THE MOTION OF A SPHERE IN A CYLINDER FILLED WITH A VISCOUS LIQUID AND WITH A DIAMETER SLIGHTLY LARGER THAN THE SPHERE DIAMETER

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Abstract

The flow around a sphere, moving under the influence of gravity in a long cylinder filled with a viscous liquid, is investigated, for the case where the inside diameter of the tube is only slightly larger than the diameter of the sphere.

For low Reynolds numbers, an extension of the Christopherson and Dowson (1) theory is obtained in the form of a first order Reynolds number correction. It is shown that the fluid exerts a first order inertial force on the ball, tending to increase the eccentricity ratio. The sideways motion of the ball due to this force has been calculated approximately.

For very high Reynolds numbers, a potential flow solution has been found in the form of a Fourier series. This solution also yields a force tending to increase the eccentricity ratio.

Experiments have been done for intermediate Reynolds numbers. The motion of the ball appears to be unstable when the Reynolds number exceeds a critical value (about 10).

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I. Introduction

In 1959, Christopherson and Dowson (1) considered the flow around a heavy sphere, descending under the influence of gravity in a vertical tube filled with a viscous fluid, the diameter of which is only slightly bigger than the sphere diameter. It was shown that the approximate flow equations for Re = 0 and the ratio of tube diameter to ball diameter going to one, result in no force perpendicular to the motion, independent of the eccentricity ratio at which the ball moves. However, there is one eccentricity ratio (about 98%) for which the rate of energy dissipation is maximum, and experiments performed by Christopherson and Dowson indicate that the sphere will always take up that eccentricity ratio, regardless of the initial conditions.

The work presented in this thesis is an effort to obtain a better understanding of the flow described above, for non-zero Reynolds numbers.

Experiments were done (see Chapter V) with a light sphere, a ping-pong ball. If the Reynolds number that characterizes the motion exceeds a certain critical value, the motion of the sphere appears to be unstable: the ball bounces against the tube wall during the upward motion and obtains a rotational speed around the vertical axis.

In order to get an idea about the character of the flow and pressure field in the region between the cylinder wall and the sphere at non-zero Reynolds number, a perturbation of the flow equations in Re* was performed. This first order inertial correction yields a force that will increase the eccentricity, but doesn't seem quite large enough to account for the Christopherson and Dowson effect. There also is a force opposing horizontal motion of the ball, which in the Stokes limit tends to infinity when the eccentricity ratio tends to one. This force will make the ball stick close to the wall, once it gets there.

Also, a potential flow solution in the region between the ball and the cylinder wall was found, as an approximation of the flow at very high Reynolds number. This approximation also results in a horizontal force, tending to increase the eccentricity ratio.

All the calculations were done with the assumption of steady flow.

II. Low Reynolds Number Expansion

For the following analysis it is assumed that the sphere moves vertically in the cylinder, at a certain constant eccentricity ratio. For that steady state condition the velocity and pressure fields can be evaluated with the eccentricity ratio as a parameter. The flow region under consideration will be the region where the distance between cylinder and sphere is of the order of the difference in the respective diameters.

Let the radius of the sphere be a, the radius of the cylinder $d = a + c_{00}$, and the eccentricity ratio e/c_{00} (see fig. 1).

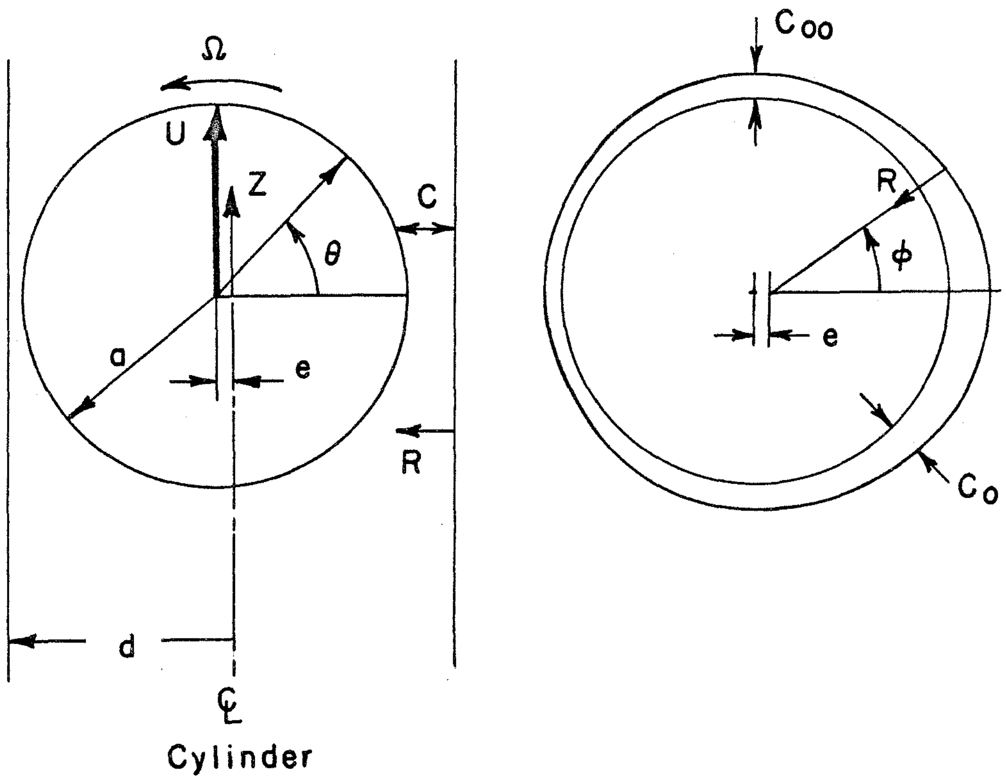

FIG. 1 DEFINITION OF THE COORDINATE SYSTEM

Essentially, a cylindrical polar coordinate system will be used. However, for convenience of integration this coordinate system will sometimes be slightly modified, so that $\frac{1}{a}\frac{\partial}{\partial\theta}$ measures the variance with the vertical coordinate, instead of $\frac{\partial}{\partial z}$. The gap width c, expressed in cylindrical polars will be

$$c = c_{00} + e\cos\varphi + a(1 - \sqrt{1-z^2/a^2})$$

For c to be of the order of c_{00}, we must require

$$z \sim O[(c_{00}a)^{1/2}]$$

so that to leading order in c_{00}/a

$$c = c_{00} + e\cos\varphi + z^2/2a \tag{1}$$

Furthermore, we define

$$c_0 = c_{00} + e\cos\varphi \tag{2}$$

and $$\tilde{c} = c_{00} + z^2/2a \tag{3}$$

The radial coordinate R will be measured from the wall. In the modified coordinate system, the expression for c will read:

$$c = c_{00} + e\cos\varphi + a(1-\cos\theta) \tag{4}$$

and $$\theta \sim O[(c_{00}/a)^{1/2}]$$

clearly $$\sin\theta = z/a \tag{5}$$

and $$\cos\theta = 1 - \frac{z^2}{2a^2} \tag{6}$$

to leading order in $\frac{c_{00}}{a}$.

As is done by Christopherson and Dowson, the axis system will be considered to move upward with the sphere with velocity U. The sphere also has an angular velocity Ω as is shown in figure 1.

The boundary conditions on the velocity are

$$
\begin{aligned}
&v_z = -U \text{ at } R = 0 && \text{(a)} \\
&v_z = \Omega a \cos\varphi \text{ at } R = c && \text{(b)} \\
&v_R = 0 \text{ at } R = 0 && \text{(c)} \\
&v_R = \Omega a \cos\varphi \frac{z}{a} \text{ at } R = c && \text{(d)} \\
&v_\varphi = 0 \text{ at } R = 0 \text{ and } R = c && \text{(e)}
\end{aligned}
\qquad (7)
$$

Since v_z in the gap region must be of order $U(\frac{a}{c_{00}})$ in order to allow the flux $-U\pi a^2$ to pass through an area of $2\pi a c_{00}$, we can rewrite 7(a) as

$$v_z = 0 \text{ at } R = 0 \qquad (7a)$$

to leading order in $\frac{c_{00}}{a}$.

Now some order of magnitude analysis can be done to obtain the right approximation of the flow equations. As we saw, v_z is of order $U\frac{a}{c_{00}}$. Then, from the continuity equation,

$$\frac{\partial v_R}{\partial R} \sim O\left[\frac{\partial v_z}{\partial z}\right]$$

with $\frac{\partial}{\partial R} \sim \frac{1}{c_{00}}$ and $\frac{\partial}{\partial z} \sim \frac{1}{(ac_{00})^{1/2}}$

so that $v_R \sim O\left[U\left(\frac{a}{c_{00}}\right)^{1/2}\right]$

From the momentum equation in the R direction, one can show in the usual way that p does not depend on R to leading order in $\frac{c_{00}}{a}$. When the inertial terms in the equations are small (the condition for which will be discussed later), $\frac{1}{\rho}\frac{\partial p}{\partial z}$ must balance the largest viscous term in the z-momentum equation, $\nu\frac{\partial^2 v_z}{\partial R^2}$, so that

$$p = O\left[\frac{\mu U a^{3/2}}{c_{00}^{5/2}}\right]$$

From the φ-momentum equation, with $\frac{\partial}{\partial\varphi} \sim \frac{e}{c_{00}}$ one then finds in the same way:

$$r_\varphi = O\left[U\left(\frac{a}{c_{00}}\right)^{1/2}\right]$$

To leading order in $\frac{c_{00}}{a}$, the v_z equation then reads

$$\rho\left(v_z\frac{\partial v_z}{\partial z} + v_R\frac{\partial v_z}{\partial R}\right) = -\frac{\partial p}{\partial z} + \frac{\mu\,\partial^2 v_z}{\partial R^2} \qquad (8)$$

The order of magnitude of the inertial terms is $\frac{\rho u^2 a^{3/2}}{c_{00}^{5/2}}$, and the viscous terms have an order of magnitude of $\frac{\mu U a}{c_{00}^3}$. The condition that the inertial terms be small compared to the viscous term is thus expressed by

$$Re^* = \frac{u a^{1/2} c_{00}^{1/2}}{\nu} << 1 \qquad (9)$$

One can assume a regular perturbation expansion in Re*:

$$fnc = fnc_0 + fnc_1 + \ldots\ldots \qquad (10)$$

where fnc stands for any of the velocity components, the pressure, or the quantities Ω and U, and $fnc_1 \sim O[fnc_0 \times Re^*]$. The zeroth order equations then reduce to the Stokes-flow equations, with a solution for the pressure field as obtained by Christopherson and Dowson. In our notation:

$$\frac{\partial p_0}{\partial z} = \frac{6\mu\Omega_0 a\cos\varphi}{c^2} - \frac{12\mu f_0}{c^3} \tag{11}$$

where $$f_0(\varphi) = \int_0^c v_{z_0}\, dR \tag{12}$$

We will proceed to find the forms of the zeroth order velocity components v_{z_0}, v_{φ_0} and v_{R_0}. From the z-momentum equation and (7a) and (7b) one obtains:

$$v_z = \frac{R^2}{c^2}\left(3\Omega_0 a\cos\varphi - \frac{6f_0}{c}\right) + \frac{R}{c}\left(\frac{6f_0}{c} - 2\Omega_0 a\cos\varphi\right) \tag{13}$$

Integration of (11) gives

$$p_0 = -12\mu f_0 g_3(z,\varphi) + 6\mu\Omega_0 a\cos\varphi\; g_2(z,\varphi) + fn(\varphi) \tag{14}$$

with

$$g_2 = \int \frac{dz}{c^2} = \frac{1}{2c_0}\,\frac{z}{c} + \sqrt{\frac{a}{2c_0^3}}\ \mathrm{arctg}\left(\frac{z}{\sqrt{2c_0 a}}\right) \tag{15}$$

and

$$g_3 = \int \frac{dz}{c^3} = \frac{1}{4c_0}\,\frac{z}{c^2} + \frac{3}{8c_0^2}\,\frac{z}{c} + \frac{3}{4}\sqrt{\frac{a}{2c_0^5}}\ \mathrm{arctg}\left(\frac{z}{\sqrt{2c_0 a}}\right) \tag{16}$$

