BUCKLING OF CYLINDRICAL SHELLS WITH RANDOM IMPERFECTIONS

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1968

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TO MY HUSBAND SAMUEL

ACKNOWLEDGMENT

The author wishes to express her gratitude and sincere appreciation to Dr. E. E. Sechler for his help, advice and encouragement which enabled her to carry out this study. The advice and suggestions of Dr. C. D. Babcock are also very much appreciated. This opportunity is also taken to thank Dr. T. K. Caughey for most helpful discussions.

Particular appreciation is expressed to my husband, Dr. S. N. Fersht, to whom this work is dedicated. His patience, understanding and encouragement were a source of inspiration during my entire engineering studies.

This study was supported in part by the National Aeronautics and Space Administration under Research Grant NsG 18-59 and this aid is gratefully acknowledged.

ABSTRACT

The buckling stability analysis of long cylindrical shells with random imperfections subjected to axial load is treated using two different approaches. The first study is based on a Lyapunov method which enables one to establish sufficient conditions for buckling stability of a long cylindrical shell with axisymmetric random imperfections. A perturbed system of equations in the neighborhood of the prebuckling solution is investigated. By reducing the problem to a system of integral equations, it is observed that the stability boundary value problem of a long shell is similar to that of a dynamical system with random parametric excitations.

Initial imperfections were assumed to have Gaussian distribution and an exponential cosine correlation function. The critical load was obtained as a function of the root mean square of the imperfections. Results obtained are qualitatively similar to those of Koiter for a periodic imperfection (Ref. 1).

The second part is based on the approximate method of truncated hierarchy. The prebuckling state of equilibrium for asymmetric imperfections is found by a successive substitution technique. A homogeneous variational system of equations is set up in order to examine the existence of bifurcation in the neighborhood of the equilibrium state. These last equations involve random parametric terms. The truncated hierarchy method is applied and characteristic equations are obtained. Various exponential cosine correlation functions associated with asymmetric imperfections are examined numerically. Qualitatively the results obtained are as anticipated.

TABLE OF CONTENTS

LIST OF SYMBOLS

a	Constant defined on Page 13
b	Constant defined on Page 13
A	Matrix defined on Page 17
B	Matrix defined on Page 6
B_i	Matrix defined on Page 10
D	$Eh^3/12(1-\nu^2)$
E	Young's modulus
F	Stress function
$\bar{F}$	Parametric coefficient matrix
f_i	Scalar function defined on Page 10
$f(\xi,\eta)$, $f_1(\xi,\eta)$, $g_1(\xi,\eta)$, $g_2(\xi,\eta)$, $h(\xi,\eta)$, $h_1(\xi,\eta)$, $h_2(\xi,\eta)$	Transfer functions
G^o	Green's function defined on Page 13
G_i	Constant matrix defined on Page 10
H^o	Green's function defined on Page 13
$F(\alpha,\beta)$, $F_1(\alpha,\beta)$, $G(\alpha,\beta)$, $G_2(\alpha,\beta)$, $H(\alpha,\beta)$, $H_1(\alpha,\beta)$, $H_2(\alpha,\beta)$	Transformed transfer functions
h	Shell thickness
I	Unit matrix
I_1, I_2, I_3	Integrals defined on Page 57

LIST OF FIGURES

LIST OF SYMBOLS (continued)

k	Non-dimensional circumferential wave number
K	Constant defined on Page 21
L	Constant defined on Page 21
M	Constant defined on Page 21
N	Constant defined on Page 21
N_x , N_y , N_{xy}	Membrane stress resultants
n	Axial wave number
P	Transformation matrix defined on Page 17
Q	Hermitian matrix defined on Page 9
$\tilde{Q}$	Green's function defined on Page 18
R_w , R_u , R_ψ	Correlation functions
R	Radius of the shell
S_n	Phase function defined on Page 60
S_u , S_ψ , $S_{u\psi}$, $S_{\psi u}$	Power spectrum functions
u	Non-dimensional radial deflection of buckled mode
V	Hermitian matrix defined on Page 9
W	Radial deflection of shell
$\bar{W}$	Initial radial deflection of shell (imperfection)
w , $\bar{w}$, $\tilde{w}$, w_0 , w_1	Non-dimensional radial deflections
x	Axial coordinate
X	Column vector with components x_i

LIST OF SYMBOLS (continued)

y	Circumferential coordinate
α, β	Independent variables in the transformed domain
γ	$\sqrt{3(1-\nu^2)}$
Γ_1, Γ_2	Functions defined on Page 15
$\delta(\alpha)$	Dirac's delta function
ε, ε_n	Parameters of the power spectrum function
$\zeta(\lambda)$	Function defined on Page 23
η	$y\sqrt{2\gamma/Rh}$
$\bar{\eta}^{(i)}$	Eigenvalues of B_i
θ, θ_n	Parameters of the power spectrum function
Θ	Transformation matrix defined on Page 21
K, K_0, K_1	Root mean squares of initial imperfections
λ	σ/σ_{cl}
Λ	Diagonal matrix defined on Page 17
$\bar{\lambda}_i$	Eigenvalues of A
μ_i	Eigenvalues of V
ν	Poisson's ratio
ξ	$x\sqrt{2\gamma/Rh}$
σ	Applied axial stress
$\sigma_{cl.}$	$Eh/R\gamma$
Φ_0, ϕ_0	Stress functions (dimensional and non-dimensional)

LIST OF SYMBOLS (continued)

ψ	Non-dimensional stress function associated with buckled mode
ϕ	Non-dimensional stress function
∇_1^4	$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2$
∇^4	$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)^2$

I. INTRODUCTION

In the last three decades it has been recognized that small geometrical imperfections are the major cause for the reduction in the buckling strength of cylindrical shells, subjected to axial loads. Particular analytical studies of the problem, using approximate techniques and considering simple periodic modes of imperfections, have been carried out by Koiter (Refs. 1, 2), Donnell and Wan (Ref. 3), Hutchinson (Ref. 4), Budiansky and Hutchinson (Ref. 5), Babcock and Sechler (Ref. 6) and others. Few attempts have been made to study problems associated with local imperfections, almost periodic and stationary random imperfections. In other words, the studies that have been carried out so far are related to ideal cases and give qualitative insight to the problem.

In the search for a more realistic description of the geometry of imperfections, it was suggested by Bolotin (Ref. 7) that the imperfection function should be considered as a random variable. By using statistical techniques based on probability distributions and their transformations one could evaluate the probabilities for buckling failure. This outlined procedure is perhaps too general and becomes impractical as the number of random variables increases.

The first attempt to select a less general class of random imperfections, assuming stationarity and ergodicity, has been made by Fraser (Ref. 8) for a beam on nonlinear elastic foundation. This problem was treated by means of equivalent linearization for small imperfections. In a recent work by Amazigo (Ref. 9), the problem of buckling of long cylindrical shells under axial load has been solved for the case of axisymmetric initial imperfections. The approximate technique of truncated hierarchies has been utilized in this solution. In both studies, an exponential cosine correlation function for the imperfections has been examined. It should be noted that the solution techniques in these two studies were based on the assumptions that the initial imperfections were small.

In the present work two different techniques have been used. The first part consists of a stability analysis which is based upon Lyapunov's direct method, and has been utilized for the axisymmetric state of imperfections. No attempt has been made to extend it to a more general state of imperfections, although it is felt that this can also be achieved. The analysis is based on a study by Caughey and Gray (Ref. 10) for dynamical systems with stationary random parametric excitations.

Considering the problem of long cylindrical shells, a particular class of random imperfections, which is of practical significance, is the stationary state of imperfections with respect to the axial variable. By expanding the imperfection function in Fourier series in the circumferential direction, one can set up the problem considering the Fourier coefficients as the random variables. These coefficients are assumed to be stationary with respect to the axial independent variable and may be cross correlated. In addition it is assumed that the joint probability distribution for these coefficients is known. Further simplification is obtained by assuming that the random variables satisfy the ergodic property.

By considering the perturbation equations of the prebuckling solution it is possible to obtain a linear system of ordinary differential equations with constant and random parametric coefficients. By disregarding the terms with parametric coefficients the system is reduced to a stable one as long as the load is below the classical buckling load.

When the parametric coefficients are included by reducing the problem into a set of integral equations it was observed that, with proper modifications, the stability analysis is similar to that of a dynamic system where the axial variable replaces the time variable. As soon as this part of the analysis is established, the application of the Lyapunov technique becomes straightforward.

Lyapunov's method yields sufficient conditions for stability, but it often occurs that this technique leads to extremely conservative conditions. One of the major problems with Lyapunov's method is that of determining the proper matrix inequalities in order to derive sharper stability conditions. This part of the problem has been handled with particular care, yet it is felt that this part is still open, as in dynamical systems, to improvement.

The present method of stability has been tested numerically for the particular case of axisymmetric random imperfections. By considering a Gaussian distribution and an exponential cosine correlation function, the critical load was obtained as a function of the root mean square of the imperfections. The curves obtained are similar to those of Koiter for the cases where the peak of the power spectrum function coincides with the frequency of the critical linear buckling mode.

Finally one should point out that the present study is perhaps only the first step in this direction. By using the same technique, sufficient conditions for stability of cylindrical shells, subjected to other types of loads, as well as deterministic, almost periodic states of imperfections, can be obtained.

The second part of the present work is based on the approximate method of truncated hierarchy. A prebuckling approximate solution is obtained by using the method of successive substitutions, which is valid under the restriction that the root mean square of the imperfections is small compared to the shell thickness. Once this part of the problem is solved one can turn to the stability analysis. In order to verify the existence of a second solution in the neighborhood of the prebuckling equilibrium state, a variational homogeneous system of equations is set up. In other words these equations will enable one to examine the existence of bifurcation. Assuming that the initial imperfections are small, the method of truncated hierarchy can be applied following (Refs. 11, 12, 9). As a result one obtains a system of integro-differential equations for the proper correlation functions. This problem is further reduced by applying double Fourier transforms which leads to a system of homogeneous equations for the proper power spectrum functions. The condition for existence of a non-trivial solution yields the desired relation for existence of bifurcation. Naturally the lowest load and the associated power spectrum mode are the final results of the present problem.

Exponential cosine correlation functions are examined numerically for combinations of asymmetric and axisymmetric modes of imperfections. The correlation function parameters are selected carefully in order to justify the applicability of the numerical results obtained. This last argument naturally is based on physical intuition rather than on experimental evidence. In a work by Arbocz and Babcock (Ref. 13) imperfections have been measured by electrical means; however, the record was too short and therefore reliable correlation functions could not be established. Although the measured results are precise and carefully obtained, the number of cross sections of the cylinder for which imperfections were measured is not sufficient for data reduction in order to set up numerically the statistical properties of the imperfections. This, for the time being, leaves only the possibility of examining known correlation functions for testing the theory. As mentioned before, the parameters in these functions are selected on the basis of intuition which really relies on speculations.

It is hoped that, in the future, the present measurement techniques will be improved considerably, and perhaps new means for the measurement of imperfections will be found. Then the present analysis could be utilized on the basis of more realistic information.

II. ALMOST SURE STABILITY OF LONG CYLINDRICAL SHELLS WITH AXISYMMETRIC RANDOM IMPERFECTIONS

1. Preliminaries

The present study treats the stability of a boundary value problem. In general Lyapunov's second method treats asymptotic stability of dynamic systems, in other words it is related to initial value problems. In order to relate the boundary value problem to an equivalent dynamic system in a steady state response or a stationary response in a statistical sense let us investigate the following system of equations.

$$\frac{d^2X}{d\xi^2} = BX + \bar{F}(\xi)X \qquad -\infty < \xi < \infty \tag{1.1}$$

where X is an N-column vector with the components x_i, $i = 1, 2, \ldots, N$, B is a constant N x N matrix and $\bar{F}(\xi)$ is an N x N matrix whose nonzero elements are stochastic processes:

$$\bar{F}(\xi) = \left[f_{ij}(\xi) \right] \tag{1.2}$$

It is assumed that the matrix B has at least one square root A,

$$A \cdot A = A^2 = B$$

the eigenvalues of which are distinct and have negative real parts.

Now consider the system of equations

$$\frac{d^2X}{d\xi^2} = A^2X \qquad -\infty < \xi < +\infty \tag{1.3}$$

with the conditions at infinity

$$X \to 0 \quad \text{as} \quad \xi \to \pm\infty \tag{1.4}$$

The solution of (1.3) as $\xi \to +\infty$ can be obtained from the equation

$$\frac{dX}{d\xi} = AX \tag{1.5}$$

Furthermore, as $\xi \to -\infty$ the solution can be obtained from

$$\frac{dX}{d\xi} = -AX \tag{1.6}$$

Equations (1.5) and (1.6) can be combined to one equation as follows

$$\frac{dX}{d|\xi|} = AX \tag{1.7}$$

The stability is defined in the sense that as $\xi \to \pm\infty$ the lateral deflection of the shell tends to zero. This is known as asymptotic stability, and the term almost sure stability is associated with it. One can therefore state that conditions (1.4) can be met if and only if there is a matrix A, the eigenvalues of which have negative real parts. This last condition together with (1.4) assures stable solutions as $|\xi| \to \infty$, or by considering (1.7) and selecting the proper A it assures that (1.7) is asymptotically stable.

Turning now to (1.1) and assuming that for $\bar{F}(\xi) = 0$ this system is stable, let it also be assumed that the elements of $\bar{F}(\xi)$, $f_{ij}(\xi)$, satisfy the following properties,

a. The processes are continuous in $-\infty < \xi < \infty$

b. The processes are strictly stationary.

c. The processes satisfy an ergodic property, guaranteeing the equality of the averages with respect to ξ and the ensemble averages.

On the basis of the assumptions with respect to A and the boundary conditions at $\xi = \pm\infty$, one can construct a Green's function matrix associated with (1.3) or (1.7)

$$G(\xi,\eta) = G(|\xi-\eta|) \tag{1.8}$$

Equations (1.1) can therefore be converted into a system of integral equations of the form,

$$X(\xi) = \frac{1}{2} A^{-1} \int_{-\infty}^{\infty} G(|\xi-\eta|)\, \bar{F}(\eta)\, X(\eta)\, d\eta \tag{1.9}$$

By observation one realizes that equations (1.9) can be obtained from the system of equations

$$\frac{dX}{d|\xi|} = AX \pm \frac{1}{2} A^{-1} \bar{F}(\xi) X \tag{1.10}$$

where the positive sign in the second term, on the right hand side, is taken as ξ increases and the negative sign as it decreases. Due to symmetry it will be sufficient to investigate the asymptotic stability of (1.10) only for $\xi \to \infty$. Hence equations (1.10) can be reduced to the form

$$\frac{dX}{d\xi} = AX + \frac{1}{2}A^{-1}\bar{F}(\xi)X \tag{1.11}$$

where a proper condition at $X_0(0) = X_0$ can be selected.

From this point, the analysis will follow Caughey and Gray (Ref. 10). If A is a stability matrix, there exists a Hermitian positive definite matrix V, such that (Ref. 14)

$$A^{*}V + VA = -I \tag{1.12}$$

where $A^{*} = \bar{A}^{T}$.

A Hermitian matrix $Q(\xi)$ can be formed as follows

$$Q(\xi) = \frac{1}{2}\left\{V^{-\frac{1}{2}}[A^{-1}\bar{F}(\xi)]^{*}V^{\frac{1}{2}} + V^{\frac{1}{2}}[A^{-1}\bar{F}(\xi)]V^{-\frac{1}{2}}\right\} \tag{1.13}$$

where $V^{\frac{1}{2}}$ and $V^{-\frac{1}{2}}$ are positive definite Hermitian matrices obtained as follows: Since V is a positive definite Hermitian matrix there exists an orthogonal transformation Θ such that

$$\Theta^{*}\Theta = I, \qquad \Theta^{*}V\Theta = \lceil \mu \rfloor$$

V possesses a unique square root $V^{\frac{1}{2}}$

$$V^{\frac{1}{2}} = \Theta \lceil \mu^{\frac{1}{2}} \rfloor \Theta^{*}$$

also

$$V^{-\frac{1}{2}} = \Theta \lceil \mu^{-\frac{1}{2}} \rfloor \Theta^{*}$$

Now, let $\|Q(\xi)\|$ be the norm of $Q(\xi)$; if $E\{\|Q(\xi)\|\}$ exists and is less than $1/\mu_{max}$, the system of equation (1.11) is almost surely stable in the large.

In the particular case that $\bar{F}(\xi)$ may be written in the form

$$\bar{F}(\xi) = \sum_{i=1}^{M} G_i f_i(\xi) \tag{1.14}$$

where G_i are constant matrices and $f_i(\xi)$ are scalar functions of ξ and $M < N^2$, it is possible to have a sharper condition of stability. If $\sum_{i=1}^{M} |\bar{\eta}^{(i)}|_{max} E\{f_i(\xi)\}$ exists and is less than $1/\mu_{max}$ then equation (1.11) is almost surely stable in the large, where $|\bar{\eta}^{(i)}|_{max}$ is the numerically largest eigenvalue of the matrix

$$B_i = \frac{1}{2}\left[V^{-\frac{1}{2}}(A^{-1}G_i)^* V^{\frac{1}{2}} + V^{\frac{1}{2}}(A^{-1}G_i)V^{-\frac{1}{2}}\right] \tag{1.15}$$

2. Basic Equations

Let a point on the cylindrical surface of radius R be specified by its axial and circumferential coordinates x and y. Due to the presence of imperfections each point is radially displaced from the cylindrical surface by $\bar{W}(x)$. It is assumed that

$$|\bar{W}(x)| \ll R \qquad |\bar{W}_{,x}| \ll 1$$

In the absence of surface loads, the equations expressing equilibrium in the x and y direction for a shallow shell involve only the membrane stress resultants N_x, N_y and N_{xy}. These equations are satisfied by introducing the stress function $F(x,y)$,

$$N_x = F_{,yy} \qquad N_y = F_{,xx} \qquad N_{xy} = -F_{,xy}$$

Let $W(x,y)$ (positive inwards) be the radial displacement of the shell. In the case of axisymmetric imperfections the functions $F(x,y)$ and $W(x,y)$ satisfy the following two nonlinear equations,

$$\frac{1}{Eh}\nabla_1^4 F + \frac{1}{R}W_{,xx} = -W_{,xx}W_{,yy} - \bar{W}_{,xx}W_{,yy} + (W_{,xy})^2 \tag{2.1}$$

$$\begin{aligned} D\nabla_1^4 W - \frac{1}{R}F_{,xx} = {} & \bar{W}_{,xx}F_{,yy} + W_{,xx}F_{,yy} + W_{,yy}F_{,xx} \\ & - 2W_{,xy}F_{,xy} \end{aligned} \tag{2.2}$$

where E is Young's modulus, ν Poisson's ratio, h is the shell thickness, Eh = membrane rigidity,

$$D = \frac{Eh^3}{12(1-\nu^2)} = \text{bending rigidity}$$

$$\nabla_1^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2$$

Equation (2.1) is the compatibility equation in membrane strains and (2.2) is the radial equilibrium equation.

