# SELF-SIMILAR ELASTODYNAMIC SOLUTIONS FOR THE PLANE WEDGE

Thesis by

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# DEDICATION

This work is dedicated to my parents, and to Nan for living with it so long.

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I am grateful to Marlys Ricards for her fine job typing the manuscript, and to Betty Wood for the excellent line drawings. Finally the financial assistance of the California Institute of Technology is much appreciated. Overall I find it hard to imagine a better technical environment in which to work.

*Felix, M. P., and A. T. Ellis, Appl. Phys. Letters, 21, 1972, p. 532.

## ABSTRACT

Wave propagation in a two-dimensional elastic wedge is fundamental to a large class of problems in elastodynamic theory, however until now analytical solutions to all but certain degenerate cases were unknown. In this thesis a general elastodynamic solution is derived for the wedge in a state of plane strain. Surface tractions are restricted to uniform normal and shear loads spreading from the wedge vertex at constant velocity. The geometry and loading then allow self-similar solutions of the governing differential equations and boundary conditions in hyperbolic and elliptic domains. Hyperbolic solutions are found in terms of the elliptic solutions by the method of characteristics, while elliptic solutions are reduced using analytic function theory to two independent Fredholm integral equations of the second kind in one dimension. Although numerical solutions are beyond the scope of the investigation, the integral equations are solvable by standard techniques. Such solutions can be used to solve a number of plane elastodynamic problems involving an edge.

# NOMENCLATURE

| | |
|---|---|
| $c_d$ | Propagation speed of dilatational waves |
| $c_r$ | Propagation speed of rotational waves |
| $c_R$ | Propagation speed of Rayleigh surface waves |
| $k$ | Ratio of the dilatational to rotational wave speeds |
| $p$ | Similarity variable, p. 12 |
| $q$ | Similarity variable, $q = kp$, p. 12 |
| $p_o$ | Value of p at the traction discontinuity |
| $p_R$ | Value of p at the Rayleigh wave, p. 29 |
| $r$ | Radial coordinate in a polar coordinate system |
| $R(p)$ | Rayleigh function, p. 29 |
| $R_m$ | Coefficients of the factored Rayleigh poles, $m = 1-4$, p. 52 |
| $S( )$ | Heaviside step function |
| $S_m$ | Coefficients of the factored traction poles, $m = 1-4$, p. 51, 52 |
| $t$ | Time |
| $T_m$ | Coefficients in the residue at the traction poles, p. 30 |
| $\underset{\sim}{u}$ | Displacement vector |
| $u_r$ | Radial component of displacement |
| $u_\theta$ | Angular component of displacement |
| $u$ | Real part of the complex variable, w, p. 39 |
| $v$ | Imaginary part of the complex variable, w, p. 39 |
| $V$ | Traction velocity, p. 5 |
| $w$ | Complex variable, p. 39 |
| $x$ | Real part of the complex variable, z, p. 39, or a Cartesian coordinate, p. 6 |

| | |
|---|---|
| $y$ | Imaginary part of the complex variable, z, p. 39, or a Cartesian coordinate, p. 6 |
| $z$ | Complex variable, p. 39, or a cylindrical polar coordinate, p. 5 |
| $\alpha$ | Complex variable, p. 17 |
| $\alpha_{\pm}$ | Characteristic variables, p. 17 |
| $\beta$ | Complex variable, p. 17 |
| $\beta_{\pm}$ | Characteristic variables, p. 17 |
| $\gamma$ | Half the wedge angle, p. 6 |
| $\Gamma(z)$ | Logarithm of the Rayleigh factor, p. 46 |
| $\delta(\ )$ | Dirac delta function |
| $\theta$ | Angular coordinate in a polar coordinate system |
| $\theta_h$ | Termination point of leading head wave, p. 21 |
| $\vartheta$ | Dilatation, p. 8 |
| $\Theta(\alpha)$ | Elliptic solution for $\vartheta$, p. 17 |
| $\Theta_{\pm}(\alpha_{\pm})$ | Hyperbolic solutions for $\vartheta$, p. 17 |
| $\lambda$ | Elastic constant, p. 8, or separation parameter, p. 35 |
| $\mu$ | Elastic constant |
| $\rho$ | Material density |
| $\sigma_{xx}$ | Components of stress tensor, $xx: \theta\theta, rr, r\theta$ |
| $\sigma_o$ | Coefficient of normal surface traction |
| $\tau_o$ | Coefficient of tangential surface traction |
| $\phi_{s,a}$ | Density function for symmetric and asymmetric loading cases, p. 53 |
| $\Phi_{s,a}$ | Residual of the factorization in the symmetric and asymmetric loading cases, p. 48 |

$\psi_{s,a}$ Density function for symmetric and asymmetric loading cases, p. 53

$\Psi_{s,a}$ Residual of the factorization in the symmetric and asymmetric loading cases, p. 49, 50

$\omega$ Magnitude of the rotation vector, p. 8

$\Omega(\beta)$ Elliptic solution for $\omega$, p. 17

$\Omega_{\pm}(\beta_{\pm})$ Hyperbolic solutions for $\omega$, p. 17

# TABLE OF CONTENTS

## INTRODUCTION

Wave propagation in a two-dimensional (plane) elastic wedge is fundamental to a large class of problems in elastodynamic theory, however analytical solutions to all but certain degenerate cases are virtually unknown. The difficulty can be attributed in general to the nonseparable nature of the boundary conditions, and as a consequence classical separation of variables or transform techniques are unsuitable.

The wedge is an idealization of geometries encountered in the theory of diffraction, waveguides, and, rather recently, steady dynamic crack propagation with branching (e.g., Knopoff [1], Miklowitz [2], Achenbach [3], respectively). Numerous applications can be found in seismology, strong-motion earthquake engineering, nondestructive evaluation, acoustic wave electrical device design, etc. With respect to analytical techniques, separable solutions are known for arbitrary angle (Kostrov [4] and Zemell [5]) in the highly idealized case that boundary conditions on both faces are mixed (i.e., normal or tangential surface traction with tangential or normal displacements, respectively, are prescribed); for the half-space and semi-infinite slit, so-called degenerate or separable geometries, realistic nonmixed boundary conditions are admissible (e.g., Lamb [6], Craggs [7], Maue [8], de Hoop [9], Miles [10]); and for the quarter space, due to equivalent half-space superpositions, mixed conditions need only be given on one face (e.g., Wright [11]). The review paper by Knopoff [1] discusses a number of mathematical techniques which have met with limited success in applications to more general cases.

A natural approach to nonseparable wedge problems is the method of self-similarity. Provided only that the geometry and boundary conditions do not introduce a characteristic length, the method yields a reduction in the number of independent variables from three to two, namely $r/t$ and $\theta$, where $r$ and $\theta$ are polar coordinates and $t$ is time. This is the technique used by Miles [10] and Craggs [7] for the degenerate cases, and Kostrov [4] for arbitrary angles with mixed boundary conditions. Knopoff discusses the technique (as the method of conical flows) and concludes, in part on the basis of Gangi's work [12], that it does not appear to allow a deterministic formulation of boundary conditions for the nonseparable problem. A paper by Achenbach and Khetan [13], as yet unpublished and just recently brought to the author's attention, treats the nonmixed wedge by similarity methods. The loading is a transient normal pressure applied uniformly to the faces and the work is a continuation of Achenbach's efforts towards an understanding of dynamic crack propagation and branching mentioned earlier. Unfortunately due to the nature of the analysis and consequent complexity of the numerical method used in the solution it appears at this stage to be of limited utility.

The aim of this thesis is to develop an elastodynamic solution for the plane wedge using the method of self-similarity. To satisfy the similarity requirement, surface tractions are restricted to uniform normal and shear loads spreading from the wedge vertex at some constant velocity. It follows that the infinitesimal dilatation and rotation are functions of $r/t$ and $\theta$ only. Practicality suggests decomposing the original load into symmetric and asymmetric components and examining the resulting problems separately. The treatment is essentially the same for each and is done concurrently in the sequel. Observe that the above loading is directly applicable to a number of interesting unsolved problems. For example, letting the traction velocity go to infinity it yields the short-time response of a two-dimensional waveguide with non-parallel faces, near the uniformly loaded end; or by the superposition of moving tractions with finite velocities to cancel the incident wave on the wedge surface, it solves the problem of the diffraction of elastic waves by a wedge shaped void.

In Sections 1-4 the wedge problem is formulated and reduced to the determination of two analytic functions over semi-infinite strips in respective complex planes. In Section 5 the stress boundary conditions are used to cast boundary values of the analytic functions in a canonical form which clearly exhibits the singular nature of the solution, e.g., the Rayleigh wave, etc., and in Section 6 the edge behavior is established. The semi-infinite strips are mapped to half-planes in Section 7 and the singular parts of the analytic functions are factored out on the basis of the canonical forms. In Section 8 determination of the unknowns remaining after the factorization is reduced to the solution of a single Fredholm integral equation of the second kind for each loading case, symmetric and asymmetric. Uniqueness of solutions is discussed in Section 9 as well as recovery of the field quantities, i.e., dilatation, rotation, stresses, and velocities. Numerical solutions of the integral equations are beyond the scope of the investigation, however they are solvable by standard techniques.

Before proceeding a few comments are in order on motivation of the thesis topic. The author's interest in the problem stems from work in experimental mechanics, in particular on the dynamics of dipping layers (in contrast to parallel layers) in earth structure idealizations. Existing techniques in elastodynamic theory have been of limited use in understanding details of the phenomenon near the edge. Typically, these structures are modeled experimentally by finite wedges and the response measured by surface transducers, photoelasticity, etc. For example, a truncated wedge of small included angle impacted at the larger end is used to model the focusing of waves from a near field seismic source (Lee and Sechler [26], Wojcik and Felix [27]). Of primary interest is the response near the wedge vertex, however it is due to a superposition of incoming as well as diffracted outgoing waves and an experimental determination of their relative strengths is formidable. Because of this and other experimental difficulties a theoretical analysis of the simplest dipping structure, namely the plane wedge of arbitrary angle, is called for. The solution given in the sequel is particularly useful in addressing this as well as many related problems in elastodynamics.

## § 1. THE PROBLEM

Consider the two-dimensional wedge shown in Figure 1, of some homogeneous, isotropic, linear elastic solid extending infinitely out of the figure (along the z axis in a cylindrical polar coordinate system). At $t = 0$ the wedge is loaded by surface tractions applied to either face, constant in the z direction so that a state of plane strain exists. The problem is to determine the system of waves generated and propagated outward from the edge.

Observe that the wedge is without a characteristic length. If the loading likewise lacks a length scale we expect aspects of the response to be similar with respect to time (dynamic or self-similarity); in which case the independent variables can be reduced from $r, \theta, t$ to $r/t, \theta$. To permit such self-similar solutions and the analytical simplifications resulting, the loading is restricted to uniform normal and shear tractions spreading along either face from the edge at constant velocity, $V$ $(0 < V < \infty)$. It is convenient at the outset to split these into symmetric and asymmetric components and solve the two problems separately. The following cases are therefore examined:

$$\theta = 0: \quad \sigma_{\theta\theta} = \mu\sigma_o S(Vt-r) \quad , \tag{1.1}$$

$$\sigma_{r\theta} = \mu\tau_o S(Vt-r) \quad , \tag{1.2}$$

$$\left.\begin{aligned} \theta = 2\gamma: \quad \sigma_{\theta\theta} &= \mu\sigma_o S(Vt-r) \quad , &(1.3)\\ \sigma_{r\theta} &= -\mu\tau_o S(Vt-r) \quad , &(1.4)\end{aligned}\right\} \quad \text{symmetric case}$$

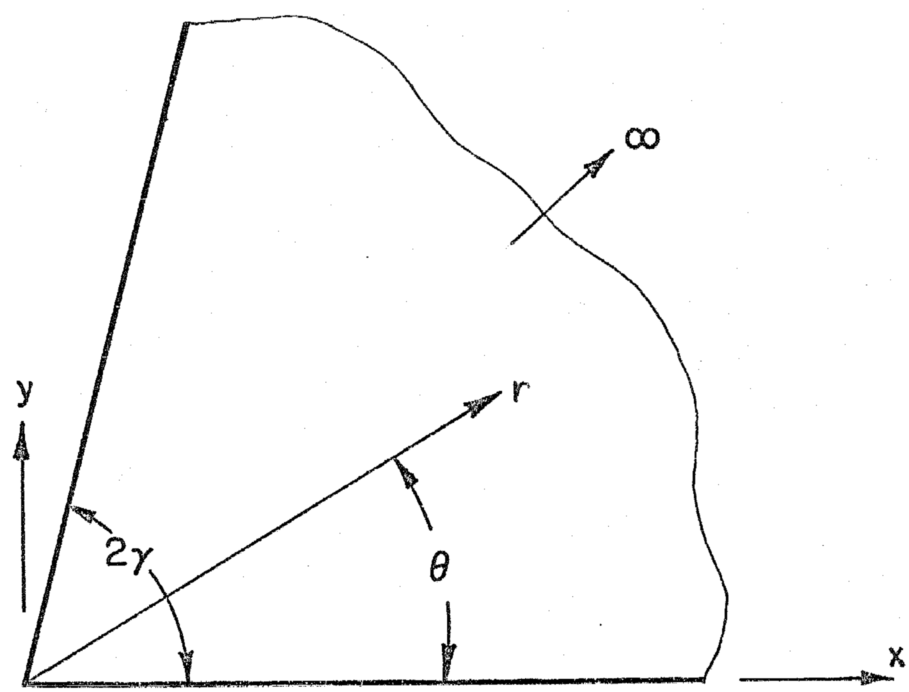


FIGURE 1

$$\left.\begin{aligned} \sigma_{\theta\theta} &= -\mu\sigma_o S(Vt-r) \ , & (1.5) \\ \sigma_{r\theta} &= \mu\tau_o S(Vt-r) \ , & (1.6) \end{aligned}\right\} \text{asymmetric case}$$

where $\mu$ is the shear modulus of the material, $\sigma_o$ at $\tau_o$ are dimensionless coefficients, and S represents the Heaviside step function, zero for negative argument and unity for positive. The decomposition yields mixed conditions on the bisecting ray as

$$\theta = \gamma : \quad \sigma_{r\theta} = u_\theta = 0 \ ; \ \text{sym. case,} \qquad (1.7)$$

$$\sigma_{\theta\theta} = u_r = 0 \ ; \ \text{asym. case,} \qquad (1.8)$$

which follow directly from the symmetry or asymmetry of the loading.

Due to the combinations of wedge angle and load velocity possible, a compromise is struck between exposition and algebraic complexity by treating a wedge with included angle, $2\gamma$, somewhat less than $\pi/2$, as in Figure 1, and limiting the surface traction velocity to $V < c_r$, where $c_r$ is the rotational wave speed. The results are readily generalized to arbitrary angle and $V \geq c_r$ as discussed in the last section.

## § 2. GOVERNING EQUATIONS AND THE SELF-SIMILARITY FORMULATION

Displacement equations governing the linearized motion of a homogeneous, isotropic, elastic solid can be written as

$$(\lambda + 2\mu)\nabla\vartheta \;-\; 2\mu\nabla\times\omega \;=\; \rho\,\ddot{\underset{\sim}{u}}\,, \tag{2.1}$$

with the dilatation, $\vartheta$, and rotation, $\underset{\sim}{\omega}$, defined by

$$\vartheta = \nabla\cdot\underset{\sim}{u}\,, \tag{2.2}$$

$$\underset{\sim}{\omega} = \tfrac{1}{2}\nabla\times\underset{\sim}{u}\,; \tag{2.3}$$

where $\underset{\sim}{u}$ is the displacement vector, $\lambda$ and $\mu$ are elastic constants, and $\rho$ is the material density. Integrating twice over time and applying quiescent initial conditions, (2.1) becomes

$$\underset{\sim}{u} = \nabla\phi + \nabla\times\underset{\sim}{\psi}\,. \tag{2.4}$$

The so-called Lamé displacement potentials, $\phi$ and $\underset{\sim}{\psi}$, are defined by

$$\phi = c_d^2\int_0^t\!\!\int_0^\tau \vartheta\, d\eta\, d\tau\;;\quad c_d^2 = (\lambda + 2\mu)/\rho\,, \tag{2.5}$$

$$\underset{\sim}{\psi} = -2c_r^2\int_0^t\!\!\int_0^\tau \underset{\sim}{\omega}\, d\eta\, d\tau\;;\quad c_r^2 = \mu/\rho\,. \tag{2.6}$$

Taking the divergence and curl of (2.1), switching the order of space and time derivatives, and applying (2.2, 3) yields

$$\nabla^2\vartheta = \frac{1}{c_d^2}\,\ddot{\vartheta}\,, \tag{2.7}$$

$$\nabla^2 \underset{\sim}{\omega} = \frac{1}{c_r^2} \ddot{\underset{\sim}{\omega}} ; \tag{2.8}$$

hence the dilatation and rotation are governed by scalar and vector wave equations, with $c_d$ and $c_r$ the respective propagation velocities of such disturbances. The same results are found for $\phi$ and $\underset{\sim}{\psi}$ by taking the divergence and curl of (2.4), differentiating (2.5, 6) twice with respect to time, substituting (2.2, 3) and combining the results, i.e.,

$$\nabla^2 \phi = \frac{1}{c_d^2} \ddot{\phi} , \tag{2.9}$$

$$\nabla^2 \underset{\sim}{\psi} = \frac{1}{c_r^2} \ddot{\underset{\sim}{\psi}} . \tag{2.10}$$

The above reduction is due to Somigliana (e.g., Miklowitz [14]) who used it to establish a one-to-one relationship between solutions of the displacement equation, (2.1), and the much simpler wave equations, (2.9, 10), through the decomposition (2.4).

Specializing these results to a state of plane strain, $u_z$ and $\partial/\partial z$ vanish, reducing the two independent components of $\underset{\sim}{\psi}$ and $\underset{\sim}{\omega}$ (because $\nabla \cdot \underset{\sim}{\psi} = \nabla \cdot \underset{\sim}{\omega} = 0$ from (2.3, 6)) to one, namely

$$\underset{\sim}{\psi} = \psi \underset{\sim}{e}_z , \quad \underset{\sim}{\omega} = \omega \underset{\sim}{e}_z , \tag{2.11}$$

and the governing equations to scalar wave equations. In polar coordinates the displacement-potential relation, (2.4), becomes

$$u_r = \phi_r + \frac{1}{r} \psi_\theta , \tag{2.12}$$

$$u_\theta = \frac{1}{r}\phi_\theta - \psi_r . \tag{2.13}$$

Introducing these into Hooke's law (e.g., Sokolnikoff [15]) yields the stress-potential relations,

$$\sigma_{\theta\theta} = \lambda\nabla^2\phi + \frac{2\mu}{r^2}[r\phi_r + \phi_{\theta\theta} + \psi_\theta - r\psi_{r\theta}] , \tag{2.14}$$

$$\sigma_{rr} = \lambda\nabla^2\phi + \frac{2\mu}{r^2}[r^2\phi_{rr} + r\psi_{r\theta} - \psi_\theta] , \tag{2.15}$$

$$\sigma_{r\theta} = \frac{\mu}{r^2}[2r\phi_{r\theta} - 2\phi_\theta + \psi_{\theta\theta} - r^2\psi_{r\theta} + r\psi_r] . \tag{2.16}$$

Note that the subscripts on u or $\sigma$ identify vector or tensor components, otherwise they imply differentiation with respect to the subscripted variable.

By differentiating (2.12-16) twice over time and substituting

$$\ddot{\phi} = c_d^2\vartheta , \quad \ddot{\psi} = -2c_r^2\omega ,$$

from (2.5, 6), the governing equations can be cast with either the displacement potentials or the dilatation and rotation as the dependent variables. Actually any time derivative of the displacement potentials is admissible. The proper choice is made by requiring the variable to exhibit self-similar solutions for the traveling step surface traction.

One simple method of establishing self-similar solutions (appropriate for the wedge problem) is to apply the principle of dimensional analysis, viz., the independent variables and parameters of the problem,

$$r,\ \theta,\ t,\ \gamma,\ \sigma_o,\ \tau_o,\ \lambda,\ \mu,\ \rho, \tag{2.17}$$

yield the following nondimensional groups,

$$r/c_d t,\ \theta,\ \gamma,\ \sigma_o,\ \tau_o,\ c_d/c_r, \tag{2.18}$$

to which self-similar solutions must conform. The similarity variable, $r/c_d t$, is due of course to the absence of a characteristic length. Such solutions are "similar" at any time and collapse onto a single "curve" by scaling the distance as above. Equivalently they are conical fields in r, $\theta$, t space, constant on rays through the origin (e.g., Keller and Blank [16]), or homogeneous functions of degree zero (e.g., Miles [10]).

