NEW METHODS OF SOLVING THE EQUATIONS FOR THE FLOW OF A COMPRESSIBLE FLUID

Thesis by

Francis Clauser

and

Milton Clauser

In Partial Fulfillment of the Requirement for the Degree

of Doctor of Philosophy

California Institute of Technology

Pasadena, California

1937

Summary.

The usual equations for the flow of a compressible fluid are non-linear in character and difficult to solve. It has been found that if the components of velocity are taken as independent variables, the resulting equations are linear. These equations are developed and new forms introduced.

A new function, Γ is introduced and a method advanced for effecting the transformation from the plane where the velocity components are coördinates, to the physical plane where x and y are coördinates.

A new way of finding plausible solutions to investigate is given and the case of flow in a corner is worked out in detail. The flow is found to have an anomalous behavior, the reason for which is explained. This solution is applied to the flow behind a curved shock wave.

We express more than the usual hollow appreciation when we sincerely thank Doctors Theodor von Kármán and Harry Bateman for laying the foundations of the authors' knowledge of compressible fluid flow and for their kind and sympathetic guidance throughout the development of this work.

Further, no small debt of gratitude is due to Miss Catharine McMillan for her part in preparing this thesis.

Francis Clauser
Milton Clauser

NEW METHODS OF SOLVING THE EQUATIONS FOR THE FLOW OF A COMPRESSIBLE FLUID

In the steady flow of an incompressible, inviscid fluid, if irrotational motion is assumed, the equations of motion reduce to the simple equation $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, where u and v are the components of velocity along the axes of x and y. This equation implies the existence of a velocity potential φ defined by the equations $u = \frac{\partial \varphi}{\partial x}$, $v = \frac{\partial \varphi}{\partial y}$. For the flow of a compressible fluid, the above reasoning is still valid if the pressure is a single valued function of the density alone.

For the case of incompressible flow, the equation of continuity, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, may be satisfied by a stream function, ψ defined by $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$. However, in the case of compressible flow, if ρ is the density, the equation of continuity is $\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$ and the stream function is now defined by $\rho u = \frac{\partial \psi}{\partial y}$ and $\rho v = -\frac{\partial \psi}{\partial x}$. In studying compressible fluids, it is convenient to introduce the two components of momentum, ρu and ρv, as new variables. However, in order to gain a symmetry that will appear later, we shall define $m = -\rho v$ $n = \rho u$. Introducing these in our equations we have

$$\left.\begin{aligned} &\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \frac{\partial m}{\partial y} = \frac{\partial n}{\partial x} \\ &\qquad\text{and the definitions} \\ &u = \frac{\partial \varphi}{\partial x}, \; v = \frac{\partial \varphi}{\partial y} \quad m = \frac{\partial \psi}{\partial x} \; n = \frac{\partial \psi}{\partial y} \end{aligned}\right\} \quad 1.$$

With the assumptions introduced above, it can be shown that the pressure and density can each be expressed as a function of the velocity, $q = \sqrt{u^2+v^2}$, only. This implies that $Q = \sqrt{m^2+n^2}$ is a function of q alone. (These relationships between p, ρ, q and Q are simply various forms of Bernoulli's equation for compressible fluids.)

If we use this fact in an attempt to find an equation for φ or ψ, we find the resulting equations are non-linear. For instance the equation for φ is $\frac{\partial}{\partial x}\left(\rho \frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho \frac{\partial \varphi}{\partial y}\right) = 0$ where ρ must be regarded as a given function of $\sqrt{u^2+v^2} = \sqrt{\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2}$. Non-linear equations are not readily amenable to analysis and very little has been accomplished in the way of direct solutions.

However, it has been found that if we use u and v as independent variables instead of x and y, the resulting equations are linear. In order to derive these equations, we eliminate dx and dy from the following equations

$$d\varphi = u\,dx + v\,dy\ ; \quad d\psi = m\,dx + n\,dy$$

This elimination is effected by solving for dx and dy in the following equations

$$\left.\begin{array}{ll} du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy & dx = \dfrac{\frac{\partial v}{\partial y}du - \frac{\partial u}{\partial y}dv}{\Delta} \\ \qquad\qquad\text{or} & dy = \dfrac{-\frac{\partial v}{\partial x}du + \frac{\partial u}{\partial x}dv}{\Delta} \\ dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy & \text{where } \Delta = \dfrac{\partial(u,v)}{\partial(x,y)} \end{array}\right\} 2.$$

If these are substituted in the above, we have

$$\left.\begin{array}{l} \Delta\, d\varphi = \left(u\frac{\partial v}{\partial y} - v\frac{\partial v}{\partial x}\right)du + \left(-u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x}\right)dv \\ \Delta\, d\psi = \left(m\frac{\partial v}{\partial y} - n\frac{\partial v}{\partial x}\right)du + \left(-m\frac{\partial u}{\partial y} + n\frac{\partial u}{\partial x}\right)dv \end{array}\right\} 3.$$

giving
$$\left.\begin{aligned}\Delta\frac{\partial\varphi}{\partial u} &= u\frac{\partial v}{\partial y} - v\frac{\partial v}{\partial x}\ ; \quad \Delta\frac{\partial\varphi}{\partial v} = -u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x}\\ \Delta\frac{\partial\psi}{\partial u} &= m\frac{\partial v}{\partial y} - n\frac{\partial v}{\partial x}\ ; \quad \Delta\frac{\partial\psi}{\partial v} = -m\frac{\partial u}{\partial y} + n\frac{\partial u}{\partial x}\end{aligned}\right\} 4.$$

If we eliminate $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ from the above with the aid of the equations $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ and $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$, we obtain the desired equations
$$\left.\begin{aligned} m\frac{\partial\varphi}{\partial u} + n\frac{\partial\varphi}{\partial v} &= v\frac{\partial\psi}{\partial v} + u\frac{\partial\psi}{\partial u}\\ u\frac{\partial\psi}{\partial m} + v\frac{\partial\psi}{\partial n} &= n\frac{\partial\varphi}{\partial n} + m\frac{\partial\varphi}{\partial m}\end{aligned}\right\} 5.$$

The first of these equations is readily obtained from the above equations but the second is more difficult. The reverse would have been true if we had used
$$\left.\begin{aligned} dm &= \frac{\partial m}{\partial x}dx + \frac{\partial m}{\partial y}dy\\ dn &= \frac{\partial n}{\partial x}dx + \frac{\partial n}{\partial y}dy\end{aligned}\right\} 6.$$

instead of equations 2. However, the second equation could have been found even more readily by noting the symmetry of u, v and φ with m, n and ψ and interchanging them in the first equation.

Equations 5 take on a particularly simple form if we use polar coördinates in the hodograph plane, i.e. $u = q\cos\theta$, $v = q\sin\theta$, $m = -Q\sin\theta$, $n = Q\cos\theta$

The equations now become *
$$\left.\frac{\partial\varphi}{\partial\theta} = \frac{q^2}{Q}\frac{\partial\psi}{\partial q} \qquad \frac{\partial\psi}{\partial\theta} = -\frac{Q^2}{q}\frac{\partial\varphi}{\partial Q}\ \right\} 7.$$

---

* During a discussion with Dr. H. Bateman, the authors learned that he had obtained the equations in the form given here. However, it is believed that the form given in equations 5 has not been given before.

These are linear equations and hence are much more readily treated than the non-linear equations found before. In these equations, q and Q are the values of the velocity and the momentum while θ is the angle of their common direction. Q will be a given function of q alone; in fact, in the case of adiabatic flow where $p = const. \rho^{\gamma}$, we have

$$Q = \rho_0 q \left[1 - \frac{\gamma-1}{2} \frac{q^2}{c_0^2}\right]^{\frac{1}{\gamma-1}} \quad \text{where} \quad \gamma = c_p/c_v,$$

the ratio of the specific heats and ρ_0 and c_0 are the density and velocity of sound at a point where $q = 0$.

Another interesting and extremely useful set of equations which have been used but little until now are obtained by introducing the variables χ and Γ by means of the contact transformations

$$\left.\begin{aligned} \chi &= ux + vy - \varphi \\ \Gamma &= mx + ny - \psi \end{aligned}\right\} \quad 8.$$

If we remember that

$$u = \frac{\partial \varphi}{\partial x} \qquad v = \frac{\partial \varphi}{\partial y} \qquad m = \frac{\partial \psi}{\partial x} \qquad n = \frac{\partial \psi}{\partial y}$$

we find the interesting relations

$$x = \frac{\partial \chi}{\partial u} \qquad y = \frac{\partial \chi}{\partial v} \qquad x = \frac{\partial \Gamma}{\partial m} \qquad y = \frac{\partial \Gamma}{\partial n} \quad 9.$$

which lead at once to the equations

$$\frac{\partial \chi}{\partial u} = \frac{\partial \Gamma}{\partial m} \qquad \frac{\partial \chi}{\partial v} = \frac{\partial \Gamma}{\partial n} \quad 10.$$

If we again go to polar coördinates in the hodograph plane, these become:

$$\left.\begin{aligned} Q \frac{\partial \chi}{\partial q} &= -\frac{\partial \Gamma}{\partial \theta} \\ q \frac{\partial \Gamma}{\partial Q} &= \frac{\partial \chi}{\partial \theta} \end{aligned}\right\} \quad 11.$$

We have found simple relations between ψ and φ and also between χ and Γ. Further, there exist relations connecting these two sets of dependent variables. First there are the relations given by combining equations 8 and 9

$$\left.\begin{aligned} \varphi &= u\frac{\partial \chi}{\partial u} + v\frac{\partial \chi}{\partial v} - \chi \\ \psi &= m\frac{\partial \Gamma}{\partial m} + n\frac{\partial \Gamma}{\partial n} - \Gamma \end{aligned}\right\} \quad 12.$$

or in the polar coördinates

$$\left.\begin{aligned} \varphi &= q\frac{\partial \chi}{\partial q} - \chi \\ \psi &= Q\frac{\partial \Gamma}{\partial Q} - \Gamma \end{aligned}\right\} \quad 13.$$

