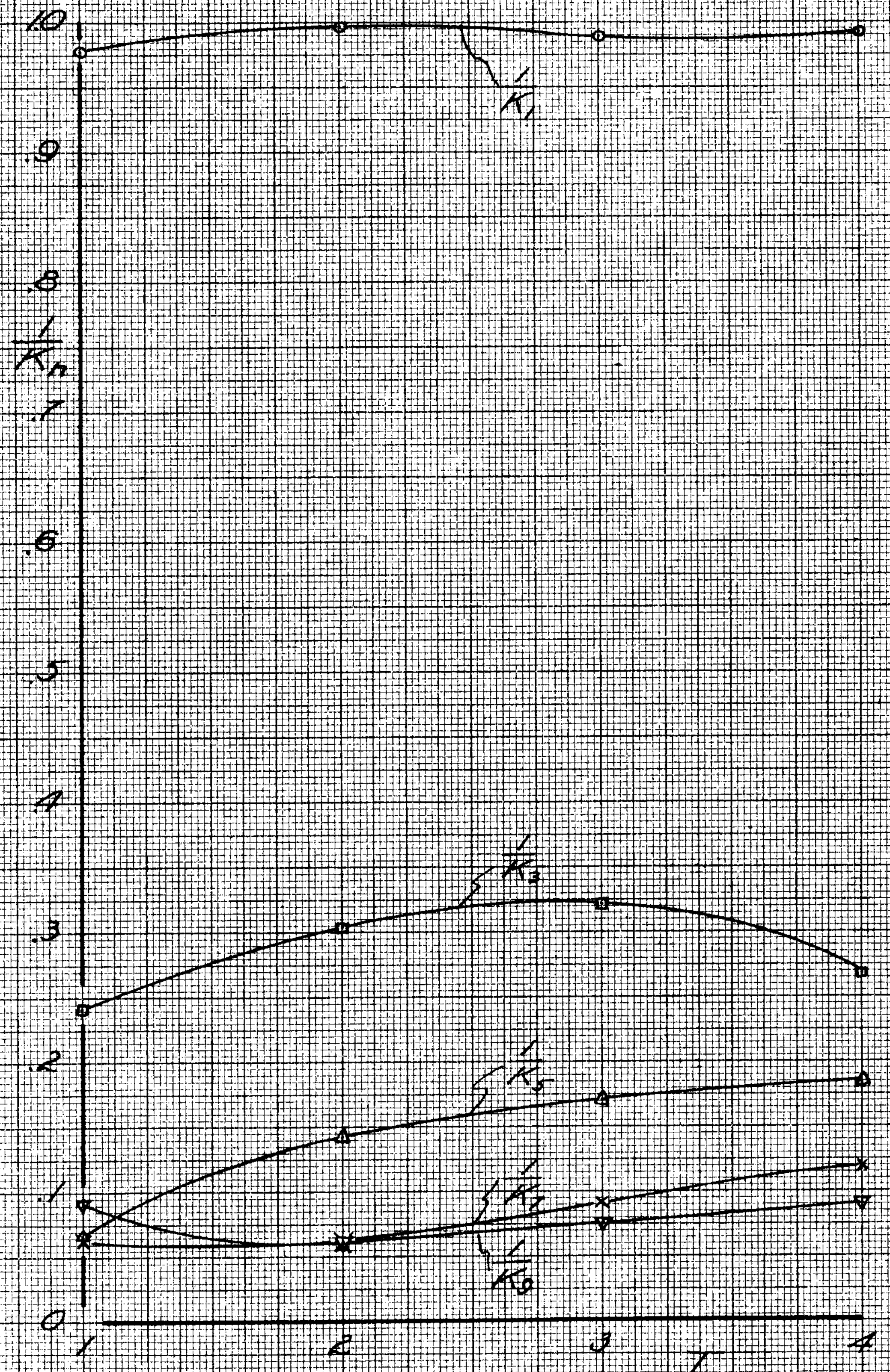
1.0
.9
.8
$\frac{1}{K_n}$
.7
.6
.5
.4
.3
.2
.1
0
1
2
3
4
T
$\frac{1}{K_1}$
$\frac{1}{K_3}$
$\frac{1}{K_5}$
$\frac{1}{K_7}$
$\frac{1}{K_9}$