By the fundamental linearity of the problem dependent variables are proportional to the applied load, i.e., $\sigma_o$ and $\tau_o$, thus to exhibit similarity solutions they must be dimensionless. Consequently $\phi$ and $\psi$ (dimensions $(\text{length})^2$) do not possess such solutions, at least for the traveling step load. An alternative is to reformulate the problem, replacing the step by the highly singular form

$$\sigma_{\theta\theta}|_{\theta=0} = \mu\sigma_o \delta'(Vt - r) ,$$

for example, where $\delta'$ is the derivative of the Dirac delta function (dimensions $(\text{length})^{-2}$). In this case $\sigma_o$ has dimensions $(\text{length})^2$ and $\phi/\sigma_o$, $\psi/\sigma_o$ are dimensionless.

Clearly then the dilatation and rotation are the appropriate dependent variables for similarity solutions of the problem (Craggs

[ 7 ]). Note that the nondimensionalized stresses and velocities (e.g., $\sigma_{\theta\theta}/\mu$, $\dot{u}_r/c_d$) are also self-similar.

Introducing the similarity variables

$$p \equiv r/c_d t, \quad q \equiv r/c_r t; \quad q = kp, \quad k \equiv c_d/c_r \tag{2.19}$$

(where for function of p or q

$$r\frac{\partial}{\partial r} = -t\frac{\partial}{\partial t} = p\frac{\partial}{\partial p} = q\frac{\partial}{\partial q})$$

reduces the wave equations on $\vartheta$ and $\omega$ to

$$p^2(1-p^2)\vartheta_{pp} + p(1-2p^2)\vartheta_p + \vartheta_{\theta\theta} = 0, \tag{2.20}$$

$$q^2(1-q^2)\omega_{qq} + q(1-2q^2)\omega_q + \omega_{\theta\theta} = 0. \tag{2.21}$$

We make note that the higher order p and q terms in the coefficients are the inertia contribution.

Making the change of dependent and independent variables in (2.12-16), the displacement-potential relations give

$$\frac{-q^2}{c_r}\frac{\partial \dot{u}_r}{\partial q} = k^2 p\vartheta_p - 2\omega_\theta, \tag{2.22}$$

$$\frac{-q^2}{c_r}\frac{\partial \dot{u}_\theta}{\partial q} = k^2\vartheta_\theta + 2q\omega_q, \tag{2.23}$$

and after some manipulation, the stress-potential relations become

$$\frac{\partial}{\partial t}\left\{\begin{array}{l} \frac{q^2}{\mu}\frac{\partial\sigma_{\theta\theta}}{\partial q} - k^2(q^2-2)\vartheta_q - \frac{4}{q}\omega_\theta \\ \frac{q^2}{\mu}\frac{\partial\sigma_{rr}}{\partial q} - ((k^2-2)q^2+2k^2)\vartheta_q + \frac{4}{q}\omega_\theta \\ \frac{q^2}{\mu}\frac{\partial\sigma_{r\theta}}{\partial q} - \frac{2k^2}{q}\vartheta_\theta + 2(q^2-2)\omega_q \end{array}\right\} = 0 \ .$$

Integrating and applying the quiescent initial conditions yields

$$\frac{q^2}{\mu}\frac{\partial\sigma_{\theta\theta}}{\partial q} = k^2(q^2-2)\vartheta_q + \frac{4}{q}\omega_\theta \ , \tag{2.24}$$

$$\frac{q^2}{\mu}\frac{\partial\sigma_{r\theta}}{\partial q} = \frac{2k^2}{q}\vartheta_\theta - 2(q^2-2)\omega_q \ , \tag{2.25}$$

$$\sigma_{rr} = 2\mu(k^2-1)\vartheta - \sigma_{\theta\theta} \ . \tag{2.26}$$

Recall, by the decomposition into even and odd loading cases, stress boundary conditions on $\theta = 0, 2\gamma$ are replaced by the simpler set, (1.1, 2, 7, 8), on $\theta = 0, \gamma$. These are written as

$$\theta = 0: \ \sigma_{\theta\theta} = \mu\sigma_o S(q_o - q) \ ; \ q_o = kp_o = V/c_r \tag{2.27}$$

$$\sigma_{r\theta} = \mu\tau_o S(q_o - q) \ , \tag{2.28}$$

$$\theta = \gamma: \ \sigma_{r\theta} = u_\theta = 0 \ ; \quad \text{sym. case,}$$

$$\sigma_{\theta\theta} = u_r = 0 \ ; \quad \text{asym. case,}$$

with similarity variables introduced by dividing the argument of S in (1.1, 2) by $c_r t$. Substituting these into the displacement and stress relations, (2.22-25), gives boundary conditions on the dilatation and rotation as

$$\theta = 0: \; -\sigma_o q_o^2 \delta(q - q_o) = k^2(q^2 - 2)\vartheta_q + \frac{4}{q}\omega_\theta , \qquad (2.29)$$

$$-\tau_o q_o^2 \delta(q - q_o) = \frac{2k^2}{q}\vartheta_\theta - 2(q^2 - 2)\omega_q , \qquad (2.30)$$

$$\theta = \gamma: \left.\begin{aligned} &\frac{2k^2}{q}\vartheta_\theta - 2(q^2 - 2)\omega_q = 0 \\ &k^2\vartheta_\theta + 2q\omega_q = 0 \end{aligned}\right\} \text{ sym. case,}$$

$$\left.\begin{aligned} &k^2(q^2 - 2)\vartheta_q + \frac{4}{q}\omega_\theta = 0 \\ &k^2 q\vartheta_q - 2\omega_\theta = 0 \end{aligned}\right\} \text{ asym. case,}$$

where $\delta$ is the delta function. Solving the homogeneous equations on $\theta = \gamma$ yields

$$\theta = \gamma: \; \vartheta_\theta = \omega_q = 0 ; \quad \text{sym. case,} \qquad (2.31)$$

$$\vartheta_q = \omega_\theta = 0 ; \quad \text{asym. case,} \qquad (2.32)$$

with the possibility of nontrivial solutions at $q = 0$ in either case.

## § 3. SELF-SIMILAR SOLUTIONS: THE REDUCED PROBLEM

The self-similar forms of the governing differential equations, (2.20, 21), are of mixed hyperbolic-elliptic type. For example, (2.20) is elliptic when $p < 1$, hyperbolic when $p > 1$, and parabolically degenerate on the unit circle, $p = 1$. In the elliptic domain, transforming the equation to canonical form by the change of variables (the so-called Chaplygin transformation),

$$\zeta = \cosh^{-1} 1/p = \log \frac{1}{p}(1 + \sqrt{1-p^2}) \ ; \ 0 < \zeta < \infty , \tag{3.1}$$

yields Laplace's equation,

$$\vartheta_{\zeta\zeta} + \vartheta_{\theta\theta} = 0 \ ;$$

while in the hyperbolic domain, the transformation

$$\xi = \cos^{-1} 1/p \ ; \ \ 0 < \xi < \pi/2 , \tag{3.2}$$

gives the wave equation

$$\vartheta_{\xi\xi} = \vartheta_{\theta\theta} \ .$$

Similar results follow for $\omega$ from (2.21). The change in character of solutions across the $q = 1$ and $p = 1$ degenerate curves partition the physical domain into three well defined regions referred to as the common hyperbolic region ($p = q/k > 1$), the composite region ($1/k < p < 1$), and the common elliptic region ($p < 1/k$) (cf. Figure 2).

Solutions for $\vartheta$ and $\omega$ are easily found from the canonical forms to be

① Common Elliptic Region
② Composite Region
③ Common Hyperbolic Region

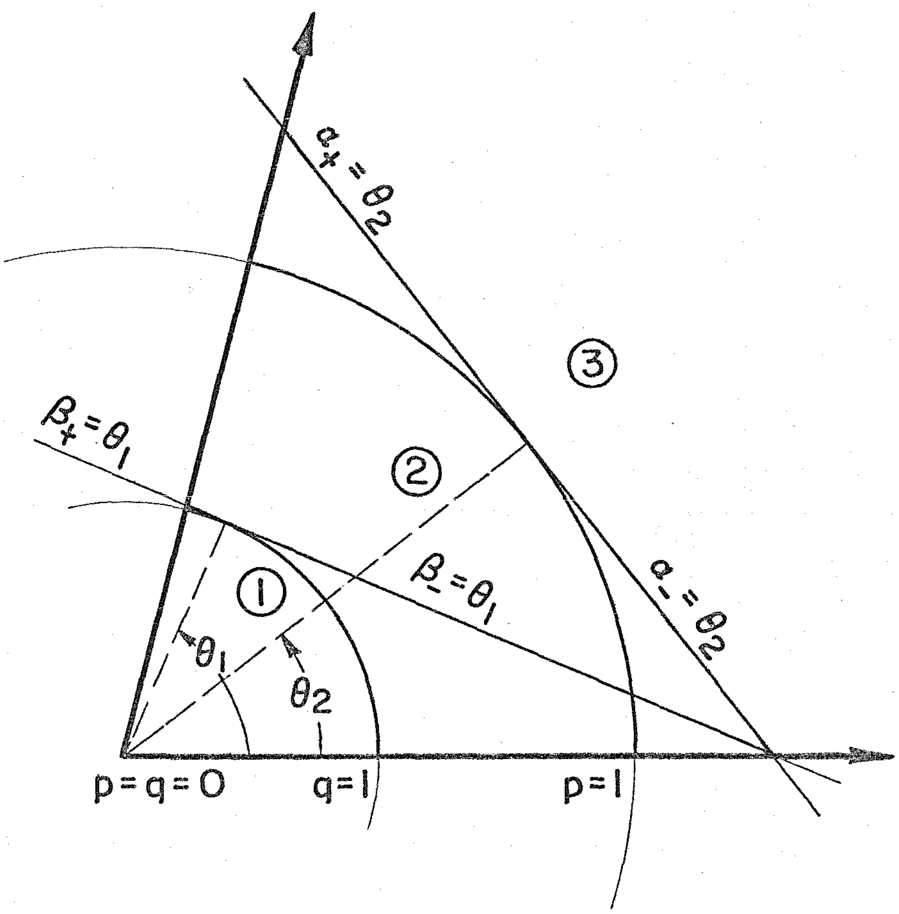


FIGURE 2

$$\vartheta(p, \theta) = \begin{cases} \Theta_+(\alpha_+) + \Theta_-(\alpha_-) \ ; \ \alpha_\pm = \theta \pm \cos^{-1} 1/p, \ p > 1, & (3.3) \\ \mathrm{Re}\, \Theta(\alpha) \ ; \ \alpha = \theta + i\cosh^{-1} 1/p, \ p < 1, & (3.4) \end{cases}$$

$$\omega(q, \theta) = \begin{cases} \Omega_+(\beta_+) + \Omega_-(\beta_-) \ ; \ \beta_\pm = \theta \pm \cos^{-1} 1/q, \ q > 1, & (3.5) \\ \mathrm{Re}\, \Omega(\beta) \ ; \ \beta = \theta + i\cosh^{-1} 1/q, \ q < 1, & (3.6) \end{cases}$$

where the positive branch of the inverse functions is taken, as in (3.1, 2). The elliptic solutions, $\Theta$ and $\Omega$, are analytic functions over strips in the complex $\alpha$ and $\beta$ planes; and the hyperbolic solutions, $\Theta_+$, $\Theta_-$, $\Omega_+$, $\Omega_-$, are distinct real functions. In a $p-\theta$ polar coordinate system, the $\alpha_\pm$ characteristics are straight lines tangent to the $p = 1$ unit circle at $\theta = \alpha_\pm$ as shown in Figure 2. Because this circle is an envelope of characteristics, it is itself a characteristic. In a $q-\theta$ coordinate system the $\beta_\pm$ characteristics are tangent to the $q = 1$ unit circle, also a characteristic.

Self-similarity and the loading decomposition have reduced the dynamical problem to determination of the above elliptic and hyperbolic solutions satisfying boundary conditions, (2.29-32), on the $\theta = 0, \gamma$ rays. These conditions involve derivatives of $\vartheta$ and $\omega$, which are found by differentiating the hyperbolic solutions directly, as

$$\vartheta_\theta = \Theta'_+(\alpha_+) + \Theta'_-(\alpha_-), \tag{3.7}$$

$$\vartheta_p = \frac{1}{p\sqrt{p^2-1}} [\Theta'_+(\alpha_+) - \Theta'_-(\alpha_-)], \tag{3.8}$$

$$\omega_\theta = \Omega'_+(\beta_+) + \Omega'_-(\beta_-) , \tag{3.9}$$

$$\omega_q = \frac{1}{q\sqrt{q^2-1}} [\Omega'_+(\beta_+) - \Omega'_-(\beta_-)] , \tag{3.10}$$

or, using (3.4, 6) and the definition for derivatives of an analytic function to write

$$\Theta'(\alpha) = \vartheta_\theta + ip\sqrt{1-p^2}\,\vartheta_p , \tag{3.11}$$

$$\Omega'(\beta) = \omega_\theta + iq\sqrt{1-q^2}\,\omega_q . \tag{3.12}$$

Eliminating $\vartheta$ and $\omega$ in (2.29-32) through the above relations yields the following reduced set of boundary conditions.

<u>$p = q/k > 1$ (Common hyperbolic region)</u>

$\theta = 0 : \quad \alpha_\pm = \pm \cos^{-1} 1/p , \quad \beta_\pm = \pm \cos^{-1} 1/q$

$$0 = \frac{k(q^2-2)}{p\sqrt{p^2-1}}[\Theta'_+(\alpha_+) - \Theta'_-(\alpha_-)] + \frac{4}{q}[\Omega'_+(\beta_+) + \Omega'_-(\beta_-)] \tag{3.13}$$

$$0 = \frac{2k^2}{q}[\Theta'_+(\alpha_+) + \Theta'_-(\alpha_-)] - \frac{2(q^2-2)}{q\sqrt{q^2-1}}[\Omega'_+(\beta_+) - \Omega'_-(\beta_-)] \tag{3.14}$$

$\theta = \gamma : \quad \alpha_\pm = \gamma \pm \cos^{-1} 1/p , \quad \beta_\pm = \gamma \pm \cos^{-1} 1/q$

$$\Theta'_+(\alpha_+) = -\Theta'_-(\alpha_-) , \quad \Omega'_+(\beta_+) = \Omega'_-(\beta_-) \; ; \quad \text{sym. case} \tag{3.15}$$

$$\Theta'_+(\alpha_+) = \Theta'_-(\alpha_-) , \quad \Omega'_+(\beta_+) = -\Omega'_-(\beta_-) \; ; \quad \text{asym. case} \tag{3.16}$$

<u>$1/k < p \leqslant 1$ (Composite region)</u>

$\theta = 0: \quad \alpha = i\cosh^{-1} 1/p\,, \quad \beta_{\pm} = \pm \cos^{-1} 1/q$

$$0 = \frac{k(q^2-2)}{p\sqrt{1-p^2}} \operatorname{Im} \Theta'(\alpha) + \frac{4}{q}[\Omega'_+(\beta_+) + \Omega'_-(\beta_-)] \tag{3.17}$$

$$0 = \frac{2k^2}{q} \operatorname{Re} \Theta'(\alpha) - \frac{2(q^2-2)}{q\sqrt{q^2-1}}[\Omega'_+(\beta_+) - \Omega'_-(\beta_-)] \tag{3.18}$$

$\theta = \gamma: \quad \alpha = \gamma + i\cosh^{-1} 1/p\,, \quad \beta_{\pm} = \gamma \pm \cos^{-1} 1/q$

$$\operatorname{Re} \Theta'(\alpha) = 0\,, \quad \Omega'_+(\beta_+) = \Omega'_-(\beta_-)\,; \quad \text{sym. case} \tag{3.19}$$

$$\operatorname{Im} \Theta'(\alpha) = 0\,, \quad \Omega'_+(\beta_+) = -\Omega'_-(\beta_-)\,; \quad \text{asym. case} \tag{3.20}$$

<u>$0 \leqslant p \leqslant 1/k$ (Common elliptic region)</u>

$\theta = 0: \quad \alpha = i\cosh^{-1} 1/p\,, \quad \beta = i\cosh^{-1} 1/q$

$$-\sigma_o q_o^2 \delta(q-q_o) = \frac{k(q^2-2)}{p\sqrt{1-p^2}} \operatorname{Im} \Theta'(\alpha) + \frac{4}{q} \operatorname{Re} \Omega'(\beta) \tag{3.21}$$

$$-\tau_o q_o^2 \delta(q-q_o) = \frac{2k^2}{q} \operatorname{Re} \Theta'(\alpha) - \frac{2(q^2-2)}{q\sqrt{1-q^2}} \operatorname{Im} \Omega'(\beta) \tag{3.22}$$

$\theta = \gamma: \quad \alpha = \gamma + i\cosh^{-1} 1/p\,, \quad \beta = \gamma + i\cosh^{-1} 1/q$

$$\operatorname{Re} \Theta'(\alpha) = \operatorname{Im} \Omega'(\beta) = 0\,; \quad \text{sym. case} \tag{3.23}$$

$$\operatorname{Im} \Theta'(\alpha) = \operatorname{Re} \Omega'(\beta) = 0\,; \quad \text{asym. case} \tag{3.24}$$

We find in the sequel that the hyperbolic solutions are easily determined from (3.13-20) by the method of characteristics, but elliptic solutions are considerably more involved.

## § 4. HYPERBOLIC SOLUTIONS: HEAD WAVES AND CYLINDRICAL WAVEFRONT BEHAVIOR

In the common hyperbolic region, as $p, q \to \infty$ corresponding to $t \to 0$ and/or $r \to \infty$, quiescent initial conditions require the dilatation and rotation to vanish. Thus from (3.3, 5),

$$\Theta_\pm(\theta \pm \pi/2) = \Omega_\pm(\theta \pm \pi/2) = 0 \; ; \quad p, q \to \infty , \tag{4.1}$$

the results of which are illustrated in Figure 3. Substituting these initial conditions into (3.13-16), solving and continuing into the interior along characteristics, it readily follows that solutions in the common hyperbolic region vanish identically, hence

$$\Theta_\pm(\alpha_\pm) = \Omega_\pm(\beta_\pm) \equiv 0 \; ; \quad p = q/k > 1 . \tag{4.2}$$

This is due of course to the absence of supersonic ($V > c_d$) boundary disturbances (recall $V < c_r$).

In the composite region, referring to Figure 4, the first $\beta_-$, $\beta_+$ characteristics on which $\omega$ does not necessarily vanish pass through $p = 1$ on $\theta = 0, 2\gamma$ and are tangent at $\theta = \theta_h$, $2\gamma - \theta_h$ respectively, where

$$\theta_h = \cos^{-1} 1/k . \tag{4.3}$$

Provided then that $2\gamma > \theta_h$ as in Figure 4, some neighborhood of the entire composite region boundary is a simple wave zone, in other words, solutions depend on a single characteristic variable, $\beta_-$ near $\theta = 0$ and $\beta_+$ near $\theta = 2\gamma$.