If we substitute these in equations 5, we find relations between ψ and χ and between φ and Γ

$$\left.\begin{aligned} v\frac{\partial \psi}{\partial v} + u\frac{\partial \psi}{\partial u} &= mu\frac{\partial^2 \chi}{\partial u^2} + (mv+nu)\frac{\partial^2 \chi}{\partial u\,\partial v} + nv\frac{\partial^2 \chi}{\partial v^2} \\ n\frac{\partial \varphi}{\partial n} + m\frac{\partial \varphi}{\partial m} &= mu\frac{\partial^2 \Gamma}{\partial m^2} + (mv+nu)\frac{\partial^2 \Gamma}{\partial m\,\partial n} + nv\frac{\partial^2 \Gamma}{\partial n^2} \end{aligned}\right\} \quad 14$$

or in polar coördinates

$$\left.\begin{aligned} -\frac{q^2}{Q}\frac{\partial \psi}{\partial q} &= q\frac{\partial^2 \chi}{\partial q\,\partial \theta} - \frac{\partial \chi}{\partial \theta} \\ -\frac{Q^2}{q}\frac{\partial \varphi}{\partial Q} &= Q\frac{\partial^2 \Gamma}{\partial \theta\,\partial Q} - \frac{\partial \Gamma}{\partial \theta} \end{aligned}\right\} \quad 15.$$

So far, in this analysis, the equations have been presented in sets of two, each set containing at least two dependent variables. However, in these sets, one independent equation can be found for each of the variables φ, ψ, χ and Γ. These independent equations can be found where u, v, m and n are variables, but the relationships between these variables proves troublesome and the most useful forms are those where q, Q and θ are variables. The desired equations are found to be *

$$\left.\begin{aligned}
&\frac{\partial}{\partial q}\left[\frac{Q^2}{q}\frac{\partial \varphi}{\partial Q}\right] + \frac{Q}{q^2}\frac{\partial^2 \varphi}{\partial \theta^2} = 0\\
&\frac{\partial}{\partial Q}\left[\frac{q^2}{Q}\frac{\partial \psi}{\partial q}\right] + \frac{q}{Q^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0\\
&q\frac{\partial}{\partial Q}\left[Q\frac{\partial \chi}{\partial q}\right] + \frac{\partial^2 \chi}{\partial \theta^2} = 0\\
&Q\frac{\partial}{\partial q}\left[q\frac{\partial \Gamma}{\partial Q}\right] + \frac{\partial^2 \Gamma}{\partial \theta^2} = 0
\end{aligned}\right\} 16$$

In obtaining solutions for these equations, it turns out to be convenient to use neither q nor Q but to introduce a dimensionless variable, τ, defined as $\tau = \frac{\gamma-1}{2} q^2/c_0^2$ where γ and c_0 are the same as previously defined. It follows immediately that

$$Q = c_0\rho_0\sqrt{\frac{2\tau}{\gamma-1}}\,(1-\tau)^{\frac{1}{\gamma-1}}$$

It will simplify matters if we put $\frac{1}{\gamma-1} = \beta$, then we have

$$\left.q = c_0\sqrt{2\beta\tau} \qquad Q = c_0\rho_0\sqrt{2\beta\tau}\,(1-\tau)^{\beta}\right\} 17$$

The following are expressions for the local velocity of sound "c", found in elementary acoustics to be $\frac{dp}{d\rho}$, and for

---

* In the previously noted discussion, Dr. Bateman mentioned having found these equations also.

the pressure and density, all as functions of τ

$$\left.\begin{aligned} c &= c_0\sqrt{1-\tau} \\ \rho &= \rho_0(1-\tau)^{\beta} \\ p &= p_0(1-\tau)^{1+\beta} \end{aligned}\right\} 18$$

It is interesting to note the range of values of τ. When $q = 0$, obviously $\tau = 0$. When the critical velocity of sound is reached, $q = c = \sqrt{\frac{2}{\gamma+1}}\, c_0$ and $\tau = \frac{\gamma-1}{\gamma+1} = \frac{1}{2\beta+1}$. When the limiting velocity, corresponding to $\rho = 0$ is attained, $q = q_{max.} = c_0\sqrt{\frac{2}{\gamma-1}}$ and $\tau = 1$. Hence, in practice, τ will range from 0 to 1 passing through the point where $q = c$ at a value of $\frac{\gamma-1}{\gamma+1} = \frac{1}{6}$ for $\gamma = 1.4$, the normal value for a diatomic gas such as air. In this case $\beta = 2\frac{1}{2}$.

Inserting the new variable τ, we have

$$\left.\begin{aligned}
&\frac{\partial}{\partial\tau}\left[\frac{2\tau(1-\tau)^{\beta+1}}{1-(2\beta+1)\tau}\frac{\partial\varphi}{\partial\tau}\right] + \frac{(1-\tau)^{\beta}}{2\tau}\frac{\partial^2\varphi}{\partial\theta^2} = 0 \\
&\frac{\partial}{\partial\tau}\left[2\tau(1-\tau)^{-\beta}\frac{\partial\psi}{\partial\tau}\right] + \frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{\partial^2\psi}{\partial\theta^2} = 0 \\
&\frac{\partial}{\partial\tau}\left[2\tau(1-\tau)^{\beta}\frac{\partial\chi}{\partial\tau}\right] + \frac{(1-\tau)^{\beta-1}\left[1-(2\beta+1)\tau\right]}{2\tau}\frac{\partial^2\chi}{\partial\theta^2} = 0 \\
&\frac{\partial}{\partial\tau}\left[\frac{2\tau}{(1-\tau)^{\beta-1}\{1-(2\beta+1)\tau\}}\frac{\partial\Gamma}{\partial\tau}\right] + \frac{1}{2\tau(1-\tau)^{\beta}}\frac{\partial^2\Gamma}{\partial\theta^2} = 0
\end{aligned}\right\} 19.$$

The first two of these equations are essentially the equations obtained by Tschaplygin.* The simultaneous sets

---

* S. Tschaplygin - Gas Jets (In Russian) Moscow Memoirs 1902

of equations with τ and θ as variables will also prove valuable in later work. They are

$$\left.\begin{aligned}
\frac{\partial\varphi}{\partial\theta} &= \frac{2\tau}{\rho_0(1-\tau)^{\beta}}\frac{\partial\psi}{\partial\tau} \\
\frac{\partial\psi}{\partial\theta} &= -\frac{2\rho_0\tau(1-\tau)^{\beta+1}}{[1-(2\beta+1)\tau]}\frac{\partial\varphi}{\partial\tau} \\
\frac{\partial\chi}{\partial\theta} &= \frac{2\tau(1-\tau)^{1-\beta}}{\rho_0[1-(2\beta+1)\tau]}\frac{\partial\Gamma}{\partial\tau} \\
\frac{\partial\Gamma}{\partial\theta} &= 2\rho_0\tau(1-\tau)^{\beta}\frac{\partial\chi}{\partial\tau}
\end{aligned}\right\} \quad 20.$$

Solutions of equations 19 for ψ and χ can be readily obtained by using expressions of the form $\psi = T_1(\tau)\,\Theta_1(\theta)$ and $\chi = T_2(\tau)\,\Theta_2(\theta)$.

The final expressions come out to be

$$\left.\begin{aligned}
\psi &= \tau^{n/2} F(a, b; c; \tau) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} n\theta \\
\chi &= \tau^{n/2} F(a' b'; c'; \tau) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} n\theta
\end{aligned}\right\} \quad 21.$$

where n may be assigned any value and

$$\left.\begin{aligned}
a + b &= n - \beta & \qquad a' &= a + \beta & \qquad \beta = \frac{1}{\gamma - 1} \\
a\,b &= -\frac{n\beta}{2}(n+1) & b' &= b + \beta & \\
c &= n + 1 & c' &= c &
\end{aligned}\right\} \quad 22$$

and $F(a, b; c; \tau)$ is the hypergeometric function defined by the series

$$F(a, b; c; \tau) = 1 + \frac{a\,b}{c}\tau + \frac{a(a+1)\,b(b+1)}{c(c+1)\;1\cdot 2}\tau^2 + \text{etc.}$$

The reason for the choice of τ as one of the independent variables becomes apparent upon inspecting the above solutions. The solutions for φ and Γ do not turn out so simply but if desired may be obtained without difficulty by using the simultaneous sets of equations (20).

However, in using these solutions, the authors found that, in general, only the two functions ψ and χ are necessary, the first to determine the streamlines and the second to effect the transformation from the hodograph plane of u and v (or τ and ϑ) to the physical plane of x and y. This latter operation is possible because of the relations $x = \frac{\partial \chi}{\partial u}$ $\quad y = \frac{\partial \chi}{\partial v}$. In fact, it is because of the ease and simplicy of this return transformation that the authors are lead to believe that the function χ (or Γ) will prove of great value in future work in compressible fluids. If only φ and/or ψ are used, as was done by Tschaplygin, the return to the plane of x and y is tedious and difficult.

The value of γ is shown, in the theory of gases, to be equal to $1 + \frac{2}{m}$ where m is the number of degrees of freedom of the molecules of the gas. The solutions given above reduce to cases of the confluent hypergeometric function if we assume the gas to be polyatomic i.e. $m = \infty$ and $\gamma = 1$. This corresponds to an isothermal type of flow. When $\gamma \rightarrow 1$, $\beta = \frac{1}{\gamma - 1} \longrightarrow \infty$ and it is necessary to define a new variable $t = \tau\beta$. The final forms of the solutions are

$$\left.\begin{aligned} \psi &= t^{n/2} F(a;c;-t) \tfrac{\sin}{\cos} n\theta \\ \chi &= t^{n/2} F(a';c';t) \tfrac{\sin}{\cos} n\theta \\ \text{where} \quad a &= \tfrac{n}{2}(1+n) \qquad a' = \tfrac{n}{2}(1-n) \\ c &= 1+n \qquad c' = 1+n \\ \text{and} \quad F(a';c';t) &= 1 + \frac{a'}{c'}t + \frac{a'(a'+1)}{c'(c'+1)1\cdot 2}t^2 + \text{etc.} \end{aligned}\right\} 23.$$

However, the authors found that no considerable simplification was attained by this specialized case and the more general solutions (21) were used.