3. Method of Solution

Equations (2.1), (2.2) admit, for an axisymmetric imperfect cylindrical shell under axial compression, an axisymmetric prebuckling solution which may be written as

$$F(x, y) = -\frac{1}{2}\sigma h y^2 + \Phi_0(x)$$

$$W(x,y) = -\frac{\nu\sigma}{E} R + W_0(x) \qquad (3.1)$$

where σ is the axial compressive stress. Substituting (3.1) into (2.1) and (2.2) yields

$$\frac{1}{Eh}\Phi_{0,xxxx} + \frac{1}{R}W_{0,xx} = 0 \qquad (3.2)$$

$$DW_{0,xxxx} + \sigma h(\bar{W}_{,xx} + W_{0,xx}) - \frac{1}{R}\Phi_{0,xx} = 0 \qquad (3.3)$$

These equations can be simplified by reducing them to a nondimensional form. Let

$$x = \sqrt{\frac{Rh}{2\gamma}}\,\xi \qquad y = \sqrt{\frac{Rh}{2\gamma}}\,\eta$$

$$W_0 = h w_0 \qquad \bar{W} = h\bar{w} \qquad \Phi_0 = \frac{\sigma_{cl} h^2 R}{2}\phi_0 \qquad (3.4)$$

$$\gamma = \sqrt{3(1-\nu^2)} \qquad \lambda = \frac{\sigma}{\sigma_{cl}} \qquad \sigma_{cl} = \frac{Eh}{R\gamma}$$

Introducing these relations into (3.2) and (3.3) yields

$$\begin{aligned} &\phi_{0,\xi\xi\xi\xi} + w_{0,\xi\xi} = 0 \\ &w_{0,\xi\xi\xi\xi} - \phi_{0,\xi\xi} + 2\lambda w_{0,\xi\xi} = -2\lambda\bar{w}_{,\xi\xi} \end{aligned} \qquad (3.5)$$

The solution of (3.5) can be written in the form

$$w_0(\xi) = \int_{-\infty}^{\infty} G^0(\xi-\eta)\,\bar{w}(\eta)\,d\eta \qquad (3.6)$$

$$\phi_0(\xi) = \int_{-\infty}^{\infty} H^\circ(\xi-\eta)\,\bar{w}(\eta)\,d\eta$$

where $G^\circ(\xi-\eta)$ and $H^\circ(\xi-\eta)$ are the Green's functions associated with the homogeneous part of (3.5). These functions are

$$G^\circ(\xi) = \frac{\lambda}{2}\, e^{-a|\xi|}\left[\frac{1}{a}\cos b\xi - \frac{1}{b}\sin b|\xi|\right]$$
$$H^\circ(\xi) = \frac{\lambda}{\sqrt{1-\lambda}}\, e^{-a|\xi|}\left[-b\cos b\xi + a\sin b|\xi|\right] \tag{3.7}$$

where

$$a = \sqrt{(1-\lambda)/2} \qquad\qquad b = \sqrt{(1+\lambda)/2}$$

This solution remains finite as long as $\lambda < 1$.

Considering the case of stationary and ergodic random imperfections with zero mean, the autocorrelation functions are defined as follows

$$R_{\bar{w}}(\xi) = E\left\{\bar{w}(\xi+\eta)\,\bar{w}(\eta)\right\} \tag{3.8}$$

$$R_{w_0}(\xi) = E\left\{w_0(\xi+\eta)\,w_0(\eta)\right\} \tag{3.9}$$

where $E\{f(\xi)\}$ is the expectation of the function f . Now

$$E\{w_0(\xi+\eta)\,w_0(\eta)\} = E\left\{\iint_{-\infty}^{\infty} G^\circ(\xi_1)\, G^\circ(\xi_2)\,\bar{w}(\eta+\xi-\xi_1)\,\bar{w}(\eta-\xi_2)\,d\xi_1\,d\xi_2\right\}$$

Introducing the expectation operator into the double integral yields

$$R_{w_0}(\xi) = \iint\limits_{-\infty}^{\infty} G^{\circ}(\xi_1)\, G^{\circ}(\xi_2)\, R_{\bar{w}}(\xi - \xi_1 + \xi_2)\, d\xi_1\, d\xi_2 \qquad (3.10)$$

This is the desired relation for the linear part of the solution.

Following Koiter (Ref. 1), the nonlinear equations (2.1), (2.2) may admit an asymmetric solution adjacent to the symmetric one which is specified by $W_1(x, y)$ and $\Phi_1(x, y)$. Hence considering

$$F(x, y) = -\frac{1}{2}\sigma h y^2 + \Phi_0(x) + \Phi_1(x, y) \qquad (3.11)$$

$$W(x, y) = -\nu \frac{\sigma}{E} R + W_0(x) + W_1(x, y) \qquad (3.12)$$

and taking into account that the deviation from the axisymmetric configuration is infinitesimal one may linearize the equations with respect to $W_1(x, y)$ and $\Phi_1(x, y)$. The compatibility condition and the equilibrium equation therefore are

$$\frac{1}{Eh}\nabla^4 \Phi_1 + \frac{1}{R} W_{1,xx} = -(W_{0,xx} + \bar{W}_{,xx})\, W_{1,yy} \qquad (3.13)$$

$$D\nabla^4 W_1 + \sigma h W_{1,xx} - \frac{1}{R}\Phi_{1,xx} = (\bar{W}_{,xx} + W_{0,xx})\Phi_{1,yy} + \Phi_{0,xx} W_{1,yy} \qquad (3.14)$$

Now, let

$$\Phi_1(x, y) = \sum \Phi_n'(x) \cos \frac{ny}{R} \qquad (3.15)$$

$$W_1(x,y) = \sum W_n^1(x) \cos \frac{ny}{R}$$

Introducing the last expressions into (3.13) and (3.14) yields

$$\frac{1}{Eh}\left(\frac{d^2}{dx^2} - \frac{n^2}{R^2}\right)^2 \Phi_n^1 + \frac{1}{R} W^1_{n,xx} = \frac{n^2}{R^2}\left(W_{o,xx} + \bar{W}_{,xx}\right) W_n^1 \tag{3.16}$$

$$D\left(\frac{d^2}{dx^2} - \frac{n^2}{R^2}\right)^2 W_n^1 + \sigma h W^1_{n,xx} - \frac{1}{R}\Phi^1_{n,xx} =$$

$$= -\frac{n^2}{R^2}\left[\left(\bar{W}_{,xx} + W_{o,xx}\right)\Phi_n^1 + \Phi_{o,xx} W_n^1\right] \tag{3.17}$$

As before, these equations can be reduced to a nondimensional form using (3.4) and the relations,

$$W_n^1 = hw \qquad \Phi_n^1 = \frac{1}{2}\sigma_{cl} h^2 \phi \qquad k^2 = \frac{h}{R} n^2$$

which is

$$\left(\frac{d^2}{d\xi^2} - k^2\right)^2 \phi + w_{,\xi\xi} = \Gamma_1(\xi)\, w \tag{3.18}$$

$$\left(\frac{d^2}{d\xi^2} - k^2\right)^2 w - \phi_{,\xi\xi} + 2\lambda w_{,\xi\xi} = -\left[\Gamma_1(\xi)\phi + \Gamma_2(\xi) w\right] \tag{3.19}$$

where

$$\Gamma_1(\xi) = k^2\left(w_o + \bar{w}\right)_{,\xi\xi} \quad ; \quad \Gamma_2(\xi) = k^2 \phi_{o,\xi\xi}$$

Now, let

$\phi = x_1 \qquad w = x_2$

$\left(\frac{d^2}{d\xi^2} - k^2\right) x_1 = x_3 \quad ; \qquad \left(\frac{d^2}{d\xi^2} - k^2\right) x_2 = x_4$

equations (3.18) and (3.14) yield

$$
\begin{aligned}
x_1'' &= k^2 x_1 + x_3 \\
x_2'' &= k^2 x_2 + x_4 \\
x_3'' &= k^2 x_3 - (k^2 x_2 + x_4) + \Gamma_1(\xi)\, x_2 \\
x_4'' &= k^2 x_4 - 2\lambda(k^2 x_2 + x_4) + k^2 x_1 + x_3 - \Gamma_1(\xi)\, x_1 + \Gamma_2(\xi)\, x_2
\end{aligned}
\tag{3.20}
$$

where prime denotes differentiation with respect to ξ .

In matrix notation (3.20) may be written as

$$X'' = A^2 X + \bar{F}(\xi)\, X \tag{3.21}$$

where

$$
A^2 = \begin{bmatrix} k^2 & 0 & 1 & 0 \\ 0 & k^2 & 0 & 1 \\ 0 & -k^2 & k^2 & -1 \\ k^2 & -2\lambda k^2 & 1 & k^2 - 2\lambda \end{bmatrix} \tag{3.22}
$$

$$\bar{F}(\xi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \cdot \Gamma_1(\xi) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \cdot \Gamma_2(\xi) \tag{3.23}$$

and

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Constructing a transformation matrix P such that

$$P^{-1} A^2 P = \Lambda^2 = \left[\diagdown \bar{\lambda}^2 \diagdown \right]$$

where Λ^2 is a diagonal matrix containing the eigenvalues of A^2, one can find the matrix A

$$A = P \Lambda P^{-1} \tag{3.24}$$

where

$$\Lambda = \left[\diagdown \bar{\lambda} \diagdown \right] \quad \text{and} \quad \mathrm{Re}\{\bar{\lambda}_i\} < 0$$

4. Derivation of Stability Condition

For the following analysis, the autocorrelation function $R_{\bar{w}}(\xi)$ will be assumed as an exponential cosine function

$$R_{\bar{w}}(\xi) = K^2 e^{-\varepsilon|\xi|} \cos\theta\xi \tag{4.1}$$

and $\bar{w}(\xi)$ will be assumed to have a Gaussian distribution. Obviously K represents the root mean square of the imperfections.

For a function $f(x)$ with Gaussian distribution

$$E\{|f(x)|\} = \sqrt{\frac{2}{\pi}}\left[R_f(0)\right]^{\frac{1}{2}} \tag{4.2}$$

In the case of the cylindrical shell,

$$E\{|\Gamma_1(\xi)|\} = \sqrt{\frac{2}{\pi}}\, k^2 \left[R_{(W_0+\bar{W}),\xi\xi}(0)\right]^{\frac{1}{2}} \tag{4.3}$$

and

$$E\{|\Gamma_2(\xi)|\} = \sqrt{\frac{2}{\pi}}\, k^2 \left[R_{W_0}(0)\right]^{\frac{1}{2}} \tag{4.4}$$

where

$$R_{(W_0+\bar{W}),\xi\xi}(0) = \int\int_{-\infty}^{\infty} \tilde{Q}(\xi_1)\,\tilde{Q}(\xi_2)\,R_{\bar{w}}(\xi_2-\xi_1)\,d\xi_1\,d\xi_2$$

$$\tilde{Q}(\xi) = \frac{e^{-a|\xi|}}{\sqrt{2}}\left[\frac{\lambda(1-2\lambda)}{\sqrt{1-\lambda}}\cos b\xi + \frac{\lambda(1+2\lambda)}{\sqrt{1+\lambda}}\sin b|\xi|\right] \tag{4.5}$$

and

$$R_{w_0}(0) = \iint_{-\infty}^{\infty} G^0(\xi_1)\, G^0(\xi_2)\, R_{\bar{w}}(\xi_2 - \xi_1)\, d\xi_1\, d\xi_2$$

After rather cumbersome integrations

$$R_{w_0}(0) = \frac{1}{4}\lambda^2 \kappa^2 \Bigg\{ \left\{ \frac{1}{\sqrt{1-\lambda}}(a+\varepsilon)(K+N) - \frac{1}{\sqrt{1+\lambda}}\left[(b+\theta)K + (b-\theta)N\right] \right\} \cdot$$

$$\cdot \left\{ \frac{1}{\sqrt{1-\lambda}}\left[(a-\varepsilon)(M+L) + (a+\varepsilon)(K+N)\right] - \frac{1}{\sqrt{1+\lambda}}\left[(b+\theta)(M+K) + (b-\theta)(L+N)\right] \right\}$$

$$+ \left\{ \frac{1}{\sqrt{1-\lambda}}\left[(b+\theta)K - (b-\theta)N\right] - \frac{1}{\sqrt{1+\lambda}}(a+\varepsilon)(N-K) \right\} \cdot$$

$$\cdot \left\{ \frac{1}{\sqrt{1-\lambda}}\left[(b+\theta)(M-K) - (b-\theta)(L-N)\right] - \frac{1}{\sqrt{1+\lambda}}\left[(a-\varepsilon)(L-M) + (a+\varepsilon)(K-N)\right] \right\}$$

$$+ \frac{1}{2}\left\{ \frac{1}{\sqrt{1-\lambda}}\ \frac{2a^2+b^2}{a(a^2+b^2)} - \frac{1}{\sqrt{1+\lambda}}\ \frac{b}{a^2+b^2} \right\} \cdot$$

$$\cdot \left\{ \frac{1}{\sqrt{1-\lambda}}\left[-(a-\varepsilon)(M+L) + (a+\varepsilon)(K+N)\right] + \frac{1}{\sqrt{1+\lambda}}\left[(b+\theta)(M-K) + (b-\theta)(L-N)\right] \right\}$$

$$+ \frac{1}{2}\left\{ \frac{1}{\sqrt{1-\lambda}}\ \frac{b}{a^2+b^2} - \frac{1}{\sqrt{1+\lambda}}\ \frac{b^2}{a(a^2+b^2)} \right\} \cdot$$

$$\cdot \left\{ \frac{1}{\sqrt{1-\lambda}}\left[(b+\theta)(M-K) + (b-\theta)(L-N)\right] - \frac{1}{\sqrt{1+\lambda}}\left[(\varepsilon-a)(M+L) + (a+\varepsilon)(K+N)\right] \right\} \Bigg\} \tag{4.6}$$

and

$$
\begin{aligned}
R_{(w_0+\bar{w}),\xi\xi}(0) = \frac{1}{4}\lambda^2\kappa^2 \Bigg\{ & \left\{\frac{1-2\lambda}{\sqrt{1-\lambda}}(a+\varepsilon)(K+N) + \frac{1+2\lambda}{\sqrt{1+\lambda}}\left[(b+\theta)K + (b-\theta)N\right]\right\} \cdot \\
& \cdot \left\{\frac{1-2\lambda}{\sqrt{1-\lambda}}\left[(a-\varepsilon)(M+L) + (a+\varepsilon)(K+N)\right] + \frac{1+2\lambda}{\sqrt{1+\lambda}}\left[(b+\theta)(M+K) + (b-\theta)(L+N)\right]\right\} \\
& + \left\{\frac{1-2\lambda}{\sqrt{1-\lambda}}\left[(b+\theta)K - (b-\theta)N\right] + \frac{1+2\lambda}{\sqrt{1+\lambda}}(a+\varepsilon)(N-K)\right\} \cdot \\
& \cdot \left\{\frac{1-2\lambda}{\sqrt{1-\lambda}}\left[(b+\theta)(M-K) - (b-\theta)(L-N)\right] + \frac{1+2\lambda}{\sqrt{1+\lambda}}\left[(a-\varepsilon)(L-M) + (a+\varepsilon)(K-N)\right]\right\} \\
& + \frac{1}{2}\left\{\frac{1-2\lambda}{\sqrt{1-\lambda}}\,\frac{2a^2+b^2}{a(a^2+b^2)} + \frac{1+2\lambda}{\sqrt{1+\lambda}}\,\frac{b}{a^2+b^2}\right\} \cdot \\
& \cdot \left\{\frac{1-2\lambda}{\sqrt{1-\lambda}}\left[-(a-\varepsilon)(M+L) + (a+\varepsilon)(K+N)\right] - \frac{1+2\lambda}{\sqrt{1+\lambda}}\left[(b+\theta)(M-K) + (b-\theta)(L-N)\right]\right\} \\
& + \frac{1}{2}\left\{\frac{1-2\lambda}{\sqrt{1-\lambda}}\,\frac{b}{a^2+b^2} + \frac{1+2\lambda}{\sqrt{1+\lambda}}\,\frac{b^2}{a(a^2+b^2)}\right\} \cdot \\
& \cdot \left\{\frac{1-2\lambda}{\sqrt{1-\lambda}}\left[(b+\theta)(M-K) + (b-\theta)(L-N)\right] + \frac{1+2\lambda}{\sqrt{1+\lambda}}\left[(\varepsilon-a)(M+L) + (a+\varepsilon)(K+N)\right]\right\}\Bigg\}
\end{aligned}
\tag{4.7}
$$

where

$$K = \frac{1}{(a+\varepsilon)^2 + (b+\theta)^2} \qquad M = \frac{1}{(a-\varepsilon)^2 + (b+\theta)^2}$$

$$N = \frac{}{(a+\varepsilon)^2 + (b-\theta)^2} \qquad L = \frac{1}{(a-\varepsilon)^2 + (b-\theta)^2}$$

To complete the stability analysis established in Section 1, one has to find the Hermitian matrix V such that

$$A^* V + VA = -I$$

where A is formulated as shown in (3.24). Then one has to find the transformation matrix Θ such that

$$\Theta^* \Theta = I \qquad \Theta^* V \Theta = \begin{bmatrix} \mu \searrow \end{bmatrix}$$

where μ_i are the eigenvalues of V. Let G_1 and G_2 be the following matrices

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{4.8}$$

Then,

$$B_1 = \frac{1}{2}\left[V^{-\frac{1}{2}}(A^{-1}G_1)^* V^{\frac{1}{2}} + V^{\frac{1}{2}}(A^{-1}G_1)V^{-\frac{1}{2}}\right] \tag{4.9}$$

and

$$B_2 = \frac{1}{2}\left[V^{-\frac{1}{2}}(A^{-1}G_2)^* V^{\frac{1}{2}} + V^{\frac{1}{2}}(A^{-1}G_2)V^{-\frac{1}{2}}\right] \tag{4.10}$$

Let $\bar{\eta}_i^{(1)}$ be the eigenvalues of B_1 and $\bar{\eta}_i^{(2)}$ the eigenvalues of B_2. The stability condition for the cylindrical shell will then be

$$|\bar{\eta}^{(1)}|_{max}\, E\{|\Gamma_1(\xi)|\} + |\bar{\eta}^{(2)}|_{max}\, E\{|\Gamma_2(\xi)|\} < \frac{1}{\mu_{max}} \tag{4.11}$$

Now from (4.3), (4.4), (4.6) and (4.7), it is easily seen that

$$E\{|\Gamma_1(\xi)|\} = C_1 \kappa$$

and

$$E\{|\Gamma_2(\xi)|\} = C_2 \kappa$$

where C_1 and C_2 are constants.

Introducing these relations into (4.11) yields

$$\kappa < \frac{1}{|\bar{\eta}^{(1)}|_{max}\, C_1 + |\bar{\eta}^{(2)}|_{max}\, C_2}\, \frac{1}{\mu_{max}} \tag{4.12}$$

This is the desired stability condition.