FIG. 14

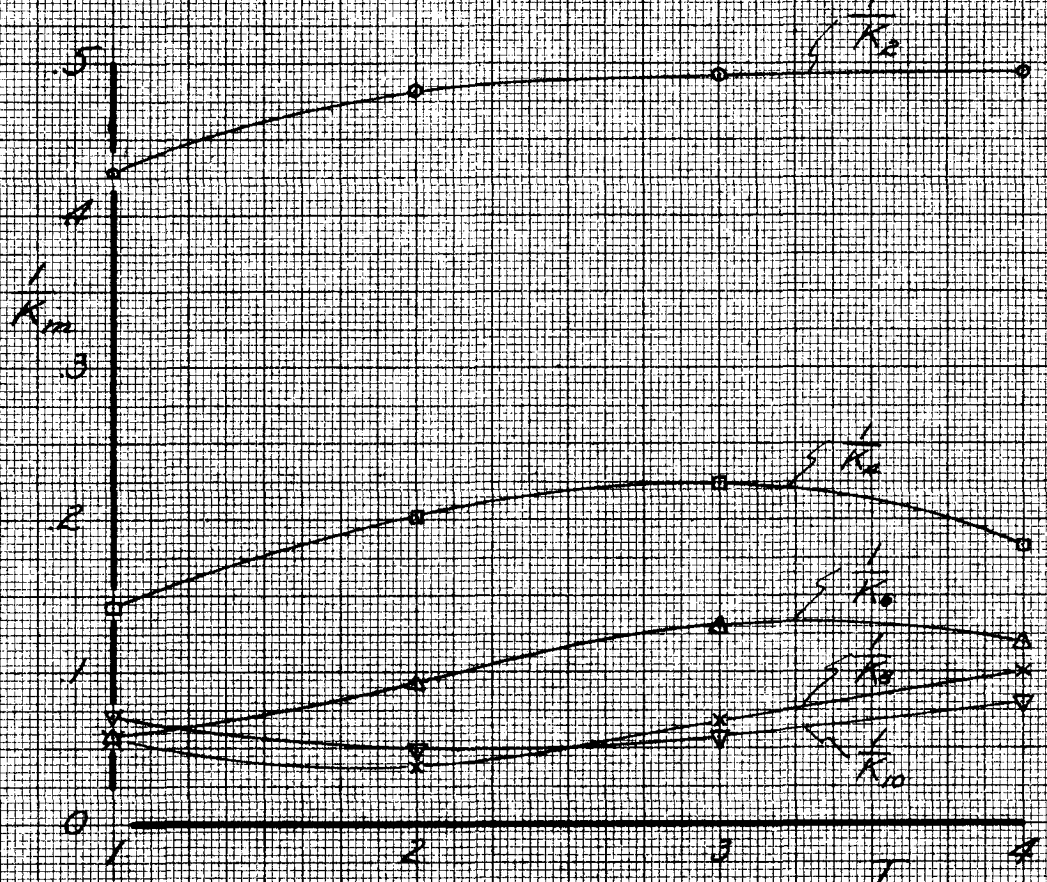

FIG. 15

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SAMPLE CALCULATIONS

(For the case: $T = 3:1$, Odd Set)

To evaluate planform coefficients, b_k, in

$$f(\theta) = \frac{c_s m_s}{c(\theta)\, m(\theta)} \sin\theta = \sum_{k=0}^{\infty} b_k \cos k\theta$$

This is done just as in reference 4, pages 3-16 and 3-18, Tables I and II, except that we are concerned with the special case of linear taper. For linear taper (assume $m = m_s$),

$$c(\theta) = c_s\left(1 - \beta \frac{y}{b/2}\right) \qquad , \; 0 \le x \le b/2 \, ,$$

where the meaning of β will soon be shown. Or,

$$c(\theta) = c_s\,(1 - \beta \cos\theta) \qquad , \; 0 \le \theta \le \frac{\pi}{2} \, .$$

Then the tip chord ($y = b/2$) is

$$c_t = c_s\,(1 - \beta) \, ,$$ and taper ratio is

$$T = \frac{c_s}{c_t} = \frac{c_s}{c_s(1-\beta)} = \frac{1}{1-\beta}$$

$$\beta = 1 - \frac{1}{T} \, .$$

$$\therefore \; f(\theta) = \frac{1}{1 - \beta\cos\theta} \sin\theta = \sum_{k=0}^{\infty} b_k \cos k\theta \, .$$

The numerical example which follows is for $T = 3$. Then $\beta = 1 - \frac{1}{3} = \frac{2}{3}$.

The following tabulation of geometric characteristics of the wing is similar to reference 4, page 3-16, Table I (Assuming $m(\theta) = m_s$).

Column	0	1	2	3	4	5	6	7	8	9	10
θ°	90	81	72	63	54	45	36	27	18	9	0
$\frac{y}{b/2} = \cos\theta$	0	.15643	.30902	.45399	.58779	.70711	.80902	.89101	.95106	.98769	1.0
$\beta\cos\theta = \frac{2}{3}\cos\theta$	0	.10429	.20601	.30266	.39186	.47141	.53935	.59401	.63404	.65846	.66667
$\frac{c(\theta)}{c_s} = 1-\beta\cos\theta$	1.0	.89571	.79399	.69734	.60814	.52859	.46065	.40599	.36596	.34154	.33333
$\sin\theta$	1.0	.98769	.95106	.89101	.80902	.70711	.58779	.45399	.30902	.15643	0
$\frac{c_s \sin\theta}{c(\theta)} = \frac{\sin\theta}{1-\beta\cos\theta}$	1.0	1.10269	1.19782	1.46514	1.33032	1.33773	1.27600	1.11823	.84441	.45801	0

