THE ROTATIONAL MOTION OF AN IDEAL FLUID
AND APPLICATION TO THE
THREE-DIMENSIONAL FLOW THROUGH AXIAL TURBOMACHINERY

Thesis by

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SUMMARY

The present paper discusses the principles and applications of an iteration method for solving certain problems involving rotational motion of an ideal fluid, such as occur in the presence of heat transfer, combustion, mechanical work processes, and non-uniform shock waves. The iteration process linearizes the essentially non-linear equations for rotational fluid motion by assuming a process for the vorticity transport, namely: the n^{th} approximation is linearized by assuming the vorticity to be transported by the $n\text{-}1^{th}$ velocity field. In some important cases, the first order solutions seem to offer considerable accuracy.

Two applications of the procedure are discussed in some detail, namely: 1) The process of straightening a non-uniform flow in a two-dimensional parallel-wall channel by means of a screen and 2) The three-dimensional flow in a multistage axial turbomachine having an infinite number of blades in each blade row. The second of these, the three-dimensional flow through a turbomachine, is given detailed analysis bearing some analogy to the Prandtl theory of finite wings. The results for the first order solution of velocity and enthalpy distributions are given explicitly and are shown to be defined by four relatively simple integrals. The cases of rotating and stationary single blade rows are evaluated completely. The general iteration process for obtaining higher approximations, utilizing the method of Green's functions, is given

in some detail.

The calculation of the flow field generated by a blade row of given geometry is illustrated by the problem of a "vortex" turbomachine operating off the design condition. The problem is found to be essentially non-linear in some respects, especially as to the approach to periodic solutions for a succession of similar stages.

I INTRODUCTION

The motion of a fluid with continuously distributed vorticity has been studied extensively only in the case of viscous fluids. The reason for this is twofold: viscosity is the most common mechanism for generating vorticity, and viscous problems often allow simplifications to be made in the mathematical procedure through deletion of minor inertia terms. The rotational motion may be generated, however, by means other than viscosity, principally through the change of enthalpy during processes involving combustion, heat transfer, and mechanical work, or through the development of an entropy gradient by means of a shock wave or heat transfer process. There is, of course, a similarity between the action of a boundary layer and a shock wave in generating the entropy gradient inasmuch as both processes are predominantly governed by viscous action. But there is also the important difference that in the case of the shock wave the rotation imparted by viscosity extends beyond the region where viscous and inertia forces are of the same order. That is, the rotation persists downstream of a shock wave of non-uniform strength while the rotation generated in a boundary layer becomes, by definition, a permanent part of this layer. The processes of heat and work addition likewise generate a rotational motion which passes beyond the range of the process itself. Thus these rotational motions of an ideal fluid possess a real significance apart from the methods of developing the rotation.

The importance of vorticity in a given problem depends essentially upon both its magnitude and distribution which in turn depend on the final solution of the problem. That is, the problem is an essentially non-linear one. It is usual, also, that the rate at which vorticity is actually generated in a given case depends upon the interaction of the fluid with a fixed set of boundaries and hence again upon the final solution. Thus the problem involves non-linearity in two distinct manners: 1) The mechanism of the vorticity transport is not known until the solution is known and 2) The rate at which vorticity is generated by fixed boundaries is not known until the solution is known. Although the lack of knowledge concerning the distribution of the vorticity is the underlying difficulty in both cases, their physical genesis is sufficiently different to justify their separation.

The process of linearization which naturally suggests itself is the a priori assumption of distribution and strength of vorticity and of the interaction between the fluid and its boundaries. A well known special example of this assumption is that of irrotational flow where the distribution of vorticity is assumed such as to vanish everywhere. With this kinematical relation, the continuity equation may be simplified so that a solution is easily achieved in many cases. This simplification may take the form of a velocity potential or may be introduced directly to obtain a linear homogeneous partial differential equation for each of the velocity components. If instead it is assumed that the vorticity components are known functions of the coordinates, the resulting equation for each velocity component is non-homogeneous and its solution presents only slightly more difficulty.

The choice of transport mechanism for the vorticity presents a real difficulty and indeed no real simplification has been accomplished if this choice can not be made with reasonable facility. The cases of parallel and cylindrical shear flow furnish clear cut examples of where the vorticity distribution may be stated explicitly and exactly. In this case the vorticity has only the component normal to the plane of the flow and its magnitude depends only on the distance measured normal to the flow. The particularly simple case where the rotation is constant throughout the entire plane has been advantageously by Tsien (Ref. 1) in computing the flow about symmetrical Joukowski profiles in a shear flow. This process appears, then, as a special case of the general procedure described here. Other cases where the distribution of vorticity is not known precisely but where the strength is known and the position may be estimated accurately are exemplified by the theory of airfoils of finite span and the theory of lightly loaded propellers. In each of these cases the vorticity is assumed to be transported by the mean fluid motion or, in the case of a moving body, is assumed to remain at the spot where it is created.

Determining the rate at which the vorticity is generated at the boundary may, however, occasion difficulties as severe as the determination of the transport mechanism inasmuch as the former determines in a large measure the ultimate strength of the distributed vorticity. In cases where the physics of the situation allows the stipulation of the vorticity variation at a boundary as well as the position of the boundary, the problem is quite simple and is illustrated by the first problem of Prandtl's wing theory. Here the distribution of the trailing vortex strength is known at its origin and it is assumed to be transported by the free stream velocity. Furthermore the position of the origin may be assumed known with sufficient accuracy for most problems although some difficulty may actually arise in computing the "downwash" velocities close to the wing. The importance of the restriction, that the boundary where the vorticity is generated be known, is illustrated more forcefully by the flow near a shock wave of non-uniform strength. Here the position and hence the strength of the shock (source of vorticity) is not known until the flow itself is known and approximation is exceedingly difficult. The more usual circumstance is, however, that the position of the boundary is known but that the strength of the vorticity generation is not known in advance. This may be illustrated classically by the second problem of Prandtl's wing theory where the geometry is known but the coupling between the geometry and the lift distribution depends upon the ultimate flow and thus complicates the direct solution.

The solution of flow processes of this non-linear type lend themselves naturally to iteration processes where in each approximation more accurate knowledge of actual vorticity transport process and the boundary interactions are employed on the basis of the previous approximation. Although the essential idea is straightforward, the mathematical procedures may become severly involved unless the iteration process is arranged with some care. Actually the possibility of generating iteration processes with a reasonably rapid convergence imposes a severe restriction on the cases which may be treated in this manner. The possible solutions are still, however, of a sufficiently great variety to possess important applications.

To complete a solution in the manner outlined, it is necessary to make logical physical or mathematical assumptions concerning the mechanism of vorticity transport and generation from the given physical data of the problem. In Chapter III, therefore, the general processes of vorticity generation and transport are discussed from a fluid mechanic and thermodynamic point of view. To illustrate the method of setting up the iteration procedure, a simple but physically significant application is discussed in Chapter IV, namely: the process of straightening an irregular velocity profile in a parallel-walled channel by means of an idealized screen.

A very striking example where the process described may be applied to provide important results is furnished by the axial turbomachine where the axial turbomachine is defined as an axially symmetric device which changes the state of the fluid through the variation of its angular momentum about the axis by means of stationary and rotating blade rows and in which the velocities are predominantly parallel to the axis. Inasmuch as the work added by the rotating blade rows is generally non-uniform, the flow is rotational and usually to such a degree as to be one of the major influences of the problem. The difficulties encountered in calculating the three-dimensional flow in turbomachines have caused a quite general adherence to the special cases in which the work is added uniformly over the radius. Then the moment of angular momentum is constant in any plane normal to the axis of symmetry and only small radial or axial velocity disturbances arise which result from fluid density adjustments. In this case the circulation imparted by each blade row is constant with radius and thus the circulation about each blade of a given row is constant along its span. Although this inflexibility presents unfavorable aerodynamic conditions at the blade root and the blade tip, these conditions need not become serious until the tip speed and the tip diameter become items of importance. This is the case in many modern applications, especially in aeronautics, and as a consequence variations of circulation along the blade length must be introduced. The rotational flow which accompanies this spanwise variation of circulation creates problems which bear similarity to the problem of finite airfoils and their solution constitutes a generalization of the three-dimensional theory of wings. The assumptions which suffice for the theory of finite wings are clearly inadequate here because the vorticity is materially transported by its own "induced" velocity. So long as these induced velocities are primarily in the tangential direction, the influence of the tangential transport may be removed by assuming an infinite number of blades in each row. With this assumption, the tangential transport of fluid elements brings them into fields which are precisely the same as those from which they left and hence has no influence on the fluid state. Furthermore in most practical cases the radial and axial velocity disturbances do not accumulate to any great proportions in comparison with the axial velocity and hence the vorticity may be assumed to be transported with the mean axial velocity. The boundary conditions at the blade row, namely the distribution of vorticity generated and the axial velocity may be determined from the desired blade loading or by an iteration process involving the blade geometry and the overall operating conditions. There are thus seen to be two main problems in the aerodynamics of turbomachine blading which bear a direct analogy to the two wing problems of Prandtl. They are then

1. Given the blade loading or the "bound vorticity", calculate the three dimensional velocity distribution, the blade shape, lift coefficients, etc.

2. Given the blade shape and the turbomachine operating conditions, calculate the velocity distribution, enthalpy distribution, blade loading, etc.

The general problem of the axial turbomachine is discussed at some length in Parts IV through VIII and in particular the first of the problems enumerated above.

II THE PRINCIPLES OF ROTATIONAL FLUID MOTION

The Space Distribution of Vorticity.- The variation in magnitude and orientation of the vorticity of a fluid medium is related to the state of the fluid through the Euler equation

$$(\bar{V}\circ\nabla)\bar{V} + \frac{1}{\rho}\operatorname{grad} p = \bar{F} \qquad 1.$$

where $\bar{V}$ is the vector velocity, p and ρ the fluid pressure and density respectively, and $\bar{F}$ is an arbitrary vector force per unit mass. By means of the identity $(\bar{V}\circ\nabla)\bar{V} = \frac{1}{2}\operatorname{grad} V^2 - \bar{V}\times\operatorname{curl}\bar{V}$ equation 1 may be rewritten to include the vorticity explicitly

$$\bar{V}\times\operatorname{curl}\bar{V} = \operatorname{grad}\frac{V^2}{2} + \frac{1}{\rho}\operatorname{grad} p - \bar{F} \qquad 2.$$

The dependence of the vorticity upon the thermodynamic state of the fluid is clarified by introducing the enthalpy h' defined as

$$h' = E + p/\rho \qquad 3.$$

and the first law of thermodynamics

$$T\,dS = dh' - dp/\rho \qquad 4.$$

where T is the temperature absolute, S the entropy and E the internal energy of the gas. By entering equations 3 and 4 into equation 2 and denoting the stagnation enthalpy $h' + \frac{V^2}{2}$ by h, the vorticity is expressed in terms of the thermodynamic variables

$$\bar{V}\times\operatorname{curl}\bar{V} = \operatorname{grad} h - T\operatorname{grad} S - \bar{F} \qquad 5.$$

where several special forms of this relation are well known. If the stagnation enthalpy is constant throughout the field and the applied forces vanish, equation 5 reduces to

$$\bar{V} \times \operatorname{curl} \bar{V} = -T \operatorname{grad} S \tag{6.}$$

which was first given explicitly by Crocco (ref. 2) and is of particular importance in the study of the non-uniform shock-wave. When the applied forces are conservative so that they may be expressed as the gradient of a potential Φ, equation 5 reduces to

$$\bar{V} \times \operatorname{curl} \bar{V} = \operatorname{grad}(h - \Phi) - T \operatorname{grad} S \tag{7.}$$

an integral of which is the well known theorem of Bjerknes. If both the enthalpy and the entropy remain constant over the field, equation 5 reduces to

$$\bar{V} \times \operatorname{curl} \bar{V} = -\bar{F} \tag{8.}$$

which is in essence a generalization of the Kutta-Joukowski theorem. In any case for which the right side of equation 5 does not vanish, the vorticity must differ from zero and the vector

$$\Lambda = \operatorname{grad} h - T \operatorname{grad} S - \bar{F}$$

is normal to the plane formed by the vorticity and the velocity vectors.

<u>The Temporal Variation of Vorticity</u>.- Information of a more nearly kinematical nature may be obtained by taking the curl of equation 5,

$$\operatorname{curl}\left(\bar{V} \times \operatorname{curl} \bar{V}\right) = \operatorname{curl} \operatorname{grad} h - \operatorname{curl}\left(T \operatorname{grad} S\right) - \operatorname{curl} \bar{F}$$

The curl of any gradient vanishes and the curl of the cross-product may be expanded and simplified through use of the continuity relation $\nabla \cdot (\rho \bar{V}) = 0$ to give

$$(\bar{V} \cdot \nabla) \frac{\bar{\omega}}{\rho} = \frac{d}{dt} \frac{\bar{\omega}}{\rho} = \left(\frac{\bar{\omega}}{\rho} \cdot \nabla\right) \bar{V} - \operatorname{grad} T \times \operatorname{grad} S - \operatorname{curl} \bar{F} \tag{9.}$$

where the various terms have simple physical interpretations.

The physical significance of the $\operatorname{curl} \bar{F}$ is clear inasmuch as any set of parallel forces generates vorticity according to their space variation, an example of which is the non-uniformly loaded lifting line. The meaning of the term $\left(\frac{\bar{\omega}}{\rho} \circ \nabla\right) \bar{V}$ is clarified by considering the velocity and vorticity vectors at a point of an incompressible fluid, and by inquiring how a particular vorticity component (fig. 1) is changed by the fluid motion. The change is accomplished in two manners: by extending the vorticity component in the direction under consideration and by rotating the other two components slightly in this direction. (Such an infinitesimal rotation had no influence on the projection of the original vorticity in its own direction.) Then a path enclosing an area A_x is chosen in the y-z plane sufficiently small that $\Gamma = \xi A_x$. If the circulation remains constant, the value of ξ may vary only through changes of the area enclosed by the path, that is, by acceleration in the x direction.

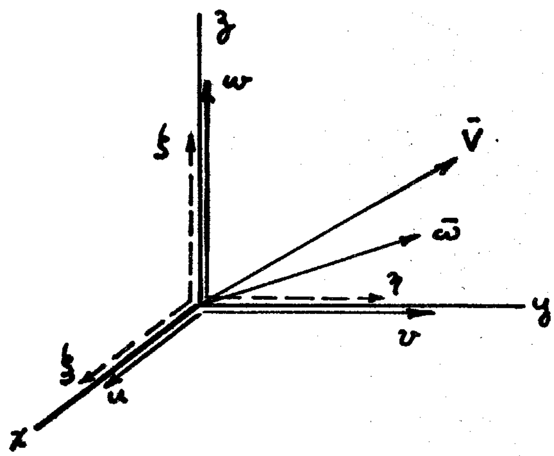

Figure 1.- Notation for vorticity and velocity components.

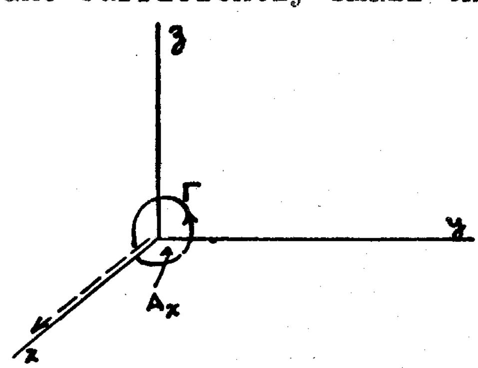

Figure 2.- Relation of vorticity and circulation.

Then

$$\Delta_1 \xi = \xi \frac{\Delta u}{u} = \xi A_x \Delta\left(\frac{1}{A_x}\right)$$

But by continuity $\Delta(u A_x) = 0$ so that

$$\Delta\left(\frac{1}{A_x}\right) = \frac{\Delta u}{u A_x}$$

and then

$$\Delta_1 \xi = \xi \frac{\Delta u}{u} = \xi \frac{\partial u}{\partial x} \Delta t$$

The variation of ξ due to the rotation of the other vorticity components in the x direction is expressed as

$$\Delta_2 \xi = \eta \beta \Delta t$$

But the angle through which the vorticity vector is turned in unit time is clearly $\beta = \frac{\partial u}{\partial y}$ so that

$$\Delta_2 \xi = \eta \frac{\partial u}{\partial y} \Delta t$$

and likewise for the component in the z direction

$$\Delta_3 \xi = \zeta \frac{\partial u}{\partial z} \Delta t$$

Thus the total time variation of the x component of vorticity becomes

$$\frac{d\xi}{dt} = \xi \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} + \zeta \frac{\partial u}{\partial z}$$

with similar expressions for the other two components. In vector form the relation becomes

$$\frac{d\bar{\omega}}{dt} = (\bar{\omega} \circ \nabla)\bar{v}$$

or in the case of variable density

$$\frac{d}{dt}\left(\frac{\bar{\omega}}{\rho}\right) = \left(\frac{\bar{\omega}}{\rho} \circ \nabla\right)\bar{v}$$

Thus the term in question represents the <u>kinematical variation of the vorticity components due to the deformation of the fluid element</u>.