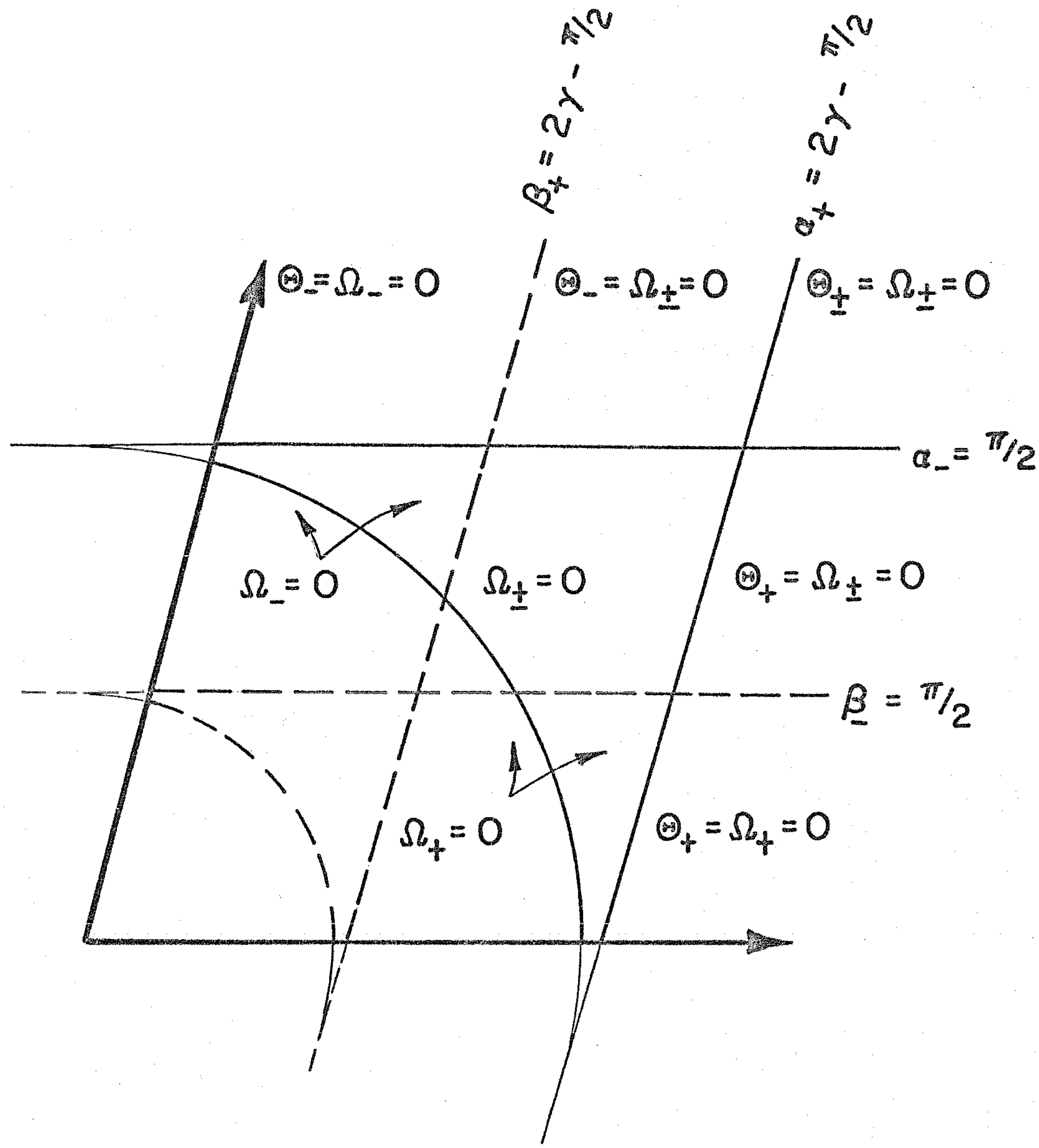


FIGURE 3

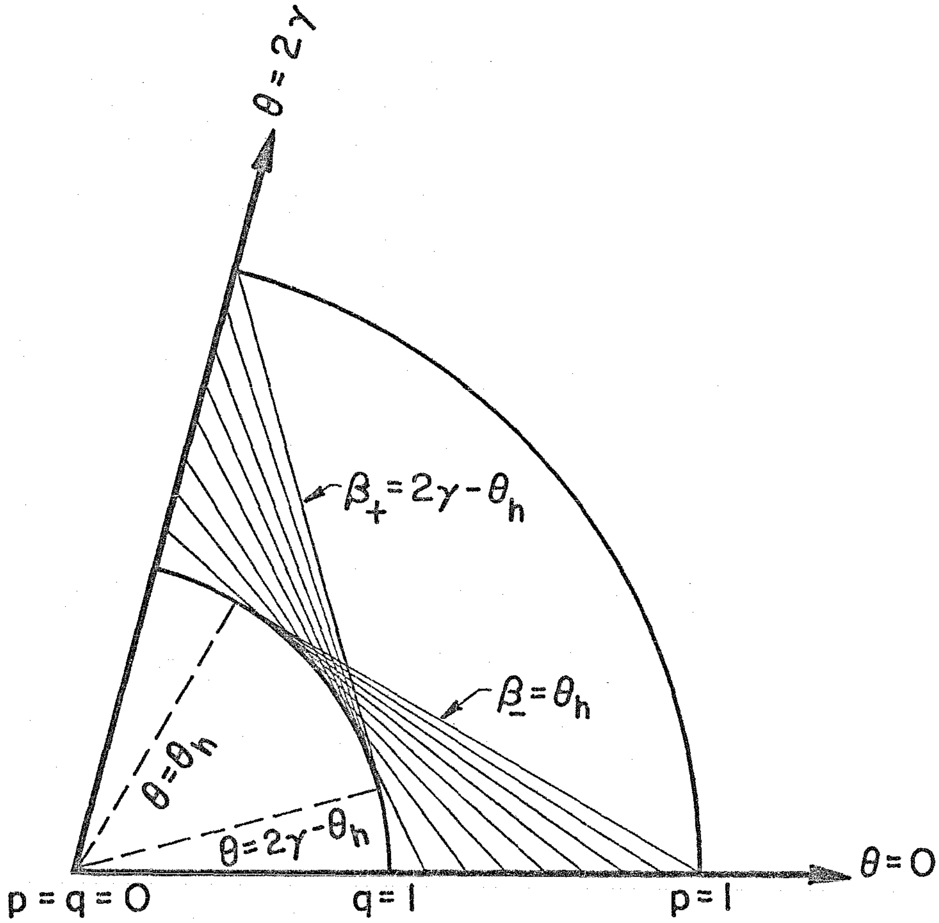


FIGURE 4

Therefore, we can set $\Omega_+(\beta_+) = 0$ in the composite boundary conditions, (3.17-18), on $\theta = 0$, giving

$$0 = \frac{k(q^2-2)}{p\sqrt{1-p^2}} \operatorname{Im} \Theta'(\alpha) + \frac{4}{q}\Omega'_-(\beta_-), \tag{4.4}$$

$$0 = \frac{2k^2}{q} \operatorname{Re} \Theta'(\alpha) + \frac{2(q^2-2)}{q\sqrt{q^2-1}} \Omega'_-(\beta_-), \tag{4.5}$$

and solve for $\Omega'_-(\beta_-)$ on the boundary as

$$\Omega'_-(\beta_-) = \frac{-k^2(q^2-2)}{4\sqrt{1-p^2}} \operatorname{Im} \Theta'(\alpha) = \frac{-k^2\sqrt{q^2-1}}{q^2-2} \operatorname{Re} \Theta'(\alpha). \tag{4.6}$$

This is continued into the composite region along characteristics by the substitution $q = 1/\cos\beta_-$ (i.e., setting $\theta = 0$ in $\beta_-$, (3.5)) whence (4.6) becomes

$$\Omega'_-(\beta_-) = \frac{-k^2 \tan\beta_-}{\tan^2\beta_- - 1} \operatorname{Re} \Theta'(i\cosh^{-1}[k\cos\beta_-]), \tag{4.7}$$

choosing the second equality. To determine $\Omega'_+(\beta_+)$ excited on the $\theta = 2\gamma$ boundary, use is made of the relationship, $\beta_+ + \beta_- = 2\gamma$, on the bisector $\theta = \gamma$ to eliminate $\beta_-$ in (4.7). Substituting the result into the conditions on $\theta = \gamma$, (3.19, 20), gives

$$\Omega'_+(\beta_+) = \mp \frac{k^2 \tan(2\gamma - \beta_+)}{\tan^2(2\gamma-\beta_+)-1} \operatorname{Re}\Theta'(i\cosh^{-1}[k\cos(2\gamma-\beta_+)]), \tag{4.8}$$

where the upper and lower signs refer to symmetric and asymmetric loading cases respectively. Thus the dilatation on the boundary determines completely the rotation in the head wave region.

These hyperbolic solutions in the composite region are commonly referred to as head waves. They are Mach envelopes of rotational disturbance excited on the boundary by the dilatation. Such solutions are readily generalized to smaller angles by including reflected head waves in the boundary conditions but this merely adds algebraic complexity and is not considered further.

Observe that on the $q = 1^+$ characteristic envelope, from (3.5),

$$\omega(1^+,\theta) = \Omega_+(\theta) + \Omega_-(\theta) .$$

Therefore, provided that the behavior of $\omega$ across the characteristic is known, values of $\omega$ on $q = 1^-$ (and likewise $\vartheta$ on $p = 1^-$) in the elliptic domain can be determined. Clearly information on the wavefront behavior is required.

To establish continuity across the characteristic envelopes recall that $\vartheta$ and $\omega$ are governed by scalar wave equations, hence the results of geometrical optics are applicable. In particular any jump or infinity across a cylindrical wavefront (with radius $R = ct$) decays like $R^{-1/2}$ due to ray divergence [17]. However, by virtue of self-similarity, such singularities must remain unchanged as the front propagates, so $\vartheta$ and $\omega$ are continuous by contradiction. The same is true for all $\theta$ derivatives because they are functions of p, q and satisfy the same scalar wave equations as $\vartheta$ and $\omega$.

Values of $\vartheta_\theta$ and $\omega_\theta$ (appropriate because $\Theta'_\pm$ and $\Omega'_\pm$ are generally known) can therefore be continued from the hyperbolic into the elliptic domains across cylindrical wavefronts as

$$\vartheta_\theta = \mathrm{Re}\,\Theta'(\theta) = \Theta'_+(\theta) + \Theta'_-(\theta), \tag{4.9}$$

$$\omega_\theta = \mathrm{Re}\,\Omega'(\theta) = \Omega'_+(\theta) + \Omega'_-(\theta), \tag{4.10}$$

with the hyperbolic solutions given by (4. 2, 6, 8). These supply necessary conditions on the elliptic solutions.

Additional details at the wavefront are found from local solutions of the governing equations, e.g.,

$$q^2(1-q^2)\omega_{qq} + q(1-2q^2)\omega_q + \omega_{\theta\theta} = 0 ,$$

in the neighborhood of $q = 1$. Introducing $q = 1 \pm \epsilon$ ; $0 \leq \epsilon << 1$, into the above gives

$$2\epsilon\omega_{\epsilon\epsilon} + \omega_\epsilon \mp \omega_{\theta\theta} \simeq 0 .$$

Separable solutions continuous across $q = 1$ have the asymptotic form,

$$\omega(1 \pm \epsilon, \theta) \simeq (b_1 + b_2^\pm \sqrt{\epsilon} + 0(\epsilon))\, B(\theta), \tag{4.11}$$

where constants $b_2^+$ and $b_2^-$ are defined on either side of the characteristic, and B is a regular function of $\theta$, i.e., possesses all $\theta$ derivatives. Similar results are found for $\vartheta$, with $p = 1 \pm \epsilon$, as

$$\vartheta(1 \pm \epsilon, \theta) \simeq (a_1 + a_2^\pm \sqrt{\epsilon} + 0(\epsilon))\, A(\theta) . \tag{4.12}$$

Substituting these local solutions into (3.5, 6) and letting $p, q \to 1^- (\epsilon \to 0)$,

$$\Theta'(\theta) = a_1 A'(\theta) - i\frac{\sqrt{2}}{2} a_2^- A(\theta) , \tag{4.13}$$

$$\Omega'(\theta) = b_1 B'(\theta) - i\frac{\sqrt{2}}{2} b_2^- B(\theta) . \tag{4.14}$$

Therefore $\Theta'(\alpha)$ and $\Omega'(\beta)$ are analytic on the $p, q = 1$ degenerate curves by the regularity of A and B.

## § 5. CANONICAL BOUNDARY VALUE REPRESENTATIONS ON $\theta = 0$ IN THE ELLIPTIC DOMAINS

The reduced boundary conditions, (3.17, 18, 21, 22), provide canonical forms for boundary value representations of $\Theta'(\alpha)$ and $\Omega'(\beta)$ on the $\theta = 0$ boundary. In the sequel these representations yield natural factorizations of $\Theta'$ and $\Omega'$ in complex half-planes (onto which the semi-infinite strips will be mapped, as in Figure 5b).

On the $\theta = 0$ boundary of the common elliptic region (3.21, 22) are equivalent to

$$-\sigma_o q_o^2 \delta(q-q_o) + i\Phi_1(p) = \frac{-ik(q^2-2)}{p\sqrt{1-p^2}}\Theta'(\alpha) + \frac{4}{q}\Omega'(\beta) , \tag{5.1}$$

$$-\tau_o q_o^2 \delta(q-q_o) + i\Psi_1(p) = \frac{2k^2}{q}\Theta'(\alpha) + \frac{i\,2\sqrt{q^2-2)}}{q\sqrt{1-q^2}}\Omega'(\beta) , \tag{5.2}$$

(by taking real parts of the above) where $\Phi_1$ and $\Psi_1$ are unknown real functions. Solving for $\Theta'$ and $\Omega'$ gives

$$\Theta'(\alpha) = \frac{-p\sqrt{1-p^2}}{kR(p)} \Big\{ (q^2-2)\Phi_1(p) + i\,2\sqrt{1-q^2}\,\Psi_1(p) + [\,i(q_o^2-2)\sigma_o - 2\sqrt{1-q_o^2}\,\tau_o]\,q_o^2\delta(q-q_o) \Big\} , \tag{5.3}$$

$$\Omega'(\beta) = \frac{q\sqrt{1-q^2}}{2R(p)} \Big\{ -i\,2\sqrt{1-p^2}\,\Phi_1(p) + (q^2-2)\Psi_1(p) + [\,2\sqrt{1-p_o^2}\,\sigma_o + i(q_o^2-2)\tau_o]\,q_o^2\delta(q-q_o) \Big\} . \tag{5.4}$$

The well-known Rayleigh function,

$$R(p) = (k^2p^2-2)^2 - 4\sqrt{1-p^2}\sqrt{1-k^2p^2} \quad , \tag{5.5}$$

has a simple zero at $p_R \equiv \frac{c_R}{c_d}$, where $c_R$ is the Rayleigh wave velocity, and double zero at the origin, with

$$R(p) \sim -2(k^2+1)p^2 \quad . \tag{5.6}$$

The poles in (5.3, 4) at $p_R$ give rise to the Rayleigh surface wave.

Delta functions in (5.3, 4) characterize the singularities of $\Theta'(\alpha)$ and $\Omega'(\beta)$ at the surface traction discontinuity as simple poles. Their residues follow from the representation. For example, writing

$$\Theta'(\alpha) = \frac{C(\alpha)}{\alpha-\alpha_o} \; ; \quad \alpha_o = i\,\zeta_o = i\cosh^{-1} 1/p_o \; , \tag{5.7}$$

and taking the limit as $\theta \to 0^+$, gives

$$\Theta'(i\,\zeta) = C(i\,\zeta)[\pi\,\delta(\zeta-\zeta_o) - i\frac{1}{\zeta-\zeta_o}] \; , \tag{5.8}$$

where the delta function is the limit of a delta convergent sequence (Gel'fand and Shilov [18]). Using the identity

$$\delta(q-q_o) = |\zeta'(q_o)|\,\delta(\zeta-\zeta_o) \; , \tag{5.9}$$

to write

$$\delta(q-q_o) = \frac{\delta(\zeta-\zeta_o)}{q_o\sqrt{1-p_o^2}} \quad , \tag{5.10}$$

then substituting (5.10) into (5.3) and comparing the result to (5.8) gives the residue of $\Theta'(\alpha)$ at $p_o$ as

$$\lim_{\alpha\to\alpha_o} (\alpha-\alpha_o)\Theta'(\alpha) = \frac{1}{\pi}\frac{p_o^2}{R(p_o)}[-i(q_o^2-2)\sigma_o + 2\sqrt{1-q_o^2}\tau_o] ,$$

$$\equiv i\sigma_o T_1 + \tau_o T_2 , \tag{5.11}$$

where the definitions of $T_1$ and $T_2$ are obvious. Similarly, the residue of $\Omega'(\beta)$ at $q_o$ is

$$\lim_{\beta\to\beta_o} (\beta-\beta_o)\Omega'(\beta) = \frac{1}{2\pi}\frac{q_o^2}{R(p_o)}[2\sqrt{1-p_o^2}\,\sigma_o + i(q_o^2-2)\tau_o] ,$$

$$\equiv \sigma_o T_3 + i\tau_o T_4 . \tag{5.12}$$

These results are equivalent to those of Cole and Huth [19] for the steady state response of the half-space to traveling surface loads.

With regards to the sufficiency of representations, (5.3, 4), we examine briefly the possibility of hidden singularities in the unknown real functions, $\Phi_1$ and $\Phi_2$. Use is made of the fact that except for isolated singular points on the boundary the real and imaginary parts of either $\Theta'$ or $\Omega'$ are complex conjugates. For example, if $\Theta'$ has a simple pole besides the traction and Rayleigh poles then depending on the character of the residue (i.e., real, imaginary, or complex) from (5.8) $\Phi_1$ in (5.3) exhibits a pole and/or delta function while $\Phi_2$ exhibits a conjugate delta function and/or pole. In other words the singularity of one completely determines the singularity of the other by virtue of the conjugacy.

The same argument applies to a simple pole in the $\Omega'$ representation, (5.4), however it is readily shown that the conjugate pair deduced from (5.3) is equivalent to the pair deduced from (5.4) if and only if the singularity occurs at the Rayleigh wave, i.e., $p = p_R$. Similar results are found for logarithmic branch points or higher order poles. In any event because the Rayleigh pole is explicitly factored in (5.3, 4) any infinity in $\Phi_1$ or $\Phi_2$ at $p_R$ can be shown to give rise to unbounded energy at the Rayleigh wave and therefore must be excluded. Note that in general any singularity worse than a simple pole is likewise inadmissible on physical grounds.

We conclude that $\Phi_1$ and $\Phi_2$ in (5.3, 4) may have at most integrable infinities on the closed interval, $0 \leqslant q \leqslant 1$, with the exception of the Rayleigh wave where they are necessarily bounded. Actually because the governing equations and boundary conditions give absolutely no indication of contrary behavior it can be safely assumed in the sequel (for the sake of simplicity) that $\Phi_1$ and $\Phi_2$ are bounded functions (except possibly at the edge as discussed in the next section).

In the composite region, boundary conditions on $\theta = 0$, given by (4.4, 5), are equivalent to

$$i\,\Phi_1(p) = \frac{-i\,k(q^2-2)}{p\sqrt{1-p^2}}\,\Theta'(\alpha) + \frac{4}{q}\,\Omega'_-(\beta) \,, \qquad (5.13)$$

$$i\Psi_1(p) = \frac{2k^2}{q}\Theta'(\alpha) + \frac{2(q^2-2)}{q\sqrt{q^2-1}}\Omega'_-(\beta) \quad , \tag{5.14}$$

(again by taking real parts). The real functions, $\Phi_1$ and $\Psi_1$, can be thought of as extensions of their namesakes in (5.1, 2). Eliminating $\Omega'_-(\beta)$ from (5.13, 14) yields

$$\Theta'(\alpha) = \frac{-p\sqrt{1-p^2}}{kR(p)} \left\{ (q^2-2)\Phi_1(p) - 2\sqrt{q^2-1}\,\Psi_1(p) \right\} \quad , \tag{5.15}$$

which is clearly the continuation of (5.3) with $\sqrt{1-q^2} \rightarrow i\sqrt{q^2-1}$ . The Rayleigh function is now complex, as

$$R(p) = (k^2p^2-2)^2 - i\,4\sqrt{1-p^2}\,\sqrt{k^2p^2-1} \quad . \tag{5.16}$$

Observe that from (5.15), or equivalently (4.6), the argument of $\Theta'(\alpha)$ on the boundary segment is

$$\mathrm{ARG}\,\Theta'(\alpha) = \tan^{-1}\frac{\mathrm{Im}\Theta'(\alpha)}{\mathrm{Re}\Theta'(\alpha)} = \tan^{-1}\frac{-4\sqrt{1-p^2}\,\sqrt{k^2p^2-1}}{(k^2p^2-2)^2} \quad . \tag{5.17}$$

In the sequel we use this information to factor out the complex behavior on the mapped segment.

The boundary value representations of $\Theta'(\alpha)$ and $\Omega'(\beta)$ given by (5.3, 4, 15) are taken as canonical forms by reason of their explicit complex and singular behavior. Additionally, for degenerate (separable) angles, $2\gamma = \pi, 2\pi$, applying simple conformal mappings, the semi-infinite strips in the $\alpha$ and $\beta$ planes go to half-planes, e.g., Figure 5b for the slit, where the representation can be continued off the real axes by inspection. This is the technique used by Craggs [7] and Miles [10] for the half-space and semi-infinite slit respectively, however their formalisms can be simplified considerably using the above canonical forms. The semi-infinite slit is the more involved of the two and is examined in the Appendix. We note in passing that the loading decomposition is superfluous for the half-space.