As is generally done in solutions of partial differential equations, general solutions may be built up by summing on the arbitrary parameter n

$$\left.\begin{aligned} \psi &= \sum_{n=-\infty}^{\infty} A_n \tau^{n/2} F(a_n, b_n; c_n; \tau) \tfrac{\sin}{\cos} n\theta \\ \chi &= \sum_{n=-\infty}^{\infty} B_n \tau^{n/2} F(a'_n, b_n; c_n; \tau) \tfrac{\sin}{\cos} n\theta \end{aligned}\right\} 24.$$

In these expressions, the subscript n denotes that the corresponding quantity is a function of n.

If the coefficients B_n are known, then the coefficients A_n may be determined by means of equations similar to (15).

However, here we are confronted with the problem common to all solutions in the hodograph plane, namely that of determining solutions (i.e. the constants A_n, B_n) which will have a meaning when transformed back into the physical plane of x and y. It occurred to the authors to compare the limits of the above solutions as they approached the case of incompressible flow with the vast quantity of solutions already obtained for incompressible flow. If the limiting values were readily comparable with known solutions, the comparison might give values for A_n and B_n which would result in solutions that might possibly have a physical interpretation. To be sure, there is no guarantee that the boundaries or types of flow would correspond, but the method should indicate likely solutions.

Since the law giving the state of the gas was assumed to be $p = \text{const.}\,\rho^{\gamma}$, it follows that the incompressible case, corresponding to constant density, is given by $\gamma \to \infty$. In this transition, we shall assume that both p and ρ remain finite. Now $c_0^2 = \gamma p_0/\rho_0$ and if we insert this in our value for τ, we find $\tau = \frac{\gamma-1}{2\gamma}\, q^2/(p_0/\rho_0)$ which remains finite as $\gamma \to \infty$. Now $\beta = \frac{1}{\gamma-1} \to 0$, and we see that $\frac{ab}{c}$, the coefficient of the second term of the hypergeometric series given in 22, which is equal to $-\frac{n\beta(n+1)}{2}$ approaches 0 as do all the rest of the terms of both hypergeometric series except for the first terms. Consequently the solutions, (24), approach expressions of the form

$$\left.\begin{aligned}\psi &\to \text{const} \sum_m A_m q^{-m} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} m\theta \\ \chi &\to \text{const} \sum_m B_m q^{-m} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} m\theta\end{aligned}\right\} \quad 25.$$

Now, normally, solutions for the incompressible case are given in the form

$$\varphi + i\psi = \sum_m C_m z^m \quad \Big\} \quad 26$$

where $z = x + iy$ and ψ is defined by $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$, that is, without the quantity ρ multiplying these components of the velocity. However, since we shall be comparing incompressible flow solutions where ρ is a constant, this difference will not cause trouble. Differentiating equation 26 shows that

$$u - iv = \sum_m m C_m z^{m-1} \quad \Big\} \quad 27.$$

Now normally this series can be inverted, giving

$$z = \sum_m D_m (u - iv)^m \quad \Big\} \quad 28.$$

(In a great many practical cases, the series (27) consists of only one or two terms, in which case, the inversion is simple.) Substituting this in equation 26 gives

$$\varphi + i\psi = \sum_m E_m (u - iv)^m = \sum_m E_m q^m (\cos m\theta - i \sin m\theta) \quad 29.$$

We now define the functions χ and Γ by the relation

$$\chi + i\Gamma = (u - iv)z - (\varphi + i\psi) \quad \text{which}$$

yields

$$\left.\begin{aligned}\chi &= ux + vy - \varphi \\ \Gamma &= -vx + uy - \psi\end{aligned}\right\} \quad 30.$$

This agrees with our previous definitions except for the now constant quantity ρ which would be needed as a factor in the last equation. If we substitute equations 28 and 29 in 30 we have

$$\chi + i\Gamma = \sum_n{}' G_n (u - iv)^n = \sum_n{}' G_n q^n (\cos n\theta - i \sin n\theta) \quad 31.$$

Inspection now shows the close resemblance of equations 39 and 31 to the limiting case of the compressible flow given by equations 25. In fact, it is by means of this close resemblance that the authors hope that likely solutions of the compressible fluids equations will be obtained.

To illustrate the method a little further, let us consider the case of the unit circular cylinder. The solution in the case of incompressible fluids is

$$\varphi + i\psi = z + \frac{1}{z}$$

$$u - iv = 1 - \frac{1}{z^2}$$

from which we find

$$\varphi + i\psi = \frac{1}{\sqrt{1-(u-iv)}} + \sqrt{1-(u-iv)}$$

$$= 2 + \frac{1}{4} q^2 e^{-2i\theta} - \frac{1}{4} q^3 e^{-3i\theta} + \frac{15}{64} q^4 e^{-4i\theta} + \text{etc}$$

$$\text{or} \quad \psi = -\frac{1}{4} q^2 \sin 2\theta + \frac{1}{4} q^3 \sin 3\theta - \frac{15}{64} q^4 \sin 4\theta \ \text{etc.}$$

Similarly

$$\chi + i\Gamma = -2\left[1 - \frac{1}{2} q e^{-i\theta} - \frac{1}{8} q^2 e^{-2i\theta} + \text{etc}\right]$$

$$\text{or} \quad \chi = -2\left[1 - \frac{1}{2} q \cos\theta - \frac{1}{8} q^2 \cos 2\theta + \text{etc}\right]$$

If the coefficients obtained from these series for ψ and χ were used to determine the corresponding compressible flow, the result might possibly be the flow around an obstacle at least vaguely resembling a sphere.[4] Unfortunately, time did not permit the authors to investigate this solution further. Instead, they have investigated another case which will now be described in detail.

Suppose, that instead of summing on n the solutions given in (21), a single fractional value of n had been chosen. A little investigation shows that the corresponding incompressible fluids solution is the case of flow in a corner. Before we proceed to discuss the solution obtained, let us first review the results obtained for the incompressible fluid. This complex potential is $\varphi + i\psi = A z^{\pi/\alpha}$ where α is the angle shown in figure 1. If we put

$$x = r\cos\omega \qquad y = r\sin\omega$$

then we have

$$\varphi = A r^{\pi/\alpha} \cos\frac{\pi\omega}{\alpha}$$
$$\psi = A r^{\pi/\alpha} \sin\frac{\pi\omega}{\alpha}$$

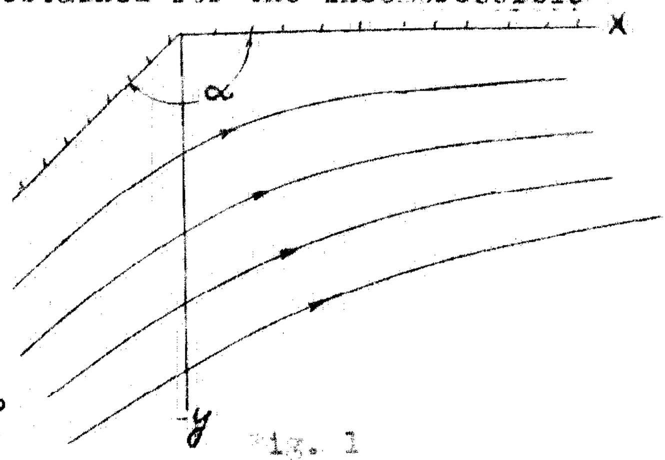


Fig. 1

The streamlines obtained from the last equations are shown in figure 1. The velocities are obtained by differentiating the complex potential and they are $u - iv = \frac{A\pi}{\alpha} z^{\frac{\pi-\alpha}{\alpha}}$.

At very great distances from the origin, the velocities become quite large. Here, we should expect to find a difference from the compressible fluids case where only finite velocities are possible. We now eliminate z and find $\varphi + i\psi$ in terms of $u - iv = q e^{-i\theta}$

$$\left.\begin{array}{l} \varphi + i\psi = A' q^{\frac{\pi}{\pi-\alpha}} e^{-\frac{i\pi\theta}{\pi-\alpha}} \\ \\ \text{whence} \quad \begin{array}{l} \varphi = A' q^{n} \cos n\theta \\ \psi = -A' q^{n} \sin n\theta \end{array} \quad \text{where} \quad n = \frac{\pi}{\pi-\alpha} \end{array}\right\} 32.$$

Plotted in the hodograph plane, the streamlines apoear as shown in figure 2. It should be remembered that the direction of flow is not given by the slope of the streamlines in the hodograph plane but by the direction of a line through the origin. Finally, we find that

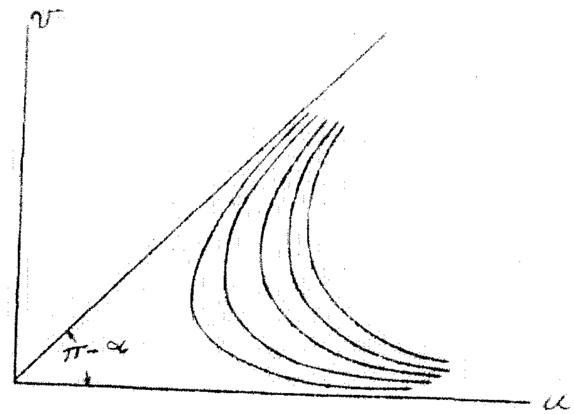


Fig. 2

$$\chi + i\Gamma = A\left(\frac{\pi-\alpha}{\alpha}\right) z^{\pi/\alpha} = A''(u-iv)^{\frac{\pi}{\pi-\alpha}}$$

$$\text{whence} \quad \left.\begin{array}{l} \chi = A'' q^{n} \cos n\theta \\ \Gamma = -A'' q^{n} \sin n\theta \end{array}\right\} 33.$$

where $n = \frac{\pi}{\pi-\alpha}$, the same as in equations 32.