5. Numerical Example

In order to evaluate the stability boundary determined in equation (4.12) a specific numerical example has been carried out. The following parameters were used in the calculation.

$$R/h = 800$$
$$\nu = 0.3$$
$$\varepsilon = 0.2$$
$$\theta = 1.0$$

The data (ε, θ) for the correlation function of the initial imperfections were selected so that the peak of the power spectrum would be in the neighborhood of the peak of the response kernel for $W_0(\xi)$. This will assure consideration of the most critical situation. The numerical evaluation determines the following relation.

$$\zeta(\lambda) = \frac{1}{|\bar{\eta}^{(1)}|_{max} C_1 + |\bar{\eta}^{(2)}|_{max} C_2} \; \frac{1}{\mu_{max}}$$

The shell will remain stable as long as $\kappa(\lambda) < \zeta(\lambda)$.

The calculation was carried out varying the wave number k in the vicinity of $k = \frac{1}{2}$. The stability boundary obtained is shown in Figure 2. This result is similar qualitatively to the deterministic cases associated with sinusoidal imperfections.

It should be pointed out that the imperfections at particular points might be higher than κ by a factor of 10 or more.

6. Concluding Remarks

The stability condition is only a sufficient criteria for the stability of the shell. The buckling problem is still open for sharper conditions, nevertheless the present condition does not require any further assumptions with respect to λ or the power spectrum functions of the initial imperfections.

It should be pointed out that, for certain particular cases, a sharper stability condition can be obtained by means of other techniques. For example, where the load is close to the linear critical one and the power spectrum function of the imperfections varies slowly in the vicinity of the axisymmetrical response function, the case can be solved in a simplified manner, considering a narrow band filter technique. The stability condition obtained will be sharper than that obtained by Lyapunov's method.

III. APPROXIMATE STABILITY ANALYSIS OF LONG CYLINDRICAL SHELLS WITH ASYMMETRIC RANDOM IMPERFECTIONS

1. Prebuckling Equilibrium with Asymmetric Imperfections

This part of the work is based on the approximate method of truncated hierarchies which has been used before for the axisymmetric case by Amazigo (Ref. 9). No attempt will be made to study the validity of this technique, however comparisons between results obtained by truncated hierarchies and those obtained by other approximate techniques, such as perturbation techniques for particular cases, are in good numerical agreement. In order to assure justification for adopting the truncated hierarchy method as used in the following, one should assume that the root mean square of the imperfections is small compared to the shell thickness. The last assumption seems to be rather restrictive, nevertheless the cases which fall into this class are of great practical significance.

Considering again the nondimensional equations of an imperfect cylindrical shell subjected to axial load,

$$\nabla^4\phi + w_{,\xi\xi} = -\gamma\left(w_{,\xi\xi}\hat{w}_{,\eta\eta} + w_{,\eta\eta}\hat{w}_{,\xi\xi} - 2w_{,\xi\eta}\hat{w}_{,\xi\eta}\right)$$

$$\nabla^4 w - \phi_{,\xi\xi} + 2\lambda\tilde{w}_{,\xi\xi} = 2\gamma\left(\phi_{,\xi\xi}\tilde{w}_{,\eta\eta} + \phi_{,\eta\eta}\tilde{w}_{,\xi\xi} - 2\phi_{,\xi\eta}\tilde{w}_{,\xi\eta}\right) \tag{1.1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}$$

$$\tilde{w}(\xi, \eta) = w(\xi, \eta) + \bar{w}(\xi, \eta)$$

$$\hat{w}(\xi, \eta) = w(\xi, \eta) + 2\bar{w}(\xi, \eta)$$

Consider a solution of equation (1.1) of the form

$$\begin{aligned} \phi(\xi, \eta) &= \phi_0(\xi, \eta) + \phi_1(\xi, \eta) \\ w(\xi, \eta) &= w_0(\xi, \eta) + w_1(\xi, \eta) \end{aligned} \tag{1.2}$$

The functions $\phi_0(\xi, \eta)$ and $w_0(\xi, \eta)$ satisfy the following system of linear equations

$$\begin{aligned} &\nabla^4 \phi_0 + w_{0,\xi\xi} = 0 \\ &\nabla^4 w_0 - \phi_{0,\xi\xi} + 2\lambda w_{0,\xi\xi} = -2\lambda \bar{w}_{,\xi\xi} \end{aligned} \tag{1.3}$$

the solutions of which are

$$w_0(\xi, \eta) = \iint_{-\infty}^{\infty} g(\xi_1, \eta_1)\, \bar{w}(\xi - \xi_1, \eta - \eta_1)\, d\xi_1\, d\eta_1$$

$$\phi_0(\xi, \eta) = \iint_{-\infty}^{\infty} h(\xi_1, \eta_1)\, \bar{w}(\xi - \xi_1, \eta - \eta_1)\, d\xi_1\, d\eta_1$$

(1.4)

$$\tilde{w}_o(\xi,\eta) = \iint_{-\infty}^{\infty} f(\xi_1,\eta_1)\,\bar{w}(\xi-\xi_1,\ \eta-\eta_1)\,d\xi_1\,d\eta_1$$

$$\hat{w}_o(\xi,\eta) = \iint_{-\infty}^{\infty} f_1(\xi_1,\eta_1)\,\bar{w}(\xi-\xi_1,\ \eta-\eta_1)\,d\xi_1\,d\eta_1$$

where the double Fourier transforms of $g(\xi,\eta)$, $h(\xi,\eta)$, $f(\xi,\eta)$ and $f_1(\xi,\eta)$ are

$$G(\alpha,\beta) = \frac{2\lambda\alpha^2(\alpha^2+\beta^2)^2}{(\alpha^2+\beta^2)^4 - 2\lambda\alpha^2(\alpha^2+\beta^2)^2 + \alpha^4}$$

$$H(\alpha,\beta) = \frac{2\lambda\alpha^4}{(\alpha^2+\beta^2)^4 - 2\lambda\alpha^2(\alpha^2+\beta^2)^2 + \alpha^4}$$

(1.5)

$$F(\alpha,\beta) = \frac{(\alpha^2+\beta^2)^4 + \alpha^4}{(\alpha^2+\beta^2)^4 - 2\lambda\alpha^2(\alpha^2+\beta^2)^2 + \alpha^4}$$

and

$$F_1(\alpha, \beta) = 2 \; \frac{(\alpha^2+\beta^2)^4 - \lambda\alpha^2(\alpha^2+\beta^2)^2 + \alpha^4}{(\alpha^2+\beta^2)^4 - 2\lambda\alpha^2(\alpha^2+\beta^2)^2 + \alpha^4}$$

One realizes that these transfer functions exist for all β, as long as $\lambda < 1$ and, in particular, for $\beta = 0$,

$$G(\alpha, 0) = \frac{2\lambda\,\alpha^2}{\alpha^4 - 2\lambda\alpha^2 + 1}$$

$$H(\alpha, 0) = \frac{2\lambda}{\alpha^4 - 2\lambda\alpha^2 + 1}$$

$$F(\alpha, 0) = \frac{\alpha^4 + 1}{\alpha^4 - 2\lambda\alpha^2 + 1} \tag{1.6}$$

and

$$F_1(\alpha, 0) = 2\,\frac{\alpha^4 - \lambda\alpha^2 + 1}{\alpha^4 - 2\lambda\alpha^2 + 1}$$

Introducing (1.2) into (1.1) and neglecting higher order terms in $\phi_1(\xi,\eta)$ and $w_1(\xi,\eta)$ as well as multiplications of sub-zero and sub-one terms lead to the equations

$$\nabla^4\phi_1 + w_{1,\xi\xi} = -\gamma\left(w_{0,\xi\xi}\,\hat{w}_{0,\xi\xi} + w_{0,\eta\eta}\,\hat{w}_{0,\xi\xi} - 2w_{0,\xi\eta}\,\hat{w}_{0,\xi\eta}\right) \tag{1.7}$$

$$\nabla^4 w_1 - \phi_{1,\xi\xi} + 2\lambda w_{1,\xi\xi} = 2\gamma\left(\phi_{0,\xi\xi}\tilde{w}_{0,\eta\eta} + \phi_{0,\eta\eta}\tilde{w}_{0,\xi\xi} - 2\phi_{0,\xi\eta}\tilde{w}_{0,\xi\eta}\right)$$

At this point consider the following useful identity

$$p_{,\xi\xi}q_{,\eta\eta} + p_{,\eta\eta}q_{,\xi\xi} - 2p_{\xi\eta}q_{\xi\eta} = \tag{1.8}$$

$$= (pq_{,\eta\eta})_{,\xi\xi} + (pq_{,\xi\xi})_{,\eta\eta} - 2(pq_{,\xi\eta})_{,\xi\eta}$$

$$= (qp_{,\eta\eta})_{,\xi\xi} + (qp_{,\xi\xi})_{,\eta\eta} - 2(qp_{,\xi\eta})_{,\xi\eta}$$

The solution of (1.7) can be formally written in the form

$$\phi_1(\xi,\eta) = \gamma\iint_{-\infty}^{\infty}\Big\{2h_1(\xi_1,\eta_1)\big[\phi_{0,\xi\xi}(\xi-\xi_1,\eta-\eta_1)\tilde{w}_{0,\eta\eta}(\xi-\xi_1,\eta-\eta_1)$$

$$+\phi_{0,\eta\eta}(\xi-\xi_1,\eta-\eta_1)\tilde{w}_{0,\xi\xi}(\xi-\xi_1,\eta-\eta_1) - 2\phi_{0,\xi\eta}(\xi-\xi_1,\eta-\eta_1)\tilde{w}_{0}(\xi-\xi_1,\eta-\eta_1)\big]$$

$$- h_2(\xi_1,\eta_1)\big[w_{0,\xi\xi}(\xi-\xi_1,\eta-\eta_1)\hat{w}_{0,\eta\eta}(\xi-\xi_1,\eta-\eta_1)$$

$$+ w_{0,\eta\eta}(\xi-\xi_1,\eta-\eta_1)\hat{w}_{0,\xi\xi}(\xi-\xi_1,\eta-\eta_1) -$$

$$- 2w_{0,\xi\eta}(\xi-\xi_1, \eta-\eta_1)\hat{w}(\xi-\xi_1, \eta-\eta_1)\Big]\Big\}\, d\xi_1\, d\eta_1$$

$$w_1(\xi,\eta) = \gamma \iint_{-\infty}^{\infty} \Big\{ 2g_1(\xi_1,\eta_1)\Big[\phi_{0,\xi\xi}(\xi-\xi_1,\eta-\eta_1)\tilde{w}_{0,\eta\eta}(\xi-\xi_1,\eta-\eta_1) \tag{1.9}$$

$$+ \phi_{0,\eta\eta}(\xi-\xi_1,\eta-\eta_1)\,\tilde{w}_{0,\xi\xi}(\xi-\xi_1,\eta-\eta_1)$$

$$- 2\phi_{0,\xi\eta}(\xi-\xi_1,\eta-\eta_1)\,\tilde{w}_{0,\xi\eta}(\xi-\xi_1,\eta-\eta_1)\Big]$$

$$- g_2(\xi_1,\eta_1)\Big[w_{0,\xi\xi}(\xi-\xi_1,\eta-\eta_1)\,\hat{w}_{0,\eta\eta}(\xi-\xi_1,\eta-\eta_1)$$

$$+ w_{0,\eta\eta}(\xi-\xi_1,\eta-\eta_1)\,\hat{w}_{0,\xi\xi}(\xi-\xi_1,\eta-\eta_1)$$

$$- 2w_{0,\xi\eta}(\xi-\xi_1,\eta-\eta_1)\,\hat{w}_{0,\xi\eta}(\xi-\xi_1,\eta-\eta_1)\Big]\Big\}\, d\xi_1\, d\eta_1$$

where the double Fourier transforms of $h_1(\xi,\eta)$, $h_2(\xi,\eta)$, $g_1(\xi,\eta)$ and $g_2(\xi,\eta)$ are

$$H_1(\alpha,\beta) = \frac{\alpha^2}{(\alpha^2+\beta^2)^4 - 2\lambda\alpha^2(\alpha^2+\beta^2)^2 + \alpha^4}$$

$$H_2(\alpha,\beta) = \frac{(\alpha^2+\beta^2)^2 - 2\lambda\alpha^2}{(\alpha^2+\beta^2)^4 - 2\lambda\alpha^2(\alpha^2+\beta^2)^2 + \alpha^4}$$

$$G_1(\alpha,\beta) = \frac{(\alpha^2+\beta^2)^2}{(\alpha^2+\beta^2)^4 - 2\lambda\alpha^2(\alpha^2+\beta^2)^2 + \alpha^4} \tag{1.10}$$

$$G_2(\alpha,\beta) = \frac{-\alpha^2}{(\alpha^2+\beta^2)^4 - 2\lambda\alpha^2(\alpha^2+\beta^2)^2 + \alpha^4}$$

From the expressions for $H_1(\alpha,\beta)$ and $G_2(\alpha,\beta)$ one can observe that, for $\beta=0$, these expressions have a singularity at $\alpha=0$. A question arises as to the existence of the formal solutions set up in (1.9). The answer to this question lies in (1.8). This nonlinear differential operation enables one to utilize only second derivatives of $h_1(\xi,\eta)$ and $g_2(\xi,\eta)$ in the following fashion

$$\iint_{-\infty}^{\infty} h(\xi_1,\eta_1)\, U_{,\xi\xi}(\xi-\xi_1,\eta-\eta_1)\, d\xi_1 d\eta_1 = \iint_{-\infty}^{\infty} h_{,\xi\xi}(\xi_1,\eta_1)\, U(\xi-\xi_1,\eta-\eta_1)\, d\xi_1 d\eta_1$$

$$\iint_{-\infty}^{\infty} h(\xi_1,\eta_1)\, U_{,\eta\eta}(\xi-\xi_1,\eta-\eta_1)\, d\xi_1 d\eta_1 = \iint_{-\infty}^{\infty} h_{,\eta\eta}(\xi_1,\eta_1)\, U(\xi-\xi_1,\eta-\eta_1)\, d\xi_1 d\eta_1$$

$$\iint_{-\infty}^{\infty} h(\xi_1,\eta_1)\, U_{,\xi\eta}(\xi-\xi_1,\eta-\eta_1)\,d\xi_1\,d\eta_1 = \iint_{-\infty}^{\infty} h_{,\xi\eta}(\xi_1,\eta_1)\, U(\xi-\xi_1,\eta-\eta_1)\,d\xi_1\,d\eta_1$$

Hence, the use of only second derivatives of the transfer functions will remove the singularities at $\beta=0$ and $\alpha=0$.

This completes the solution to the second order of approximation. A higher order of approximation can be achieved by proceeding further with the successive substitutions, which will not be sought here.

2. Variational Equations and Stability Analysis

With the assumption that a solution of equations (1.1) can be found to a satisfactory order of approximation, one can consider the stability problem by seeking the possibility of admittance of a second solution. In other words, a variational equation will be set up in the neighborhood of the existing solution to verify the existence of bifurcation.

Returning to equations (1.1) and, assuming that they admit a second solution specified by $u(\xi,\eta)$ and $\psi(\xi,\eta)$, the deviation of which from the basic solution is small, one may linearize the equations with respect to $u(\xi,\eta)$ and $\psi(\xi,\eta)$. The compatibility equation and equilibrium equation therefore take the form

$$\nabla^4\psi + u_{,\xi\xi} = -2\gamma(\tilde{w}_{,\xi\xi}u_{,\eta\eta} + \tilde{w}_{,\eta\eta}u_{,\xi\xi} - 2\tilde{w}_{,\xi\eta}u_{,\xi\eta})$$

$$\nabla^4 u - \psi_{,\xi\xi} + 2\lambda u_{,\xi\xi} = 2\gamma(\phi_{,\xi\xi}u_{,\eta\eta} + \phi_{,\eta\eta}u_{,\xi\xi} - 2\phi_{,\xi\eta}u_{,\xi\eta} + \tilde{w}_{,\xi\xi}\psi_{,\eta\eta} + \tilde{w}_{,\eta\eta}\psi_{,\xi\xi} - 2\tilde{w}_{,\xi\eta}\psi_{,\xi\eta}) \qquad (2.1)$$

where $\tilde{w} = \tilde{w}_0 + \tilde{w}_1$ and $\phi = \phi_0 + \phi_1$.