From symmetry arguments one can show that in this linear case $\partial p_0/\partial\varphi = 0$ at $z = 0$, so that the integration constant

$$fn(\varphi) = 0 \tag{17}$$

Then v_{φ_0} follows from

$$\frac{\mu\partial^2 v_{\varphi_0}}{\partial R^2} = \frac{1}{a}\frac{\partial p_0}{\partial\varphi} \quad \text{(18) and (7e)}$$

$$v_{\varphi_0} = \left(3\Omega_0\cos\varphi\frac{\partial g_2}{\partial\varphi} - 3\Omega_0\sin\varphi g_2 - \frac{6f_0}{a}\frac{\partial g_3}{\partial\varphi} - \frac{6g_3}{a}\frac{\partial f_0}{\partial\varphi}\right)(R^2 - Rc) \tag{18}$$

and v_{R_0} from

$$\frac{\partial v_{R_0}}{\partial R} = -\frac{\partial v_{z_0}}{\partial z} - \left(\frac{1}{a}\frac{\partial v_{\varphi_0}}{\partial\varphi}\right) \quad \text{and (7c, d)}$$

$$v_{R_0} = \left(\frac{2\Omega_0 a\cos\varphi}{c^3} - \frac{6f_0}{c^4}\right)\frac{z}{a}R^2 + \left(\frac{6f_0}{c^3} - \frac{\Omega_0 a\cos\varphi}{c^2}\right)\frac{z}{a}R \tag{19}$$

According to Christopherson and Dowson, one can express f_0 and Ω_0 in terms of the eccentricity e and the difference in weight of the fluid displaced and the sphere, m'g. One obtains:

$$f_0 = \frac{m'gV_2c_0}{9\pi\mu a^{5/2}}E(\varphi) \tag{20}$$

and

$$\Omega_0 = \frac{em'gc_{00}^{1/2}}{3\sqrt{2}\,\pi\mu a^{7/2}I_1} \tag{21}$$

with

$$E(\varphi) = \frac{c_0^{3/2}}{\pi} + \frac{ec_{00}^{1/2}\cos\varphi}{I_1} \tag{22}$$

and

$$I_1 = \int_{-\pi}^{\pi}\frac{\cos^2\varphi}{(1+\frac{e}{c_{00}}\cos\varphi)^{1/2}}\,d\varphi \tag{23}$$

Then, one can express the Stokes approximation of the velocity components in the following way:

$$v_{z_0} = k_1(z,\varphi)R + k_2(z,\varphi)R^2 \tag{24}$$

$$v_{\varphi_0} = \ell_1(z,\varphi)R + \ell_2(z,\varphi)R^2 \tag{25}$$

$$v_{R_0} = h_1(z,\varphi)R^2 + h_2(z,\varphi)R^3 \tag{26}$$

with the functions $k_{1,2}$, $\ell_{1,2}$ and $h_{1,2}$ defined by (13) and (18) through (23).

We are now ready to find the first order inertial corrections fnc_1 (10); those for the pressure and the z-velocity component follow from:

$$\frac{-\partial p_1}{\partial z} + \mu \frac{\partial^2 v_{z_1}}{\partial R^2} = \rho\left(v_{z_0}\frac{\partial v_{z_0}}{\partial z} + v_{R0}\frac{\partial v_{z_0}}{\partial R}\right) \tag{27}$$

The boundary conditions on v_{z_1} are:

$$\left.\begin{aligned} v_{z_1} &= 0 \text{ at } R = 0 && \text{(a)} \\ v_{z_1} &= \Omega_1 a\cos\varphi \text{ at } R = c && \text{(b)} \end{aligned}\right\} \tag{28}$$

For (23a) to be true, one must require that

$$\frac{c_{00}}{a} << Re^* << 1$$

After substitution of (24), (25), (26), equation (27) reads

$$\frac{-\partial p_1}{\partial z} + \mu \frac{\partial^2 v_{z_1}}{\partial R^2} = \rho\left\{q_4(z,\varphi)R^4 + q_3(z,\varphi)R^3 + q_2(z,\varphi)R^2\right\} \tag{29}$$

with $q_2(z,\varphi) = h_1k_1 + k_1\frac{\partial k_1}{\partial z}$ (30)

$$q_3(z,\varphi) = h_2 k_1 + 2h_1 k_2 + k_1 \frac{\partial k_2}{\partial z} + k_2 \frac{\partial k_1}{\partial z} \tag{31}$$

$$q_4(z,\varphi) = 2h_2 k_2 + k_2 \frac{\partial k_2}{\partial z} \tag{32}$$

Integrating (29) twice over R, and using (28), v_{z_1} can be written as follows

$$v_{z_1} = \frac{R}{30}(R^5 - c^5)q_4 + \frac{R}{20}(R^4 - c^4)q_3 + \frac{R}{12}(R^3 - c^3)\, q_2 + \frac{\partial p_1}{\partial z}\,\frac{R}{2}(R-c) + \Omega_1 \; a\cos\varphi \,\frac{R}{c} \tag{33}$$

Defining

$$\int_0^c v_{z_1}\, dR = f_1(\varphi) \tag{34}$$

one can obtain an expression for $\frac{\partial p_1}{\partial z}$, similar to the expression for $\frac{\partial p_0}{\partial z}$, (11):

$$\frac{1}{\mu}\,\frac{\partial p_1}{\partial z} = \frac{6\Omega_1 a\cos\varphi}{c^2} - \frac{12 f_1}{c^3} - \frac{1}{\nu}\left(\frac{c^4}{7}\, q_4 - \frac{c^3}{5}\, q_3 - \frac{3c^2}{10}\, q_2\right) \tag{35}$$

To facilitate the integration of this expression, the functions $q_i(z,\varphi)$ can be written in terms of $Q_j(\varphi)\,\frac{z}{c^k}$

$$q_2 = -Q_1(\varphi)\frac{z}{c^3} + 6Q_2(\varphi)\frac{z}{c^4} - 8Q_3(\varphi)\frac{z}{c^5} \tag{36}$$

$$q_3 = 4Q_1(\varphi)\frac{z}{c^4} - 4Q_2(\varphi)\frac{z}{c^5} + 16Q_3(\varphi)\frac{z}{c^6} \tag{37}$$

$$q_4 = -3Q_1(\varphi)\frac{c}{c^5} + 10Q_2(\varphi)\frac{z}{c^6} - 8Q_3(\varphi)\frac{z}{c^7} \tag{38}$$

with

$$Q_1(\varphi) = \frac{e^2 (m'g)^2 c_{00}}{9\pi^2 \mu^2 a^6 I_1^2}\; \cos^2\varphi \tag{39}$$

$$Q_2(\varphi) = \frac{e(m'g)^2 c_{00}^{1/2} c_0}{9\pi^2 \mu^2 a^6 I_1} \cos\varphi \; E(\varphi) \tag{40}$$

$$Q_3(\varphi) = \frac{(m'g)^2 c_0^{\,2}}{9\pi^2 \mu^2 a^6} E^2(\varphi) \tag{41}$$

Then (35) can be written as

$$\frac{1}{\mu}\frac{\partial p_1}{dz} = \frac{6\Omega_1 a\cos\varphi}{c^2} - \frac{12 f_{01}}{c^3} - \frac{1}{\nu}\left(\frac{-5}{70} Q_1 \frac{z}{c} + \frac{17}{7} Q_2 \frac{z}{c^2} + \right.$$
$$\left. + \frac{16}{35} Q_3 \frac{z}{c^3}\right) \tag{42}$$

Integration of this equation however will give an integration constant which may depend on φ, $fn_1(\varphi)$. This function can be determined by evaluating the pressure on the top and bottom of the sphere ($\theta = \pm \frac{\pi}{2}$), and requiring that the pressure there be independent of φ.

In order to accomplish that it is necessary to use the modified coordinate system (with $a\theta$ instead of z) and the Sommerfeld transformation, as was done by Christopherson and Dowson to find the zeroth order pressure at $\theta = \pm \pi/2$. In the modified coordinate system, expression (42) reads

$$\frac{1}{\mu a}\frac{\partial p_1}{\partial\theta} = \frac{6\Omega_1 a\cos\varphi}{c^2} - \frac{12 f_1}{c^3} - \frac{1}{\nu}\left(\frac{1}{14} Q_1 \frac{a\sin\theta}{c} + \right.$$
$$\left. + \frac{17}{7} Q_2 \frac{a\sin\theta}{c^2} - \frac{16}{35} Q_3 \frac{a\sin\theta}{c^3}\right) \tag{43}$$

For the Sommerfeld transformation we define

$$n = \frac{a}{a+c_0} \sim 1 - \frac{c_0}{a} \tag{44}$$

To leading order in $\frac{c_{00}}{a}$ one obtains

$$c = a(1-n\cos\theta) \tag{45}$$

The Sommerfeld transformation defines a new angle γ

$$1-n\cos\theta = \frac{1-n^2}{1+n\cos\gamma} \tag{46}$$

One can use the expressions obtained with this transformation all the way to $\gamma = \pm\pi$ (corresponding to $\theta = \pm\pi/2$) since, to leading order in c_{00}/a, $|\gamma|$ assumes the value of π for every $|\theta| > O[(\frac{c_{00}}{a})^{1/2}]$. This comes down to neglecting changes in the pressure field outside the region $|\theta| \sim O[(\frac{c_{00}}{a})^{1/2}]$ compared to changes in this region.

Then, from (46) one obtains

$$\frac{d\theta}{d\gamma} = \frac{(1-n^2)^{1/2}}{1+n\cos\gamma} \tag{47}$$

and

$$\sin\theta = \frac{(1-n^2)^{1/2}\sin\gamma}{1+n\cos\gamma} \tag{48}$$

Equation (43) transforms into

$$\frac{\partial p_1}{\partial\gamma} = \frac{6\mu\Omega_1\cos\alpha\,(1+n\cos\gamma)}{(1-n^2)^{3/2}} - \frac{12\mu f_1(1+n\cos\gamma)^2}{a^2(1-n^2)^{5/2}} + $$
$$-\frac{\mu}{\nu}\left(\frac{1}{14}\frac{Q_1 a\sin\gamma}{1+n\cos\gamma} + \frac{17}{7}\frac{Q_2\sin\gamma}{1-n^2} + \frac{16}{35a}\frac{Q_3\sin\gamma\,(1+n\cos\gamma)}{(1-n^2)^2}\right) \tag{49}$$

which can be integrated to give

$$p_1 = \frac{6\mu\Omega_1 \cos\varphi(\gamma+\sin\gamma)}{(1-n^2)^{3/2}} - \frac{3\mu f_1(4\gamma+8n\sin\gamma+2n^2\gamma+n^2\sin 2\gamma)}{a^2(1-n^2)^{5/2}}$$

$$+ \frac{\mu}{\nu}\left\{\frac{1}{14}\,\frac{Q_1 a}{n}\,\ell n\left(\frac{1+n\cos\gamma}{1-n}\right) + \frac{17}{7}\,\frac{Q_2\cos\gamma}{1-n^2} + \right. \tag{50}$$

$$\left. + \frac{16}{35a}\,\frac{Q_3(\cos\gamma+\frac{n}{4}\cos 2\gamma)}{(1-n^2)^2}\right\} + fn_1(\varphi)$$

The first two terms of this expression are odd in γ, all the other terms are even. Hence, to compute the pressure difference $(p_1)_{\gamma=-\pi} - (p_1)_{\gamma=\pi}$, only the first two terms are of importance, and the same applies to the torque resulting from the viscous shear stress. Using those two terms then, one can proceed to compute the values of Ω_1 and U_1, similar to the way in which Christopherson and Dowson computed Ω_0 and U_0. This determines f_1 in such a way that the contribution to the pressure from the first two terms is independent of φ at $\gamma = \pm\pi$.

