SOME PROBLEMS CONCERNING

THE ROTATIONAL MOTION OF A PERFECT FLUID

Thesis by

Frank E. Marble

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology
Pasadena, California
1948

ACKNOWLEDGEMENTS

It is a pleasure to acknowledge the considerable interest and assistance offered by members of the faculty in Aeronautics, Mathematics, and Applied Mechanics during the course of my research. In connection with the boundary value problems encountered, Professors De Prima, Lagerstrom and Erdelyi generously devoted considerable time to several stimulating discussions which contributed materially to my understanding of these problems.

In particular I should like to express my deep appreciation to Professor Hans Wolfgang Liepmann, advisor to my research, whose insistence upon high scientific standards as well as his friendship and personal encouragement have had a marked influence upon all of my work at the California Institute of Technology.

My grateful thanks are also extended to the National Research Council for making my continuation of graduate work possible by the generous grant of a Predoctoral Fellowship.

The rather extensive calculations of Part IV were competently undertaken by the staff of the 10 foot GALCIT Wind Tunnel. The difficult and tedious task of typing the thesis was excellently done by Mrs. Beverley Cottingham and Miss Alberta Pampeyan.

SUMMARY

In an effort to obtain some understanding of the processes involved in the rotational motion of a perfect fluid several particular linearized examples of rotational flow are solved in detail. The first part discusses some types of boundary value problem which arise. The solution of the non-linear partial differential equation by a particular iteration process is considered and the process is shown to converge for an extended version of the problem when the vorticity distribution is sufficiently smooth. The first step of the iteration process may constitute a good approximation in these cases and is taken as the basis of linearized solutions studied in the remainder of the work.

The process of straightening a non-uniform velocity profile by means of an idealized screen is considered in Part II as a problem in rotational motion of an ideal fluid with the screen replaced by an appropriate non-conservative force field. The detailed solution is given for both the linearized problem and the second approximation. The complete second order correction is less than 6 percent of the local velocity given by the linear solution for a rather severe case. The corrections arising from the various physical processes involved are analyzed and found to exceed 6 percent in some cases but are inherently compensating.

The two-dimensional rotational flow about a closed body is

obtained in Part III by utilizing the Green's function method of solving the inhomogeneous differential equation involved. The conformal transformation which maps the given contour into a circle is used to find the appropriate Green's function for the contour. Solutions are then written down for any body, the Riemann mapping function of which is known. The Blasius force and moment formula are extended to include the case of general rotational motion, the relations of Kuo appearing as special forms where the vorticity distribution is uniform.

In the final part the theory of the three-dimensional flow through an axial turbomachine, associated with variation of circulation along the blade length, is described as an extension of the classical theory of finite wings and is simplified to a problem in axially symmetric rotational fluid motion by considering an infinite number of blades in each row. The linearized problem is solved for the radial, tangential, and axial velocity components induced by a single row of stationary or rotating blades with finite chord and prescribed loading. The particular case for which the blade chord approaches zero, and the tangential velocity changes discontinuously, is associated with the theory of the Prandtl lifting line for finite wings. The complete solution is given for a single stationary or rotating blade row of given loading with a hub/tip ratio of 0.6 and blade aspect ratio of 2. The corresponding discontinuous approximation is compared with the more nearly exact solution and is shown to constitute a useful approximation to the solution for a finite blade chord when the discontinuity is located appropriately. An exponential approximation for the velocity components, deduced from the analysis, allows rapid estimation of the rate at which the equilibrium velocity profiles develop ahead of and behind a blade row and, using the superposition principle, provides a simple means of approximating the velocity distribution in a multistage turbomachine and of discussing mutual interference of blade rows.

TABLE OF CONTENTS

INTRODUCTION

The growth of the classical hydrodynamics in the middle of the nineteenth century owed much of its vigor to the realization that the assumption of irrotationality led to a set of linear problems similar to those of potential theory which had been developed to a considerable degree. The physical implications of irrotationality have proven reasonably valid, except for the boundary layer on a solid surface, for a great variety of problems and have served particularly well in aeronautical applications. However the recent developments in high speed flight, reaction propulsion, combustion in a moving fluid, etc. have given new impetus to the study of rotational fluid motion.

From a scientific point of view the principal interest in rotational flow lies in the fact that it is a non-linear problem the behavior of whose solutions is not well understood physically or mathematically. Inasmuch as there is, at the present time, little chance of solving the non-linear problem with any degree of generality, the solutions investigated are of a linearized type. In fact the present work consists essentially in a wholesale exploitation of a particular method of linearization where the transport of vorticity is fixed according to the streamlines of the irrotational solution.

Clearly the most interesting of the problems involved in rotational fluid motion are left untouched by such an analysis.

However it is felt that considerable familiarity with both the mathematics and physics of the problem is to be gained by considering these linearized examples in some detail. Furthermore such a linearized analysis serves the purpose of isolating those features of rotational flow which are essentially non-linear in much the same manner as investigation of subsonic and supersonic linearized flow has tended to emphasize the properties of transonic flow which are essentially non-linear. Because of the abundance of examples from which to choose, problems treated in the present work have been restricted to those which have some interesting and familiar physical counterpart. This is done not solely to insure technical interest but also because the physical guidance necessary in working with approximate methods comes much more freely from familiar physical situations.

The work is presented in four nearly independent portions connected only by the general subject matter and the nature of linearization employed in the analysis. As a consequence it is more readily accessible for one interested in a particular problem than if it had been made artificially continuous. The first part discusses the general problem of plane and axially symmetric rotational motion with and without a non-conservative force field. The solution of the problem by iteration is described, the first step of which is the linearized solution used in the remaining portion of the paper. The second part applies the first and second approximations to the problem of straightening a non-uniform flow by means of an idealized screen while Part III considers the general plane rotational flow about arbitrary closed contours. The final portion deals with the flow through an axial turbomachine as a problem in axially symmetric rotational flow with a non-conservative force field. Solutions for the linearized velocity field are obtained with reasonable simplicity.

I. - ON THE APPROXIMATE SOLUTION OF SOME BOUNDARY VALUE PROBLEMS IN THE ROTATIONAL MOTION OF A PERFECT FLUID

ON THE APPROXIMATE SOLUTION OF SOME BOUNDARY VALUE PROBLEMS IN THE ROTATIONAL MOTION OF A PERFECT FLUID

1. Introduction:- The mathematical difficulties encountered in solving problems concerning rotational fluid motion result from the non-linearity of both the equations of motion and the boundary conditions. The non-linearity in the equations of motion arises physically from the dependence of the solution upon the distribution of vorticity which is determined, in turn, by the manner in which the vorticity is transported by the final velocity field. Non-linearity of the boundary conditions appears when the rate of vorticity generation is determined by the interaction of final flow with given solid boundaries instead of through the action of known forces.

Unfortunately the physical problem is not completely clear in general as to just what boundary conditions are prescribed. It is obvious, for example, that the vorticity may be prescribed upstream on a transversal to the streamlines and that certain other information on the flow may be given at the upstream boundary of the region concerned. However it is not quite certain what, if anything, can physically be prescribed far downstream or, in fact, just what downstream means in come cases.

Physically it is of considerable importance to determine a reasonable linear approximation to the solution of rotational flow problems. It is clear that the necessary condition for linearization is to neglect the interaction between the vorticity and the velocity induced by this vorticity which, in turn, implies finding a manner of prescribing a priori a vorticity distribution which is a good approximation to the final one. The fact that this is not always possible suggests an iteration procedure of solution a few steps of which may lead to a satisfactory approximation.

Hence the problems which are of both physical and mathematical interest are to find, for a particular statement of the physical problem, conditions under which

a) A solution of the problem exists.

b) The solution is unique.

c) The solution may be found by a particular iteration process.

For the non-linear equations involved in rotational flow this is not a simple matter for although it seems reasonable on physical grounds that a solution exists, it is equally plausible that it is a discontinuous one, or one corresponding to instability. Furthermore it is quite obvious that the solutions are not unique under some circumstances although features of the problem on which the uniqueness depends are not easily singled out. Consequently these questions are investigated in the following section only to the extent of showing some of the interesting complications which may arise. In order that the iteration and approximation have physical significance, the process of vorticity generation and transformation is considered first following which the equations describing plane rotational flow are written down. The solutions for several statements of the boundary value problem are then discussed and an iteration process is developed the first step of which is denoted the linearized approximation for the rotational flow problem. A similar program is undertaken for the case of axially rotational flow with and without a tangential velocity component.

2. Generation and Transformation of Vorticity in a Perfect Fluid: -

The steady motion of a perfect fluid is governed by the Eulerian equation of motion

$$u_j u_{i,j} + \frac{1}{\rho} p_{,i} = F_i \qquad i,j = 1,2,3 \qquad 2.1$$

and the continuity equation

$$u_{i,i} = 0 \qquad 2.2$$

where the u_i are the three velocity components in the cartesian coordinate system x_i , p and ρ are the pressure and density, and F_i the components of force per unit mass. The process of generating and transforming vorticity is shown most clearly by taking the curl of the equations of motion and using equation 2.2

$$\frac{d \xi_i}{dt} \equiv u_j \xi_{i,j} = \xi_j u_{i,j} + \varepsilon_{ijk} F_{j,k} \qquad 2.3$$

where the $\xi_i = \varepsilon_{ijk} u_{j,k}$ are the vorticity components in the x_i coordinate system. Now a small closed path in a fluid possesses a circulation corresponding to the number of vortex lines passing through it or, in other words, a circulation equal to the product of the area enclosed by the path and the mean value of the vorticity normal to this area. These vortex lines constitute a vortex tube the circulation of which is unchanged through all deformations of the fluid except in the presence of a non-conservative force field. Deformations which extend or contract the vortex tube along its length will increase or decrease the vorticity correspondingly as the path around the vortex tube contracts or stretches. All other deformations merely change the local orientation of the tube. This process is expressed mathematically by equation 2.2.

In the case of conservative forces $\epsilon_{ijk} F_{j,k}$ vanishes and equation 2.3 describes the progressive distortion of a rotational fluid motion by the rate-of-deformation tensor $u_{i,j}$. Clearly the diagonal terms $u_{i,i}$ represent the rate of increase of the length of the fluid filaments in the x_i direction and $\xi_i u_{i,i}$ denotes the rate of increase of ξ_i resulting from this stretching. The terms $u_{i,j}$ $(i \neq j)$ represent, on the other hand, rates of rotation of planes normal to x_i and hence the rates of rotation of vortex filaments normal to these planes. Consequently the terms $\xi_j u_{i,j}$ $(i \neq j)$ denote the kinematic rates of turning the ξ_j vorticity component into the x_i direction. If, in particular, the flow is two dimensional and the force field is conservative, the rate of change of vorticity along a streamline vanishes and the vorticity is a property of the streamlines.

In a non-conservative force field a rotational motion will be generated from an initially irrotational one or the vorticity of an existing rotational field will be modified. For if C is a path with tangent vector l_i enclosing an area S , the curl of the force vector is

$$\epsilon_{ijk} F_{j,k} = \lim_{S \to 0} \frac{1}{S} \oint F_i \, dl_i \qquad 2.4$$

where the integral represents the unbalanced moment per unit mass tending to set up a rotation of the individual fluid elements.

The conditions necessary for linearizing the rotational motions follow directly from equation 2.3 by considering a known initial rotational motion with velocity and vorticity components U_i and Z_i respectively, to be perturbed by another flow u_i, ζ_i resulting from either a change in vorticity far upstream or an additional set of forces. Then

$$U_j(Z_i+\zeta_i)_{,j} + u_j(Z_i+\zeta_i)_{,j} = (Z_j+\zeta_j)U_{i,j} + (Z_j+\zeta_j)u_{i,j} + \varepsilon_{ijk}F_{j,k} \qquad 2.5$$

which expresses the fact that the initial and perturbation vorticities are transported and transformed separately by the initial and perturbation velocities. The expression is linearized by deleting the transport and deformation of the perturbation vorticity by the velocity associated with perturbation vorticity. Neglecting this interaction

$$U_j(Z_i+\zeta_i)_{,j} + u_j Z_{i,j} = (Z_j+\zeta_j)U_{i,j} + Z_j u_{i,j} + \varepsilon_{ijk}F_{j,k} \qquad 2.6$$

and the validity of the linearization depends upon the magnitude and smoothness of the perturbation as well as on the nature of the initial flow.

The equations of motion which correspond to this linearized vorticity transport process are

$$\left.\frac{P}{\rho}\right)_{,i} = \varepsilon_{ijk}\left(U_j Z_k + U_j \zeta_k + u_j Z_k\right) + F_i \qquad 2.7$$

where P, the local stagnation pressure is defined in the linearized form as

$$P = p + \frac{\rho}{2}\left(U_j U_j + 2 U_j u_j\right) \qquad 2.8$$

and p is the local static pressure.

<u>3. Plane Rotational Motion:</u> - The rotational motion of a perfect fluid confined to a plane is particularly simple because the vorticity is transported unmodified along the streamlines. In this case equations 2.3 reduce to the single relation

$$u_i \xi_{3,i} = F_{2,1} - F_{1,2} \equiv \mathcal{F}(x_i, u_i) \qquad 3.1$$

If now the stream function ψ is introduced with the properties $\psi_{,1} = -u_2$, $\psi_{,2} = u_1$, and lines $\psi =$ constant are denoted streamlines, then equation 3.1 becomes

$$w \frac{\partial}{\partial s}(\xi) \equiv -w \frac{\partial}{\partial s} \nabla^2 \psi = \mathcal{F}(x_i, \psi_{,i}) \qquad 3.2$$

where w is the magnitude of the velocity vector $\sqrt{u_1^2 + u_2^2}$ and s is the arc length measured along a streamline. This equation is clearly non-linear and care must be exercized in specifing the boundary conditions. For consider the Eulerian equations of motion and the constant equations which describe the flow

$$\begin{aligned} u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + \frac{1}{\rho} \frac{\partial p}{\partial x_1} &= F_1 \\ u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} + \frac{1}{\rho} \frac{\partial p}{\partial x_2} &= F_2 \\ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} &= 0 \end{aligned} \qquad 3.3$$

and consider values of $u_1(\lambda)$, $u_2(\lambda)$, $p(\lambda)$, prescribed along a curve $x_1(\lambda)$, $x_2(\lambda)$ with parameter λ . Along this curve the derivitives are related

$$\frac{\partial u_1}{\partial x_1}\frac{dx_1}{d\lambda} + \frac{\partial u_1}{\partial x_2}\frac{dx_2}{d\lambda} = \frac{du_1(\lambda)}{d\lambda}$$

$$\frac{\partial u_2}{\partial x_1}\frac{dx_1}{d\lambda} + \frac{\partial u_2}{\partial x_2}\frac{dx_2}{d\lambda} = \frac{du_2(\lambda)}{d\lambda}$$

$$\frac{\partial p}{\partial x_1}\frac{dx_1}{d\lambda} + \frac{\partial p}{\partial x_2}\frac{dx_2}{d\lambda} = \frac{dp(\lambda)}{d\lambda} \qquad 3.4$$

Hence unique algebraic evaluation of the first velocity and pressure derivitives from equations 3.3 and 3.4 is possible if and only if

$$\begin{vmatrix} u_1 & u_2 & 0 & 0 & \frac{1}{\rho} & 0 \\ 0 & 0 & u_1 & u_2 & 0 & \frac{1}{\rho} \\ 1 & 0 & 1 & 0 & 0 & 0 \\ \dot{x}_1 & \dot{x}_2 & 0 & 0 & 0 & 0 \\ 0 & \rho & \dot{x}_1 & \dot{x}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dot{x}_1 & \dot{x}_2 \end{vmatrix} \neq 0 \qquad 3.5$$

But if the curve along which these values are prescribed is a streamline, an additional restriction is imposed, namely that the velocity be tangential to the curve. This relation

$$\begin{vmatrix} u_1 & u_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} = 0 \qquad 3.6$$

assures the vanishing of the determinant so that unique solutions do not exist in the neighborhood of a streamline along which initial values are given. Physically this is clear, for if velocities and pressure are given along a streamline the neighboring flow is not determined unless the velocity gradient normal to the streamline, i.e. the vorticity, is prescribed also. Consequently the streamlines are characteristics of the set of differential equations 3.3 and hence of equation 3.2. The boundary conditions must be prescribed on transversals to the streamlines and certain of these, namely the vorticity, upon paths which cut the streamlines only once.