Substituting equations (1.5) and (1.9) into (2.1) yields

$$\begin{aligned}
\nabla^4\psi + u_{,\xi\xi} = -2\gamma\Big\{ & u_{,\eta\eta}\iint_{-\infty}^{\infty} f^{''}(\xi_1,\eta_1)\,\bar{w}(\xi-\xi_1,\eta-\eta_1)\,d\xi_1\,d\eta_1 \\
& + u_{,\xi\xi}\iint_{-\infty}^{\infty} f^{\cdot\cdot}(\xi_1,\eta_1)\,\bar{w}(\xi-\xi_1,\eta-\eta_1)\,d\xi_1\,d\eta_1 \\
& - 2u_{,\xi\eta}\iint_{-\infty}^{\infty} f^{|\cdot}(\xi_1,\eta_1)\,\bar{w}(\xi-\xi_1,\eta-\eta_1)\,d\xi_1\,d\eta_1
\end{aligned}$$

$$\begin{aligned}
+2\gamma\Big\{\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty}\Big\{ & \big[u_{,\eta\eta}(\xi,\eta)g_1^{''}(\xi_1,\eta_1) + u_{,\xi\xi}(\xi,\eta)g_1^{\cdot\cdot}(\xi_1,\eta_1) - 2u_{,\xi\eta}(\xi,\eta)g_1^{|\cdot}(\xi_1,\eta_1)\big]\cdot \\
& \cdot\big[h^{''}(\xi_2,\eta_2)f^{\cdot\cdot}(\xi_3,\eta_3) + h^{\cdot\cdot}(\xi_2,\eta_2)f^{''}(\xi_3,\eta_3) - 2h^{|\cdot}(\xi_2,\eta_2)f^{|\cdot}(\xi_3,\eta_3)\big] \\
& -\tfrac{1}{2}\big[u_{,\eta\eta}(\xi,\eta)g_2^{''}(\xi_1,\eta_1) + u_{,\xi\xi}g_2^{\cdot\cdot}(\xi_1,\eta_1) - 2u_{,\xi\eta}(\xi,\eta)g_2^{|\cdot}(\xi_1,\eta_1)\big]\cdot
\end{aligned}$$

$$\cdot\left[f_1^{\prime\prime}(\xi_2,\eta_2)\, g^{\cdot\cdot}(\xi_3,\eta_3) + f_1^{\cdot\cdot}(\xi_2,\eta_2)\, g^{\prime\prime}(\xi_3,\eta_3) - 2f_1^{\prime\cdot}(\xi_2,\eta_2)\, g^{\prime\cdot}(\xi_3,\eta_3)\right]\Big\}\cdot$$

$$\cdot\bar{w}(\xi-\xi_1-\xi_2,\, \eta-\eta_1-\eta_2)\,\bar{w}(\xi-\xi_1-\xi_3,\, \eta-\eta_1-\eta_3)\, d\xi_1 d\eta_1\, d\xi_2 d\eta_2\, d\xi_3 d\eta_3\Big\}\Bigg\}$$

$$(2.2)$$

and

$$\nabla^4 u - \psi_{,\xi\xi} + 2\lambda u_{,\xi\xi} = 2\gamma\Bigg\{u_{,\eta\eta}\iint_{-\infty}^{\infty} h^{\prime\prime}(\xi_1,\eta_1)\,\bar{w}(\xi-\xi_1,\, \eta-\eta_1)\, d\xi_1 d\eta_1$$

$$+ u_{,\xi\xi}\iint_{-\infty}^{\infty} h^{\cdot\cdot}(\xi_1,\eta_1)\,\bar{w}(\xi-\xi_1,\, \eta-\eta_1)\, d\xi_1 d\eta_1$$

$$- 2u_{,\xi\eta}\iint_{-\infty}^{\infty} h^{\prime\cdot}(\xi_1,\eta_1)\,\bar{w}(\xi-\xi_1,\, \eta-\eta_1)\, d\xi_1 d\eta_1$$

$$+ \psi_{,\eta\eta}\iint_{-\infty}^{\infty} f^{\prime\prime}(\xi_1,\eta_1)\,\bar{w}(\xi-\xi_1,\, \eta-\eta_1)\, d\xi_1 d\eta_1$$

$$+ \psi_{,\xi\xi}\iint_{-\infty}^{\infty} f^{\cdot\cdot}(\xi_1,\eta_1)\,\bar{w}(\xi-\xi_1,\, \eta-\eta_1)\, d\xi_1 d\eta_1$$

$$- 2\psi_{,\xi\eta}\iint_{-\infty}^{\infty} f^{\prime\cdot}(\xi_1,\eta_1)\,\bar{w}(\xi-\xi_1,\, \eta-\eta_1)\, d\xi_1 d\eta_1 \quad +$$

$$+2\gamma\Big\{\int\!\!\int_{-\infty}^{\infty}\int\!\!\int_{-\infty}^{\infty}\int\!\!\int_{-\infty}^{\infty}\Big\{\Big[u_{,\eta\eta}(\xi,\eta)\,h_1''(\xi_1,\eta_1)+u_{,\xi\xi}(\xi,\eta)\,h_1^{\bullet\bullet}(\xi_1,\eta_1)-2u_{,\xi\eta}(\xi,\eta)\,h_1^{\prime\bullet}(\xi_1,\eta_1)$$

$$+\psi_{,\eta\eta}(\xi,\eta)\,g_1''(\xi_1,\eta_1)+\psi_{,\xi\xi}(\xi,\eta)\,g_1^{\bullet\bullet}(\xi_1,\eta_1)-2\psi_{,\xi\eta}(\xi,\eta)\,g_1^{\prime\bullet}(\xi_1,\eta_1)\Big]\cdot$$

$$\cdot\Big[h''(\xi_2,\eta_2)\,f^{\bullet\bullet}(\xi_3,\eta_2)+h^{\bullet\bullet}(\xi_2,\eta_2)\,f''(\xi_3,\eta_3)-2h^{\prime\bullet}(\xi_2,\eta_2)\,f^{\prime\bullet}(\xi_3,\eta_3)\Big]$$

$$-\frac{1}{2}\Big[u_{,\eta\eta}(\xi,\eta)\,h_2''(\xi_1,\eta_1)+u_{,\xi\xi}(\xi,\eta)\,h_2^{\bullet\bullet}(\xi_1,\eta_1)-2u_{,\xi\eta}(\xi,\eta)\,h_2^{\prime\bullet}(\xi_1,\eta_1)$$

$$+\psi_{,\eta\eta}(\xi,\eta)\,g_2''(\xi_1,\eta_1)+\psi_{,\xi\xi}(\xi,\eta)\,g_2^{\bullet\bullet}(\xi_1,\eta_1)-2\psi_{,\xi\eta}(\xi,\eta)\,g_2^{\prime\bullet}(\xi_1,\eta_1)\Big]\cdot$$

$$\cdot\Big[f_1''(\xi_2,\eta_2)\,g^{\bullet\bullet}(\xi_3,\eta_3)+f_1^{\bullet\bullet}(\xi_2,\eta_2)\,g''(\xi_3,\eta_3)-2f_1^{\prime\bullet}(\xi_2,\eta_2)\,g^{\prime\bullet}(\xi_3,\eta_3)\Big]\Big\}\cdot$$

$$\cdot\,\bar{w}(\xi-\xi_1-\xi_2,\eta-\eta_1-\eta_2)\,\bar{w}(\xi-\xi_1-\xi_3,\eta-\eta_1-\eta_3)\,d\xi_1\,d\eta_1\,d\xi_2\,d\eta_2\,d\xi_3\,d\eta_3\Big\}\Big\}$$

(2.3)

where prime denotes differentiation with respect to the first argument in the function and dot denotes differentiation with respect to the second argument in the function.

Let the following correlation functions now be defined

$$R_u(\rho,\tau) = E\{u(\xi+\rho,\eta+\tau)\,u(\xi,\eta)\}$$

$$R_\psi(\rho,\tau) = E\{\psi(\xi+\rho,\eta+\tau)\,\psi(\xi,\eta)\}$$

$$R_{u\psi}(\rho,\tau) = E\{u(\xi+\rho,\eta+\tau)\,\psi(\xi,\eta)\}$$

$$R_{\psi u}(\rho,\tau) = E\{\psi(\xi+\rho,\eta+\tau)\,u(\xi,\eta)\} \tag{2.4}$$

and

$$R_{11}(\rho,\tau;\mu,\varsigma) = E\{u(\xi+\rho,\eta+\tau)\,u(\xi,\eta)\,\bar{w}(\xi+\mu,\eta+\varsigma)\}$$

$$R_{12}(\rho,\tau;\mu,\varsigma) = E\{u(\xi+\rho,\eta+\tau)\,\psi(\xi,\eta)\,\bar{w}(\xi+\mu,\eta+\varsigma)\}$$

$$R_{21}(\rho,\tau;\mu,\varsigma) = E\{\psi(\xi+\rho,\eta+\tau)\,u(\xi,\eta)\,\bar{w}(\xi+\mu,\eta+\varsigma)\}$$

$$R_{22}(\rho,\tau;\mu,\zeta) = E\left\{\psi(\xi+\rho,\eta+\tau)\,\psi(\xi,\eta)\,\bar{w}(\xi+\mu,\eta+\zeta)\right\}$$

(2.5)

where the double integrals associated with the expectation are taken over the repeated variables.

At this point one should also consider the proper approximation for truncation in the technique to be used. The correlation discard approximation in the process of closing the hierarchy in a typical case is

$$E\left\{\bar{w}(\xi+\mu,\eta+\zeta)\,\bar{w}(\xi+\mu_1,\eta+\zeta_1)\,u(\xi+\mu_1,\eta+\zeta_1)\,u(\xi,\eta)\right\}$$

$$\doteqdot E\left\{\bar{w}(\xi+\mu,\eta+\zeta)\,\bar{w}(\xi+\mu_1,\eta+\zeta_1)\right\}\,E\left\{u(\xi+\mu_1,\eta+\zeta_1)\,u(\xi,\eta)\right\}$$

$$= R_{\bar{w}}(\mu-\mu_1,\zeta-\zeta_1)\,R_u(\mu_1,\zeta_1)$$

(2.6)

Although approximations of this kind have been applied in different physical problems, their validity has not been established. Nevertheless, for the case of buckling one can find in the work of Amazigo (Ref. 9) a comparison between truncated hierarchies and perturbation solutions in simple deterministic buckling problems. For reasonably small imperfections the agreement between the two techniques is good and therefore this technique will be adopted for the problem of shells for the case of small imperfections as compared to the shell thickness.

Equations (2.2) and (2.3) can be written for the point $(\xi+\bar{\xi}, \eta+\bar{\eta})$, in which case all differentiations are applied with respect to the proper arguments. Doing so, and multiplying equation (2.2) (written for the point $(\xi+\bar{\xi}, \eta+\bar{\eta})$) by $\psi(\xi,\eta)$, taking the expectation of the result and using the correlation discard approximation yields the following equation

$$\nabla^4 R_{\psi}(\bar{\xi},\bar{\eta}) + R_{u\psi,\xi\xi}(\bar{\xi},\bar{\eta}) = -2\gamma \left\{ \iint_{-\infty}^{\infty} \left\{ f^{\prime\prime}(\xi_1,\eta_1) R_{12,\zeta\zeta}(\bar{\xi},\zeta;\bar{\xi}-\xi_1,\bar{\eta}-\eta_1)\Big|_{\zeta=\bar{\eta}} \right.\right.$$

$$+ f^{\cdot\cdot}(\xi_1,\eta_1) R_{12,\mu\mu}(\mu,\bar{\eta};\bar{\xi}-\xi_1,\bar{\eta}-\eta_1)\Big|_{\mu=\bar{\xi}} - 2 f^{\prime\cdot}(\xi_1,\eta_1) R_{12,\mu\zeta}(\mu,\zeta;\bar{\xi}-\xi_1,\bar{\eta}-\eta_1)\Big|_{\substack{\mu=\bar{\xi}\\ \zeta=\bar{\eta}}} \left.\right\} d\xi_1 d\eta_1$$

$$+2\gamma \iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty} \left\{ \left[R^{\cdot\cdot}_{\psi u}(\bar{\xi},\bar{\eta}) g_1^{\prime\prime}(\xi_1,\eta_1) + R^{\prime\prime}_{\psi u}(\bar{\xi},\bar{\eta}) g_1^{\cdot\cdot}(\xi_1,\eta_1) - 2 R^{\prime\cdot}_{\psi u}(\bar{\xi},\bar{\eta}) g_1^{\prime\cdot}(\xi_1,\eta_1) \right] \cdot \right.$$

$$\cdot \left[h^{\prime\prime}(\xi_2,\eta_2) f^{\cdot\cdot}(\xi_3,\eta_3) + h^{\cdot\cdot}(\xi_2,\eta_2) f^{\prime\prime}(\xi_3,\eta_3) - 2 h^{\prime\cdot}(\xi_2,\eta_2) f^{\prime\cdot}(\xi_3,\eta_3) \right] -$$

$$-\frac{1}{2}\left[R^{\cdot\cdot}_{u\psi}(\bar{\xi},\bar{\eta})\,g_2^{''}(\xi_1,\eta_1)+R^{''}_{u\psi}(\bar{\xi},\bar{\eta})g_2^{\cdot\cdot}(\xi_1,\eta_1)-2R^{'\cdot}_{u\psi}(\xi,\eta)g_2^{'\cdot}(\xi_1,\eta_1)\right]\cdot$$

$$\cdot\left[f_1^{''}(\xi_2,\eta_2)\,g^{\cdot\cdot}(\xi_3,\eta_3)+f_1^{\cdot\cdot}(\xi_2,\eta_2)g^{''}(\xi_3,\eta_3)-2f_1^{'\cdot}(\xi_2,\eta_2)\,g^{'\cdot}(\xi_3,\eta_3)\right]\Big\}\cdot$$

$$\cdot R_{\bar{w}}(\xi_3-\xi_2,\eta_3-\eta_2)\,d\xi_1 d\eta_1\;d\xi_2 d\eta_2\;d\xi_3\,d\eta_3\Big\}$$

(2.7)

Considering equation (2.2) again at the point $(\xi+\bar{\xi},\ \eta+\bar{\eta})$, multiplying by $u(\xi,\eta)$ and taking the expectation of the resulting equation yields

$$\nabla^4 R_{\psi u}(\bar{\xi},\bar{\eta})+R_{u,\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\eta})=-2\gamma\left\{\iint_{-\infty}^{\infty}\Big\{f^{''}(\xi_1,\eta_1)\,R_{11,\zeta\zeta}(\bar{\xi},\zeta\,;\bar{\xi}+\xi_1,\bar{\eta}+\eta_1)\Big|_{\zeta=\bar{\eta}}\right.$$

$$+f^{\cdot\cdot}(\xi_1,\eta_1)R_{11,\mu\mu}(\mu,\bar{\eta}\,;\bar{\xi}+\xi_1,\bar{\eta}+\eta_1)\Big|_{\mu=\bar{\xi}}-2f^{'\cdot}(\xi_1,\eta_1)R_{11,\mu\zeta}(\mu,\zeta\,;\bar{\xi}+\xi_1,\bar{\eta}+\eta_1)\Big|_{\substack{\mu=\bar{\xi}\\ \zeta=\bar{\eta}}}\Big\}d\xi_1\,d\eta_1$$

$$+2\gamma\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty}\Big\{\Big[R^{\cdot\cdot}_{u}(\bar{\xi},\bar{\eta})g_1^{''}(\xi_1,\eta_1)+R^{''}_{u}(\bar{\xi},\bar{\eta})g_1^{\cdot\cdot}(\xi_1,\eta_1)-2R^{'\cdot}_{u}(\bar{\xi},\bar{\eta})g^{'\cdot}(\xi_1,\eta_1)\Big]\cdot$$

$$\cdot\left[h''(\xi_2,\eta_2)f^{\cdot\cdot}(\xi_3,\eta_3)+h^{\cdot\cdot}(\xi_2,\eta_2)f''(\xi_3,\eta_3)-2h'^{\cdot}(\xi_2,\eta_2)f'^{\cdot}(\xi_3,\eta_3)\right]$$

$$-\frac{1}{2}\left[R_u^{\cdot\cdot}(\bar\xi,\bar\eta)g_2''(\xi_1,\eta_1)+R_u''(\bar\xi,\bar\eta)g_2^{\cdot\cdot}(\xi_1,\eta_1)-2R_u'^{\cdot}(\bar\xi,\bar\eta)g_2'^{\cdot}(\xi_1,\eta_1)\right]\cdot$$

$$\cdot\left[f_1''(\xi_2,\eta_2)g^{\cdot\cdot}(\xi_3,\eta_3)+f_1^{\cdot\cdot}(\xi_2,\eta_2)g''(\xi_3,\eta_3)-2f'^{\cdot}(\xi_2,\eta_2)g'^{\cdot}(\xi_3,\eta_3)\right]\Big\}\cdot$$

$$\cdot\, R_{\bar w}(\xi_3-\xi_2,\eta_3-\eta_2)\,d\xi_1\,d\eta_1\,d\xi_2\,d\eta_2\,d\xi_3\,d\eta_3\ \Big\}$$

(2.8)

The same procedure as described for equation (2.2) will be followed for equation (2.3), considering correlation discard approximation, the following two equations will result

$$\nabla^4 R_u(\bar\xi,\bar\eta)-R_{\psi u,\bar\xi\bar\xi}(\bar\xi,\bar\eta)+2\lambda R_{u,\bar\xi\bar\xi}=2\gamma\left\{\iint_{-\infty}^{\infty}\left\{h''(\xi_1,\eta_1)R_{11,\zeta\zeta}(\bar\xi,\zeta;\bar\xi-\xi_1,\bar\eta-\eta_1)\right|_{\zeta=\bar\eta}\right.$$

$$+h^{\cdot\cdot}(\xi_1,\eta_1)R_{11,\mu\mu}(\mu,\bar\eta;\bar\xi-\xi_1,\bar\eta-\eta_1)\Big|_{\mu=\bar\xi}-2h'^{\cdot}(\xi_1,\eta_1)R_{11,\mu\zeta}(\mu,\zeta;\bar\xi-\xi_1,\bar\eta-\eta_1)\Big|_{\substack{\mu=\bar\xi\\ \zeta=\bar\eta}}+$$

$$+f''(\xi_1,\eta_1)R_{21,\zeta\zeta}(\bar{\xi},\zeta;\bar{\xi}-\xi_1,\bar{\eta}-\eta_1)\Big|_{\zeta=\bar{\eta}}+f^{\cdot\cdot}(\xi_1,\eta_1)R_{21,\mu\mu}(\mu,\bar{\eta};\bar{\xi}-\xi_1,\bar{\eta}-\eta_1)\Big|_{\mu=\bar{\xi}}$$

$$-2f'^{\cdot}(\xi_1,\eta_1)R_{21,\mu\zeta}(\mu,\zeta;\bar{\xi}-\xi_1,\bar{\eta}-\eta_1)\Big|_{\substack{\mu=\bar{\xi}\\ \zeta=\bar{\eta}}}\Big\}\,d\xi_1\,d\eta_1$$

$$+2\gamma\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty}\Big\{\Big[R_{u}^{\cdot\cdot}(\bar{\xi},\bar{\eta})h_1''(\xi_1,\eta_1)+R_{u}''(\bar{\xi},\bar{\eta})h_1^{\cdot\cdot}(\xi_1,\eta_1)-2R_{u}'^{\cdot}(\bar{\xi},\bar{\eta})h_1'^{\cdot}(\xi_1,\eta_1)$$

$$+R_{\psi u}^{\cdot\cdot}(\bar{\xi},\bar{\eta})g_1''(\xi_1,\eta_1)+R_{\psi u}''(\bar{\xi},\bar{\eta})g_1^{\cdot\cdot}(\xi_1,\eta_1)-2R_{\psi u}'^{\cdot}(\bar{\xi},\bar{\eta})g_1'^{\cdot}(\xi_1,\eta_1)\Big]\cdot$$

$$\cdot\Big[h''(\xi_2,\eta_2)f^{\cdot\cdot}(\xi_3,\eta_3)+h^{\cdot\cdot}(\xi_2,\eta_2)f''(\xi_3,\eta_3)-2h'^{\cdot}(\xi_2,\eta_2)f'^{\cdot}(\xi_3,\eta_3)\Big]$$

$$-\frac{1}{2}\Big[R_{u}^{\cdot\cdot}(\bar{\xi},\bar{\eta})h_2''(\xi_1,\eta_1)+R_{u}''(\bar{\xi},\bar{\eta})h_2^{\cdot\cdot}(\xi_1,\eta_1)-2R_{u}'^{\cdot}(\bar{\xi},\bar{\eta})h_2'^{\cdot}(\xi_1,\eta_1)$$

$$+R_{\psi u}^{\cdot\cdot}(\bar{\xi},\bar{\eta})g_2''(\xi_1,\eta_1)+R_{\psi u}''(\bar{\xi},\bar{\eta})g_2^{\cdot\cdot}(\xi_1,\eta_1)-2R_{\psi u}'^{\cdot}(\bar{\xi},\bar{\eta})g_2'^{\cdot}(\xi_1,\eta_1)\Big]\cdot$$

$$\cdot\left[f_1^{\prime\prime}(\xi_2,\eta_2)\,g^{\cdot\cdot}(\xi_3,\eta_3)+f_1^{\cdot\cdot}(\xi_2,\eta_2)\,g^{\prime\prime}(\xi_3,\eta_3)-2f_1^{\prime\cdot}(\xi_2,\eta_2)\,g^{\prime\cdot}(\xi_3,\eta_3)\right]\Big\}\cdot$$

$$\cdot\, R_{\bar{w}}(\xi_3-\xi_2,\eta_3-\eta_2)\,d\xi_1\,d\eta_1\,d\xi_2\,d\eta_2\,d\xi_3\,d\eta_3\Bigg\} \qquad (2.9)$$