The influence of the term $\text{grad}\,T \times \text{grad}\,S$ may be illustrated by an example. When a fluid remains stationary in the vicinity of a heated plate, the temperature and entropy gradients are parallel and, as is intuitively obvicus, no vorticity is generated. Now let the fluid be accelerated in the x direction by means of an isentropic expansion, so that the temperature gradient changes its direction (fig. 3) while the entropy gradient remains essentially unchanged. Then inasmuch as the term $\text{grad}\,T \times \text{grad}\,S$ does not vanish, equation 9 indicates that vorticity is being generated. The physical basis for this phenomenon is then quite clear; for since each fluid element expands along

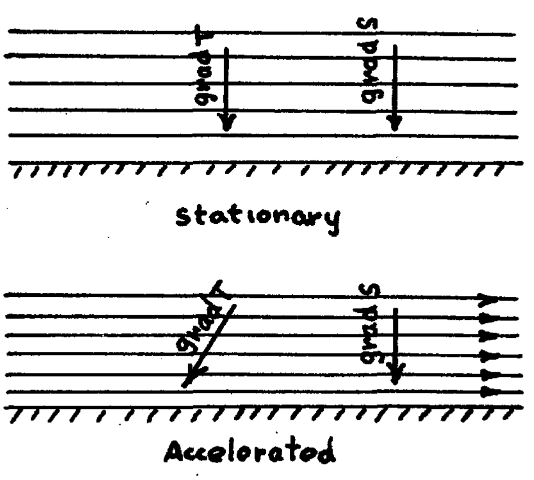

Figure 3.- Temperature and Entropy Gradients in Stationary and accelerated Fluid.

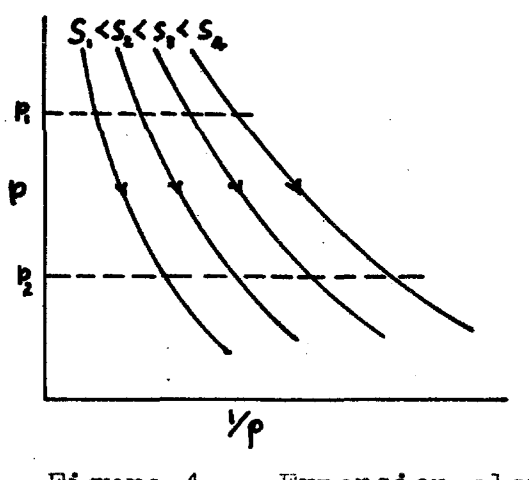

Figure 4.- Expansion along various isentropes.

its own isentrope (fig. 4), the lamina near the wall attain greater specific volume and accelerate more rapidly. Through this generation of vorticity, the velocity profile undergoes changes as shown in figure 5. It is by this process, for example, that the vorticity in a boundary layer is actually reduced by a strong negative pressure gradient. The entropy gradient in a boundary layer is directed toward the wall and, in the same manner as before, the fluid of higher entropy is accelerated more rapidly than that of low entropy so that the vorticity is reduced.

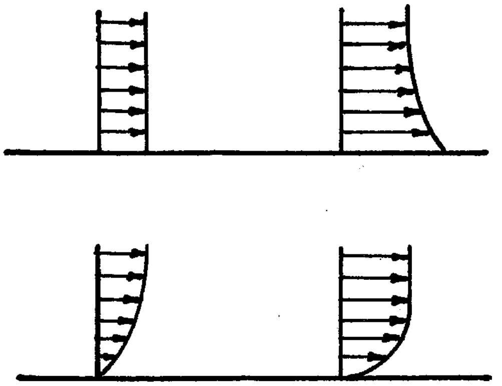

Figure 5.- Vorticity variations caused by acceleration.

That such a change of vorticity should take place under an expansion is quite clear, for any parallel flow of constant enthalpy becomes irrotational if expanded to its maximal velocity.

III FLOW THROUGH AN IDEALIZED RESISTANCE

The following application of the iteration procedure for solving certain problems of rotational flow involves all of the characteristic difficulties but retains essential simplicity and therefore serves as a useful illustration. The problem is that of the flow of an ideal incompressible fluid in a two-dimensional channel with parallel walls, where the fluid posessés a distorted profile which is straightened by an idealized screen. The corresponding mathematical problem is then as follows:

<u>The Mathematical Problem</u>.- Consider a fluid motion governed by the equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad 10a$$

$$v\zeta = -\frac{\partial h}{\partial x} + F_x \qquad 10b$$

$$-u\zeta = -\frac{\partial h}{\partial y} + F_y \qquad 10c$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad 10d$$

and subjected to the boundary conditions

$$v = 0 \quad ; \quad y = 0, \ell \qquad 11a$$

$$\left.\begin{aligned} v &= \frac{\partial v}{\partial x} = 0 \\ u &= u(-\infty, y) \\ h &= h(-\infty, y) \\ \zeta &= \zeta(-\infty, y) \end{aligned}\right\} x = -\infty \qquad 11b$$

$$v = \frac{\partial v}{\partial x} = 0 \quad ; \quad x = +\infty \qquad 11c$$

$$\left.\begin{aligned} u(0-,y) &= u(0+,y) \\ v(0-,y) &= v(0+,y) \\ h(0-,y) - h(0+,y) &= k(u^2+v^2) \end{aligned}\right\} x = 0 \qquad 11d$$

The geometrical arrangement and nomenclature is shown in figure 6.

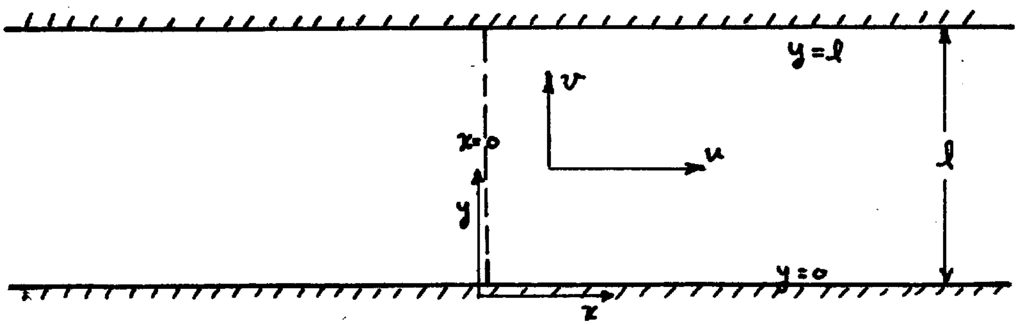

Figure 6.- Geometry and Nomenclature for flow through an idealized screen.

The Modified Integral-Differential Relations.- From equations 10a and 10b it follows that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial f}{\partial x} \qquad 12$$

which is valid as long as the differentiation indicated is possible. This may then be transformed into the related integral equation

$$v = \int_{-\infty}^{+\infty}\int_{0}^{l} \frac{\partial f}{\partial \alpha}\, G(\alpha,\beta;x,y)\, d\alpha\, d\beta \qquad 13$$

where α, β are the running coordinates corresponding to the x and y directions respectively, and the Green's function $G(\alpha,\beta;x,y)$ satisfies the boundary conditions 11a, 11b, 11c, on v and has a logarithmic singularity at $x=\alpha$; $y=\beta$. The Green's function may be written as the infinite series

$$G(\alpha,\beta;x,y) = \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin\frac{n\pi\beta}{l} \sin\frac{n\pi y}{l}\, e^{-\frac{n\pi}{l}|x-\alpha|} \qquad 14$$

The horizontal velocity component then follows directly from equation 10a, the continuity equation

$$u = \int_{-\infty}^{+\infty}\int_{0}^{l} \frac{\partial \zeta}{\partial \alpha} H(\alpha,\beta;x,y)\, d\alpha\, d\beta + Y(y) \qquad 15$$

where the new function $H(\alpha,\beta;x,y)$ is defined by the series

$$\left| H(\alpha,\beta;x,y) \right| = \left| \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin\frac{n\pi\beta}{l} \cos\frac{n\pi y}{l}\, e^{-\frac{n\pi}{l}|x-\alpha|} \right| \qquad 16$$

The drop in total head in passing the screen becomes now according to the boundary condition 11d

$$\Delta h = k(u^2+v^2) \qquad 17$$

so that if the enthalpy or total pressure is known just upstream of the screen, it is known downstream of the screen also.

The manner in which the enthalpy content is transported follows from the equations 10b and 10c. Upon elimination of the vorticity ζ, it follows that

$$v\left(\frac{\partial h}{\partial y} - F_y\right) + u\left(\frac{\partial h}{\partial x} - F_x\right) = 0$$

so that in the absence of external forces

$$\frac{\partial h}{\partial x} + \frac{v}{u}\frac{\partial h}{\partial y} = 0 \qquad 18$$

But if the variation of enthalpy along an arbitrary path in the fluid be considered, then

$$dh = \left(\frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\frac{dy}{dx}\right)dx$$

This agrees with equation 18 if the path along which the enthalpy variation is considered is a streamline, that is $\frac{dy}{dx} = \frac{v}{u}$, and if the variation of enthalpy along this path vanishes. Thus in the absence of external forces, the enthalpy is transported unchanged along the streamlines. This result is in agreement, of course, with equation 9 of Chapter II. Thus by calculation of the stream function

$$\psi = \int u\,dy - v\,dx \tag{19}$$

the distribution of enthalpy is known throughout the fluid. For the boundary condition at $x = -\infty$ associates an enthalpy value with each streamline and this association is maintained up to the screen. Now the jump of enthalpy across the screen is known and hence another boundary condition on the enthalpy is known just downstream of the screen. This enthalpy distribution now associates another value of enthalpy with each streamline, usually different from the first, and the association is maintained downstream of the screen. This condition may be expressed mathematically as follows:

$$\left.\begin{aligned} \psi(-\infty, y) &= g(y) \\ h(-\infty, y) &= \frac{p_0}{\rho} + \frac{U^2}{2} = \frac{p_0}{\rho} + \frac{1}{2} f^2(y) \end{aligned}\right\} \tag{20}$$

Therefore $h(-\infty, y) = \varphi[\psi(-\infty, y)]$ and inasmuch as the enthalpy is transported along the streamlines,

$$h(x, y) = \varphi[\psi(x, y)] \tag{21}$$

upstream of the screen. Immediately downstream of the screen the enthalpy may be computed as

$$h(0+,y) = h(0-,y) - k^2(u^2+v^2)$$

Then again $h(0+,y) = \bar{\varphi}\left[\psi(0+,y)\right]$ so that downstream of the screen

$$h(x,y) = \bar{\varphi}\left[\psi(x,y)\right] \qquad 22$$

The knowledge of the enthalpy distribution now allows the calculation of the vorticity distribution, for according to equation 10b

$$\zeta = \frac{1}{u}\frac{\partial h}{\partial y}$$

at points other than the screen. The x derivitive of this, however, is exactly the unknown function in the integral equations 13 and 14.

The Iteration Procedure.- It appears that the relations just developed allow the construction of an iteration process in the following manner

$$(h_1)_n - (h_2)_n = k\left(u_{n-1}^2 + v_{n-1}^2\right) \qquad 23a$$

$$\psi_n(x,y) = \int u_{n-1}\,dy - v_{n-1}\,dx \qquad 23b$$

$$\left.\begin{aligned} h_n(x,y) &= \varphi\left[\psi_{n-1}(x,y)\right] \\ &= \varphi\left[\psi_{n-1}(-\infty,y)\right] = h(-\infty,y) \\ &= \varphi\left[\psi_{n-1}(0+,y)\right] = h(0+,y) \end{aligned}\right\} \qquad 23c$$

$$\zeta_n = \frac{1}{u_{n-1}}\frac{\partial h_{n-1}}{\partial y} \qquad 23d$$

$$v_n = \int_{-\infty}^{+\infty}\int_{0}^{l} \left.\frac{\partial \zeta}{\partial \alpha}\right)_n G(\alpha,\beta;x,y)\,d\alpha\,d\beta \qquad 23e$$

$$u_n = \int_{-\infty}^{+\infty}\int_{0}^{l} \left.\frac{\partial \zeta}{\partial \alpha}\right)_n H(\alpha,\beta;x,y)\,d\alpha\,d\beta + \gamma_n(y) \qquad 23f$$

The zeroth approximation is then essentially one which satisfies the boundary conditions on u and v and in which the pressure drop across the screen vanishes; that is

$$(h_1)_0 - (h_2)_0 = 0 \qquad 24a$$

$$\psi_0(x,y) = \text{const.} \qquad 24b$$

$$h_0(x,y) = h(-\infty,y) \qquad 24c$$

$$\zeta_0(x,y) = \zeta(-\infty,y) \qquad 24d$$

$$v_0(x,y) = 0 \qquad 24e$$

$$u_0(x,y) = u(-\infty,y) \qquad 24f$$

The first approximation follows directly from the above values through the application of equations 23:

$$(h_0)_1 - (h_0)_1 = k\,(U_0^2) \qquad 25a$$

$$\psi_1(x,y) = U_0 y \qquad 25b$$

$$h_1(x,y) = \begin{cases} h_0(x,y) \;; & x<0 \\ h_0(x,y) - k(U_0^2) \;; & x>0 \end{cases} \qquad 25c$$

$$\zeta_1(x,y) = \begin{cases} \zeta_0(x,y) \;; & x<0 \\ \zeta_0(x,y) - 2k\dfrac{\partial U_0}{\partial y} \;; & x>0 \end{cases} \qquad 25d$$

$$v_1(x,y) = \int_0^l [\zeta_1]_{\alpha=0}\, G(\alpha,\beta;x,y)\, d\beta \qquad 25e$$

$$u_1(x,y) = \int_0^l [\zeta]_{\alpha=0}\, H(\alpha,\beta;x,y)\, d\beta + r(y) \qquad 25f$$

The higher approximations follow in a direct manner, each involving progressively more calculation. By virtue of the Green's function solution, much of the work may be carried out numerically once the Green's functions have been computed.

The first approximation constitutes the solution linearized by the assumption that the vorticity is transported by the velocity given at $x = -\infty$, that is, by the velocity corresponding to the zeroth approximation. In general, then, <u>the nth approximation constitutes a solution linearized by the assumption that the vorticity is transported by the velocity field corresponding to the n-1 th approximation</u>.

This property is characteristic of the present method. The first approximation remains somewhat arbitrary, and intuitively, the rapidity of convergence and perhaps even the convergence itself, depends upon the choice of the zeroth approximation. In many cases it will suffice to choose as the zeroth approximation, the potential flow corresponding to the same boundary conditions.

IV FLOW THROUGH AN AXIAL TURBOMACHINE

Using the example of the last chapter as a model, it is now possible to set up the solution for the flow through a multistage axial turbomachine. The axial turbomachine will be considered (fig. 7) to have inner and outer boundaries consisting of coaxial circular cylinders and to possess consecutive rows of either stationary or rotating airfoil-shaped blades, the blades in any one row being identical. It is the function of the moving blades to add energy to the fluid by means of the forces exerted by the blades on the fluid; and it is the function of the stationary blades to direct the air in such a manner, and by means of a similar set of forces, as to allow the most favorable action of the moving blades. In modern practice it is usual, with the possible exception of the first and last stages, that consecutive blade rows impart opposite angular momenta to the fluid.

<u>The Physical Problems</u>.- There are two basic problems involved in axial turbomachines which bear close analogy to two of the problems of Prandtl in the theory of wings of finite span. These are

1. Given the blade loading or the "bound vorticity", calculate the three-dimensional velocity distribution, the blade shape, lift coefficient, etc.

2. Given the blade shape and turbomachine operating conditions, calculate the velocity distribution, enthalpy rise, blade loading, etc.