Now we turn to the solution for the compressible fluids case. We have

$$\left.\begin{array}{l} \psi = B\,\tau^{n/2} F(a,b;c;\tau) \sin n\theta \\ \chi = B'\tau^{n/2} F(a',b';c';\tau) \cos n\theta \end{array}\right\} 34.$$

where we shall use the hint given by equations 32 and put

$n = \frac{\pi}{\pi - \alpha}$. Using the expressions given in 22, we find that

$$
\left.
\begin{aligned}
a &= \frac{n}{2} - \frac{\beta}{2} - \frac{\sqrt{n^2 + 2n^2\beta + \beta^2}}{2} & a' &= \frac{n}{2} + \frac{\beta}{2} - \frac{\sqrt{n^2 + 2n^2\beta + \beta^2}}{2} \\
b &= \frac{n}{2} - \frac{\beta}{2} + \frac{\sqrt{n^2 + 2n^2\beta + \beta^2}}{2} & b' &= \frac{n}{2} + \frac{\beta}{2} + \frac{\sqrt{n^2 + 2n^2\beta + \beta^2}}{2} \\
c &= n + 1 & c' &= n + 1
\end{aligned}
\right\} \quad 35
$$

with $n = \frac{\pi}{\pi - \alpha}$

Throughout the rest of this work, γ will be assumed to have the normal value for air of 1.4 and $\beta = 2\frac{1}{2}$
We can now plot a, b, c, a', b', and c' as functions of α , the angle of the corner. These are shown in figure 3.

If either of the first two parameters of the hypergeometric function are negative integers, the series is no longer infinite, but becomes a polynomial. Now figure 3 shows that both a and a' pass through negative integral values and hence by chosing suitable values of α , either ψ or χ could be made to be a finite polynomial in τ . Such a choice would greatly simplify calculation, but the question immediately rises as to whether these special values might not give types of flow differing a great deal from that of the more general case. The authors chose a special value of α to perform the preliminary investigation and then studied the differences from the more general case and these results are given later in this work.

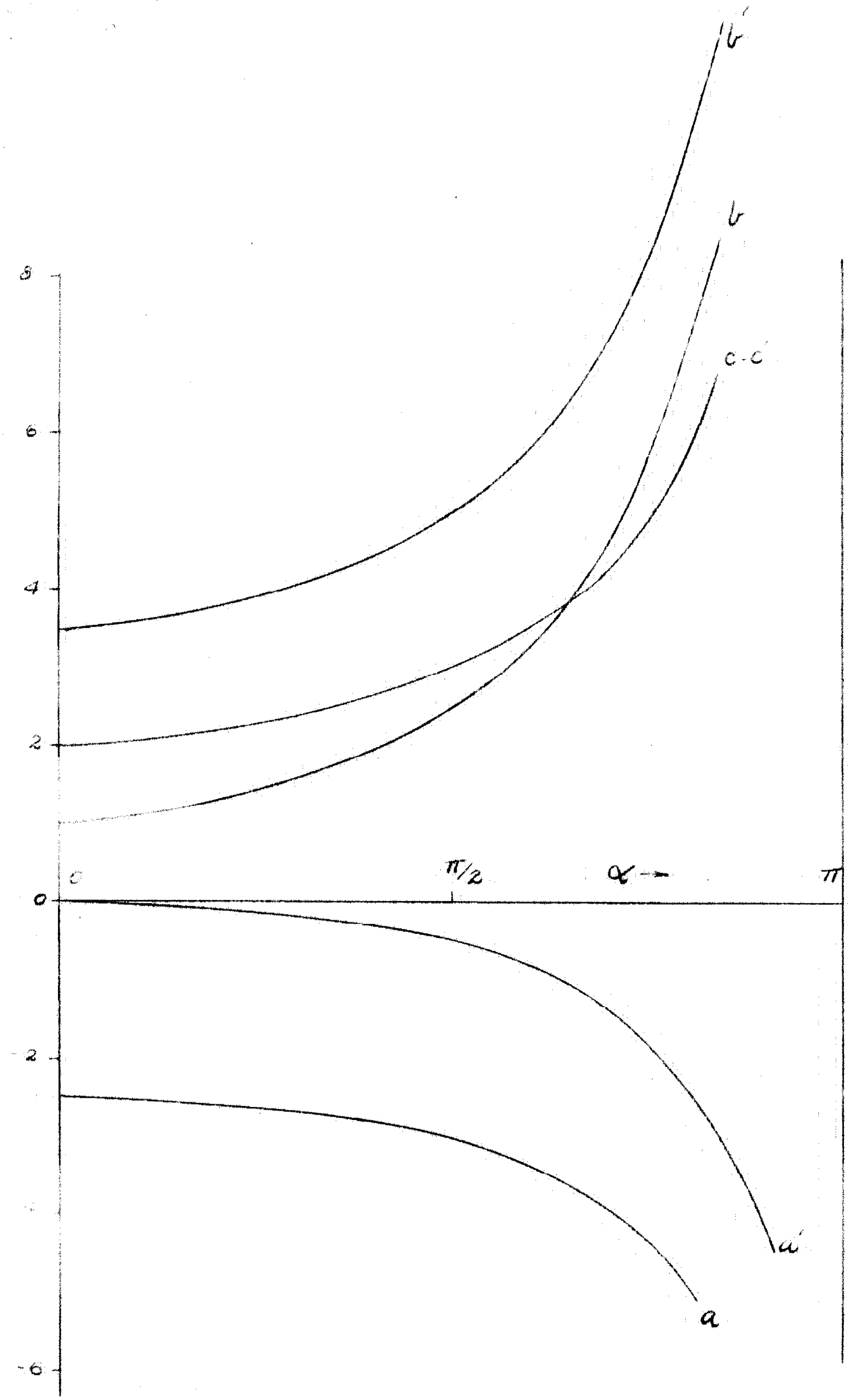


Fig. 3

The value of α decided upon was $\alpha = 9\pi/14$, which gives $a = -3\tfrac{1}{2}$, $b = 3.8$, $c = c' = 3.8$, $a' = -1$, $b' = 6.3$. These values lead to an infinite series for the hypergeometric series involved in the ψ function, but the hypergeometric function of the χ function is a polynomial of the first degree. These functions are:

$$\left.\begin{aligned} F(a,b;c;\tau) &= 1 + \frac{(-3\tfrac{1}{2})(3.8)}{3.8}\tau + \frac{(-3\tfrac{1}{2})(-2\tfrac{1}{2})(3.8)(4.8)}{(3.8)(4.8)(1)(2)}\tau^2 + \text{etc} \\ &= 1 - 3\tfrac{1}{2}\tau + \frac{(3\tfrac{1}{2})(2\tfrac{1}{2})}{2}\tau^2 - \frac{(3\tfrac{1}{2})(2\tfrac{1}{2})(1\tfrac{1}{2})}{6}\tau^3 + \text{etc} \\ &= 1 - 3.5\tau + 4.375\tau^2 - 2.1875\tau^3 + .2734375\tau^4 + \text{etc} \\ \text{and } F(a',b';c';\tau) &= 1 + \frac{(-1)(6.3)}{3.8}\tau \\ &= 1 - 1.658\tau \end{aligned}\right\} \tag{36}$$

In discussing this solution, we shall first discuss the results in the hodograph plane with the ψ function as our principal tool. Then we shall discuss the transformation to the physical plane, using the χ function to effect this transformation.

The stream function can be written in the form

$$\left.\begin{aligned} \psi &= T_1(\tau)\sin\frac{14\theta}{5} \\ \text{where } T_1(\tau) &= \tau^{m/2} F(a;b;c;\tau) \\ &= \tau^{7/5}\left[1 - 3.5\tau + 4.375\tau^2 - 2.1875\tau^3 + \text{etc}\right] \end{aligned}\right\} \tag{37}$$

A streamline can be plotted up by assigning a given value to ψ and varying τ and θ according to equations 37. In practice, the way this was done was to plot $T_1(\tau)$, as a function of τ (shown in figure 4), then choose a range of values of τ, find the values of $T_1(\tau)$ from figure 4 and solve for the corresponding values of θ by means of equations 37 written in the form

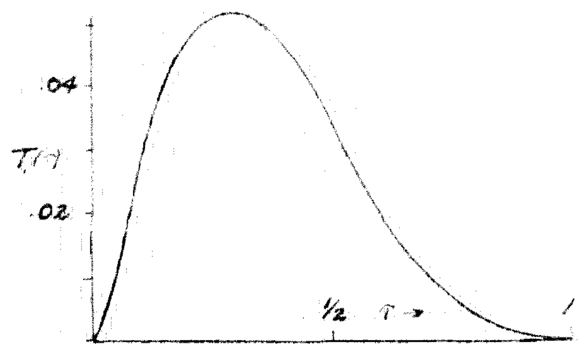


Fig. 4

$$\sin \frac{14\theta}{5} = \psi / T_1(\tau)$$

This process can be repeated to find different streamlines corresponding to different values of ψ. The resulting streamlines are shown in figure 5. Comparing figure 5 with figure 2, we see that near the origins the streamlines are similar as we should expect since the effects of compressibilty are small for small velocities. However, figure 5 shows that the velocities no longer become very large but are limited by the value $q = q_{max}$. corresponding to $\tau = 1$. Furthermore, the streamlines are closed curves passing smoothly through the critical velocity of sound.