and

$$\nabla^4 R_{u\psi}(\bar{\xi},\bar{\eta})-R_{\psi,\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\eta})+2\lambda R_{u\psi,\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\eta})=$$

$$2\gamma\Bigg\{\iint_{-\infty}^{\infty}\Big\{h^{\prime\prime}(\xi_1,\eta_1)\,R_{12,\zeta\zeta}(\bar{\xi},\zeta;\bar{\xi}-\xi_1,\bar{\eta}-\eta_1)\Big|_{\zeta=\bar{\eta}}$$

$$+h^{\cdot\cdot}(\xi_1,\eta_1)\,R_{12,\mu\mu}(\mu,\bar{\eta};\bar{\xi}-\xi_1,\bar{\eta}-\eta_1)\Big|_{\mu=\bar{\xi}}$$

$$-2h^{\prime\cdot}(\xi_1,\eta_1)\,R_{12,\mu\zeta}(\mu,\zeta;\bar{\xi}-\xi_1,\bar{\eta}-\eta_1)\Big|_{\substack{\mu=\bar{\xi}\\ \zeta=\bar{\eta}}}$$

$$+f''(\xi_1,\eta_1)R_{22,\varsigma\varsigma}(\bar{\xi},\varsigma\ ;\bar{\xi}-\xi_1,\ \bar{\eta}-\eta_1)\Big|_{\varsigma=\bar{\eta}}$$

$$+f^{\cdot\cdot}(\xi_1,\eta_1)R_{22,\mu\mu}(\mu,\bar{\eta}\ ;\bar{\xi}-\xi_1,\ \bar{\eta}-\eta_1)\Big|_{\mu=\bar{\xi}}$$

$$-2f^{\prime\cdot}(\xi_1,\eta_1)R_{22,\mu\varsigma}(\mu,\varsigma;\ \bar{\xi}-\xi_1,\ \bar{\eta}-\eta_1)\Big|_{\substack{\mu=\bar{\xi}\\ \varsigma=\bar{\eta}}}$$

$$+2\gamma\iint\limits_{-\infty}^{\infty}\iint\limits_{-\infty}^{\infty}\iint\limits_{-\infty}^{\infty}\Big\{\Big[R^{\cdot\cdot}_{u\psi}(\bar{\xi},\bar{\eta})h_1''(\xi_1,\eta_1)+R''_{u\psi}(\bar{\xi},\bar{\eta})h_1^{\cdot\cdot}(\xi_1,\eta_1)-2R^{\prime\cdot}_{u\psi}(\bar{\xi},\bar{\eta})h_1^{\prime\cdot}(\xi_1,\eta_1)$$

$$+R^{\cdot\cdot}_{\psi}(\bar{\xi},\bar{\eta})g_1''(\xi_1,\eta_1)+R''_{\psi}(\bar{\xi},\bar{\eta})g_1^{\cdot\cdot}(\xi_1,\eta_1)-2R^{\prime\cdot}(\bar{\xi},\bar{\eta})g_1^{\prime\cdot}(\xi_1,\eta_1)\Big]\cdot$$

$$\cdot\Big[h''(\xi_2,\eta_2)f^{\cdot\cdot}(\xi_3,\eta_3)+h^{\cdot\cdot}(\xi_2,\eta_2)f''(\xi_3,\eta_3)-2h^{\prime\cdot}(\xi_2,\eta_2)f^{\prime\cdot}(\xi_3,\eta_3)\Big]$$

$$-\frac{1}{2}\Big[R^{\cdot\cdot}_{u\psi}(\bar{\xi},\bar{\eta})h_2''(\xi_1,\eta_1)+R''_{u\psi}(\bar{\xi},\bar{\eta})h^{\cdot\cdot}(\xi_1,\eta_1)-2R^{\prime\cdot}_{u\psi}(\bar{\xi},\bar{\eta})h^{\cdot\cdot}(\xi_1,\eta_1)$$

$$+R^{\cdot\cdot}_{\psi}(\bar{\xi},\bar{\eta})g_2''(\xi_1,\eta_1)+R''_{\psi}(\bar{\xi},\bar{\eta})g_2^{\cdot\cdot}(\xi_1,\eta_1)-2R^{\prime\cdot}_{\psi}(\bar{\xi},\bar{\eta})g^{\prime\cdot}(\xi_1,\eta_1)\Big]\cdot$$

$$\cdot\left[f_1''(\xi_2,\eta_2)\,g^{\cdot\cdot}(\xi_3,\eta_3)+f_1^{\cdot\cdot}(\xi_2,\eta_2)\,g''(\xi_3,\eta_3)-2f_1^{\prime\cdot}(\xi_2,\eta_2)\,g^{\prime\cdot}(\xi_3,\eta_3)\right]\Big\}\cdot$$

$$\cdot R_{\bar{w}}(\xi_3-\xi_2,\eta_3-\eta_2)\,d\xi_1\,d\eta_1\,d\xi_2\,d\eta_2\,d\xi_3\,d\eta_3\Bigg\}$$

(2.10)

To obtain the expressions for $R_{11}(\bar{\xi},\bar{\eta};\mu,\zeta)$, $R_{12}(\bar{\xi},\bar{\eta};\mu,\zeta)$, $R_{21}(\bar{\xi},\bar{\eta};\mu,\zeta)$ and $R_{22}(\bar{\xi},\bar{\eta};\mu,\zeta)$ one has to multiply equations (2.2) and (2.3) written out at the point $(\xi+\bar{\xi},\ \eta+\bar{\eta})$, by $u(\xi,\eta)\,\bar{w}(\xi+\mu,\ \eta+\zeta)$, take the expectation with respect to ξ and η, use the correlation discard approximation and neglect higher order terms. This will give results in the form of the following equations

$$\nabla^4 R_{11}(\bar{\xi},\bar{\eta};\mu,\zeta)-R_{21,\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\eta};\mu,\zeta)+$$

$$+2\lambda R_{11,\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\eta};\mu,\zeta)=$$

$$2\gamma\iint_{-\infty}^{\infty}\Big\{R_u^{\cdot\cdot}(\bar{\xi},\bar{\eta})\,h''(\xi_1,\eta_1)+R_u''(\bar{\xi},\bar{\eta})\,h^{\cdot\cdot}(\xi_1,\eta_1)-2R_u^{\prime\cdot}(\bar{\xi},\bar{\eta})\,h^{\prime\cdot}(\xi_1,\eta_1)+$$

$$+ R_{\psi u}^{\cdot\cdot}(\bar{\xi},\bar{\eta})\, f''(\xi_1,\eta_1) + R_{\psi u}''(\bar{\xi},\bar{\eta})\, f^{\cdot\cdot}(\xi_1,\eta_1) - 2R_{\psi u}^{\prime\cdot}(\bar{\xi},\bar{\eta})\, f^{\prime\cdot}(\xi_1,\eta_1) \Big\} \cdot$$

$$\cdot\, R_{\bar{w}}(\bar{\xi}-\xi_1-\mu,\ \bar{\eta}-\eta_1-\zeta)\, d\xi_1\, d\eta_1$$

(2.11)

$$\nabla^4 R_{21}(\bar{\xi},\bar{\eta};\mu,\zeta) + R_{11,\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\eta};\mu,\zeta) = -2\gamma \iint_{-\infty}^{\infty} \Big\{ f''(\xi_1,\eta_1)\, R_{u}^{\cdot\cdot}(\bar{\xi},\bar{\eta}) +$$

$$f^{\cdot\cdot}(\xi_1,\eta_1)\, R_{u}''(\bar{\xi},\bar{\eta}) - 2f^{\prime\cdot}(\xi_1,\eta_1)\, R^{\prime\cdot}(\bar{\xi},\bar{\eta}) \Big\}\, R_{\bar{w}}(\bar{\xi}-\xi_1-\mu,\ \bar{\eta}-\eta_1-\zeta)\, d\xi_1\, d\eta_1$$

(2.12)

$$\nabla^4 R_{12}(\bar{\xi},\bar{\eta};\mu,\zeta) - R_{22,\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\eta};\mu,\zeta) + 2\lambda R_{12,\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\eta};\mu,\zeta) =$$

$$= 2\gamma \iint_{-\infty}^{\infty} \Big\{ R_{u\psi}^{\cdot\cdot}(\bar{\xi},\bar{\eta})\, h''(\xi_1,\eta_1) + R_{u\psi}''(\bar{\xi},\bar{\eta})\, h^{\cdot\cdot}(\xi_1,\eta_1) - 2R_{u\psi}^{\prime\cdot}(\bar{\xi},\bar{\eta})\, h^{\prime\cdot}(\xi_1,\eta_1) +$$

$$+ R_{\psi}^{\cdot\cdot}(\bar{\xi},\bar{\eta})\, f''(\xi_1,\eta_1) + R''_{\psi}(\bar{\xi},\bar{\eta})\, f^{\cdot\cdot}(\xi_1,\eta_1) - 2R_{\psi}^{\prime\cdot}(\bar{\xi},\bar{\eta})\, f'^{\cdot}(\xi_1,\eta_1)\Big\} \cdot$$

$$\cdot\, R_{\bar{w}}(\bar{\xi}-\xi_1-\mu\,;\,\bar{\eta}-\eta_1-\zeta)\, d\xi_1\, d\eta_1 \tag{2.13}$$

and

$$\nabla^4 R_{22}(\bar{\xi},\bar{\eta}\,;\mu,\zeta) + R_{12,\xi\xi}(\bar{\xi},\bar{\eta}\,;\mu,\zeta) = -2\gamma \iint_{-\infty}^{\infty} \Big\{ f''(\xi_1,\eta_1) R_{u\psi}^{\cdot\cdot}(\bar{\xi},\bar{\eta})$$

$$+ f^{\cdot\cdot}(\xi_1,\eta_1) R''_{u\psi}(\bar{\xi},\bar{\eta}) - 2f'^{\cdot}(\xi_1,\eta_1) R_{u\psi}^{\prime\cdot}(\bar{\xi},\bar{\eta})\Big\} R_{\bar{w}}(\bar{\xi}-\xi_1-\mu,\, \bar{\eta}-\eta_1-\zeta)\, d\xi_1\, d\eta_1 \tag{2.14}$$

Formal solution of equations (2.11), (2.12), (2.13) and (2.14) yields

$$R_{11}(\bar{\xi},\bar{\eta}\,;\mu,\zeta) = 2\gamma \Big\{ \iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty} \Big\{ g_1(\xi_3,\eta_3)\Big[R_u^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)\, h''(\xi_2,\eta_2)$$

$$+ R''_u(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)\, h^{\cdot\cdot}(\xi_2,\eta_2) - 2R_u^{\prime\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)\, h'^{\cdot}(\xi_2,\eta_2)$$

$$+ R_{\psi u}^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)\, f''(\xi_2,\eta_2) + R''_{\psi u}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)\, f^{\cdot\cdot}(\xi_2,\eta_2)$$

$$-2R^{\dot{|}}_{\psi u}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{|\cdot}(\xi_2,\eta_2)\Big]-g_2(\xi_3,\eta_3)\Big[R_{\ddot{u}}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{''}(\xi_2,\eta_2)$$

$$+R_u^{''}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\cdot\cdot}(\xi_2,\eta_2)-2R^{|\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{|\cdot}(\xi_2,\eta_2)\Big]\cdot$$

$$\cdot R_{\bar{w}}(\bar{\xi}-\xi_2-\xi_3-\mu,\bar{\eta}-\eta_2-\eta_3-\zeta)\,d\xi_2\,d\eta_2\,d\xi_3\,d\eta_3$$

(2.15)

$$R_{12}(\bar{\xi},\bar{\eta};\mu,\zeta)=2\gamma\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty}\Big\{g_1(\xi_3,\eta_3)\Big[R_{\ddot{u}\psi}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{''}(\xi_2,\eta_2)+R^{''}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{\cdot\cdot}(\xi_2,\eta_2)$$

$$-2R^{\dot{|}}_{u\psi}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{|\cdot}(\xi_2,\eta_2)+R_{\ddot{\psi}}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{''}(\xi_2,\eta_2)$$

$$+R_{\psi}^{''}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\cdot\cdot}(\xi_2,\eta_2)-2R^{|\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{|\cdot}(\xi_2,\eta_2)\Big]$$

$$-g_2(\xi_3,\eta_3)\Big[R_{\ddot{u}\psi}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{''}(\xi_2,\eta_2)+R_{u\psi}^{''}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\cdot\cdot}(\xi_2,\eta_2)$$

$$-2R^{\dot{|}}_{u\psi}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{|\cdot}(\xi_2,\eta_2)\Big]\Big\}R_{\bar{w}}(\bar{\xi}-\xi_2-\xi_3-\mu,\bar{\eta}-\eta_2-\eta_3-\zeta)\,d\xi_2\,d\eta_2\,d\xi_3\,d\eta_3$$

(2.16)

$$R_{21}(\bar{\xi},\bar{\eta};\mu,\zeta) = 2\gamma \iint\limits_{-\infty}^{\infty} \iint\limits_{-\infty}^{\infty} \Big\{ h_1(\xi_3,\eta_3)\Big[R_u^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{\prime\prime}(\xi_2,\eta_2) + R_u^{\prime\prime}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{\cdot\cdot}(\xi_2,\eta_2)$$

$$-2R_u^{\prime\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{\prime\cdot}(\xi_2,\eta_2) + R_{\psi u}^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\prime\prime}(\xi_2,\eta_2)$$

$$+R_{\psi u}^{\prime\prime}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\cdot\cdot}(\xi_2,\eta_2) - 2R_{\psi u}^{\prime\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\prime\cdot}(\xi_2,\eta_2)\Big]$$

$$-h_2(\xi_3,\eta_3)\Big[R_u^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\prime\prime}(\xi_2,\eta_2) + R_u^{\prime\prime}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\cdot\cdot}(\xi_2,\eta_2)$$

$$-2R_u^{\prime\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\prime\cdot}(\xi_2,\eta_2)\Big]\Big\} R_{\bar{w}}(\bar{\xi}-\xi_3-\xi_2-\mu,\bar{\eta}-\xi_3-\xi_2-\zeta)\,d\xi_2\,d\eta_2\,d\xi_3\,d\eta_3$$

(2.17)

$$R_{22}(\bar{\xi},\bar{\eta};\mu,\zeta) = 2\gamma \iint\limits_{-\infty}^{\infty} \iint\limits_{-\infty}^{\infty} \Big\{ h_1(\xi_3,\eta_3)\Big[R_{u\psi}^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{\prime\prime}(\xi_2,\eta_2) + R_{u\psi}^{\prime\prime}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{\cdot\cdot}(\xi_2,\eta_2)$$

$$-2R_{u\psi}^{\prime\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{\prime\cdot}(\xi_2,\eta_2) + R_{\psi}^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{\prime\prime}(\xi_2,\eta_2)$$

$$R_{\psi}^{\prime\prime}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{\cdot\cdot}(\xi_2,\eta_2) - 2R_{\psi}^{\prime\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{\prime\cdot}(\xi_2,\eta_2)\Big]$$

$$-h_2(\xi_3,\eta_3)\Big[R_{u\psi}^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\prime\prime}(\xi_2,\eta_2) + R_{u\psi}^{\prime\prime}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\cdot\cdot}(\xi_2,\eta_2)$$

$$-2R_{u\psi}^{\prime\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\prime\cdot}(\xi_2,\eta_2)\Big]\Big\} R_{\bar{w}}(\bar{\xi}-\xi_3-\xi_2-\mu,\bar{\eta}-\eta_3-\eta_2-\zeta)\,d\xi_2\,d\eta_2\,d\xi_3\,d\eta_3$$

(2.18)

Upon proper differentiation of expressions (2.15), (2.16), (2.17) and (2.18), setting up the necessary arguments and introducing into equations (2.7), (2.8), (2.9) and (2.10) one obtains

$$\nabla^4 R_\psi(\bar\xi,\bar\eta) + R_{u\psi,\bar\xi\bar\xi}(\bar\xi,\bar\eta) =$$

$$= -4\gamma^2 \int_{-\infty}^{\infty}\!\!\int\!\!\int_{-\infty}^{\infty}\!\!\int\int_{-\infty}^{\infty}\!\!\int \Big\{\Big\{\Big[f''(\xi_1,\eta_1)g_1^{\cdot\cdot}(\xi_3,\eta_3) + f^{\cdot\cdot}(\xi_1,\eta_1)g_1''(\xi_3,\eta_3) - 2f'^{\cdot}(\xi_1,\eta_1)g_1'^{\cdot}(\xi_3,\eta_3)\Big]\cdot$$

$$\Big[R_{u\psi}^{\cdot\cdot}(\bar\xi-\xi_3,\bar\eta-\eta_3)h''(\xi_2,\eta_2) + R_{u\psi}''(\bar\xi-\xi_3,\bar\eta-\eta_3)h^{\cdot\cdot}(\xi_2,\eta_2)$$

$$-2R_{u\psi}'^{\cdot}(\bar\xi-\xi_3,\bar\eta-\eta_3)h'^{\cdot}(\xi_2,\eta_2) + R_\psi^{\cdot\cdot}(\bar\xi-\xi_3,\bar\eta-\eta_3)f''(\xi_2,\eta_2)$$

$$R_\psi''(\bar\xi-\xi_3,\bar\eta-\eta_3)f^{\cdot\cdot}(\xi_2,\eta_2) - 2R_\psi'^{\cdot}(\bar\xi-\xi_3,\bar\eta-\eta_3)f'^{\cdot}(\xi_2,\eta_2)\Big]$$

$$-\Big[f''(\xi_1,\eta_1)g_2^{\cdot\cdot}(\xi_3,\eta_3) + f^{\cdot\cdot}(\xi_1,\eta_1)g_2''(\xi_3,\eta_3) - 2f'^{\cdot}(\xi_1,\eta_1)g_2'^{\cdot}(\xi_3,\eta_3)\Big]\cdot$$

$$\cdot\Big[R_{u\psi}^{\cdot\cdot}(\bar\xi-\xi_3,\bar\eta-\eta_3)f''(\xi_2,\eta_2) + R_{u\psi}''(\bar\xi-\xi_3,\bar\eta-\eta_3)f^{\cdot\cdot}(\xi_2,\eta_2) -$$

$$-2R^{\prime\cdot}_{u\psi}(\bar{\xi}-\xi_3,\ \bar{\eta}-\eta_3)f^{\prime\cdot}(\xi_2,\eta_2)\Big]\Big\}R_{\bar{w}}(\xi_1-\xi_2-\xi_3,\ \eta_1-\eta_2-\eta_3)$$

$$+\Big\{\Big[h^{\prime\prime}(\xi_2,\eta_2)f^{\cdot\cdot}(\xi_3,\eta_3)+h^{\cdot\cdot}(\xi_2,\eta_2)f^{\prime\prime}(\xi_3,\eta_3)-2h^{\prime\cdot}(\xi_2,\eta_2)f^{\prime\cdot}(\xi_3,\eta_3)\Big]\cdot$$