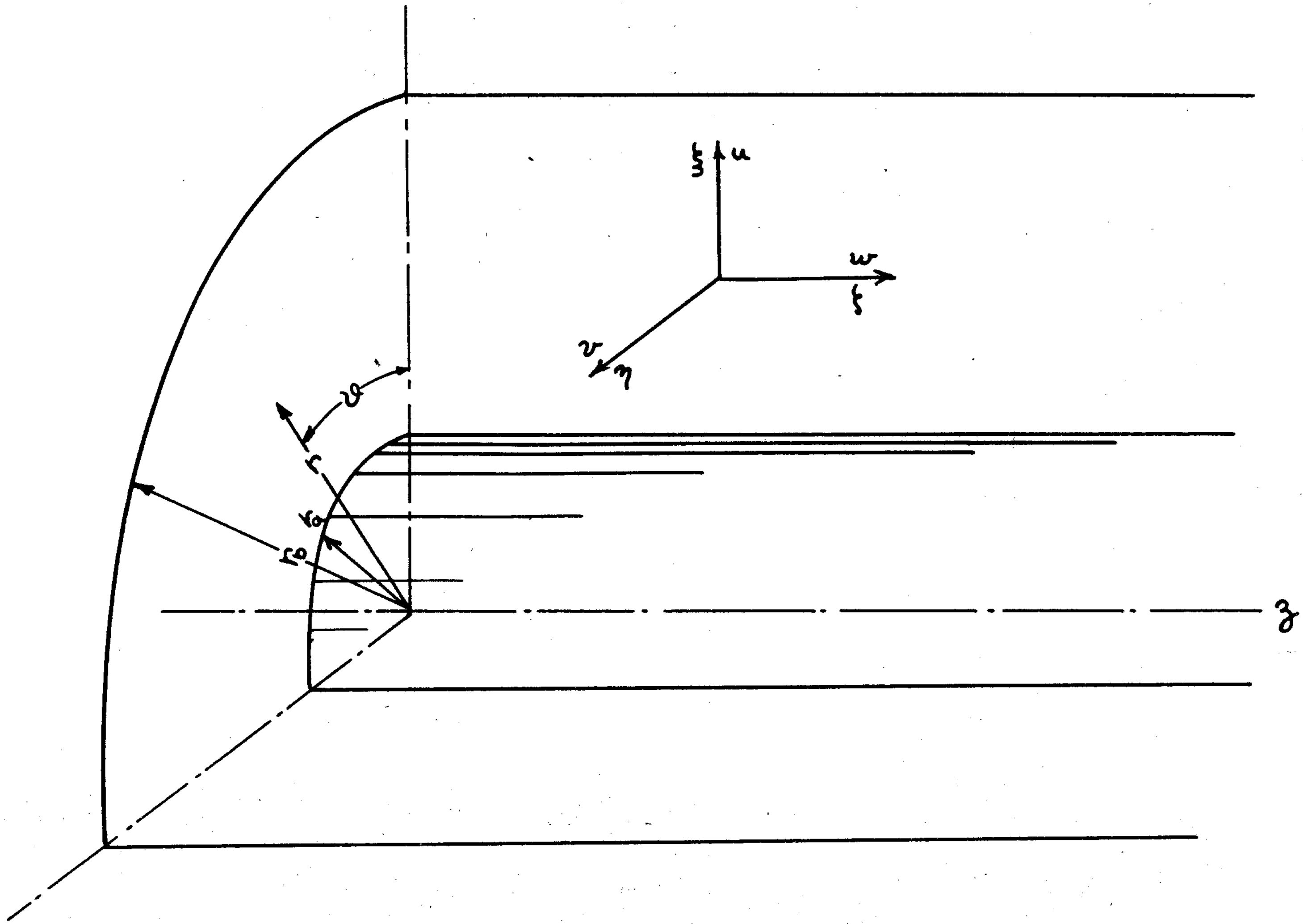

Figure 7. Coordinate system, velocity component, and vorticity component designations for flow between concentric circular cylinders.

The Mathematical Problem.- In setting up the corresponding mathematical problem it is assumed further that

1. All variations with respect to the angle about the axis vanish.

2. The applied force in the radial direction vanishes.

3. The fluid is non-viscous and incompressible.

The first assumption has the physical counterpart of an infinite number of blades in each blade row; the second, that the normal to the blade surface lies substantially tangent to the cylinder through the point; the third is explicit. It may now be asked whether the problem, thus simplified, makes physical sense. It is clear that it does make sense so long as the three-dimensional flow resulting from the variation along the blade of the rate at which work is added is the phenomenon of interest. For this the results should be substantially correct and the true approach to reality will of course depend upon how closely the above assumptions are fulfilled in any particular case.

The flow process for each of the problems is governed by:

The continuity equation

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \qquad \text{26a}$$

The equations of motion

$$v\zeta - w\eta = \frac{\partial h}{\partial r} \qquad \text{26b}$$

$$w\xi - u\zeta = -F_{\vartheta} \qquad \text{26c}$$

$$u\eta - v\xi = -F_z + \frac{\partial h}{\partial z} \qquad 26d$$

The definition of the vorticity components

$$\xi = -\frac{\partial v}{\partial z} \qquad 26e$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \qquad 26f$$

$$\zeta = \frac{\partial v}{\partial r} + \frac{v}{r} \qquad 26g$$

The definition of the stagnation enthalpy for an incompressible fluid

$$h = \frac{p}{\rho} + \frac{1}{2}(u^2+v^2+w^2)$$

The boundary conditions for the two problems differ only at the blade surfaces and these will be treated separately. The boundary conditions common to the two problems are the following

$$u = 0 ; \qquad r = r_a, r_b \qquad 27a$$

$$\left.\begin{aligned} u &= \frac{\partial u}{\partial z} = 0 \\ v &= v(r,-\infty) \\ w &= w(r,-\infty) \neq 0 \\ \xi &= 0 \\ \eta &= \eta(r,-\infty) \\ \zeta &= \zeta(r,-\infty) \\ h &= h(r,-\infty) \\ p &= p(r,-\infty) \end{aligned}\right\} \quad z = -\infty \qquad 27b$$

$$u = \frac{\partial u}{\partial z} = 0 \qquad z = +\infty \qquad 27c$$

In the case of an infinite number of blades, that is, where the forces have uniform tangential distribution, a simple relation exists between the blade geometry, the direction of the velocity, and the direction of the blade force at each point of the blade row. This relation is: the force shall be normal to the blade surface and the velocity shall be tangential to the blade surface. Hence the velocities v and w may be considered as the local velocity components or as the direction numbers of the blade surface with respect to axes fixed in the blade.

For the stationary blade the condition that the force be normal to the blade surface is

$$w F_z + v F_\vartheta = 0$$

If the blade row moves with a tangential velocity ωr, the condition is modified to

$$w F_z + (v - \omega r) F_\vartheta = 0$$

If it be required that the axial and radial velocity components remain continuous throughout the fluid field (admitting discontinuities in only the tangential velocity) the boundary conditions at the blades become

$$(v - \omega r) F_\vartheta + w F_z = 0$$

$$\left(\omega = \text{angular velocity of the blade row}\right) \qquad 27d$$

$$u, w \text{ continuous}$$

<u>The Integral-Differential Relations for the First Problem</u>.- To proceed with the solution of the first problem, the partial differential equations 26a and 26f are transformed into a pair

of integral equations. By differentiating 26a with respect to r , 26f with respect to z and adding, there results the non-homogeneous partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\frac{u}{r} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial \eta}{\partial z} \qquad 28$$

which may be transformed into the integral equation

$$u = \int_{-\infty}^{+\infty}\int_{r_a}^{r_b} \frac{\partial \eta}{\partial \beta} G(\alpha, \beta; r, z)\, d\alpha\, d\beta \qquad 29$$

for the radial velocity. From equation 29 and the continuity equation 26a follows the related integral equation for the axial velocity

$$w = \int_{-\infty}^{+\infty}\int_{r_a}^{r_b} \frac{\partial \eta}{\partial \beta} H(\alpha, \beta; r, z)\, d\alpha\, d\beta \quad + \gamma \qquad 30$$

The Green's function $G(\alpha, \beta; r, z)$ satisfies the boundary conditions 27a, 27b, 27c on u and is a solution to the homogeneous equation related to equation 28. The function $H(\alpha, \beta; r, z)$ is derived simply from $G(\alpha, \beta; r, z)$ and γ , as yet unknown, must be determined so as to satisfy the conditions on w . The forms of the Green's functions will be determined later.

Inasmuch as the Stokes stream function depends only on the axial and radial velocities and is independent of the axially symmetric tangential velocity, it may be calculated directly from the values of the radial and axial velocities. Thus

$$\psi(r,z) = \int ru\,dz - rw\,dr \qquad 31$$

Because of the axial symmetry of the flow, the tangential velocity enjoys a certain degree of independence from the other velocity components and hence a simpler determination. By substituting from equations 26e and 26g the values of ξ and ζ into the equation for tangential equilibrium, 26c, it follows that

$$\frac{u}{w}\zeta - \xi = \frac{1}{w}F_\vartheta$$

or

$$\frac{u}{w}\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) + \frac{\partial v}{\partial z} = \frac{1}{w}F_\vartheta$$

Multiplying by the radius r, this relation assumes the form

$$\frac{u}{w}\frac{\partial}{\partial r}(vr) + \frac{\partial}{\partial z}(vr) = \frac{rF_\vartheta}{w} \qquad 32$$

But the total variation of vr may be written

$$d(vr) = \left(\frac{\partial(vr)}{\partial r}\frac{dr}{dz} + \frac{\partial(vr)}{\partial z}\right)dz$$

where the left side of the equation 32 is recognized as the variation of vr in the direction $\frac{dr}{dz} = \frac{u}{w}$; that is, along a stream surface $\psi(r,z)$ constant. Then

$$\left.\frac{d(vr)}{dz}\right|_{\psi=\text{const.}} = \frac{rF_\vartheta}{w} \qquad 33$$

which states that the rate-of-change of angular momentum along the stream surface is equal to the moment of the applied force. The tangential velocity component then follows directly through integration of equation 33

$$v = v(r,-\infty) + \frac{1}{r}\int_{-\infty}^{+z} \frac{rF_\theta}{w}\,dz \qquad 34$$

where the integration is carried out along a stream surface.

The enthalpy distribution may be calculated by means of a similar device, for along any stream surface the change of stagnation enthalpy is

$$dh = \left(\frac{\partial h}{\partial r}\frac{dr}{dz} + \frac{\partial h}{\partial z}\right)dz$$

$$= \left(\frac{u}{w}\frac{\partial h}{\partial r} + \frac{\partial h}{\partial z}\right)dz$$

The values of $\frac{\partial h}{\partial r}$ and $\frac{\partial h}{\partial z}$ are given, however, by equations 26b and 26d. Substituting these into the above relation gives

$$dh = \left(\frac{vu}{w}\zeta - u\eta + u\eta - v\xi + F_z\right)dz$$

$$= \left(\frac{v}{w}(u\zeta - w\xi) + F_z\right)dz$$

so that

$$dh = \left(\frac{v}{w}F_\theta + F_z\right)dz \qquad 35$$

$$\psi = \text{constant}$$

The two force components F_θ and F_z are generated by the blades and therefore are not independent but are subject to the boundary conditions that their vector sum be normal to the blade surface and hence to the relative velocity. For a fixed blade row, condition 27d reduces to $vF_\vartheta + wF_z = 0$ so that

$$dh = \frac{1}{w}\left(vF_\vartheta + wF_z\right)dz = 0 \qquad 36$$

Thus it follows that <u>for forces imposed by a fixed blade row or in the absence of forces, the stagnation enthalpy remains constant along any stream surface</u>. When the blade row rotates with an angular velocity ω, however, the enthalpy variation along a stream surface is

$$dh = \frac{\omega r}{w} F_\vartheta \, dz \qquad 37$$

which is the intuitively obvious relation that the work done per unit mass is equal to the product of the torque and the angular velocity of the impeller. But the value of $\frac{rF_\vartheta}{w}$ is given by equation 33 and therefore

$$dh = \omega \left.\frac{d(rv)}{dz}\right|_{\psi = \mathrm{const.}} dz$$

Thus by integrating along a stream surface ψ = constant, the enthalpy distribution becomes

$$h = h(r, -\infty) + \int_{-\infty}^{z} \omega \frac{d(rv)}{dz} dz \qquad 38$$

where the expression is left in the integral form because the angular velocity ω is a discontinuous function of z.

Equations 26b and 26d have still not been employed independently of each other, hence equation 26b may be solved for η as

$$\eta = \frac{1}{w}\left(v\zeta - \frac{\partial h}{\partial r}\right)$$

and thus

$$\frac{\partial \eta}{\partial z} = \frac{1}{w}\frac{\partial}{\partial z}(v\zeta) - \frac{1}{w}\frac{\partial^2 h}{\partial r \partial z} + \frac{\eta}{w^2}\frac{\partial w}{\partial z}\frac{\partial h}{\partial r} \qquad 39$$

It is quite sufficient to leave the result in this form but it will prove advisable to introduce some modifications. Differentiating equation 26d with respect to r gives

$$\frac{\partial^2 h}{\partial r \partial z} = \frac{\partial}{\partial r}(u\eta - v\xi) + \frac{\partial}{\partial r}F_z \qquad 40$$

But from the boundary condition 27d and equation 26c

$$\frac{\partial F_z}{\partial r} = -\frac{\partial}{\partial r}\frac{1}{w}(v - \omega r)F_\theta$$

$$= \frac{\partial}{\partial r}\frac{1}{w}(v - \omega r)(w\xi - u\zeta)$$

$$\frac{\partial F_z}{\partial r} = \frac{\partial}{\partial r}(v - \omega r)\xi - \frac{\partial}{\partial r}\frac{u}{w}(v - \omega r)\zeta \qquad 41$$

Thus using 40 and 41, equation 39 becomes

$$\frac{\partial \eta}{\partial z} = \frac{1}{w}\left(\frac{\partial}{\partial z}(v\zeta) + \frac{\partial}{\partial r}(\omega r\xi)\right)$$

$$+\frac{1}{w}\frac{\partial}{\partial r}\frac{u}{w}(v - \omega r)\zeta - \frac{1}{w}\frac{\partial}{\partial r}u\eta + \frac{\eta}{w^2}\frac{\partial w}{\partial z}\frac{\partial h}{\partial r} \qquad 42$$

<u>Solution of First Problem by Iteration</u>.- The original set of differential equations 26 and the boundary conditions 27 has now been transformed into a set of integral and differential equations which are suitable as the basis of an iteration process. Collecting the transformed equations, the iteration process may be arranged in the following manner:

$$\left.\frac{\partial \eta}{\partial z}\right)_n = \frac{1}{w_{n-1}}\Bigg\{\frac{\partial}{\partial z}\, v\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)_{n-1} - \frac{\partial}{\partial r}\left(wr\,\frac{\partial v}{\partial z}\right)_{n-1}$$

$$+\frac{\partial}{\partial r}\left(\left(\frac{u}{w}\right)_{n-1}\left\{(v_{n-1}-wr)\xi_{n-1}-\eta_{n-1}w_{n-1}\right\}\right) \qquad 43a$$

$$+\left.\frac{\eta}{w}\right)_{n-1}\left.\frac{\partial w}{\partial z}\right)_{n-1}\left.\frac{\partial h}{\partial r}\right)_{n-1}\Bigg\}$$

$$u_n = \int_{-\infty}^{+\infty}\int_{r_a}^{r_b}\left.\frac{\partial \eta}{\partial \beta}\right)_n G(\alpha,\beta;r,z)\,d\alpha\,d\beta \qquad 43b$$

$$w_n = \int_{-\infty}^{+\infty}\int_{r_a}^{r_b}\left.\frac{\partial \eta}{\partial \beta}\right)_n H(\alpha,\beta;r,z)\,d\alpha\,d\beta + Y \qquad 43c$$

$$\xi_n = -\frac{\partial v_{n-1}}{\partial z} \qquad 43d$$

$$\eta_n = \frac{\partial u_{n-1}}{\partial z} - \frac{\partial w_{n-1}}{\partial r} \qquad 43e$$

$$\zeta_n = \frac{\partial v_{n-1}}{\partial r} + \frac{v_{n-1}}{r} \qquad 43f$$

$$\psi_n = \int r u_{n-1}\,dz - r w_{n-1}\,dr \qquad 43g$$

$$v_n = \frac{1}{r}\lambda\left(\psi_{n-1}(r,z)\right) + \frac{1}{r}\int_{-\infty \; \psi_{n-1} = \text{const.}}^{+z} \frac{r F_\vartheta}{w_{n-1}}\, dz \tag{43h}$$

$$\begin{aligned} h_n &= \varphi\left(\psi_{n-1}(r,z)\right) \quad ; \qquad z < \bar{z} \\ &= \varphi\left(\psi_{n-1}(r,z)\right) + \int_{\bar{z} \; \psi = \text{const.}}^{z} \omega \frac{d(rv)}{dz}\, dz \; ; \quad \bar{z} \le z \le \hat{z} \\ &= \hat{\varphi}\left(\psi_{n-1}(r,z)\right) \qquad \hat{z} < z \end{aligned} \tag{43i}$$

To illustrate the use of the iteration procedure, the basic, zeroth, and first approximations will be tabulated. To begin the process by obtaining the basic set of conditions, assume that the forces F_ϑ and F_z vanish. Then

$$\begin{aligned} \frac{\partial \eta}{\partial z} &= 0 \\ u &= 0 \\ w &= w(r, -\infty) \\ \xi &= 0 \\ \eta &= \eta(r, -\infty) \\ \zeta &= \zeta(r, -\infty) \\ \psi &= \text{———} \\ h &= h(r, -\infty \\ v &= v(r, -\infty) \end{aligned} \tag{44}$$