Therefore, $fn_1(\varphi)$ is determined by the condition that the part of p_1 even in γ, be also independent of φ at $\gamma = \pm\pi$. Hence

$$fn_1(\varphi) = \frac{\mu}{\nu}\left\{\frac{17}{7}\,\frac{Q_2}{1-n^2} + \frac{16}{35a}\,Q_3\,\frac{1-\frac{n}{4}}{(1-n^2)^2}\right\} \tag{51}$$

Defining the part of p_1 that is even in γ as P_1, we then obtain

$$P_1 = \frac{\mu}{\nu}\left\{\frac{1}{14}\,\frac{Q_1 a}{n}\,\ell n\left(\frac{1+n\cos\gamma}{1-n}\right) + \frac{17}{7}\,Q_2\,\frac{\cos\gamma+1}{1-n^2} + \right.$$

$$\left. + \frac{16}{35a}\,Q_3\,\frac{\cos\gamma+1+\frac{n}{4}(\cos 2\gamma-1)}{(1-n^2)^2}\right\} \tag{52}$$

Clearly, only this part of the pressure will contribute to any force perpendicular to the velocity U. Such a force will be expressed by

$$F = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \{P_1 \cos\varphi a d\varphi\} \cos\theta a d\theta \tag{53}$$

and will be destabilizing (increasing the eccentricity ratio) if $F > 0$.

Now

$$\cos\theta d\theta = \frac{(1-n^2)^{1/2}(\cos\gamma+n)}{(1+n\cos\gamma)^2} d\gamma$$

and the integration limits on γ are $-\pi$ and π.

It is not possible to perform the γ integration analytically, but to get an idea about the magnitude of the force, we can transform expression (52) back to the z-variable and integrate over z in the region of interest, say for z from $-(c_{00}a)^{1/2}$ to $(c_{00}a)^{1/2}$. The Sommerfeld transformation formula (46) can be written as

$$\frac{1-n^2}{1+n\cos\gamma} = \frac{c}{a} \tag{54}$$

Furthermore, to leading order in $\frac{c_{00}}{a}$,

$$1-n^2 = \frac{2c_0}{a} \tag{55}$$

Then

$$\frac{1+n\cos\gamma}{1-n} = \frac{a}{2c} \tag{56}$$

$$\frac{\cos\gamma+1}{1-n^2} = \frac{n\cos\gamma+1+O(\frac{c_{00}}{a})}{1-n^2}$$

or

$$\frac{\cos\gamma+1}{1-n^2} = \frac{a}{c} \tag{57}$$

and

$$\cos\gamma + 1 + \frac{n}{4}(\cos 2\gamma - 1) = \frac{n}{2}\cos^2\gamma + \cos\gamma + \frac{2-n}{2}$$

$$= \frac{1}{2n}\left(n^2\cos^2\gamma + 2n\cos\gamma + 1 + O\left(\frac{c_{00}}{a}\right)\right)$$

so that

$$\frac{\cos\gamma + 1 + \frac{n}{4}(\cos 2\gamma - 1)}{(1-n^2)^2} = \frac{1}{2n}\frac{a^2}{c^2} \tag{58}$$

Expression (52) then transforms into

$$P_1 = \frac{\mu}{\nu}\left\{\frac{-1}{14} Q_1 \; a\,\ell n\left(\frac{2c}{a}\right) + \frac{17}{7} Q_2 \frac{a}{c} + \frac{8}{35} Q_3 \frac{a}{c^2}\right\} \tag{59}$$

to leading order in c_{00}/a, and

$$F_i \sim 2 \int_0^{(c_{00}a)^{1/2}} \left\{\int_0^{2\pi} P_1 \cos\varphi \, a d\varphi\right\} dz \tag{60}$$

Still, the φ integration cannot be done analytically, except asymptotically, for $e/c_{00} << 1$. We will proceed to evaluate F_i for that case, to leading order in e/c_{00}.

Straightforward manipulation shows that

$$I_1 = \int_{-\pi}^{\pi} \frac{\cos^2\varphi}{\left(1 + \frac{e}{c_{00}}\cos\varphi\right)^{1/2}} d\varphi = \pi + O[e/c_{00}] \tag{61}$$

$$Q_1 = O[(e/c_{00})^2] \tag{62}$$

$$\frac{Q_2}{c} = \frac{e(m'g)^2 c_{00}{}^3 \cos\varphi}{9\pi^4\mu^2 a^6 (c_{00} + z^2/2a)}\left\{1 + O[e/c_{00}]\right\} \tag{63}$$

$$\frac{Q_3}{c} = \frac{(m'g)^2 c_{00}^{\,5}\left(1+7\,\dfrac{e}{c_{00}}\cos\varphi - 2\,\dfrac{e}{c_{00}+z^2/2a}\cos\varphi\right)}{9\pi^4\mu^2 a^6 (c_{00}+z^2/2a)^2}\left\{1+O[e/c_{00}]\right\} \tag{64}$$

When one computes F_i to leading order in e/c_{00}, clearly only the last two terms in the right-hand side of (59) will contribute. Performing the φ integration, one finds

$$F_i = 2\,\frac{\mu}{\nu}\,\frac{e(m'g)^2 c_{00}^{\,3}}{9\pi^2 a^5 \mu^2}\left\{\frac{17}{7}\int_0^{(c_{00}a)^{1/2}} \frac{dz}{c_{00}+\dfrac{z^2}{2a}} + \frac{8c_{00}}{5}\int_0^{(c_{00}a)^{1/2}} \frac{dz}{\left(c_{00}+\dfrac{z^2}{2a}\right)^2} + \frac{-16c_{00}^{\,2}}{35}\int_0^{(c_{00}a)^{1/2}} \frac{dz}{\left(c_{00}+\dfrac{z^2}{2a}\right)^3}\right\} \tag{65}$$

or:

$$F_i = \rho\,\frac{e(m'g)^2 c_{00}^{\,3}}{9\pi^2 a^5 \mu^2}\left[\frac{232}{315}\left(\frac{a}{c_{00}}\right)^{1/2} + \frac{214\sqrt{2}}{35}\left(\frac{a}{c_{00}}\right)^{1/2}\operatorname{arctg}\left(\frac{1}{\sqrt{2}}\right)\right] \tag{66}$$

to leading order in e/c_{00}.

One can conclude that for small e/c_{00} the force due to the first order inertial effects will always tend to increase the eccentricity ratio.

For e/c_{00} of order one, the integration has been performed numerically, and the results are listed below.

$$F = \frac{\rho (m'g)^2 c_{00}^{\ 3}}{9\pi^2 \mu^2 a^4} \cdot 2 \int_0^{(c_{00}a)^{\frac{1}{2}}} \left[\int_0^{2\pi} \left\{ -\frac{1}{14} \frac{(e/c_{00})^2}{I_1^{\ 2}} \ell n \left(1 + \frac{e}{c_{00}} \cos\varphi + \right. \right. \right.$$

$$\left. + \frac{z^2}{2ac_{00}} \right) \cos^3 \varphi +$$

$$+ \frac{17}{7} \frac{e/c_{00}}{I_1} \frac{\left(1 + \frac{e}{c_{00}} \cos\varphi\right) \left\{ \frac{(1 + \frac{e}{c_{00}} \cos\varphi)^{3/2}}{\pi} + \frac{\frac{e}{c_{00}} \cos\varphi}{I_1} \right\} }{1 + \frac{e}{c_{00}} \cos\varphi + \frac{z^2}{2ac_{00}}} \cos^2\varphi +$$

$$\left. \left. + \frac{8}{35} \frac{\left(1 + \frac{e}{c_{00}} \cos\varphi\right)^2 \left\{ \frac{(1 + \frac{e}{c_{00}} \cos\varphi)^{3/2}}{\pi} + \frac{\frac{e}{c_{00}} \cos\varphi}{I_1} \right\}^2 }{\left(1 + \frac{e}{c_{00}} \cos \varphi + \frac{z^2}{2ac_{00}}\right)^2} \cos\varphi \right\} d\varphi \right] dz \tag{67}$$

$$\text{or:} \quad F_i = \frac{\rho (m'g)^2 c_{00}^{\ 3}}{9\pi^2 \mu^2 a^4} \tilde{F} \tag{68}$$

For $a = 2$, $c_{00} = 0.1$:

e/c_{00}	$\tilde{F}$	e/c_{00}	$\tilde{F}$
0.10	0.086	0.91	0.744
0.20	0.174	0.92	0.741
0.30	0.263	0.93	0.737
0.40	0.354	0.94	0.731
0.50	0.447	0.95	0.722
0.60	0.540	0.96	0.709
0.70	0.629	0.97	0.691
0.80	0.705	0.98	0.663
0.90	0.744	0.99	0.617

e/c_{00}	$\tilde{F}$
0.991	0.610
0.992	0.603
0.993	0.595
0.994	0.588
0.995	0.582
0.996	0.563
0.997	0.548
0.998	0.529
0.999	0.503

The computed values of $\tilde{F}$ are plotted in graph 1.

One can express the order of magnitude of F in terms of ρ, μ, U, c_{00} and a. The balance of buoyancy and pressure forces requires that

$$m'g = O[p_0 a^2]$$

so

$$\left(\frac{m'g}{\mu}\right)^2 = O\left[\frac{U^2 a^7}{c_{00}^5}\right]$$

$$F_i = O\left[\rho\left(\frac{m'g}{\mu}\right)^2 \frac{c_{00}^3}{a^4} (c_{00}a)^{1/2}\right]$$

$$F_i = O\left[\rho U^2 a^2 \left(\frac{a}{c_{00}}\right)^{3/2}\right] \tag{69}$$

From expression (33) for v_{z_1} one can also obtain the first order inertial correction of the viscous shear force on the sphere, and its component F_ν perpendicular to the velocity U. Only the part of $\partial v_{z_1}/\partial R$ that is odd in z will contribute to F_ν and after differentiating (33) with respect to R and substituting (36), (37) and (38) F_ν

will be expressed as

$$F_\nu = \int_{-(c_{00}a)^{\frac{1}{2}}}^{+(c_{00}a)^{\frac{1}{2}}} \left\{ \int_0^{2\pi} \mu \frac{\partial v_{z_1}}{\partial R} \cos\varphi a d\varphi \right\} \frac{z}{a} dz$$

$$= \frac{2\mu}{\nu} \int_0^{(c_{00}a)^{\frac{1}{2}}} \left\{ \int_0^{2\pi} \left(\frac{Q_1}{20} z + \frac{71}{30} Q_2 \frac{z}{c} - \frac{2}{15} Q_3 \frac{z}{c^3} \right) \cos\varphi a d\varphi \right\} \frac{z}{a} dz \tag{70}$$

F_ν will tend to increase $|e|$ if positive. However, it can readily be shown that

$$\frac{F_\nu}{F_i} = O\left[\left(\frac{c_{00}}{a}\right)^{\frac{1}{2}} \right] \tag{71}$$

so that to leading order in c_{00}/a the sign of the pressure force F_i determines the question of whether the first order inertial forces act to increase or to decrease $|e|$.

III. Approximate Sideways Motion of the Ball Under Influence of the Inertial Force

If at any time during the vertical motion the ball takes up an eccentric position in the tube, the inertial force F_i, calculated in Chapter II, will act to accelerate the ball in a horizontal direction, such that the eccentricity ratio is increased. If u is the horizontal velocity of the ball obtained by the action of F_i, there will be a force F_p, opposing the horizontal motion which for small enough u will be linear in u. The condition to be satisfied for F_p to be linear in u is that the leading order equations governing the flow be the Stokes equations. In that case F_p can be obtained from the pressure field around the ball caused by pure horizontal motion of the ball (figure 2).