## § 6. EDGE BEHAVIOR

Typically with physical problems involving an edge, conditions on the behavior of solutions as $r \to 0$ must be established to insure uniqueness. Therefore, before proceeding with the elliptic solutions, we investigate the asymptotic behavior of $\vartheta$ and $\omega$ at the edge. A straightforward approach is to solve local forms of the governing equations and boundary conditions. The results so obtained are compatible with the self-similar solutions and reduced boundary conditions.

Provided $c_r t \gg r$, in particular as $r \to 0$ at the edge, p and q are small. Expanding the coefficients of the governing equations, (2.20, 21), to lowest order in p and q gives

$$p^2 \vartheta_{pp} + p\vartheta_p + \vartheta_{\theta\theta} \simeq 0 \,, \qquad (6.1)$$

$$p^2 \omega_{pp} + p\omega_p + \omega_{\theta\theta} \simeq 0 \,, \qquad (6.2)$$

where the neglected terms are due to inertia. Consequently we expect solutions of (6.1, 2) to yield quasi-static behavior. Expanding the boundary conditions, (2.24, 25), in a similar fashion gives, on $\theta = 0,\ 2\gamma$

$$k^2 p\vartheta_p \simeq 2\omega_\theta \,, \qquad (6.3)$$

$$k^2 \vartheta_\theta \simeq -2p\omega_p \,, \qquad (6.4)$$

which we recognize as the Cauchy-Riemann equations in polar coordinates, p and $\theta$ .

Separable solutions of (6.1, 2) satisfying (6.3, 4) are

$$\vartheta(p,\theta) \simeq p^{\lambda}(a_1 \cos\lambda\theta + a_2 \sin\lambda\theta) \ , \qquad (6.5)$$

$$\omega(p,\theta) \simeq \frac{k^2}{2} p^{\lambda}(a_1 \sin\lambda\theta - a_2 \cos\lambda\theta) \ , \qquad (6.6)$$

or, for the separation parameter, $\lambda$ (not to be confused with the elastic constant used in § 2), vanishing

$$\vartheta(p,\theta) \simeq b_1\theta + b_2 \ell n p \ , \qquad (6.7)$$

$$\omega(p,\theta) \simeq \frac{k^2}{2} (b_2\theta - b_1 \ell n p) \ , \qquad (6.8)$$

where $a_1$, $a_2$, $b_1$, $b_2$ are arbitrary real constants. These satisfy the Cauchy-Riemann equations, (6.3, 4), for arbitrary $\theta$, and so asymptotically $\frac{2}{k^2}\omega$ is the harmonic conjugate of $\vartheta$. Also, substituting the Cauchy-Riemann equations into (2.22, 23) gives

$$\frac{-q^2}{c_r} \frac{\partial \dot{u}_r}{\partial_q} = \frac{r}{c_r^2} \ddot{u}_r \simeq 0 \ , \qquad (6.9)$$

$$\frac{-q^2}{c_r} \frac{\partial \dot{u}_\theta}{\partial q} = \frac{r}{c_r^2} \ddot{u}_\theta \simeq 0 \ , \qquad (6.10)$$

the expected result that inertia effects are negligible for $r \ll c_r t$ . We conclude that a neighborhood of the edge (expanding uniformly with time) moves at some constant velocity.

To determine the order of the power singularity in (6.5, 6) we resort to the stress equations of motion [15], written as

$$r\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{r\theta}}{\partial\theta} + \sigma_{rr} - \sigma_{\theta\theta} = \rho r\ddot{u}_r \ ,$$

$$r\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{\partial\sigma_{\theta\theta}}{\partial\theta} + 2\sigma_{r\theta} = \rho r\ddot{u}_\theta \ .$$

Recalling that $r\frac{\partial}{\partial r} = p\frac{\partial}{\partial p}$ for functions of $r/t$, and substituting (6.9, 10), these reduce to the static equilibrium equations,

$$p\frac{\partial\sigma_{rr}}{\partial p} + \frac{\partial\sigma_{r\theta}}{\partial\theta} + \sigma_{rr} - \sigma_{\theta\theta} \simeq 0 \ , \tag{6.11}$$

$$p\frac{\partial\sigma_{r\theta}}{\partial p} + \frac{\partial\sigma_{\theta\theta}}{\partial\theta} + 2\sigma_{r\theta} \simeq 0 \ , \tag{6.12}$$

with the compatibility equation, $\nabla^2\vartheta = 0$, implicit in (6.1). Substituting (6.5) into (2.26), gives

$$\sigma_{rr} = p^{\lambda}(c_1\cos\lambda\theta + c_2\sin\lambda\theta) - \sigma_{\theta\theta} \ ,$$

with

$$c_{1,2} = 2\mu(k^2 - 1)\, a_{1,2} \ ;$$

introducing this into (6.11), eliminating $\sigma_{r\theta}$ or $\sigma_{\theta\theta}$ between the two equations, and solving, yields

$$\sigma_{\theta\theta} \simeq p^{\lambda}\hat{A}(\theta) \ , \tag{6.13}$$

$$\sigma_{r\theta} \simeq p^{\lambda}\hat{B}(\theta) \ , \tag{6.14}$$

where

$$\hat{A}(\theta) = \frac{-(\lambda+1)(\lambda+2)}{\lambda^2}[c_1 \cos\lambda\theta + c_2 \sin\lambda\theta] + c_3 \cos(\lambda+2)\theta$$

$$+ c_4 \sin(\lambda+2)\theta \quad , \tag{6.15}$$

$$\hat{B}(\theta) = \frac{(\lambda+1)(\lambda+2)}{\lambda^2}[(-c_1 + \frac{2}{\lambda} c_2)\sin\lambda\theta + (\frac{2}{\lambda}c_1 + c_2)\cos\lambda\theta]$$

$$+ \frac{1}{\lambda}[(\lambda+2)c_3 - 2c_4)\sin(\lambda+2)\theta - (2c_3+(\lambda+2)c_4)\cos(\lambda+2)\theta] \tag{6.16}$$

Boundary conditions on $\theta = 0, 2\gamma$ may be taken as

$$\frac{\partial\sigma_{\theta\theta}}{\partial p} = \frac{\partial\sigma_{r\theta}}{\partial p} = 0 \quad , \tag{6.17}$$

which require

$$\hat{A}(0) = \hat{B}(0) = \hat{A}(2\gamma) = \hat{B}(2\gamma) = 0 \quad .$$

The conditions on $\theta = 0$ give

$$c_3 = \frac{(\lambda+1)(\lambda+2)}{\lambda^2}c_1 \ , \quad c_4 = \frac{\lambda+1}{\lambda}c_2 \ , \tag{6.18}$$

while those on $\theta = 2\gamma$ yield a homogeneous system of equations for $c_1$ and $c_2$. For a nontrivial solution, the determinant of the system vanishes leaving a transcendental equation for $\lambda$, namely

$$\sin^2 2\gamma(\lambda+1) = (\lambda+1)^2 \sin^2 2\gamma \ . \tag{6.19}$$

This equation and its admissible solutions are well known from static theory [20], [21]. In particular $\lambda = -1/2$ when $\gamma = \pi$ (slit) and increases monotonically to $\lambda = 0$ when $\gamma = \pi/2$ (half-space).

For the reentrant wedge, $\pi < 2\gamma \leq 2\pi$, we therefore expect power singularities in the dilatation and rotation, but for smaller angles the behavior is given by (6.7, 8), with at most a logarithmic singularity at the edge. We mention in passing that attempting to solve for constants $b_1$ and $b_2$ in (6.7, 8) using the static equations, (6.11, 12), as above gives in general only the trivial solution (exceptions are at $2\gamma = \pi/2, \pi$).

To determine the behavior of $\Theta'(\alpha)$ and $\Omega'(\beta)$ we substitute the edge solutions, (6.5-8), into asymptotic forms of (3.5, 6), giving for $p, q \to 0$

$$\Theta'(\alpha) \sim \begin{cases} -i\lambda 2^{\lambda}(a_1 + i a_2)\, e^{i\lambda\alpha} \; ; \quad 2\gamma > \pi \; , & (6.20) \\ b_1 + i b_2 \; ; \quad 2\gamma \leq \pi \; , & (6.21) \end{cases}$$

$$\Omega'(\beta) \sim \begin{cases} \lambda \dfrac{k^{2-\lambda}}{2^{1-\lambda}} (a_1 - i a_2) e^{i\lambda\beta} \; ; \quad 2\gamma > \pi \; , & (6.22) \\ -i\dfrac{k^2}{2} (b_1 + i b_2) \; ; \quad 2\gamma \leq \pi \; . & (6.23) \end{cases}$$

With the edge behavior known we are in a position to solve uniquely for $\Theta'(\alpha)$ and $\Omega'(\beta)$.

## § 7. FACTORIZATIONS OF $\Theta'(\alpha(z))$ AND $\Omega'(\beta(w))$ IN THE TRANSFORMED ELLIPTIC DOMAINS

In preparation for determining the analytic functions, $\Theta'(\alpha)$ and $\Omega'(\beta)$, the semi-infinite strips in the range $0 < \theta < \gamma$ are mapped onto the upper half of complex z and w planes by the conformal transformations,

$$z = x + iy = 1/\cos\frac{\pi\alpha}{\gamma} \quad , \tag{7.1}$$

$$w = u + iv = 1/\cos\frac{\pi\beta}{\gamma} \quad . \tag{7.2}$$

The succession of mappings from the physical domain, to the semi-infinite strips, to the z and w planes, is illustrated in Figures 5a, b. On the real axes (the mapped elliptic domain boundaries) we have

$$1/x = \cos\frac{\pi\theta}{\gamma}\cosh\left(\frac{\pi}{\gamma}\cosh^{-1} 1/p\right) \quad , \tag{7.3}$$

$$1/u = \cos\frac{\pi\theta}{\gamma}\cosh\left(\frac{\pi}{\gamma}\cosh^{-1} 1/q\right) \quad . \tag{7.4}$$

Points $x_o$, $x_R$, $u_o$, $u_R$ in Figure 5b are the images of $p_o$, $p_R$, $q_o$, $q_R$ (i.e., the traction and Rayleigh poles) on $\theta = 0$, while subscripts a-g, used primarily in the sequel, refer to images of A-G in Figure 5a (e.g., $x_b$ on the real axis in the z plane).

The conformal mapping preserves boundary values, so collecting boundary conditions on $\Theta'(\alpha)$ from the previous sections gives conditions on $\Theta'(\alpha(x))$ as:

$x \geq 1$ , $x \leq -1$ $(p = 1$ , $0 \leq \theta \leq \gamma)$

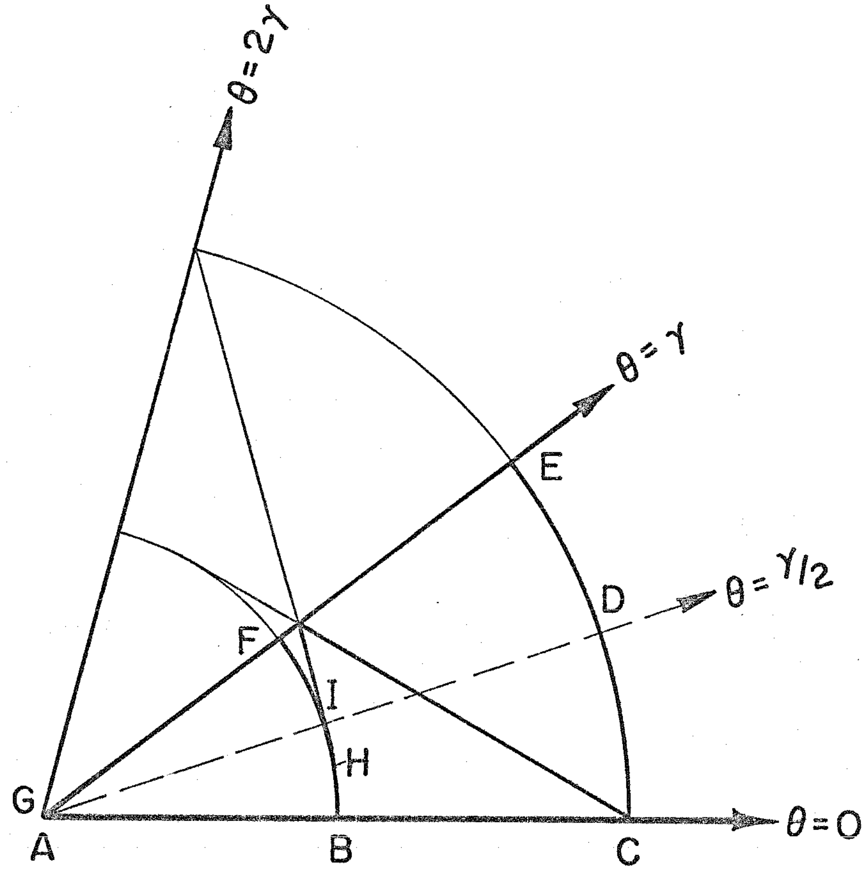


FIGURE 5a

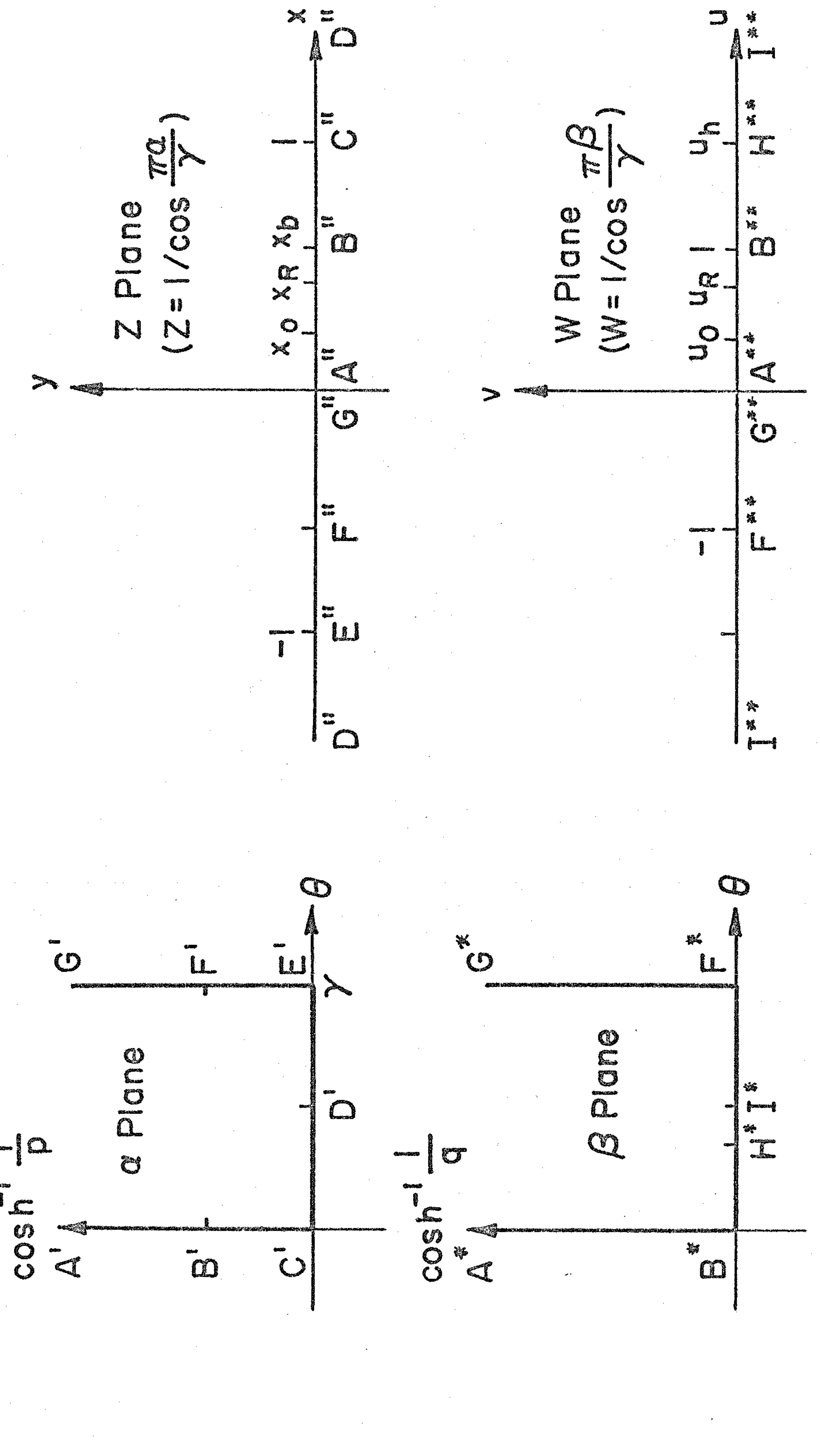
Z Plane
(Z = 1/cos πα/γ)
W Plane
(W = 1/cos πβ/γ)
α Plane
β Plane
cosh⁻¹ 1/p
cosh⁻¹ 1/q
x
y
u
v
θ
γ
x₀ x_R x_b
u₀ u_R 1 u_h
-1
1
A″ B″ C″ D″ E″ F″ G″
A** B** F** G** H** I**
A′ B′ C′ D′ E′ F′ G′
A* B* F* G* H* I*

FIGURE 5b

$$\mathrm{Re}\,\Theta'(\alpha(x)) = 0 \quad , \tag{7.5}$$

from (4.9) and (4.2), where

$$\alpha(x) = \theta(x) = \frac{\gamma}{\pi} \cos^{-1} 1/x \quad ; \tag{7.6}$$

$-1<x<0 \ (\theta = \gamma\ ,\ 0<p<1)$

$$\mathrm{Re}\,\Theta'(\alpha(x)) = 0 \quad ; \quad \text{sym. case,} \tag{7.7}$$

$$\mathrm{Im}\,\Theta'(\alpha(x)) = 0 \quad ; \quad \text{asym. case,} \tag{7.8}$$

from (3.19, 20, 23, 24) where

$$\alpha(x) = \gamma + i\cosh^{-1} 1/p(x) = \gamma + i\frac{\gamma}{\pi}\cosh^{-1} 1/x \quad ; \tag{7.9}$$

$0 \leqslant x<1 \ (\theta = 0\ ,\ 0 \leqslant p<1)$

$$\Theta'(\alpha(x)) = \frac{-p(x)\sqrt{1-p^2(x)}}{kR(p(x))} \Bigg\{ (q^2(x)-2)\,\Phi_1(p(x)) + i\,2\sqrt{1-q^2(x)}\;\Psi_1(p(x)) + [\,i\,(q_o^2-2)\,\sigma_o - 2\sqrt{1-q_o^2}\;\tau_o]\,q_o^2\,\delta(q(x)-q_o) \Bigg\} \quad , \tag{7.10}$$

from (5.3, 15) with

$$1/p(x) = k/q(x) = \cosh(\gamma/\pi\ \cosh^{-1} 1/x) \quad . \tag{7.11}$$

Similarly, boundary values on $\Omega'(\beta(u))$ are:

$u \geqslant 1\ ,\ u \leqslant -1 \ (q=1\ ,\ 0 \leqslant \theta \leqslant \gamma)$

$$\mathrm{Re}\,\Omega'(\beta(u)) = \begin{cases} H(\theta(u)) \ ; & 1 \leqslant u < u_h \ , \\ H(\theta(u)) \pm H(2\gamma-\theta(u)) \ ; & u \geqslant u_h\ ,\ u \leqslant -1 \ , \end{cases} \tag{7.12}$$

with

$$H(\theta) = \frac{-k^2 \tan\theta}{\tan^2\theta - 1} \; \mathrm{Re}\, \Theta'(i \cosh^{-1}(k\cos\theta)) \quad , \qquad (7.13)$$

from (4.10, 7, 8), where

$$\beta(u) = \theta(u) = \frac{\gamma}{\pi} \cos^{-1} 1/u \quad ; \qquad (7.14)$$