Next, we fill out table III of reference 4, page 3-18: "Computation of Planform Coefficients, C_{2n}" by using values of the last line in the table just completed. Note that C_{2n} in the notation of reference 4 is equivalent to our b_k, that is, $b_0 \equiv C_0$, $b_2 \equiv C_2$, ..., $b_{20} = C_{20}$. The odd b_k's are zero because the planform is symmetrical about the root chord. This computation yields as the last line of Table III the following values:

$$
\begin{aligned}
2b_0 &= +\ 2.12607 \quad , \quad b_0 = +1.06304 \\
b_4 &= -\ .37018 \\
b_6 &= -\ .07917 \\
b_8 &= -\ .12335 \\
b_{10} &= -\ .04018 \\
b_{12} &= -\ .01536 \\
b_{14} &= -\ .05957 \\
b_{16} &= -\ .02082 \\
b_{18} &= +\ .00287 \\
2b_{20} &= -\ .06665 \quad , \quad b_{20} = -.03333
\end{aligned}
$$

Next, we fill out the matrix $[k_{ij}]$ of equation (7). The result is as follows:

$[k_{ij}] =$

Row ↓ / Column →	1	2	3	4	5	6	7	8	9	10
1	+2.45002	0	+.04623	0	−.29101	0	+.04418	0	−.08317	0
2	0	+1.24813	0	−.12239	0	−.12342	0	−.01950	0	−.05400
3	+.01541	0	+.73508	0	−.06687	0	−.11000	0	−.02127	0
4	0	−.06120	0	+.56236	0	−.07094	0	−.08871	0	−.00490
5	−.05820	0	−.04012	0	+.43325	0	−.06172	0	−.06212	0
6	0	−.04114	0	−.04730	0	+.35691	0	−.04406	0	−.05823
7	+.00631	0	−.04714	0	−.04408	0	+.31223	0	−.04330	0
8	0	−.00487	0	−.04435	0	−.03305	0	+.26836	0	−.04085
9	−.00924	0	−.00709	0	−.03451	0	−.03368	0	+.23591	0
10	0	−.01080	0	−.00196	0	−.03494	0	−.03268	0	+.21594

As explained in the discussion in the body of the thesis, the above matrix can be separated into two matrices having only odd or even subscripts of $C_m^{(n)}$, respectively, as follows:

ODD SET

	1	3	5	7	9
1	+2.45002	+.04623	−.29101	+.04418	−.08317
3	+.01541	+.73508	−.06687	−.11000	−.02127
5	−.05820	−.04012	+.43325	−.06172	−.06212
7	+.00631	−.04714	−.04408	+.31223	−.04330
9	−.00924	−.00709	−.03451	−.03368	+.23591

EVEN SET

	2	4	6	8	10
2	+1.24813	−.12239	−.12342	−.01950	−.05400
4	−.06120	+.56236	−.07094	−.08871	−.00490
6	−.04114	−.04730	+.35691	−.04406	−.05823
8	−.00487	−.04435	−.03305	+.26836	−.04085
10	−.01080	−.00196	−.03494	−.03268	+.21594

The method of solving the odd matrix will be indicated. Identically the same procedure is followed for the even matrix. We use only three decimal places for $c_m^{(n)}$ in the beginning, later go to four, and finally to five as the convergence warrants.

ODD MATRIX

$$\begin{bmatrix} +2.45002, & +.04623, & \cdots \\ +.01541, & \cdots & \\ \vdots & \cdots & \\ \vdots & \cdots & \\ -.00924, & \cdots & \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{Bmatrix} 2.45002 \\ .01541 \\ -.05820 \\ .00631 \\ -.00924 \end{Bmatrix} = 2.45002 \begin{Bmatrix} 1.000 \\ .006 \\ -.024 \\ .003 \\ -.004 \end{Bmatrix}$$

The next step gives:

$$\begin{bmatrix} 2.45002, & +.04623, & \cdots \\ .01541, & \cdots & \\ \vdots & \cdots & \\ \vdots & \cdots & \\ -.00924, & \cdots & \end{bmatrix} \begin{Bmatrix} 1.000 \\ .006 \\ -.024 \\ .003 \\ -.004 \end{Bmatrix} = \begin{Bmatrix} 2.457 \\ .021 \\ -.068 \\ .008 \\ -.009 \end{Bmatrix} = 2.457 \begin{Bmatrix} 1.000 \\ .009 \\ -.028 \\ .003 \\ -.004 \end{Bmatrix}$$

etc.

However, for actual calculation it is more convenient to tabulate as follows: (Read left to right)

1st STEP

TRY↓	1	3	5	7	9	$\{N_i\}$	$\{C_i^{(1)}\}$
1	2.45002	0	0	0	0	2.45002	1.000
0	.01541	0	0	0	0	.01541	.006
0	−.05820	0	0	0	0	−.05820	−.024
0	.00631	0	0	0	0	.00631	.003
0	−.00924	0	0	0	0	−.00924	−.004

2ND STEP

1	3	5	7	9	$\{N_i\}$	$\{C_i^{(1)}\}$
2.450	.000	.007	.000	.000	2.457	1.000
.015	.004	.002	.000	.000	.021	.009
−.058	.000	−.010	.000	.000	−.068	−.028
.006	.000	.001	.001	.000	.008	.003
−.009	.000	.001	.000	−.001	−.009	−.004