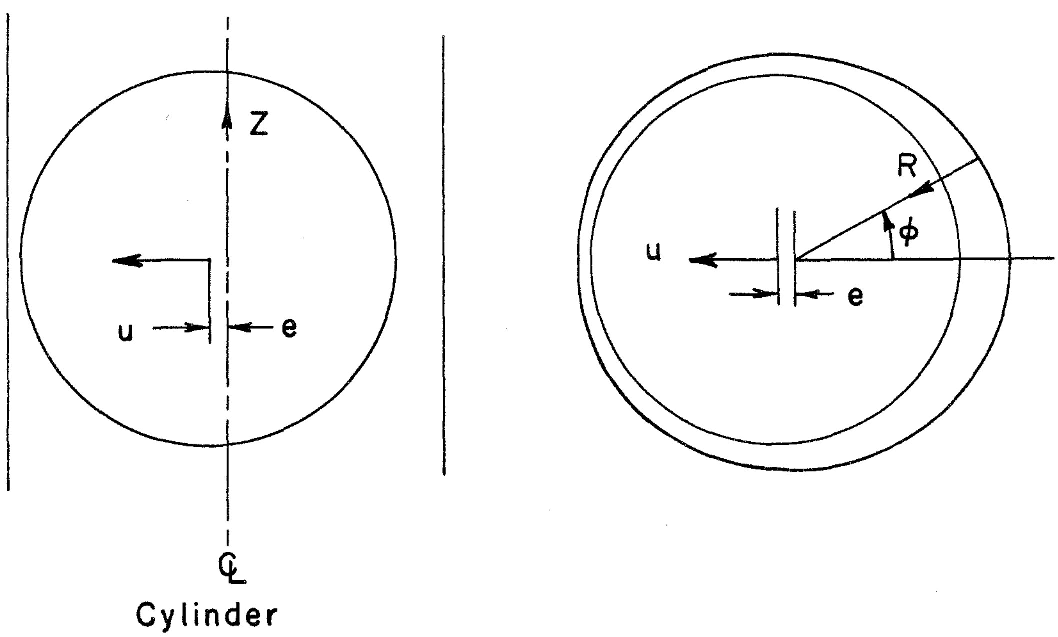

FIG. 2 SIDEWAYS MOTION OF THE SPHERE

The boundary conditions on the velocity are

$$\left.\begin{aligned} v_R &= u\cos\varphi \text{ at } R = c && \text{(a)} \\ v_R &= 0 \quad \text{at } R = 0 && \text{(b)} \end{aligned}\right\} \qquad (72)$$

$$\left.\begin{aligned} v_\varphi &= u\sin\varphi \text{ at } R = c && \text{(a)} \\ v_\varphi &= 0 \quad \text{at } R = 0 && \text{(b)} \end{aligned}\right\} \qquad (73)$$

$$v_z = 0 \text{ at } R = c \text{ and } R = 0 \qquad (74)$$

where the flow region under consideration is again

$$z = O[(c_{00}a)^{\frac{1}{2}}]$$

The typical Reynolds number of this motion is

$$Re = O[v_z \frac{\partial v_z}{\partial z}] \;/\; O[\nu \frac{\partial^2 v_z}{\partial R^2}]$$

or $$Re = O[\frac{v_z c_{00}^{3/2}}{\nu a^{\frac{1}{2}}}]$$

From the continuity equation it will be shown that

$$v_z = O\left[\frac{u a^{\frac{1}{2}}}{c_{00}^{\frac{1}{2}}}\right]$$

so that

$$Re = \frac{u c_{00}}{\nu}$$

If $$Re = \frac{u c_{00}}{\nu} \ll 1 \quad \text{and } Re^* = \frac{U(c_{00}a)^{\frac{1}{2}}}{\nu} \ll 1$$

F_p will be linear in u and can be calculated from the Stokes equations with boundary conditions as stated above, since the Strouhal number of the motion is of the same order as the Reynolds number.

To leading order in c_{00}/a, the flow equations then are

$$\frac{\partial v_R}{\partial R} + \frac{1}{a}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} = 0 \tag{75}$$

$$\frac{1}{a}\frac{\partial p}{\partial \varphi} = \mu \frac{\partial^2 v_\varphi}{\partial R^2} \tag{76}$$

$$\frac{\partial p}{\partial z} = \mu \frac{\partial^2 v_z}{\partial R^2} \tag{77}$$

$$\frac{\partial p}{\partial R} = \mu \frac{\partial^2 v_R}{\partial R^2} \tag{78}$$

with boundary conditions (72) through (74).

With $R = O[c_{00}]$, $z = O[(c_{00}a)^{\frac{1}{2}}]$ and the boundary conditions (72) on v_R, one obtains easily from (75)

$$v_z = O\left[\frac{ua^{\frac{1}{2}}}{c_{00}^{\frac{1}{2}}}\right] \tag{79}$$

Then, from (77) it follows that

$$\frac{\partial p}{\partial z} = O\left[\frac{\mu u a^{\frac{1}{2}}}{c_{00}^{5/2}}\right] \tag{80}$$

and from (87)

$$\frac{\partial p}{\partial R} = O\left[\frac{\mu u}{c_{00}^{2}}\right] \tag{81}$$

Hence the pressure does not depend on R to leading order in c_{00}/a. The leading order term of p will depend on φ at $z = 0$, but must be independent of φ at the top and bottom of the sphere and must consequently be a function of both z and φ. With $\partial/\partial\varphi$ of order e/c_{00}, it can then be shown from (80) that

$$\frac{1}{a}\frac{\partial p}{\partial \varphi} = O\left[\frac{\mu u}{c_{00}^{2}}\right] \tag{82}$$

and the φ-momentum equation (76) then gives

$$v_{\varphi} = O[u] \tag{83}$$

To leading order in c_{00}/a, the continuity equation (75) reduces to

$$\frac{\partial v_R}{\partial R} + \frac{\partial v_z}{\partial z} = 0 \tag{84}$$

which can be integrated over R from R = 0 to R = c to yield

$$\int_0^c \frac{\partial v_z}{\partial z}\, dR = -u\cos\varphi \tag{85}$$

using (72).

Physically, the integrated form of the continuity equation has the following significance: consider a volume element extending from the ball to the tubewall, with width $a d\varphi$ and height dz (figure 3)

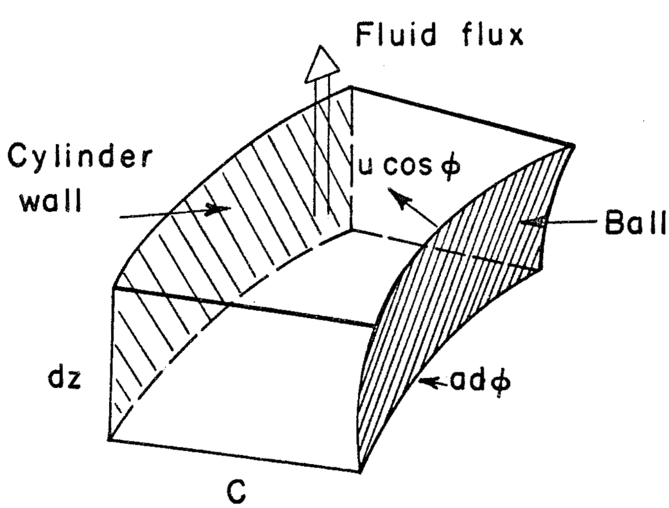

FIG. 3

When the ball moves closer to the wall, fluid has to be spilled out of the contracting volume, and since $\frac{1}{a}\frac{\partial v_\varphi}{\partial \varphi} = O[\frac{c_{00}}{a}\frac{\partial v_z}{\partial z}]$ all the fluid will be spilled in the z direction, to leading order in c_{00}/a.

Equation (77) can now readily be integrated to give v_z. Using the boundary conditions, (74):

$$v_z = \frac{1}{2\mu}\frac{\partial p}{\partial z}(R^2 - Rc) \tag{86}$$

Substituting this in (85) and again using the fact that $v_z = 0$ at $R = c$, one obtains after performing the integration

$$\frac{1}{12\mu}\frac{\partial}{\partial z}(\frac{\partial p}{\partial z}c^3) = u\cos\varphi \tag{87}$$

or

$$\frac{\partial p}{\partial z} = \frac{12\mu u\cos\varphi z}{c^3} \tag{88}$$

To find the integration constant $C(\varphi)$ that results from integrating (88) over z, one has to use the same method as employed for integration of equation (42) (see Chapter II). In spherical coordinates, (88) reads

$$\frac{1}{a}\frac{\partial p}{\partial \theta} = \frac{12\mu u\cos\varphi\; a\sin\theta}{c^3} \tag{89}$$

and using the Sommerfeld transformation as defined in (44) through (46)

$$\frac{dp}{d\gamma} = \frac{12\mu u\cos\varphi}{a}\frac{\sin\gamma(1+n\cos\gamma)}{(1-n^2)^2} \tag{90}$$

which can be integrated to give

$$p = \frac{-12\mu u\cos\varphi}{a} \cdot \frac{\cos\gamma + \frac{n}{4}\cos 2\gamma}{(1-n^2)^2} + C(\varphi) \tag{91}$$

Obviously p is an even function of γ, and the integration constant $C(\varphi)$ is determined by the condition that p does not depend on φ at $\gamma = \pm\,\pi$. Hence

$$C(\varphi) = \frac{12\mu u\cos\varphi}{a} \cdot \frac{\frac{n}{4} - 1}{(1-n^2)^2}$$

so that

$$p = \frac{-12\mu u\cos\varphi}{a} \cdot \frac{(\cos\gamma+1) + \frac{n}{4}(\cos 2\gamma - 1)}{(1-n^2)^2}$$

and using expression (58)

$$p = \frac{-6\mu u a\cos\varphi}{c^2} = \frac{-6\mu u a\cos\varphi}{\left(c_{00} + \frac{z^2}{2a} + e\cos\varphi\right)^2} = \frac{-6\mu u a\cos\varphi}{(\tilde{c} + e\cos\varphi)^2} \tag{92}$$

to leading order in c_{00}/a.

The force resulting from this pressure distribution will approximately be

$$F_p = -6\mu u a^2 \int_{-(c_{00}a)^{\frac{1}{2}}}^{(c_{00}a)^{\frac{1}{2}}} \int_{-\pi}^{\pi} \frac{\cos^2\varphi\, d\varphi}{(\tilde{c} + e\cos\varphi)^2}\, dz$$

or

$$F_p = -12\mu u a^2 \int_0^{(c_{00}a)^{\frac{1}{2}}} \left[\frac{dz}{\tilde{c}^2} \int_{-\pi}^{\pi} \frac{\cos^2\varphi\, d\varphi}{(1 + \frac{e}{\tilde{c}}\cos\varphi)^2}\right] \tag{93}$$

The φ integration can be performed to give

$$\int_{-\pi}^{\pi} \frac{\cos^2\varphi}{(1+\frac{e}{\tilde{c}}\cos\varphi)^2}\, d\varphi = \frac{\tilde{c}^2}{e^2(1-e^2/\tilde{c}^2)} \left\{ \frac{-\frac{e}{\tilde{c}}\sin\varphi}{1+\frac{e}{\tilde{c}}\cos\varphi} + (1-\frac{e^2}{\tilde{c}})\varphi + (2\frac{e^2}{\tilde{c}^2}-1)D \right\} \Bigg|_{-\pi}^{\pi}$$

with

$$D = \frac{2}{\sqrt{1-\frac{e^2}{\tilde{c}^2}}} \operatorname{arctg}\left[\left(\frac{1-\frac{e}{\tilde{c}}}{1+\frac{e}{\tilde{c}}}\right)^{\frac{1}{2}} \operatorname{tg}\frac{\varphi}{2}\right]$$

or

$$\int_{-\pi}^{\pi} \frac{\cos^2\varphi\, d\varphi}{(1+\frac{e}{\tilde{c}}\cos\varphi)^2} = 2\pi\left[\frac{\tilde{c}^2}{e^2} + \frac{2-\frac{\tilde{c}^2}{e^2}}{(1-\frac{e^2}{\tilde{c}^2})^{3/2}}\right] \tag{94}$$

and

$$F_p = -24\pi\mu u a^2 \int_0^{(c_{00}a)^{\frac{1}{2}}} \left[\frac{1}{e^2} + \frac{2}{\tilde{c}^2} - \frac{1}{e^2}\, \frac{1}{(1-e^2/\tilde{c}^2)^{3/2}}\right] dz \tag{95}$$

The z-integration cannot be performed analytically, but one can get the order of magnitude of F_p by setting $\tilde{c} = c_{00}$.