4. <u>The Boundary Value Problems:</u> - Clearly the boundary value problem for rotational flow may take one of several forms and the first example will consist in the following physical problem: Consider the flow between two arbitrary walls when the stream function and the vorticity are prescribed on an upstream transversal and the flow direction is prescribed downstream, find the stream function throughout the domain $\mathcal{R}$.

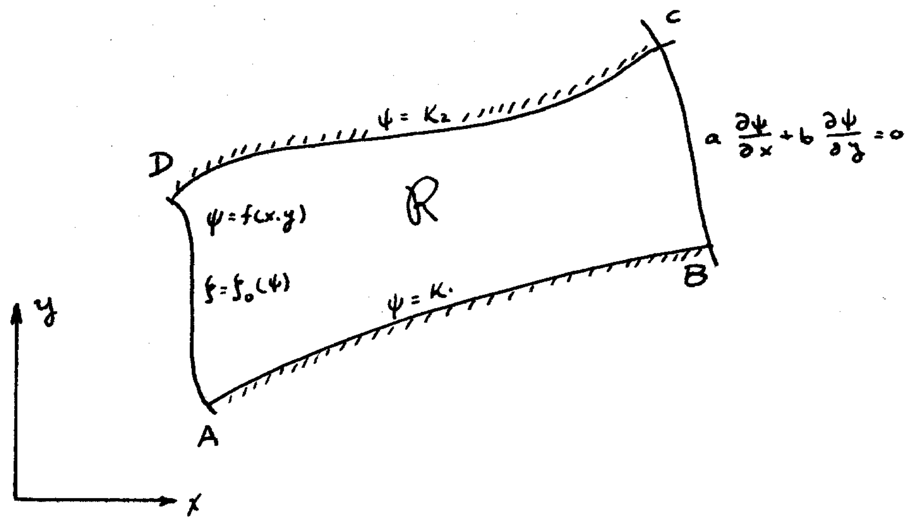

Figure I.1

Within the scope of perfect fluids this corresponds to the following mathematical problem

$$
\begin{aligned}
&w \frac{\partial}{\partial s} \nabla^2 \psi = 0 \\
&\zeta = \zeta_0(\psi) \text{ on } AD; \ |\zeta_0| \le m \qquad K_1 \le \psi \le K_2 \\
&\psi = K_1, K_2 \text{ on } AB, DC \ ; \quad K_2 > K_1 \\
&\psi = f(x,y) \text{ on } AD \\
&a(x,y) \frac{\partial \psi}{\partial x} + b(x,y) \frac{\partial \psi}{\partial y} = 0 \quad \text{on } BC
\end{aligned}
\tag{4.1}
$$

where S is the arc-length measured along a streamline. If it is assumed that no stagnation points (i.e. point where $w = 0$) will occur except on the solid boundaries, then the differential equation is easily integrated and the boundary condition on ζ employed to give an equivalent problem

$$\begin{aligned} \nabla^2 \psi &= -\zeta_0(\psi) \\ \psi &= K_1, K_2 \quad \text{on } AB, DC \\ \psi &= f(x,y) \quad \text{on } DA \\ a\frac{\partial \psi}{\partial x} + b\frac{\partial \psi}{\partial y} &= 0 \quad \text{on } BC \end{aligned} \qquad 4.2$$

Here the differential equation expresses the fact that the vorticity reamins constant along the streamlines and hence depends only on the streamfunction ψ . However ζ_0 is defined physically only for $K_2 \geq \psi \geq K_1$ so that the differential equation $\nabla^2\psi = -\zeta_0(\psi)$ is defined only over that part of the region R for which ψ lies within this range. Physically this means that conditions may arise where, for example, closed streamlines appear within the region (Figure I.1) for which no vorticity value is prescribed by the initial values ζ_0 . But this clearly involves at least one

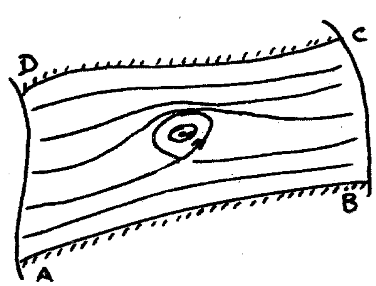

Figure I.2

stagnation point within the fluid field and hence is related to the original assumption that the gradient of the stream function has no zeros within R.

For purposes of solution it will be sufficient to extend the definition of the vorticity on the streamlines to cover all possible values of the stream function by a function $\bar{\zeta}_o(\psi)$ which agrees with $\zeta_o(\psi)$ in the range $K_2 \geq \psi \geq K_1$ and is bounded and continuous elsewhere. Then the solution of the problem may be divided into two parts: the solution of the homogeneous equation with inhomogeneous boundary conditions, and the solution of the inhomogeneous equation with the homogeneous boundary conditions. If ψ_o is denoted the solution of the elliptic equation with mixed boundary conditions

$$\begin{aligned} &\nabla^2\psi_o = 0 \\ &\psi_o = K_1, K_2 \quad \text{on } AB, DC \\ &\psi_o = f(x.y) \quad \text{on } AD \\ &a\frac{\partial\psi_o}{\partial x} + b\frac{\partial\psi_o}{\partial y} = 0 \quad \text{on } BC \end{aligned} \qquad 4.3$$

and if $\psi = \psi_o + \bar{\psi}$, then $\bar{\psi}$ is the solution of the problem with homogeneous boundary conditions.

$$\begin{aligned} &\nabla^2\bar{\psi} = -\bar{\zeta}_o(\psi_o + \bar{\psi}) \\ &\bar{\psi} = 0 \quad \text{on } CDAB \\ &a\frac{\partial\bar{\psi}}{\partial x} + b\frac{\partial\bar{\psi}}{\partial y} = 0 \quad \text{on } BC \end{aligned} \qquad 4.4$$

But these mixed boundary conditions are still those of an elliptic equation so that if $G(x.y;\xi.\eta)$ denotes the appropriate Green's function, the problem 4.4 may be transformed into the non-linear integral equation

$$\bar{\psi} = \iint_{R} G(x.y;\xi.\eta)\,\bar{\zeta}_0(\psi_0+\bar{\psi})\,d\xi\,d\eta \qquad 4.5$$

Consequently equation 4.3 and 4.5 constitute a problem equivalent, under the assumption of non vanishing gradient ψ and the extension of ζ_0 , to the problem 4.1.

The non-linear integral equation 4.5 may be replaced by a sequence of linear problems

$$\begin{aligned} \bar{\psi}_1 &= \iint_{R} G(x.y;\xi.\eta)\,\bar{\zeta}_0(\psi_0)\,d\xi\,d\eta \\ &\vdots \\ \bar{\psi}_{n+1} &= \iint G(x.y;\xi.\eta)\,\bar{\zeta}_0(\psi_0+\bar{\psi}_n)\,d\xi\,d\eta \\ &\vdots \end{aligned} \qquad 4.6$$

if the iterated stream functions $\bar{\psi}_n$ form a convergent sequence. Now this iteration process has a very simple physical interpretation with regard to the linearization of rotational flow mentioned in paragraph 2. The first approximation $\bar{\psi}_1$ corresponds to the stream function of the velocity field induced by assuming the initial vorticity to be transported along streamlines ψ_0 of the irrotational flow. This corresponds, clearly, to neglecting the interaction between the vorticity transport and the velocities induced by this vorticity. Then higher linear approximations, say $\bar{\psi}_{n+1}$, corre-ponds to assuming the vorticity to be transported by the basic irrotational flow and the n^{th} approximation to the rotational part of the flow.

If the Green's function $G(x,y;\xi,\eta)$ has the property that

$$\left| \iint_{R} G(x,y;\xi,\eta)\, d\xi\, d\eta \right| < M \tag{4.7}$$

for all values of x,y in R, then the $\bar{\psi}_n$ are uniformly bounded

$$|\bar{\psi}_n| \leq M m \qquad ; \quad m \geq \max|\bar{\zeta}_0|$$

Furthermore considering the absolute value of the difference between any two successive approximations

$$\begin{aligned} |\psi_{n+1}-\psi_n| &= \left| \iint_{R} G(x,y;\xi,\eta)\left(\bar{\zeta}_0(\psi_0+\bar{\psi}_n)-\bar{\zeta}_0(\psi_0+\bar{\psi}_{n-1})\right) d\xi\, d\eta \right| \\ &\leq M \max\left|\frac{d\bar{\zeta}_0}{d\psi}\right| |\psi_n-\psi_{n-1}| \\ &\leq \left(M \max\left|\frac{d\bar{\zeta}_0}{d\psi}\right|\right)^n |\bar{\psi}_1| \end{aligned} \tag{4.8}$$

and this inequality holds uniformly for all x,y in R. Hence if $\max\left|\frac{d\bar{\zeta}_0}{d\psi}\right|$ is sufficiently small so that

$$M \max\left|\frac{d\bar{\zeta}_0}{d\psi}\right| < 1$$

the sequence ψ_n of iterated stream functions converges to a solution of 4.5 and this solution is unique. Furthermore if this solution is such that $\psi_0+\bar{\psi} = \psi$ has a non-vanishing gradient at all points and involves only values of ψ

$$K_2 \geq \psi \geq K_1$$

the result is a solution of 4.1.

The restrictions which it has been necessary to impose on the problem are of particular physical significance and interest. A point at which the velocity or the gradient of ψ vanishes corresponds to a saddle point in the region of flow.

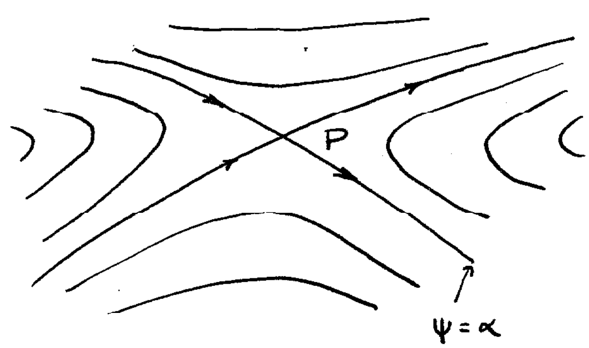

Figure I.3

A fluid element moving along the streamlines $\psi = \alpha$ toward P never reaches P because of the vanishing velocity. Hence the process of vorticity transport does not describe conditions at the point P since no change of prescription on the streamline $\psi = \alpha$ can modify the vorticity at P. Consequently P is a singular point of the vorticity distribution so that if the iteration process of solution just described leads to internal stagnation points it is certainly not unique. If, however, it does not lead to stagnation points and ranges over only the restricted range of ψ, then it is the only such solution. If, on the other hand, the process leads to a velocity field with internal stagnation points it is no indication that a solution free from these singularities does not exist, for clearly a singularity may arise in the course of the iteration and remain there. The occurance of such a phenomenom does, however, suggest that the singularity-free flow might be unstable to perturbations of sufficient magnitude to create an internal stagnation point momentarily.

The condition that the initial vorticity distribution be sufficiently smooth, i.e. $\max\left|\frac{d\zeta_0}{d\psi}\right|$ be sufficiently small, leads to the possibility of a first order approximation or truly linearized solution. For the difference between the first and second approximations to the rotational part of the stream functions may be bounded uniformly in x, y

$$\left|\bar{\psi}_2 - \bar{\psi}_1\right| \leq M \max\left|\frac{d\zeta_0}{d\psi}\right|\left|\bar{\psi}_1\right| \qquad 4.9$$

so that if $\max\left|\frac{d\zeta_0}{d\psi}\right|$ is a sufficiently small quantity the second alteration to the flow pattern produces changes of ψ of second order with respect to the first. This is actually then a perturbation solution where

$$\psi \approx \psi_0 + \psi_1$$

with any desired accuracy for a sufficiently small disturbance $\left|\frac{d\xi_0}{d\psi}\right|$.

Another point of some interest may be brought out by considering a slightly modified version of problem 4.1

$$
\begin{aligned}
&\omega \frac{\partial}{\partial s} \nabla^2 \psi = 0 \\
&\xi = \xi_0(x,y) \quad \text{on } AD \quad |\xi_0| \le m \\
&\psi = K_1, K_2 \quad \text{on} \quad AB, DC; \quad K_2 > K_1 \\
&a \frac{\partial \psi}{\partial x} + b \frac{\partial \psi}{\partial y} = 0 \quad \text{on } BC \\
&c \frac{\partial \psi}{\partial x} + d \frac{\partial \psi}{\partial y} = 0 \quad \text{on } AD
\end{aligned}
\qquad 4.10
$$

The formal procedure and restrictions follow similarly as before but whereas previously the curve AD was known to be a transversal because the stream function was prescribed monotonically on it, this is no longer clear and it is possible that at any point in the iteration procedure the streamlines may loop back and cross the initial curve again thereby invalidating the assumption that it is a transversal.

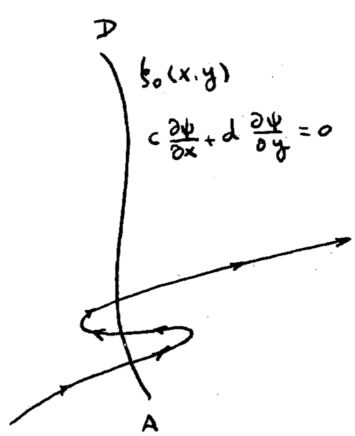

Figure I.4

The conditions under which this phenomenom occurs are related to all boundary values prescribed and not merely those on the initial transversal. An investigation shows that the physical situations related to the mathematical problem 4.10 do not make sense with the vorticity prescribed as a function of position along the initial curve but rather as a function of ψ itself. For example in the case of a channel with two sets of guide vanes which prescribe the direction of the velocity, it is a

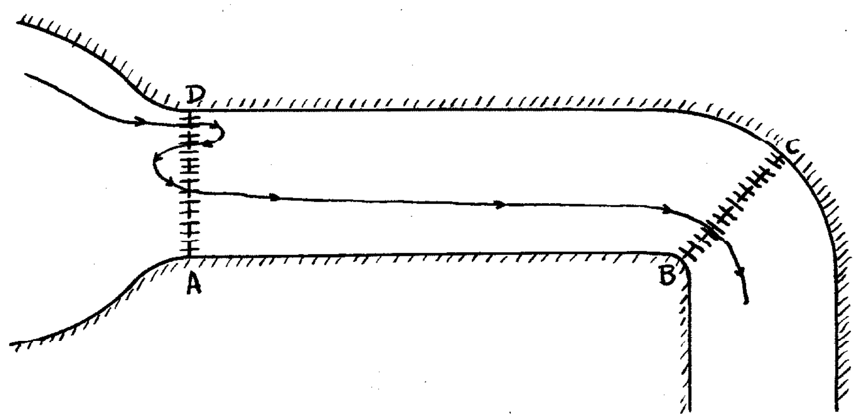

Figure I.5

distinct physical possibility that the presence of the downstream set of vanes may cause a reversal of flow in the first set of vanes without making the flow physically invalid. But it is equally clear that in such a case the vorticity must be prescribed on the streamlines from conditions other than the point or points at which it crosses the plane of the guide vanes. It is clear, therefore, that replacing the condition $\zeta_0 = \zeta_0(x,y)$ in problem 4.10 by the condition $\zeta_0 = f_0(\psi)$ gives a problem making physical sense and offering no essential complications over those offered by problem 4.1.