3RD STEP

1	3	5	7	9	$\{N_i\}$	$\{C_i^{(1)}\}$
2.4500	.0004	.0081	.0001	.0003	2.4589	1.0000
.0154	.0066	.0019	−.0003	.0001	.0237	.0112
−.0582	−.0004	−.0121	−.0002	.0002	−.0707	−.0287
.0063	−.0004	.0012	.0009	.0002	.0082	.0033
−.0092	−.0001	.0010	−.0001	−.0009	−.0093	−.0038

4TH STEP

1	3	5	7	9	$\{N_i\}$	$\{C_i^{(1)}\}$
2.4500	.0005	.0084	.0001	.0003	2.4593	1.0000
.0154	.0082	.0019	−.0004	.0001	.0252	.0102
−.0582	−.0004	−.0124	−.0002	.0002	−.0710	−.0289
.0063	−.0005	.0013	.0010	.0002	.0083	.0034
−.0092	−.0001	.0010	−.0001	−.0009	−.0093	−.0038

5TH STEP

$$\begin{bmatrix} 2.45002 & .00047 & .00841 & .00015 & .00032 \\ .01541 & .00750 & .00193 & \overline{.00037} & .00008 \\ \overline{.05820} & \overline{.00041} & \overline{.01252} & \overline{.00021} & .00024 \\ .00631 & \overline{.00048} & .00127 & .00106 & .00016 \\ \overline{.00924} & \overline{.00007} & .00100 & \overline{.00011} & \overline{.00090} \end{bmatrix} \begin{Bmatrix} 2.45937 \\ .02455 \\ \overline{.07110} \\ .00832 \\ \overline{.00932} \end{Bmatrix} \begin{Bmatrix} 1.00000 \\ .00998 \\ \overline{.02891} \\ .00338 \\ \overline{.00379} \end{Bmatrix}$$

make a guess for next step, since trend is well established: Try →

$$\begin{Bmatrix} 1.0 \\ .00983 \\ \overline{.02892} \\ .00336 \\ \overline{.00378} \end{Bmatrix}$$

$$\begin{bmatrix} 2.45002 & .00045 & .00842 & .00015 & .00031 \\ .01541 & .00723 & .00193 & \overline{.00037} & .00008 \\ \overline{.05820} & \overline{.00039} & \overline{.01253} & \overline{.00021} & .00023 \\ .00631 & \overline{.00046} & .00127 & .00105 & .00016 \\ \overline{.00924} & \overline{.00007} & .00100 & \overline{.00011} & \overline{.00089} \end{bmatrix} \begin{Bmatrix} 2.45935 \\ .02428 \\ \overline{.07110} \\ .00833 \\ \overline{.00931} \end{Bmatrix} \begin{Bmatrix} 1.0 \\ .00987 \\ \overline{.02891} \\ .00339 \\ \overline{.00379} \end{Bmatrix}$$

$$\begin{bmatrix} 2.45002 & .00046 & .00841 & .00015 & .00032 \\ .01541 & .00726 & .00193 & \overline{.00037} & .00008 \\ \overline{.05820} & \overline{.00040} & \overline{.01253} & \overline{.00021} & .00024 \\ .00631 & \overline{.00047} & .00127 & .00106 & .00016 \\ \overline{.00924} & \overline{.00007} & .00100 & \overline{.00011} & \overline{.00089} \end{bmatrix} \begin{Bmatrix} 2.45936 \\ .02431 \\ \overline{.07110} \\ .00833 \\ \overline{.00931} \end{Bmatrix} \begin{Bmatrix} 1.0 \\ .00988 \\ \overline{.02891} \\ .00339 \\ \overline{.00379} \end{Bmatrix}$$

Upon once more repeating the matrix multiplication, the identical values of $\{C_i^{(1)}\}$ result, indicating that the process has converged. Therefore we see that $\ell_1 = 2.45936$, and

$$c_1^{(1)} = 1.00000$$
$$C_3^{(1)} = .00988$$
$$C_5^{(1)} = -.02891$$
$$C_7^{(1)} = .00339$$
$$C_9^{(1)} = \overline{.00379}$$

We now use the orthogonality property to eliminate the fundamental solution.

$$\sum_{m=1}^{9} m C_m^{(i)} C_m^{(j)} = 0 \quad , \text{ for } i \neq j$$

$$\therefore \quad C_1^{(j)} = -3C_3^{(1)}C_3^{(j)} - 5C_5^{(1)}C_5^{(j)} - 7C_7^{(1)}C_7^{(j)} - 9C_9^{(1)}C_9^{(j)} \quad , j \neq 1$$

$$= \overline{.02964}\,C_3^{(j)} + .14455\,C_5^{(j)} - .02373\,C_7^{(j)} + .03411\,C_9^{(j)} \quad , j \neq 1.$$