Likewise for the zeroth approximation, the application of the equations 43 gives, allowing the forces to act

$$
\begin{aligned}
\left.\frac{\partial \eta}{\partial z}\right)_0 &= 0 \\
u_0 &= 0 \\
w_0 &= w(r,-\infty) \\
\eta_0 &= \eta(r,-\infty) \\
\xi_0 &= 0 \\
\zeta_0 &= \zeta(r,-\infty) \\
\psi_0 &= -\int r\,w(r,-\infty)\,dr \qquad (45) \\
h_0 &= h(r,-\infty) \; ; \quad z < \bar{z} \\
&= h(r,-\infty) + \int_{\bar{z}}^{z} \frac{\omega r}{w(r,-\infty)} F_\theta \, dz \; ; \quad \bar{z} \le z \le \hat{z} \\
&= h(r,-\infty) + \int_{\bar{z}}^{\hat{z}} \frac{\omega r}{w(r,-\infty)} F_\theta \, dz \; ; \quad \hat{z} < z \\
v_0 &= v(r,-\infty) + \int_{-\infty}^{z} \frac{F_\theta}{w(r,-\infty)} \, dz
\end{aligned}
$$

The first approximatio.. then assumes form in all of the quantities:

$$\left.\frac{\partial \eta}{\partial z}\right)_1 = \frac{1}{w(r,-\infty)} \left\{ \frac{\partial}{\partial z}\left(v_0\left(\frac{\partial v_0}{\partial r} + \frac{v_0}{r}\right)\right) - \frac{\partial}{\partial r}\left(w r \frac{\partial v_0}{\partial z}\right) \right\}$$

$$u_1 = \int_{-\infty}^{\infty}\int_{r_a}^{r_b} \left.\frac{\partial \eta}{\partial z}\right)_1 G(\alpha,\beta;r,z)\, d\alpha\, d\beta$$

$$w_1 = \int_{-\infty}^{+\infty}\int_{r_a}^{r_b} \left.\frac{\partial \eta}{\partial z}\right)_1 H(\alpha,\beta;r,z)\, d\alpha\, d\beta + \gamma$$

$$\xi_1 = -\frac{\partial v_0}{\partial z}$$

$$\eta_1 = -\frac{\partial}{\partial r} w(r,-\infty) \tag*{46}$$

$$\zeta_1 = \frac{\partial v_0}{\partial r} + \frac{v_0}{r}$$

$$\psi_1 = -\int r\, w(r,-\infty)\, dr$$

$$h_1 = h_0$$

$$v_1 = v_0$$

From this point, the continuation of the iteration is straightforward and obvious but becomes very laborious as higher approximations are attempted. In the general case, numerical methods must be used, of course, and here the Green's functions perform the great simplification. For the Green's functions need be computed only once inasmuch as they do not change from one approximation to the other. The distribution function under the integral changes, but in a manner which may be computed easily.

The Second Problem.- The procedure for the second problem, that is the case in which the blade contour is given, differs only in a few details from that for the first problem. The main difference between the two is the rapidity of convergence; the approximation for the second problem must be carried essentially one iteration farther than that for the first problem to achieve a given accuracy.

The given data in the second problem is then

$$f(r,z) = \frac{F_z}{F_\theta} = \frac{v - \omega r}{w}$$

or the equivalent. That is, the slope of the surface of a typical blade is a known function of r and z. The tangential velocity may then be expressed in the form

$$v = \omega r + w f(r,z) \qquad 47$$

This relation is now used both as a determination of the tangential velocity and in the relation for determining the variation of stagnation enthalpy. Then equations 45h and 45i will be replaced by the equations

$$v_n = \frac{1}{r}\lambda(\psi_{n-1}(r,z))$$
outside blade row

43h'

$$v_n = \omega r + w_{n-1} f(r,z)$$
inside blade row

$$h_n = \varphi(\psi_{n-1}(r,z))$$
outside blade row

43i'

$$h_n = \varphi(\psi_{n-1}(r,z)) + \int_{\bar{z}}^{z}\left(2\omega^2 r + \frac{\partial}{\partial z} w_{n-1}\, r f(r,z)\right) dz$$
inside blade row

It is worth noting here that because of the difference in the definition of the tangential velocity inside and outside of a blade row, there will usually exist a discontinuity in the tangential velocity where the two regions join. This corresponds to a vortex sheet normal to the axis of symmetry and is accompanied by a discontinuity in the enthalpy when the blade row which the fluid enters is a moving row. It will be seen later that this vortex sheet, in the case of an infinite blade number, corresponds to the increase of vorticity distribution in the neighborhood of the leading edge of a thin airfoil normally observed when the angle of attack is increased.

V DETERMINATION OF THE GREEN'S FUNCTIONS

The Green's functions for the radial and axial velocity disturbances are determined by the two partial differential equations satisfied by the velocity components

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2} = \frac{\partial \eta}{\partial z} \qquad 28$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \qquad 26a$$

and by the boundary conditions imposed upon these velocity components

$$\left.\begin{aligned} u &= \frac{\partial u}{\partial z} = 0 \\ w &= w(r, -\infty) \end{aligned}\right\} \quad z = -\infty$$

$$u = 0 \qquad\qquad r = r_a ; r_b \qquad 27'$$

$$u = \frac{\partial u}{\partial z} = 0 \qquad\qquad z = \infty$$

The non-homogeneous equation 28 will be solved by considering first a discontinuous variation of η , so that the equation becomes homogeneous in the regions between the discontinuities, and then allowing the discontinuities to coalesce into a continuous variation. These discontinuities correspond to vortex sheets normal to the axis of symmetry and of strength which varies with the radius. This procedure is of particular interest not only as a means of obtaining an insight into the nature of the solution, but also to obtain relations describing the flow near these discontinuities. It will be remembered that such discontinuities can occur in the second problem at the leading edge of each blade row.

Solution of the Homogeneous Equation for the Radial Velocity Component.- The homogeneous partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2} = 0$$

has a general solution of the form

$$u = \sum_{n=1}^{\infty} [\bar{A}_n J_1(\epsilon_n r) + \bar{B}_n Y_1(\epsilon_n r)][a_n e^{\epsilon_n z} + b_n e^{-\epsilon_n z}]$$

where $J_1(\epsilon_n r)$ and $Y_1(\epsilon_n r)$ denote Bessel functions of the first order, of the first and second kinds respectively, and of real argument. The characteristic values ϵ_n must be determined to satisfy the conditions that the radial velocity vanish at the inner and outer radii of the passage. The constants $\bar{A}_n$, $\bar{B}_n$, a_n, b_n, are then available for satisfying the boundary conditions at $\pm\infty$ and at the discontinuities. Inasmuch as there are two radii at which the radial velocity must vanish, the problem of determining the characteristic values of ϵ_n is considerably simplified by writing the solution in the form

$$u = \sum_{n=1}^{\infty} U_1(\epsilon_n r)\left(A_n e^{\epsilon_n z} + B_n e^{-\epsilon_n z}\right) \qquad 47$$

where

$$U_1(\epsilon_n r) = J_1(\epsilon_n r)\, Y_1(\epsilon_n r_a) - J_1(\epsilon_n r_a)\, Y_1(\epsilon_n r) \qquad 48$$

and r_a and r_b are the inner and outer radii respectively of the region. Inasmuch as the value of $U_1(\epsilon_n r)$ is identically zero, the characteristic values of the problem are determined from the condition that the function vanish at the outer radius, that is

$$U_1(\epsilon_n r_b) = J_1(\epsilon_n r_b)\,Y_1(\epsilon_n r_a) - J_1(\epsilon_n r_a)\,Y_1(\epsilon_n r_b) = 0 \qquad 49$$

Approximate values for the first six roots of this equation are tabulated in reference 3.

Solution for a Finite Number of Discontinuities.- To complete the solution for a finite number of vortex sheets normal to the axis of symmetry, the distance along the z axis is divided into $m+1$ regions by m discontinuities occurring at $z = a_1, a_2, a_3, \cdots a_k, \cdots a_m$ where each vortex sheet corresponds to a local distribution of $\xi(r)$ and ξ vanishes elsewhere. Each vortex sheet will, in general, correspond to a discontinuity in the tangential vorticity component η, namely $[\eta]_k$. (See Figure 8.)

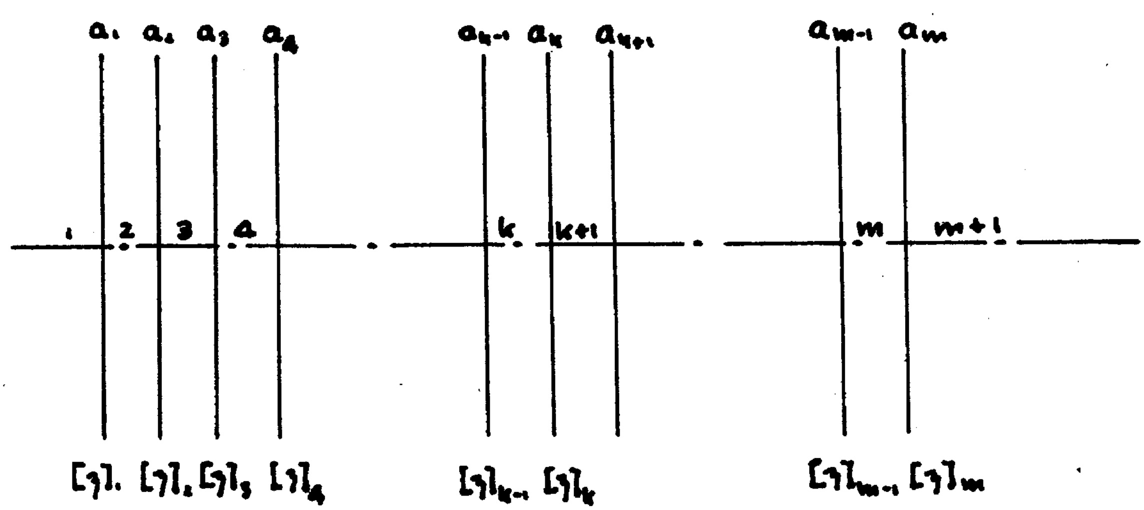

Figure 8.- Notation for the Finite Number of Discontinuities.

Thus a solution

$$u^k = \sum U_1(\varepsilon_n r)\left(A_n^k e^{\varepsilon_n z} + B_n^k e^{-\varepsilon_n z}\right)$$

holds for each of the $m+1$ regions. In addition to the known jump of the tangential vorticity across each of the discontinuities, the boundary conditions 27d state that the axial and tangential velocity components shall be continuous across each vortex sheet. Therefore the following three relations hold at the k^{th} vortex sheet:

$$u^k\Big]_{z=a_k} = u^{k+1}\Big]_{z=a_k} \tag{50a}$$

$$w^k\Big]_{z=a_k} = w^{k+1}\Big]_{z=a_k} \tag{50b}$$

$$[\eta]_k = \left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right]_k = \left[\frac{\partial u}{\partial z}\right]_{z=a_k} = \frac{\partial u^{k+1}}{\partial z}\Big)_{z=a_k} - \frac{\partial u^k}{\partial z}\Big)_{z=a_k} \tag{50c}$$

The vanishing of the term $\left[\frac{\partial w}{\partial r}\right]$ in the condition 50c is implied by the continuity of the axial velocity, 50b. Inasmuch as 50a is an identity for all radii, the following relation exists between the coefficients

$$A_n^{k+1} e^{\varepsilon_n a_k} + B_n^{k+1} e^{-\varepsilon_n a_k} = A_n^k e^{\varepsilon_n a_k} + B_n^k e^{-\varepsilon_n a_k} \tag{51}$$

Equation 50c provides a further relation between the coefficients

$$\sum U_1(\varepsilon_n r)\left\{\left(\varepsilon_n A_n^{k+1} e^{\varepsilon_n a_k} - \varepsilon_n A_n^k e^{\varepsilon_n a_k}\right) - \left(\varepsilon_n B_n^{k+1} e^{-\varepsilon_n a_k} - \varepsilon_n B_n^k e^{-\varepsilon_n a_k}\right)\right\} = [\eta]_{a_k} \tag{52}$$

whereupon substituting from equation 51 into 52, there results

$$\sum U_1(\varepsilon_n r)\left\{\varepsilon_n A_n^{k+1} e^{\varepsilon_n a_k} - \varepsilon_n A^k e^{\varepsilon_n a_k}\right\} = \frac{1}{2}[\eta]_{a_k} \tag{53}$$

If the quantity $[\eta]_{a_k}$ is treated as a known function of the radius, the expression for the constant term may be determined through the orthogonal properties of the Bessel functions $U_1(\varepsilon_n r)$. Thus if ν_n^2 is defined as the norm of the function $U_1(\varepsilon_n r)$,

$$\nu_n^2 = \int_{r_a}^{r_b} r U_1^2(\varepsilon_n r)\,dr = r_b^2 U_0^2(\varepsilon_n r_b) - r_a^2 U_0^2(\varepsilon_n r_a)$$

The difference equation between the coefficients A_n^k becomes then

$$A_n^{k+1} - A_n^k = \frac{e^{-\varepsilon_n a_k}}{2\nu_n^2 \varepsilon_n} \int_{r_a}^{r_b} r U_1(\varepsilon_n r)\,[\eta]_k\,dr \qquad 54$$

and likewise for the coefficients B_n^k,

$$B_n^{k+1} - B_n^k = \frac{-e^{\varepsilon_n a_k}}{2\nu_n^2 \varepsilon_n} \int_{r_a}^{r_b} r U_1(\varepsilon_n r)\,[\eta]_k\,dr \qquad 55$$

Equations 54 and 55 constitute recurrence relations for the constants A_n^k and B_n^k so that it is necessary to stipulate only one of each in order to determine the entire set. By inspection of equation 47, however, it is evident that for the radial velocity to remain finite at large distances upstream and downstream from the terminal discontinuities, $A_n^{m+1} \equiv B_n^1 = 0$. The coefficients A_n^k may then be tabulated:

$$A_n^{m+1} = 0$$

$$A_n^{m} = -\frac{e^{-\varepsilon_n a_m}}{2\gamma_n^2 \varepsilon_n}\int_{r_a}^{r_b} r U_1(\varepsilon_n r)\,[\eta]_m\,dr$$

$$A_n^{m-1} = -\frac{e^{-\varepsilon_n a_{m-1}}}{2\gamma_n^2 \varepsilon_n}\int_{r_a}^{r_b} r U_1(\varepsilon_n r)\,[\eta]_{m-1}\,dr - \frac{e^{-\varepsilon_n a_m}}{2\gamma_n \varepsilon_n}\int_{r_a}^{r_b} r U_1(\varepsilon_n r)\,[\eta]_{m-1}\,dr \qquad 56$$

$$A_n^{m-p} = -\sum_{i=0}^{p}\frac{e^{-\varepsilon_n a_{m-i}}}{2\gamma_n^2 \varepsilon_n}\int_{r_a}^{r_b} r U_1(\varepsilon_n r)[\eta]_{m-i}\,dr$$

Similarly the coefficients B_m^k may be tabulated:

$$B_n^1 = 0$$

$$B_n^2 = -\frac{e^{\varepsilon_n a_1}}{2\gamma_n^2 \varepsilon_n}\int_{r_a}^{r_b} r U_1(\varepsilon_n r)\,[\eta]_1\,dr$$

$$B_n^3 = -\frac{e^{\varepsilon_n a_1}}{2\gamma_n^2 \varepsilon_n}\int_{r_a}^{r_b} r U_1(\varepsilon_n r)\,[\eta]_1\,dr - \frac{e^{\varepsilon_n a_2}}{2\gamma_n^2 \varepsilon_n}\int_{r_a}^{r_b} r U_1(\varepsilon_n r)\,[\eta]_2\,dr \qquad 57$$

$$B_n^q = -\sum_{j=1}^{q}\frac{e^{\varepsilon_n a_{j-1}}}{2\gamma_n^2 \varepsilon_n}\int_{r_a}^{r_b} r U_1(\varepsilon_n r)\,[\eta]_{j-1}\,dr$$

Now if $[\eta]_i$ is assumed to be known or is calculated by the iteration process described previously, then equation 47 together with the constants given by 56 and 57 provides a complete solution for the radial velocity component.