$$\cdot\Big[R^{\cdot\cdot}_{u\psi}(\bar{\xi},\bar{\eta})\,g_1^{\prime\prime}(\xi_1,\eta_1)+R^{\prime\prime}_{u\psi}(\bar{\xi},\bar{\eta})\,g_1^{\cdot\cdot}(\xi_1,\eta_1)-2R^{\prime\cdot}_{u\psi}(\bar{\xi},\bar{\eta})\,g_1^{\prime\cdot}(\xi_1,\eta_1)\Big]$$

$$-\frac{1}{2}\Big[f_1^{\prime\prime}(\xi_2,\eta_2)\,g^{\cdot\cdot}(\xi_3,\eta_3)+f_1^{\cdot\cdot}(\xi_2,\eta_2)\,g^{\prime\prime}(\xi_3,\eta_3)-2f^{\prime\cdot}(\xi_2,\eta_2)\,g^{\prime\cdot}(\xi_1,\eta_1)\Big]\cdot$$

$$\cdot\Big[R^{\cdot\cdot}_{u\psi}(\bar{\xi},\bar{\eta})\,g_2^{\prime\prime}(\xi_1,\eta_1)+R^{\prime\prime}_{u\psi}(\bar{\xi},\bar{\eta})\,g_2^{\cdot\cdot}(\xi_1,\eta_1)-2R^{\prime\cdot}_{u\psi}(\bar{\xi},\bar{\eta})\,g_2^{\prime\cdot}(\xi_1,\eta_1)\Big]\Big\}\cdot$$

$$\cdot\,R_{\bar{w}}(\xi_3-\xi_2,\ \eta_3-\eta_2)\Big\}\,d\xi_1\,d\eta_1\,d\xi_2\,d\eta_2\,d\xi_3\,d\eta_3 \qquad (2.19)$$

$$\nabla^4 R_{u\psi}(\bar{\xi},\bar{\eta}) - R_{\psi,\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\eta}) + 2\lambda R_{u\psi,\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\eta}) =$$

$$= 4\gamma^2 \iint\limits_{-\infty}^{\infty} \iint\limits_{-\infty}^{\infty} \iint\limits_{-\infty}^{\infty} \Big\{\Big\{\big[h^{\prime\prime}(\xi_1,\eta_1) g_1^{\cdot\cdot}(\xi_3,\eta_3) + h^{\cdot\cdot}(\xi_1,\eta_1) g_1^{\prime\prime}(\xi_3,\eta_3) - 2h^{\prime\cdot}(\xi_1,\eta_1) g_1^{\prime\cdot}(\xi_3,\eta_3)$$

$$+ f^{\prime\prime}(\xi_1,\eta_1) h_1^{\cdot\cdot}(\xi_3,\eta_3) + f^{\cdot\cdot}(\xi_1,\eta_1) h_1^{\prime\prime}(\xi_3,\eta_3) - 2f^{\prime\cdot}(\xi_1,\eta_1) h_1^{\prime\cdot}(\xi_3,\eta_3)\big] \cdot$$

$$\cdot \big[R_{u\psi}^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)\, h^{\prime\prime}(\xi_2,\eta_2) + R_{u\psi}^{\prime\prime}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)\, h^{\cdot\cdot}(\xi_2,\eta_2)$$

$$-2R_{u\psi}^{\prime\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)\, h^{\prime\cdot}(\xi_2,\eta_2) + R_{\psi}^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3) f^{\prime\prime}(\xi_2,\eta_2)$$

$$+ R_{\psi}^{\prime\prime}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3) f^{\cdot\cdot}(\xi_2,\eta_2) - 2R_{\psi}^{\prime\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3) f^{\prime\cdot}(\xi_2,\eta_2)\big]$$

$$-\big[h^{\prime\prime}(\xi_1,\eta_1) g_2^{\cdot\cdot}(\xi_3,\eta_3) + h^{\cdot\cdot}(\xi_1,\eta_1) g_2^{\prime\prime}(\xi_3,\eta_3) - 2h^{\prime\cdot}(\xi_1,\eta_1) g_2^{\prime\cdot}(\xi_3,\eta_3)$$

$$+ f^{\prime\prime}(\xi_1,\eta_1) h_2^{\cdot\cdot}(\xi_3,\eta_3) + f^{\cdot\cdot}(\xi_1,\eta_1) h_2^{\prime\prime}(\xi_3,\eta_3) - 2f^{\prime\cdot}(\xi_1,\eta_2) h_2^{\prime\cdot}(\xi_3,\eta_3)\big] \cdot$$

$$\cdot \big[R_{u\psi}^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3) f^{\prime\prime}(\xi_2,\eta_2) + R_{u\psi}^{\prime\prime}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3) f^{\cdot\cdot}(\xi_2,\eta_2) -$$

$$- 2R_{u\psi}^{\prime\cdot}(\bar\xi-\xi_3\ ,\ \bar\eta-\eta_3)\, f^{\prime\cdot}(\xi_2,\eta_2)\Big]\Big\} \cdot R_{\bar w}(\xi_1-\xi_2-\xi_3\, ,\ \eta_1-\eta_2-\eta_3)$$

$$+\Big\{\Big[h''(\xi_2,\eta_2)\, f^{\cdot\cdot}(\xi_3,\eta_3) + h^{\cdot\cdot}(\xi_2,\eta_2)\, f''(\xi_3,\eta_3) - 2h^{\prime\cdot}(\xi_2,\eta_2)\, f^{\prime\cdot}(\xi_3,\eta_3)\Big]\cdot$$

$$\cdot\Big[R_{u\psi}^{\cdot\cdot}(\bar\xi,\bar\eta)\, h_1''(\xi_1,\eta_1) + R_{u\psi}''(\bar\xi,\bar\eta)\, h_1^{\cdot\cdot}(\xi_1,\eta_1) - 2R_{u\psi}^{\prime\cdot}(\bar\xi,\bar\eta)\, h_1^{\prime\cdot}(\xi_1,\eta_1)$$

$$+ R_{\psi}^{\cdot\cdot}(\bar\xi,\bar\eta)\, g_1''(\xi_1,\eta_1) + R_{\psi}''(\bar\xi,\bar\eta)\, g_1^{\cdot\cdot}(\xi_1,\eta_1) - 2R_{\psi}^{\prime\cdot}(\bar\xi,\bar\eta)\, g_1^{\prime\cdot}(\xi_1,\eta_1)\Big]$$

$$-\frac{1}{2}\Big[f_1''(\xi_2,\eta_2)\, g^{\cdot\cdot}(\xi_3,\eta_3) + f_1^{\cdot\cdot}(\xi_2,\eta_2)\, g''(\xi_1,\eta_1) - 2 f_1^{\prime\cdot}(\xi_2,\eta_2)\, g^{\prime\cdot}(\xi_1,\eta_1)\Big]\cdot$$

$$\cdot\Big[R_{u\psi}^{\cdot\cdot}(\bar\xi,\bar\eta)\, h_2''(\xi_1,\eta_1) + R_{u\psi}''(\bar\xi,\bar\eta)\, h_2^{\cdot\cdot}(\xi_1,\eta_1) - 2R_{u\psi}^{\prime\cdot}(\bar\xi,\bar\eta)\, h_2^{\prime\cdot}(\xi_1,\eta_1)$$

$$+ R_{\psi}^{\cdot\cdot}(\bar\xi,\bar\eta)\, g_2''(\xi_1,\eta_1) + R_{\psi}''(\bar\xi,\bar\eta)\, g_2^{\cdot\cdot}(\xi_1,\eta_1) - 2R_{\psi}^{\prime\cdot}(\bar\xi,\bar\eta)\, g_2^{\prime\cdot}(\xi_1,\eta_1)\Big]\Big\}\cdot$$

$$\cdot\, R_{\bar w}(\xi_3-\xi_2\ ,\ \eta_3-\eta_2)\ \Big\}\ d\xi_1\, d\eta_1\, d\xi_2\, d\eta_2\, d\xi_3\, d\eta_3$$

(2.20)

$$\nabla^4 R_{\psi u}(\bar{\xi},\bar{\eta}) + R_{u,\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\eta}) =$$

$$= -4\gamma^2 \iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty} \Big\{\Big\{\Big[f''(\xi_1,\eta_1)g_1^{\cdot\cdot}(\xi_3,\eta_3)+f^{\cdot\cdot}(\xi_1,\eta_1)g_1''(\xi_3,\eta_3)-2f'^{\cdot}(\xi_1,\eta_1)g_1'^{\cdot}(\xi_3,\eta_3)\Big]\cdot$$

$$\cdot\Big[R_u^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h''(\xi_2,\eta_2)+R_u''(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{\cdot\cdot}(\xi_2,\eta_2)$$

$$-2R_u'^{\cdot}(\bar{\xi}-\xi_2,\bar{\eta}-\eta_2)h'^{\cdot}(\xi_2,\eta_2) + R_{\psi u}^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f''(\xi_2,\eta_2)$$

$$R_{\psi u}''(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\cdot\cdot}(\xi_2,\eta_2)-2R_{\psi u}'^{\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f'^{\cdot}(\xi_2,\eta_2)\Big]$$

$$-\Big[f''(\xi_1,\eta_1)g_2^{\cdot\cdot}(\xi_3,\eta_3)+f^{\cdot\cdot}(\xi_1,\eta_1)g_2''(\xi_3,\eta_3)-2f'^{\cdot}(\xi_1,\eta_1)g_2'^{\cdot}(\xi_3,\eta_3)\Big]\cdot$$

$$\cdot\Big[R_u^{\cdot\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f''(\xi_2,\eta_2) + R_u''(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\cdot\cdot}(\xi_2,\eta_2)$$

$$-2R_u'^{\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f'^{\cdot}(\xi_2,\eta_2)\Big]\Big\} R_{\bar{w}}(\xi_1-\xi_2-\xi_3,\eta_1-\eta_2-\eta_3)+$$

$$+\Big\{\Big[h''(\xi_2,\eta_2)f^{\cdot\cdot}(\xi_3,\eta_3)+h^{\cdot\cdot}(\xi_2,\eta_2)f''(\xi_3,\eta_3)-2h'^{\cdot}(\xi_2,\eta_2)f'^{\cdot}(\xi_3,\eta_3)\Big]\cdot$$

$$\cdot\Big[R_u^{\cdot\cdot}(\bar\xi,\bar\eta)g_1''(\xi_1,\eta_1)+R_u''(\bar\xi,\bar\eta)g_1^{\cdot\cdot}(\xi_1,\eta_1)-2R_u'^{\cdot}(\bar\xi,\bar\eta)g_1'^{\cdot}(\xi_1,\eta_1)\Big]$$

$$-\tfrac{1}{2}\Big[f_1''(\xi_2,\eta_2)g^{\cdot\cdot}(\xi_3,\eta_3)+f_1^{\cdot\cdot}(\xi_2,\eta_2)g''(\xi_3,\eta_3)-2f_1'^{\cdot}(\xi_2,\eta_2)g'^{\cdot}(\xi_3,\eta_3)\Big]\cdot$$

$$\cdot\Big[R_u^{\cdot\cdot}(\bar\xi,\bar\eta)g_2''(\xi_1,\eta_1)+R_u''(\bar\xi,\bar\eta)g_2^{\cdot\cdot}(\xi_1,\eta_1)-2R_u'^{\cdot}(\bar\xi,\bar\eta)g_2'^{\cdot}(\xi_1,\eta_1)\Big]\Big\}\cdot$$

$$\cdot R_{\bar w}(\xi_3-\xi_2,\eta_3-\eta_2)\Big\}\,d\xi_1\,d\eta_1\,d\xi_2\,d\eta_2\,d\xi_3\,d\eta_3 \qquad (2.21)$$

and

$$\nabla^4 R_u(\bar\xi,\bar\eta)-R_{\psi u,\bar\xi\bar\xi}(\bar\xi,\bar\eta)+2\lambda R_{u,\bar\xi\bar\xi}(\bar\xi,\bar\eta)=$$

$$=4\gamma^2\iint\limits_{-\infty}^{\infty}\iint\limits_{-\infty}^{\infty}\iint\limits_{-\infty}^{\infty}\Big\{\Big\{\Big[h''(\xi_1,\eta_1)g_1^{\cdot\cdot}(\xi_3,\eta_3)+h^{\cdot\cdot}(\xi_1,\eta_1)g_1''(\xi_3,\eta_3)-2h'^{\cdot}(\xi_1,\eta_1)g'^{\cdot}(\xi_3,\eta_3)$$

$$+f''(\xi_1,\eta_1)h_1^{\cdot\cdot}(\xi_3,\eta_3)+f^{\cdot\cdot}(\xi_1,\eta_1)h_1''(\xi_3,\eta_3)-2f'^{\cdot}(\xi_1,\eta_1)h_1'^{\cdot}(\xi_3,\eta_3)\Big]\cdot$$

$$\cdot\left[R\ddot{u}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h''(\xi_2,\eta_2)+R_u''(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{\cdot\cdot}(\xi_2,\eta_2)\right.$$

$$-2R_u^{|\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)h^{|\cdot}(\xi_2,\eta_2)+R_{\psi\ddot{u}}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f''(\xi_2,\eta_2)$$

$$\left.+R_{\psi u}''(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\cdot\cdot}(\xi_2,\eta_2)-2R_{\psi u}^{|\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{|\cdot}(\xi_2,\eta_2)\right]$$

$$-\left[h''(\xi_1,\eta_1)g_2^{\cdot\cdot}(\xi_3,\eta_3)+h^{\cdot\cdot}(\xi_1,\eta_1)g_2''(\xi_3,\eta_3)-2h^{|\cdot}(\xi_1,\eta_1)g_2^{|\cdot}(\xi_3,\eta_3)\right.$$

$$\left.+f''(\xi_1,\eta_1)h_2^{\cdot\cdot}(\xi_3,\eta_3)+f^{\cdot\cdot}(\xi_1,\eta_1)h_2''(\xi_3,\eta_3)-2f^{|\cdot}(\xi_1,\eta_1)h_2^{|\cdot}(\xi_3,\eta_3)\right]\cdot$$

$$\cdot\left[R\ddot{u}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f''(\xi_2,\eta_2)+R_u''(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{\cdot\cdot}(\xi_2,\eta_2)\right.$$

$$\left.\left.-2R_u^{|\cdot}(\bar{\xi}-\xi_3,\bar{\eta}-\eta_3)f^{|\cdot}(\xi_2,\eta_2)\right]\right]\Big\}R_{\bar{w}}(\xi_1-\xi_2-\xi_3,\eta_1-\eta_2-\eta_3)$$

$$+\Big\{\left[f^{\cdot\cdot}(\xi_3,\eta_3)h''(\xi_2,\eta_2)+f''(\xi_3,\eta_3)h^{\cdot\cdot}(\xi_2,\eta_2)-2f^{|\cdot}(\xi_3,\eta_3)h^{|\cdot}(\xi_2,\eta_2)\right]\cdot$$

$$\cdot\Big[R_u^{\cdot\cdot}(\bar\xi,\bar\eta)\,h_1^{\prime\prime}(\xi_1,\eta_1) + R_u^{\prime\prime}(\bar\xi,\bar\eta)\,h_1^{\cdot\cdot}(\xi_1,\eta_1) - 2R_u^{\prime\cdot}(\bar\xi,\bar\eta)\,h_1^{\prime\cdot}(\xi_1,\eta_1)$$

$$+ R_{\psi u}^{\cdot\cdot}(\bar\xi,\bar\eta)\,g_1^{\prime\prime}(\xi_1,\eta_1) + R_{\psi u}^{\prime\prime}(\bar\xi,\bar\eta)\,g_1^{\cdot\cdot}(\xi_1,\eta_1) - 2R_{\psi u}^{\prime\cdot}(\bar\xi,\bar\eta)\,g_1^{\prime\cdot}(\xi_1,\eta_1)\Big]$$

$$-\Big[f_1^{\prime\prime}(\xi_2,\eta_2)\,g^{\cdot\cdot}(\xi_3,\eta_3) + f_1^{\cdot\cdot}(\xi_2,\eta_2)\,g^{\prime\prime}(\xi_3,\eta_3) - 2f_1^{\prime\cdot}(\xi_2,\eta_2)\,g^{\prime\cdot}(\xi_3,\eta_3)\Big]\cdot$$

$$\cdot\Big[R_u^{\cdot\cdot}(\bar\xi,\bar\eta)\,h_2^{\prime\prime}(\xi_1,\eta_1) + R_u^{\prime\prime}(\bar\xi,\bar\eta)\,h_2^{\cdot\cdot}(\xi_1,\eta_1) - 2R_u^{\prime\cdot}(\bar\xi,\bar\eta)\,h_2^{\prime\cdot}(\xi_1,\eta_1)$$

$$+ R_{\psi u}^{\cdot\cdot}(\bar\xi,\bar\eta)\,g_2^{\prime\prime}(\xi_1,\eta_1) + R_{\psi u}^{\prime\prime}(\bar\xi,\bar\eta)\,g_2^{\cdot\cdot}(\xi_1,\eta_1) - 2R_{\psi u}^{\prime\cdot}(\bar\xi,\bar\eta)\,g_2^{\prime\cdot}(\xi_1,\eta_1)\Big]\Big]\Big\}\cdot$$

$$\cdot R_{\bar w}(\xi_3-\xi_2,\ \eta_3-\eta_2)\Big\}\ d\xi_1\, d\eta_1\, d\xi_2\, d\eta_2\, d\xi_3\, d\eta_3 \qquad (2.22)$$

One can observe that the four equations obtained consist of two identical sets of equations for different sets of correlation functions. It will therefore be sufficient to concentrate in the following analysis on one set of equations, namely (2.21) and (2.22).