Let us throw out the first row of the odd matrix and substitute for $C_1^{(j)}$, where $j=3$. The new matrix becomes:

$$\begin{bmatrix} (.73508 - .02964 \times .01541), & (\overline{.06687} + .14455 \times .01541), & (\overline{.11000} - .02373 \times .01541), & (\overline{.02127} + .03411 \times .01541) \\ (\overline{.04012} + .02964 \times .05820), & (.43325 - .14455 \times .05820), & (\overline{.06172} + .02373 \times .05820), & (\overline{.06212} - .03411 \times .05820) \\ (-.04714 - .02964 \times .00631), & (\overline{.04408} + .14455 \times .00631), & (.31223 - .02373 \times .00631), & (\overline{.04330} + .03411 \times .00631) \\ (-.00709 + .02964 \times .00924), & (\overline{.03451} - .14455 \times .00924), & (\overline{.03368} + .02373 \times .00924), & (.23591 - .03411 \times .00924) \end{bmatrix}$$

$$
= \begin{bmatrix} +.73462 & -.06464 & -.11037 & -.02074 \\ -.03839 & +.42484 & -.06034 & -.06411 \\ -.04733 & -.04317 & +.31208 & -.04308 \\ -.00682 & -.03585 & -.03346 & +.23559 \end{bmatrix}
$$

We shall now solve this square matrix of four rows. Try

$$
{}_0\left\{ C_i^{(3)} \right\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

The method of tabulation used on pgs. 52 and 53 will no longer be used. Instead the matrix multiplication will be indicated.

$$
\begin{array}{c} \\ 3 \\ 5 \\ 7 \\ 9 \end{array}
\begin{array}{c} \begin{array}{cccc} 3 & 5 & 7 & 9 \end{array} \\ \begin{bmatrix} +.73462 & -.06464 & -.11037 & -.02074 \\ -.03839 & +.42484 & -.06034 & -.06411 \\ -.04733 & -.04317 & +.31208 & -.04308 \\ -.00682 & -.03585 & -.03346 & +.23559 \end{bmatrix} \end{array}
\begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} .73462 \\ -.03839 \\ -.04733 \\ -.00682 \end{Bmatrix} = .73462 \begin{Bmatrix} 1.000 \\ -.052 \\ -.064 \\ -.009 \end{Bmatrix}
$$

$$
\Big[\text{Same matrix as above} \uparrow \Big] \begin{Bmatrix} 1.0 \\ -.052 \\ -.064 \\ -.009 \end{Bmatrix} = \begin{Bmatrix} .745 \\ -.055 \\ -.065 \\ -.005 \end{Bmatrix} = .745 \begin{Bmatrix} 1.000 \\ -.074 \\ -.087 \\ -.007 \end{Bmatrix}
$$

$$
\Big[\text{" \quad " \quad " \quad "} \Big] \overset{\text{TRY}}{\begin{Bmatrix} 1.0 \\ -.100 \\ -.110 \\ -.004 \end{Bmatrix}} = \begin{Bmatrix} .753 \\ -.073 \\ .077 \\ .000 \end{Bmatrix} = .753 \begin{Bmatrix} 1.000 \\ -.097 \\ -.102 \\ 0 \end{Bmatrix}
$$

$$
\Big[\text{" \quad " \quad " \quad "} \Big] \begin{Bmatrix} 1.0 \\ -.097 \\ -.102 \\ 0 \end{Bmatrix} = \begin{Bmatrix} .752 \\ -.073 \\ -.075 \\ -.001 \end{Bmatrix} = .752 \begin{Bmatrix} 1.000 \\ -.097 \\ -.100 \\ -.001 \end{Bmatrix}
$$

Change to 4 decimals:

$$
\Big[\text{" \quad " \quad " \quad "} \Big] \begin{Bmatrix} 1.0 \\ -.097 \\ -.100 \\ -.001 \end{Bmatrix} = \begin{Bmatrix} .7519 \\ -.0736 \\ -.0743 \\ -.0002 \end{Bmatrix} = .7519 \begin{Bmatrix} 1.0000 \\ -.0979 \\ -.0988 \\ -.0003 \end{Bmatrix}
$$

$$\left[\begin{matrix} '' & '' & '' & '' \end{matrix}\right]\begin{Bmatrix}1.0\\ -.0985\\ -.0987\\ 0\end{Bmatrix}=\begin{Bmatrix}.7519\\ -.0742\\ -.0738\\ 0\end{Bmatrix}=.7519\begin{Bmatrix}1.0\\ -.0987\\ -.0982\\ 0\end{Bmatrix}$$

$$\left[\begin{matrix} '' & '' & '' & '' \end{matrix}\right]\overset{\text{TRY}}{\begin{Bmatrix}1.0\\ -.0990\\ -.0978\\ \end{Bmatrix}}=\begin{Bmatrix}.7518\\ -.0746\\ -.0735\\ 0\end{Bmatrix}=.7518\begin{Bmatrix}1.0\\ -.09923\\ -.09777\\ 0\end{Bmatrix}$$

$$\left[\begin{matrix} '' & '' & '' & '' \end{matrix}\right]\begin{Bmatrix}1.0\\ -.09923\\ -.09777\\ 0\end{Bmatrix}=\begin{Bmatrix}.75182\\ -.07465\\ -.07356\\ .00001\end{Bmatrix}=.75182\begin{Bmatrix}1.0\\ -.09929\\ -.09784\\ .00001\end{Bmatrix}$$