The corresponding solution for the axial velocity is found by integrating equation 26a and by determining the arbitrary functions by condition 50a: that the axial velocity be continuous across each of the vortex sheets. Then

$$w = -\int \left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) dz + \gamma(r)$$

and by using the solution just obtained for the radial velocity component,

$$w^k = -\int_{-\infty}^{z} \sum_{n=1}^{\infty} \left(U_1'(\varepsilon_n r) + \frac{U_1(\varepsilon_n r)}{\varepsilon_n r}\right)\left(\frac{A_n^k}{\varepsilon_n} e^{\varepsilon_n z} + \frac{B_n^k}{\varepsilon_n} e^{-\varepsilon_n z}\right) dz + \gamma^k(r)$$

$$= -\sum_{n=1}^{\infty} U_0(\varepsilon_n r)\left(A_n^k e^{\varepsilon_n z} - B_n^k e^{-\varepsilon_n z}\right) + \gamma^k(r) \qquad 58$$

By equating axial velocities across the vortex sheet, $w^{k+1} = w^k$, the recursion formula for the function is obtained

$$\gamma^{k+1}(r) - \gamma^k(r) = \sum_{n=1}^{\infty} U_0(\varepsilon_n r)\left\{(A_n^k - A_n^{k+1})e^{\varepsilon_n a_n} - (B_n^k - B_n^{k+1})e^{-\varepsilon_n a_n}\right\} \qquad 59$$

where the A_n and B_n values are given by equations 56 and 57. Inasmuch as $B_n^1 = 0$, the value of $Y^1(r)$ is determined from the initial axial velocity far upstream, $Y^1(r) = w(r, -\infty)$. The functions of integration are tabulated below

$$Y^1(r) = w(r, -\infty)$$

$$Y^2(r) = w(r, -\infty) + \sum_{n=1}^{\infty} U_0(\varepsilon_n r)\left\{(A_n^2 - A_n^1)\, e^{\varepsilon_n a_1} - B_n^2\, e^{-\varepsilon_n a_1}\right\}$$

$$Y^3(r) = w(r, -\infty) + \sum_{n=1}^{\infty} U_0(\varepsilon_n r)\left\{(A_n^2 - A_n^1)\, e^{\varepsilon_n a_1} - B_n^2\, e^{-\varepsilon_n a_1}\right\} \qquad 60$$

$$\vdots \qquad + \sum_{n=1}^{\infty} U_0(\varepsilon_n r)\left\{(A_n^3 - A_n^2)\, e^{\varepsilon_n a_2} - (B_n^3 - B_n^2)\, e^{-\varepsilon_n a_2}\right\}$$

$$Y^k(r) = w(r, -\infty) + \sum_{j=1}^{k-1} \sum_{n=1}^{\infty} U_0(\varepsilon_n r)\left\{(A_n^{j+1} - A_n^j)\, e^{\varepsilon_n a_j} - (B_n^{j+1} - B_n^j)\, e^{-\varepsilon_n a_j}\right\}$$

Equations 58 and 60 therefore complete the solution for the axial velocity disturbances caused by a series of discontinuous changes in the tangential vorticity. The results for a finite number of discontinuities may be put in a more convenient form. If the values for the coefficients A_n^k and B_n^k are entered into equation 47, the solution for the radial velocity component becomes

$$u^k = -\sum_{n=1}^{\infty} U_1(\varepsilon_n r)\left\{\sum_{i=0}^{m-k} \frac{e^{-\varepsilon_n a_{m-i}}}{2\gamma_n^2 \varepsilon_n} \int_{r_a}^{r_b} e^{\varepsilon_n \mathfrak{z}}\, r\, U_1(\varepsilon_n r)\, [\eta]_{m-i}\, dr \right.$$

$$\left. + \sum_{j=1}^{k} \frac{e^{\varepsilon_n a_{j-1}}}{2\gamma_n^2 \varepsilon_n} \int_{r_a}^{r_b} e^{-\varepsilon_n \mathfrak{z}}\, r\, U_1(\varepsilon_n r)\, [\eta]_{j-1}\, dr \right\}$$

so that

$$u = -\sum_{i=0}^{m} \int_{r_a}^{r_b} \frac{[\eta]_i}{r} \left\{ \sum_{n=1}^{\infty} \frac{r\alpha\, U_1(\varepsilon_n r)\, U_1(\varepsilon_n \alpha)}{2\, \varepsilon_n\, \nu_n^2}\, e^{-\varepsilon_n |z - a_i|} \right\} d\alpha \qquad 61$$

Here the term in curly brackets represents a sort of Green's function except that one variable of summation has discrete instead of continuous values. The velocity pattern associated with each discontinuity is clearly symmetrical with respect to that discontinuity and the influence of each discontinuity dies off exponentially both upstream and downstream. It is also to be noted that the same expression holds for all values of k by virtue of writing the absolute value of the argument of the of the exponential.

The same process may be undertaken for the axial velocity to give

$$w^k - w(r,-\infty) = \sum_{j=0}^{m} \int_{r_a}^{r_b} [\eta]_j \left\{ \sum_{n=1}^{\infty} \frac{\alpha\, U_1(\varepsilon_n \alpha)\, U_0(\varepsilon_n r)}{2\, \varepsilon_n\, \nu_n^2}\, e^{-\varepsilon_n |z - a_j|} \right\} d\alpha$$

$$+ 2\sum_{j=0}^{m} \int_{r_a}^{r_b} [\eta]_j \left\{ \sum_{n=0}^{\infty} \frac{\alpha\, U_1(\varepsilon_n \alpha)\, U_0(\varepsilon_n r)}{2\, \varepsilon_n\, \nu_n^2} \left(1 - e^{-\varepsilon_n |z - a_j|}\right) \right\} d\alpha \qquad 62$$

<u>The Green's Function Solution for the Radial and Axial Velocities</u>.- In the forms 61 and 62, the expressions for the radial and axial velocities, terms in curly brackets exhibit, in each case, the nature of a Green's function. In this form the transition to a continuous distribution of vorticity be-

comes quite obvious. Thus $a_j \to \beta$ and becomes the running coordinate in the direction of the axis of symmetry. Likewise $[\eta]_j$ becomes $\frac{\partial \eta}{\partial \beta} d\beta$. Therefore the relations for the continuous distribution of bound vorticity (or blade forces) become

$$u = -\int_{-\infty}^{\infty}\int_{r_a}^{r_b} \frac{\partial \eta}{\partial \beta} G(\alpha,\beta;r,z)\, d\alpha\, d\beta \qquad 63$$

$$w - w(r,-\infty) = \int_{-\infty}^{\infty}\int_{r_a}^{r_b} \frac{\partial \eta}{\partial \beta} H(\alpha,\beta;r,z)\, d\alpha\, d\beta + 2\int_{-\infty}^{\infty}\int_{r_a}^{r_b} \frac{\partial \eta}{\partial \beta} F(\alpha,\beta;r,z)\, d\alpha\, d\beta \qquad 64$$

where $G(\alpha,\beta;r,z)$, $H(\alpha,\beta;r,z)$, and $F(\alpha,\beta;r,z)$ represent the Green's functions of the problem. These are given explicitly by the relations

$$G(\alpha,\beta;r,z) = \sum_{n=1}^{\infty} \frac{\alpha\, U_1(\varepsilon_n r)\, U_1(\varepsilon_n \alpha)}{2\,\nu_n^2\, \varepsilon_n}\, e^{-\varepsilon_n |z-\beta|} \qquad 65a$$

$$H(\alpha,\beta;r,z) = \sum_{n=1}^{\infty} \frac{\alpha\, U_1(\varepsilon_n \alpha)\, U_0(\varepsilon_n r)}{2\,\nu_n^2\, \varepsilon_n}\, e^{-\varepsilon_n |z-\beta|} \qquad 65b$$

$$F(\alpha,\beta;r,z) = \sum_{n=1}^{\infty} \frac{\alpha\, U_1(\varepsilon_n \alpha)\, U_0(\varepsilon_n r)}{2\,\nu_n^2\, \varepsilon_n}\left(1 - e^{-\varepsilon_n |z-\beta|}\right) \qquad 65c$$

VI SOLUTIONS IN THE FIRST APPROXIMATION

An inspection of the equations 43 involved in the iteration procedure and of equations 46 for the first approximation indicates that the errors involved in the first approximation are of the order $\frac{u}{w}$ as compared to unity if the distribution of velocity is substantially smooth, that is if the first derivatives of the velocities are of the same order as the total velocity variation divided by the characteristic length $(r_b - r_a)$. More precisely, the first approximation is in error because the vorticity, enthalpy, moment of momentum, etc., are assumed to follow coaxial circular cylinders instead of the true stream surfaces. For small radial velocities this first approximation should give information of both qualitative and quantitative accuracy and, because of its simplicity relative to higher order approximations, merits more detailed consideration. Only the problem of the first kind will be considered, that is, the problem in which the blade forces or the blade vorticity is given.

The Modified First Order Solution.- The first order solution may be given in an arrangement differing slightly from that given in the iteration process but more convenient for application

$$v = v(r,-\infty) + \int_{-\infty}^{z} \frac{1}{w(r,-\infty)} F_\theta \, dz$$

$$= v(r,-\infty) - \int_{-\infty}^{z} \xi \, dz \qquad 67a$$

$$\xi = -\frac{\partial v}{\partial z} \qquad 67b$$

$$\zeta = \frac{\partial v}{\partial r} + \frac{v}{r} \qquad 67c$$

$$\frac{\partial \eta}{\partial z} = \frac{1}{w(r,-\infty)}\left\{\frac{\partial}{\partial z}(v\zeta) - \frac{\partial}{\partial r}(\omega r \xi)\right\} \qquad 67d$$

$$u = \int_{-\infty}^{\infty}\int_{r_a}^{r_b} \frac{\partial \eta}{\partial \beta} \sum_{n=1}^{\infty} \frac{\alpha U_1(\varepsilon_n r) U_1(\varepsilon_n \alpha)}{2\varepsilon_n \nu_n^2} e^{-\varepsilon_n |z-\beta|} \, d\alpha \, d\beta \qquad 67e$$

$$w - w(r,-\infty) = \int_{-\infty}^{\infty}\int_{r_a}^{r_b} \frac{\partial \eta}{\partial \beta} \sum_{n=1}^{\infty} \frac{\alpha U_1(\varepsilon_n \alpha) U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} e^{-\varepsilon_n |z-\beta|} \, d\alpha \, d\beta$$

$$+ 2\int_{-\infty}^{\infty}\int_{r_a}^{r_b} \frac{\partial \eta}{\partial \beta} \sum_{n=1}^{\infty} \frac{\alpha U_1(\varepsilon_n \alpha) U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} \left(1 - e^{-\varepsilon_n |z-\beta|}\right) d\alpha \, d\beta \qquad 67f$$

$$h = h(r,-\infty) - \int_{-\infty}^{z} \omega r \xi \, dz \qquad 67g$$

The expressions for the Green's functions have been included explicitly in the formulas to allow for the case where interchanging the order of summation and integration allows a simple quadrature. Then the resulting series will be considerably more rapidly convergent than the series for the Green's function and the amount of numerical work is reduced to a minimum. In general, of course, the simple quadrature is not possible.

<u>Separation of the Vorticity Distribution</u>.- In most cases of practical importance the initial tangential velocity vanishes ($v(r,-\infty)=0$), the initial axial velocity distribution is uniform ($w(r,-\infty)=w_0$), and the initial enthalpy distribution is constant ($h(r,-\infty)=h_0$). Furthermore, inasmuch as the blade vorticity or the blade forces may be stipulated at will in the first problem, it is usual that these quantities be arranged independently in the radial and axial direction. This means, physically, that the vorticity patterns are similar in all planes normal to the axis of symmetry and that the vorticity distribution are similar in all coaxial cylinders. Mathematically this allows the tangential velocity to be written in the form $v = w_0 R \cdot Z$ where R and Z denote respectively functions of r and z alone. Then equations 67 assume the form

$$\xi = -w_0 R \cdot Z' \qquad 68$$

$$v = w_0 R \cdot Z \qquad 69$$

$$\zeta = w_0 \left(R' + \frac{R}{r}\right) Z \qquad 70$$

The relation for the radial velocity component may then be written in the form

$$u = \int_{-\infty}^{\infty}\int_{r_a}^{r_b} G(\alpha,\beta;r,z)\, w_0 \left\{\left(2RZ - \frac{\omega r}{w_0}\right)\left(R' + \frac{R}{r}\right) Z'\right\} d\alpha\, d\beta$$

or inserting the expression for the Green's function

$$\frac{u}{w_0} = \sum_{n=1}^{\infty} \frac{2U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} \int_{r_a}^{r_b} R(R'\alpha + R)U_1(\varepsilon_n\alpha)\,d\alpha \int_{-\infty}^{\infty} zz' e^{-\varepsilon_n|\zeta-\beta|}\,d\beta$$

$$- \frac{\omega r_b}{w_0} \sum_{n=1}^{\infty} \int_{r_a}^{r_b} (R'\alpha + R)\frac{\alpha}{r_b} U_1(\varepsilon_n\alpha)\,d\alpha \int_{-\infty}^{\infty} z' e^{-\varepsilon_n|\zeta-\beta|}\,d\beta \qquad 71$$

Likewise the axial velocity may be written as

$$\frac{w}{w_0} - 1 = 2\sum_{n=1}^{\infty} \frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} \int_{r_a}^{r_b} R(R'\alpha + R)U_1(\varepsilon_n\alpha)\,d\alpha \left\{ \int_{-\infty}^{\infty} zz' e^{-\varepsilon_n|\zeta-\beta|}\,d\beta \right.$$

$$\left. -2\int_{-\infty}^{\zeta} zz' e^{-\varepsilon_n|\zeta-\beta|}\,d\beta + z^2(\zeta) \right\}$$

$$- \frac{\omega r_b}{w_0} \sum_{n=1}^{\infty} \frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} \int_{r_a}^{r_b} (R'\alpha + R)\frac{\alpha}{r_b} U_1(\varepsilon_n\alpha)\,d\alpha \left\{ \int_{-\infty}^{\infty} z' e^{-\varepsilon_n|\zeta-\beta|}\,d\beta \right.$$

$$\left. -2\int_{-\infty}^{\zeta} z' e^{-\varepsilon_n|\zeta-\beta|}\,d\beta + z(\zeta) \right\} \qquad 72$$

Finally the enthalpy change is

$$h - h_0 = \int_{-\infty}^{\zeta} \omega r R z'\,d\zeta = \sum \omega R r\,[z] \qquad 73$$

where the summation is carried out over the moving blade rows. It is observed that the expressions for the radial and axial velocities introduce two new characteristic integrals of each of two kinds. These integrals are defined in the following manner

$$T_n^{(1)} = \int_{r_a}^{r_b} R\,(R'\alpha + R)\, U_1(\varepsilon_n \alpha)\, d\alpha \qquad 74$$

$$T_n^{(2)} = \int_{r_a}^{r_b} \frac{\alpha}{r_b}\,(R'\alpha + R)\, U_1(\varepsilon_n \alpha)\, d\alpha \qquad 75$$

$$V_n^{(1)}(\zeta) = \int_{-\infty}^{\zeta} z\,z'\, e^{-\varepsilon_n |\zeta - \beta|}\, d\beta \qquad 76$$

$$V_n^{(1)} \equiv V_n^{(1)}(\infty)$$

$$V_n^{(2)}(\zeta) = \int_{-\infty}^{\zeta} z'\, e^{-\varepsilon_n |\zeta - \beta|}\, d\beta \qquad 77$$

$$V_n^{(2)} \equiv V_n^{(2)}(\infty)$$

The integrals $V_n^{(1)}$ and $V_n^{(2)}$ exhibit the same inherent difficulty in evaluation as did the Green's functions, that is, the integral must be evaluated separately upstream and downstream of the point $\beta = \zeta$. In practical cases they must also be evaluated separately in each of the regions where has a particular functional representation.

By means of these integrals, the expressions for radial and axial velocities are simplified to the following form

$$\frac{u}{w_0} = 2\sum_{n=1}^{\infty}\frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2}\,T_n^{(1)}V_n^{(1)} - \frac{\omega r_b}{w_0}\sum_{n=1}^{\infty}\frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2}\,T_n^{(2)}V_n^{(2)} \tag{78}$$

$$\frac{w}{w_0} - 1 = 2\sum_{n=1}^{\infty}\frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2}\,T_n^{(1)}\left\{V_n^{(1)} - 2V_n^{(2)}(\zeta) + Z^2(\zeta)\right\}$$

$$- \frac{\omega r_b}{w_0}\sum_{n=1}^{\infty}\frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2}\,T_n^{(2)}\left\{V_n^{(2)} - 2V_n^{(2)}(\zeta) + Z(\zeta)\right\} \tag{79}$$

<u>The Use of Asymptotic Expansions</u>.- The evaluation of the integrals $T_n^{(1)}$, $T_n^{(2)}$ is always possible by graphical means if analytical procedures become unduly complex. In this form, however, it is difficult to proceed with an analytical discussion of the solutions without actually carrying out laborious general calculations. A large part of this difficulty is due to the Bessel's functions $U_1(\varepsilon_n r)$ and $U_0(\varepsilon_n r)$ which arise naturally in the solution of the problem. These functions are not tabulated and only the first six zeros are tabulated (ref. 3). Therefore the characteristic values are not easily found. A crude method may be employed which utilises the asymptotic expansion of the functions.