Upon applying double Fourier transform using proper convolution relations, the following equations are obtained

$$(\alpha^2+\beta^2)^2 S_{\psi u}(\alpha,\beta) - \alpha^2 S_u(\alpha,\beta) = - S_u(\alpha,\beta) I_1(\alpha,\beta) - S_{\psi u}(\alpha,\beta) I_2(\alpha,\beta)$$

$$\left[(\alpha^2+\beta^2)^2 - 2\lambda\alpha^2\right] S_u(\alpha,\beta) + \alpha^2 S_{\psi u}(\alpha,\beta) = \tag{2.23}$$

$$= S_u(\alpha,\beta) I_3(\alpha,\beta) + S_{\psi u}(\alpha,\beta) I_1(\alpha,\beta)$$

where

$$S_u(\alpha,\beta) = \iint_{-\infty}^{\infty} R_u(\xi,\eta)\, e^{-i(\alpha\xi+\beta\eta)}\, d\xi\, d\eta$$

$$\tag{2.24}$$

$$S_{\psi u}(\alpha,\beta) = \iint_{-\infty}^{\infty} R_{\psi u}(\xi,\eta)\, e^{-i(\alpha\xi+\beta\eta)}\, d\xi\, d\eta$$

and

$$I_1(\alpha,\beta) = \frac{\gamma^2}{\pi^2}\iint_{-\infty}^{\infty}(\alpha\beta_1-\alpha_1\beta)^4\, F(\alpha_1,\beta_1)\big[H(\alpha_1,\beta_1)\, G_1(\alpha-\alpha_1,\ \beta-\beta_1)$$

$$+F(\alpha_1,\beta_1)\, H_1(\alpha-\alpha_1,\beta-\beta_1)\big]\, S_{\bar{w}}(\alpha_1,\beta_1)\, d\alpha_1\, d\beta_1$$

$$I_2(\alpha,\beta) = \frac{\gamma^2}{\pi^2}\iint_{-\infty}^{\infty}(\alpha\beta_1-\alpha_1\beta)^4 \; F^2(\alpha_1,\beta_1)\cdot$$

$$\cdot\, G_1(\alpha-\alpha_1,\beta-\beta_1)\, S_{\bar{w}}(\alpha_1,\beta_1)\, d\alpha_1\, d\beta_1$$

(2.25)

$$I_3(\alpha,\beta) = \frac{\gamma^2}{\pi^2}\iint_{-\infty}^{\infty}(\alpha\beta_1-\alpha\beta_1)^4\Big\{\big[H^2(\alpha_1,\beta_1)-F^2(\alpha_1,\beta_1)\big]G_1(\alpha_1-\alpha,\beta_1-\beta)$$

$$+2F(\alpha_1,\beta_1)\big[H(\alpha_1,\beta_1)+\lambda F(\alpha_1,\beta_1)\big]H_1(\alpha_1-\alpha,\beta_1-\beta)\Big\}S_{\bar{w}}(\alpha_1,\beta_1)\,d\alpha_1 d\beta_1$$

Equations (2.23) are a system of linear homogeneous equations in $S_u(\alpha,\beta)$ and $S_{\psi u}(\alpha,\beta)$ for all (α,β). A necessary condition for the existence of a non-trivial solution for these functions is the vanishing of the determinant of coefficients. One therefore obtains,

$$(\alpha^2+\beta^2)^2 - 2\lambda\alpha^2 + \frac{\alpha^4}{(\alpha^2+\beta^2)^2} = I_3(\alpha,\beta) - I_2(\alpha,\beta)$$

$$+\frac{2\alpha^2}{(\alpha^2+\beta^2)^2}\big[I_1(\alpha,\beta)+\lambda I_2(\alpha,\beta)\big] + \frac{1}{(\alpha^2+\beta^2)^2}\big[I_2(\alpha,\beta)\,I_3(\alpha,\beta)-I_1^2(\alpha,\beta)\big]$$

(2.26)

Equation (2.26) is an implicit relation between λ , α and β . Naturally the lowest value of λ is to be sought. Formally, by minimizing λ with respect to α and β , will result two more relations which will uniquely determine the minimum value of λ as well as the corresponding α and β at which it will occur.

Since $I_1(\alpha,\beta)$, $I_2(\alpha,\beta)$ and $I_3(\alpha,\beta)$ involve λ in an implicit form in a rather complex fashion, the treatment of the solution from here on will be numerical rather than proceeding with cumbersome analytical relations. Finally one should point out that by minimizing λ with respect to α and β , one commits himself to a particular solution of the variational equation represented by a double simple harmonical mode. Since the variational equations are introduced in order to investigate the existence of a second solution in the neighborhood of a given solution, and since one is concerned with double continuous spectrums, it is possible to choose any arbitrary non-trivial spectrum mode. Naturally the double periodic spectrum modes are associated with the lowest value of λ .

3. Particular Cases

Prior to considering numerical examples of particular cases, one should investigate the power spectrum function $S_{\bar{w}}(\alpha,\beta)$ of the imperfections. This function is dependent on the two arguments α and β which can be either discrete or continuous.

Considering first the argument β , representing a circular modified frequency in the circumferential direction, it is obvious that this variable is discrete, and can only be of the form

$$\beta = \beta_n = n\,\Delta\beta \qquad n = 0, 1, 2, \ldots$$

For $n \gg 1$, β can be considered as a continuous variable for all practical numerical computations, this will be followed only by insignificant numerical errors. The axially symmetric mode of imperfections is naturally the case of $\beta = 0$.

Hence, since any general state of imperfections can be expanded in Fourier series in the circumferential direction in the form

$$\bar{w}(\xi, \eta) = \sum_{n=0}^{\infty} W_n(\xi) \sin\left[r_n \eta + S_n(\xi)\right] \tag{3.1}$$

where

$$r_n = n r \qquad n = 0, 1, 2, \ldots$$

$$r = \sqrt{h/(2\gamma R)}$$

and, since the concept of correlation function can be utilized not only with stationary random functions but also with periodic functions, it will be useful to examine the correlation function in the case of the function (3.1)

$$R_{\bar{w}}(\mu,\zeta) = \lim_{L\to\infty} \lim_{M\to\infty} \frac{1}{4LM} \int_{-L}^{L}\int_{-M}^{M} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \bar{W}_n(\xi)\, W_m(\xi+\mu) \sin[r_n\eta + S_n(\xi)] \cdot$$

$$\cdot \sin[r_m(\eta+\zeta) + S_m(\xi+\mu)]\, d\xi\, d\eta$$

$$= \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \Big\{ \bar{W}_0(\xi)\, \bar{W}_0(\xi+\mu) + \frac{1}{2}\sum_{n=1}^{\infty} \bar{W}_n(\xi)\, \bar{W}_n(\xi+\mu) \cdot$$

$$\cdot \cos[r_n\zeta + S_n(\xi+\mu) - S_n(\xi)] \Big\}\, d\xi$$

$$= R_{\bar{w}_0}(\mu) + \sum_{n=1}^{\infty} R_{\bar{w}_n}(\mu,\zeta) \qquad (3.2)$$

The last expression leads to some interesting conclusions; for cases where the phase $S_n(\xi)$ is constant (independent of ξ), the correlation function $R_{\bar{w}_n}(\mu,\zeta)$ takes the particular form

$$R_{\bar{w}_n}(\mu,\zeta) = \frac{1}{2} E_\xi \Big\{ \bar{W}_n(\xi)\, \bar{W}_n(\xi+\mu) \Big\} \cos r_n \zeta \qquad (3.3)$$

which physically means that there are no "torsional imperfections" present in the shell. On the other hand, for the more general case where the phase S_n is a function of ξ, "torsional imperfections" are present in the shell and the correlation function $R_{\bar{w}}(\mu,\zeta)$ takes a more complicated form.

In the following analysis it is assumed that S_n is independent of ξ . Furthermore, exponential cosine correlation functions for $\bar{W}_n(\xi)$ are examined numerically. The last functions are often used in control systems, and seem to be acceptable for stationary states of random imperfections (Ref. 8, 9). However, the selection of parameters in this particular correlation function is of major significance, since imperfections may occur in a certain range of frequencies. Particular attention should be paid to those modes of the power spectrum which are most effective in reducing the buckling strength yet remain reasonably practical.

Since, in the following the power spectrum $S_{\bar{W}}(\alpha, \beta)$ will be needed, one should note that the double Fourier transform of expression (3.2) for constant S_n yields

$$S_{\bar{W}}(\alpha,\beta) = 2\pi\left\{ \bar{S}_{\bar{W}_0}(\alpha)\delta(\beta) + \frac{1}{4}\sum_{n=1}^{\infty} \bar{S}_{\bar{W}_n}(\alpha)\,\delta(|\beta| - r_n) \right\} \tag{3.4}$$

where $\bar{S}_{\bar{W}_n}(\alpha)$, $n = 0, 1, 2, \cdots$ is the Fourier transform of the correlation function

$$\bar{R}_{\bar{W}_n}(\mu) = E_\xi\left\{ \bar{W}_n(\xi)\ \bar{W}_n(\xi+\mu)\right\} \qquad n = 0, 1, 2, \cdots$$

which in the following analysis will be taken in the form

$$\bar{R}_{\bar{W}_n}(\mu) = K_n^2\, e^{-\varepsilon_n|\mu|} \cos\theta_n\mu \tag{3.5}$$

The corresponding power spectrum $\bar{S}_{\bar{W}_n}(\alpha)$ will therefore be

$$S_{\bar{w}_n}(\alpha) = \frac{2K_n^2 \varepsilon_n (\alpha^2 + \varepsilon_n^2 + \theta_n^2)}{\alpha^4 + 2(\varepsilon_n^2 - \theta_n^2)\alpha^2 + (\varepsilon_n^2 + \theta_n^2)^2} \qquad (3.6)$$

Typical power spectrum curves are presented in Figure 1.

At this point let us turn to particular cases .

(i) Axisymmetric Imperfections

First, considering the case of axisymmetric imperfections for which

$$S_{\bar{w}}(\alpha, \beta) = 2\pi \bar{S}_{\bar{w}_o}(\alpha)\, \delta(\beta) \qquad (3.7)$$

Upon introducing this power spectrum into (2.25) one obtains

$$I_1^{\circ}(\alpha, \beta) = \frac{2\gamma^2}{\pi} \beta^4 \int_{-\infty}^{\infty} \alpha_1^4 F(\alpha_1, 0) \big[H(\alpha_1, 0)\, G_1(\alpha_1 - \alpha, \beta)$$

$$+ F(\alpha_1, 0) H_1(\alpha_1 - \alpha, \beta) \big] \bar{S}_{\bar{w}_o}(\alpha_1)\, d\alpha_1$$

$$I_2^{\circ}(\alpha, \beta) = \frac{2\gamma^2}{\pi} \beta^4 \int_{-\infty}^{\infty} \alpha_1^4 F^2(\alpha_1, 0)\, G_1(\alpha_1 - \alpha, \beta)\, \bar{S}_{\bar{w}_o}(\alpha_1)\, d\alpha_1$$

(3.8)

$$I_3^{\circ}(\alpha, \beta) = \frac{2\gamma^2}{\pi} \beta^4 \int_{-\infty}^{\infty} \Big\{ \big[H^2(\alpha_1, 0) - F^2(\alpha_1, 0) \big] G_1(\alpha_1 - \alpha, \beta) +$$

$$+ 2F(\alpha_1, o)\left[H(\alpha_1, o) + \lambda F(\alpha_1, o)\right] H_1(\alpha_1 - \alpha, \beta)\, \bar{S}_{\bar{w}_o}(\alpha_1)\, d\alpha_1$$

Introducing these expressions into (2.26) yields the final desired form for the characteristic equation, in the axisymmetric case. A comparison between the equation obtained and the one obtained by Amazigo (Ref. 9) reveals a slight difference. However, a numerical comparison between the results obtained from the two equations is almost in perfect agreement.

The integrals were evaluated numerically by means of Simpson's quadrature which has the feature of selecting the proper size of integration subintervals according to the desired number of significant figures. Minimization of λ as α and β vary continuously has been carried out numerically. With the assumption that the number of waves in the circumferential direction is large enough to justify continuous variation of β, which not always was the case, the final minimum values have been obtained.

From the expression for the power spectrum (3.6) one realizes that, for the axisymmetric case, the mode shape of $\bar{S}_{\bar{w}_o}(\alpha)$ is dependent on two parameters, namely, ε_o, θ_o. The case $\varepsilon_o = 0$ and $\theta_o = 1$ is the deterministic case of a simple periodic imperfection. The present analysis has been examined numerically, in addition to the deterministic case, for $\theta_o = 1$ and various ε_o between 0 and 1. These results are presented in Figure 3. In the particular case of axisymmetric random imperfections it was expected that the characteristic equation will be reduced to the one obtained in (Ref. 9). It turned out that a slightly different characteristic equation was obtained. Nevertheless, applying the present numerical integration technique for the integral terms in both cases revealed that quantitatively the results obtained from both characteristic equations were identical. However, these numerical results were not in agreement with those obtained in (Ref. 9). As explained in this reference the integrals have been evaluated numerically using calculus of residues. Since this technique has not been given in detail, it was impossible to investigate further the cause for the discrepancy and no further comments can be made.

(ii) Asymmetric Imperfections

Another case of practical significance is the one where, in addition to the axisymmetric mode of random imperfection, an asymmetric mode of random imperfection is present. Let the power spectrum for this case be

$$S_{\bar{w}}(\alpha,\beta) = 2\pi\left[\bar{S}_{\bar{w}_0}(\alpha)\,\delta(\beta) + \frac{1}{4}\bar{S}_{\bar{w}_1}(\alpha)\,\delta(|\beta|-k)\right] \tag{3.9}$$

Introducing this power spectrum into (2.25) yields

$$I_1(\alpha,\beta) = I_1^{\circ}(\alpha,\beta) + \frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty}\Big\{(\alpha k-\alpha_1\beta)^4\; F(\alpha_1,k)\cdot$$

$$\cdot\left[G_1(\alpha-\alpha_1,\beta-k)H(\alpha_1,k)+H_1(\alpha-\alpha_1,\beta-k)F(\alpha_1,k)\right] +$$

$$+(\alpha k + \alpha_1 \beta)^4 F(\alpha_1, k)\left[G_1(\alpha - \alpha_1,\ \beta + k)\ H(\alpha_1, k)\right.$$

$$\left.\left. + H_1(\alpha - \alpha_1,\ \beta + k)\, F(\alpha_1, k)\right]\right\} S_{\bar{w}_1}(\alpha_1)\, d\alpha_1$$

$$I_2(\alpha, \beta) = I_2^{\circ}(\alpha, \beta) + \frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} F^2(\alpha_1, k)\left[(\alpha k - \alpha_1 \beta)^4\, G_1(\alpha - \alpha_1, \beta - k)\right.$$

$$\left. + (\alpha k + \alpha_1 \beta)^4\, G_1(\alpha - \alpha_1, \beta + k)\right] \bar{S}_{\bar{w}_1}(\alpha_1)\, d\alpha_1$$

$$(3.10)$$

$$I_3(\alpha, \beta) = I_3^{\circ}(\alpha, \beta) + \frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \left\{ (\alpha k - \alpha_1 \beta)^4 \left\{ \left[H^2(\alpha, k) - F^2(\alpha, k) \right] G_1(\alpha_1 - \alpha, \beta - k) \right.\right.$$

$$\left. + 2F(\alpha_1, k)\left[H(\alpha_1, k) + \lambda F(\alpha_1, k)\right] H_1(\alpha_1 - \alpha, \beta - k) \right\}$$

$$+ (\alpha k + \alpha_1 \beta)^4 \left\{ \left[H^2(\alpha_1, k) - F^2(\alpha_1, k) \right] G_1(\alpha_1 - \alpha, \beta + k) \right.$$

$$\left.\left. + 2F(\alpha_1, k)\left[H(\alpha_1, k) + \lambda F(\alpha_1, k)\right] H_1(\alpha_1 - \alpha, \beta + k) \right\} \right\} \bar{S}_{\bar{w}_1}(\alpha_1)\, d\alpha_1$$

For the case where the instability occurs at $\beta = k$

$$I_1(\alpha,k) = I_1^\circ(\alpha,k) + \frac{\gamma^2 k^4}{2\pi}\int_{-\infty}^{\infty}\Big\{F(\alpha_1,k)\,H(\alpha_1,k)\big[(\alpha_1-\alpha)^4 G_1(\alpha_1-\alpha,0)$$

$$+ (\alpha_1+\alpha)^4 G_1(\alpha_1-\alpha,2k)\big] + F^2(\alpha_1,k)\big[H_1(\alpha_1-\alpha,0)(\alpha_1-\alpha)^4$$

$$+ (\alpha_1+\alpha)^4 H_1(\alpha_1-\alpha,2k)\big]\Big\}\,\bar{S}_{\bar{w}_1}(\alpha_1)\,d\alpha_1$$

$$I_2(\alpha,k) = I_2^\circ(\alpha,k) + \frac{\gamma^2 k^4}{2\pi}\int_{-\infty}^{\infty}F^2(\alpha_1,k)\big[(\alpha_1-\alpha)^4\,G_1(\alpha_1-\alpha,0)$$

$$+(\alpha_1+\alpha)^4 G_1(\alpha_1-\alpha,2k)\big]\,\bar{S}_{\bar{w}_1}(\alpha_1)\,d\alpha_1 \tag{3.11}$$

$$I_3(\alpha,k) = I_3^\circ(\alpha,k) + \frac{\gamma^2 k^4}{\pi}\int_{-\infty}^{\infty}\Big\{\big[H^2(\alpha_1,k) - F^2(\alpha_1,k)\big]\big[(\alpha_1-\alpha)^4 G_1(\alpha_1-\alpha,0)$$

$$+ (\alpha_1+\alpha)^4 G_1(\alpha_1-\alpha,2k)\big] + F(\alpha_1,k)\big[H(\alpha_1,k) + \lambda F(\alpha_1,k)\big]\cdot$$

$$\cdot\big[(\alpha_1-\alpha)^4 H_1(\alpha_1-\alpha,0) + (\alpha_1+\alpha)^4 H(\alpha_1-\alpha,2k)\big]\Big\}\,\bar{S}_{\bar{w}_1}(\alpha_1)\,d\alpha_1 .$$

Introducing expressions (3.8) properly into (3.11) and the resulting expressions into (2.26) yields the final desired form for the characteristic equation for this particular asymmetric case, where the instability mode occurs at $\beta = k$. λ , in this case is minimized with respect to α .

The integrals (3.10) or (3.11) were also evaluated numerically using Simpson's rule. Minimization of λ has been carried out numerically. The various results are presented in Figures 4 and 5. The family of curves obtained are qualitatively as anticipated. Quantitatively these results are valid for small imperfections which, in most practical applications, are the case. As the imperfections become in magnitude of the order of the shell thickness the present approach is no longer valid and other techniques will have to be sought. This is naturally coupled also into questions of validity of the equations of the shell and remains for future investigations.

Perhaps of all known techniques for stability analysis it seems that the Lyapunov approach is the most powerful tool for establishing sufficient conditions to such questions. This will involve the application of the theory of functionals.

4. Concluding Remarks

The method of truncated hierarchy proved to be a powerful tool in the stability analysis of the cylindrical shell with small imperfections. Although this technique is limited to a narrow class of imperfections, it is this class which is of major concern in engineering applications. Any attempt to adopt this technique for moderate imperfections will be followed by cumbersome computations associated with higher hierarchies. Perhaps a more difficult task would be to justify the results obtained. This naturally suggests the examination of other techniques which are not based on a direct solution of the equations, yet rather investigate the properties of the solutions. As pointed out before, the Lyapunov analysis is one way to approach this problem.

In conclusion, the analysis presented in Part III and in particular the characteristic equation obtained, are sufficient to establish the buckling load in practical applications for cylindrical shells with stationary random imperfections. The particular numerical cases considered in this work are only the first step in investigating numerically the nature of the problem where a more complicated state of imperfections is concerned. Finally the present study can easily be extended to the buckling problem where, in addition to axial load, the shell is internally pressurized.