We shall now consider briefly what would have happened to this picture in the hodograph plane if a special value had not been chosen for a'. Referring to figure 4, we see that $T_1(\tau) = 0$ for $\tau = 1$. Investigation showed that this was not generally true. For values of α between 0 and π, two other general shapes were possible, namely (1)

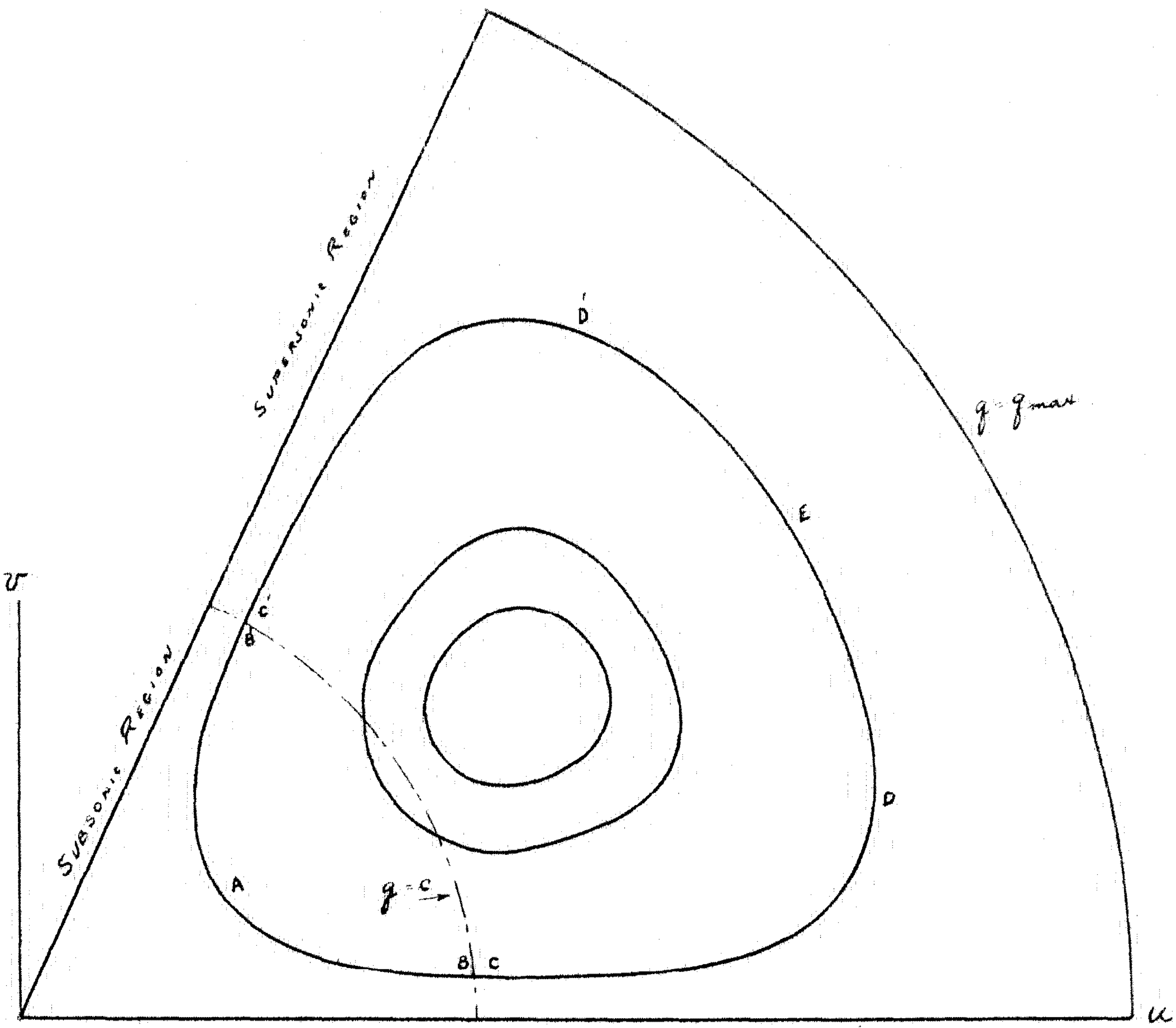


Fig. 5

and (2) of figure 6. The end point $T_1(\tau=1)$ can be found as a function of α by means of the relation $F(a,b;c;1)=$

$$\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

and is shown in figure 7.

These different shapes for the function T_1 give different patterns for the streamlines in the hodograph plane. Type (1) of figure 6 leads to a streamline picture similar to that shown in figure 8, while type (2) corresponds to that in figure 9. In both cases, we have some streamlines which are no longer closed but end at $q = q_{max}$ corresponding to $\tau = 1$. For these streamlines the flow starts off with zero density.

We shall return to a discussion of these cases later, but now we turn our attention to the transformation to the

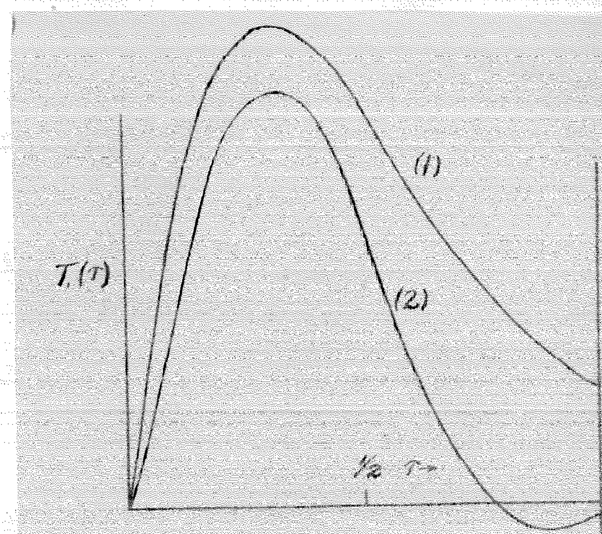


Fig. 6

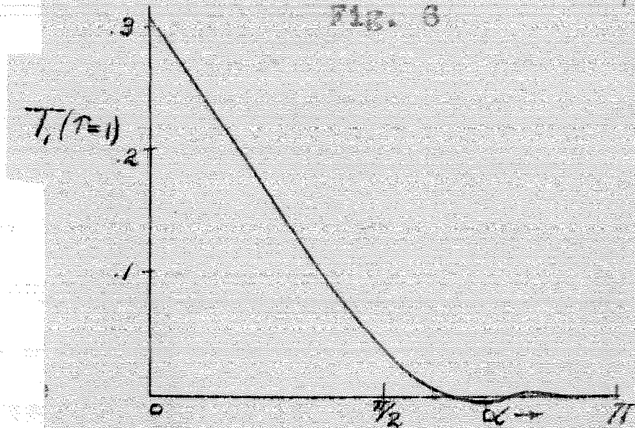


Fig. 7

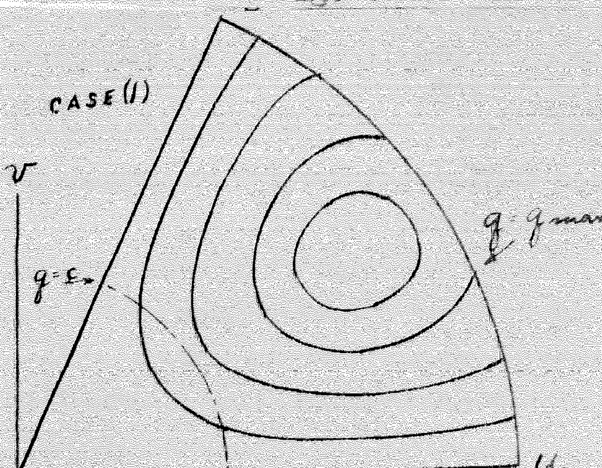


FIG 8

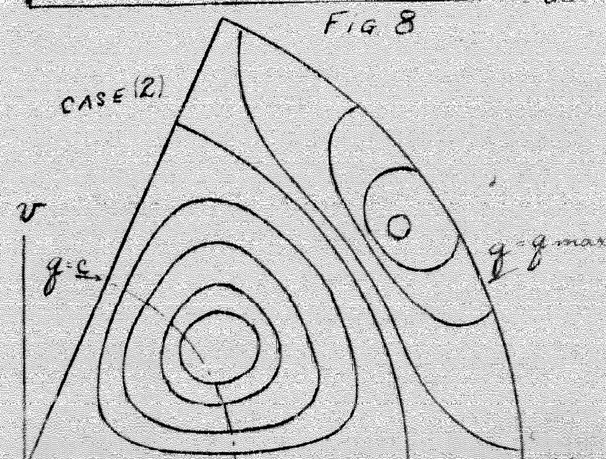


FIG 9

physical plane of x and y. As previously mentioned, this transformation will be effected by means of the relations $x = \frac{\partial \chi}{\partial u}$ $y = \frac{\partial \chi}{\partial v}$ which in terms of τ and θ are

$$\left.\begin{aligned} x &= \frac{1}{c_0}\sqrt{\frac{2}{\beta}}\left[\sqrt{\tau}\,\frac{\partial \chi}{\partial \tau}\cos\theta - \frac{1}{2\sqrt{\tau}}\,\frac{\partial \chi}{\partial \theta}\sin\theta\right] \\ y &= \frac{1}{c_0}\sqrt{\frac{2}{\beta}}\left[\sqrt{\tau}\,\frac{\partial \chi}{\partial \tau}\sin\theta + \frac{1}{2\sqrt{\tau}}\,\frac{\partial \chi}{\partial \theta}\cos\theta\right] \end{aligned}\right\} 38$$

As would be expected from dimensional considerations, there are no data in the problem from which we could determine a scale of lengths. Hence, we shall drop the factor $\frac{1}{c_0}\sqrt{\frac{2}{\beta}}$ in equations 38. Substitution of the expression for from equations 34 in equations 38 leads to the following result.

$$\left.\begin{aligned} x &= F_1(\tau)\cos\theta\cos n\theta + F_2(\tau)\sin\theta\sin n\theta \\ y &= F_1(\tau)\sin\theta\cos n\theta - F_2(\tau)\cos\theta\sin n\theta \\ \text{where}\quad F_1(\tau) &= \frac{n}{2}\,\tau^{\frac{n-1}{2}} F(a', b', c', \tau) + \tau^{\frac{n+1}{2}}\frac{dF}{d\tau}(a', b', c', \tau) \end{aligned}\right\} 39.$$

$$F_2(\tau) = \frac{n}{2}\,\tau^{\frac{n-1}{2}} F(a', b', c', \tau)$$

If we insert the numerical of our case of $\alpha = \frac{9\pi}{14}$, these expressions become

$$\left.\begin{aligned} x &= F_1(\tau)\cos\theta\cos\frac{14\theta}{5} + F_2(\tau)\sin\theta\sin\frac{14\theta}{5} \\ y &= F_1(\tau)\sin\theta\cos\frac{14\theta}{5} - F_2(\tau)\cos\theta\sin\frac{14\theta}{5} \\ \text{where}\quad F_1(\tau) &= \tau^{9/10}(1 - 2.842\,\tau) \\ F_2(\tau) &= \tau^{9/10}(1 - 1.658\,\tau) \end{aligned}\right\} 40.$$

The functions $\bar{F}_1(\tau)$ and $\bar{F}_2(\tau)$ are shown in figure 10.