It is interesting to note the influence of the hyperbolic nature of rotational flow in the difficulties encountered in discussing possible solutions. The streamlines are true hyperbolic characteristics and the terminal streamlines on the initial curve cut out a domain of pseudo-dependence in the sense that the vorticity

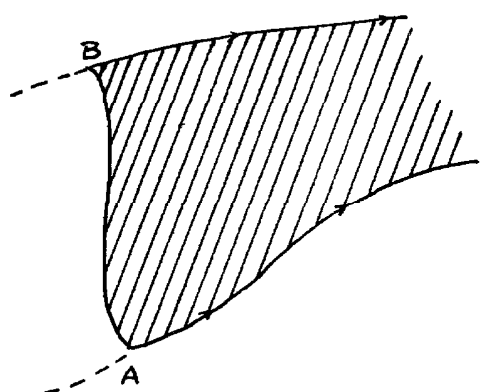

Figure I.6

is determined only in this region. The cases for which the iteration process holds are those where this domain coincides with $\mathcal{R}$. In a manner similar to that for non-linear hyperbolic equations, the initial value problem has a unique solution in the neighborhood of the initial curve.

Considerations so far have been restricted to the cases where the force field is conservative and hence does not change the rotational character upon a streamline. Many of the most interesting problems, however, deal with a non-conservative field of forces which depends not only on position but also upon the local values of the derivatives of ψ. This problem may be posed in the form

$$\begin{aligned} w \frac{\partial}{\partial s} \nabla^2 \psi &= - \mathcal{F}\left(x, y, \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right) \\ \xi_0 &= \bar{\xi}_0(\psi) \quad \text{on } AD \\ \psi &= K_1, K_2 \quad \text{on } AB, CD \\ \psi &= f(x, y) \quad \text{on } AD \\ a\frac{\partial \psi}{\partial x} + b\frac{\partial \psi}{\partial y} &= 0 \quad \text{on } BC \end{aligned} \qquad 4.11$$

where for $\left|\frac{\partial \psi}{\partial x}\right| < N$; $\left|\frac{\partial \psi}{\partial y}\right| < N$ the function $\mathcal{F}$ vanishes outside of a closed domain $D \subset R$ and is bounded within this domain. With the provision that $w \neq 0$ in R, the differential equation and the boundary conditions on ξ_0 may be transformed into an integro-differential equation

$$\begin{aligned} \nabla^2 \psi &= - \bar{\xi}_0(\psi) - \int_{\psi}^{x,y} \frac{\mathcal{F}\left(x, y; \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right)}{\sqrt{\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2}}\, ds \\ &= - \bar{\xi}_0(\psi) - \int_{\psi}^{x,y} \mathcal{F}'\left(x, y; \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right) ds \end{aligned} \qquad 4.12$$

where s is the arc length measured along a streamline and the integration is carried out along a streamline ψ = constant from the initial curve to the point x, y. As in the previous example if ψ_0 denotes the solution of the homogeneous problem 4.3 with inhomogeneous boundary conditions, then $\psi = \psi_0 + \bar{\psi}$ where $\bar{\psi}$ is a solution of

$$\nabla^2\bar{\psi} = -\bar{\xi}_0(\psi) - \int_{\psi_0+\bar{\psi}}^{x,y} \mathcal{F}'\left(x, y, \frac{\partial\, \psi_0+\bar{\psi}}{\partial x}, \frac{\partial\, \psi_0+\bar{\psi}}{\partial y}\right) ds$$

$$\bar{\psi} = 0 \quad \text{on} \quad CDAB \qquad 4.13$$

$$a\frac{\partial\psi}{\partial x} + b\frac{\partial\psi}{\partial y} = 0 \quad \text{on } BC$$

This, in turn, is easily transformed formally to an integral equation

$$\bar{\psi} = \iint_{R} G(x,y;\xi,\eta)\left\{ \bar{\xi}_0(\psi) + \int_{\psi_0+\bar{\psi}}^{\xi,\eta} \mathcal{F}'\left(\xi,\eta; \frac{\partial\psi_0+\bar{\psi}}{\partial\xi}, \frac{\partial\psi_0+\bar{\psi}}{\partial\eta}\right) ds \right\} d\xi\, d\eta \qquad 4.14$$

where $G(x,y;\xi,\eta)$ is the appropriate Green's function. The solution 4.14 may be found as the limit of a sequence of linear problems

$$\bar{\psi}_1 = -\iint_{R} G(x,y;\xi,\eta)\left\{ \bar{\xi}_0(\psi_0) + \int_{\psi_0}^{\xi,\eta} \mathcal{F}'\left(\xi,\eta, \frac{\partial\psi_0}{\partial\xi}, \frac{\partial\psi_0}{\partial\eta}\right) ds \right\} d\xi\, d\eta$$

$$\vdots$$

$$\bar{\psi}_{n+1} = -\iint_{R} G(x,y;\xi,\eta)\left\{ \bar{\xi}_0(\psi_0+\bar{\psi}_n) + \int_{\psi_0+\bar{\psi}_n}^{\xi,\eta} \mathcal{F}'\left(\xi,\eta, \frac{\partial\,\psi_0+\bar{\psi}_n}{\partial\xi}, \frac{\partial\,\psi_0+\bar{\psi}_n}{\partial\eta}\right) ds \right\} d\xi\, d\eta \qquad 4.15$$

$$\vdots$$

if the limit exists. But even though the function $\mathcal{F}'\left(\xi,\eta, \frac{\partial\,\psi_0+\bar{\psi}_n}{\partial x}, \frac{\partial\,\psi_0+\bar{\psi}_n}{\partial y}\right)$ is bounded in D and vanishes outside of D, it can not easily be shown that the integral $\int_{\psi_n}^{x,y} \mathcal{F}'\, ds$ is finite for all n and all x, y

in D. It is not clear, for instance that the path of integration within D remains of finite length for there appears no reason why, upon successive iterations, the particular streamline may not wind about in the region as long as it pleases.

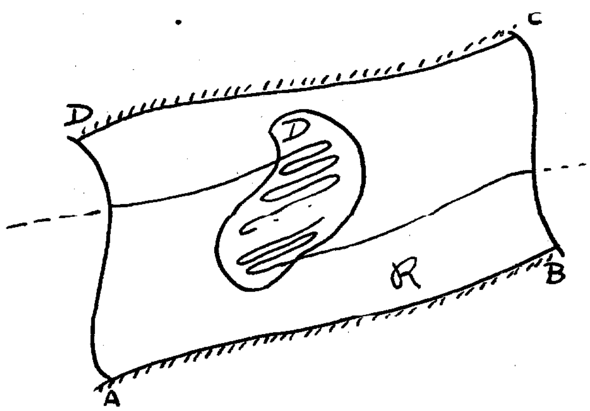

Figure I.7

This implies, however, that the streamline is either reentrant, that is closed in D, or comes arbitrarily close to one point an infinite number of times. In either case an internal stagnation point occurs. The difference between the limitations on this solution and on that where only initial vorticity was present lies in the existence of the limit of the iteration process. When only initial vorticity was present the process could be made convergent by extending ζ_0 to $\bar{\zeta}_0$ and then the final solutions checked for singularities. In the present case, however, the iteration process is not known to be convergent even for the extended problem. It seems physically probable, however, that the sequence 4.13 will converge if the forces are small enough and the distribution smooth enough.

5. <u>The Distribution of Energy and Pressure:</u> - The local values of the static pressure p and the energy $\frac{P}{\rho} \equiv \frac{p}{\rho} + \frac{u_i u_i}{2}$ may easily be found by considering the manner in which the energy is transported and changed by the flow process. Scalar multiplication of equation 2.1 by the velocity gives the result

$$\left(\frac{u_j u_j}{2}\right)_{,i} u_i + \frac{1}{\rho} p_{,i} u_i = F_i u_i$$

or

$$u_i \left(\frac{u_j u_j}{2} + p\right)_{,i} \equiv u_i \left(\frac{P}{\rho}\right)_{,i} = F_i u_i \qquad 5.1$$

where $u_i \left(\frac{P}{\rho}\right)_{,i}$ represents the rate of change of energy in passing with the stream along a streamline ψ = constant. Clearly then when the forces vanish, the energy remains constant along a streamline and its value along each streamline can be determined, except for a constant, from equations 2.1 and the initial values of the vorticity. Consequently the derivatives of the total energy distribution are

$$\left.\frac{P}{\rho}\right)_{,i} = - \psi_{,i} \, \zeta_0(\psi) \qquad 5.2$$

from which the energy distribution $h(\psi)$ follows directly and also the static pressure distribution from the definition of the energy per unit mass.

Where the forces per unit mass F_i do not vanish identically equation 5.1 gives its variation to be equal to the rate of work done on the fluid by this set of forces. That is

$$w \frac{\partial}{\partial s}\left(\frac{P}{\rho}\right) = F_i u_i \qquad 5.3$$

which may be integrated directly to give

$$\frac{P(x_1, x_2)}{\rho} = \frac{P_0}{\rho}(\psi) + \int_{\psi}^{x_1, x_2} \frac{u_i F_i}{w} ds \qquad 5.4$$

Thus the energy at a point consists of the sum of that transported from the initial line according to equations 5.2 and that generated by the force field as the fluid passes along the known streamlines to the point x_1, x_2 , corresponding to the integral in equation 5.4. Again the local static pressure follows from the known velocity components and the energy distribution calculated above.

6. Axially Symmetric Rotational Motion: - It is to be expected that plane rotational motion will prove to be a special case of axially symmetric rotational flow. The manner of treating the latter bears much resemblance to that for plane flow but involves complications both from the less simple geometry and from the additional degree of freedom: the tangential velocity. Consider the flow to be described in a cylindrical coordinate system r, ϑ, z, with corresponding velocity components u, v, w. The existence of the tangential velocity does not obviate the use of the stream function with the properties

$$u = \frac{1}{r}\frac{\partial\psi}{\partial z} \quad ; \quad w = -\frac{1}{r}\frac{\partial\psi}{\partial r} \tag{6.1}$$

where because of the complete axial symmetry, the continuity equation

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{6.2}$$

and hence the stream function does not involve the tangential velocity v. The motion is governed by the Eulerian equations of motion

$$\begin{aligned} v\zeta - w\eta &= -F_r + \frac{\partial h}{\partial r} \\ w\xi - u\zeta &= -F_\theta \\ u\eta - v\xi &= -F_z + \frac{\partial h}{\partial r} \end{aligned} \tag{6.3}$$

where the vorticity components are defined as

$$\xi = -\frac{\partial v}{\partial z}$$
$$\eta = \frac{1}{r}\frac{\partial^2\psi}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) \qquad 6.4$$
$$\zeta = \frac{\partial v}{\partial r} + \frac{v}{r} = \frac{1}{r}\frac{\partial (rv)}{\partial r}$$

In analogy with the case of plane motion, the component of equation 2.3 normal to the plane through the axis of symmetry is considered

$$\frac{\partial}{\partial z}(v\zeta - w\eta) - \frac{\partial}{\partial r}(u\eta - v\xi) = \frac{\partial F_z}{\partial r} - \frac{\partial F_r}{\partial z} \qquad 6.5$$

where from the continuity equation 6.2 it follows that

$$u\frac{\partial\eta}{\partial r} - \frac{u\eta}{r} + w\frac{\partial\eta}{\partial z} = \left(\frac{\partial v\xi}{\partial r} + \frac{\partial v\zeta}{\partial z}\right) + \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \qquad 6.6$$

But if it is noted that $r\frac{d}{dz}\left(\frac{\eta}{r}\right) = \left(\frac{\partial\eta}{\partial r} - \frac{\eta}{r}\right)\frac{dr}{dz} + \frac{\partial\eta}{\partial z}$ it is clear that if the differentiation is carried out along a stream surface where $\frac{dr}{dz} = \frac{u}{w}$, the right side of equation 6.6 may be re-written to give

$$\left(w\frac{\partial}{\partial z} + u\frac{\partial}{\partial r}\right)\left(\frac{\eta}{r}\right) = V\frac{d}{ds}\left(\frac{\eta}{r}\right) = \frac{1}{r}\left(\frac{\partial v\xi}{\partial r} + \frac{\partial v\zeta}{\partial z}\right) + \frac{1}{r}\left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right) \qquad 6.7$$

Equation 6.7 may be integrated along a stream surface ψ = constant and, employing the velocity and vorticity components in terms of the stream function, the integro -differential equation becomes

$$\frac{1}{r}\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right) = \frac{\eta_0}{r_0}(\psi)\cdot\frac{1}{r} - r\int_{\psi}^{r,z}\frac{\frac{\partial v\xi}{\partial r} + \frac{\partial v\zeta}{\partial z}}{V}\,ds - r\int_{-\infty}^{r,z}\frac{\mathrm{curl}_\theta F}{V}\,ds \qquad 6.8$$

This relation bears close analogy with that obtained for plane flow except for the terms involving the tangential velocity. Inasmuch as the differential operator on the left denotes the tangential vorticity at any point of a stream surface, the first term on the right represents the transformation of the initial vorticity. Because of axial symmetry any deformation of a tangential vortex ring consists only in changing its radius and hence in changing its length. But according to the effect of stretching a vortex filament described in paragraph 2, this deformation increases the vorticity of the vortex line according to the ratio of the instantaneous and initial radii. The third term on the right of equation 6.8 corresponds to the summation of the tangential vorticity generated along each stream surface by the non-conservative force field and to its modification by the ratio of the instantaneous radius to the radius at which it was generated. For zero tangential velocity the remaining integral vanishes so that plane and axially symmetric rotational flow become formally identical. The nature of the second integral on the right side of equation 6.8 may be seen by writing the numerator of the integrand in the form

$$\frac{\partial}{\partial r} v \xi + \frac{\partial}{\partial z} v \zeta = \frac{\partial}{\partial z} \left(\frac{v^2}{r} \right)$$

which represents the curl of the centrifugal force and depends only on the tangential velocity. Consequently the second integral is similar to the third except that the force field is generated by the centrifugal field rather than by the prescribed forces.

Because of the axial symmetry, the tangential velocity enjoys a degree of independence from the other velocity components. For according to the second equation of motion 6.3

$$\left(u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}\right) r v = r F_\vartheta \qquad 6.9$$

where if the differentiation is carried out along a stream surface

$$V \frac{d}{ds}(rv) = F_\vartheta \qquad 6.10$$

which states the physically obvious result that the rate of change of moment of momentum with time is equal to the moment of the tangential force. Upon integration along a streamline

$$v = v_0\left(\frac{r_0}{r}\right) + \int_{\psi}^{r,z} \frac{r F_\vartheta}{V}\, ds \qquad 6.11$$

As in the case of plane rotational motion, the stream surfaces are again characteristic surfaces and the boundary conditions must be arranged so that the initial vorticity components are prescribed only once on each of these surfaces. It is to be noted that the initial tangential velocity of equation 6.11 is determined, except for a constant, by the axial vorticity component.