The use of asymptotic expansions need not be restricted to calculation procedures but may be introduced into the general relations to simplify the analysis in special cases. The accuracy of the asymptotic expansions for the Bessel functions improves, of course, with increase in the argument. The minimum magnitude of the argument $\varepsilon_n r$ occurring in any case depends upon the ratio of outer to inner radius and is quite large when this ratio does not exceed 2.50. Within this range small errors are incurred by retaining only the first terms of each of the asymptotic expansions. This limit on the radius ratio admits most of the axial compressor and turbine applications but clearly excludes the problem of the ducted propeller.

These expressions for the Bessel functions of the zeroth and first order and first and second kinds are

$$\lim_{\varepsilon_n \to \infty} J_0(\varepsilon_n r) = \frac{1}{\sqrt{\pi \varepsilon_n r}} \left(\cos(\varepsilon_n r) + \sin(\varepsilon_n r) \right)$$

$$\lim_{\varepsilon_n \to \infty} Y_0(\varepsilon_n r) = \frac{1}{\sqrt{\pi \varepsilon_n r}} \left(\sin(\varepsilon_n r) - \cos(\varepsilon_n r) \right)$$

$$\lim_{\varepsilon_n \to \infty} J_1(\varepsilon_n r) = \frac{1}{\sqrt{\pi \varepsilon_n r}} \left(\sin(\varepsilon_n r) - \cos(\varepsilon_n r) \right) \qquad 80$$

$$\lim_{\varepsilon_n \to \infty} Y_1(\varepsilon_n r) = \frac{-1}{\sqrt{\pi \varepsilon_n r}} \left(\sin(\varepsilon_n r) + \cos(\varepsilon_n r) \right)$$

Thus the limiting forms of the functions $U_1(\varepsilon_n r)$ and $U_0(\varepsilon_n r)$ become

$$\lim_{\varepsilon_n \to \infty} U_0(\varepsilon_n r) = \frac{-2}{\pi \sqrt{\varepsilon_n r \cdot \varepsilon_n r_a}} \cos \varepsilon_n (r - r_a)$$

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$$\lim_{\varepsilon_n \to \infty} U_1(\varepsilon_n r) = \frac{-2}{\pi \sqrt{\varepsilon_n r \cdot \varepsilon_n r_a}} \sin \varepsilon_n (r - r_a)$$

The characteristic values, ε_n , are determined from the condition that $U_1(\varepsilon_n r_b)$ vanish for all values of n. In the asymptotic expression, these values may be written down explicitly as

$$\varepsilon_n = \frac{n\pi}{r_b - r_a} \; ; \qquad n = 1, 2, 3, \; \ldots\ldots$$

The square of the norm may also be easily determined

$$\lim_{\varepsilon_n \to \infty} \nu_n^2 = \lim_{\varepsilon_n \to \infty} \int_{r_a}^{r_b} r\, U_1^2(\varepsilon_n r)\, dr = \frac{4}{\varepsilon_n^2 \pi r_a} \int_{r_a}^{r_b} \sin^2 \varepsilon_n (r_b - r_a)\, dr$$

$$= \frac{2(r_b - r_a)}{\pi^2 r_a \varepsilon_n^2} \qquad 82$$

The asymptotic approximation to the fundamental integrals for the radial and axial velocity components may be written down directly:

$$T_n^{(1)} = \int_0^1 \sqrt{\frac{r_b - r_a}{\alpha}}\, R(R'\alpha + R) \sin n\pi\bar{\alpha}\, d(n\pi\bar{\alpha}) \qquad 84$$

$$T_n^{(2)} = \int_0^1 \frac{\rho - 1}{\rho} \sqrt{\frac{\alpha}{r_b - r_a}}\, (R'\alpha + R) \sin n\pi\bar{\alpha}\, d(n\pi\bar{\alpha}) \qquad 85$$

$$V_n^{(1)}(\zeta) = \int_{-\infty}^{\bar{\zeta}} 2 Z' (r_b - r_a)\, e^{-n\pi|\bar{\zeta} - \bar{\zeta}|}\, d(n\pi\bar{\zeta}) \qquad 86$$

$$V_n^{(2)}(\bar{z}) = \int_{-\infty}^{\bar{z}} Z'(r_b - r_a)\, e^{-n\pi|\bar{z}-\bar{\zeta}|}\, d(n\pi\bar{\zeta}) \qquad 87$$

where ρ denotes the radius ratio $\frac{r_b}{r_a}$ and the variables with bars denote division by the blade length, e.g. $\bar{r} = \frac{r - r_a}{r_b - r_a}$; $\bar{z} = \frac{z}{r_b - r_a}$. The expressions for the radial and axial velocities may now be written as trigonometric series with variable coefficients:

$$\frac{u}{w_o} = 2\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\right)^3\sqrt{\frac{r_b - r_a}{r}}\,\sin n\pi\bar{r}\, T_n^{(1)} V_n^{(1)} - \frac{\omega r_b}{w_o}\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\right)^3\sqrt{\frac{r_b - r_a}{r}}\,\sin n\pi\bar{r}\, T_n^{(2)} V_n^{(2)} \qquad 88$$

$$\frac{w}{w_o} - 1 = 2\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\right)^3\sqrt{\frac{r_b - r_a}{r}}\,\cos n\pi\bar{r}\, T_n^{(1)}\left\{V_n^{(1)} - 2V_n^{(1)}(\bar{z}) + n\pi Z^2(\bar{z})\right\}$$

$$- \frac{\omega r_b}{w_o}\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\right)^3\sqrt{\frac{r_b - r_a}{r}}\,\cos n\pi\bar{r}\, T_n^{(2)}\left\{V_n^{(2)} - 2V_n^{(2)}(\bar{z}) + n\pi Z(\bar{z})\right\} \qquad 89$$

These expressions are of more use than mere approximations to the more nearly precise ones involving the Bessel functions. The Bessel functions are tabulated in conventional tables, references 3 and 4 , to values of the argument not exceeding 16.00. For arguments exceeding this value, the first terms of the asymptotic expansions provide an excellent approximation and indeed the most direct means of extending the calculations to high arguments, i.e., to high characteristic values. Inasmuch as the same problem is encountered in calculating the higher terms of the Green's functions, the asymptotic forms of the Green's functions are given below:

$$G(\alpha,\beta;r,z) = \sum_{n=1}^{\infty} \frac{\sqrt{r\alpha}}{n\pi} e^{-n\pi|\bar{z}-\bar{\beta}|} \sin n\pi\bar{r} \sin n\pi\bar{\alpha}$$

$$H(\alpha,\beta;r,z) = \sum_{n=1}^{\infty} \frac{1}{n\pi}\sqrt{\frac{\alpha}{r}}\, e^{-n\pi|\bar{z}-\bar{\beta}|} \cos n\pi\bar{r} \sin n\pi\bar{\alpha} \qquad 89$$

$$F(\alpha,\beta;r,z) = \sum_{n=1}^{\infty} \frac{1}{n\pi}\sqrt{\frac{\alpha}{r}}\left(re^{-n\pi|\bar{z}-\bar{\beta}|}\right) \cos n\pi\bar{r} \sin n\pi\bar{\alpha}$$

<u>Solution for Vanishing Inner Radius</u>.- An important special case of the foregoing general analysis is that where the inner radius becomes vanishingly small. This corresponds to the ducted or shrouded propeller with small driving shaft. For vanishing inner radius, the Bessel function of the first kind automatically satisfies the condition that the radial velocity be zero at the axis of symmetry. Since for integral order, the positive and negative orders of Bessel's functions are not linearly independent, the solution will involve only Bessel functions of the first kind and the first (positive) order. The resulting radial and axial velocity distributions then reduce to

$$u(r,z) = \int_{-\infty}^{\infty}\int_{0}^{r_b} \frac{\partial\eta}{\partial\beta} \sum_{n=1}^{\infty} \frac{\alpha J_1(\varepsilon_n\alpha) J_1(\varepsilon_n r)}{2\varepsilon_n\gamma_n^2}\, e^{-\varepsilon_n|z-\beta|}\, d\alpha\, d\beta \qquad 90$$

$$w(r,z) - w(r,-\infty) = \int_{-\infty}^{\infty}\int_{0}^{r_b} \frac{\partial \eta}{\partial \beta} \sum_{n=1}^{\infty} \frac{\alpha\, J_1(\varepsilon_n \alpha)\, J_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2}\, e^{-\varepsilon_n |z-\beta|}\, d\alpha\, d\beta$$

$$+ 2\int_{-\infty}^{z}\int_{0}^{r_b} \frac{\partial \eta}{\partial \beta} \sum_{n=1}^{\infty} \frac{\alpha\, J_1(\varepsilon_n \alpha)\, J_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} \left(1 - e^{-\varepsilon_n |z-\beta|}\right) d\alpha\, d\beta \qquad 91$$

All other relations may be simplified for the vanishing inner radius simply by replacing $U_1(\varepsilon_n r)$ and $U_0(\varepsilon_n r)$ by $J_1(\varepsilon_n r)$ and $J_0(\varepsilon_n r)$ respectively. This case has the distinct advantage of allowing direct use of tabulated values of the functions as well as characteristic values.

VII FIRST APPROXIMATION SOLUTIONS FOR A SINGLE STATIONARY OR ROTATING BLADE ROW

As an illustration of solutions in the first approximation the problem of the single stationary or rotating blade row brings out most of the interesting features of the analysis and physical interpretation. The blade loading or vorticity distribution will be assumed of such a form that the method of the characteristic integrals may be applied. The solution resulting from the consideration of an actual chord-wise distribution of vorticity may then be compared with the corresponding solution for the case where the tangential velocity is imparted discontinuously, that is, where the blade row is replaced by a vortex sheet normal to the axis of symmetry. This comparison corresponds to the analogous comparison of the downwash generated by an actual wing with distributed vorticity and that associated with the lifting line approximation.

The Distribution of Vorticity.- Consider the spanwise and chordwise distribution of vorticity given by the relation

$$\zeta = \zeta_0 \left(1 + \left(\frac{r}{k}\right)^2\right)\left(\frac{z}{c} + \frac{1}{2}\right)^{-\frac{1}{2}} \qquad 92$$

which is typical of a load distribution giving significant radial velocity components and producing a marked distortion of the axial velocity. The chordwise distribution of vorticity decreases sharply toward the trailing edge of the airfoil and has an infinity of low order at the leading edge.

At any axial position the vorticity increases with the blade radius parabolically from a value of $\xi_0(1+(\frac{r_a}{k})^2)(\frac{z}{c}+\frac{1}{2})^{-\frac{1}{2}}$ at the blade root to $\xi_0(1+(\frac{r_b}{k})^2)(\frac{z}{c}+\frac{1}{2})^{-\frac{1}{2}}$ at the blade tip. To give the distribution more physical significance, the spanwise distribution of circulation may be calculated directly. The circulation about a corresponding physical blade is identical with the sum of all radial vorticity included in a wedge of thickness equal to the blade chord and of vertex angle $\frac{2\pi}{n}$ where n is the number of physical blades. Then the circulation is

$$\Gamma(r) = \int_{-\frac{c}{2}}^{\frac{c}{2}}\int_{0}^{\frac{2\pi}{n}} \xi\, r\, d\vartheta\, dz$$

which may be evaluated in terms of the assumed vorticity distribution.

$$\Gamma(r) = \xi_0 c\int_{-\frac{1}{2}}^{+\frac{1}{2}} (1+(\tfrac{r}{k})^2)(\tfrac{z}{c}+\tfrac{1}{2})^{-\frac{1}{2}} d(\tfrac{z}{c})\, r\, d\vartheta$$

$$= \frac{4\xi_0\pi r}{n}\,(1+(\tfrac{r}{k})^2) \qquad 93$$

The first approximations to the tangential velocity and the vorticity components corresponding to this may now be computed for use in evaluating the characteristic integrals. The tangential velocity in each of the three regions, consisting in that upstream of the blade row, that within the blade passage, and that downstream of the blade row is then

$$v = 0 \qquad\qquad z < -c/2$$

$$v = \int_{-\frac{c}{2}}^{z} \zeta_0 \left(1+\left(\frac{r}{k}\right)^2\right)\left(\frac{z}{c}+\frac{1}{2}\right)^{-\frac{1}{2}} dz$$

$$= 2\zeta_0 c\left(1+\left(\frac{r}{k}\right)^2\right)\left(\frac{z}{c}+\frac{1}{2}\right)^{\frac{1}{2}} \qquad -\frac{c}{2} \le z \le \frac{c}{2} \qquad 94$$

$$v = 2\zeta_0 c\left(1+\left(\frac{r}{k}\right)^2\right) \qquad\qquad z > \frac{c}{2}$$

The distribution of axial vorticity shed from the blades is also easily calculated for each of the three regions

$$\zeta = \frac{\partial v}{\partial r} + \frac{v}{r} = 0 \qquad\qquad z < -\frac{c}{2}$$

$$= \zeta_0 c\left(\frac{z}{c}+\frac{1}{2}\right)^{-\frac{1}{2}}\left(1+2\left(\frac{r}{k}\right)^2\right); \qquad -\frac{c}{2} \le z \le \frac{c}{2} \qquad 95$$

$$= \zeta_0 c/r\left(1+2\left(\frac{r}{k}\right)^2\right) \qquad\qquad z > \frac{c}{2}$$

The Characteristic Integrals.- The values of each of the characteristic integrals, equations 74 through 77, may be calculated through use of the foregoing results. The functions R and Z are easily recognized from the expression for the tangential velocity to be

$$R = \left(1+\left(\frac{r}{k}\right)^2\right) \qquad 96$$

$$Z = \frac{2\zeta_0 c}{\omega_0}\left(\frac{z}{c}+\frac{1}{2}\right)^{\frac{1}{2}} \qquad 97$$

Consider first the integral $T_n^{(1)}$. From equation 74 it follows that

$$T_n^{(1)} = \int_{r_a}^{r_b} \left(1+\left(\frac{r}{k}\right)^2\right)\left(2\left(\frac{r}{k}\right)^2+1+\left(\frac{r}{k}\right)^2\right) U_1(\varepsilon_n r)\, dr$$

$$= \frac{1}{\varepsilon_n}\int_{r_a}^{r_b} \left(1+\left(\frac{r}{k}\right)^2\right)\left(1+3\left(\frac{r}{k}\right)^2\right) U_1(\varepsilon_n r)\, d(\varepsilon_n r)$$

which may be integrated directly as

$$T_n^{(1)} = \frac{1}{\varepsilon_n}\left[1-4\left(\frac{r}{k}\right)^2-3\left(\frac{r}{k}\right)^4+\frac{24}{\varepsilon_n^2 k^2}\left(\frac{r}{k}\right)^2\right] U_0(\varepsilon_n r)\Bigg|_{r_a}^{r_b}$$

$$= \frac{1}{\varepsilon_n}\left(1+\left(\frac{24}{(\varepsilon_n k)^2}-4\right)\left(\frac{r}{k}\right)^2-3\left(\frac{r}{k}\right)^4\right) U_0(\varepsilon_n r)\Bigg|_{r_a}^{r_b}$$

Inserting the limits of integration, the final form of the integral T_n^{1} becomes

$$T_n^{(1)} = \frac{1}{\varepsilon_n}\left\{\left(\frac{24}{(\varepsilon_n k)^2}-4\right)\left(\left(\frac{r_b}{k}\right)^2 U_0(\varepsilon_n r_b)-\left(\frac{r_a}{k}\right)^2 U_0(\varepsilon_n r_a)\right)+\left(U_0(\varepsilon_n r_b)-U_0(\varepsilon_n r_a)\right)\right. \qquad 98$$

$$\left. +3\left(\left(\frac{r_b}{k}\right)^4 U_0(\varepsilon_n r_b)-\left(\frac{r_a}{k}\right)^4 U_0(\varepsilon_n r_a)\right)\right\}$$

The integral $T_n^{(2)}$ may be evaluated in a similar manner although the result is not quite so convenient inasmuch as it is not in a closed form. From equation 75

$$T_n^{(2)} = \int_{r_a}^{r_b} \left(\frac{r}{r_b}\right)(R'r+R)\, U_1(\varepsilon_n r)\, dr$$

$$= \frac{k}{r_b}\int_{r_a}^{r_b} \left(\frac{r}{k}\right)\left(1+3\left(\frac{r}{k}\right)^2\right) U_1(\varepsilon_n r)\, dr$$

Through repeated integration by parts

$$T_n^{(2)} = \frac{k}{\varepsilon_n r_b}\left\{ \left(\frac{r}{k}\right)^3 U_o(\varepsilon_n r)\Big|_{r_a}^{r_b} - 2\int_{\varepsilon_n r_a}^{\varepsilon_n r_b} (\varepsilon_n r)\, U_1(\varepsilon_n r)\, d(\varepsilon_n r) \right\}$$

$$= \frac{k}{\varepsilon_n r_b}\left\{ \left(\frac{r}{k}\right)^3 U_o(\varepsilon_n r) - \frac{2}{(\varepsilon_n k)^2} U_o(\varepsilon_n r)\left(1 + \frac{1}{(\varepsilon_n r)^2} + \frac{2}{(\varepsilon_n r)^4} + \cdots\right)\right\}\Bigg|_{r_a}^{r_b}$$

so that the final representation becomes

$$T_n^{(2)} = \frac{k}{\varepsilon_n r_b}\left\{ \left(\frac{r_b}{k}\right)^3 - \frac{2}{(\varepsilon_n k)^2}\frac{r_b}{k}\, g(\varepsilon_n r_b)\right\} U_o(\varepsilon_n r_b)$$

$$- \frac{k}{\varepsilon_n r_b}\left\{ \left(\frac{r_a}{k}\right)^3 + \frac{2}{(\varepsilon_n k)^2}\frac{r_a}{k}\, g(\varepsilon_n r_a)\right\} U_o(\varepsilon_n r_a) \qquad 99$$

$$g(\lambda) = 1 + \lambda^{-2} + 2\lambda^{-4} + \cdots$$

The integrals $V_n^{(1)}(z)$ and $V_n^{(2)}(z)$ must be evaluated in three cases according to whether z is upstream, within, or downstream of the blade row.