$$\left[\begin{matrix} '' & '' & '' & '' \end{matrix}\right]\overset{\text{TRY}}{\begin{Bmatrix}1.0\\ -.99350\\ -.97890\\ .00001\end{Bmatrix}}=\begin{Bmatrix}.75184\\ -.07469\\ -.07359\\ +.00002\end{Bmatrix}=.75184\begin{Bmatrix}1.0\\ -.09934\\ -.09788\\ .00003\end{Bmatrix}$$

$$\left[\begin{matrix} '' & '' & '' & '' \end{matrix}\right]\begin{Bmatrix}1.0\\ -.09934\\ -.09788\\ .00003\end{Bmatrix}=\begin{Bmatrix}.75184\\ -.07468\\ -.07359\\ .00003\end{Bmatrix}=.75184\begin{Bmatrix}1.0\\ -.09933\\ -.09788\\ .00004\end{Bmatrix}$$

$$\left[\begin{matrix} '' & '' & '' & '' \end{matrix}\right]\begin{Bmatrix}1.0\\ -.09933\\ -.09788\\ .00004\end{Bmatrix}=\begin{Bmatrix}.75184\\ -.07468\\ -.07359\\ .00003\end{Bmatrix}=.75184\begin{Bmatrix}1.0\\ -.09933\\ -.09788\\ .00004\end{Bmatrix}$$

The last operation introduced no change, indicating convergence.

$\therefore\ l_3 = .75184$

$$\begin{cases} c_3^{(3)} = 1.0\\ c_5^{(3)} = -.09933\\ c_7^{(3)} = -.09788\\ c_9^{(3)} = .00004\end{cases}$$

From the orthogonality equation of before, written for $j=3$,

$$C_1^{(3)} = -.02964\,C_3^{(3)} + .14455\,C_5^{(3)} - .02373\,C_7^{(3)} + .03411\,C_9^{(3)}$$
$$= -.02964 - .01436 + .00232 + 0$$
$$\therefore\ C_1^{(3)} = -.04168$$

Now let us check the 3^{rd} set in the original odd matrix:

$$\begin{bmatrix} \overset{+}{2.45002}, & \overset{+}{.04623}, & \cdots, & \cdots, & \cdots \\ .01541, & .73508, & \cdots, & \cdots, & \cdots \\ \bar{.05820}, & \bar{.04012}, & \cdots, & \cdots, & \cdots \\ .00631, & \bar{.04714}, & \cdots, & \cdots, & \cdots \\ \bar{.00924}, & \bar{.00709}, & \cdots, & \cdots, & \cdots \end{bmatrix} \begin{pmatrix} \bar{.04168} \\ 1.00000 \\ \bar{.09933} \\ \bar{.09788} \\ .00004 \end{pmatrix} = \begin{pmatrix} \bar{.03130} \\ .75185 \\ \bar{.07468} \\ \bar{.07358} \\ .00004 \end{pmatrix} = .75185 \begin{pmatrix} \bar{.04163} \\ 1.00000 \\ \bar{.09933} \\ \bar{.09787} \\ .00005 \end{pmatrix}$$

We see that the 3^{rd} set satisfies the original odd matrix.

We now proceed to solve for the 5^{th} set:
Our orthogonality equation, for $j=5$, is

$$C_1^{(5)} = -.02964\,C_3^{(5)} + .14455\,C_5^{(5)} - .02373\,C_7^{(5)} + .03411\,C_9^{(5)} \qquad Eq.\ (I)$$

Also, orthogonality gives

$$1C_1^{(3)}C_1^{(5)} + 3C_3^{(3)}C_3^{(5)} + 5C_5^{(3)}C_5^{(5)} + 7C_7^{(3)}C_7^{(5)} + 9C_9^{(3)}C_9^{(5)} = 0 \qquad Eq.\ (II)$$

We substitute the expression for $C_1^{(5)}$ of eq. (I) into eq. (II): This gives

$$C_3^{(5)} = .16749\,C_5^{(5)} + .22796\,C_7^{(5)} + .00035\,C_9^{(5)} \qquad Eq.\ (III)$$

We substitute for $C_3^{(5)}$ in the matrix of four rows and columns, and, throwing out the top row, we get:

$$\begin{bmatrix} +.41841, & -.06909, & -.06412 \\ -.05110, & +.30129, & -.04310 \\ -.03699, & -.03501, & +.23559 \end{bmatrix}$$

Solving this matrix by the usual method (illustrated above), we get: $l_5 = .44787$, and

$$C_5^{(5)} = 1.0$$
$$C_7^{(5)} = -.31252$$
$$C_9^{(5)} = -.12269$$

Substituting for $C_5^{(5)}$, $C_7^{(5)}$, and $C_9^{(5)}$ in eq. (III), we get

$$C_3^{(5)} = +.09621 \quad ,$$

and substituting these four values into eq (I), we get

$$C_1^{(5)} = +.14494$$

Checking the 5th set in the original odd matrix, we get

$$\begin{bmatrix} 2.45002, & .04623, & \cdots \\ .01541, & \cdots \\ -.05820, & \cdots \\ .00631, & \cdots \\ -.00924, & \cdots \end{bmatrix} \begin{Bmatrix} .14494 \\ .09621 \\ 1.0 \\ -.31252 \\ -.12269 \end{Bmatrix} = \begin{Bmatrix} .06494 \\ .04307 \\ .44786 \\ -.13998 \\ .05494 \end{Bmatrix} = .44786 \begin{Bmatrix} .14500 \\ .09617 \\ 1.0 \\ -.31255 \\ -.12267 \end{Bmatrix}$$

which checks O.K.