7. The Boundary Value Problems: - In studying some of the boundary value problems which arose in plane rotational motion they were divided into two main classes accordingly as the auxiliary force field was or was not conservative. In axially symmetric rotational flow the same division may logically be made but in addition the effects of the force field must be considered in the part resulting directly from the forces and that resulting from the centrifugal forces set up by the tangential velocity which, in turn, is caused by the tangential forces.

In the case of axially symmetric rotational flow with neither forces nor tangential velocity, the problem is

$$\sqrt{\left(\frac{\partial\psi}{r\partial z}\right)^2+\left(\frac{\partial\psi}{r\partial r}\right)^2}\;\frac{\partial}{\partial s}\left(\frac{1}{r}\nabla^2\psi\right)=0$$

$$\begin{aligned}
&\psi = K_1, K_2 \text{ on } AB, CD \;;\; K_2 > K_1\\
&\psi = f(r,z) \text{ on } AD\\
&a\frac{\partial\psi}{\partial r}+b\frac{\partial\psi}{\partial z}=0 \text{ on } BC\\
&\frac{\eta}{r} = \frac{\eta_0}{r_0}(\psi) \text{ on } AD;\; \left|\frac{\eta_0}{r_0}\right| < m \;;\; K_2 \geq \psi \geq K_1
\end{aligned} \qquad 7.1$$

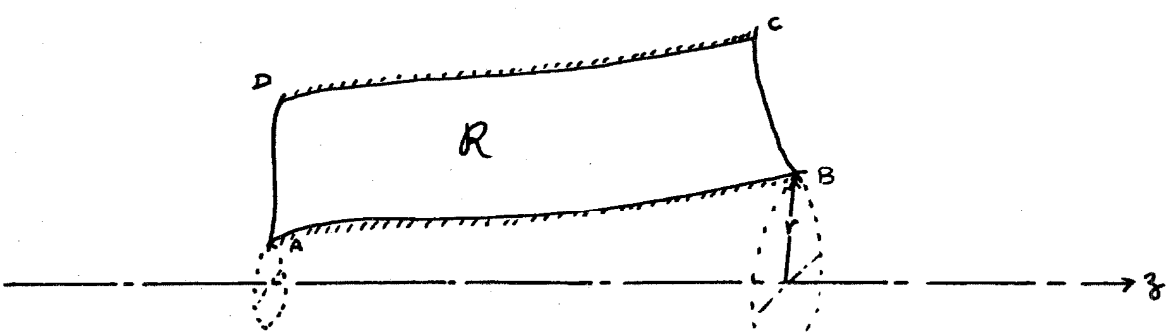

Figure I.8

This corresponds, for example to the physical problem of the longitudinal motion of a solid of revolution in an axially symmetric jet of varying entropy. Clearly the problem is formally the same as problem 4.1. Thus if $\sqrt{\left(\frac{\partial\psi}{r\partial r}\right)^2+\left(\frac{\partial\psi}{r\partial z}\right)^2} \neq 0$ the solution may be represented in the form $\psi = \psi_0 + \bar{\psi}$ where

$$\begin{aligned} \nabla^2\psi_0 &= 0 \\ \psi_0 &= K_1, K_2 \quad \text{on } AB, CD \\ \psi_0 &= f(r,z) \quad \text{on } AD \\ a\frac{\partial\psi}{\partial r} + b\frac{\partial\psi}{\partial z} &= 0 \quad \text{on } BC \end{aligned} \tag{7.2}$$

and

$$\bar{\psi} = \iint_R \alpha\, G(r,z;\alpha,\beta)\, \frac{\eta_0}{r_0}(\psi_0+\bar{\psi})\, d\alpha\, d\beta \tag{7.3}$$

where $G(r,z;\alpha,\beta)$ is the appropriate Green's function satisfying the homogeneous boundary conditions. Equation 7.2 is simply an elliptic problem with mixed boundary conditions whereas 7.3 represents the limit of the sequence

$$\begin{aligned} \bar{\psi}_1 &= \iint_R \alpha\, G(r,z;\alpha,\beta)\, \frac{\eta_0}{r_0}(\psi_0)\, d\alpha\, d\beta \\ &\vdots \\ \bar{\psi}_{n+1} &= \iint_R \alpha\, G(r,z;\alpha,\beta)\left(\frac{\eta_0}{r_0}\right)(\psi_0+\bar{\psi}_n)\, d\alpha\, d\beta \end{aligned} \tag{7.4}$$

if the limit exists and the limiting solution involves only values of ψ over which $\frac{\eta_o}{v_o}(\psi)$ was defined. The linearized solution corresponding to the first step in this iteration process gives good approximation to the true result for $\left|\frac{d}{d\psi}\frac{\eta_o}{v_o}(\psi)\right|$ sufficiently small.

The case of axially symmetric rotational flow with non-conservative forces and no tangential velocity is illustrated by the axially symmetrically distorted flow through a straightening screen. The problem is clearly given for non-vanishing velocity at all points

$$\nabla^2\psi = -r\frac{\eta_o}{v_o}(\psi) - r\int_{\psi}^{r,z}\frac{\left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)}{\sqrt{\left(\frac{\partial\psi}{r\partial z}\right)^2 + \left(\frac{\partial\psi}{r\partial r}\right)^2}}\,ds$$

$$\begin{aligned} &\psi = K_1, K_2 \quad \text{on } AB, CD \\ &\psi = f(r,z) \quad \text{on } AD \\ &\frac{\eta}{r} = \frac{\eta_o}{v_o}(\psi) \quad \text{on } AD; \quad \left|\frac{\eta_o}{v_o}\right| < m \;; \quad K_2 \geq \psi \geq K_1 \\ &a\frac{\partial\psi}{\partial r} + b\frac{\partial\psi}{\partial z} = 0 \quad \text{on } BC \end{aligned} \qquad 7.5$$

when F_r, F_z are functions of $r, z, \frac{\partial\psi}{\partial r}, \frac{\partial\psi}{\partial z}$ and vanish outside of a domain $D \subset R$. Again this is formally identical with problem 4.12 plane flow and exhibits all of the similar difficulties when it is attempted to demonstrate existence or find conditions on F_r, F_z under which solutions do exist.

The final and most involved case is that where the tangential force differs from zero and the radial and axial forces constitute a non-conservative field. This type of flow is thoroughly exemplified by the flow through any axially symmetrical pump or compressor with an infinite number of blades. Another special example, where the radial and axial forces are absent is provided by the motion of a body of revolution in the slipstream of a propeller. The mathematical problem now involves both the stream function and the tangential velocity component $v(r,z)$. Thus it becomes

$$\nabla^2\psi = -r\frac{\eta_0}{r_0}(\psi) - r\int_{\psi}^{r,z}\frac{\frac{\partial}{\partial z}\left(\frac{v^2(r,z)}{r}\right)ds}{\sqrt{\left(\frac{\partial\psi}{r\partial r}\right)^2+\left(\frac{\partial\psi}{r\partial z}\right)^2}} - r\int_{\psi}^{r,z}\frac{\left(\frac{\partial F_r}{\partial z}-\frac{\partial F_z}{\partial r}\right)}{\sqrt{\left(\frac{\partial\psi}{r\partial r}\right)^2+\left(\frac{\partial\psi}{r\partial z}\right)^2}}\,ds$$

$$v(r,z) = \frac{v_0r_0(\psi)}{r} + \int_{\psi}^{r,z}\frac{rF_\theta}{\sqrt{\left(\frac{\partial\psi}{r\partial r}\right)^2+\left(\frac{\partial\psi}{r\partial z}\right)^2}}\,ds$$

$$\psi = K_1, K_2 \quad \text{on} \quad AB, CD \qquad K_2 > K_1$$ 7.6

$$\psi = f(r,z) \quad \text{on} \quad AD$$

$$\frac{\eta_0}{r_0} = \frac{\eta_0}{r_0}(\psi) \quad \text{on} \quad AD \qquad \left|\frac{\eta_0}{r_0}\right| < m \ ; \quad K_2 \geq \psi \geq K_1$$

$$v_0r_0 = v_0r_0(\psi) \quad \text{on} \quad AD \qquad |v_0r_0| < t \ ; \quad K_2 \geq \psi \geq K_1$$

$$a\frac{\partial\psi}{\partial r} + b\frac{\partial\psi}{\partial z} = 0 \quad \text{on} \quad BC$$

where F_r , F_θ , F_z are prescribed functions of $r, z, \frac{\partial\psi}{\partial r}, \frac{\partial\psi}{\partial z}$ which vanish outside of a closed domain $D \subset R$ and have at most integrable singularities in r,z within D . This clearly requires an iteration on both ψ and v inasmuch as each formula involves the other as a variable. Therefore if again the solution ψ is expressed $\psi = \psi_0 + \bar{\psi}$ where ψ_0 is a solution of problem 7.2 and

$$\bar{\psi} = \iint_R G(r,z;\alpha,\beta)\left\{ -\alpha \frac{\eta_0}{v_0}(\psi) - \alpha \int_{\psi}^{\alpha,\beta} \frac{\frac{\partial}{\partial\beta}\left(\frac{v^2(\alpha,\beta)}{\alpha}\right)}{\sqrt{\left(\frac{\partial\psi}{\alpha\partial\alpha}\right)^2 + \left(\frac{\partial\psi}{\alpha\partial\beta}\right)^2}}\, ds \right.$$

$$\left. -\alpha \int_{\psi}^{\alpha,\beta} \frac{\left(\frac{\partial F_\alpha}{\partial\beta} - \frac{\partial F_\beta}{\partial\alpha}\right)}{\sqrt{\left(\frac{\partial\psi}{\alpha\partial\alpha}\right)^2 + \left(\frac{\partial\psi}{\alpha\partial\beta}\right)^2}}\, ds \right\} d\alpha\, d\beta \qquad 7.6$$

$$\equiv \iint_R G(r,z;\alpha,\beta)\, \mathcal{F}\left(\alpha,\beta,\psi,\frac{\partial\psi}{\partial\alpha},\frac{\partial\psi}{\partial\beta}, v(\alpha,\beta)\right) d\alpha\, d\beta$$

where $G(r,z;\alpha,\beta)$ is the appropriate Green's function associated with the homogeneous boundary conditions, the iteration process is

$$\bar{\psi}_1 = \iint_R G(r,z;\alpha,\beta)\, \mathcal{F}\left(\alpha,\beta,\psi_0,\frac{\partial\psi_0}{\partial\alpha},\frac{\partial\psi_0}{\partial\beta},0\right) d\alpha\, d\beta$$

$$v_1 = \frac{v_0 r_0(\psi_0)}{r} + \int_{\psi_0}^{r,z} \frac{r F_\theta\left(r,z,\frac{\partial\psi_0}{\partial r},\frac{\partial\psi_0}{\partial z}\right)}{\sqrt{\left(\frac{\partial\psi_0}{r\partial r}\right)^2 + \left(\frac{\partial\psi_0}{r\partial z}\right)^2}}\, ds$$

$$\vdots$$

$$\bar{\psi}_{n+1} = \iint_R G(r,z;\alpha,\beta)\, \mathcal{F}\left(\alpha,\beta,\psi_0+\bar{\psi}_n,\frac{\partial\,\psi_0+\bar{\psi}_n}{\partial\alpha},\frac{\partial\,\psi_0+\bar{\psi}_n}{\partial\beta}, v_n\right) d\alpha\, d\beta \qquad 7.7$$

$$v_{n+1} = \frac{v_0 r_0(\psi_0+\bar{\psi}_n)}{r} + \int_{\psi_0+\bar{\psi}_n}^{r,z} \frac{r F_\theta\left(r,z,\frac{\partial\,\psi_0+\bar{\psi}_n}{\partial r},\frac{\partial\,\psi_0+\bar{\psi}_n}{\partial z}\right)}{\sqrt{\left(\frac{\partial\,\psi_0+\bar{\psi}_n}{r\partial r}\right)^2 + \left(\frac{\partial\,\psi_0+\bar{\psi}_n}{r\partial z}\right)^2}}\, ds$$

$$\vdots$$

II. - THE EFFECT OF AN IDEALIZED SCREEN ON THE NON-UNIFORM FLOW IN A TWO-DIMENSIONAL PARALLEL WALLED CHANNEL

THE EFFECT OF AN IDEALIZED SCREEN ON THE NON-UNIFORM FLOW IN A TWO-DIMENSIONAL PARALLEL WALLED CHANNEL

I. Introduction: - The problem of straightening a non-uniform flow in a channel is encountered not only in wind tunnels but in the design of any duct system which is required to deliver fluid with a uniform velocity profile. Screens, honeycombs, and similar arrangements are the most usual devices for accomplishing this although considerations have been given (Ref. 1) to the use of freely rotating windmills to counteract the disturbance imparted by the driving fan and the natural velocity profile set up by the channel walls.

If the effects of turbulence and viscous shear are neglected (Ref. 2) the fundamental action of the screen is to redistribute and to reduce the local vorticity of the fluid which approaches the screen. Consequently, if the walls are parallel, the general reduction of vorticity and vorticity gradients will promote a uniform velocity profile. As pointed out previously by Batchelor (Ref. 3) the manner of generating this vorticity need not be considered in detail but is simply related to the local pressure difference across the screen or, as will be shown, may be replaced by a distribution of forces which are dependent upon the local velocity components at the screen.

The effect of a screen on the overall change of velocity profile from a station far upstream to one far downstream has been considered by Collar (Ref. 4) and by Batchelor (Ref. 3) who arrive at similar results by making similar but not identical approximations on the magnitude of the velocity component parallel to the plane of the screen. In neither case is an attempt made to correct for this velocity component by a higher approximation. Another feature which has not been previously investigated in any approximation is the rate at which these changes actually take place, that is the nature of the flow field in the transition range in the neighborhood of the screen. This is of considerable practical importance in the actual application of screens for straightening purposes particularly where several screens are to be installed in tandem.

The problem is of considerable theoretical interest, however, because it affords possibly the simplest case for detailed investigation of a rotational flow process by the method of iteration. In this case all of the complicating difficulties are present but in a very much simplified form so that the nature and magnitude of all of the non-linear effects may be observed and investigated with comparative ease.

In the present section the general physical and mathematical problems are formulated for the flow through an idealized screen and the general linear approximation to the solution is written down. The simplicity of this solution allows the second approximation to be carried through in detail for one particular case and an analysis of the second order correction terms is made. Finally a method is discussed by means of which the first approximation may be improved. The determination of the force distribution at the screen is improved by considering the flow far downstream of the screen in a manner similar to that in which the inverse problem of wing theory is solved by considering the flow in the Trefftz plane.