We are now in a position to plot the streamlines of figure 5 in the plane of x and y. These are shown in figures 11 and 12.

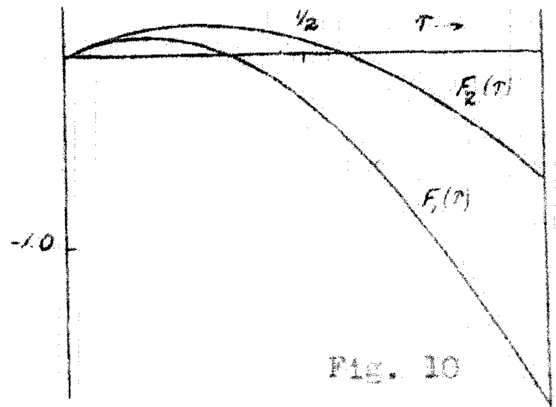
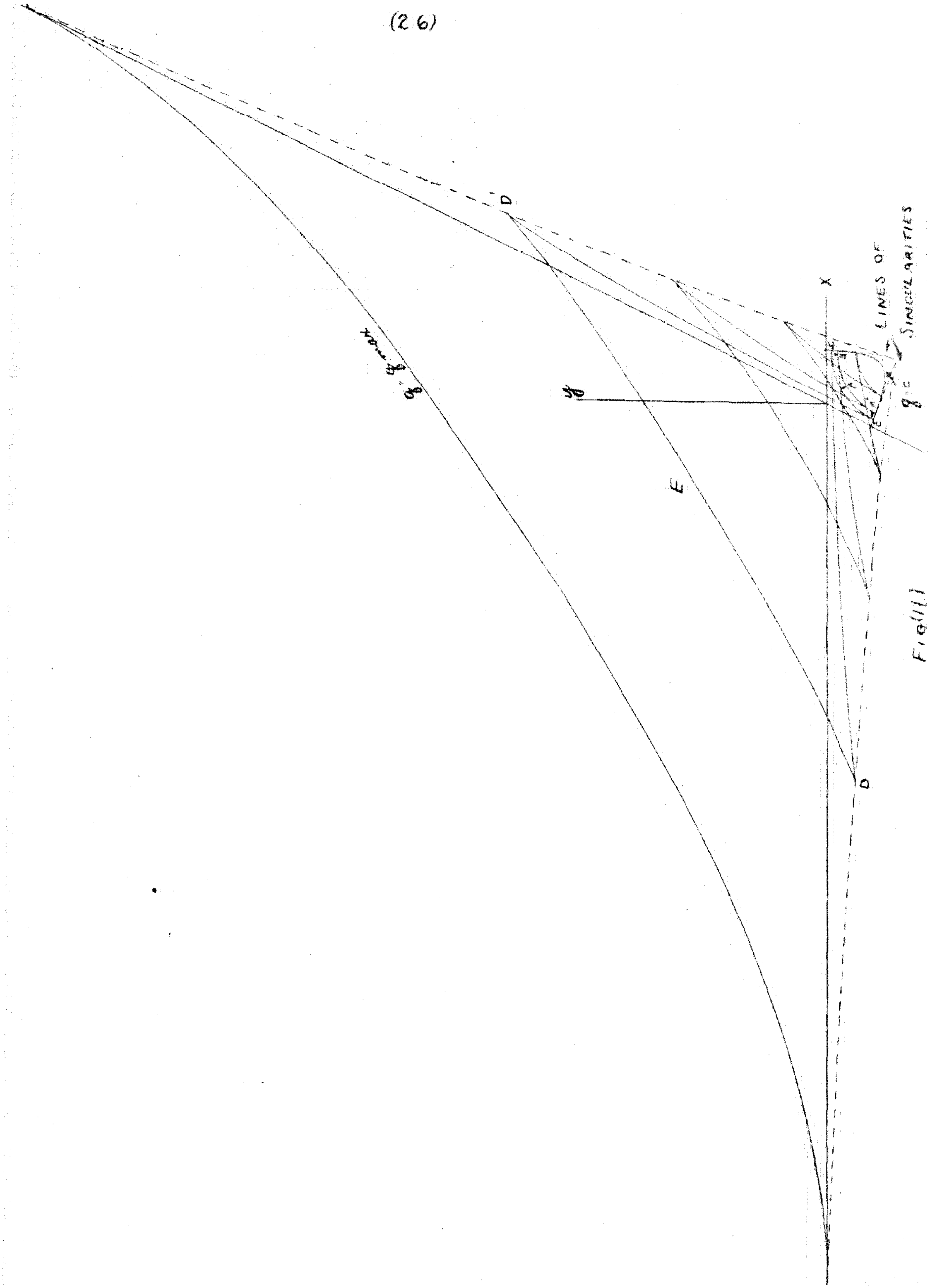


Fig. 10

The results were most unexpected and gave the authors no small amount of worry at first. The streamlines appeared to overlap and to double back on themselves in a most disconcerting manner. Let us trace the course of a typical streamline comparing the corresponding points of figures 5 and 11. Let us start at point A in the subsonic region of figure 5. In figure 11, the point A is located in the region in the lower right hand corner. As we pass along the streamline, we pass the point B at which the sonic velocity is reached. So far all is well. However as we progress a small finite distance further we arrive at the point C where the streamline touches the dotted line (labeled "line of singularities") of figure 11. Here the streamline doubles back on itself without a corresponding discontinuity in τ or θ, indicating that the direction of the streamline has not suddenly changed by 180°. The streamline now goes back through the region from which it has just come, crossing one of the walls of the corner which was previously a boundary of the flow. It next arrives at the point D on another line of singularities



Fig(11)

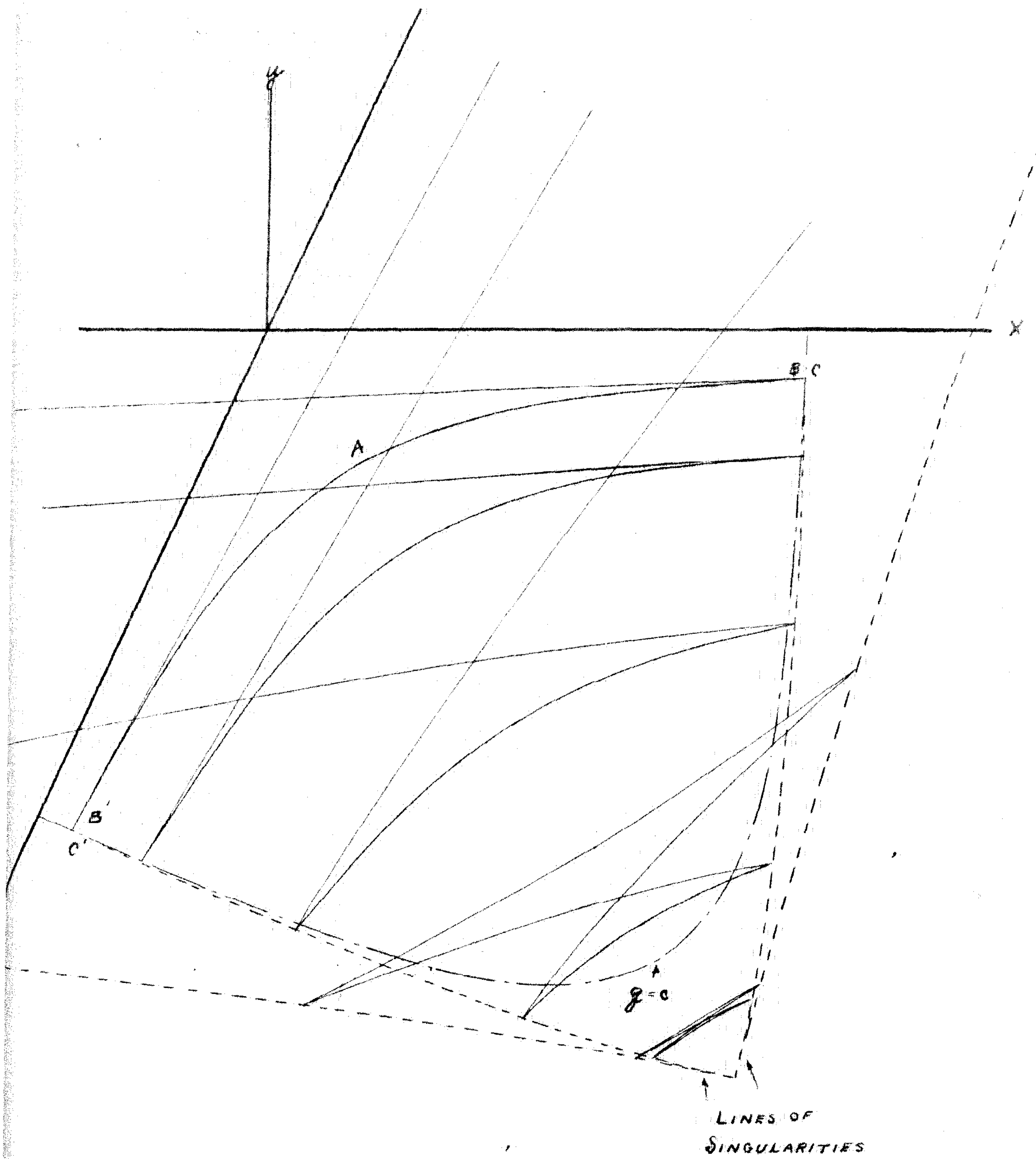


FIG. 12

where it again doubles back on itself. From here it crosses the axis of symmetry at E and repeats the above process symmetrically at the points D' C' and B'. In passing from D to D', it should be noted that the streamline crossed the extensions of both walls of the corner. The details of the flow in the subsonic region are shown to a greater scale in figure 12.