We now proceed to find the 7th set:

The orthogonality equation for $n=7$ is

$$C_1^{(7)} = -.02964 C_3^{(7)} + .14455 C_5^{(7)} - .02373 C_7^{(7)} + .03411 C_9^{(7)} \qquad (I').$$

Equation (III) written for $n=7$ is

$$C_3^{(7)} = .16749\, C_5^{(7)} + .22796\, C_7^{(7)} + .00035\, C_9^{(7)} \qquad (III').$$

Also, orthogonality gives

$$C_5^{(7)} = -\frac{1}{5C_5^{(5)}} \left(1 C_1^{(5)} C_1^{(7)} + 3 C_3^{(5)} C_3^{(7)} + 7 C_7^{(5)} C_7^{(7)} + 9 C_9^{(5)} C_9^{(7)} \right) \qquad (IV).$$

Substituting (I′) and (III′) into (IV), we get

$$C_5^{(7)} = +.41950\, C_7^{(7)} + .21685\, C_9^{(7)} \qquad (IV').$$

We substitute for $C_5^{(7)}$ in the square matrix of three rows, dropping the first row, to get

$$\begin{bmatrix} +.27985, & -.05418 \\ -.05053, & +.22757 \end{bmatrix}$$

Solving this by the matrix method, we get

$$\ell_7 = .31220$$

$$\begin{cases} C_7^{(7)} = 1.0 \\ C_9^{(7)} = -.59709 \end{cases}$$

From equation (IV′) we find $C_5^{(7)} = +.29002$

" " (III′) " " $C_3^{(7)} = +.27633$

" " (I′) " " $C_1^{(7)} = -.01037$

Checking the 7^{th} set in the original odd matrix, we get:

$$\begin{bmatrix} 2.45002, & .04623, & \cdots \\ .01541, & \cdots & \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} ^{-}.01037 \\ ^{+}.27633 \\ ^{+}.29002 \\ ^{+}1.0 \\ ^{-}.59709 \end{Bmatrix} = \begin{Bmatrix} ^{-}.00320 \\ .08627 \\ .09053 \\ .31220 \\ ^{-}.18641 \end{Bmatrix} = .31220 \begin{Bmatrix} ^{-}.01025 \\ .27633 \\ .28997 \\ 1.0 \\ ^{-}.59709 \end{Bmatrix} \text{ which checks O.K.}$$

To find the 9^{th} set:

$$l_7 + l_9 = a + b$$

$$.31220 + l_9 = .27985 + .22757 \quad \text{, (from square 2-row matrix).}$$

$$\therefore \; l_9 = .19522$$

Substituting for l_9 and for $C_9^{(9)} (=1.0)$ in either row of the matrix (say the top row), we get

$$(.27985 - l_9)\, C_7^{(9)} - .05418\, C_9^{(9)} = 0$$

$$\therefore \; C_7^{(9)} = .64020$$

from equation (IV') written for $n=9$,

$$C_5^{(9)} = .41950\, C_7^{(9)} + .21685\, C_9^{(9)} = ^{+}.48541$$

" " (III') written for $n=9$: $C_3^{(9)} = ^{+}.22744$

" " (I') " " " : $C_1^{(9)} = ^{+}.08235$

To check 9^{th} set in the original odd matrix:

$$\begin{bmatrix} \\ \text{Original odd matrix} \\ \\ \end{bmatrix} \begin{Bmatrix} .08235 \\ .22744 \\ .48541 \\ .64020 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} .01612 \\ .04431 \\ .09476 \\ .12499 \\ .19523 \end{Bmatrix} = .19523 \begin{Bmatrix} .08257 \\ .22696 \\ .48538 \\ .64022 \\ 1.0 \end{Bmatrix}$$

Let us also check the sum of the diagonals of the original odd matrix against the sum of the l_n's:

$$\left.\begin{aligned} a+b+c+d+e &= 4.16649 \\ l_1+l_3+l_5+l_7+l_9 &= 4.16649 \end{aligned}\right\} \text{Checks O.K.}$$

We find also: $\sum_{m=1}^{9} m\left(C_m^{(n)}\right)^2$ to be:

for $n=1$, $1.01026 = K_1$
" $n=3$, $3.11815 = K_3$
" $n=5$, $5.86793 = K_5$
" $n=7$, $10.85842 = K_7$
" $n=9$, $13.05615 = K_9$