Then:

$$F_p = \frac{-24\pi\mu u a^{5/2}}{c_{00}^{3/2}} \left[\frac{c_{00}^2}{e^2} + \left(2-\frac{c_{00}^2}{c^2}\right)\frac{1}{(1-e^2/c_{00}^2)^{3/2}}\right] \tag{96}$$

F_p is negative, and hence opposing the sideways motion for all e/c_{00},

e. g. $\quad F_p = \dfrac{-12\pi\mu u a^{5/2}}{c_{00}^{3/2}} \quad$ for $\quad e/c_{00} = 0$

and F_p tends to $-\infty$ when e/c_{00} tends to one, and one can write

$$F_p = \frac{-24\pi\mu u a^{5/2}}{c_{00}^{3/2}} A_p(e/c_{00}) \tag{97}$$

The inertial force was found in Chapter II to be

$$F_i = \frac{\rho(m'g)^2 c_{00}^3}{9\pi^2 \mu^2 a^4} \tilde{F}(e/c_{00})$$

where $\tilde{F}$ represents an integral over z and φ, and has the dimension of length. According to Christopherson and Dowson, one can express $m'g/\mu$ in terms of the vertical velocity U:

$$\frac{m'g}{\mu} = \frac{U 9\pi^2 a^{7/2}}{\sqrt{2}\, c_{00}^{5/2}} \frac{1}{\frac{I_2}{\pi} + \frac{e^2 \pi}{c_{00}^2 I_1}} \tag{98}$$

with I_1 as defined in (23) and

$$I_2 = \int_{-\pi}^{\pi} (1 + \frac{e}{c_{00}} \cos\varphi)^{5/2} \, d\varphi \tag{99}$$

Now define

$$A_i(e/c_{00}) = \frac{\tilde{F}}{(c_{00} a)^{\frac{1}{2}}} \left(\frac{1}{\frac{I_2}{\pi} + \frac{e^2 \pi}{c_{00}^2 I_1}} \right)^2 \tag{100}$$

then F_i can be written as

$$F_i = \frac{9\pi^2}{2} \frac{\rho U^2 a^{7/2}}{c_{00}^{3/2}} A_i(e/c_{00}) \tag{101}$$

The equation for the horizontal motion of the ball under the action of F_i and F_p will then be

$$m_b \frac{du}{dt} = F_i - F_p$$

or

$$m_b \frac{du}{dt} = \frac{9\pi^2}{2} \frac{\rho U^2 a^{7/2}}{c_{00}^{3/2}} A_i(e/c_{00}) - \frac{24\pi\mu u a^{5/2}}{c_{00}^{3/2}} A_p(e/c_{00}) \quad (102)$$

Since A_i and A_p are nonlinear functions of e, and $u = de/dt$, equation (102) is in fact a nonlinear second order differential equation in e(t), which cannot be solved exactly. However, it is possible to get the order of magnitude of u by solving (102) as if A_i and A_p were constants. The solution of the then resulting linear equation can easily be found to be

$$u = \frac{A_i}{A_p} \frac{9\pi}{48} \frac{\rho U^2 a}{\mu} \left\{ 1 - \exp\left(- \frac{A_p}{m_b} \frac{24\pi\mu a^{5/2}}{c_{00}^{3/2}} t \right) \right\} \quad (103)$$

if the ball is assumed to be at rest at $t = 0$.

When the value of e/c_{00} tends to one, the value of A_i remains rather small (about 0.1), but as we saw the value of A_p will tend to infinity. It follows that the horizontal velocity of the ball will tend to zero when the eccentricity ratio tends to unity, which means that the ball will be virtually locked in place close to the wall.

To find the order of magnitude of time t needed for the ball to travel horizontally over a distance of order c_{00}, one can take the order of magnitude of u to be $\frac{A_i}{A_p} \frac{9\pi}{48} \frac{\rho U^2 c_{00}}{\mu}$. Typical values of A_i and A_p can be taken as

$$A_i = 0.06 \quad \text{and} \quad A_p = 1$$

(approximate values at $e/c_{00} = 0.5$). So t follows from

$$t \times 0.04 \frac{\rho U^2}{\mu} = \frac{c_{00}}{a}$$

or

$$t \sim \frac{c_{00}}{a} \frac{25\nu}{U^2} \tag{104}$$

In that time, the ball travels vertically over a distance s

$$s = \frac{25\nu}{U} \frac{c_{00}}{a}$$

If s/a is of order unity, one can conclude that the ball reaches the wall while it travels vertically over a distance of order a, and then stays close to the wall by virtue of an infinite Stokes drag force in a horizontal direction. However

$$\frac{s}{a} = \frac{\nu}{Ua} \; 25 \; \frac{c_{00}}{a} \tag{105}$$

is of order 50 for most of the experiments performed by Christopherson and Dowson.

When the Reynolds number increases, the character of the flow will change, but it seems reasonable to assume that the inertial force keeps growing till it reaches the potential flow limit, and that s/a decreases when Re* increases. At the value of Re* where in the experiments the ball starts bouncing against the tube wall, the Reynolds number is too high to make s/a from expression (105) a fair measure of the distance travelled in a vertical direction during the time needed for the ball to travel a distance of order c_{00} in a horizontal direction. There is a Reynolds number associated with s/a, namely

$$\mathrm{Re}' = \frac{Ua}{\nu}\frac{a}{c_{00}} = \mathrm{Re}^*\left(\frac{a}{c_{00}}\right)^{3/2}$$

but its values at the stage where experimentally the bouncing instability occurs are spread over a range of 10^2.

IV. Potential Flow Solution

If the Reynolds number as defined in Chapter II is very high one can assume that the vorticity of the flow is confined in boundary layers on the sphere and cylinder wall, thin compared to the gap width. In that case, the main flow in the gap region may be considered to be potential flow, and the pressure distribution around the sphere can be derived from the potential flow solution.

The condition for potential flow is that the boundary layer thickness δ is small compared to c_{00}. So

$$\frac{\delta}{c_{00}} \sim \frac{\nu}{U(ac_{00})^{\frac{1}{2}}} << 1$$

or

$$\frac{Ua}{\nu} >> \left(\frac{a}{c_{00}}\right)^{\frac{1}{2}} \tag{106}$$

The problem then reduces to

$$\nabla^2\phi = 0 \tag{107}$$

where

$$\underline{v} = \underline{\nabla}\,\phi \tag{108}$$

with the boundary condition

$$\phi_n = 0 \text{ at } R = 0, \; R = c$$

or

$$\left.\begin{aligned} \phi_R &= 0 \text{ at } R = 0 \\ \phi_R &= \frac{z}{a}\,\phi_2 \text{ at } R = c \end{aligned}\right\} \tag{109}$$

From the continuity equation (107) it follows that

$$\frac{\partial\phi_z}{\partial z} \quad \text{and} \quad \frac{\partial\phi_R}{\partial R}$$

are of the same order. Hence, like in the low Reynolds-number case

$$\phi_z = O\left[\phi_R\left(\frac{a}{c_{00}}\right)^{\frac{1}{2}}\right] \tag{110}$$

However, to satisfy irrotationality we must have

$$\frac{\partial\phi_z}{\partial R} = \frac{\partial\phi_R}{\partial z} \tag{111}$$

or

$$\phi_z = O\left[\phi_R\left(\frac{c_{00}}{a}\right)^{\frac{1}{2}}\right]$$

unless ϕ_z is not a function of R to first order in $(\frac{c_{00}}{a})$, in which case (111) can be satisfied by the next term in an expansion in $\frac{c_{00}}{a}$ for ϕ_z. Hence, restricting the analysis to the terms of leading order in c_{00}/a, like in the low Reynolds number case,

$$\frac{\partial}{\partial R}(\phi_z) = 0 \tag{112}$$

must be satisfied by a valid solution of the potential equation.

We will now proceed to find such a solution, in the form of a Fourier series, the coefficients of which are series as well. The boundary conditions (109) are clearly satisfied by a "similarity solution":

$$\phi_R = \frac{R}{c}\,\frac{z}{a}\,\phi_z \tag{113}$$

Now (112) and (113) imply

$$\frac{\partial}{\partial R}\left(\frac{ca}{Rz}\phi_R\right) = \frac{ca}{Rz}\phi_{RR} - \frac{ca}{R^2 z}\phi_R = 0 \qquad (114)$$

The potential equation (107) reads

$$\phi_{RR} - \frac{1}{d-R}\phi_R + \frac{1}{(d-R)^2}\phi_{\varphi\varphi} + \phi_{zz} = 0$$

or to leading order in c_{00}/a

$$\phi_{RR} + \phi_{zz} + \frac{1}{a^2}\phi_{\varphi\varphi} = 0 \qquad (115)$$

From (113), (114), (115) one then finds

$$\frac{1}{R}\phi_z + \frac{ca}{Rz}\phi_{zz} + \frac{c}{Rza}\phi_{\varphi\varphi} = 0 \qquad (116)$$

The angular velocity component ϕ_φ must be symmetrical and periodical in φ, so that one can write ϕ_φ in the form of a Fourier sinus series

$$\phi_\varphi = \sum_{n=1}^{\infty} g_n(R,z)\sin n\varphi \qquad (117)$$

and consequently

$$\phi = -\sum_{n=1}^{\infty} g_n(R,z)\frac{\cos n\varphi}{n} + g_0(R,z). \qquad (118)$$

Substituting (118) into (116) one obtains

$$\frac{1}{R}\left(-\sum_{n=1}^{\infty}\frac{\partial g_n}{\partial z}\frac{\cos n\varphi}{n} + \frac{\partial g_0}{\partial z}\right) + \frac{(\tilde{c}+e\cos\varphi)a}{Rz}\left(-\sum_{n=1}^{\infty}\frac{\partial^2 g_n}{\partial z^2}\frac{\cos n\varphi}{n} + \right.$$

$$\left. + \frac{\partial^2 g_0}{\partial z^2}\right) + \frac{\tilde{c}+e\cos\varphi}{Rza}\sum_{n=1}^{\infty} g_n n\cos n\varphi = 0 \qquad (119)$$

It is now clear that the last term in (119) is of order c_{00}/a times the other terms and hence can be neglected. With

$$\cos\varphi\cos n\varphi = \frac{1}{2}\left\{\cos(n+1)\varphi + \cos(n-1)\varphi\right\}$$

Equation (119) then reads

$$-\sum_{n=1}^{\infty}\frac{\partial g_n}{\partial z}\frac{\cos n\varphi}{n} + \frac{\partial g_0}{\partial z} - \frac{\tilde{c}a}{z}\sum_{n=1}^{\infty}\frac{\partial^2 g_n}{\partial z^2}\frac{\cos n\varphi}{n} +$$

$$+\frac{\tilde{c}a}{z}\frac{\partial^2 g_0}{\partial z^2} - \frac{ea}{z}\sum\frac{\partial^2 g_n}{\partial z^2}\frac{1}{2n}\left\{\cos(n+1)\varphi + \cos(n-1)\varphi\right\} +$$

$$+\frac{ea}{z}\frac{\partial^2 g_0}{\partial z^2}\cos\varphi = 0 \qquad (120)$$

Equating the coefficients of $\cos n\varphi$ to zero we obtain for $n = 0$

$$\frac{\partial g_0}{\partial z} + \frac{\tilde{c}a}{z}\frac{\partial^2 g_0}{\partial z^2} - \frac{ea}{2z}\frac{\partial^2 g_1}{\partial z^2} = 0 \qquad (121)$$

For $n = 1$

$$\frac{\partial g_1}{\partial z} + \frac{\tilde{c}a}{z}\frac{\partial^2 g_1}{\partial z^2} = \frac{e}{\tilde{c}}\frac{\tilde{c}a}{z}\frac{\partial^2 g_0}{\partial z^2} - \frac{1}{4}\frac{e}{\tilde{c}}\left[\frac{\tilde{c}a}{z}\frac{\partial^2 g_2}{\partial z^2}\right] \qquad (122)$$

and the general equation, for $n > 1$

$$\frac{\partial g_n}{\partial z} + \frac{\tilde{c}a}{z}\frac{\partial^2 g_n}{\partial z^2} = -\frac{e}{\tilde{c}}\left[\frac{\tilde{c}a}{z}\frac{n}{2(n-1)}\frac{\partial^2 g_{n-1}}{\partial z^2} + \frac{\tilde{c}a}{z}\frac{n}{2(n+1)}\frac{\partial^2 g_{n+1}}{\partial z^2}\right]$$