<u>2. The Physical and Mathematical Problems:</u> - In order to formulate a definite physical problem consider a two-dimensional parallel-walled channel of infinite length, and with walls separated by a distance l. The general geometry and location of the coordinate system are shown in figure II.1. Physically it is supposed that

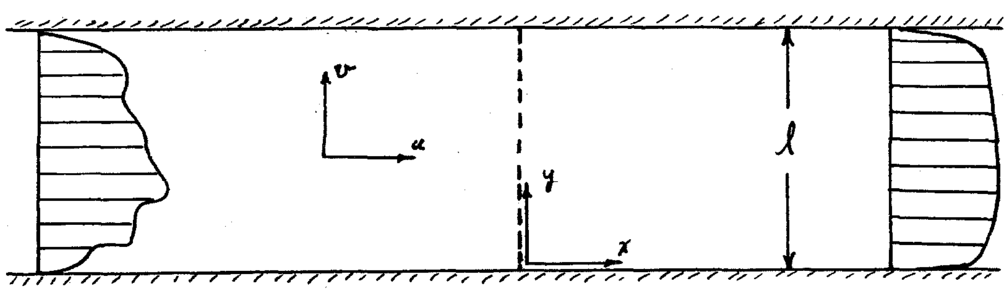

Figure II.1

the velocity profile is given far upstream of the screen, that is to say the distribution of velocity parallel to the walls is a known function of the distance from the channel floor and the velocity normal to the walls vanishes. Far downstream it is supposed, on physical grounds, that the flow becomes parallel to the walls so that the vertical velocity vanishes. It is to be emphasized that this is a definite assumption which has rather important consequences, namely that it rules out the case where the flow becomes unstable in space and tends toward some sort of turbulent motion far downstream. All that can be said strictly of the flow far downstream is that a certain quantity of fluid and momentum are transported over any cross section and that the velocity normal to the wall must vanish at the wall. This point of view emphasizes the hyperbolic character of the problem whereas the point of view to be taken in the present analysis, namely that the velocity direction can be prescribed far downstream, tends to emphasize the elliptic nature. As far as the screen is concerned it will be assumed that the pressure loss across the screen is proportional to the local velocity, that is that it is essentially a laminar resistance. The physical function of the screen is to set up a certain horizontal force field which depends upon the local flow and is generally a non-conservative field.

The corresponding mathematical problem is governed by the Eulerian equation of motion and the continuity equation

$$v \zeta = \frac{\partial}{\partial x}\left(\frac{P}{\rho}\right) - F_x \qquad 2.1$$

$$u \zeta = -\frac{\partial}{\partial y}\left(\frac{P}{\rho}\right) + F_y \qquad 2.2$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad 2.3$$

where u , v are the velocity components in the x and y directions respectively, $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the local vorticity and P the stagnation pressure is defined as $P \equiv \rho\left(\frac{u^2+v^2}{2}\right) + p$ where p is the local static pressure. If ψ is defined as the usual two-dimensional stream function, then by taking the partial derivitive of 2.1 by y, the partial derivitive of 2.2 by x, adding, and applying the continuity equation 2.3 it follows that

$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\nabla^2\psi = \frac{\partial F_x}{\partial y}-\frac{\partial F_y}{\partial x} \qquad 2.4$$

This is the fundamental differential equation of the problem and states that the variation of vorticity along a streamline results only from the action of the force components imposed by the screen. Now according to the physical conditions imposed upon the problem

$$F_x = \begin{cases} ku & ; \; x=0 \\ 0 & ; \; x\neq 0 \end{cases} \qquad 2.5$$

$$F_y = 0$$

where k is a constant depending on the nature of the screen and the fluid. The boundary conditions are essentially prescribed on the velocities, namely that the vertical velocity component vanishes both upstream and downstream as well as upon both solid boundaries, and the horizontal velocity is prescribed upstream. The homogeneous boundary conditions on the vertical velocity suggest a formulation of the problem using v as the dependent variable. This can be done by integrating equation 1.4 along a streamline and differentiating by x. Then the problem may be stated in the form

$$\nabla^2 v = -\frac{\partial}{\partial x}\zeta_0(\psi) + \frac{\partial}{\partial x}\int_{\psi}^{x,y}\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]^{-\frac{1}{2}}\frac{\partial F_x}{\partial y}\,ds$$

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

$$\left.\begin{matrix} v=0 \\ u=u(-\infty,y) \end{matrix}\right\} x=-\infty \qquad v=0,\; y=0,l \qquad v=0;\; x=+\infty \qquad 2.6$$

$$F_x = \begin{Bmatrix} ku; & x=0 \\ 0; & x\neq 0 \end{Bmatrix} \qquad \psi = \int u\,dy - v\,dx$$

where $\zeta_0(\psi)$ indicates that the initial vorticity is transported along streamlines and the initial vorticity is clearly

$$\zeta_0 = -\frac{\partial}{\partial y}\, u(-\infty, y) \qquad 2.7$$

Now if $G(x,y;\xi,\eta)$ is the appropriate Green's function satisfying the homogeneous boundary conditions on v, then the problem becomes

$$v = \iint_{R} G(x,y;\xi,\eta)\left[-\frac{\partial}{\partial \xi}\,\zeta_0(\psi) + \frac{\partial}{\partial \xi}\int_{\psi}^{\xi,\eta}\left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2\right]^{-\frac{1}{2}}\frac{\partial F_x}{\partial \eta}\, ds\right] d\xi\, d\eta$$

$$u - u(-\infty, y) = -\int_{-\infty}^{x}\left(\frac{\partial v}{\partial y}\right) d\xi + r(y)$$

$$2.8$$

where R indicates the infinite strip $0 \le y \le l$.

3. Solution of the Linearized Problem: - The mathematical problem 2.8 may be linearized by approximating the stream function occurring within the integrand to be the stream function corresponding to the flow undisturbed by the screen. Then if U_0 represents the mean horizontal velocity component the linearized problem is

$$v_1 = \int_0^l G(x,y;0,\eta)\,\frac{k}{U_0}\left(\frac{\partial}{\partial\eta}u(-\infty,\eta)\right)d\eta \qquad 3.1$$

$$u_1 - u(-\infty,y) = -\int_{-\infty}^{x}\left(\frac{\partial v_1}{\partial y}\right)d\xi + r(y)$$

The Green's function for the problem 2.8 may be written as the infinite series

$$G(x,y;\xi,\eta) = \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin\frac{n\pi\eta}{l}\sin\frac{n\pi y}{l}\,e^{-\frac{n\pi}{l}|x-\xi|} \qquad 3.2$$

which may clearly be summed to give

$$G(x,y;\xi,\eta) = \frac{1}{4\pi}\log\left\{\frac{1-2e^{-\pi\left|\frac{x-\xi}{l}\right|}\cos\pi\left(\frac{y+\eta}{l}\right)+e^{-2\pi\left|\frac{x-\xi}{l}\right|}}{1-2e^{-\pi\left|\frac{x-\xi}{l}\right|}\cos\pi\left(\frac{y-\eta}{l}\right)+e^{-2\pi\left|\frac{x-\xi}{l}\right|}}\right\} \qquad 3.3$$

Using one form of the Green's function or the other the velocity components are easily evaluated for any particular initial upstream velocity profile. With the representation 3.2 it is convenient to assume that the velocity profile may be expressed as

$$u(-\infty,y) = U_0\left(1+\sum_{n=1}^{\infty} b_n \cos\frac{n\pi y}{l}\right) \qquad 3.4$$

where only the cosine terms are required since the interval $0, l$ is a half period. Then the vertical velocity component is

$$\frac{v_1(x,y)}{U_0} = \sum_{n=1}^{\infty} \frac{k}{n\pi U_0} \sin\frac{n\pi y}{l}\, e^{-\left|\frac{n\pi x}{l}\right|} \int_0^l \sum_{m=1}^{\infty} \frac{m\pi b_m}{l} \sin\frac{m\pi \eta}{l} \sin\frac{n\pi \eta}{l}\, d\eta$$

which is easily evaluated as

$$\frac{v_1(x,y)}{U_0} = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{k}{2U_0} b_n \sin\frac{n\pi y}{l}\, e^{\frac{n\pi x}{l}} & x \le 0 \\ \displaystyle\sum_{n=1}^{\infty} \frac{k}{2U_0} b_n \sin\frac{n\pi y}{l}\, e^{-\frac{n\pi x}{l}} & x \ge 0 \end{cases} \qquad 3.5$$

The corresponding horizontal velocity components then follow according to 3.1 by direct integration

$$u_1(x,y) - u(-\infty,y) = \begin{cases} \displaystyle -\sum_{n=1}^{\infty} \frac{k}{2} b_n \cos\frac{n\pi y}{l}\, e^{\frac{n\pi x}{l}} + \gamma_1 \\ \displaystyle\sum_{n=1}^{\infty} \frac{k}{2} b_n \cos\frac{n\pi y}{l}\, e^{-\frac{n\pi x}{l}} + \gamma_2 \end{cases} \qquad 3.6$$

where γ_1 follows from the initial conditions and γ_2 from the condition that the two solutions join continuously at $x=0$. Therefore

$$Y_1 = 0$$

$$Y_2 = -U_0 \sum_{n=1}^{\infty} \frac{k}{U_0} b_n \cos \frac{n\pi y}{l} \qquad 3.7$$

The distribution of axial velocity may then be written in the form

$$\frac{u_1(x,y)}{U_0} = \begin{cases} 1 + \sum\limits_{n=1}^{\infty} \left(1 - \frac{k}{2U_0} e^{\frac{n\pi x}{l}}\right) b_n \cos \frac{n\pi y}{l} & x \leq 0 \\ 1 + \sum\limits_{n=1}^{\infty} \left(1 - \frac{k}{U_0} + \frac{k}{2U_0} e^{-\frac{n\pi x}{l}}\right) b_n \cos \frac{n\pi y}{l} & x \geq 0 \end{cases} \qquad 3.8$$

The general velocity distribution is then completely determined by the Fourier coefficients of initial velocity distribution and by the parameter $\frac{k}{2U_0}$. The physical significance of this parameter is seen if the pressure difference across the screen is written as $\frac{\Delta p}{\rho} = k U_0 = \frac{U_0^2}{2} C_D$. Hence $\frac{k}{2U_0} \sim \frac{C_D}{4}$ where C_D is the conventional drag or pressure loss coefficient for the screen. For the velocity profile far downstream, $u(\infty, y)$, this linear approximation gives

$$\frac{u_1(\infty, y) - u(-\infty, y)}{u(-\infty, y) - U_0} = -\frac{k}{2U_0} = -\frac{C_D}{4} \qquad 3.9$$

It is sufficient to consider in detail only the two simple cases where the initial velocity profile is symmetrical and where it is unsymmetrical.

Case 1. - Symmetrical Distribution: $u(-\infty, y) = U_0\left(1 - a\cos\frac{2\pi y}{\ell}\right)$

$$\left.\begin{aligned} \frac{v_1(x.y)}{U_0} &= -\left(\frac{k}{2U_0}\right) a \sin\frac{2\pi y}{\ell}\, e^{\frac{2\pi x}{\ell}} \\ \frac{u_1(x.y)}{U_0} &= 1 - \left(1 - \frac{k}{2U_0}\, e^{\frac{2\pi x}{\ell}}\right) a\cos\frac{2\pi y}{\ell} \end{aligned}\right\} x \leq 0$$

$$\left.\begin{aligned} \frac{v_1(x.y)}{U_0} &= -\left(\frac{k}{2U_0}\right) a \sin\frac{2\pi y}{\ell}\, e^{-\frac{2\pi x}{\ell}} \\ \frac{u_1(x.y)}{U_0} &= 1 - \left(1 - 2\left(\frac{k}{2U_0}\right) + \frac{k}{2U_0}\, e^{-\frac{2\pi x}{\ell}}\right) a\cos\frac{2\pi y}{\ell} \end{aligned}\right\} x \geq 0 \tag{3.10}$$

Case 2. - Asymmetrical Distribution: $u(-\infty, y) = U_0\left(1 + a\cos\frac{\pi y}{\ell}\right)$

$$\left.\begin{aligned} \frac{v_1(x.y)}{U_0} &= \left(\frac{k}{2U_0}\right) a \sin\frac{\pi y}{\ell}\, e^{\frac{\pi x}{\ell}} \\ \frac{u_1(x.y)}{U_0} &= 1 + \left(1 - \frac{k}{2U_0}\, e^{\frac{\pi x}{\ell}}\right) a\cos\frac{\pi y}{\ell} \end{aligned}\right\} x \leq 0$$

$$\left.\begin{aligned} \frac{v_1(x.y)}{U_0} &= \left(\frac{k}{2U_0}\right) a \sin\frac{\pi y}{\ell}\, e^{-\frac{\pi x}{\ell}} \\ \frac{u_1(x.y)}{U_0} &= 1 + \left(1 - 2\left(\frac{k}{2U_0}\right) + \frac{k}{2U_0}\, e^{-\frac{\pi x}{\ell}}\right) a\cos\frac{\pi y}{\ell} \end{aligned}\right\} x \geq 0 \tag{3.11}$$

It is to be noted in general that, according to this first linear approximation, the distribution of vertical velocity is symmetrical about the screen. Consequently the variations in axial velocity resulting from these vertical velocity components take place one half on each side of the screen. Thus the velocity profile variation is half accomplished upstream of the screen by the pressure field built up by the flow through the screen or from the point of view taken in the analysis, by the force field which has replaced the screen.

Since the vertical velocity components are potential solutions except on the y axis, it is clear that the solutions may be expressed conveniently as the sum of potential and non-potential portions. Using the complex notation $z = x + iy$ it follows from 3.5 and 3.8 that

$$\frac{u_1 - (u(-\infty, y) + U_0) - i v_1}{U_0} = -\frac{k}{2U_0} \sum_{n=1}^{\infty} b_n e^{\frac{n\pi z}{l}} \; ; \quad R(z) < 0$$

$$\frac{u_1 - (1 - \frac{k}{2U_0})(u(-\infty, y) + U_0) - i v_1}{U_0} = -\frac{k}{2U_0} \sum_{n=1}^{\infty} b_n e^{-\frac{n\pi z}{l}} \; ; \quad R(z) > 0 \qquad 3.12$$

Consequently, as is obvious from the manner in which the problem was linearized, the non-potential portion of the flow ahead of the screen consists in the velocity induced by the initial vorticity while the non-potential portions downstream of the screen consists in the velocity associated with the initial vorticity and the increment of vorticity generated by the screen.

4. Solution of the Second Approximation: - Since the linearized first order solutions of the channel problem have such a simple form, the calculation of the second approximation is not at all a prohibitive task. It will be carried out now for the particularly simple asymmetric case, the first order solutions of which are given by equations 3.11. According to the general statement of the problem 2.8, it will be necessary first to calculate the stream function corresponding to the first order solution, this gives

$$\psi_1 = U_0 \left\{ y + \frac{l}{\pi} a \left(1 - \frac{k}{2U_0} e^{\frac{\pi x}{l}}\right) \sin \frac{\pi y}{l} \right\} \qquad ; \; x \leq 0$$

$$= U_0 \left\{ y + \frac{l}{\pi} a \left(1 - 2\frac{k}{2U_0} + \frac{k}{2U_0} e^{\frac{-\pi x}{l}}\right) \sin \frac{\pi y}{l} \right\} \qquad ; \; x \geq 0 \tag{4.1}$$

Clearly the second order approximation of the force exerted by the screen is

$$F_x = k\, U_0 \left(1 + \left(1 - \frac{k}{2U_0}\right) a \cos \frac{\pi y}{l}\right) \tag{4.2}$$

The second approximation corresponding to problem 3.1 then, from 2.8

$$v_2 = - \int_0^l \int_{-\infty}^{\infty} G(x,y;\xi,\eta) \frac{\partial}{\partial \xi} \psi_1(\xi,\eta)\, d\xi\, d\eta$$

$$+ \int_0^l \int_{-\infty}^{\infty} G(x,y;\xi,\eta) \left[\frac{\partial}{\partial \xi} \int_{\psi_1}^{\xi} \frac{\pi k}{l} \left(-1 + \frac{k}{2U_0}\right) a \cos \frac{\pi \eta}{l}\, d\xi \right] d\xi\, d\eta \tag{4.3}$$

$$u_2 - u(-\infty, y) = -\int_{\psi_1}^{x} \frac{\partial v_2}{\partial y}\, d\xi \quad + r(y) \tag{4.4}$$

The Green's function is not changed for the new problem, but the initial vorticity is transported along first order streamlines. Furthermore the forces exerted by the screen have been changed and the vorticity generated by them is transported along the first order streamlines.