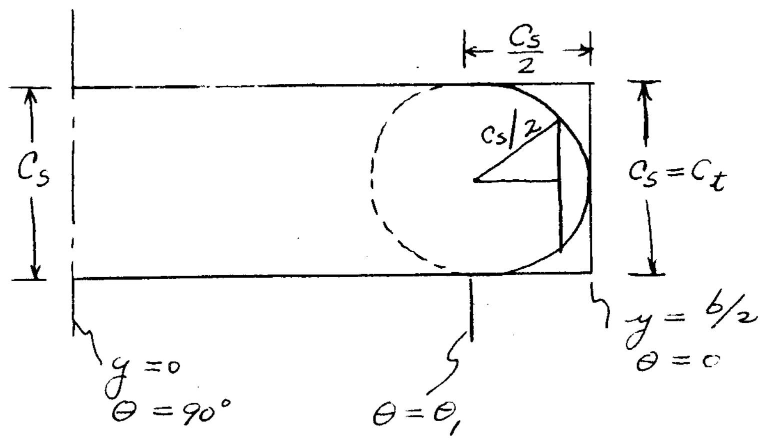
We now tabulate all 25 - $C_m^{(n)}$'s
and 5 - l_n's
and 5 - $\sum_{m=1}^{9} m\left(C_m^{(n)}\right)^2$
and $\varphi_n(\theta) = \sum_{m=1}^{9} C_m^{(n)} \sin m\theta$, as follows:

SUMMARY OF SAMPLE CALCULATIONS

$C_m^{(n)}\downarrow$	$n=1$	$n=3$	$n=5$	$n=7$	$n=9$
$C_1^{(n)}$	1.0	$\overline{.04168}$	$^{+}.14494$	$\overline{.01037}$	$^{+}.08235$
$C_3^{(n)}$	.00988	1.0	.09621	.27633	.22744
$C_5^{(n)}$	$\overline{.02891}$	$\overline{.09933}$	1.0	.29002	.48541
$C_7^{(n)}$	.00339	$\overline{.09788}$	$\overline{.31252}$	1.0	.64020
$C_9^{(n)}$	$\overline{.00379}$	.00004	$\overline{.12269}$	$\overline{.59709}$	1.0
l_n	2.45936	.75184	.44787	.31220	.19522
$\sum_{m=1}^{9} m\left(C_m^{(n)}\right)^2$	1.01026	3.11815	5.86793	10.85842	13.05615

$\theta^\circ =$ / $\varphi_n\downarrow$	10°	20°	30°	40°	50°	60°	70°	80°	90°
φ_1	.15589	.32427	.47951	.65791	.79373	.89398	.94616	.95650	.95403
φ_3	.32473	.69101	.97839	.96957	.57845	$\overline{.03483}$	$\overline{.59694}$	$\overline{.93743}$	$\overline{1.04309}$
φ_5	.42292	.91680	.94763	.14226	$\overline{.84898}$	$\overline{1.01112}$	$\overline{.20231}$	.80910	1.23856
φ_7	.70114	1.16416	.01324	$\overline{.85136}$	.91309	.60586	1.16480	$\overline{.40509}$	$\overline{1.59377}$
φ_9	2.10144	1.11467	$\overline{.80877}$	$\overline{.54659}$	.60946	.20536	$\overline{.63027}$	$\overline{.02279}$	.70012

The tip shape was chosen semi-circular for a wing of aspect ratio = 6.



$$A\!\!\!R = \frac{b}{c_s} = 6 \quad , \quad \cos\theta = \frac{y}{b/2}$$

$$\cos\theta_1 = \frac{b/2 - c_s/2}{b/2} = 1 - \frac{c_s}{b} = 1 - \frac{1}{6} = \frac{5}{6}$$

Therefore,

for $0 \leq \cos\theta \leq \frac{5}{6}$, $\quad c(\theta) = c_s \quad , \therefore \quad \frac{c_s \sin\theta}{c(\theta)} = \sin\theta$

for $\frac{5}{6} \leq \cos\theta \leq 1$,

$$c(\theta) = 2\left[\left(\frac{c_s}{2}\right)^2 - \left(\frac{b}{2}\cos\theta - \frac{5}{6}\frac{b}{2}\right)^2\right]^{1/2}$$

$$= \left[c_o^2 - b^2\left(\cos\theta - \frac{5}{6}\right)^2\right]^{1/2}$$

$$\frac{c_s \sin\theta}{c(\theta)} = \frac{c_s \sin\theta}{\left[c_s^2 - b^2\left(\cos\theta - \frac{5}{6}\right)^2\right]^{1/2}}$$

$$= \frac{\sin\theta}{\left[1 - \left(\frac{b}{c_s}\right)^2\left(\cos\theta - \frac{5}{6}\right)^2\right]^{1/2}}$$

$$= \frac{\sin\theta}{\left[1 - (6\cos\theta - 5)^2\right]^{1/2}}$$

$$= \frac{\sin\theta}{\left[-36\cos^2\theta + 60\cos\theta - 24\right]^{1/2}}$$

$$\therefore \frac{C_s \sin\theta}{C(\theta)} = \frac{\sin\theta}{\left[12\left(-3\cos^2\theta + 5\cos\theta - 2\right)\right]^{1/2}}, \text{ for } \frac{5}{6} \le \cos\theta \le 1 .$$

The tip ends at $\theta_1 = \cos^{-1} 5/6$.
For any other aspect ratio, tip shape will be elliptical.