(123)

The conditions under which this system is to be solved are:

(a) $\phi_z \neq$ function (R)

(b) the flux condition $\int_0^{2\pi}\phi_z\tilde{c}a\,d\varphi = -U\pi a^2$ (124)

(c) $\phi_\varphi = 0$ if $e = 0$, so

$$g_n = 0 \text{ if } e = 0 \text{ for all } n \geqslant 1 \tag{125}$$

The equations (121) through (123) form a system of an infinite number of equations with an infinite number of unknown functions g_n. The system can be solved, as will be demonstrated, under the assumption that

$$g_n = O\left[\frac{e}{c_{00}} \; g_{n-1}\right] \tag{126}$$

Condition (125) implies that g_1 must be the particular solution of (122), and g_n the particular solution of (123) for $n > 1$, so that

$$g_n = O\left[\frac{e}{c_{00}} (g_{n-1}, \; g_{n+1})\right]$$

for $n \geqslant 1$. Hence the assumption (126) is consistent with the system. For $e = 0$, the full solution can simply be obtained from equation (121), which then reads

$$\frac{\partial g_{0,0}}{\partial z} + \frac{\tilde{c}a}{z} \; \frac{\partial^2 g_{0,0}}{\partial z^2} = 0 \tag{127}$$

The meaning of the second index in $g_{0,0}$ will become clear later. Writing (127) as

$$\frac{\partial^2 g_{0,0}/\partial z^2}{\partial g_{0,0}/\partial z} = \frac{-z/a}{c_{00}+z^2/2a}$$

it is clear that

$$\frac{\partial g_{0,0}}{\partial z} = \frac{A}{c_{00}+z^2/2a} = \frac{A}{\tilde{c}} \tag{128}$$

where the constant A is to be evaluated from the flux condition (124).

For non-zero eccentricity one has to solve the complete system (121) through (123). For small values of e/c_{00}, one can find a first approximation of $\partial g_1/\partial z$, by substituting (128) into (122):

$$\frac{\partial g_{1,0}}{\partial z} + \frac{\tilde{c}a}{z} \frac{\partial^2 g_{1,0}}{\partial z^2} = \frac{-eA}{\tilde{c}^2} \tag{129}$$

the particular solution of which is easily found to be

$$\frac{\partial g_{1,0}}{\partial z} = \frac{eA}{\tilde{c}^2} \tag{130}$$

The first correction of $\partial g_0/\partial z$ can then be obtained from (121), after substitution of (130):

$$\frac{\partial g_{0,1}}{\partial z} + \frac{\tilde{c}a}{z} \frac{\partial^2 g_{0,1}}{\partial z^2} = - \frac{e^2 A}{\tilde{c}^3} \tag{131}$$

Again $\partial g_{0,1}/\partial z = 0$ for $e = 0$, so $\partial g_{0,1}/\partial z$ must be the particular solution of (131):

$$\frac{\partial g_{0,1}}{\partial z} = \frac{e^2 A}{2\tilde{c}^3} \tag{132}$$

In the same way one can continue to compute the first and next approximations of g_n, and the forms that one obtains suggest a general solution of the form:

$$\frac{\partial g_n}{\partial z} = \sum_{\ell=0}^{\infty} \frac{\partial g_{n,\ell}}{\partial z} = \sum_{\ell=0}^{\infty} k_{n,\ell} \frac{A}{\tilde{c}} \left(\frac{e}{\tilde{c}}\right)^{n+2\ell} \tag{133}$$

The validity of this expression can easily be shown by induction from equations (121) through (123), and in doing so one obtains recursive relations for the constants $k_{n,\ell}$:

$$k_{0,\ell} = \frac{1}{2} k_{1,\ell-1} \tag{134}$$

$$k_{1,\ell} = k_{0,\ell} - \frac{1}{4} k_{2,\ell-1} \tag{135}$$

$$k_{n,\ell} = \frac{-n}{2(n-1)} k_{n-1,\ell} - \frac{n}{2(n+1)} k_{n+1,\ell-1} \tag{136}$$

for $n > 1$

Using (118) and (133), one then can write the z-component of the velocity as

$$\phi_z = -\sum_{n=1}^{\infty} \left[\frac{\cos n\varphi}{n} \sum_{\ell=0}^{\infty} \left\{ k_{n,\ell} \frac{A}{\tilde{c}} \left(\frac{e}{\tilde{c}} \right)^{n+2\ell} \right\} \right] + \sum_{\ell=0}^{\infty} k_{0,\ell} \frac{A}{\tilde{c}} \left(\frac{e}{\tilde{c}} \right)^{2\ell} \tag{137}$$

For such a solution to exist, $k_{n,\ell}$ must be bounded for either one of the indices n, ℓ tending to infinity while the other one is fixed, since $e/\tilde{c} < 1$. The details of the convergence of the series in the right hand side of (137) are given in the Appendix.

Finally, with $k_{0,0} = 1$, and all other $k_{n,\ell}$'s known from (134) through (136), A can be determined by the flux condition (124). Straightforward integration of $\phi_z \tilde{c}a$ gives

$$A = \frac{-Ua}{2} \frac{1}{\sum_{\ell=0}^{\infty} k_{0,\ell} \left(\frac{e}{\tilde{c}}\right)^{2\ell}} \tag{138}$$

The pressure distribution can then be obtained from Bernoulli's equation

$$-\frac{1}{\rho} \frac{\partial p}{\partial z} = v_z \frac{\partial v_z}{\partial z} \tag{139}$$

since $v_R \frac{\partial v_R}{\partial R}$ and $\frac{v\varphi}{a} \frac{\partial v_z}{\partial \varphi}$ are of order $\frac{c_{00}}{a}$ times $v_z \frac{\partial v_z}{\partial z}$.

So $$p + \frac{\rho v_z^2}{2} = \text{const}$$

along streamlines and since all streamlines come from the undisturbed, φ independent region, one can define

$$p = \frac{-\rho v_z^2}{2} \tag{140}$$

with v_z as in (137).

The resulting force on the sphere can be approximated as before

$$F = \int_{-(c_{00}a)^{\frac{1}{2}}}^{(c_{00}a)^{\frac{1}{2}}} \left\{ \int_0^{2\pi} p\cos\varphi \; a d\varphi \right\} dz \tag{141}$$

How many terms one wants to keep in v_z depends on the value of e/c_{00} and on the required accuracy. For instance if $e/c_{00} \ll 1$ and one wants F to leading order in e/c_{00}, one easily obtains from the general formulae

$$v_z = \frac{-Ua}{2\tilde{c}} + \frac{e}{\tilde{c}} \frac{Ua}{2\tilde{c}} \cos\varphi \tag{142}$$

$$p = \frac{-\rho U^2 a^2}{8\tilde{c}^2} + \frac{\rho U^2 a^2 e}{4\tilde{c}^3} \cos\varphi \tag{143}$$

and

$$F = \frac{\rho U^2 a^3 \pi e}{4} \left[\frac{17}{36} \frac{a^{1/2}}{c_{00}^{5/2}} + \frac{3\sqrt{2}}{4} \frac{a^{1/2}}{c_{00}^{5/2}} \operatorname{arctg} \frac{1}{\sqrt{2}} \right] \tag{144}$$

V. Experimental Work

a) Apparatus

The apparatus used for the experimental work consisted of a long glass tube, vertically mounted in a frame, in such a way that it could be turned 180° about a horizontal axis.

Three different tubes were used, two of which (tubes number I and II) were standard tubes with an unknown tolerance on the inside diameter. Tube number III was a precision bore tube, with a maximum tolerance of 0.1% on the inside diameter. The values of length and average inside diameter of the tubes are listed in Table VI.

A number of ping-pong balls were used as spheres. The sphericity of the balls was crudely checked with a micrometer and only those with a deviation of sphericity of less than 3% were used. Balls number 4 and 5, used in the precision bore tube, had a deviation of sphericity of less than 0.5%. The values of the diameters of the balls are listed in Table VII.

To change the upward velocity of the balls in the tube without having to change the viscosity of the fluid, the weight of some of the balls was increased by inserting solder wire through a little hole in the ball side , which was then sealed. This of course changes the inertial moments of the balls, but the flow in the gap region and the forces acting on the ball will still be the same.

When a ping-pong ball remains for some time in the fluid, the material of the ball side will absorb some of the fluid. The increase of weight caused by this effect during the time needed to run the experiment at a certain weight of the ball appears to be negligible, even more so when the weight is already increased by the inserted solder wire. However, in the experiments with the precision bore tube, the clearance ratio c/a was very small and the motion seemed almost critical* already with an unweighted ball. Therefore, the effect of fluid absorption could be used to increase the weight of the ball and decrease its upward velocity slightly. After every measurement, the ball was left in the tube and the experiment was repeated after some time. The absorption of fluid did not appear to have a measurable effect on the sphere radius a.

The fluids used in the tube were mixtures of various amounts of glycerine with water. The viscosity of each mixture was determined with a precision viscometer of the falling ball type, accurate up to 1%. The specific gravity of each mixture was measured with a pycnometer and a precision balance. The balance was also used to determine the weight of the spheres for every run, except in the case of the experiment in the precision bore tube.

b) The process and the observed phenomena

The lower end of the tube was closed with a rubber plug and the tube was filled with fluid. The viscosity and specific gravity of the fluid were determined beforehand. Then the sphere was inserted at the upper end of the tube, which was then closed and the tube was turned around. Intervals of 5 cm had been marked on the tube wall, so that the ascent velocity could be measured.

In the experiments with tubes I and II, the tube was turned

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*See paragraph b.

three more times at each weight of the ball, so that an average over four measurements could be taken. After that the ball was taken out again, more solder wire was inserted, and the experiment was repeated.

For the experiment with the precision bore tube, only two measurements at a time were done, to make the time, needed for the measurements at a certain weight and hence the variance in weight smaller. The ball was now left in the tube, and the experiment was repeated after roughly half an hour.

When the Reynolds number was higher than a critical value (Re^*_{cr}), the ball obtained a spin velocity about the vertical axis, the direction of which appeared to be random. Also, it bounced from wall to wall during the motion. The spin velocity was measured by timing a certain number of complete rotations of the ball, and the bouncing frequency by timing a certain number of times that the ball bounces off the same side of the tubewall. When $Re^* < Re^*_{cr}$ the motion appears to be smooth. Both the spin and the bouncing disappear, and the ball goes up in a straight line.

When the Reynolds number is very high, or the ball very light, the bouncing frequency is too high to be measured. In the tables of measurements in paragraph c of this chapter one can see that the bouncing frequency for balls 2 and 3 could only be measured when the weight of the ball was roughly over 20 grams. For balls 4 and 5, in the precision bore tube, the bouncing frequency was too high to be measured all the way to the critical Reynolds number, where the bouncing disappeared. With Re^* tending towards Re^*_{cr}

from above the bouncing frequency decreases, to go to zero rather suddenly when $Re^* < Re^*_{cr}$. The spin velocity first becomes very erratic in magnitude, sometimes even reversing its direction during the upward motion. It disappears completely with the bouncing, for $Re^* < Re^*_{cr}$.

In all cases where there is spin, the spin velocity seems to increase when the ball bounces against the tubewall, and to decrease again in between walls. However, due to the method of measuring the spin velocity, this observation could not be established as a fact.

A possible explanation of this behavior is that the ball rolls over the wall as it were, while it bounces off the wall, which will be true if the direction of approach does not coincide with the "direction" of the eccentricity ($\varphi = \pi$ in figure 1), see figure 4.