$-1 \leq u < 0 \quad (\theta = \gamma \;,\; 0 < q \leq 1)$

$$\mathrm{Im}\, \Omega'(\beta(u)) = 0 \quad ; \quad \text{sym. case,} \qquad (7.15)$$

$$\mathrm{Re}\, \Omega'(\beta(u)) = 0 \quad ; \quad \text{asym. case,} \qquad (7.16)$$

from (3.23, 24), with

$$\beta(u) = \gamma + i \frac{\gamma}{\pi} \cosh^{-1} 1/u \quad ; \qquad (7.17)$$

$0 \leq u < 1 \quad (\theta = 0, \; 0 \leq q < 1)$

$$\Omega'(\beta(u)) = \frac{q(u)\sqrt{1-q^2(u)}}{2R(p(u))} \left\{ -i\, 2\sqrt{1-p^2(u)}\; \Phi_1(p(u)) + (q^2(u)-2)\, \Psi_1(p(u)) \right.$$
$$\left. + \left[ 2\sqrt{1-p_o^2}\; \sigma_o + i\,(q_o^2 - 2)\, \tau_o \right] q_o^2\, \delta(q(u)-q_o) \right\} , \qquad (7.18)$$

from (5.4), where

$$1/q(u) = 1/kp(u) = \cosh\left(\frac{\gamma}{\pi} \cosh^{-1} 1/u\right) \quad . \qquad (7.19)$$

This boundary behavior is illustrated in Figure 6. Notice that on the characteristic envelopes, CE and BF in Figure 5a,

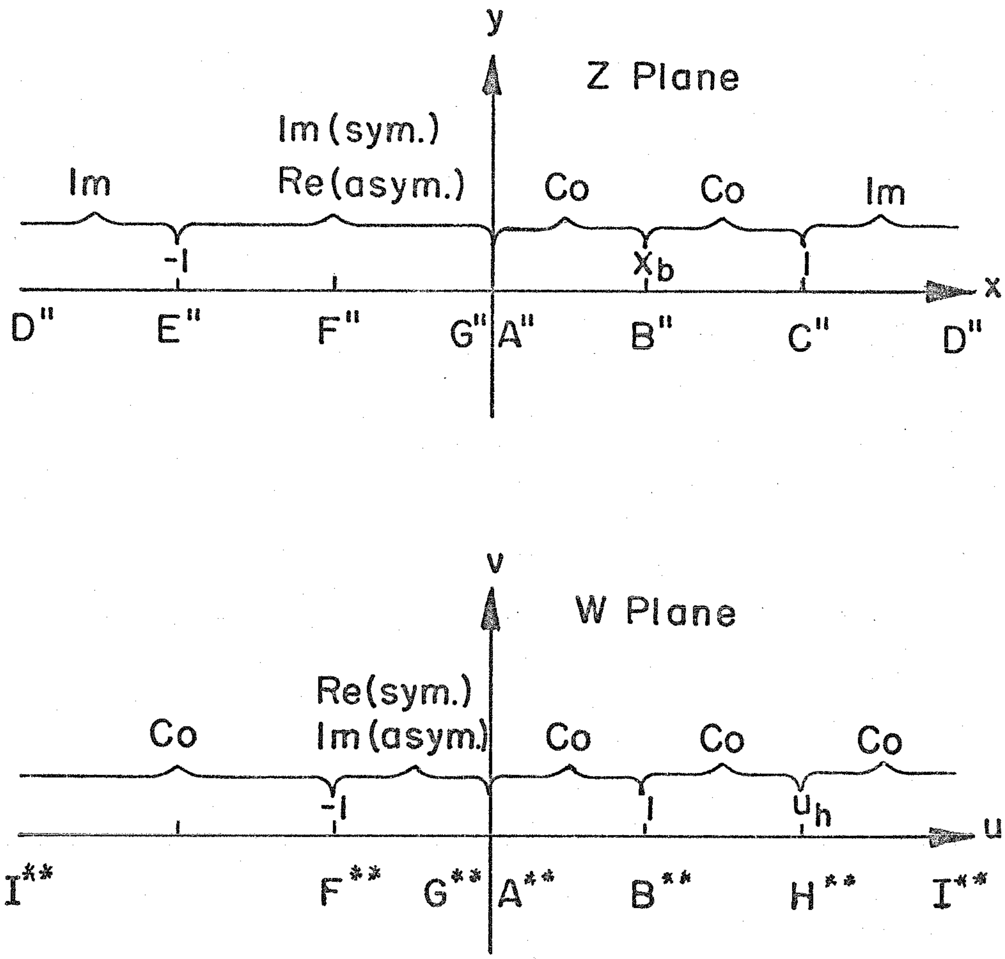
y
Z Plane
Im (sym.)
Re (asym.)
Im
Co
Co
Im
-1
$x_b$
1
x
D"
E"
F"
G"
A"
B"
C"
D"
v
W Plane
Re(sym.)
Im (asym.)
Co
Co
Co
Co
-1
1
$u_h$
u
I**
F**
G**
A**
B**
H**
I**

FIGURE 6

applying results of the wavefront expansion, i.e., (4.13, 14), $\Theta'(\theta)$ and $\Omega'(\theta)$ are bounded there, including the endpoints. Hence at infinity in the z and w planes (images of D or I in Figure 5a) $\Theta'$ and $\Omega'$ go to imaginary and complex constants respectively (recall, from (4.2, 9) Re $\Theta'$ vanishes). Additionally, solutions at branch points corresponding to B, C, E, F, and H are bounded, at least from one side, in which case analytic function theory guarantees boundedness from both sides. This excludes, for example, logarithmic or algebraically unbounded branch points (e.g., $1/\sqrt{z}$). Therefore transitions from real or imaginary to complex along the boundary in the z or w planes are continuous at images of B, C, E, F, and H.

At the origin the asymptotic edge solutions have both $\Theta'$ and $\Omega'$ going to complex constants as in (6.21, 23), but applying (7.7, 8) or (7.15, 16) requires that $b_1 = 0$ for the symmetric and $b_2 = 0$ for the asymmetric case; namely, for $z, w \to 0$

$$\Theta'(\alpha(z)) \sim \begin{cases} i b_2 \; ; & \text{sym. case,} \\ b_1 \; ; & \text{asym. case,} \end{cases} \tag{7.20}$$

$$\Omega'(\beta(w)) \sim \begin{cases} \frac{k^2}{2} b_2 \; ; & \text{sym. case,} \quad (7.21) \\ -i\frac{k^2}{2} b_1 \; ; & \text{asym. case.} \quad (7.21) \end{cases}$$

We now use these boundary value representations to factor out the explicit boundary behavior of $\Theta'(\alpha(z))$ and $\Omega'(\beta(w))$ (i.e., branch points, poles, etc.).

Beginning with $\Theta'(\alpha(z))$, the complex behavior of the Rayleigh function in (7.10) (cf., (5.15, 16)) is factored by a sectionally holomorphic function with line of discontinuity (branch cut) on $x_b < x < 1$, and equal to $R(p(x))$ there as $y \to 0^+$. In other words, we seek a function analytic in the upper half-plane and real or imaginary on the real axis, except for the segment $x_b < x < 1$ on which it is complex and equal to $R(p(x))$. This is easily found using limiting values ($y \to 0^+$) of the Cauchy type integral,

$$\Gamma(z) = \frac{1}{\pi}\int_{x_b}^{1} \frac{D(s)}{s-z}\,ds \quad , \qquad (7.22)$$

to represent the principal value of the logarithm of $R(p(x))$ in the range $x_b < x < 1$ as

$$\log R(p(x)) = \mathrm{Log}\,|R(p(x))| + i\tan^{-1}\frac{-4\sqrt{1-p^2(x)}\,\sqrt{k^2p^2(x)-1}}{(k^2p^2(x)-2)^2}$$

$$= \lim_{y\to 0^+}\Gamma(z) \equiv \Gamma^+(x) \quad ,$$

$$= \frac{1}{\pi}\int_{x_b}^{1} \frac{D(s)}{s-x}\,ds + i\,D(x) \quad ,$$

with the last integral taking its principal value at $s = x$. Equating real and imaginary parts gives

$$D(x) = \tan^{-1} \frac{-4\sqrt{1-p^2(x)}\,\sqrt{k^2p^2(x)-1}}{(k^2p^2(x)-2)^2} \quad , \tag{7.23}$$

$$\Gamma^P(x) \equiv \frac{1}{\pi}\int_{x_b}^{1} \frac{D(s)}{s-x}\, ds = \text{Log}\,|R(p(x))| \ ,$$

$$= \frac{1}{2}\,\text{Log}\,[\,(k^2p^2(x)-2)^4 + 16(1-p^2(x)(k^2p^2(x)-1)] \quad , \tag{7.24}$$

where the P superscript denotes a principal value. The Rayleigh factor is then

$$e^{\Gamma(z)} = \exp[\frac{1}{\pi}\int_{x_b}^{1} \frac{D(s)}{s-z}\, ds\,] \quad , \tag{7.25}$$

analytic everywhere except on the real segment, $x_b \leq x \leq 1$, (i.e., sectionally holomorphic) and $O(1/z)$ at infinity.

Removing the complex behavior on $x_b < x < 1$ and the branch point at $z = 1$ (from the $\sqrt{1-p^2(x)}$ term in (7.10)) gives

$$\Phi_2(z) = \frac{e^{\Gamma(z)}\Theta'(\alpha(z))}{\sqrt{1-z}} \quad , \tag{7.26}$$

with the square root positive for positive argument, that is, at $z = x < 1$. This convention is applied to all subsequent square root branch points, with branch cuts taken on the real axis. The analytic function, $\Phi_2(z)$, is complex on $0 < x < x_b$, real on $x \geq x_b$, and $O(\frac{1}{\sqrt{z}})$ at infinity. Referring to Figure 6, for the symmetric case $\Phi_2$ is imaginary on $x<0$, and for the asymmetric case it has a branch point at $z=-1$ like $\sqrt{1+z}$ (rather than $1/\sqrt{1+z}$, by continuity).

It is convenient in either case to remove the Rayleigh and traction poles by subtraction, yielding the factorizations,

$$\sqrt{z}\,\Phi_s(z) = \frac{e^{\Gamma(z)}\Theta'(\alpha(z))}{\sqrt{1-z}} - \frac{iR_1\sqrt{x_b-z} + R_2\sqrt{z}}{z-x_R} - \frac{\sigma_o S_1\sqrt{z} + i\tau_o S_2\sqrt{x_b-z}}{z-x_o}, \tag{7.27}$$

$$\Phi_a(z) = \frac{e^{\Gamma(z)}\Theta'(\alpha(z))}{\sqrt{1-z^2}} - \frac{iR_1\sqrt{z/(x_b-z)}+R_2}{z-x_R} - \frac{\sigma_o S_1 + i\tau_o S_2\sqrt{z/(x_b-z)}}{z-x_o}. \tag{7.28}$$

The surviving factors, $\Phi_{s,a}$ (symmetric and asymmetric respectively), are real on the real axis except on $0<x<x_b$ where they are complex. The square root branch points are chosen to make the real and imaginary boundary behavior consistent and to match the order at infinity. Solving for $\Theta'(\alpha(z))$ gives, for the symmetric case,

$$\Theta'(\alpha(z)) = e^{-\Gamma(z)}\sqrt{1-z}\left[\sqrt{z}\,\Phi_s(z) + \frac{iR_1\sqrt{x_b-z} + R_2\sqrt{z}}{z-x_R} + \frac{\sigma_o S_1\sqrt{z} + i\tau_o S_2\sqrt{x_b-z}}{z-x_o}\right], \tag{7.29}$$

and for the asymmetric case,

$$\Theta'(\alpha(z)) = e^{-\Gamma(z)}\sqrt{1-z^2}\left[\Phi_a(z) + \frac{i R_1\sqrt{z/(x_b-z)} + R_2}{z - x_R} + \frac{\sigma_o S_1 + i\tau_o S_2\sqrt{z/(x_b-z)}}{z - x_o}\right], \tag{7.30}$$

where the unknown factors are necessarily $O(1/z)$ at infinity. To exhibit the asymptotic results, (7.20), at the origin, clearly for $x \to 0^+$

$$\mathrm{Re}\ \Phi_s(x) = o(1/\sqrt{x}),\ \mathrm{Im}\ \Phi_s(x) = O(1/\sqrt{x}), \tag{7.31}$$

$$\mathrm{Re}\ \Phi_a(x) = O(1),\ \mathrm{Im}\ \Phi_a(x) = o(1). \tag{7.32}$$

Also, from the $\sqrt{1-q^2(x)}$ term in (7.10), $\mathrm{Im}\ \Theta'(\alpha(x))$ vanishes at $x_b$ like $\sqrt{x_b-x}$, whence as $x \to x_b^-$

$$\mathrm{Im}\ \Phi_s(x) = O(\sqrt{x_b-x}), \tag{7.33}$$

$$\mathrm{Im}\ \Phi_a(x) = O(1/\sqrt{x_b-x}). \tag{7.34}$$

Factorization of $\Omega'(\beta(w))$ are obtained in an analogous fashion as

$$\sqrt{1+w}\ \Psi_s(w) = \frac{\Omega'(\beta(w))}{\sqrt{1-w}} - \frac{R_3\sqrt{1+w} + i R_4\sqrt{w}}{w - u_R} - \frac{i\sigma_o S_3\sqrt{w} + \tau_o S_4\sqrt{1+w}}{w - u_o}, \tag{7.35}$$

$$\sqrt{w}\;\Psi_a(w) = \frac{\Omega'(\beta(w))}{\sqrt{1-w}} - \frac{R_3\sqrt{w} + i R_4\sqrt{1+w}}{w - u_R}$$

$$- \frac{i\sigma_o S_3\sqrt{1+w} + \tau_o S_4\sqrt{w}}{w - u_o} \quad , \tag{7.36}$$

where both residual factors are complex on $u > 0$ , $u < -1$ while $\Psi_s$ is real on $-1 \leqslant u < 0$ and $\Psi_a$ is imaginary there. Solving for $\Omega'(\beta(w))$, the symmetric case gives

$$\Omega'(\beta(w)) = \sqrt{1-w}\left[\sqrt{1+w}\;\Psi_s(w) + \frac{R_3\sqrt{1+w} + i R_4\sqrt{w}}{w - u_R}\right.$$

$$\left. + \frac{i\sigma_o S_3\sqrt{w} + \tau_o S_4\sqrt{1+w}}{w - u_o}\right] \quad , \tag{7.37}$$

and for the asymmetric case,

$$\Omega'(\beta(w)) = \sqrt{1-w}\left[\sqrt{w}\;\Psi_a(w) + \frac{R_3\sqrt{w} + i R_4\sqrt{1+w}}{w - u_R}\right.$$

$$\left. + \frac{i\sigma_o S_3\sqrt{1+w} + \tau_o S_4\sqrt{w}}{w - u_o}\right] \quad , \tag{7.38}$$

where again the unknown factors are $O(1/z)$ at infinity. As $u \to 0^+$ the asymptotic solutions, (7.20), require

$$\operatorname{Re}\Psi_s(u) = O(1) \;,\quad \operatorname{Im}\Psi_s(u) = o(1) \;, \tag{7.39}$$

$$\operatorname{Re}\Psi_a(u) = o(1/\sqrt{u}),\quad \operatorname{Im}\Psi_a(u) = O(1/\sqrt{u}). \tag{7.40}$$

Continuity at the $w = -1$ branch point and (7.12, 15, 16) yield for $u \to -1^-$

$$\mathrm{Re}\,\Psi_s(u) = o(1/\sqrt{-1-u}) \quad , \quad \mathrm{Im}\,\Psi_s(u) = O(1/\sqrt{-1-u}) \quad , \tag{7.41}$$

$$\mathrm{Re}\,\Psi_a(u) = O(1) \quad , \quad \mathrm{Im}\,\Psi_a(u) = o(1) \quad . \tag{7.42}$$

The residue at $x_o$, hence $S_1$ and $S_2$, is found from (7.29, 30) using (5.11) in

$$\lim_{z \to x_o} (z-x_o)\Theta'(\alpha(z)) = (i\,\sigma_o T_1 + \tau_o T_2) \lim_{\substack{z \to x_o \\ (\alpha \to \alpha_o)}} \frac{z - x_o}{\alpha - \alpha_o} \quad ,$$

$$= \frac{\pi}{\gamma} (-\sigma_o T_1 + i\,\tau_o T_2)\, x_o \sqrt{1-x_o^2} \quad ,$$

to be

$$S_1 = \frac{-\pi}{\gamma} T_1 e^{\Gamma(x_o)} \begin{cases} \sqrt{x_o(1+x_o)} & ; \text{ sym. case,} \\ x_o & ; \text{ asym. case,} \end{cases} \tag{7.43}$$

$$S_2 = \frac{\pi}{\gamma} T_2 \sqrt{x_o}\, e^{\Gamma(x_o)} \begin{cases} \sqrt{x_o(1+x_o)/(x_b-x_o)} & ; \text{ sym. case,} \\ \sqrt{x_b-x_o} & ; \text{ asym. case.} \end{cases} \tag{7.49}$$

Similarly $S_3$ and $S_4$ are found from (7.37, 38) using (5.12), as

$$S_3 = \frac{\pi}{\gamma} T_3 \sqrt{u_o} \begin{cases} \sqrt{1+u_o} & \text{sym. case,} \\ \sqrt{u_o} & \text{asym. case,} \end{cases} \tag{7.45}$$

$$S_4 = \frac{-\pi}{\gamma} T_4 \sqrt{u_o} \begin{cases} \sqrt{u_o} & \text{sym. case,} \\ \sqrt{1+u_o} & \text{asym. case.} \end{cases} \tag{7.46}$$

In order that the residues of $\Theta'(\alpha(z))$ and $\Omega'(\beta(\omega))$ in the factorization be consistent with (7.10) and (7.18) at the Rayleigh pole,

$$R_3 = -R_1 \frac{k^2(q_R^2-2)}{4\sqrt{1-q_R^2}} \sqrt{u_R/x_R}\ e^{-\Gamma(x_R)} \quad .$$

$$\begin{cases} \sqrt{u_R/x_R}\ \sqrt{(x_b-x_R)(1+x_R)} & \text{sym. case,} \\ 1/\sqrt{x_b-x_R} & \text{asym. case,} \end{cases} \tag{7.47}$$

$$R_4 = R_2 \frac{k\sqrt{1-P_R^2}}{q_R^2-2} \sqrt{u_R/x_R}\ e^{-\Gamma(x_R)} \quad .$$

$$\begin{cases} \sqrt{(1+u_R)(1+x_R)} & \text{sym. case,} \\ \sqrt{u_R/x_R} & \text{asym. case .} \end{cases} \tag{7.48}$$

The factorizations of $\Theta'(\alpha(z))$ and $\Omega'(\beta(w)$ are now complete. There remains the determination of residual factors $\Phi_{s,a}(z)$ and $\Psi_{s,a}(w)$. In the next section this is reduced to the solution of regular Fredholm integral equations of the second kind.

## § 8. INTEGRAL EQUATIONS ON THE RESIDUAL FACTORS

Recall that the residual factors in $\Theta'(\alpha(z))$ and $\Omega'(\beta(w))$ are analytic in the extended planes with the exception of branch cuts on the real axes, and with at most integrable singularities at the branch points. Consequently they are sectionally holomorphic functions, equivalent to Cauchy type integrals over segments of the real axes.

We therefore express $\Phi_{s,a}$ and $\Psi_{s,a}$ as

$$\Phi_s(z) = \frac{1}{\pi}\int_0^{x_b} \sqrt{\frac{x_b - s}{s}}\;\frac{\phi_s(s)}{s-z}\,ds\;, \tag{8.1}$$

$$\Phi_a(z) = \frac{1}{\pi}\int_0^{x_b} \sqrt{\frac{s}{x_b - s}}\;\frac{\phi_a(s)}{s-z}\,ds\;, \tag{8.2}$$

$$\Psi_s(w) = \frac{1}{\pi}\int_L \sqrt{\frac{s}{1+s}}\;\frac{\psi_s(s)}{s-w}\,ds\;, \tag{8.3}$$

$$\Psi_a(w) = \frac{1}{\pi}\int_L \sqrt{\frac{1+s}{s}}\;\frac{\psi_a(s)}{s-w}\,ds\;, \tag{8.4}$$

with the weighted density functions, $\phi_{s,a}$ and $\psi_{s,a}$ (henceforth referred to as the density functions), assumed Hölder continuous on the open interval. The integration interval, L, is given by

$$L : s \text{ from } -\infty \text{ to } -1,\; 0 \text{ to } \infty\;. \tag{8.5}$$

These representations clearly have the proper order at infinity. Boundary values as $y, v \to 0^+$ on the paths of integration are

$$\Phi_s^+(x) = i\sqrt{(x_b - x)/x}\ \phi_s(x) + \Phi_s^P(x) \ , \tag{8.6}$$

$$\Phi_a^+(x) = i\sqrt{x/(x_b - x)}\ \phi_a(x) + \Phi_a^P(x) \ , \tag{8.7}$$

$$\Psi_s^+(u) = i\sqrt{u/(1+u)}\ \psi_s(u) + \Psi_s^P(u) \ , \tag{8.8}$$

$$\Psi_a^+(u) = i\sqrt{(1+u)/u}\ \psi_a(u) + \Psi_a^P(u) \ , \tag{8.9}$$

where as before the P superscript means a principal value.