Next came the problem of explaining this apparent tomfoolery. Investigation showed that the doubling back could be traced to the maxima of the curves of figure 10 and that where these cusp-like singularities occurred, dx and dy were both zero. This indicated that there was a break-down of the mapping between the hodograph plane and the physical plane as expressed by the condition $\frac{\partial(x,y)}{\partial(u,v)} = 0$ or in terms of the χ function, $\frac{\partial^2\chi}{\partial u^2}\frac{\partial^2\chi}{\partial v^2} - \left(\frac{\partial^2\chi}{\partial u \partial v}\right)^2 = 0$ If the variables τ and θ are introduced in place of u and v, after considerable simplification and rearrangement, this condition becomes,

$$\left[\frac{1}{2\tau}\frac{\partial^2\chi}{\partial\theta^2} + \frac{\partial\chi}{\partial\tau}\right]\left[\frac{\partial\chi}{\partial\tau} + 2\tau\frac{\partial^2\chi}{\partial\tau^2}\right] = \left[\frac{1}{2\tau}\frac{\partial\chi}{\partial\theta} - \frac{\partial^2\chi}{\partial\theta\partial\tau}\right]^2 \Bigg\} 41.$$

If we use the differential equation for χ in the form

$$\frac{1}{2\tau}\frac{\partial^2\chi}{\partial\theta^2} + \frac{\partial\chi}{\partial\tau} = \frac{(1-\tau)}{(2\beta+1)\tau-1}\left[\frac{\partial\chi}{\partial\tau} + 2\tau\frac{\partial^2\chi}{\partial\tau^2}\right] \Bigg\} 42.$$

this equation becomes

$$\left[\frac{1}{2\tau}\frac{\partial^2\chi}{\partial\theta^2} + \frac{\partial\chi}{\partial\tau}\right] = \pm\sqrt{\frac{1-\tau}{(2\beta+1)\tau-1}}\left[\frac{1}{2\tau}\frac{\partial\chi}{\partial\theta} - \frac{\partial^2\chi}{\partial\theta\partial\tau}\right] \Bigg\} 43.$$

Inserting the expressions 34 and using the value $\alpha = \frac{9\pi}{14}$, this can be put in the form

$$\tan \frac{14\theta}{5} = \pm \frac{\sqrt{(1-\tau)(6\tau-1)}}{1-3.5\tau} \qquad 44$$

These lines of the singularities have been plotted in the hodograph plane in figure 13 and have already been mentioned as the dotted lines in figures 11 and 12.

To gain a clearer insight into what is behind all of this, let us find the slope of the streamlines in the hodograph plane where they cross these lines of singularities. (It should be noted that here, as previously, these conditions are first worked out making no assumptions as to the form of the general function χ and hence are applicable to <u>all</u> cases of irrotational inviscid flow.) By combining equations 11 and 13 and introducing τ in place of q and Q we find

$$\left.\begin{aligned} \frac{\partial \psi}{\partial \tau} &= \rho_0 (1-\tau)^{\beta} \left[\frac{\partial^2 \chi}{\partial \theta \partial \tau} - \frac{1}{2\tau} \frac{\partial \chi}{\partial \theta} \right] \\ \frac{\partial \psi}{\partial \theta} &= \rho_0 (1-\tau)^{\beta} \left[\frac{\partial^2 \chi}{\partial \theta^2} + 2\tau \frac{\partial \chi}{\partial \tau} \right] \end{aligned}\right\} \quad 45$$

Now along a streamline

$$d\psi = \frac{\partial \psi}{\partial \tau} d\tau + \frac{\partial \psi}{\partial \theta} d\theta = 0$$

or for the polar slope, we have

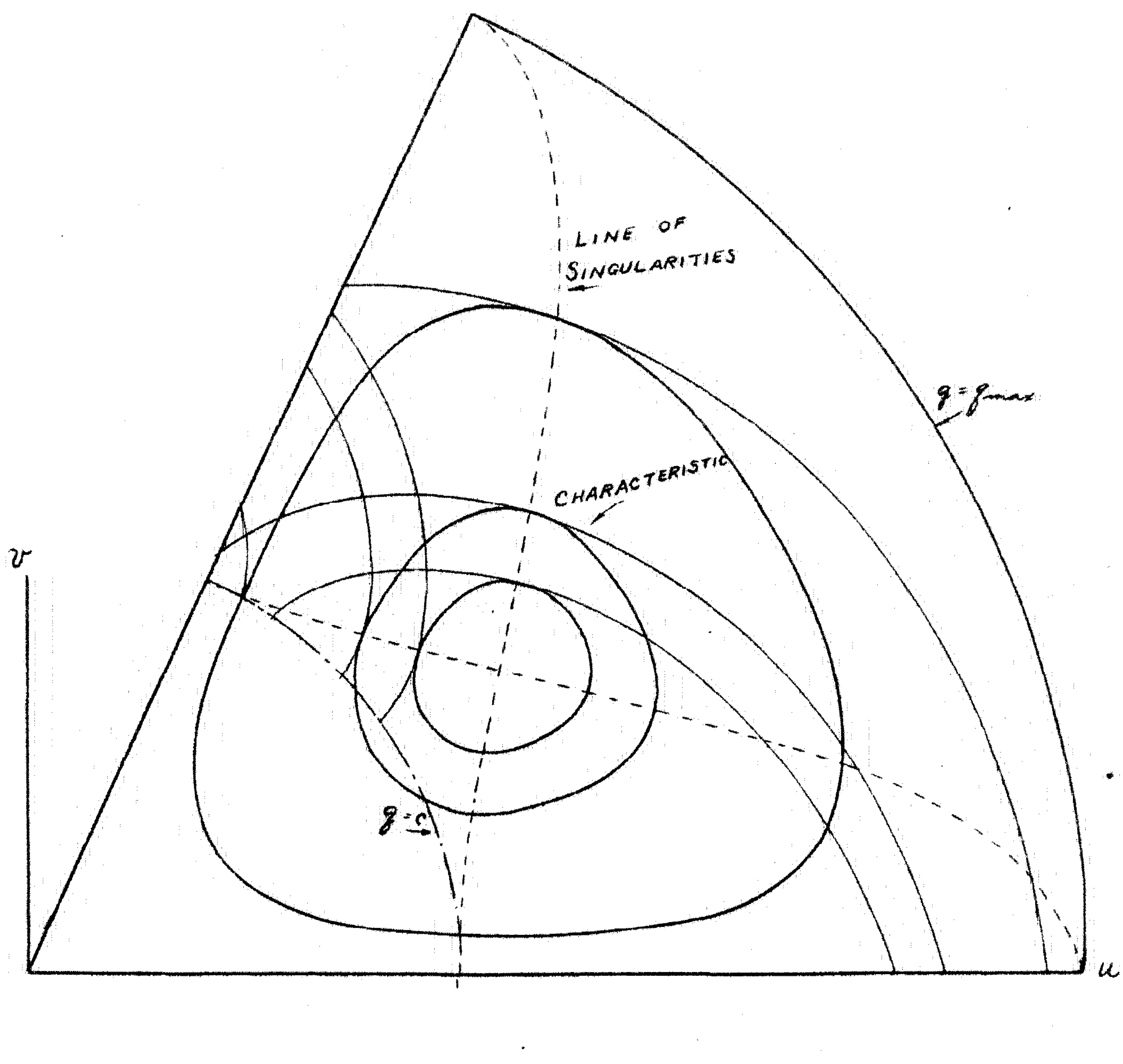


Fig. 13

$$\frac{d\tau}{\tau\, d\vartheta} = -\frac{\frac{\partial \psi}{\tau \partial \vartheta}}{\frac{\partial \psi}{\partial \tau}} = \frac{\frac{1}{2\tau}\frac{\partial^2 \chi}{\partial \theta^2} + \frac{\partial \chi}{\partial \tau}}{\frac{1}{2\tau}\frac{\partial \chi}{\partial \theta} - \frac{\partial^2 \chi}{\partial \theta \partial \tau}} \qquad 46.$$

Where the streamline crosses a line of singularities given by equation 43 this expression for the slope becomes

$$\frac{d\tau}{\tau\, d\vartheta} = \pm\sqrt{\frac{1-\tau}{(2\beta+1)\tau-1}} \qquad 47.$$

which is simply the slope of the characteristics of the differential equations for the flow of a compressible fluid. This means that the singularities are the points at which the streamlines are tangent to the characteristics. To those who are familiar with the fundamental nature of the characteristics of a partial differential equation of the hyperbolic type, this fact explains to a great degree the possibility of the existence of a flow pattern which is unheard of in the solutions of the normal elliptic type potential flow equation.

To those unfamiliar with the theory of characteristics, let us say just a word. The differential equation for the characteristics are the same for all 4 of equations 19 and is essentially equation 47. Integration of equation 47 gives epicycloids, portions of which are shown in figure 13. In the Prandtl-Meyer type of flow*

---

* cf. Th. Meyer - Forschungsarbeit, Heft 62-1908

it was found that if a streamline coincided with a characteristic, a line in the hodograph plane mapped into a complete region in the physical plane. Further, in the theory of partial differential equations, the characteristics constitute an essential tool in the solution of equations of the hyperbolic type such as the wave equation. An account of the mathematical theory is given in Riemann-Weber, Differentialgleichungen der Physik Bd I, S. 505 Braunschweig 1925. The application of the method of characteristics to the solution of problems in supersonic flow is discussed by Busemann in Handbuch der Experimentalphysik IV 1. Teil S. 421 Akademische Verlagsgesellschaft MBH Leipzig 1931.

Now let us reëxamine our flow pattern. Figure 13 shows that the lines of singularities divide the sector into four regions. One of these regions (that which includes the subsonic portion of the field) touches both of the radial lines which bound the sector and which correspond to the walls of the corner. Two of the regions touch one of these bounding lines each, while the fourth region has no line of contact with either bounding line. In the physical plane of figure 11, these four regions map into four regions which overlap one another. The first region maps into the expected flow in a corner. The second and third regions contain a flow which heeds the presence of one wall but crosses the other wall, while the flow in the fourth region crosses both walls.