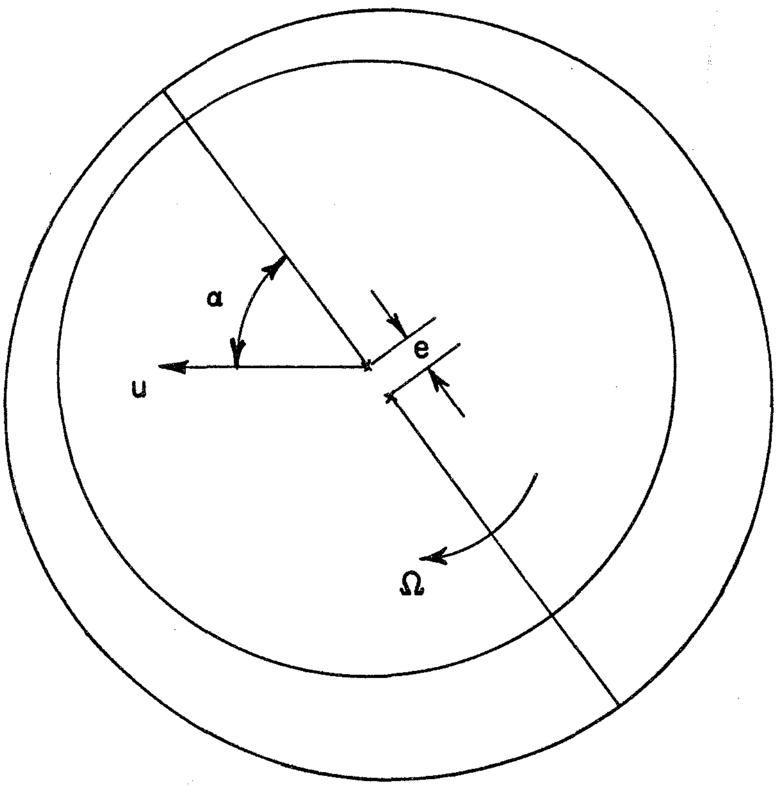

FIG. 4

It is clear that the magnitude of such a spin-torque will depend on the velocity of the ball in the $z = 0$ plane and on the angle α between the direction of approach and $\varphi = \pi$. Since the velocity in the horizontal plane depends on the vertical velocity U, this interpretation is consistent with the decrease in Ω with U on the average.

The occurrence of the bouncing instability at a certain value of Re^* can probably be explained by the fact that the inertial force as computed in Chapter II, has to grow as a function of Re^*, the effective Reynolds number in the gap region. At the critical value of Re^*, this inertial force gives enough momentum to the ball to make it bounce against the wall before it is brought to rest by the force, opposing the horizontal motion.

The experiments, due to the lack of accuracy resulting from their simplicity have to be considered as qualitative rather than quantitative. However, they show that at an "intermediate" Reynolds number, where theoretical analysis would involve the full nonlinear Navier-Stokes equations, an instability of the described nature sets in.

Finally, we have to point out that in the experiments I through III, the freedom of spin about a horizontal axis was suppressed by the presence of a weight (the solder wire) inside the ball. But the experiments IV and V, in which spherical symmetry was preserved, suggest that this is not a very important factor for the occurrence of the instability.

c) Data

In the following tables, the weight of the ball is listed together with the average values of the upward velocity V, the spin about the vertical axis Ω and the bouncing frequency f_b at that weight. Also which tube and ball were used for the specific experiment is indicated, and the values of the viscosity ν and density ρ of the fluid used.

I Tube I, ball 1, $\nu = 2.1 \times 10^{-2}$ cm^2/sec, $\rho = 1.12$ gm/cm^3

m_{ball} (gm)	U(cm/sec)	Ω (rad/sec)	f_b(#/sec)	$c_{00}/a = 8.5\%$
27.190	2.04	0.50	1.55	$c_{00} = 0.16$ cm
28.176	1.71	0.31	1.22	
28.765	1.56	0.12 *	0.90	
28.972	1.45	0.28 *	0.81	
29.157	1.35	0.29	0.77	
29.384	1.17	0.29	0.67	
29.544	1.06	0.26	0.62	
29.645	0.975	0.24	0.58	
29.747	0.862	0.18	0.53	
29.807	0.819	0.17	0.51	
29.897	0.725	0.15	0.48	
29.956	0.640	0.11	0.44	
30.008	0.610	0.11	0.42	
30.081	0.492	0	0.00	

* Rather erratic, deviation from average value above 15%

II Tube II, ball 2; $\nu = 4.35 \times 10^{-2}$ cm^2/sec; $\rho = 1.162$ gm/cm^3

m_{ball} (gm)	U(cm/sec)	Ω(rad/sec)	f_b(#/sec)	$c_{00}a = 10\%$
2.490	5.45	2.31	-	$c_{00} = 0.19$ cm
2.795	5.42	2.29	-	
3.413	5.50	1.77	-	
4.356	5.43	2.02	-	
5.309	5.33	2.00	-	
6.577	5.13	1.84	-	
8.138	4.91	1.65	-	
9.715	4.76	1.67	-	
11.288	4.65	1.31	-	
13.197	4.53	1.15	-	
15.090	4.21	0.87	-	
16.972	3.96	0.60	-	
18.874	3.65	0.41*	-	
20.761	3.37	0.25	2.40**	
22.665	3.04	0.14	2.15	
24.590	2.71	0.33	1.96	
26.448	2.36	0.17	1.74	
27.166	2.14	0.25	1.61	
27.792	1.98	0.17	1.49	
28.230	1.84	0.14	1.42	
28.588	1.63	0.043	1.30	
28.872	1.59	0.027	1.25	
29.324	1.44	0.0	1.12	
29.633	1.34	0.028	1.11	
29.912	1.24	0.0	1.04	
30.180	0.94	0.0	1.00	
30.288	0.82	0.0	0	

*Spin becomes very erratic; deviation of 20% and more from average value

**For higher velocities, f_b too high to be accurately measurable

III Tube II, ball 3; $\nu = 4.35 \cdot 10^{-2}$ cm^2/sec; $\rho = 1.162$ gm/cm^3

$c_{00}/a = 6\%$

$c_{00} = 0.18$ cm

m_{ball}	U	Ω	f_b
2.326	5.78	2.49	-
3.094	5.57	2.48	-
3.726	5.58	2.24	-
5.632	5.40	2.02	-
6.580	5.22	1.80	-
7.842	5.17	1.79	-
9.422	5.00	1.61	-
11.010	4.84	1.55	-
12.593	4.66	1.15	-
14.486	4.47	1.03	-
16.383	4.18	0.70	-
18.299	3.94	0.53*	-
20.222	3.50	0.37	-
22.111	3.21	0.24	2.23
24.356	2.68	0.12	1.84
26.325	2.33	0.16	1.56
27.057	2.04	0.13	1.49
27.612	1.82	0.093	1.35
28.200	1.81	0.10	1.30
28.606	1.75	0.094	1.26
28.989	1.60	0.043	1.18
29.273	1.46	0.00	1.14
29.791	1.12	0.083	1.05
30.066	1.03	0.081	0.939
30.370	0.853	0.00	0.893
30.431	0.775	0.00	0.0

Tube III

Measurements in the precision tube;

IV: Ball 4, $\nu = 1.28 \times 10^{-2}$ cm^2/sec; $\rho = 1.078$ gm/cm^3;

$c_{00} = 0.024$ cm $c_{00}/a = 1.3\%$

wobble disappears at U = 0.54 cm/sec

V: Ball 5, $\nu = 0.81 \times 10^{-2}$ cm^2/sec, $\rho = 1.035$ gm/cm^3; c/a =

$c_{00} = 0.018$ cm $c_{00}/a = 0.95\%$

wobble disappears at U = 0.29 cm/sec

VI:

Tube no.	d
I	2.05
II	2.075
III	1.907

VII:

Ball no.	a
1	1.890
2	1.886
3	1.894
4	1.883
5	1.889

Critical Reynolds numbers

Experiment #	Re^*_{cr}
I	12.9
II	11.3
III	10.1
IV	9.0
V	6.6

The experimental data are plotted in graph 2, giving Ω and f_b as a function of Re^*.

Appendix

The convergence of the potential flow solution

Consider $k_{n,\ell}$ to be an element of a two-dimensional matrix with n denoting the row number, and ℓ the column number. Repeated executing of relation (136) then gives a relation for element $k_{n,\ell}$ in terms of the first (n+1) elements in the preceding column:

$$k_{n,\ell+1} = \frac{-n}{2(n-1)} k_{n-1,\ell-1} - \frac{n}{2(n+1)} k_{n+1,\ell}$$

$$= (-\frac{1}{2}) \frac{n}{n-1} \left[\frac{-(n-1)}{2(n-2)} k_{n-2,\ell+1} - \frac{n-1}{2n} k_{n,\ell}\right] + (-\frac{1}{2}) \frac{n}{n+1} k_{n+1,\ell}$$

$$= (-\frac{1}{2})^2 \frac{n}{n-2} k_{n-2,\ell+1} + (\frac{-1}{2})^2 \frac{n}{n} k_{n,\ell} + (-\frac{1}{2}) \frac{n}{n+1} k_{n+1,\ell}$$

and after j steps

$$k_{n,\ell+1} = (-\frac{1}{2})^j \frac{n}{n-j} k_{n-j,\ell+1} + (-\frac{1}{2})^j \frac{n}{n-j+2} k_{n-j+2,\ell}$$

$$+ (-\frac{1}{2})^{j-1} \frac{n}{n-j+3} k_{n-j+3,\ell} +$$

$$+ ---- (-\frac{1}{2})^2 k_{n,\ell} + (-\frac{1}{2}) \frac{n}{n+1} k_{n+1,\ell} \qquad \text{(I)}$$

Let n-j = 1 and use (134) and (135) combined:

$$k_{1,\ell+1} = \frac{1}{2} k_{1,\ell} - \frac{1}{4} k_{2,\ell-1}$$

to obtain from (I)

$$k_{n,\ell+1} = (-\frac{1}{2})^{n+1} nk_{1,\ell} + \sum_{j=1}^{n+1} (-\frac{1}{2})^{n-j+2} \frac{n}{j} k_{j,\ell} \qquad \text{(II)}$$

for $n \geq 1$ and all ℓ.

To evaluate $k_{n,0}$ for $n \geq 1$, (135) and (136) can be written as

$$k_{1,0} = k_{0,0} \tag{III}$$

$$k_{n,0} = -\frac{1}{2}\,\frac{n}{n-1}\,R_{n-1,0} \tag{IV}$$

Then $k_{1,0} = 1$ (see (128)) and (IV) can be written as

$$k_{n,0} = (-\frac{1}{2})^{j}\,\frac{n}{n-j}\,k_{n-j,0}$$

$$= (-\frac{1}{2})^{n-1}\,nk_{1,0}$$

or $$k_{n,0} = (-\frac{1}{2})^{n-1}\,n \tag{V}$$

With expressions (134), (II) and (V) one can then evaluate all the matrix elements $k_{n,\ell}$, column by column.