Applying results in Muskhelishvili [22] for the evaluation of Cauchy type line integrals near the endpoints, the singular integrals in (8.6-9) are all bounded at the endpoints provided the density functions are bounded and nonvanishing there. For z or w approaching an endpoint from off the integration path the integrals are unbounded like the Cauchy kernals, e.g., $\sqrt{(x_b - x)/x}\ \phi_s(x)$ in (8.1). It follows then that these representations satisfy conditions (7.31-34) and (7.39-42) on the residual factors, with $\phi_{s,a}$ and $\psi_{s,a}$ bounded and nonvanishing at the endpoints.

The factorizations, with unknowns represented by (8.1-4), exhibit all the explicit behavior of the boundary value representations but as of yet still do not satisfy boundary conditions on $0 < x < x_b$ and $u < -1$, $u > 0$. Necessary conditions on the density functions are found by substituting (8.6-9) into the factorizations, (7.29, 30) and (7.37, 38), evaluated on the real axes, and applying boundary conditions. The two loading cases are considered separately below. Because forms of the equations are similar in either case, to simplify the notation the same symbols are used although their definitions generally differ. No confusion should arise from this convention.

SYMMETRIC LOADING CASE

Introducing (8.6, 7) into the $\Theta'(\alpha(z))$ representation, (7.29), evaluated on $0 \leq x \leq x_b$, gives

$$\mathrm{Re}\,\Theta'(\alpha(x)) = e^{-\Gamma(x)}\sqrt{x(1-x)}\left[\Phi_s^P(x) + \frac{R_2}{x-x_R} + \frac{\sigma_o S_1}{x-x_o}\right.$$

$$+ R_1\sqrt{(x_b-x_R)/x_R}\;\pi\,\delta(x-x_R)$$

$$\left. + \tau_o S_1\sqrt{(x_b-x_o)/x_o}\;\pi\,\delta(x-x_o)\right] , \tag{8.10}$$

$$\mathrm{Im}\,\Theta'(\alpha(x)) = e^{-\Gamma(x)}\sqrt{(x_b-x)(1-x)}\left[\phi_s(x) + \frac{R_1}{x-x_R} + \frac{\tau_o S_2}{x-x_o}\right.$$

$$- R_2\sqrt{x_R/(x_b-x_R)}\;\pi\,\delta(x-x_R)$$

$$\left. - \sigma_o S_1\sqrt{x_o/(x_b-x_o)}\;\pi\,\delta(x-x_o)\right] . \tag{8.11}$$

Similarly, on $0 \leq u \leq 1$, (8.8, 9) into (7.37, 38) gives

$$\text{Re}\ \Omega'(\beta(u)) = \sqrt{1-u^2}\left[\Psi_s^P(u) + \frac{R_3}{u-u_R} + \frac{\tau_o S_4}{u-u_o} + R_4\sqrt{u_R/(1+u_R)}\ \pi\delta(u-u_R) + \sigma_o S_3\sqrt{u_o/(1+u_o)}\ \pi\delta(u-u_o)\right] , \quad (8.12)$$

$$\text{Im}\ \Omega'(\beta(u)) = \sqrt{u(1-u)}\left[\psi_s(u) + \frac{R_4}{u-u_R} + \frac{\sigma_o S_3}{u-u_o} - R_3\sqrt{(1+u_R)/u_R}\ \pi\,\delta(u-u_R) - \tau_o S_4\sqrt{(1+u_o)/u_o}\ \pi\,\delta(u-u_o)\right] . \quad (8.13)$$

The delta functions come from boundary values at the traction and Rayleigh poles as in (5.7, 8).

Substituting these into the original reduced boundary conditions on $\theta = 0$, $q = kp < 1$, (3.21, 22), with $\alpha, \beta \rightarrow \alpha(x), \beta(u(x))$, yields coupled singular integral equations on $\phi_s(x)$ and $\psi_s(u)$ over $0 \leqslant x \leqslant x_b$, $0 \leqslant u \leqslant 1$ as

$$\phi_s(x) = f(x)\,\Psi_s^P(u(x)) + R_1 f_1(x) + \tau_o f_2(x) , \quad (8.14)$$

$$\psi_s(u(x)) = g(x)\Phi_s^P(x) + R_2 g_1(x) + \sigma_o g_2(x) , \quad (8.15)$$

where, from (7.11, 19), x and u are related by

$$1/u(x) = \cosh\left\{\frac{\pi}{\gamma}\cosh^{-1}\left[\frac{1}{k}\cosh\left(\frac{\gamma}{\pi}\cosh^{-1}\frac{1}{x}\right)\right]\right\} , \quad (8.16)$$

and

$$f(x) = \frac{-4\sqrt{1-p^2(x)}}{k^2(q^2(x)-2)} \sqrt{\frac{1-u^2(x)}{(x_b-x)(1-x)}} \; e^{\Gamma(x)} \quad , \tag{8.17}$$

$$= \frac{-4\sqrt{1-p^2(x)}\ \sqrt{1-q^2(x)}}{(q^2(x)-2)^2} \sqrt{\frac{x(1+u(x))}{u(x)(x_o-x)}} \ \frac{1}{g(x)} \quad , \tag{8.18}$$

$$f_1(x) = \frac{-1}{x-x_R} + \frac{(R_3/R_1)\,f(x)}{u(x)-u_R} \quad , \tag{8.19}$$

$$g_1(x) = \frac{g(x)}{x-x_R} - \frac{R_4/R_2}{u(x)-u_R} \quad , \tag{8.20}$$

$$f_2(x) = \frac{-S_2}{x-x_o} + \frac{S_4 f(x)}{u(x)-u_o} \quad , \tag{8.21}$$

$$g_2(x) = \frac{S_1 g(x)}{x-x_o} - \frac{S_3}{u(x)-u_o} \quad . \tag{8.22}$$

Observe that (8.17-22) have removable singularities only and that $f(x)$ and $g(x)$ are nonvanishing.

In the w plane, on $u<-1$, $u>1$ (i.e., the map of the $q = 1$ characterstic envelope), substituting (8.8, 9) into the factorization gives

$$\mathrm{Re}\,\Omega'(\beta(u)) = \mathrm{sgn}\,(u)\sqrt{u(u-1)} \left[ \psi_s(u) + \frac{R_4}{u-u_R} + \frac{\sigma_o S_3}{u-u_o} \right] \quad , \tag{8.23}$$

$$\mathrm{Im}\,\Omega'(\beta(u)) = -\mathrm{sgn}\,(u)\sqrt{u^2-1} \left[ \Psi_s^P(u) + \frac{R_3}{u-u_R} + \frac{\tau_o S_4}{u-u_o} \right] \quad , \tag{8.24}$$

where the sgn function gives the sign of its argument. Replacing

Re $\Omega'$ in (7.12) by the above and solving for the density function, the transformed conditions on the characteristic envelope yield

$$\psi_s(u) = \frac{-R_4}{u-u_R} - \frac{\sigma_o S_3}{u-u_o} + \frac{1}{\sqrt{u(u-1)}} \begin{cases} H(\theta(u)) \ ; \ 1<u<u_h \ , \\ H(\theta(u)) + H(2\gamma-\theta(u)) \ ; \ u \geq u_h \ , \\ -H(\theta(u)) - H(2\gamma-\theta(u)) \ ; \ u<-1 \ . \end{cases} \quad (8.25)$$

Substituting the factorization of $\Theta'(\alpha(x))$ into the definition of $H(\theta)$, (7.13), and introducing a superscript notation to designate contributions on $\beta_+$ and $\beta_-$ characteristics in the physical domain (namely, the $2\gamma-\theta(u)$ or $\theta(u)$ arguments respectively in (8.25)), on $1 \leq u \leq u_h$

$$\psi_s(u) = h^1(u)\Phi_s(x^1(u) + R_1 h_1^1(u) + R_2 h_2^1(u) + \sigma_o h_3^1(u) + \tau_o h_4^1(u) \ , \quad (8.26)$$

and on $u<-1$, $u>u_h$

$$\psi_s(u) = h^1(u)\Phi_s(x^1(u)) + h^2(u)\Phi_s(x^2(u)) + R_1[h_1^1(u) + h_1^2(u)]$$
$$+ R_2[h_2^1(u) + h_2^2(u)] + \sigma_o[h_3^1(u) + h_3^2(u)]$$
$$+ \tau_o[h_4^1(u) + h_4^2(u)] \ , \quad (8.27)$$

with

$$h^n(u) = -\mathrm{sgn}(u) \frac{k^2((q^n(u))^2-2)\sqrt{(q^n(u))^2-1}}{\sqrt{((q^n(u))^2-2)^4 + 16(1-(p^n(u))^2)(1-(q^n(u))^2)}} \cdot$$
$$\sqrt{\frac{x^n(u)(1-x^n(u))}{u(u-1)}} \ ; \quad (8.28)$$

$$h_1^n(u) = \sqrt{\frac{x^n(u)-x_b}{x^n(u)}}\ \frac{h^n(u)}{x^n(u)-x_R} \quad , \tag{8.29}$$

$$h_2^1(u) = \frac{h^1(u)}{x^1(u)-x_R} - \frac{R_4/R_2}{u-u_R} \quad , \tag{8.30}$$

$$h_2^2(u) = \frac{h^2(u)}{x^2(u)-x_R} \quad , \tag{8.31}$$

$$h_3^1(u) = \frac{S_1 h^1(u)}{x^1(u)-x_o} - \frac{S_3}{u-u_o} \quad , \tag{8.32}$$

$$h_3^2(u) = \frac{S_1 h^2(u)}{x^2(u)-x_o} \quad , \tag{8.33}$$

$$h_4^n(u) = S_2\sqrt{\frac{x^n(u)-x_b}{x^n(u)}}\ \frac{h^n(u)}{x^n(u)-x_o} \quad , \tag{8.34}$$

where n = 1, 2 and

$$1/x^n(u) = \cosh\left(\frac{\pi}{\gamma}\cosh^{-1} 1/p^n(u)\right) \tag{8.35}$$

$$1/p^n(u) = k/q^n(u) = \begin{cases} k\cos\theta(u) \ ; \ n = 1 \ , \\ \\ k\cos(2\gamma - \theta(u)) \ ; \ n = 2 \ , \end{cases} \tag{8.36}$$

$$\theta(u) = \frac{\gamma}{\pi}\cos^{-1} 1/u \ .$$

It follows that $h^n(u)$, $h^n_m(u)$, $m = 1, 2, 3, 4$, $n = 1, 2$, are bounded (the singularity in $h^1(u)$ at $u = 1$ is removable because $q^1(u) = 1$) and $O(1/u)$ at infinity.

To understand the notation consider a point $w = u$ on the real axis with $|u| > 1$, whence u is an image of some point $\theta(u)$ on the characteristic envelope in the physical domain. Tangent to the envelope at $\theta(u)$ are $\beta_{\pm} = \theta(u)$ characteristics, illustrated in Figure 7 for $u = u_h$, $\infty$, $-1$ corresponding to $\theta(u) = \theta_h$, $\gamma/2$, $\gamma$. The $\beta_-$ characteristics pass through $p = p^1(u)$ on $\theta = 0$, with image $x^1(u)$ in the z plane; while $\beta_+$ are continued off $\theta = \gamma$ as $\beta_- = 2\gamma - \theta(u)$ and intersect $p = p^2(u)$, the image of $x^2(u)$. Superscripts 1 or 2 in (8.26, 27) et seq. therefore designate contributions "carried" on $\beta_-$ or $\beta_+$ head waves respectively. From Figure 7 it is clear that superscript 2 quantities are only defined for $u \geq u_h$, $u < -1$. Note that if $\gamma > \theta_h$, then the head waves do not overlap (as they do in Figure 4) and (8.27) is unnecessary. For smaller angles (with multiple head wave reflections) the contribution from reflected head waves can be accounted for as in (8.27) by defining higher order superscripted quantitites (e.g., $h^3(u)$, etc.) over appropriate segments of the u axis.

Examining (8.15, 26, 27), the $\psi_s(u)$ density function is determined completely in terms of integrals on $\phi_s(x)$, known smooth functions, and unknown constants, $R_1$ and $R_2$. By evaluating $\Psi^P_s(u(x))$ in (8.15) we obtain the sought after equation on the density function, $\phi_s(x)$, for satisfaction of the boundary conditions.

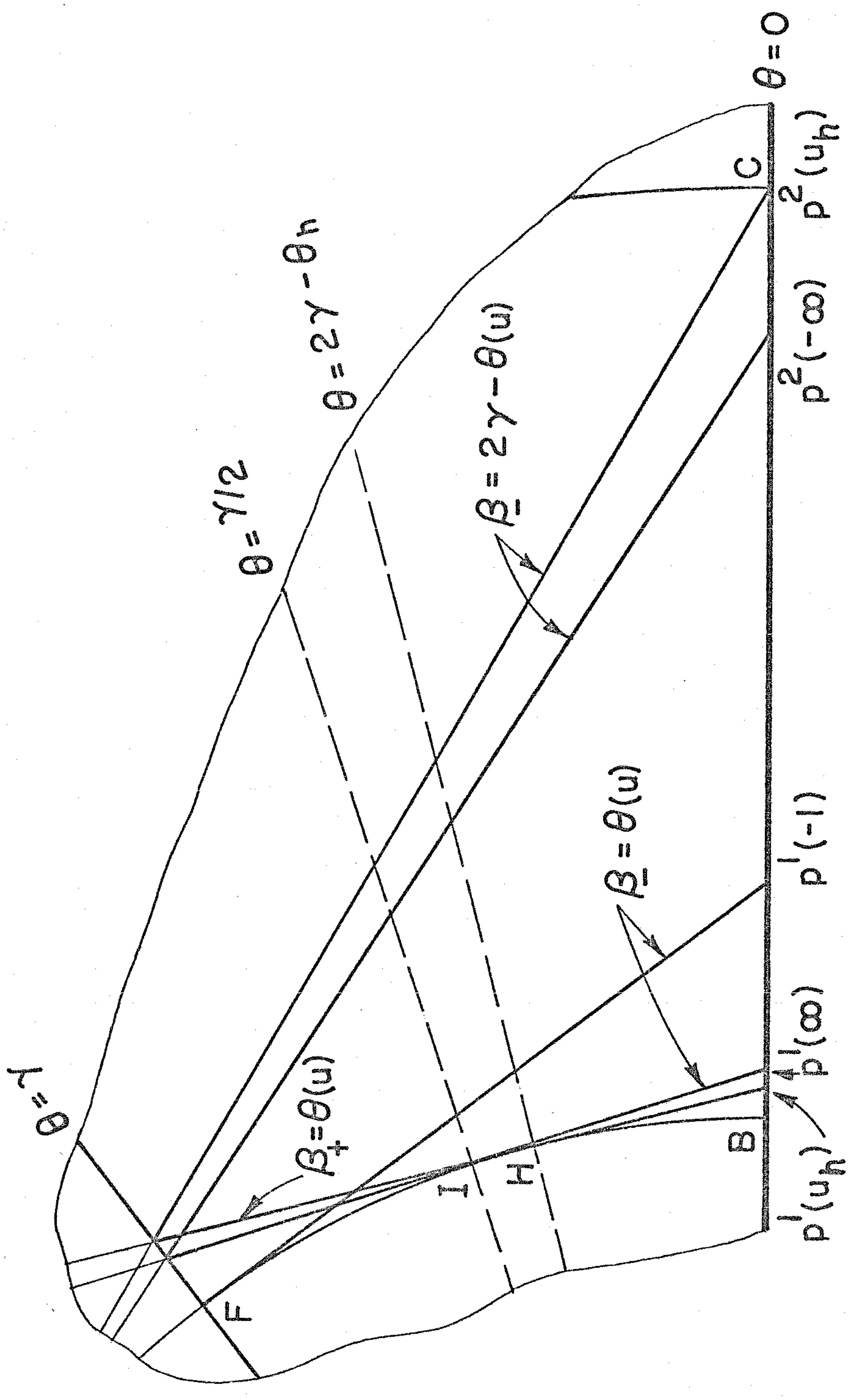


FIGURE 7

Consider $\Psi_s^P$ in (8.15) written as

$$\Psi_s^P(u) = \frac{1}{\pi}\int_{-\infty}^{-1} \sqrt{\frac{s}{1+s}}\,\frac{\psi_s(s)}{s-u}\,ds + \frac{1}{\pi}\int_{0}^{1} \sqrt{\frac{s}{1+s}}\,\frac{\psi_s(s)}{s-u}\,ds$$

$$+ \frac{1}{\pi}\int_{1}^{u_h} \sqrt{\frac{s}{1+s}}\,\frac{\psi_s(s)}{s-u}\,ds + \frac{1}{\pi}\int_{u_h}^{\infty} \sqrt{\frac{s}{1+s}}\,\frac{\psi_s(s)}{s-u}\,ds\ , \qquad (8.37)$$

where $0 \leqslant u \leqslant 1$ so the second integral is a principal value. The aim is to evaluate each of these using (8.15, 26, 27). First note that because $\psi_s(1)$ in (8.15, 26) does not necessarily vanish, the second and third integrals have logarithmic singularities in general as $u \to 1^-$, which cancel in the sum. Such singularities could easily be removed by subtraction, for example

$$\int_{1}^{u_h} \sqrt{\frac{s}{1+s}}\,\frac{\psi_s(s)}{s-u} = \int_{1}^{u_h} \sqrt{\frac{s}{1+s}}\,\frac{\psi_s(s)-\psi_s(1)}{s-u}\,ds$$

$$+ \psi_s(1)\int_{1}^{u_h} \sqrt{\frac{s}{1+s}}\,\frac{ds}{s-u}\ ,$$

but for the sake of simplicity they are carried through.

Defining the two integration paths,

$L^1$ : s from $-\infty$ to $-1$, 1 to $\infty$, (8.38)

$L^2$ : s from $-\infty$ to $-1$, $u_h$ to $\infty$, (8.39) consistent with the previous superscript notation, and substituting (8.26, 27) into the integrals in (8.37) over $L^1$ yields

$$\frac{1}{\pi}\int_{L^1}\sqrt{\frac{s}{1+s}}\,\frac{\psi_s(s)}{s-u}\,ds = \frac{1}{\pi}\int_{L^1}\sqrt{\frac{s}{1+s}}\,\frac{h^1(s)\,\Phi_s(x^1(s))}{s-u}\,ds$$

$$+\frac{1}{\pi}\int_{L^2}\sqrt{\frac{s}{1+s}}\,\frac{h^2(s)\,\Phi_s(x^2(s))}{s-u}\,ds \tag{8.40}$$

$$+ R_1[H_1^1(u) + H_1^2(u)] + R_2[H_2^1(u) + H_2^2(u)]$$

$$+ \sigma_o[H_3^1(u) + H_3^2(u)] + \tau_o[H_4^1(u) + H_4^2(u)]$$

where

$$H_m^n(u) = \frac{1}{\pi}\int_{L^n}\sqrt{\frac{s}{1+s}}\,\frac{h_m^n(s)}{s-u}\,ds \ , \tag{8.41}$$

with $m = 1$-$4$, $n = 1, 2$.