Case I. $z < -\frac{c}{2}$

$$V_n^{(1)}(z) = 0 \qquad 100$$

$$V_n^{(1)} = \int_{-\infty}^{\infty} z z' e^{-\varepsilon_n |z-\beta|} d\beta = \frac{\zeta_o^2 c^2}{w_o^2}\int_{-\frac{c}{2}}^{+\frac{c}{2}} \frac{1}{c}\, e^{-\varepsilon_n(\beta - z)} d\beta$$

$$= \frac{\zeta_o^2 c^2}{w_o^2}\, e^{\varepsilon_n z}\,\frac{\sinh \frac{\varepsilon_n c}{2}}{\frac{\varepsilon_n c}{2}} \qquad 101$$

Case II. $-\frac{c}{2} \leq z \leq \frac{c}{2}$

$$V_n^{(1)}(z) = \int_{-\infty}^{z} zz' e^{-\varepsilon_n|z-\beta|} d\beta = \frac{\zeta_0^2 c^2}{w_0^2} \int_{-\frac{c}{2}}^{z} \frac{1}{c} e^{-\varepsilon_n(\beta - z)} d\beta$$

$$= \frac{\zeta_0^2 c^2}{w_0^2} \frac{1}{\varepsilon_n c} \left(e^{\varepsilon_n (z + \frac{c}{2})} - 1 \right)$$ 102

$$V_n^{(1)} = \int_{-\infty}^{\infty} zz' e^{-\varepsilon_n|z-\beta|} d\beta = \frac{\zeta_0 c^2}{w_0^2} \int_{-\frac{c}{2}}^{z} \frac{1}{c} e^{-\varepsilon_n(\beta - z)} d\beta$$

$$+ \frac{\zeta_0^2 c^2}{w_0^2} \int_{z}^{c/2} \frac{1}{c} e^{\varepsilon_n(\beta - z)} d\beta$$

$$= \frac{\zeta_0^2 c^2}{w_0^2} \frac{1}{\varepsilon_n c} \left(e^{\varepsilon_n c/2} \cosh(\varepsilon_n z) - 2 \right)$$ 103

Case III. $z > \frac{c}{2}$

$$V_n^{(1)}(z) = \int_{-\infty}^{z} zz' e^{-\varepsilon_n|z-\beta|} d\beta = \int_{-\infty}^{\infty} zz' e^{-\varepsilon_n|z-\beta|} d\beta = V_n^{(1)}$$ 104

$$V_n^{(1)} = \int_{-\infty}^{\infty} z z' e^{-\epsilon_n|\zeta-\beta|} d\beta = -\frac{\zeta_o^2 c^2}{\omega_o^2} e^{-\epsilon_n \zeta} \frac{\sinh \frac{\epsilon_n c}{2}}{\frac{\epsilon_n c}{2}} \qquad 105$$

The integrals $V_n^{(2)}(\zeta)$ and $V_n^{(2)}$ must be evaluated in a similar manner except that the integration is not so direct. The integrals involved are of the type

$$\int \left(\frac{\beta}{c} + \frac{1}{2}\right)^{-\frac{1}{2}} e^{-\epsilon_n(\beta-\zeta)} d\beta = e^{\epsilon_n \zeta} \int \left(\frac{\beta}{c} + \frac{1}{2}\right)^{-\frac{1}{2}} e^{-\epsilon_n \beta} d\beta$$

$$= e^{\epsilon_n(\zeta + \frac{c}{2})} \cdot c \int \left(\frac{\beta}{c} + \frac{1}{2}\right)^{-\frac{1}{2}} e^{-\epsilon_n\left(\frac{\beta}{c} + \frac{1}{2}\right)} d\left(\frac{\beta}{c} + \frac{1}{2}\right)$$

$$= e^{\epsilon_n(\zeta + \frac{c}{2})} \sqrt{\frac{c}{\epsilon_n}} \int \left[c\epsilon_n\left(\frac{\beta}{2} + \frac{1}{2}\right)\right]^{-\frac{1}{2}} e^{-\epsilon_n c\left(\frac{\beta}{c} + \frac{1}{2}\right)} d\, \epsilon_n c\left(\frac{\beta}{c} + \frac{1}{2}\right)$$

Inasmuch as the integration over β extends from $-\frac{c}{2}$ to ζ the variable $\frac{\beta}{2} + \frac{1}{2}$ is always positive. Therefore the substituation $\lambda^2 = \epsilon_n c\left[\frac{\beta}{2} + \frac{1}{2}\right]$ is appropriate which gives

$$\int \left(\frac{\beta}{c} + \frac{1}{2}\right)^{-\frac{1}{2}} e^{-\epsilon_n(\beta-\zeta)} d\beta = 2\sqrt{\frac{c}{\epsilon_n}}\, e^{\epsilon_n(\zeta + \frac{c}{2})} \int e^{-\lambda^2} d\lambda \qquad 106$$

Therefore the results the integrations involved in evaluating $V_n^{(2)}(\zeta)$ and $V_n^{(2)}$ will be given in terms of integrals related to the incomplete error function.

Case I. $\zeta < -\frac{c}{2}$

$$V_n^{(2)}(\zeta) = \int_{-\infty}^{\zeta} Z' e^{-\varepsilon_n|\zeta-\beta|} d\beta \quad = 0 \qquad 107$$

$$V_n^{(2)} = \int_{-\infty}^{\infty} Z' e^{-\varepsilon_n|\beta-\zeta|} d\beta = \int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{\xi_0}{w_0}\left(\frac{\beta}{c}+\frac{1}{2}\right)^{-\frac{1}{2}} e^{-\varepsilon_n(\beta-\zeta)} d\beta$$

$$= \frac{2\xi_0}{w_0}\sqrt{\frac{c}{\varepsilon_n}}\, e^{\varepsilon_n(\zeta+\frac{c}{2})}\int_0^1 e^{-\lambda^2} d\lambda \qquad 108$$

Case II. $-\frac{c}{2} \leq \zeta \leq \frac{c}{2}$

$$V_n^{(2)}(\zeta) = \int_{-\infty}^{\zeta} Z' e^{-\varepsilon_n|\zeta-\beta|} d\beta = \int_{-\frac{c}{2}}^{\zeta} \frac{\xi_0}{w_0}\left(\frac{\beta}{c}+\frac{1}{2}\right)^{-\frac{1}{2}} e^{-\varepsilon_n(\beta-\zeta)} d\beta$$

$$= 2\,\frac{\xi_0}{w}\sqrt{\frac{c}{\varepsilon_n}}\, e^{\varepsilon_n(\zeta+\frac{c}{2})}\int_0^{\sqrt{\varepsilon_n(\zeta+\frac{c}{2})}} e^{-\lambda^2} d\lambda \qquad 109$$

$$V_n^{(2)} = \int_{-\infty}^{\infty} Z' e^{-\varepsilon_n|\zeta-\beta|} d\beta = \int_{-c/2}^{\zeta} \frac{\xi_0}{w_0}\left(\frac{\beta}{c}+\frac{1}{2}\right)^{-\frac{1}{2}} e^{-\varepsilon_n(\beta-\zeta)} + \int_{\zeta}^{c/2} \frac{\xi_0}{w_0}\left(\frac{\beta}{c}+\frac{1}{2}\right)^{-\frac{1}{2}} e^{\varepsilon_n(\beta-\zeta)} d\beta$$

$$= \frac{2\xi_0}{w_0}\sqrt{\frac{c}{\varepsilon_n}}\left\{ e^{\varepsilon_n(\zeta+\frac{c}{2})}\int_0^{\sqrt{\varepsilon_n(\zeta+\frac{c}{2})}} e^{-\lambda^2} d\lambda + e^{-\varepsilon_n(\zeta+\frac{c}{2})}\int_{\sqrt{\varepsilon_n(\zeta+\frac{c}{2})}}^{\sqrt{\varepsilon_n c}} e^{\lambda^2} d\lambda \right\} \qquad 110$$

Case III. $\zeta > \frac{c}{2}$

$$V_n^{(2)}(\zeta) = \int_{-\infty}^{\zeta} Z' e^{-\varepsilon_n|\zeta-\beta|} d\beta = \int_{-\infty}^{\infty} Z' e^{-\varepsilon_n|\zeta-\beta|} d\beta = V_n^{(2)} \tag{111}$$

$$V_n^{(2)} = \int_{-\frac{c}{2}}^{+\frac{c}{2}} \frac{\xi_o}{w_o} \left(\frac{\beta}{c}+\frac{1}{2}\right)^{-\frac{1}{2}} e^{-\varepsilon_n(\zeta-\beta)} d\beta$$

$$= \frac{2\xi_o}{w_o}\sqrt{\frac{c}{\varepsilon_n}}\; e^{-\varepsilon_n(\zeta+\frac{c}{2})} \int_0^1 e^{\lambda^2} d\lambda \tag{112}$$

Hence the explicit solution for any of the three regions of flow, i.e., upstream, within, or downstream of the blade row, may be written down through insertion of the appropriate values of the above integrals into equations 78 and 79

$$\frac{u}{w_o} = 2\sum_{n=1}^{\infty} \frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(1)} V_n^{(1)} - \frac{\omega r_b}{w_o} \sum_{n=1}^{\infty} \frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(2)} V_n^{(2)} \tag{78}$$

$$\frac{w}{w_o} - 1 = 2\sum_{n=1}^{\infty} \frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(1)} \left\{ V_n^{(1)} - 2V_n^{(1)}(\zeta) + Z^2(\zeta) \right\}$$

$$- \frac{\omega r_b}{w_o} \sum_{n=1}^{\infty} \frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(2)} \left\{ V_n^{(2)} - 2V_n^{(2)}(\zeta) + Z(\zeta) \right\} \tag{79}$$

Solution for the Vortex Sheet.- To obtain the corresponding solution for a vortex sheet, it need only be noted that the jump of the tangential vorticity may be written in the form

$$[\eta] = \frac{1}{w_0} v_0 \left(\frac{\partial v_0}{\partial r} + \frac{v_0}{r} \right) - \frac{\partial}{\partial r} (\omega r v_0) \qquad 113$$

and then apply the procedures considered in Part V. The appropriate constants A_n^1, A_n^2, B_n^1, and B_n^2 may be calculated directly from equations 56 and 57 through use of the equation 113 and the relations already calculated in the consideration of the solution of the problem with chordwise vorticity distribution. Then

$$A_n^1 = -\frac{1}{2\nu_n^2 \varepsilon_n} \left(\frac{\xi_0 c}{w_0} \right)^2 \int_{r_a}^{r_b} R(rR'+R) U_1(\varepsilon_n r)\, dr + \frac{1}{2\varepsilon_n \nu_n^2} \int_{r_a}^{r_b} \frac{\xi_0 c}{w_0} \frac{\omega r_b}{w_0} (R'r+R) \frac{r}{r_b} U_1(\varepsilon_n r)\, dr$$

$$= -\left(\frac{\xi_0 c}{w_0} \right)^2 \frac{1}{2\varepsilon_n \nu_n^2} T_n^{(1)} + \frac{\omega r_b}{w_0} \left(\frac{\xi_0 c}{w_0} \right) \frac{1}{2\varepsilon_n \nu_n^2} T_n^{(2)} \qquad 114$$

$$A_n^2 = 0$$

and likewise

$$B_n^1 = 0$$

$$B_n^2 = -\frac{1}{2\nu_n^2 \varepsilon_n} \left(\frac{\xi_0 c}{w_0} \right)^2 T_n^{(1)} + \frac{1}{2\varepsilon_n \nu_n^2} \frac{\omega r_b}{w_0} \left(\frac{\xi_0 c}{w_0} \right) T_n^{(2)} \qquad 115$$

The solutions for the radial velocity component upstream and downstream of the discontinuity may then be written down directly in the form

$$\frac{u}{w_o} = -\left(\frac{\xi_o c}{w_o}\right)^2 \sum_{n=1}^{\infty} \frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(1)} e^{\varepsilon_n z} \qquad z<0$$

$$+ \frac{\xi_o c}{w_o} \frac{\omega r_b}{w_o} \sum_{n=1}^{\infty} \frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(2)} e^{\varepsilon_n z} \qquad 116$$

$$\frac{u}{w_o} = -\left(\frac{\xi_o c}{w_o}\right)^2 \sum_{n=1}^{\infty} \frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(1)} e^{-\varepsilon_n z} \qquad z>0$$

$$+ \frac{\xi_o c}{w_o} \frac{\omega r_b}{w_o} \sum_{n=1}^{\infty} \frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(2)} e^{-\varepsilon_n z} \qquad 117$$

Clearly the radial velocity component is symmetrical about the discontinuity so that it is to be expected that half of the change in axial velocity distribution will take place upstream of the discontinuity and half downstream. This result is in agreement with the well known theory of the actuator disc.

To determine the axial velocities, only the values of the functions $Y(r)$ need be calculated in addition to the information already available. From equation 59 it follows that

$$Y^1 = w_o$$

$$Y^2 = w_o + 2\left(\frac{\xi_o c}{w_o}\right)^2 \sum_{1}^{\infty} \frac{U_o(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(1)}$$

$$-2 \frac{\xi_o c}{w_o} \frac{\omega r_b}{w_o} \sum_{n=1}^{\infty} \frac{U_o(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(2)} \qquad 118$$

Thus the axial velocity components in each of the two regions may be written down directly as

$$\frac{w}{w_0}-1 = \left(\frac{\xi_0 c}{w_0}\right)^2 \sum_{n=1}^{\infty} \frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(1)} e^{\varepsilon_n \zeta} + \frac{\xi_0 c}{w_0}\,\frac{\omega r_b}{w_0} \sum_{n=1}^{\infty} \frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(2)} e^{\varepsilon_n \zeta} \qquad \zeta<0$$

$$\begin{aligned}\frac{w}{w_0}-1 = &-\left(\frac{\xi_0 c}{w_0}\right)^2 \sum_{n=1}^{\infty} \frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(1)} e^{-\varepsilon_n \zeta} \\ &+\left(\frac{\xi_0 c}{w_0}\right)\left(\frac{\omega r_b}{w_0}\right) \sum_{n=1}^{\infty} \frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(2)} e^{-\varepsilon_n \zeta} \\ &+2\left(\frac{\xi_0 c}{w_0}\right)^2 \sum_{n=1}^{\infty} \frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(1)} \\ &-2\left(\frac{\xi_0 c}{w_0}\right)\left(\frac{\omega r_b}{w_0}\right) \sum_{n=1}^{\infty} \frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(2)} \end{aligned} \qquad \zeta>0 \qquad 119$$

Inspection of these relations verifies the assertion that the distortion of axial velocity distribution is half accomplished by the time the fluid has reached the plane of the discontinuity.

<u>Comparison of the Two Solutions</u>.- The comparison of the solution involving chordwise distribution of vorticity with that assuming a tangential discontinuity is simple, and hence particularly illuminating, in the case of a stationary blade row. Inasmuch as the radial velocities are symmetrical with respect to the center of the blade row in each distribution, only the case where $\zeta > 0$ need be discussed. Here the two expressions for the radial velocity may be written explicitly in the form

$$\frac{u}{w_o} = \left(\frac{\xi_o c}{w_o}\right)^2 \sum_{n=1}^{\infty} \frac{U_1(\varepsilon_n r)}{2\nu_n^2 \varepsilon_n} T_n^{(1)} \frac{\sinh \frac{\varepsilon_n c}{2}}{\frac{\varepsilon_n c}{2}} e^{-\varepsilon_n z} \qquad 121$$

Finite Blade Chord

$$\frac{u}{w_o} = \left(\frac{\xi_o c}{w_o}\right)^2 \sum_{n=1}^{\infty} \frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(1)} e^{-\varepsilon_n z} \qquad 122$$

Tangential Discontinuity

Thus the expression associated with the blade of finite chord differs from that associated with the vortex sheet only by the insertion of the factor

$$\frac{\sinh \frac{\varepsilon_n c}{2}}{\frac{\varepsilon_n c}{2}}$$

Furthermore the expression for the blade of finite chord reduces to that for the vortex sheet as $c \to 0$, provided that $\xi_o c \sim$ blade circulation remains constant during the process.