E. g.,

$$k_{n,1} = (-\frac{1}{2})^{n+1}\,nk_{1,0} + \sum_{j=1}^{n+1}(-\frac{1}{2})^{n-j+2}\,\frac{n}{j}\,(-\frac{1}{2})^{j-1}\,j$$

$$= (-\frac{1}{2})^{n+1}\,n + (-\frac{1}{2})^{n+1}\,(n+1)n$$

or $$k_{n,1} = (-\frac{1}{2})^{n+1}\,n(n+2) \quad \text{for} \quad n \geq 1 \tag{VI}$$

Next $$k_{n,2} = (-\frac{1}{2})^{n+1}\,n\,k_{1,1} + \sum_{j=1}^{n+1}(-\frac{1}{2})^{n-j+2}\,\frac{n}{j}\,(-\frac{1}{2})^{j+1}\,j(j+2)$$

$$= 3n(-\frac{1}{2})^{n+3} + (-\frac{1}{2})^{n+3}\left\{2n(n+1) + n\sum_{j=1}^{n+1} j\right\}$$

Now $$\sum_{j=1}^{m} j^{k} = \frac{B_{k+1}(m+1)-B_{k+1}(0)}{k+1} \tag{VII}$$ (see [2])

Where B_{k+1} is the Bernoulli polynomial of $k+1^{st}$ order,

$$B_{k+1}(x) = \sum_{i=0}^{k+1} b_{k+1,i}\, x^i \qquad \text{(VIII), with } b_{m,m} = 1$$

For $k = 1$ one obtains

$$\sum_{j=1}^{n+1} j = \frac{B_2(n+2)-B_2(0)}{2} = \frac{(n+1)(n+2)}{2}$$

and hence

$$k_{n,2} = (-\tfrac{1}{2})^{n+3}\ \frac{n}{2}\left\{n^2 + 7n + 12\right\} \qquad \text{(IX)}$$

The expressions (V), (VI) and (IX) suggest a general form for $k_{n,\ell}$

$$k_{n,\ell} = (-\tfrac{1}{2})^{n-1+2\ell}\ \frac{n}{\ell!}\left\{n^\ell + a_{\ell,1}\, n^{\ell-1} + \text{----} + a_{\ell,i}\, n^{\ell-i} + \text{---} + a_{\ell,\ell}\right\} \qquad \text{(X)}$$

Expression (X) can be shown to be valid by induction; substitute (X) into (II) to find

$$k_{n,\ell+1} = (-\tfrac{1}{2})^{n+1}\ n(-\tfrac{1}{2})^{2\ell} + \frac{1}{\ell!}\left\{1 + \sum_{i=1}^{\ell} a_{\ell,i}\right\} +$$

$$+ \sum_{j=1}^{n+1}\left\{(-\tfrac{1}{2})^{n-j+2}\ \frac{n}{j}\ (-\tfrac{1}{2})^{j-1+2\ell}\ \frac{j}{\ell!}\,(j^\ell + \text{---} + a_{\ell,i}\, j^{\ell-i} + \text{---} + a_{\ell,\ell})\right\}$$

$$= (-\tfrac{1}{2})^{n-1+2(\ell+1)}\ \frac{n}{\ell!}\Bigg[\left\{1 + \sum_{i=1}^{\ell} a_{\ell,i}\right\}$$

$$+ \sum_{j=1}^{n+1}(j^\ell + \text{----} + a_{\ell,i}\, j^{\ell-i} + \text{----} + a_{\ell,\ell})\Bigg]$$

Using (VII) one obtains

$$k_{n,\ell+1} = (-\frac{1}{2})^{n-1+2(\ell+1)} \frac{n}{\ell!} \left[\left\{ 1+\sum_{i=1}^{\ell} a_{\ell,i} \right\} + \frac{B_{\ell+1}(n+2)-B_{\ell+1}(0)}{\ell+1} + \right.$$

$$+a_{\ell,1} \frac{B_{\ell}(n+2)-B_{\ell}(0)}{\ell} + ----+a_{\ell,i} \frac{B_{\ell-i+1}(n+2)-B_{\ell-i+1}(0)}{\ell-i+1} +$$

$$\left. + (n+1)a_{\ell,\ell} \right]$$

Only the Bernoulli polynomial of order $\ell+1$ will give a $n^{\ell+1}$ term. Since $b_{\ell+1,\ell+1} = 1$, one finds

$$k_{n,\ell+1} = (-\frac{1}{2})^{n-1+2(\ell+1)} \frac{n}{(\ell+1)!} \left[n^{\ell+1} + a_{\ell+1,1} n^{\ell} + ----+a_{\ell+1,\ell+1} \right] \tag{XI}$$

where the $a_{\ell+1,i}$'s can be found by evaluation of the Bernoulli polynomials, and consequently (X) is true.

Clearly, for fixed ℓ and large enough n

$$k_{n,\ell} < (-\frac{1}{2})^{n-1+2\ell} \frac{n^{\ell+1}}{\ell!} \cdot n \tag{XII}$$

which tends to zero for n tending to infinity.

The convergence for ℓ tending to infinity and fixed n is harder to prove since one must evaluate all the $a_{\ell,j}$'s, for which it is necessary to know the values of the coefficients of the Bernoulli polynomials $b_{i,j}$. A general form for $b_{i,j}$ is not known. However, if we assume that none of the $a_{\ell,j}$'s grow faster than $\ell!$, $k_{n,\ell}$ is bounded for ℓ tending to infinity and n fixed, and the series in the right hand side of (137) is convergent.

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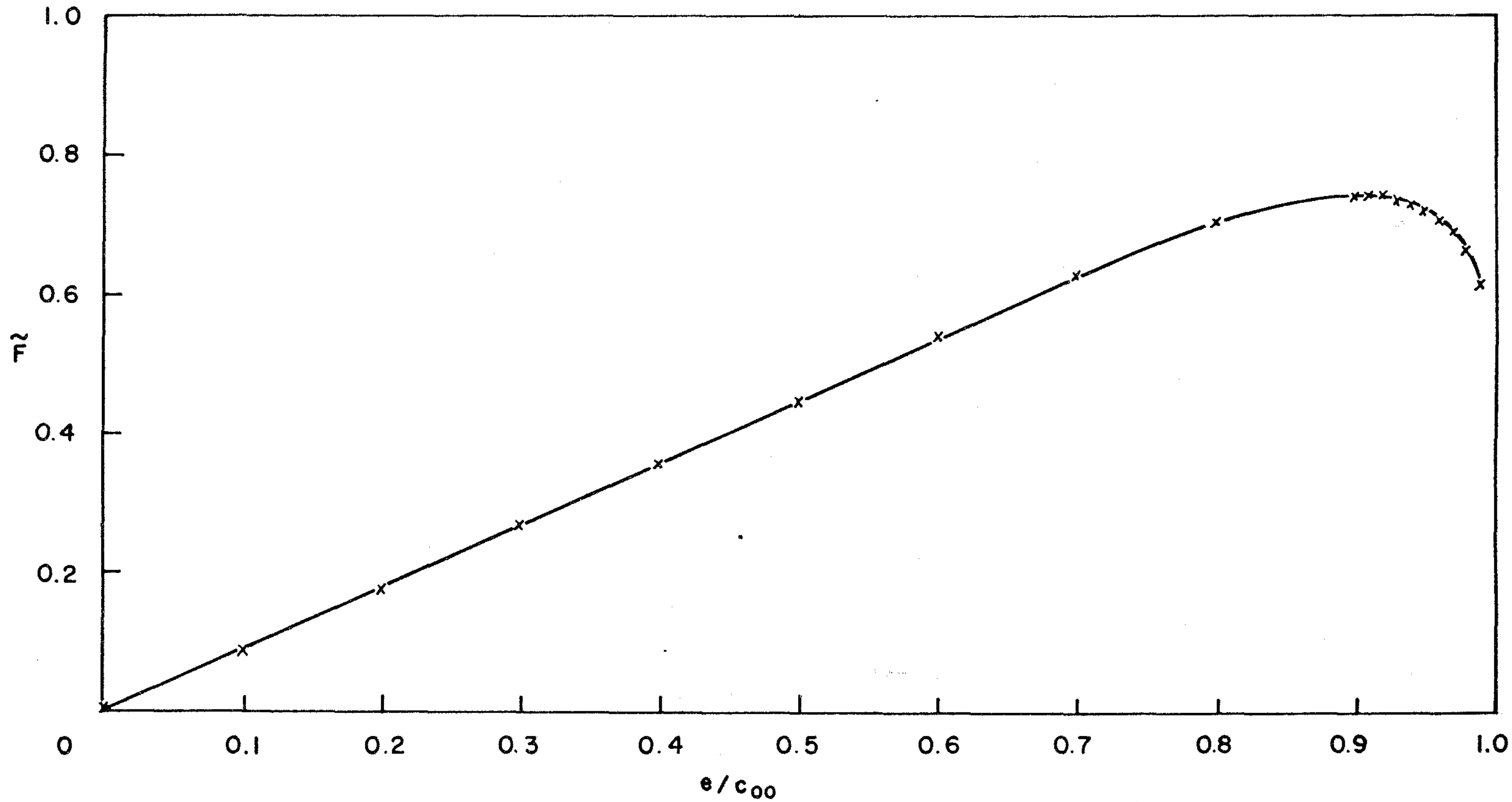
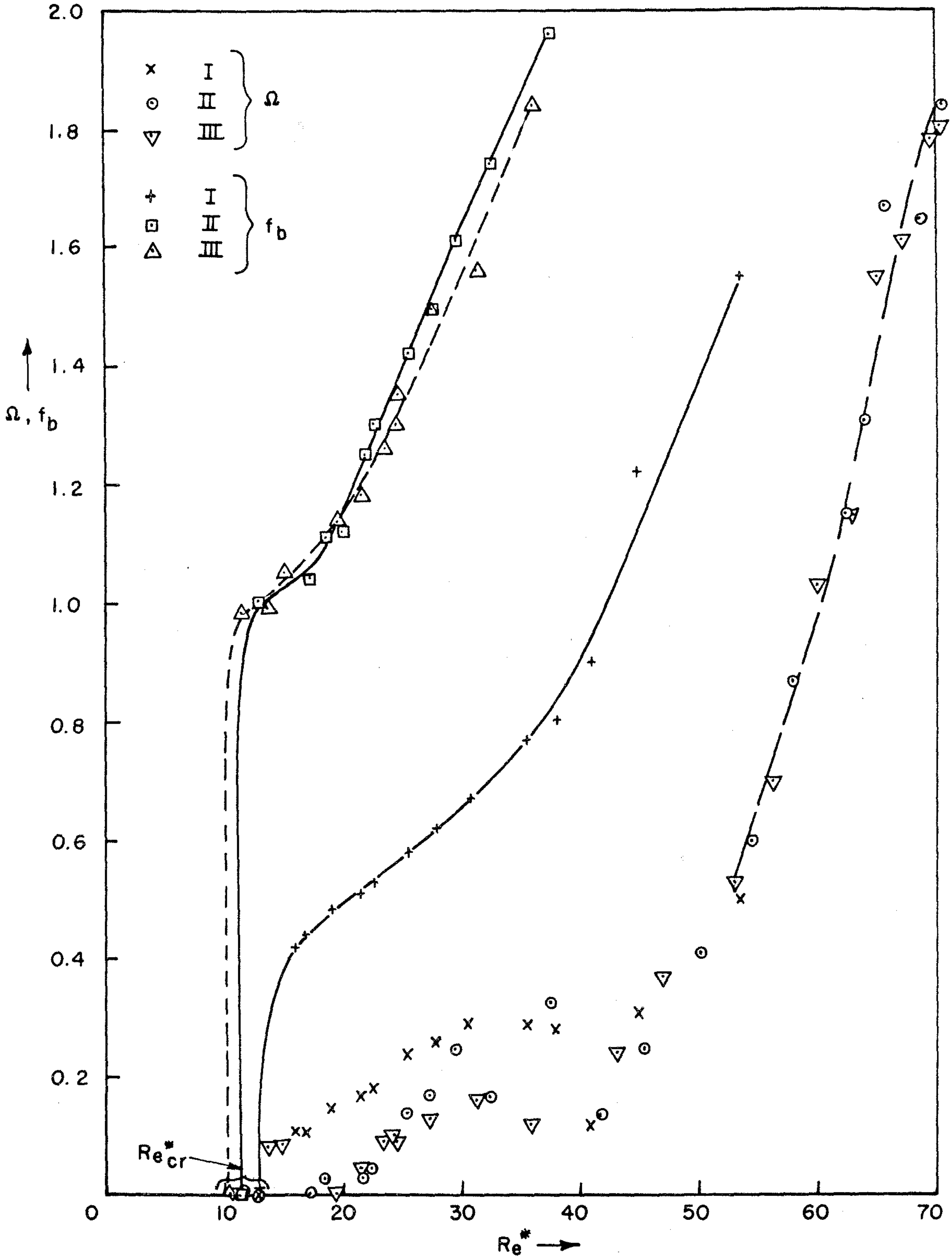
2. Milton Abramowitz and Irene A. Stegun (ed.): "Handbook of Mathematical Functions." Dover Publications, Inc., New York, 1965, p. 804.



GRAPH I INERTIAL FORCE AS FUNCTION OF ECCENTRICITY RATIO: $\tilde{F}\,(e/c_{00})$, SEE (67) (68)



GRAPH 2 Ω AND f_b AS FUNCTIONS OF R_e^*