So what we have obtained is in reality four distinct flow patterns, every one of which is possible in itself, but whose hodograph patterns unite to form a continuous pattern. The question arises, now that we have four "pieces" of flow patterns with streamlines ending apparently in space, can these solutions be extended so as to give physically conceivable types of flow? The mathematical theory of characteristics answers yes, it is quite possible to add on to any one of these solutions; all that is necessary is to know the shape of one bounding wall for any given addition.

We now return to the problem of the limitations added by the selection of a special value of α. First, our selection gave χ as a finite polynomial in τ. It is easy to verify that if χ had not been a finite polynomial, the resulting hypergeometric series would have been divergent at $\tau = 1$. Since all of the derivatives of χ would have been likewise divergent at $\tau = 1$, this implies that x and y would become infinite for $\tau = 1$. This means that the solution would have extended to infinity instead of being confined to the finite part of the physical plane as it did in the case illustrated. Furthermore, in connection with figures 8 and 9 the streamlines that start with maximum velocity and zero density would come in from infinity. The lines of singularities for these two cases are shown in figures 14 and 15. In figure 14, the sector is still divided into four regions. However in figure 15, the sector is

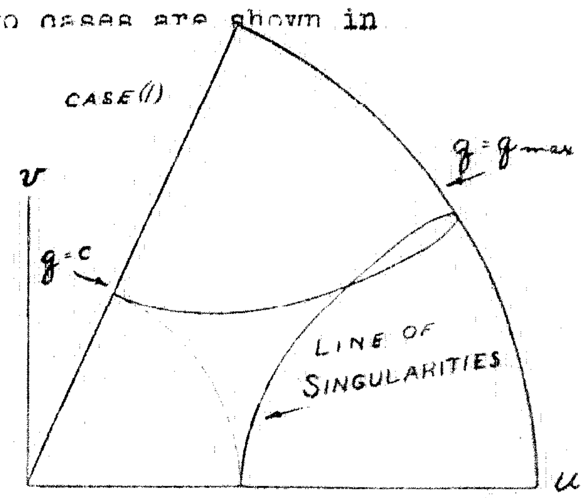


FIG 14

divided into seven regions. Unfortunately, time did not permit an extensive investigation into the mapping of these cases in the plane of x and y. However it was found that

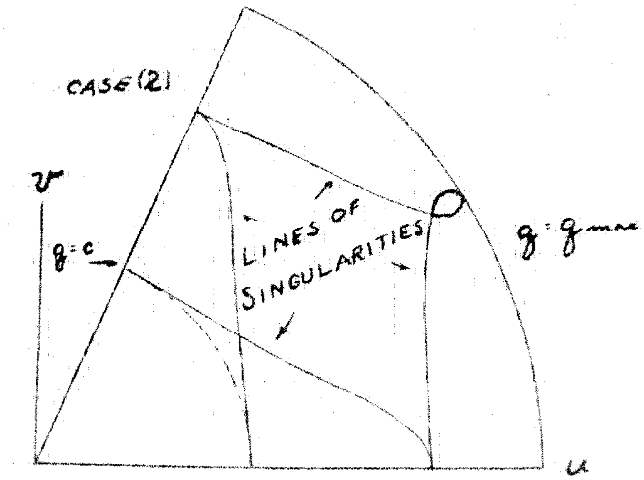


Fig. 15

in all cases the mapping of the first region, which includes the subsonic portion of the field, was quite similar to the case illustrated.

The solution which has been presented here lends itself to an interesting application. In fact, this application was the original problem which the authors set out to solve.

Meyer*, in treating the case of an oblique shock wave, found that, with a given angle of flow after the shock, there was a limiting supersonic velocity, below which, his solution was no longer applicable. An examination of photographs of projectiles

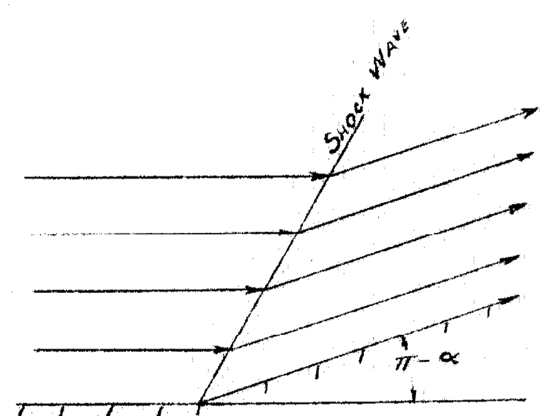


Fig. 16

in flight showed that for high velocities an oblique shock came off directly from the point of the projectile, while for

---

* Th. Meyer. loc. cit.

velocities only slightly greater than that of sound, the shock wave stood out in front of the projectile and was curved as shown in figure 17. It is known that for a normal shock such as exists directly in front of the projectile, the velocity behind the shock is subsonic, in fact the product of the velocitys in front and behind are equal to the square of the velocity of sound. Hence, since the velocity in front in greater than the sonic velocity, the velocity behind is subsonic. This suggests that we apply our solution for the flow in a corner to the region behind the shock wave. The difficulty is that the fluid, when it passes through the curved shock front gains a rotation or vorticity which is not a property of the flow which was previously illustrated. At first it was thought that the rotation would be small and could be neglected. However, subsequent investigation showed that this is not the case. The next step was to try a flow in <u>front</u> of the shock wave which had a distribution of vorticity such that after the fluid has passed through the shock, it will emerge without rotation. Such a flow is physically possible and while it is not quite the solution originally sought, it is believed that it will give a good idea of what happens in the desired flow.

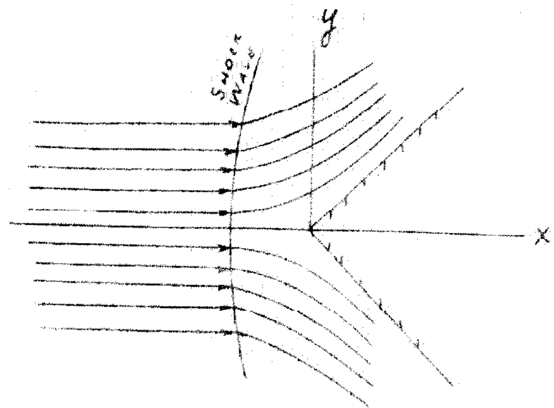


Fig. 17

The authors have not been able completely to carry out the process of fitting a shock wave to their solution, but will outline the elements in the method of such a solution.

In the first place, we shall have to consider six variables along the shock, namely x, y, U (the velocity in front of the shock, which will vary, due to the assumed vorticity) τ , θ (characteristic of the flow after the shock) and ϖ, the angle of inclination of the shock. Our equations 39 furnish us x and y as functions of τ and θ , which we shall write

$$\left.\begin{aligned} x &= f(\tau,\theta) \\ y &= g(\tau,\theta) \end{aligned}\right\} 48$$

In Busemann's article (see page 32) on page 436 is found the relation

$$v^2 = (U-u)^2 \frac{u - \frac{w^{*2}}{U}}{\left(\frac{w^{*2}}{U} + \frac{2}{\gamma+1} U\right) - u}$$

where u and v are the components of velocity behind the shock and w* is the critical velocity of sound. In our notation, this becomes

$$\left.\tau \sin^2\theta = \left(T^{1/2} - \tau^{1/2}\cos\theta\right)^2 \frac{\tau^{1/2}\cos\theta - \frac{\gamma-1}{\gamma+1}\frac{1}{T^{1/2}}}{\frac{\gamma-1}{\gamma+1}\frac{1}{T^{1/2}} + \frac{2T^{1/2}}{\gamma+1} - \tau^{1/2}\cos\theta}\right\} 49$$

where $T = \frac{U^2}{q_{max}^2}$

Furthermore, we shall have the relation

$$\cot \omega = \frac{v}{U-u} = \frac{\tau^{1/2} \sin \theta}{T^{1/2} - \tau^{1/2} \cos \theta} \quad \Big\} \; 50.$$

Lastly, we shall have

$$\left(\frac{dy}{dx}\right)_{\text{along the shock}} = \tan \omega \quad \Big\} \; 51.$$

From these five equations, we should be able to find any one of the six variables as a function of any other variable such as x as a function of y and thus determine the shape of the shock wave.

However, the analysis is quite complex and has not been carried out, but it is comparatively easy to find an expression for the radius of curvature of the shock directly in front of the projectile.

The expression is

$$\frac{R}{X_0} = \left[\frac{\gamma-1}{\gamma+1}\,\frac{1}{\tau_0} - 1\right]\left[1 - \frac{14}{5}\,\frac{F_2(\tau_0)}{F_1(\tau_0)}\right]$$

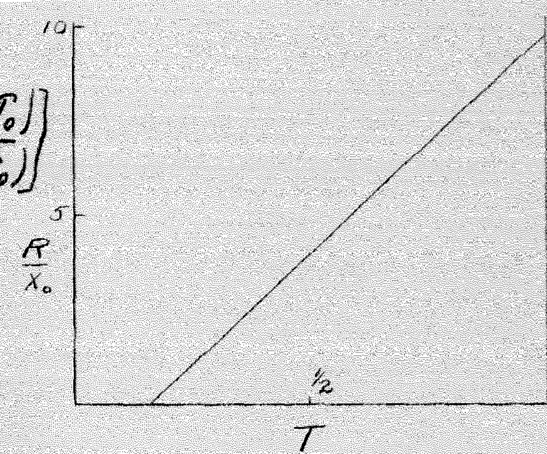


Fig. 18

where R is the radius of curvature, X_0 is the distance from the point of the projectile to the shock and τ_0 is the value of τ just behind the center of the shock. As mentioned above, we have the relation $Uu = w^{*2}$ just behind the center of the shock. This leads to the relation

$$\tau_o = \left(\frac{\gamma - 1}{\gamma + 1}\right)^2 \frac{1}{T} \qquad \text{where} \qquad T = \frac{U_2^2}{g_{max}}$$

In figure 18, $\frac{R}{x_o}$ has been plotted as a function of T .
No experimental work has been found as yet which might be used to check these predictions.