It is convenient to define $H_m^n(u)$ by integrals over finite paths on $x^1(s)$ and $x^2(s)$ through the change of variables, (8.35) et seq. In terms of $x^1$ and $x^2$, $L^1$ and $L^2$ are equivalent to

$$L^1 : x^1(s) \text{ from } x_b \text{ to } x_f \ , \tag{8.42}$$

$$L^2 : x^2(s) \text{ from } 1 \text{ to } x_f \ , \tag{8.43}$$

where $x_f$ is the image of $p^1(-1)$ in Figure 7, i.e.,

$$x_f = \cosh[\frac{\pi}{\gamma} \cosh^{-1}(k \cos\gamma)] \quad . \tag{8.44}$$

Changing the variable of integration gives

$$H_m^n(x) \equiv H_m^n(u(x)) = \frac{1}{\pi} \int_{L^n} \sqrt{\frac{u(s')}{1+u(s')}} \frac{M(x,s')\, h_m^n(u(s'))}{s'-x} \, ds' \quad , \tag{8.45}$$

where

$$M(x,s') = \frac{s'-x}{u(s')-u(x)} \frac{du(s')}{ds'} \quad , \tag{8.46}$$

with $s' = x^n(s)$, $u(s') = s$. Note, as $s' \to x_f^-$, $u(s') \to -1^-$ and the integrand is unbounded but integrable. Also, as $x \to x_b$, $H_m^1(x)$ is logarithmically unbounded. The expression for $u(s')$ follows from (8.35) et seq. as

$$1/u(s') = \cos \left\{ \frac{\pi}{\gamma} \cos^{-1}[ \frac{1}{k} \cosh(\frac{\gamma}{\pi} \cosh^{-1} 1/s')] \right\} \quad , \tag{8.47}$$

hence

$$\frac{du(s')}{ds'} = \frac{u(s')\sqrt{u^2(s')-1}}{ks'\sqrt{1-(s')^2}} \frac{\sinh(\frac{\gamma}{\pi} \cosh^{-1} 1/s')}{\sin(\frac{\gamma}{\pi} \cos^{-1} 1/u(s'))} \quad . \tag{8.48}$$

Applying the same change of integration variable to the integrals in (8.40) and introducing $\Phi_s(s')$ from (8.1), we find

$$\frac{1}{\pi}\int_{L^n}\sqrt{\frac{s}{1+s}}\,\frac{h^n(s)\,\Phi_s(x^n(s))}{s-u}\,ds = \frac{1}{\pi^2}\int_{L^n}ds'\int_0^{x_b}\sqrt{\frac{u(s')}{1+u(s')}}\sqrt{\frac{x_b-s''}{s''}}\,\cdot$$

$$\frac{M(x,s')h^n(u(s'))\phi_s(s'')}{(s'-x)(s''-s)}\,ds''. \quad (8.49)$$

Interchanging the order of integration in (8.49) allows (8.40) to be written as

$$\frac{1}{\pi}\int_{L^1}\sqrt{\frac{s}{1+s}}\,\frac{\psi_s(s)}{s-u}\,ds = \int_0^{x_b}\left[\frac{H^1(x,s'')+H^2(x\ s'')}{\sqrt{s''}}\right]\phi_s(s'')\,ds''$$

$$+ R_1[H_1^1(x)+H_1^2(x)] + R_2[H_2^1(x)+H_2^2(x)]$$

$$+ \sigma_o[H_3^1(x)+H_3^2(x)] + \tau_o[H_4^1(x)+H_4^2(x)]\ , \quad (8.50)$$

where

$$H^n(x_1s'') = \frac{\sqrt{x_b-s''}}{\pi^2}\int_{L^n}\sqrt{\frac{u(s')}{1+u(s')}}\,\frac{M(x,s')\,h^n(u(s'))}{(s'-x)(s''-s')}\,ds'\ . \quad (8.51)$$

A sufficient condition for validity of the above interchange is that the integrand of (8.49) be integrable over the s', s'' rectangle. Observing that

$$0 \leqslant x,\ s' \leqslant x_b \text{ and } \begin{cases} x_b \leqslant s'' \leqslant x_f\ ;\ n = 1 \\ \\ x_f \leqslant s'' \leqslant 1\ ;\ n = 2 \end{cases}\ ,$$

for n = 2 it is certainly integrable, and likewise for n = 1 provided $x < x_b$. When $x = x_b$ (8.49) is logarithmically unbounded and the question of interchange is meaningless.

To complete the evaluation of $\Psi_s^P(u)$ we substitute $\psi_s(u)$ from (8.15) into the principal value integral in (8.37) giving

$$\frac{1}{\pi}\int_0^1 \sqrt{\frac{s}{1+s}}\,\frac{\psi_s(s)}{s-u(x)}\,ds = \frac{1}{\pi}\int_0^1 \sqrt{\frac{s}{1+s}}\,\frac{g(x(s))\,\Phi_s^P(x(s))}{s-u(x)}\,ds$$

$$+ R_2 G_1(x) + \sigma_o G_2(x) \ , \qquad (8.52)$$

where

$$G_\ell(x) = \frac{1}{\pi}\int_0^1 \sqrt{\frac{s}{1+s}}\,\frac{g_\ell(x(s))}{s-u(x)}\,ds \ , \qquad (8.53)$$

and $\ell = 1, 2$. Replacing $\Phi_s^P(x(s))$ by its singular integral representation and making a change of integration variables, the repeated singular integral in (8.52) becomes

$$\frac{1}{\pi}\int_o^1 \sqrt{\frac{s}{1+s}}\,\frac{g(x(s))\,\Phi_s^P(x(s))}{s-u(x)}\,ds = \frac{1}{\pi^2}\int_0^{x_b} ds' \int_0^{x_b} \sqrt{\frac{u(s')}{1+u(s')}}\sqrt{\frac{x_b-s}{s''}} \ .$$

$$\frac{M(x,s')\,g(s')\,\phi_s(s'')}{(s'-x)(s''-s')}\,ds'' \ , \qquad (8.54)$$

where $M(x_1 s')$ is given by (8.46) and $u(s')$ by (8.16) with

$$\frac{du(s')}{ds'} = \frac{u(s')\sqrt{1-u^2(s')}\,\sqrt{1-(p(s'))^2}}{s'\sqrt{1-s')^2}\sqrt{1-(q(s'))^2}} \ . \qquad (8.55)$$

Applying the Poincaré-Bertrand transformation formula for repeated singular integrals (e.g., Muskhelishvili [22]) to (8.54), the order of integration can be switched yielding

$$\frac{1}{\pi}\int_0^1 \sqrt{\frac{s}{1+s}}\;\frac{g(x(s))\,\Phi_s^P(x(s))}{s-u(x)}\,ds = -\sqrt{\frac{u(x)(x_b-x)}{x(1+u(x))}}\;M(x,x)\,g(x)\,\phi_s(x) + \int_0^{x_b} G(x,s'')\,\frac{\phi_s(s'')}{\sqrt{s''}}\,ds'' , \quad (8.56)$$

where

$$G(x,s'') = \frac{\sqrt{x_b-s''}}{\pi^2}\int_0^{x_b}\sqrt{\frac{u(s')}{1+u(s')}}\;\frac{M(x,s')\,g(s')}{(s'-x)(s''-s')}\,ds' , \quad (8.57)$$

and we note that $M(x,x) = 1$. Writing

$$\frac{1}{(s'-x)(s''-s')} = \frac{1}{s''-x}\left(\frac{1}{s'-x} - \frac{1}{s'-s''}\right) ,$$

$G(x,s'')$ becomes

$$G(x,s'') = \frac{1}{\pi^2}\,\frac{\sqrt{x_b-s''}}{s''-x}\left[\int_0^1 \sqrt{\frac{s}{1+s}}\;\frac{g(x(s))}{s-u(x)}\,ds - \int_0^{x_b}\sqrt{\frac{u(s')}{1+u(s')}}\;\frac{M(x,s')\,g(s')}{s'-s''}\,ds'\right] , \quad (8.58)$$

where it is natural to evaluate the first integral as shown rather than over s'. In reference to the applicability of the Poincaré-Bertrand formula to (8.54) with a square root infinity at an end-point, Muskhelishvili's proof of the formula can easily be shown to apply.

Finally, substituting (8.56) into (8.52), and (8.50, 52) into (8.37), we obtain an integral equation of the second kind on $\phi_s(x)$ over $0 \leq x < x_b$ :

$$\frac{R(p(x))}{(q^2(x)-2)^2}\,\phi_s(x) = \int_0^{x_b} \frac{J(x,s)}{\sqrt{s}}\,\phi_s(s)\,ds + R_1[f(x)(H_1^1(x)+H_1^2(x))+f_1(x)]$$

$$+ R_2 f(x)[H_2^1(x)+H_2^2(x)+G_1(x)]$$

$$+ \sigma_o f(x)[H_3^1(x)+H_3^2(x)+G_2(x)]$$

$$+ \tau_o[f(x)(H_4^1(x)+H_4^2(x))+f_2(x)] \quad ,$$

$$\equiv I(x) + R_1F_1(x) + R_2F_2(x)$$

$$+ \sigma_o F_3(x) + \tau_o F_4(x) \quad , \tag{8.59}$$

where

$$J(x,s) = f(x)[H^1(x,s)+H^2(x,s)+G(x,s)] \quad , \tag{8.60}$$

and the definitions of $I(x)$ and $F_i(x)$, $i = 1\text{-}4$, are obvious. The right hand side of (8.59) has a removable logarithmic singularity at $x = x_b$, otherwise the terms are continuous. The only non-removable singularity of the kernel is the algebraic infinity due to $1/\sqrt{s}$ .

There remain two unknown constants, $R_1$ and $R_2$, in (8.51) which are related to the residue at the Rayleigh pole as in (7.27, 28). These are evaluated quite naturally by means of the Rayleigh function multiplying $\phi_s(x)$ on the left hand side. From (5.5) et seq., $R(p(x))$ vanishes at $x = 0, x_R$ but $\phi_s(x)$ is bounded at these points, thus $x = 0, x_R$ are irregular points of the integral equation. However these irregularities are removable by the proper choice of $R_1$ and $R_2$. Setting $x = 0, x_R$ yields two equations for their determination, namely

$$F_1(0)R_1 + F_2(0)R_2 = -[I(0) + \sigma_o F_3(0) + \tau_o F_4(0)] \quad , \tag{8.61}$$

$$F_1(x_R)R_1 + F_2(x_R)R_2 = -[I(x_R) + \sigma_o F_3(x_R) + \tau_o F_4(x_R)] \quad . \tag{8.62}$$

Provided only that

$$D \equiv F_1(0)F_2(x_R) - F_1(x_R)F_2(0) \neq 0 \quad , \tag{8.63}$$

then $R_1$ and $R_2$ are

$$-R_1 = \int_0^{x_b} \frac{J_1(s)}{\sqrt{s}} \phi_s(s)\,ds + \sigma_o D_{13} + \tau_o D_{14} \quad , \tag{8.64}$$

$$R_2 = \int_0^{x_b} \frac{J_2(s)}{\sqrt{s}} \phi_s(s)\,ds + \sigma_o D_{23} + \tau_o D_{24} \quad , \tag{8.65}$$

where

$$J_i \equiv \frac{1}{D}\left[F_i(0)\, J(x_R, s) - F_i(x_R)\, J(0, s)\right] \quad , \tag{8.66}$$

$$D_{ij} \equiv \frac{1}{D}\left[F_i(0)\, F_j(x_R) - F_i(x_R)\, F_j(0)\right] \quad , \tag{8.67}$$

with $i = 1, 2$, $j = 3, 4$. Replacing $R_1$ and $R_2$ in (8.59) by (8.64, 65) gives

$$\frac{R(p(x))}{(q^2(x)-2)^2}\, \phi_s(x) = \int_0^{x_b} \frac{K(x, s)}{\sqrt{s}}\, \phi_s(s)\, ds + \sigma_o \Sigma(x) + \tau_o T(x) \quad , \tag{8.18}$$

where

$$K(x, s) = J(x, s) - F_1(x)\, J_1(s) + F_2(x)\, J_2(s) \quad , \tag{8.69}$$

$$\Sigma(x) = -D_{13} F_1(x) + D_{23} F_2(x) + F_3(x) \quad , \tag{8.70}$$

$$T(x) = -D_{14} F_1(x) + D_{24} F_2(x) + F_4(x) \quad . \tag{8.71}$$

## ASYMMETRIC LOADING CASE

Derivation of the integral equation on $\phi_a(x)$ is analogous to the above case, hence only the essential steps are described below.

For this case the real and imaginary parts of $\Theta'(\alpha(x))$ and $\Omega'(\beta(u))$ on $0 \leqslant x \leqslant x_b$, $0 \leqslant u \leqslant 1$ are

$$\mathrm{Re}\,\Theta'(\alpha(x)) = e^{-\Gamma(x)}\sqrt{1-x^2}\,\Big[\Phi_a^P(x) + \frac{R_2}{x-x_R} + \frac{\sigma_o S_1}{x-x_o}$$

$$+ R_1\sqrt{x_R/(x_b-x_R)}\;\pi\delta(x-x_R)$$

$$+ \tau_o S_2\sqrt{x_o/(x_b-x_o)}\;\pi\delta(x-x_o)\Big] , \tag{8.72}$$

$$\mathrm{Im}\,\Theta'(\alpha(x)) = e^{-\Gamma(x)}\sqrt{\frac{x(1-x^2)}{x_b-x}}\,\Big[\phi_a(x) + \frac{R_1}{x-x_R} + \frac{\tau_o S_2}{x-x_o}$$

$$- R_2\sqrt{(x_b-x_R)/x_R}\;\pi\,\delta(x-x_R)$$

$$- \sigma_o S_1\sqrt{(x_b-x_o)/x_o}\;\pi\delta(x-x_o)\Big] , \tag{8.73}$$

$$\mathrm{Re}\,\Omega'(\beta(u)) = \sqrt{u(1-u)}\,\Big[\Psi_a^P(u) + \frac{R_3}{u-u_R} + \frac{\tau_o S_4}{u-u_o}$$

$$+ R_4\sqrt{(1+u_R)/u_R}\;\pi\delta(u-u_R)$$

$$+ \sigma_o S_3\sqrt{(1+u_o)/u_o}\;\pi\delta(u-u_o)\Big] , \tag{8.74}$$

$$\mathrm{Im}\,\Omega'(\beta(u)) = \sqrt{1-u^2}\,\Big[\psi_a(u) + \frac{R_4}{u-u_R} + \frac{\sigma_o S_3}{u-u_o}$$

$$- R_3\sqrt{u_R/(1+u_R)}\;\pi\delta(u-u_R)$$

$$- \tau_o S_4\sqrt{u_o/(1+u_o)}\;\pi\delta(u-u_o)\Big] . \tag{8.75}$$

Applying the reduced boundary conditions to these yields the singular integral equations,

$$\phi_a(x) = f(x)\,\Psi_a^P(u(x)) + R_1 f_1(x) + \tau_o f_2(x) , \tag{8.76}$$

$$\psi_a(u(x)) = g(x)\,\Phi_a^P(x) + R_2 g_1(x) + \sigma_o g_2(x) \quad , \tag{8.77}$$

where $u(x)$ and $f_i$, $g_i$, $i = 1,2$ are given by (8.16, 19-22), but $f(x)$ and $g(x)$ become

$$f(x) = \frac{-4\sqrt{1-p^2(x)}}{k^2(q^2(x)-2)} \sqrt{\frac{u(x)(1-u(x))(x_b-x)}{x(1-x^2)}}\; e^{\Gamma(x)} \quad , \tag{8.78}$$

$$= \frac{-4\sqrt{1-p^2(x)}\,\sqrt{1-q^2(x)}}{(q^2(x)-2)^2} \sqrt{\frac{u(x)(x_b-x)}{x(1+u(x))}}\; \frac{1}{g(x)} \quad . \tag{8.79}$$

Note that $f(x)$ has a simple zero at $x_b$ (where $u(x_b) = 1$).

On $u < -1$, $u > 1$ the real and imaginary parts of $\Omega'(\beta(u))$ are

$$\mathrm{Re}\,\Omega'(\beta(u)) = \mathrm{sgn}\,(u)\sqrt{u^2-1}\left[\psi_a(u) + \frac{R_4}{u-u_R} + \frac{\sigma_o S_3}{u-u_o}\right] , \tag{8.80}$$

$$\mathrm{Im}\,\Omega'(\beta(u)) = -\mathrm{sgn}\,(u)\sqrt{u(u-1)}\left[\Psi_a^P(u) + \frac{R_3}{u-u_R} + \frac{\tau_o S_4}{u-u_o}\right] . \tag{8.81}$$

Applying (8.80, 81) to the boundary condition on the characteristic envelope, (7.12), $\psi_a(u)$ on $1 < u \leq u_h$ is written as

$$\psi_a(u) = h^1(u)\,\Phi_a(x^1(u)) + R_1 h_1^1(u) + R_2 h_2^1(u) + \sigma_o h_3^1(u)$$
$$+ \tau_o h_4^1(u) \quad , \tag{8.82}$$

and on $u < -1$, $u > u_h$,

$$\psi_a(u) = h^1(u)\,\Phi_a(x^1(u)) + h^2(u)\,\Phi_a(x^2(u)) + R_1[h_1^1(u) + h_1^2(u)]$$

$$+ R_2[h_2^1(u) + h_2^2(u)] + \sigma_o[h_3^1(u) + h_3^2(u)]$$

$$+ \tau_o[h_4^1(u) + h_4^2(u)] \quad , \tag{8.83}$$

with $h_2^n$ and $h_3^n$, $n = 1, 2$, given by (8.30-33) and

$$h^n(u) = -\mathrm{sgn}(u)\frac{k^2((q^n(u))^2-2)\sqrt{(q^n(u))^2-1}}{\sqrt{(q^n(u))^2-2)^4 + 16(1-p^n(u))^2(1-(q^n(u))^2)}}$$

$$\sqrt{\frac{1-(x^n(u))^2}{u^2-1}} \quad ; \tag{8.84}$$

$$h_1^n(u) = \sqrt{\frac{x^n(u)}{x^n(u)-x_b}}\;\frac{h^n(u)}{x^n(u)-x_R} \quad , \tag{8.85}$$

$$h_4^n(u) = S_2\sqrt{\frac{x^n(u)}{x^n(u)-x_b}}\;\frac{h^n(u)}{x^n(u)-x_o} \quad , \tag{8.86}$$

where $x^n(u)$ is as previously defined in (8.35, 36) et seq. Note that $h_1^1(u)$, $h_4^1(u)$, and $\Phi_a(x^1(u))$ are all unbounded like $1/\sqrt{x^1(u)-x_b}$ as $u \to 1^+$ (where $x^1(1) = x_b$) however the singularity is removable in (8.82).

Substituting $\psi_a(u)$ from (8.77, 82, 83) into $\Psi_a^P(u)$, with the integration path divided as in (8.37), yields

$$\frac{1}{\pi}\int_{L^1}\sqrt{\frac{1+s}{s}}\;\frac{\psi_a(s)}{s-u}\,ds = \int_0^{x_b}\frac{[H^1(x,s'')+H^2(x,s'')]}{\sqrt{x_b-s''}}\,\phi_a(s'')\,ds''$$

$$+\;R_1[H_1^1(x)+H_1^2(x)] \;+\; R_2[H_2^1(x)+H_2^2(x)]$$

$$+\;\sigma_o[H_3^1(x)+H_3^2(x)] \;+\; \tau_o[H_4^1(x)+H_4^2(x)]\;, \qquad (8.87)$$

$$\frac{1}{\pi}\int_0^1\sqrt{\frac{1+s}{s}}\;\frac{\psi_a(s)}{s-u}\,ds = -\sqrt{\frac{x(1+u(x))}{u(x)(x_b-x)}}\;g(x)\,\phi_a(x)$$

$$+\int_0^{x_b}\frac{G(x,s'')}{\sqrt{x_b-s''}}\,\phi_a(s'')\,ds''$$

$$+\;R_2G_1(x) \;+\; \sigma_oG_2(x)\;, \qquad (8.88)$$

where

$$H^n(x,s'') = \frac{\sqrt{s''}}{\pi^2}\int_{L^n}\sqrt{\frac{1+u(s')}{u(s')}}\;\frac{M(x,s')\,h^n(u(s'))}{(s'-x)(s''-s')}\,ds'\;, \qquad (8.89)$$

$$H_m^n(x) = \frac{1}{\pi}\int_{L^n}\sqrt{\frac{1+u(s')}{u(s')}}\;\frac{M(x,s')\,h_m^n(u(s'))}{s'-x}\,ds'\;, \qquad (8.90)$$

$$G(x, s'') = \frac{1}{\pi^2} \frac{\sqrt{s''}}{s''-x} \left[ \int_0^1 \sqrt{\frac{1+s}{s}} \frac{g(x(s))}{s-u(x)} ds - \int_0^{x_b} \sqrt{\frac{1+u(s')}{u(s')}} \cdot \frac{M(x, s') g(s')}{s'-s''} ds' \right] , \tag{8.91}$$

$$G_\ell(x) = \frac{1}{\pi} \int_0^1 \sqrt{\frac{1+s}{s}} \frac{g_\ell(x(s))}{s-u(x)} ds , \tag{8.92}$$

and the other terms are as previously defined. The integration interchanges leading to (8.87) and (8.88) are valid provided $x \neq x_b$. The difficulty at $x_b$ in this case is the removable algebraic singularity (in addition to the removable logarithmic singularity as in the symmetric case).