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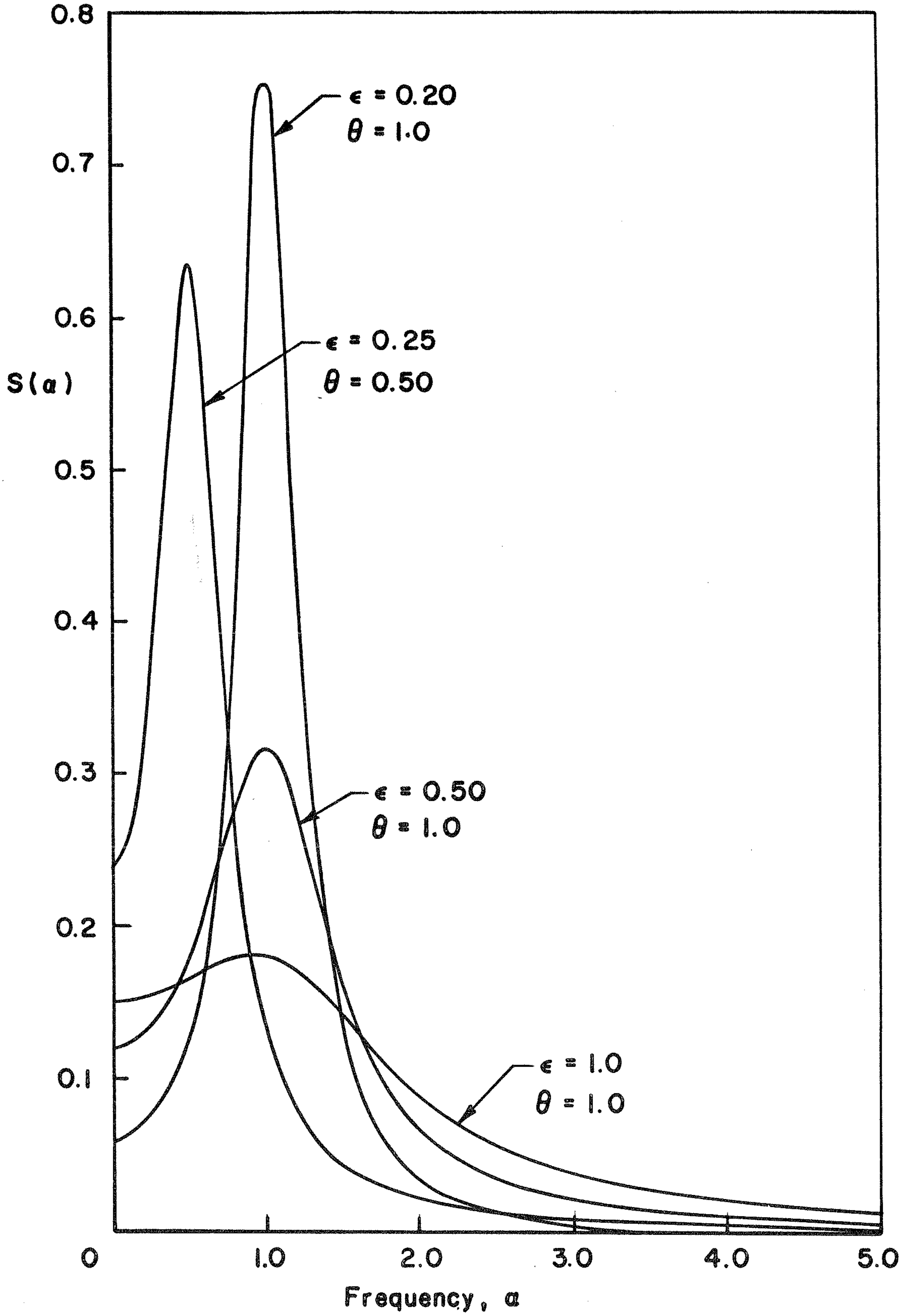

FIG. I TYPICAL POWER SPECTRUM CURVES

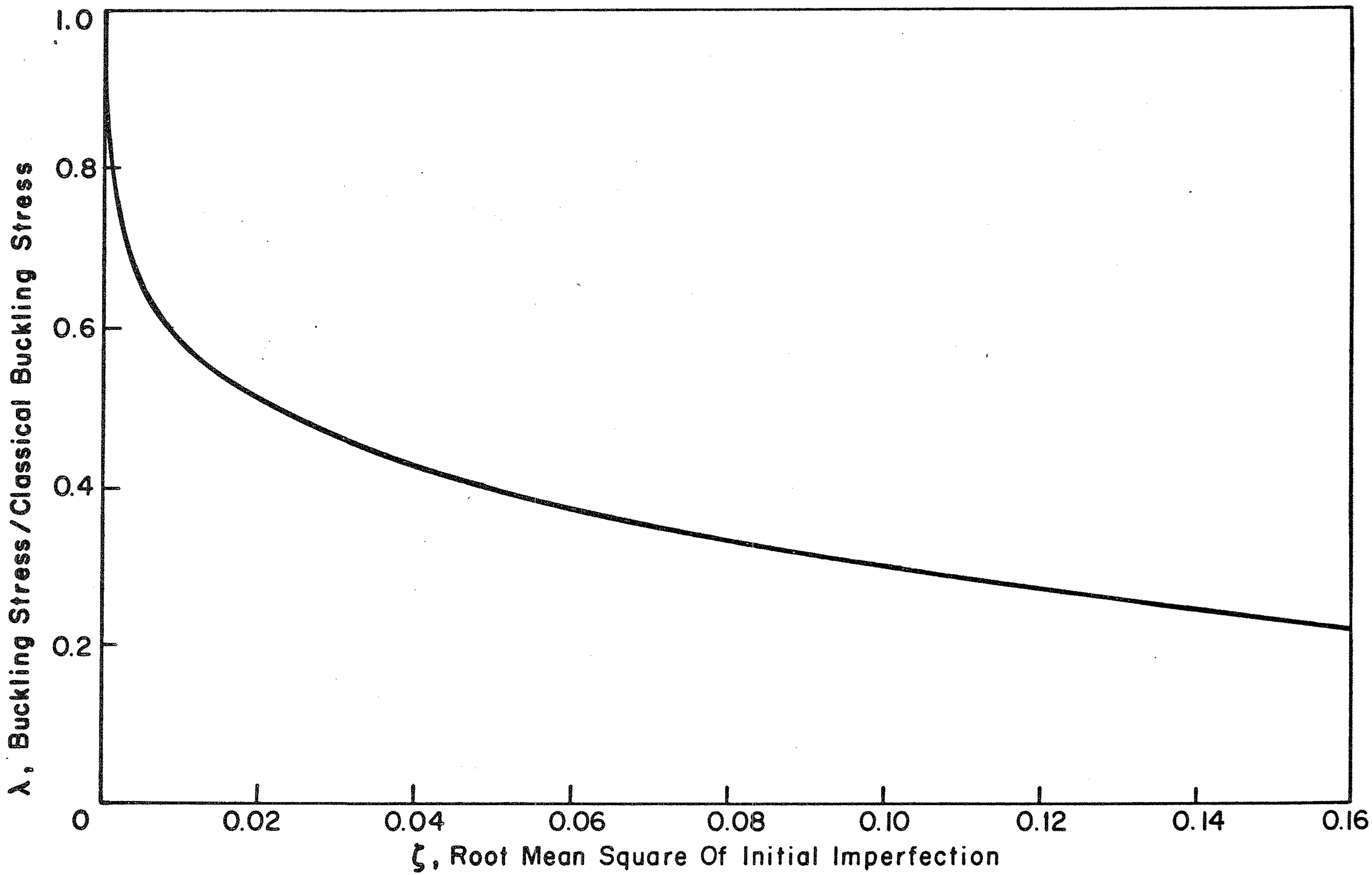

FIG. 2 STABILITY BOUNDARY FOR A CYLINDRICAL SHELL WITH AXISYMMETRIC IMPERFECTIONS

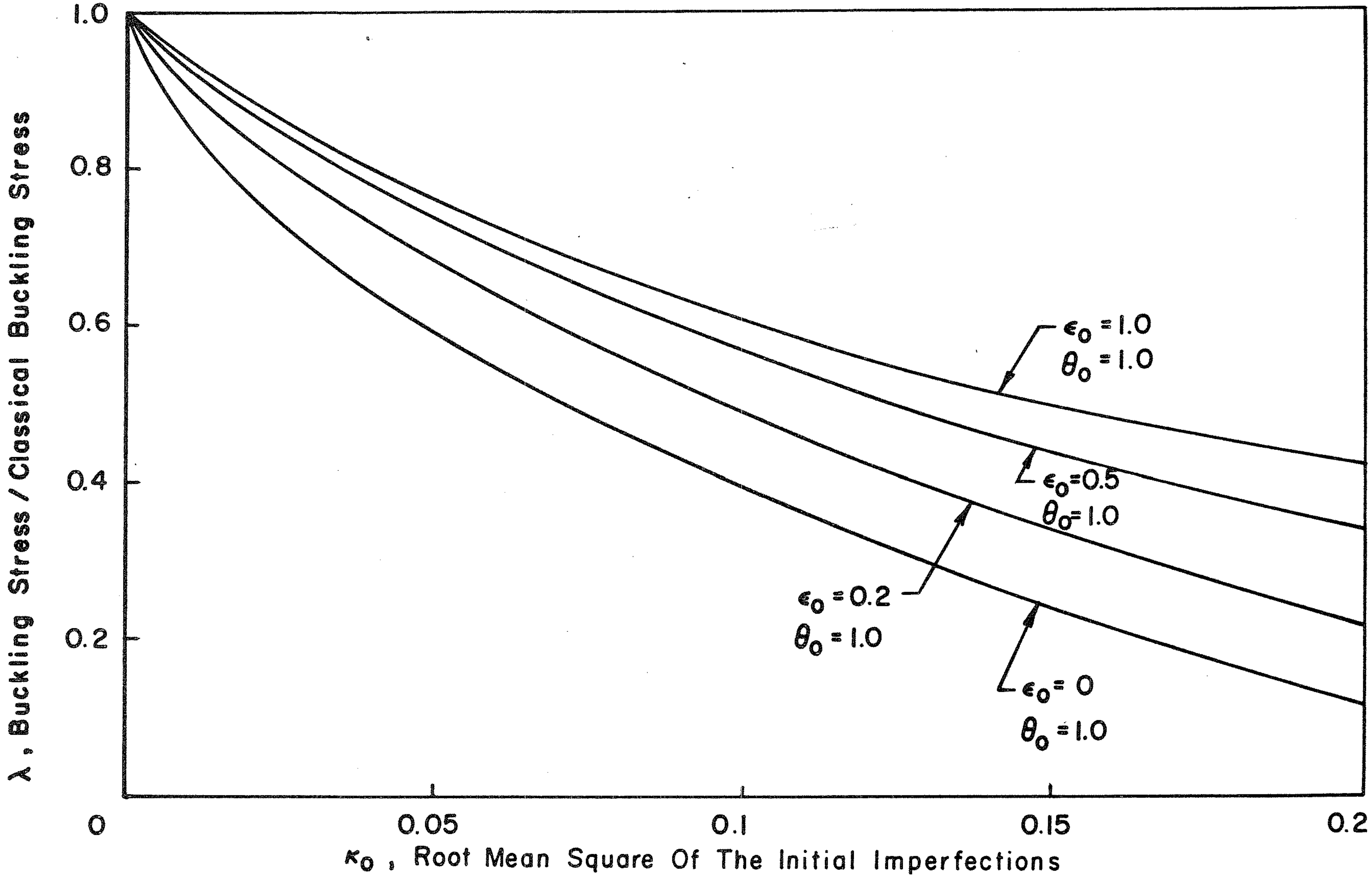

FIG. 3 BUCKLING STRENGTH DEPENDENCE FOR DIFFERENT POWER SPECTRUM PARAMETERS

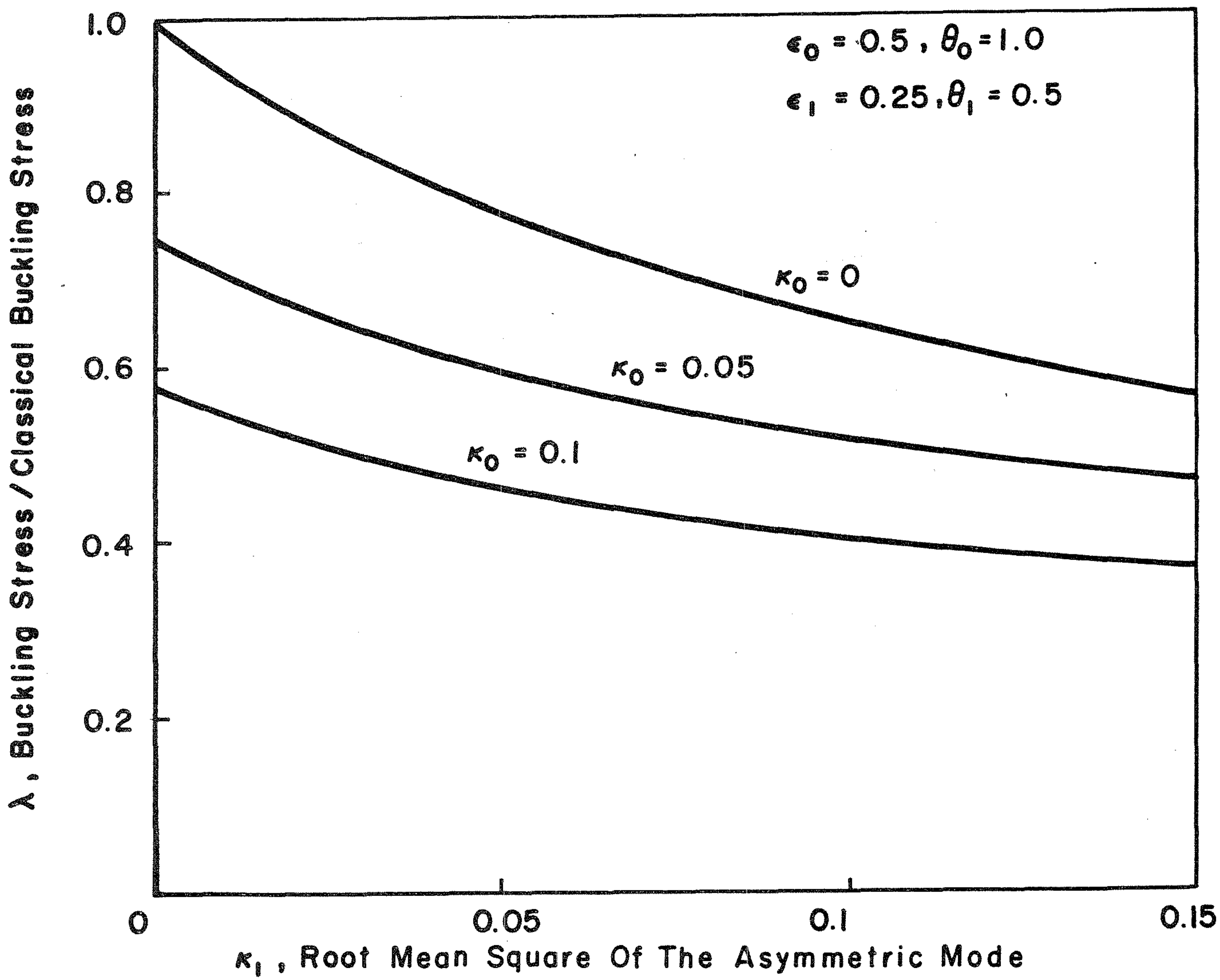

FIG. 4 BUCKLING STRENGTH DEPENDENCE FOR DIFFERENT κ_0

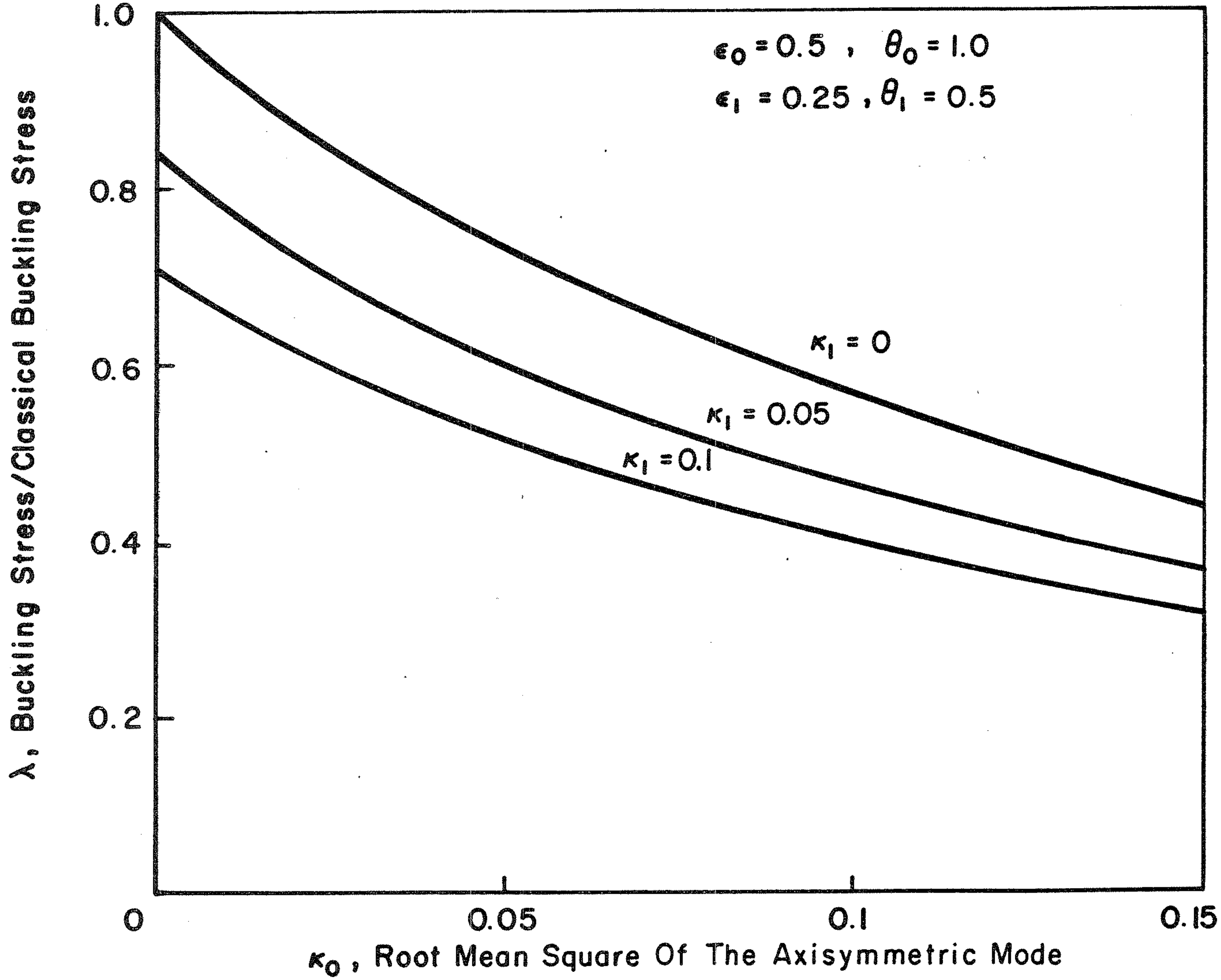

FIG. 5 BUCKLING STRENGTH DEPENDENCE FOR DIFFERENT κ_1