The value of $\frac{\partial \zeta_0(\psi)}{\partial \xi}$ is most easily calculated by considering $\frac{d\zeta_0}{d\psi_1}\,\frac{d\psi_1}{d\xi}$ where ζ_0 is a known function of ψ_1. Consequently upstream of the screen

$$\frac{\partial \zeta_0(\psi_1)}{\partial \xi} = \left.\frac{d\zeta_0}{d\psi_1}\right|_{\xi=-\infty} \frac{\partial \psi_1}{\partial \xi} = \left(\frac{\pi}{\ell}\right)^2 \frac{a\cos\frac{\pi y}{\ell}}{1+a\cos\frac{\pi\eta}{\ell}}\left(-\frac{ka}{2}e^{\frac{\pi\xi}{\ell}}\sin\frac{\pi\eta}{\ell}\right) \tag{4.5}$$

and downstream of the screen

$$\frac{\partial \zeta_0(\psi_1)}{\partial \xi} = \left.\frac{d\zeta_0}{d\psi_1}\right|_{\xi=0+} \frac{\partial \psi_1}{\partial \xi} = \left(\frac{\pi}{\ell}\right)^2 \frac{a\cos\frac{\pi y}{\ell}}{1+a\cos\frac{\pi\eta}{\ell}}\left(-\frac{ka}{2}e^{-\frac{\pi\xi}{\ell}}\sin\frac{\pi\eta}{\ell}\right) \tag{4.6}$$

The second integral in equation 4.3 represents the vorticity generated by the action of the new force system

$$\left(-ka\frac{\pi}{\ell}\sin\frac{\pi\eta}{\ell}\right) + \left(\frac{k}{2U}ka\frac{\pi}{\ell}\sin\frac{\pi\eta}{\ell}\right)$$

which denotes the initial rate of vorticity production plus the increment due to the change of local velocity. The values of the terms in the second integral then become

$$\frac{\partial}{\partial \xi}\int_{\psi_1}^{\xi} \frac{1}{U_0}\,\mathrm{curl}\,F\,d\xi \equiv \frac{\partial}{\partial \psi_1}\int_{\psi_1} \frac{1}{U_0}\,\mathrm{curl}\,F\,d\xi \quad \frac{\partial \psi_1}{\partial \xi}$$

$$= \frac{ka\left(\frac{\pi}{l}\right)^2 \frac{k}{2U_0}\cos\frac{\pi\eta}{l}\sin\frac{\pi\eta}{l}e^{-\frac{\pi\xi}{l}}}{1+a\left(1-\frac{k}{2U_0}\right)\cos\frac{\pi\eta}{l}} - \frac{ka\left(\frac{\pi}{l}\right)^2\left(\frac{k}{2U_0}\right)^2\cos\frac{\pi\eta}{l}\sin\frac{\pi\eta}{l}e^{-\frac{\pi\xi}{l}}}{1+a\left(1-\frac{k}{2U_0}\right)\cos\frac{\pi\eta}{l}} \qquad 4.7$$

where the first term represents the influence of vorticity generated by the first order approximation to the forces and transported along the first order streamlines and the second represents the effect of the vorticity generated by the increment of force and transported also along the first order streamlines. The second order solution follows then from substituting the known Green's function and the results of 4.5, 4.6, 4.7 into the integral relation 4.3 and carrying out the indicated integration. This tedious but, for the most part, straightforward computation gives the following results for the vertical velocities:

$$\frac{v_2}{U_0} = \frac{k}{2U_0}\,a\sin\frac{\pi y}{l}e^{\frac{\pi x}{l}} - \left(\frac{k}{2U_0}\right)^2 a\sin\frac{\pi y}{l}e^{\frac{\pi x}{l}}$$

$$+ \frac{2}{\pi}a\frac{k}{2U_0}\left\{e^{\frac{\pi x}{l}}\ \frac{1-\frac{\pi x}{l}}{2}I_1(a)\sin\frac{\pi y}{l} + \sum_{n=2}^{\infty}\left(\frac{1}{n^2-1}e^{\frac{\pi x}{l}} - \frac{1}{n(n^2-1)}e^{\frac{n\pi x}{l}}\right)I_n(a)\sin\frac{n\pi y}{l}\right\} \qquad 4.8$$

$$- \frac{2}{\pi}a\left(\frac{k}{2U_0}\right)^2 \sum_1 \frac{1}{n(n+1)} I_n\left(a+\frac{kn}{2U_0}\right)e^{\frac{n\pi x}{l}}\sin\frac{n\pi y}{l}$$

$$x \le 0$$

and

$$\frac{v_2}{U_0} = \frac{k}{2U_0}\, a \sin\frac{\pi y}{l}\, e^{-\frac{\pi x}{l}} - \left(\frac{k}{2U_0}\right)^2 a \sin\frac{\pi y}{l}\, e^{\frac{\pi x}{l}}$$

$$+ \frac{2}{\pi} a \frac{k}{2U_0}\left\{ e^{-\frac{\pi x}{l}}\, \frac{\left(1+\frac{\pi x}{l}\right)}{2}\, I_1(a) \sin\frac{\pi y}{l} + \sum_{n=2}^{\infty}\left(\frac{1}{n^2-1}\, e^{-\frac{\pi x}{l}} - \frac{1}{n(n^2-1)}\, e^{-\frac{n\pi x}{l}}\right) I_n(a) \sin\frac{n\pi y}{l}\right\}$$

$$- \frac{2}{\pi} a \left(\frac{k}{2U_0}\right)^2 \left\{ e^{-\frac{\pi x}{l}}\left(\frac{1}{2} + \frac{\pi x}{l}\right) I_1\!\left(a + \frac{ka}{2U_0}\right) \sin\frac{\pi y}{l} + \sum_{n=2}^{\infty}\left(\frac{2n}{n^2-1}\, e^{-\frac{\pi x}{l}} - \frac{1}{n-1}\, e^{-\frac{n\pi x}{l}}\right) I_n\!\left(a + \frac{ka}{2U_0}\right) \sin\frac{n\pi y}{l}\right\} \qquad 4.9$$

$$x \geq 0$$

where

$$I_n(\lambda) = \int_0^{\pi} \frac{\lambda \cos\vartheta \sin\vartheta \sin n\vartheta}{1 + \lambda\cos\vartheta}\, d\vartheta \qquad 4.9$$

and has the values

$$I_0 = 0 \qquad\qquad I_{n+1} + \frac{2}{\lambda} I_n + I_{n-1} = 0 \;;\quad n \neq 2$$

$$I_1 = \frac{\pi}{2} - \frac{\pi}{\lambda^2}\left(1 - \sqrt{1-\lambda^2}\right) \qquad\qquad = \frac{\pi}{2} \;;\quad n = 2 \qquad 4.10$$

The first term of each of the expressions 4.8 and 4.9 is the first approximation solution while the second term represents the effect of transporting the additional vorticity due to the increment of force along the old streamlines, that is, along horizontal lines. The third term represents the change of vertical velocity due to transporting the vorticity of the first approximation along the first order streamlines while the fourth term denotes the change of vertical velocity due to transporting the additional vorticity along the first order streamlines.

The calculation of the axial velocity from the known values of the vertical velocity is straightforward and the unknown functions are determined from the initial conditions and the condition of continuity at the screen. The axial velocities arranged in a similar fashion with regard to the significance of the terms are

$$\begin{aligned}\frac{u_2}{U_0} &= \left\{1 + \left(1 - \frac{k}{2U_0} e^{\frac{\pi x}{\ell}}\right) a \cos\frac{\pi y}{\ell}\right\} - \left(\frac{k}{2U_0}\right)^2 e^{\frac{\pi x}{\ell}} a \cos\frac{\pi y}{\ell} \\ &- \frac{2}{\pi} a \frac{k}{2U_0} \left\{\left(1 - \frac{1}{2}\frac{\pi x}{\ell}\right) I_1(a) e^{\frac{\pi x}{\ell}} \cos\frac{\pi y}{\ell} + \sum_{n=2}^{\infty} I_n(a) \left[\frac{n}{n^2-1} e^{\frac{\pi x}{\ell}} - \frac{1}{n(n^2-1)} e^{\frac{n\pi x}{\ell}}\right] \cos\frac{n\pi y}{\ell}\right\} \\ &+ \frac{2}{\pi} a \left(\frac{k}{2U_0}\right)^2 \sum_{n=1}^{\infty} I_n\left(a + \frac{ka}{2U_0}\right) \frac{1}{n(n+1)} e^{\frac{n\pi x}{\ell}} \cos\frac{n\pi y}{\ell} \qquad x \leq 0\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\frac{u_2}{U_0} &= \left\{1 + \left(1 - 2\frac{k}{2U_0} + \frac{k}{2U_0} e^{\frac{-\pi x}{\ell}}\right) a \cos\frac{\pi y}{\ell}\right\} - \left(\frac{k}{2U_0}\right)^2 e^{\frac{-\pi x}{\ell}} a \cos\frac{\pi y}{\ell} \\ &+ \frac{2}{\pi} a \left(\frac{k}{2U_0}\right) \left\{\left[\left(1 + \frac{1}{2}\frac{\pi x}{\ell}\right) e^{\frac{-\pi x}{\ell}} - 2\right] I_1(a) \cos\frac{\pi y}{\ell} + \sum_{n=2}^{\infty} \left(\frac{n}{n^2-1} e^{\frac{-\pi x}{\ell}} - \frac{1}{n(n^2-1)} e^{\frac{-n\pi x}{\ell}} - 2\right) I_n(a) \cos\frac{n\pi y}{\ell}\right\} \\ &- \frac{2}{\pi} a \left(\frac{k}{2U_0}\right)^2 \left\{\left[e^{-\frac{\pi x}{\ell}} \left(\frac{3}{2} + \frac{\pi x}{\ell}\right) - 2\right] I_1\left(a + \frac{ka}{2U_0}\right) \cos\frac{\pi y}{\ell} + \sum_{n=2}^{\infty} I_n\left(a + \frac{ka}{2U_0}\right) \left(\frac{2n}{n^2-1} e^{\frac{-\pi x}{\ell}} - \frac{1}{n(n-1)} e^{\frac{-n\pi x}{\ell}} - 2\right) \cos\frac{n\pi y}{\ell}\right\} \qquad x \geq 0\end{aligned} \tag{4.12}$$

The first and second approximations to the axial and vertical velocities have been calculated for an initial velocity profile having what is considered to be a rather severe disturbance, namely: $\frac{k}{2U_0} = 0.25$ and $a = 0.2$ corresponding to a 20% initial velocity distortion and a screen giving a pressure drop of one dynamic pressure. The first order solution follows simply from equations 3.11 with the proper numerical values inserted. The items of main interest are the various second order correction terms enumerated above. These relative corrections are shown as functions of position in figures II.2, II.3 and II.4 in terms of the corresponding first approximation velocity at the same point. The quantities $\frac{\Delta_1 v}{v_1}$, $\frac{\Delta_2 v}{v_1}$, $\frac{\Delta_3 v}{v_1}$ denote respectively the products of the second, third, and fourth terms of equation 4.8, 4.9 by $\frac{U_0}{v_1}$ where v_1 is the corresponding first approximation velocity at the same point; $\frac{\Delta_1 u}{u_1}$, $\frac{\Delta_2 u}{u_1}$, $\frac{\Delta_3 u}{u_1}$ denote similarly the products of the second, third and fourth terms of equations 4.11, 4.12 by $\frac{U_0}{u_1}$. The relative corrections denoted by the various terms are:

$\Delta_1(\ )$: Transport of second order vorticity increment along the initial streamlines.

$\Delta_2(\ )$: Transport of initial vorticity along the first order streamlines.

$\Delta_3(\ )$: Transport of the second order vorticity increment along the first order streamlines.

The corrections $\Delta_1(\ )$ are shown in figure II.2 for both the vertical and horizontal velocity components. The second order vorticity is created by the interaction of the first order velocity profile distortion with the screen and consequently is proportional to first order vorticity field generated by the initial velocity profile. As a result the corresponding relative correction to the vertical velocity component is a constant $(.25)^2$ or 6.25% of the first order vertical velocity. Similarly second order correction to the horizontal velocity component is proportional to the horizontal velocity disturbances induced by the first order velocity field. The division by the first order velocity field which includes the mean horizontal velocity as well as the disturbance velocity. As a consequence the curves representing the relative error have the spread shown in figure II.2. It should be noted however, that the second order vorticity field has a strength proportional to the square of the first order field so that if the latter is a small quantity the second order correction will be reasonably small.

The corrections $\Delta_2(\ )$ shown in figure II.3 are not of such a simple variety. Considering first the correction to the vertical velocity component, this quantity is caused by the transport of the initial vorticity with the first order vertical velocity component. Near the bottom of the channel this motion of the vorticity tends to take it out of the region where it is distorting the velocity profile and hence relieves the distortion even more than the first order solution would indicate. This is indicated by the increased vertical velocity and decreased horizontal velocity near the bottom of the channel. Near the top of the channel, the vorticity of opposite sign flow into the region where it is causing the velocity distortion and hence aggrevate the distortion. This is reflected in the negative vertical velocity induced against the general trend of the stream and the corresponding reduction of horizontal velocity near the channel top. The composite result of these two motions is to accelerate the stream slightly in the central portion of the channel.

The corrections $\Delta_3(\,)$, figure II.4 are essentially the most complex of all. It is clear that the vorticity distribution is modified only downstream of the screen by this correction. Furthermore the process bears the same relation to the correction in figure II.2 as the correction in figure II.3 did to the first order velocity disturbances with the exception here that the vorticity here is of the opposite sense. Consequently for points downstream of the screen, the correction $\Delta_3(\,)$ acts like a reflection of correction $\Delta_2(\,)$ about the center line of the channel. As a consequence the horizontal velocity is retarded at the center of the channel. In a general qualitative sense, corrections $\Delta_2(\,)$ and $\Delta_3(\,)$ act so as to cancel each other downstream of the screen.

5. The Direct Calculation of the Forces: - The determination of the second order corrections in the last paragraph have shown that the largest errors are incurred by the approximation made in estimating the forces exerted on the fluid by the screen. Inasmuch as this error is due directly to the approximation of the axial velocity at the screen, it appears possible to make marked improvement in the linear solution by a closer estimation of the velocity as the screen. By using the fact that the linearized solution shows one-half of the axial velocity variation to have taken place by the time the fluid encounters the screen, the velocity at the screen may be calculated by a device very similar to the calculation of downwash from the velocities induced at the Trefftz plane. Then having estimated this velocity, the force estimation in the linearized solution may be improved.