The dependence of this factor upon the geometry of the system is clarified by inserting the asymptotic approximations for the characteristic values. That is $\varepsilon_n \to \frac{n\pi}{r_b - r_a}$. Then $\frac{\varepsilon_n c}{2} \to \frac{n\pi}{2}\frac{c}{r_b - r_a} = \frac{n\pi}{2\,AR}$ where AR denotes the aspect ratio of the blades. Also the parameter $\varepsilon_n z \to n\pi \frac{z}{r_b - r_a}$ and hence is a measure of the distance from the blade center line in terms of the blade length. Therefore the factor

$$\frac{\sinh \frac{\varepsilon_n c}{2}}{\frac{\varepsilon_n c}{2}} \longrightarrow \frac{\sinh \frac{n\pi}{2AR}}{\frac{n\pi}{2AR}}$$

is indicative of the rate of decay of the n^{th} Bessel component of the disturbance generated by the blade of finite chord in comparison with the rate of decay of the same component generated by the vortex sheet. The asymptotic values of this expression are tabulated below for various blade aspect ratios and component orders.

$n \downarrow$ / AR $\rightarrow$	5	3	1
1	1.016	1.045	1.460
2	1.063	1.192	3.69
3	1.155	1.45	11.67
4	1.235	1.92	41.3

The influence of the chordwise vorticity distribution is predominently small for the high aspect ratios and the components of low order. For medium and small aspect ratios as well as for cases of sharply distorted flow patterns where the higher order Bessel components are of considerable magnitude, the chordwise load distribution will merit consideration.

VIII THE SECOND PROBLEM:

VORTEX COMPRESSOR OFF DESIGN CONDITION

The approximate analysis involving only one iteration becomes exact for the case where the circulation is constant along the blade length because then the field is essentially irrotational and there are no variations of the axial velocity. Inasmuch as the solution of the second problem converges more slowly than does the solution of the first problem, it is appropriate to investigate the former for a case where the approximations are known to be relatively good. Such an example is furnished by the vortex compressor operating off the design condition where by "vortex" compressor the case of constant circulation along the blade length is implied. As the operation of the machine departs from the conditions for which it was designed, the circulation varies <u>slightly</u> along the blade length and the motion of the fluid departs <u>slightly</u> from irrotational.

<u>The Boundary Conditions at the Blades</u>.- Inasmuch as the circulation about each blade and hence about the turbomachine axis is independent of the radius, for the design condition, the tangential velocity component may be written

$$v = v^* = \frac{\Gamma^*(z)}{2\pi r} \qquad 123$$

and the axial velocity is a constant.

$$w = w^* \qquad 124$$

The starred quantities (Γ^*, etc.) denote the values of the variables at the design conditions. The geometry of the guiding surfaces with respect to a set of axes fixed in the blade row under consideration is then given by

$$\left(\frac{v}{w}\right)^* = \frac{\Gamma^*(z)}{2\pi r w^*} \qquad 125$$

The tangent to the surface of a stationary blade is then given explicitly by equation 125 and is, of course, independent of the speed of rotation. For the rotating blade rows the situation is more involved. The tangential velocity with respect to the rotating wheel is then at design condition

$$v_w = \frac{\Gamma^*(z)}{2\pi r} - \omega^* r \qquad 126$$

inasmuch as the tangential velocities of the fluid and the wheel are considered positive in the same direction. The actual geometry of the rotating blade row is

$$\left(\frac{v_w}{w}\right)^* = \frac{\frac{\Gamma^*}{2\pi r} - \omega^* r}{w^*} \qquad 127$$

which gives the tangent of the blade angle with respect to the rotor itself. Inasmuch as it has been assumed that there are an infinite number of blades in each blade row, the direction of the velocity vector is uniquely determined by the blade contours (equations 125 and 127) at conditions slightly different from design. Hence for the stationary blade row

$$\frac{v}{w} = \left(\frac{v}{w}\right)^* = \frac{\Gamma^*(z)}{2\pi r w^*}$$

and the tangential velocity becomes

$$v = \frac{w}{w^*} \frac{\Gamma^*(z)}{2\pi r} \tag{128}$$

For the rotating blade row, similar conditions prevail and in particular

$$\left(\frac{v_w}{w}\right) = \frac{v - \omega r}{w} = \left(\frac{v_w}{w}\right)^* = \frac{\frac{1}{2\pi r}(\Gamma^*(z)) - \omega^* r}{w^*}$$

so that the tangential velocity in the rotating blade row may be written as

$$v = \frac{w}{w^*}\left(\frac{\Gamma^*(z)}{2\pi r} - \omega^* r\right) + \omega r \tag{129}$$

To specify further the velocity outside of the blade rows, it is assumed, according to the first approximation, that the fluid leaves each blade tangentially and continues with the same moment of momentum until its tangential velocity is changed through contact with the succeeding blade row. Now at the design condition the fluid from one blade row meets the succeeding blades tangentially and hence suffers no abrupt tangential acceleration. When the machine operates at conditions other than design, however, the fluid is forced to change direction suddenly and in the case of an infinite number of blades there exists a tangential discontinuity or vortex sheet at the leading edge of each blade row. Consequently the vorticity associated with the blade is modified not only by the equations 128 and 129 but also by the abrupt increase of vorticity at the leading edge. This is the counterpart, for the case of an infinite number of blades, of the normal increase of vorticity at the nose of a thin airfoil with increase of angle of attack.

Calculation of the Vorticity Distribution.- For calculation only equation 129 need be considered since it reduces to equation 128 by deleting the angular velocity. Starting with this expression for the tangential velocity and using the first order assumption that $\frac{\partial w}{\partial z}$ is small, the radial and axial vorticity components become

$$\xi = -\frac{\partial v}{\partial z} \sim \frac{w}{w_*} \frac{1}{2\pi r} \frac{\partial \Gamma^*}{\partial z} \qquad 130$$

$$\zeta = \frac{\partial v}{\partial r} + \frac{v}{r} = 2\left(\omega - \frac{w}{w_*}\omega^*\right) \qquad 131$$

To determine the radial velocity it is necessary to find the value $\frac{\partial \eta}{\partial z}$. Within the blade channel this function is in first approximation

$$\frac{\partial \eta}{\partial z} = \frac{1}{w}\left(\frac{\partial}{\partial z}(v\zeta) + \frac{\partial}{\partial r}(wr\xi)\right)$$

$$= \frac{2}{w}\frac{\partial}{\partial z}\left\{\left(\omega - \frac{w}{w_*}\omega^*\right)\left[\frac{w}{w_*}\frac{\Gamma^*(z)}{2\pi r} + \left(\omega - \omega^*\frac{w}{w_*}\right)r\right]\right\}$$

$$= \frac{\left(\omega - \frac{w}{w_*}\omega^*\right)}{\pi r w^*}\frac{\partial \Gamma^*}{\partial z} \qquad 132$$

inasmuch as $\frac{\partial}{\partial r}(wr\xi) = 0$. For the discontinuity at the leading edge of the blade row the function $\frac{\partial \eta}{\partial z}$ must be replaced by $[\eta]$. However

$$[\eta] = [\frac{v\xi}{w} - \frac{1}{w}\frac{\partial h}{\partial r}]$$

$$= \frac{1}{w}[v\xi] - \frac{1}{w}[\frac{\partial h}{\partial r}] \qquad 133$$

Thus in the form used previously there is the correspondence

$$\frac{\partial}{\partial z}(v\xi) \longrightarrow [v\xi] = [v(\frac{\partial v}{\partial r} + \frac{v}{r})]$$

and

$$\frac{\partial}{\partial r}(wr\xi) \longrightarrow -[\frac{\partial h}{\partial r}] = -\frac{\partial}{\partial r}[h] - \frac{\partial}{\partial r} r\omega[v]$$

assuming that the enthalpy distribution is continuous both upstream and downstream of the discontinuity of tangential velocity. The magnitude of the jump may be computed from the tangential velocities given according to equation 129. The velocity leaving the stationary blade row is

$$v = \frac{w}{w^*}\left(\frac{\Gamma^*(z)}{2\pi r}\right) \qquad 134$$

while that entering the following rotating blade row is simply

$$v = \frac{w}{w^*}\left(\frac{\Gamma^*(z)}{2\pi r} - \omega^* r\right) + \omega r \qquad 135$$

It is clear that $\Gamma^*(z)$ remains constant in the space between the blade rows so that it follows from equation 135 that

$$[v\zeta] = 2r\left(\omega - \omega^* \frac{w}{w^*}\right)^2 \qquad 136$$

in a similar manner it follows that

$$\left[\frac{\partial h}{\partial r}\right] = \frac{\partial}{\partial r}\left\{(r^2\omega)\left(\omega - \omega^* \frac{w}{w^*}\right)\right\}$$

$$= 2r\omega\left(\omega - \omega^* \frac{w}{w^*}\right) \qquad 137$$

Then from equation 133 the jump in tangential vorticity component becomes

$$[\eta] = \frac{-2r\omega^*\left(\omega - \omega^* \frac{w}{w^*}\right)}{w} \qquad 138a$$

From the preceeding analysis it is clear that just the opposite is true when passing from a rotor to a stator, that is

$$[\eta] = \frac{2r\omega^*\left(\omega - \omega^* \frac{w}{w^*}\right)}{w} \qquad 138b$$

Equations 132 and 138 now give the distribution of the disturbance in a vortex turbomachine operating off the design condition.

<u>The Radial and Axial Velocities</u>.- Using these values in the integral equation for the radial velocities, the radial velocity becomes

$$u = -\int_{-\infty}^{\infty}\int_{r_a}^{r_b} \frac{\left(\omega - \frac{w}{w^*}\omega^*\right)}{\pi w^* \alpha} \frac{\partial \Gamma^*}{\partial \zeta} \sum_{n=1}^{\infty} \frac{\alpha U_1(\varepsilon_n r) U_1(\varepsilon_n \alpha)}{2\varepsilon_n \nu_n^2} e^{-\varepsilon_n|\zeta - \beta|} \, d\alpha \, d\beta$$

$$- \sum_{i=1}^{m} \int_{r_a}^{r_b} \frac{2\omega^*\alpha\left(\omega - \omega^* \frac{w}{w^*}\right)(-1)^i}{w} \sum_{n=1}^{\infty} \frac{\alpha U_1(\varepsilon_n r) U_1(\varepsilon_n \alpha)}{2\varepsilon_n \nu_n^2} e^{-\varepsilon_n|\zeta - \beta|} \, d\alpha \, d\beta \qquad 139$$

The difference between the actions of the discontinuity sheets and the blades themselves is particularly worthy of note. The discontinuities occurring at the leading edge of consecutive blade row are of opposite sense and of very nearly the same magnitude. Consequently the radial and axial velocity changes induced by each are of opposite direction and as a result there is only a small cumulative effect due to the variation in strength of the discontinuities. The first term, however, which represents the continuous influence due to the change of the effective angle of the blades, differs from zero only for the rotating blade rows and these terms are always of the same sign. Therefore they contribute to a cumulative distortion of the velocity profiles, particularly the axial velocity. (Rigorously, the preceeding calculation will be in error as soon as $\frac{\partial w}{\partial z}$ becomes of appreciable size).

To complete the solution, the equations for the axial velocity are given below

$$w = \int_{-\infty}^{\infty}\int_{r_a}^{r_b} \frac{\omega - \omega^* \frac{w}{w^*}}{\pi\, \omega^* \alpha}\, \frac{\partial \Gamma^*}{\partial \beta} \sum_{n=1}^{\infty} \frac{\alpha U_0(\varepsilon_n r) U_1(\varepsilon_n \alpha)}{2\, \varepsilon_n \nu_n^2}\, e^{-\varepsilon_n |z-\beta|}\, d\alpha\, d\beta$$

$$+ 2\int_{-\infty}^{z}\int_{r_a}^{r_b} \frac{\omega - \omega^* \frac{w}{w^*}}{\pi\, \omega^* \alpha}\, \frac{\partial \Gamma^*}{\partial \beta} \sum_{n=1}^{\infty} \frac{\alpha U_0(\varepsilon_n r) U_1(\varepsilon_n \alpha)}{2\, \varepsilon_n \nu_n^2} \left(1 - e^{-\varepsilon_n |z-\beta|}\right) d\alpha\, d\beta$$

$$+ \sum_{j=1}^{m} \int_{r_a}^{r_b} \frac{2\omega^* \alpha \left(\omega - \frac{w}{w^*}\omega^*\right)(-1)^j}{w} \sum_{n=1}^{\infty} \frac{\alpha U_1(\varepsilon_n \alpha) U_0(\varepsilon_n r)}{2\, \varepsilon_n \nu_n^2}\, e^{-\varepsilon_n |z-a_j|}\, d\alpha \qquad 140$$

$$+ \sum_{j=1}^{m} \int_{r_a}^{r_b} \frac{2\omega^* \alpha \left(\omega - \frac{w}{w^*}\omega^*\right)(-1)^j}{w} \sum_{n=1}^{\infty} \frac{\alpha U_1(\varepsilon_n \alpha) U_0(\varepsilon_n r)}{2\, \varepsilon_n \nu_n^2} \left(1 - e^{-\varepsilon_n |z-a_j|}\right) d\alpha$$

<u>The Essential Non-Linearity</u>.- The radial and axial velocity components, given by equations 139 and 140 respectively, exhibit the non-linearity of the boundary conditions with particular force in the term $\left(\omega - \frac{w}{w^*}\omega^*\right)$ within the integral. The influence of the non-linearity may best be shown by an attempt to solve a particular problem of physical importance, namely:

Consider a vortex turbomachine which consists of consecutive similar stages, that is $\frac{\partial \Gamma^*}{\partial z}$ is a periodic function of known period. When operating at a condition different from the design point, does the flow pattern become periodic in z and if so how rapidly does it approach this periodic state?

If the problem is linearized by taking the ratio $\frac{w}{w^*}$ equal to the quotient of the actual to design flow rate, that is, neglecting the variation of axial velocity within the integral, then the "weight function" of the integral equations becomes a constant. The periodicity is clearly not obtained except for the case where $\frac{\omega}{\omega^*} = \frac{w}{w^*}$. This condition represents one which is <u>dynamically similar</u> to the design condition. This indicates that the equations, even in their approximate form, are exact in the limiting cases. Except for these cases, however, the axial velocity profile becomes continuously distorted until even in the crudest approximation, the radial variation of axial velocity can not be neglected in the integrand. Thus the approach to the periodic solution is completely dependent upon the variation of axial velocity within the integral, as intuitively it must, since this term represents the change of blade angle of attack due to the change of operating conditions. By using the Green's functions it is possible to arrive at a reasonably accurate solution by going to higher order solutions.

The cumulative axial velocity disturbance is given by the second integral and the second summation in equation 140. In principle, then, these two terms may be represented in the integral form

$$\frac{\tilde{w}}{w^*} = \int_{-\infty}^{z}\int_{r_a}^{r_b}\left(\frac{\omega}{\omega^*} - \frac{w}{w^*}\right) L(r,z)\, K(\alpha,\beta;r,z)\, d\alpha\, d\beta \qquad 141$$

If this possesses a solution periodic in z, the other terms of equation 140 will likewise have periodic solutions. But the period in z is known to be (say) T where T is the distance along the z axis between similar positions on consecutive stages. Furthermore $L(r,z)$ must be of this same period and hence for periodicity

$$\int_0^T \int_{r_a}^{r_b} \left(\frac{\omega}{\omega^*} - \frac{w}{w^*}\right) L(\alpha, \bar{z}+\beta)\, K(\alpha, \bar{z}+\beta; r, z)\, d\alpha\, d\beta = 0 \qquad 142$$

so that from the integral equation

$$\frac{w}{w^*}(\bar{z}) = \frac{w}{w^*}(\bar{z}+T) \qquad 143$$

This, together with the auxiliary condition that

$$\frac{Q}{Q} = \frac{1}{w^*(r_b^2 - r_a^2)} \int_{r_a}^{r_b} w\, r\, dr \qquad 144$$

constitute the existence conditions for a periodic solution. The actual existence is clearly not dependent upon the form of L so long as it is periodic, but only on the expression $\frac{\omega}{\omega^*} - \frac{w}{w^*}$ and the kernel of the integral equation.

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