Substituting $\Psi_a^P(u)$ from (8.87, 88) into (8.76) yields a second kind integral equation on $\phi_a(x)$ over $0 \leq x < x_b$:

$$\frac{R(p(x))}{(q^2(x)-x)^2} \phi_a(x) = \int_0^{x_b} \frac{J(x, s)}{\sqrt{x_b - s}} \phi_a(s) ds$$

$$+ R_1[f(x)(H_1^1(x) + H_1^2(x)) + f_1(x)]$$

$$+ R_2 f(x)[H_2^1(x) + H_2^2(x) + G_1(x)]$$

$$+ \sigma_o f(x)[H_3^1(x) + H_3^2(x) + G_2(x)]$$

$$+ \tau_o [ f(x) (H_4^1(x) + H_4^2(x)) + f_2(x) ] \quad ,$$

$$\equiv I(x) + R_1 F_1(x) + R_2 F_2(x)$$

$$+ \sigma_o F_3(x) + \tau_o F_4(x) \quad , \tag{8.92}$$

where J(x, s) is given by (8.60). Because f(x) has a simple zero at $x_b$, multiplication of $\Psi_a^P(u(x))$ in (8.76) by f(x) has effectively cancelled the removable singularities at $x_b$. The only remaining singularity is the integrable infinity in the kernel due to $1/\sqrt{x_b - s}$ .

The unknown constants, $R_1$ and $R_2$, are determined as in the previous case (cf., (8.62-68)) to be

$$-R_1 = \int_0^{x_b} \frac{J_1(s)}{\sqrt{x_b - s}} \phi_a(s)\, ds + \sigma_o D_{13} + \tau_o D_{14} \quad , \tag{8.94}$$

$$R_2 = \int_0^{x_b} \frac{J_2(s)}{\sqrt{x_b - s}} \phi_a(s)\, ds + \sigma_o D_{23} + \tau_o D_{24} \quad . \tag{8.95}$$

Substituting then back into (8.93) yields the final form of the integral equation,

$$\frac{R(p(x))}{(q^2(x) - 2)^2} \phi_a(x) = \int_0^{x_b} \frac{K(x, s)}{\sqrt{x_b - s}} \phi_a(s)\, ds + \sigma_o \Sigma(x) + \tau_o T(x) \quad , \tag{8.96}$$

where K(x, s), Σ(x), and T(x) are as previously defined by (8.69-71).

## § 9. DISCUSSION AND GENERALIZATION OF THE SOLUTIONS

Despite the removable irregularities and weak endpoint singularities in the kernels, the integral equations for the density functions are of the Fredholm type. Consequently the Fredholm Alternative (e.g., Mikhlin [23]) applies and either unique solutions exist or the homogeneous equations obtained by setting $\sigma_o = \tau_o = 0$, i.e., zero surface traction, have nontrivial solutions. Applying the Fredholm theorems the latter alternative implies that there are no solutions, otherwise they would exist for particular values of $\sigma_o$ and $\tau_o$ violating the linearity of the problem.

We use the above results to examine the question of uniqueness. Recall from our discussion of the canonical forms in § 5, particularly for $\Theta'(\alpha)$ on $0 \leq q \leq 1$, that besides the explicit singular and complex behavior of $\Theta'$ in (5.3) integrable infinities (i.e., algebraic branch points) are admissible. Although the density functions, $\phi_{s,a}$, are assumed Hölder continuous in § 8 they can in fact be singular (but integrable) in such a way that an integrable infinity of $\Theta'(\alpha)$ (relegated to the residual factors in the factorizations) can be represented as in (8.1, 2). Such cases are discussed by Muskhelishvili [ 22 ] in the context of the Hilbert problem with discontinuous or singular coefficients. It follows that the Cauchy integrals in (8.12) are sufficient to represent the residual factors, moreover because the factorizations themselves are a result of necessary conditions, (7.29, 30) and (8.1, 2) are both necessary and sufficient to represent $\Theta'(\alpha(z))$. Therefore in order for solutions to exist a necessary and sufficient condition is that the integral equations have solutions. Hence if the elasticity problem possesses solutions they follow uniquely from the integral equations by virtue of the Fredholm Alternative. This argument is by no means rigorous (e.g., we have ignored the possibility of essential singularities), however because the problem gives no indication of contrary behavior there is little doubt that the integral equations do indeed possess solutions.

Solving the integral equations, for example by reducing them to a system of linear algebraic equations using approximate quadrature formula compatible with the endpoint singularities of the kernels (e.g, Kantorovich and Krylov [24] and Krylov [25]), then the analytic functions, $\Theta'(\alpha)$ and $\Omega'(\beta)$, are known and the head waves follow from (4.7, 8). Substituting these into (3.10-12) and the result into (2.22-26) the p derivatives of the field quantities are determined. Convenient forms for evaluating these derivatives are

$$\vartheta_p = \frac{\operatorname{Im}\Theta'(\alpha)}{p\sqrt{1-p^2}} \tag{9.1}$$

$$\omega_p = \begin{cases} \dfrac{1}{p\sqrt{q^2-1}} \left[ \Omega'_+(\beta_+) - \Omega'_-(\beta_-) \right] \quad ; \quad q > 1 \,, & (9.2) \\ \\ \dfrac{\operatorname{Im} \Omega'(\beta)}{p\sqrt{1-q^2}} \quad ; \quad q < 1 \,, & (9.3) \end{cases}$$

$$\frac{\partial \dot{u}_r}{\partial p} = \frac{-c_d}{p} \vartheta_p + \frac{2c_d}{q^2} \cdot \begin{cases} \left[ \Omega'_+(\beta_+) + \Omega'_-(\beta_-) \right] \; ; \; q > 1 \,, & (9.4) \\ \operatorname{Re} \Omega'(\beta) \; ; \; q < 1 \,, & (9.5) \end{cases}$$

$$\frac{\partial \dot{u}_\theta}{\partial p} = \frac{-c_d}{p^2} \operatorname{Re} \Theta'(\alpha) - \frac{2c_d}{q} \omega_q \,, \qquad (9.6)$$

$$\frac{\partial \sigma_{\theta\theta}}{\partial p} = \frac{\mu}{p} \left[ \frac{2}{c_r} \frac{\partial \dot{u}_r}{\partial q} + k^2 q \vartheta_q \right] \,, \qquad (9.7)$$

$$\frac{\partial \sigma_{r\theta}}{\partial p} = \frac{-2\mu}{p} \left[ \frac{1}{c_r} \frac{\partial \dot{u}_\theta}{\partial q} + q\omega_q \right] . \qquad (9.8)$$

Therefore to recover the response at $\theta$ and $p = r/c_r t$ we integrate the above on the $\theta$ ray from $p = 1^+$ in the common hyperbolic region (where the solutions vanish) to p, e.g.,

$$\dot{u}_r(r, \theta, t) = \int_1^{r/c_d t} \frac{\partial \dot{u}_r}{\partial s} \, ds \; .$$

Such integrals must in general be evaluated numerically.

A simple example is the dilatational wavefront behavior at $p = 1^-$ (cf. (4.12)). From the analyticity of $\Theta'(\alpha(z))$ on $x<1$, $x>1$ $\Theta'(\alpha)$ is analytic on $p = 1$ and can be expanded in a Taylor series. Setting $\Omega'_\pm(\beta_\pm) = 0$ in (9.2, 4), retaining only the first term in the $\Theta'(\alpha)$ expansion (i.e., $\Theta'(\theta)$ which is purely imaginary), and integrating,

$$\left.\begin{array}{l} \vartheta(r,\theta,t) \\ \\ -\dfrac{1}{c_r}\dot{u}_r(r,\theta,t) \\ \\ \dfrac{1}{\mu(k^2-1)}\sigma_{\theta\theta}(r,\theta,t) \end{array}\right\} \sim -\sqrt{2}\ \mathrm{Im}\ \Theta'(\theta)\sqrt{1-r/(c_d t)} \tag{9.9}$$

with $\dot{u}_\theta$ and $\sigma_{r\theta}$ $o(1-r/c_d t)$.

On the $\theta = 0$ surface, setting $\Omega'_+(\beta_+) = 0$ and evaluating $\dfrac{\partial\sigma_{\theta\theta}}{\partial p}$ and $\dfrac{\partial\sigma_{r\theta}}{\partial p}$, the surface velocities become

$$\frac{\partial\dot{u}_r}{\partial p} = \frac{-c_d}{2}\left[\sigma_o q_o \delta(q-q_o) + \frac{k^2\,\mathrm{Im}\,\Theta'(\alpha)}{\sqrt{1-p^2}}\right], \tag{9.10}$$

$$\frac{\partial\dot{u}_\theta}{\partial p} = \begin{cases} \dfrac{c_d\Omega'_-(\beta_-)}{\sqrt{q^2-1}} \quad ; \quad q>1\ , & (9.11) \\ \\ \dfrac{c_d}{2}\left[\tau_o q_o \delta(q-q_o) - \dfrac{2\,\mathrm{Im}\,\Omega'(\beta)}{\sqrt{1-q^2}}\right] \quad ; \quad q<1\ , & (9.12) \end{cases}$$

and from (2.26) the unknown stress is

$$\sigma_{rr} = 2\mu(k^2-1)\vartheta - \mu\sigma_o S(q-q_o) \quad . \tag{9.13}$$

To examine surface behavior near the traction and Rayleigh poles we expand (9.1, 3, 10, 11) near the poles, replace Im $\Theta'(\alpha)$ and Im $\Omega'(\beta)$ by (8.11, 13, 73, 75), and integrate directly across the singularity. Integrals of the delta function, in (8.11) for example, give a simple jump while principal value integrals of the pole give a logarithmic infinity. This is of course equivalent to contour integration around the pole in the complex domain.

Applying the above at the traction discontinuity $\vartheta$, $\sigma_{rr}$, and $\dot{u}_r$ have simple jumps proportional to $\sigma_o$ and logarithmic infinites proportional to $\tau_o$, and similarly for $\omega$ and $u_\theta$ with the roles of $\sigma_o$ and $\tau_o$ switched. This behavior is analogous to the half-space result.

At the Rayleigh wave $\vartheta$, $\sigma_{rr}$, and $\dot{u}_r$ have jumps proportional to $R_2$ and infinities proportional to $R_1$, and vice versa for $\omega$ and $\dot{u}_\theta$. In contrast to results for the half-space, because $R_1$ and $R_2$ in (8.64, 65) depend on both $\sigma_o$ and $\tau_o$, the Rayleigh wave response has a jump and infinity for either component of applied surface traction. For example, in Craggs' [ 7 ] solution for applied normal traction on the half-space $\vartheta$, $\sigma_{rr}$, and $\dot{u}_r$ jump at the Rayleigh wave while $\omega$ and $\dot{u}_\theta$ are smooth.

The method of solution developed in this work is easily generalized to arbitrary wedge angle. For smaller angles, with $2\gamma < \theta_h$, reflected head waves must be included in the treatment of the composite region as mentioned in §4. In practice this requires no more than careful bookkeeping. For larger angles the details simplify somewhat because the head wave overlap region, which necessitates the superscript notation in (8.27, 83), shrinks and eventually vanishes when $\gamma \geqslant \theta_h$ (cf. (4.3)). The net effect of arbitrary angle is to shift the $u_h$ branch point in Figure 5b but with the method otherwise unchanged. When $2\gamma > \pi$ and the wedge is reentrant the density functions must exhibit the edge singularity given in (6.20, 22). In principle this does not cause any difficulties except when $2\gamma = 2\pi$, i.e., the slit, in which case the problem is solvable by more direct methods as discussed in the Appendix.

To generalize the method to arbitrary traction velocity we consider the locus of traction poles as V increases through $c_r$ and $c_d$. Referring to Figure 4, for $c_r \leqslant V \leqslant c_d$ the traction pole in $\Theta'(\alpha)$ still occurs on $\theta = 0$ but moves to $p_o = V/c_d$ in the composite region. The head waves essentially transfer the traction discontinuity on the $\beta_- = -\cos^{-1} 1/q_o$ characteristic (called the primary rotational wave) to the $q = 1$ cylindrical wavefront at the point of termination, $\theta_o = -\cos^{-1} 1/q_o$, where $\Omega'(\beta)$ then exhibits the traction pole. Therefore as V increases through $c_r$ the traction pole in $\Omega'(\beta)$ migrates from the $\theta = 0$ boundary along the cylindrical wavefront. For smaller wedge angles the primary wave may reflect off the $\theta = 2\gamma$ boundary and in fact undergo multiple reflections before terminating. Similarly, as V increases through $c_d$ the traction pole in $\Theta'(\alpha)$ migrates from the $\theta = 0$ boundary along the $p = 1$ cylindrical wavefront (e.g., Figure 2). In any event the complete system of primary waves in the common hyperbolic and composite regions, including residues at the wavefront traction poles, is readily found using the method of characteristics. In terms of the factorization for $\Theta'(\alpha(z))$ and $\Omega'(\beta(w))$ the poles are removed in the same manner with only their location and residue altered. Note that when the traction discontinuity and Rayleigh wave coincide, from (5. 3, 4, 11, 12) the singularities in $\Theta'(\alpha)$ and $\Omega'(\beta)$ behave like double poles in which case the method breaks down. This is attributable to deficiencies in linear elasticity theory rather than the self-similar formulation because transform solutions for the half-space exhibit the same pathology.

In conclusion we observe that although the general wedge solution presented here is considerably more involved than that for the degenerate case of a half-space, the qualitative wavefront and surface behavior is much the same. This is implicit in the canonical boundary value representations used to deduce the factorizations. The canonical forms are in fact the key to the method.

## APPENDIX: THE SEMI-INFINITE SLIT

For the sake of completeness the semi-infinite slit will be examined in some detail. A natural point of departure from the general analysis is the conformal mappings to half-planes in § 7. Substituting $\gamma = \pi$ in the boundary relations, (7.3, 4), between x and p and u and q gives

$$x = \begin{cases} p \; ; & x > 0 \\ -p \; ; & x < 0 \end{cases} \quad , \quad u = \begin{cases} q \; ; & u > 0 \\ -q \; ; & u < 0 \end{cases} \quad , \qquad \text{(A1)}$$

whence we obtain the simple relationship,

$$u = kx \quad . \qquad \text{(A2)}$$

The canonical boundary value representations become

$$\Theta'(\alpha(x)) = \frac{-x\sqrt{1-x^2}}{kR(x)} \Big\{ (k^2x^2-2)\,\Phi_1(x) + i\,2\sqrt{1-k^2x^2}\;\Psi_1(x)$$
$$+ \left[ i\,(kx_o^2-2)\sigma_o - 2\sqrt{1-k^2x_o^2}\;\tau_o \right] kx_o^2\,\delta(x-x_o) \Big\} \qquad \text{(A3)}$$

$$\Omega'(\beta(u)) = \frac{u\sqrt{1-u^2}}{2R(u/k)} \Big\{ -i\frac{2}{k}\sqrt{k^2-u^2}\;\;\Phi_1(u/k) + (u^2-2)\,\Psi_1(u/k)$$
$$+ \left[ \frac{2}{k}\sqrt{k^2-u_o^2}\;\sigma_o - i\,(u_o^2-2)\tau_o \right] u_o^2\;\delta(u-u_o) \Big\} \qquad \text{(A4)}$$

valid on $0 < x$, $u < 1$. Note that the Rayleigh function,

$$R(x) = (k^2x^2-2)^2 - 4\sqrt{1-x^2}\;\sqrt{1-k^2x^2} \quad , \qquad \text{(A5)}$$

is sectionally holomorphic with branch cuts over $1/k \leq |x| \leq 1$ on the real axis, zeros at $x = \pm x_R$, and $O(x^4)$ at infinity. It can be factored by applying the same technique used to determine the Rayleigh factor, $e^{\Gamma(x)}$ in (7.25), giving

$$R(x) = k^4 x^4 (x^2 - x_R^2)\, e^{\Gamma_+(x)}\, e^{\Gamma_-(x)} \tag{A6}$$

where $e^{\Gamma_+(x)} \equiv e^{\Gamma(x)}$ is analytic in the left half-plane with branch cut on $1/k \leq x \leq 1$, while $e^{\Gamma_-(x)}$ is analytic in the right half-plane with branch cut on $-1 \leq x \leq -1/k$. This is the factorization used by deHoop [9] in his solution of the slit by means of the Wiener-Hopf technique.

In order to solve the problem we attempt to find $\Phi_1(x)$ and $\Psi_1(x)$ in (A3, 4) which exhibit the traction poles, the proper order at infinity (so that $\Theta'(\alpha(x))$ and $\Omega'(\beta(u))$ are $O(1)$), and cancel terms in the Rayleigh factorization, (A6), incompatible with the behavior shown in Figure 6 (e.g., the Rayleigh pole and complex behavior on $x, u < 0$). Substituting (A6) into (A3, 4) two forms are found by inspection, namely

$$\Phi_1(x) = \begin{cases} \sigma_o S_1 e^{\Gamma_-(x)} \sqrt{\dfrac{x}{1+x}}\; \dfrac{x+x_R}{x-x_o} & ; \text{ sym. case,} \\ 0 ; & \text{asym. case,} \end{cases} \tag{A7}$$

$$\Psi_1(x) = \begin{cases} 0 ; & \text{sym. case,} \\ \tau_o S_2 e^{\Gamma_-(x)} \sqrt{\dfrac{x}{1+kx}}\; \dfrac{x+x_R}{x-x_o} & ; \text{ asym. case.} \end{cases} \tag{A8}$$

A solution for symmetric loading is then

$$\Theta'(\alpha(z)) = \frac{-\sigma_o S_1}{k^5} \frac{\sqrt{1-z}\,(k^2 z^2 - 2)\, e^{-\Gamma(z)}}{\sqrt{z}\,(z-x_o)(z-x_R)} , \tag{A9}$$

$$\Omega'(\beta(w)) = \frac{-\sigma_o S_1}{k} \frac{i\sqrt{k-w}\,\sqrt{1-w^2}\, e^{-\Gamma(w/k)}}{\sqrt{w}\,(w-u_o)(w-u_R)} , \tag{A10}$$

and for asymmetric loading,

$$\Theta'(\alpha(z)) = \frac{-\tau_o S_2}{k^5} \frac{i\sqrt{1-z^2}\,\sqrt{1-kz}\, e^{-\Gamma(z)}}{\sqrt{z}\,(z-x_o)(z-x_R)} , \tag{A11}$$

$$\Omega'(\beta(w)) = \frac{\tau_o S_2}{2\sqrt{k}} \frac{\sqrt{1-w}\,(w^2-2)\, e^{-\Gamma(w/k)}}{\sqrt{w}\,(w-u_o)(w-u_R)} , \tag{A12}$$

where, using the equations preceeding (7. 43) to evaluate the residues and substituting $T_1$ and $T_2$ from (5. 11, 12),

$$S_1 = \frac{-k^5}{\pi} x_o^{7/2} \sqrt{1+x_o} \frac{e^{\Gamma(x_o)}(x_o - x_R)}{R(x_o)} , \tag{A13}$$

$$S_2 = \frac{-2k^5}{\pi} x_o^{7/2} \sqrt{1+kx_o} \frac{e^{\Gamma(x_o)}(x_o - x_R)}{R(x_o)} . \tag{A14}$$

Clearly these solve the slit problem for the symmetric normal load and asymmetric shear load. Generalizing them to arbitrary load velocity they are in fact the solutions found by de Hoop [9] and Miles [10] for the two-dimensional problem of a slit diffracting an incoming plane wave. This follows by observing that traveling surface tractions as considered here can be applied to the slit faces to satisfy the boundary conditions thereby canceling reflections of the incident plane wave and reducing the diffraction problem to a simple superposition.

Before concluding we note that the symmetric shear load is amenable to a more general approach, similar to but much simpler than that used for arbitrary angle. Although the details will not be reproduced here it turns out that the resulting integral equations can be solved in closed form. The same approach applied to the asymmetric normal load yields the trivial solution only, however it can be shown that such a loading on the slit is actually a contact problem so there is no contradiction.

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