Far downstream of the screen the vertical velocity component dies off exponentially and consequently the vorticity distribution is given by

$$\zeta(\infty, y) = - \frac{\partial u(\infty, y)}{\partial y} \qquad 5.1$$

From the second equation of motion 2.2 however, it follows that this vorticity component may be expressed in the form

$$\zeta(\infty, y) = - \frac{1}{u(\infty, y)} \frac{\partial}{\partial y} \frac{P}{\rho}(\infty, y) \qquad 5.2$$

that is, it is directly related to the vertical distribution of total pressure. But according to the observation that the vorticity and hence the total pressure are transported along streamlines, then according to the linearized solution, the total pressure at a given y coordinate far downstream differs from the total pressure at the same value of y far upstream only by the pressure loss across the screen at the same value of y. Hence

$$\frac{P}{\rho}(\infty,y) = \frac{P}{\rho}(-\infty,y) - \Delta\frac{P}{\rho}(y) \qquad 5.3$$

where the local loss of pressure across the screen $\Delta\frac{P}{\rho}(y)$ is given by

$$\Delta\frac{P}{\rho}(y) = k\,(u(0,y)) \qquad 5.4$$

and $u(0,y)$ is the local horizontal velocity component. This local velocity consists in the initial velocity plus one-half of the over-all change of velocity or the mean of the upstream and downstream velocities

$$u(0,y) = \frac{1}{2}\Big[u(-\infty,y) + u(\infty,y)\Big] \qquad 5.5$$

Consequently from the foregoing relations the equation for the horizontal velocity far downstream is

$$u(\infty,y)\frac{\partial}{\partial y}u(\infty,y) = \frac{\partial}{\partial y}\left[\frac{u^2(-\infty,y)}{2} - \frac{k}{2}\Big(u(-\infty,y)+u(\infty,y)\Big)\right]$$

or upon integration

$$\Big[u(\infty,y)-u(-\infty,y)\Big]^2 + 2\,\big(u(-\infty,y)-k\big)\Big[u(\infty,y)-u(-\infty,y)\Big] - \big(2k\,u(-\infty,y)+D\big) = 0 \qquad 5.6$$

where D is a constant of integration to be evaluated from the condition of continuity.

This relation determines the velocity far downstream in terms of the initial velocity and the screen characteristics and may be solved exactly or in various approximations. If it is assumed, for example, that $u(\infty,y)+u(-\infty,y)\approx 2U_0$ and that $u(-\infty,y)+k\approx U_0$ it follows upon applying the continuity integral relation that

$$\frac{u_1(\infty,y)-u_1(-\infty,y)}{u(-\infty,y)-U_0} = -\frac{k}{2U_0} \qquad 5.7$$

which is identical with the approximation found in the linearized solution of paragraph 3. This can be rewritten in the more convenient form

$$\frac{u_1(\infty,y)-U_0}{u_1(-\infty,y)-U_0} = 1-\frac{k}{2U_0} \qquad 5.8$$

If, in a similar fashion only the squares of the terms $(u(\infty,y)-U_0)$ and $u(-\infty,y)-U_0$ are systematically neglected, then the relation corresponding to 5.8 is

$$\frac{u_1(\infty,y)-U_0}{u_1(-\infty,y)-U_0} = \frac{1-\frac{k}{2U_0}}{1+\frac{k}{2U_0}} \qquad 5.9$$

which is identical with the result obtained for the overall effect of the screen by Collar (Ref. 4) and Batchelor (Ref. 3). It is possible, of course, to solve the quadratic 5.6 exactly if it is desirable, so that if

$$\alpha \equiv \frac{u(\infty,y) - U_0}{u(-\infty,y) - U_0} \quad ; \quad \beta \equiv \frac{u(-\infty,y) - k}{u(-\infty,y) - U_0} \quad ; \quad \gamma = \frac{2k\, u(-\infty,y) + D}{(u(-\infty,y) - U_0)^2}$$

then

$$\begin{aligned} \alpha - 1 &= -\beta\left(1 - \sqrt{1 + \frac{\gamma}{\beta^2}}\right) \\ &= \beta\left(\frac{1}{2}\frac{\gamma}{\beta^2} - \frac{1}{8}\left(\frac{\gamma}{\beta^2}\right)^2 + \cdots\right) \end{aligned} \qquad 5.10$$

where the value of the constant D may be determined to suit the particular approximation used. From this general relation it is clear that the value of F_χ, equation 2.5 may be approximated with better accuracy

$$F_\chi = \begin{cases} k\left(\frac{1+\alpha}{2}\, u(-\infty,y) + \frac{1-\alpha}{2}\, U_0\right) & ; \quad x = 0 \\ 0 & ; \quad x \neq 0 \end{cases} \qquad 5.11$$

The corresponding linearized problem is, similar to problem 3.1,

$$\begin{aligned} v_1 &= \int_0^l G(x,y;0,\eta)\, \frac{k}{U_0}\left(\frac{1+\alpha}{2}\right) \frac{\partial}{\partial \eta}\, u(-\infty,\eta)\, d\eta \\ u_1 - u(-\infty,y) &= -\int_{-\infty}^{x} \frac{\partial v_1}{\partial y}\, d\xi \end{aligned} \qquad 5.12$$

REFERENCES

1. Collar, A. R.: The Use of a Freely Rotating Wind Mill to Improve the Flow in a Wind Tunnel. Aeronautical Research Council Reports and Memoranda 1866. 1939.

2. Dryden, Hugh, L. and Schubauer, G. B.: The Use of Damping Screens for the Reduction of Wind Tunnel Turbulence. Journal of the Aeronautical Sciences, Vol. 14, No. 4, pp. 221-228. April 1947.

3. Batchelor, G. K.: On the Concept and Properties of the Idealized Hydrodynamic Resistance. Australian Council for Aeronautics, Report ACA-13. January 1945.

4. Collar, A. R.: The Effect of a Guaze on the Velocity Distribution in a Uniform Duct. Aeronautical Research Council Reports and Memoranda 1867. 1939.

III. - THE ROTATIONAL FLOW OF A PERFECT FLUID ABOUT A CLOSED BOUNDARY

THE ROTATIONAL FLOW OF A PERFECT FLUID ABOUT A CLOSED BOUNDARY

1. Introduction: - The simplest case of the rotational motion of a perfect fluid is that where no non-conservative force field is applied to the fluid and as a consequence the initial vorticity prescribed far upstream is merely transported along the streamlines. This process is well illustrated by the rotational flow about an arbitrary closed contour, for example an airfoil moving in the wake of some blunt body where the wake may be idealized to a rotational velocity field. Examples of this type with considerable technological importance have become more abundant in internal aerodynamics where large energy and velocity gradients are relatively common and the problem of computing pressure distribution around and forces acting on bodies in this rotational flow must be considered.

The motion of bodies through a perfect fluid with constant vorticity has been considered by Ray (Ref. 1), Tsien (Ref. 2), Kuo (Ref. 3), and Richardson (Ref. 4). This case of constant vorticity is a particularly simple singular case because inasmuch as the vorticity is uniform, the transport mechanism is of no consequence and the exact problem reduces simply to the solution of the Poisson equation with a constant inhomogenous part. The results, however, are of considerable interest, especially the generalized Blasius equations which were extended by Kuo to the case of shear flow or uniform vorticity.

When the vorticity is not uniformly distributed, the problem can not be solved in general but may be reduced with considerable accuracy to a linearized problem when the vorticity is either small or does not differ greatly from uniform. In the following analysis and discussion this linearized solution is developed for a general class of closed contours, actually the class which may be mapped conformally on to a circle. The corresponding Blasius force and moment relations are written down for the more general rotational flow field.

<u>2. Plane Rotational Motion about a Contour:</u> - In the absence of a non-conservative force field the plane rotational motion of a prefect fluid is described by the second order, non-linear partial differential equation

$$\nabla^2 \psi = - \zeta_0 (\psi) \qquad 2.1$$

where ψ is the stream function with the usual properties

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = - \frac{\partial \psi}{\partial x} \qquad 2.2$$

u and v being the velocity components in the cartesian coordinate system x, y . The function ζ_0 gives the distribution of vorticity over the streamlines according to the values prescribed far upstream and the relation 2.1 has the physical meaning that the vorticity $\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ remains constant along any particular streamline. The problem of finding the rotational flow about a given closed boundary consists then in solving equation 2.1 consistent with the conditions

a) The value of ψ is constant along the given contour $\mathcal{C}$.

b) The flow far upstream of the contour consists in a parallel rotational motion.

c) The flow at large distances from the contour is not disturbed.

Because of the non-linearity of equation 2.1 it is not likely that exact solutions will be found for any very general contours $\mathcal{C}$.

Consequently if, as will be the case in most physical problems encountered, the velocities induced by the vorticity are small compared with those induced by the main stream, then a good linearized approximation to the solution of equation 2.1 will be obtained by assuming the vorticity to be transported along the streamlines of the corresponding irrotational flow.

If the velocity far upstream is $U(y)$, then this may be expressed in the form

$$U(y) = U_0 + u_0(y) \tag{2.3}$$

where

$$U_0 \equiv \lim_{A \to \infty} \int_{-A}^{A} U(y)\, dy \tag{2.4}$$

The linearized solution will then consist in two parts $\psi = \psi_0 + \psi_1$ where ψ_0 is the potential part of the solution such that

$$\begin{aligned} \nabla^2 \psi_0 &= 0 \\ \frac{\partial \psi_0}{\partial y} &= U_0 \quad at \quad x = -\infty \\ \psi_0 &= 0 \quad on \quad \mathcal{C} \end{aligned} \tag{2.5}$$

and ψ_1 is the solution if the inhomogeneous problem

$$\begin{aligned} \nabla^2 \psi_1 &= - \zeta_0(\psi_0) \\ \frac{\partial \psi_1}{\partial y} &= u(y) \ ; \ x = -\infty \\ \zeta_0 &= - \frac{\partial u(y)}{\partial y} \ ; \ x = -\infty \\ \psi_1 &= 0 \quad on \quad \mathcal{C} \end{aligned} \tag{2.6}$$

Problem 2.5 is solved easily by the well known methods of function theory while the inhomogeneous equation of problem 2.6 may be solved by the method of Green's functions. Now if $G(x,y,\xi,\eta)$ is the Green's function which gives the value of ψ induced at the point x,y by a unit of vorticity at the point ξ,η such that the value of ψ is maintained zero on the contour $\mathcal{C}$

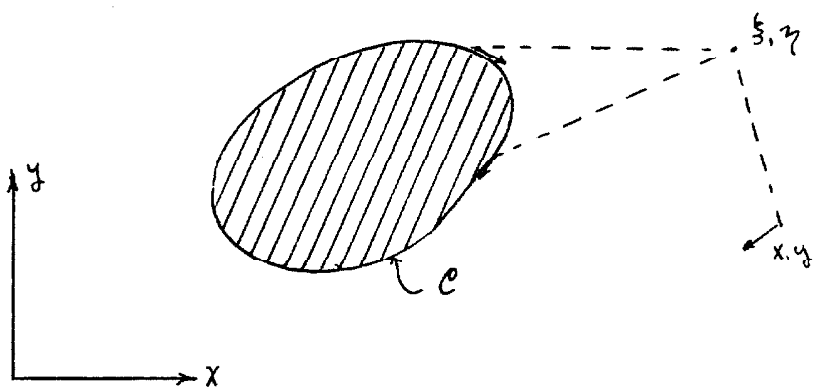

Figure III.1

then the solution of problem 2.6 may be written simply

$$\psi_1 = -\iint_{D} \zeta_0(\psi_0(\xi,\eta))\, G(x,y;\xi,\eta)\, d\xi\, d\eta \qquad 2.7$$

where the integration is extended over that portion of the plane exterior to the contour $\mathcal{C}$. The solution of 2.7 is then essentially that of determining the Green's functions $G(x,y;\xi,\eta)$ for the appropriate contour.

The linearized solution satisfies completely the boundary conditions on the original non-linear problem but as has been pointed out, it does not satisfy the differential equation 2.1 which says that streamlines and lines of constant vorticity should coincide. Consequently the situation is somewhat as shown in figure III.2

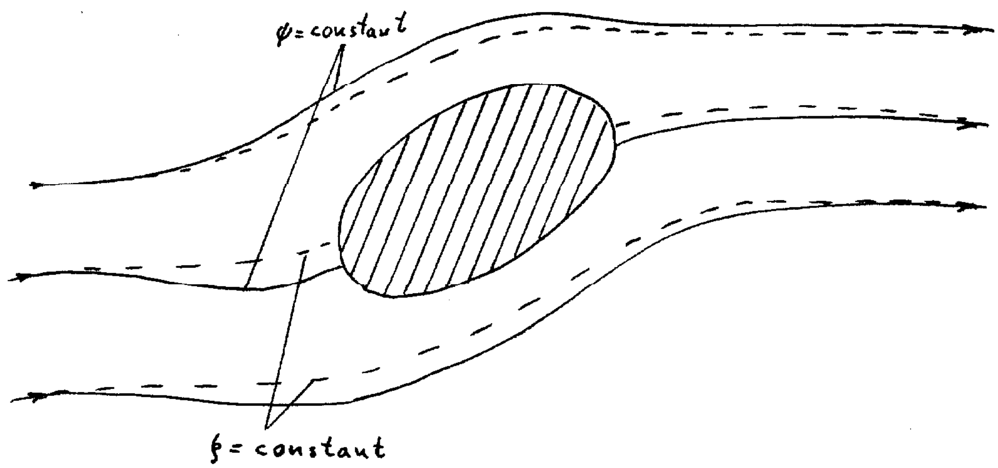

Figure III.2

If the deviations of the two sets of curves are slight, that is if the streamlines of the irrotational and rotational flow are nearly the same, then the approximation should be quite sufficient for technical purposes.

<u>3. The Green's Functions and the Formal Solution:</u> - The representation of the Green's functions may be written down directly for some particularly simple contours. For example in the case of a plane wall

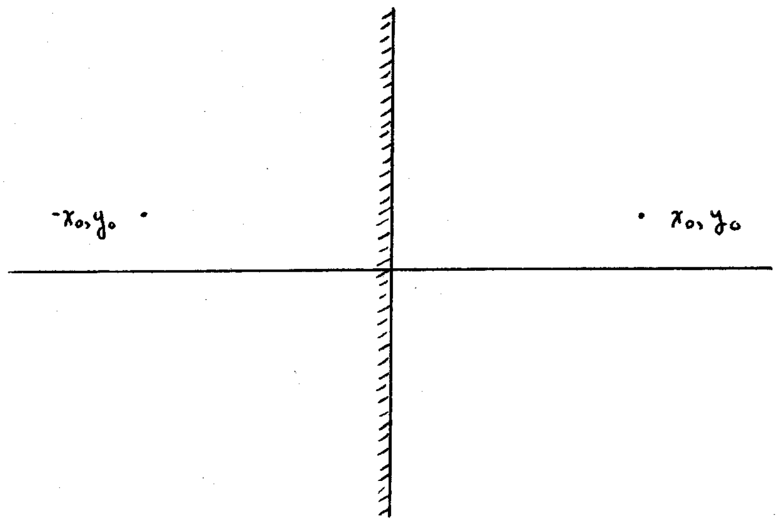

Figure III.3

the Green's function corresponds to the real part of the function

$$\mathcal{G} = \frac{1}{2\pi} \log\left(\frac{z - z_0}{z + \bar{z}_0}\right) \qquad 3.1$$

where $z \equiv x + iy$; z_0 is the location of the source of the disturbance and $\bar{z}_0$ is the complex conjugate. This satisfies the condition that $\Re(\mathcal{G})$ vanishes on the boundary and has a logarithmic singularity at the point $z = z_0$. Furthermore $\mathcal{G}$ is an analytic function of z so that this property remains under conformal transformation of the z plane. For example consider the homographic transformation

$$\lambda = \mu + i\nu = \frac{az + b}{cz + d} \qquad 3.2$$

in the particular instance where

$$a = -b = \alpha \qquad ; \quad c = d = 1 \tag{3.3}$$

so that

$$\frac{\lambda}{\alpha} = \frac{z-1}{z+1} \tag{3.4}$$

where α is a real positive constant. Then the imaginary axis of the z plane is mapped into a circle of radius α about the origin of the λ plane. Then since

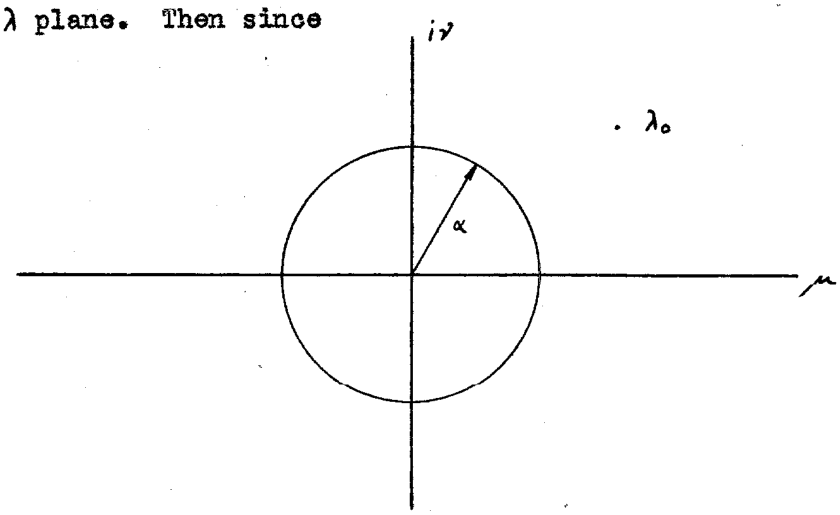

Figure III.4

z may be expressed as

$$z = \frac{\lambda+\alpha}{\lambda-\alpha} \tag{3.5}$$

it follows upon direct substitution in 3.1 that the $\mathcal{G}$ function becomes

$$\mathcal{G}(\lambda) = \frac{1}{2\pi} \log\left\{ \alpha \frac{\lambda_0 - \lambda}{\bar{\lambda}_0 \lambda - \alpha^2} \right\} \tag{3.6}$$

In case the boundary region is a circle, the value of

$$\mathcal{R}(\mathcal{L}(\lambda)) \equiv \frac{1}{2\pi} \mathcal{R} \log \left\{ \alpha \frac{\lambda_0 - \lambda}{\bar{\lambda}_0 \lambda - \alpha^2} \right\} \tag{3.7}$$

may be substituted for $G(x,y;\xi,\eta)$ in equation 2.7 and the integral formula reduces to that of Poisson. Furthermore the irrotational portion of the flow about the circular contour is known, for if W_0 is the complex potential function, then

$$W_0 = \varphi_0 + i\psi_0 = U_0 \left(\lambda + \frac{\alpha^2}{\lambda} \right) \tag{3.8}$$

Consequently if $\zeta_0(\psi)$ represents the distribution of vorticity far ahead of the circle, the linear approximation for the rotational flow is

$$\psi = \psi_0 + \psi_1 = U_0 \mu \left(1 + \frac{\alpha^2}{\mu^2 + \nu^2} \right)$$

$$+ \frac{1}{2\pi} \iint_D \zeta_0\left(U_0 \mu \left(1 - \frac{\alpha^2}{\mu^2 + \nu^2} \right) \right) \mathcal{R} \log \left\{ \alpha \frac{\lambda - \lambda_0}{\bar{\lambda}_0 \lambda - \alpha^2} \right\} d\mu \, d\nu \tag{3.9}$$

Numerical solution then requires, formally, only the integration of 3.9.

Now that the $\mathcal{L}(\lambda)$ is known for the circular contours, it is a simple matter to find $\mathcal{L}(z)$ and hence the Green's function $G(a,b;\alpha,\beta)$ for any other contour whose conformal transformation into the circle is known. For if this transformation is given by

$$z = f(\lambda) \tag{3.10}$$

with an inverse

$$\lambda = F(z) \tag{3.11}$$

then the potential for the irrotational part of the flow is

$$W_1(z) = U_0\left(\lambda(z) + \frac{\alpha^2}{\lambda(z)}\right)$$
$$= U_0\left(F(z) + \frac{\alpha^2}{F(z)}\right) \tag{3.12}$$

so that

$$\psi_0 = Im\left\{U_0\left(F(z) + \frac{\alpha^2}{F(z)}\right)\right. \tag{3.13}$$

Furthermore the function $\mathcal{L}$ becomes

$$\mathcal{L}(z) = \frac{1}{2\pi}\log\left\{\alpha\frac{F(z_0) - F(z)}{\bar{F}(z_0)F(z) - \alpha^2}\right\} \tag{3.14}$$

and the Green's function is the real part of this. Then in analogy with 3.9 the stream function for the rotational flow about the new contour is

$$\psi \equiv \psi_0 + \psi_1 = Im\left\{U_0\left(F(z) + \frac{\alpha^2}{F(z)}\right)\right\}$$

$$+ \frac{1}{2\pi}\iint \zeta_0\left(Im\left\{U_0 F(z) + \frac{U_0\alpha^2}{F(z)}\right\}\right) R\log\left\{\alpha\frac{F(z) - F(z_0)}{\bar{F}(z_0)F(z) - \alpha^2}\right\} d\tilde{\alpha}\, d\beta \tag{3.15}$$

As a result of the relationship between the conformal transformation of a given contour into a circle and the Green's function for this contour, all of the highly developed techniques and methods of approximation are available for computing the Green's functions and hence the rotational flow in any case for which the transformation exists. It is clear also that a similar method may be used for boundaries other than closed contours; for example, in the analysis of the progression of a rotational velocity profile in a divergent channel. Such flows may easily be analyzed by using the principles indicated in the foregoing examples and applying a Schwarz-Christoffel transformation to the function 3.1 in order to find the Green's function of the problem.

In connection with the problem of airfoils with the Kutta condition must be employed to fix the circulation it is impossible to fix the circulation when the irrotational part of the flow is first calculated. As a consequence, the streamlines over which the initial vorticity is transported are known only to a parameter Γ, the circulation required to satisfy the Kutta condition. Therefore this parameter appears both within the integrand and in the stream functions of the irrotational part of the flow (see equation 3.15). As a rule, this expression will involve Γ in a rather complex manner after integration so that the evaluation may be carried out numerically.

4. The Force and Moment on a Solid Body: - The force and moment exerted on a body by the rotational flow of a perfect fluid are calculated by considering the flux of momentum across and the pressure distribution on a contour which completely encloses the body in question

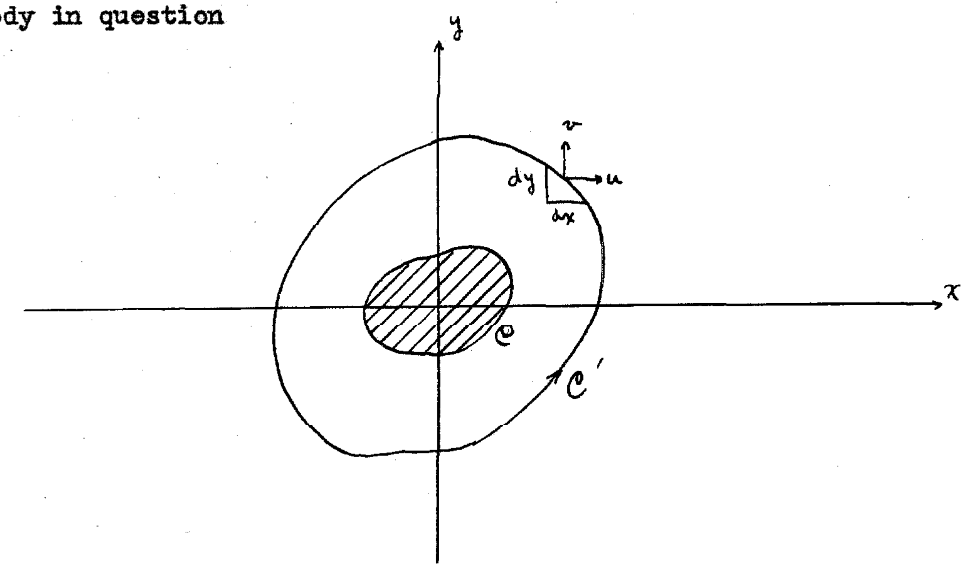

Figure III.5

The increment of the x force component X caused by the momentum transport and pressure at a given element of the contour is given by

$$(u\,dy - v\,dx)\rho u + p\,dy = d(-X)$$

so that the X component of force is obtained by integrating this expression about the contour

$$X = -\oint_{C'} (u\,dy - v\,dx)\rho u + p\,dy \qquad 4.1$$

Now the local static pressure p is related to the local values of the x and y velocity components and the local value of the total pressure or the Bernoulli function $\mathcal{B}(x,y)$,

$$p + \frac{\rho}{2}(u^2+v^2) = \mathcal{B}(x,y) \tag{4.2}$$

where it is known that both the vorticity and the Bernoulli function remain constant on streamlines and, to the present approximation, on streamlines corresponding to the irrotational flow. Consequently the value of $\mathcal{B}(x,y)$ may be evaluated from the condition prescribed far upstream of the body and then taken to depend on only the stream function of the irrotational velocity field.

Neglecting an arbitrary constant, the Bernoulli function far upstream is

$$\mathcal{B}(-\infty,y) = \frac{\rho}{2}\left(U-u(-\infty,y)\right)^2$$

But the distribution of the irrotational stream function at is simply

$$\psi_o(-\infty,y) = Uy$$

so that the Bernoulli function is written in general as

$$B(x,y) \equiv B(\psi_o) = \frac{\rho}{2}\left(U+u\left(-\infty, \frac{\psi_o(x,y)}{U}\right)\right)^2 \tag{4.3}$$

Consequently the X force component may be written

$$X = -\oint (u\,dy - v\,dx)\rho u + \frac{\rho}{2}\left(\left[U+u\left(-\infty, \frac{\psi_o(x,y)}{U}\right)\right]^2 - u^2 - v^2\right)dy \tag{4.4}$$

and in a similar manner the Y force component becomes

$$Y = -\oint (u\,dy - v\,dx)\rho v - \frac{\rho}{2}\left(\left[U + u\left(-\infty, \frac{\psi_0(x,y)}{U}\right)\right]^2 - u^2 - v^2\right)dx \tag{4.5}$$

Writing this force in the complex form it follows that

$$X - iY = +\frac{i\rho}{2}\oint (u - iv)^2 (dx + i\,dy) - \frac{\rho}{2}\oint\left[U + u\left(-\infty, \frac{\psi(x,y)}{U}\right)\right]^2 (dy + i\,dx) \tag{4.6}$$

The velocities u and v may each be expressed as the sum of irrotational and rotational components such that if $u - iv = (u_0 - iv_0) + (u_1 - iv_1)$ then

$$u_0 + iv_0 = \frac{dW_0}{dz} \tag{4.7}$$

and

$$u_1 - iv_1 = \frac{\partial \psi_1}{\partial y} + i\frac{\partial \psi_1}{\partial x} \tag{4.8}$$

where the stream function ψ_1 is determined from the general integral formula 3.15. Then the force law may be written

$$X - iY = \frac{i\rho}{2}\oint \left(\frac{dW_0}{dz}\right)^2 dz + i\rho\oint \frac{dW_0}{dz}\left(\frac{\partial \psi_1}{\partial y} + i\frac{\partial \psi_1}{\partial x}\right)dz$$

$$+ \frac{i\rho}{2}\oint\left(\frac{\partial \psi_1}{\partial y} + i\frac{\partial \psi_1}{\partial x}\right)^2 dz - \frac{i\rho}{2}\oint\left[U + u\left(-\infty, \frac{\psi_0(x,y)}{U}\right)\right]^2 (dx - i\,dy) \tag{4.9}$$

which is a generalization of Blasius first law. When the motion is irrotational, the last three integrals vanish and the formula is reduced to the usual statement. When the rotational part is the particular distribution of constant vorticity, the stream function for the rotational part of the flow may be written

$$\psi_1 = -\frac{k^2}{2} y^2 \qquad 4.10$$

where k is the constant value of the vorticity. In this case the result simplifies directly to

$$X - iY = \frac{i\rho}{2} \oint \left(\frac{dw_0}{dz} \right)^2 dz + ik\rho \, \mathrm{Im} \oint z \, \frac{dw_0}{dz} \, dz \qquad 4.11$$

which is identical with the result obtained by Tsien (Ref. 2) and Kuo (Ref. 3).

The expression for the moment on the body may be written down in a similar fashion

$$M = \oint \rho u (u\,dy - v\,dx) y + \rho v (v\,dx - u\,dy) x + \int p (x\,dx + y\,dy)$$

$$= -\frac{\rho}{2} R \oint (u - iv)^2 z \, dz + \frac{\rho}{2} \oint \left[U + u\left(-\infty, \frac{\psi_0(x,y)}{U}\right) \right]^2 (x\,dx + y\,dy) \qquad 4.12$$

Again separating the velocities into their irrotational and rotational components the moment becomes

$$M = -\frac{\rho}{2}\,\mathcal{R}\oint\left(\frac{dw_0}{dz}\right)^2 z\,dz - \rho\,\mathcal{R}\oint\frac{dw_0}{dz}\left(\frac{\partial\psi_1}{\partial y} + i\,\frac{\partial\psi_1}{\partial x}\right) z\,dz$$

$$-\frac{\rho}{2}\,\mathcal{R}\oint\left(\frac{\partial\psi_1}{\partial y} + i\,\frac{\partial\psi_1}{\partial x}\right)^2 z\,dz + \frac{\rho}{2}\oint\left[U + u\left(-\infty, \frac{\psi_0(x,y)}{U}\right)\right]^2 (x\,dx + y\,dy) \qquad 4.13$$

which reduces again to the usual formula for irrotational flow and to the Tsien and Kuo results for the case of shear flow.

REFERENCES

1. Ray, Manohar: Motion of an Infinite Elliptic Cylinder in Fluids with Constant Vorticity. Proceedings Royal Society of London. Series A, Vol. 158, 1937. pp. 522-535.

2. Tsien, Hsue-Shen: Symmetrical Joukowsky Airfoils in Shear Flow. Quarterly of Applied Mathematics. Vol. I, No. 2, July 1943. pp. 130-148.

3. Kuo, Y. H.: On the Force and Moment Acting on a Body in Shear Flow. Quarterly of Applied Mathematics. Vol. I, No. 3, October 1943. pp. 273-275.

4. Richardson, M.: The Pressure Distribution on a Body in Shear Flow. Quarterly of Applied Mathematics. Vol. III, No. 2, July 1945. pp. 175-178.

IV. - THE FLOW OF A PERFECT FLUID THROUGH AN AXIAL TURBOMACHINE WITH PRESCRIBED BLADE LOADING

THE FLOW OF A PERFECT FLUID THROUGH AN AXIAL TURBOMACHINE WITH PRESCRIBED BLADE LOADING

1. Introduction: - Those aerodynamic problems of axial-flow turbomachine theory which are subject to treatment by the theory of perfect fluids may be classified under the following two general problems:

1. Given the blade loading, blade speed, and the fluid state far ahead of all blades, determine the three-dimensional velocity field, blade shape, and distribution of energy in the fluid.
2. Given the blade shape, blade speed, and the fluid state far ahead of all blades, determine the three-dimensional velocity field, blade loading, and the distribution of energy in the fluid.

These are designated respectively the inverse and direct problems of turbomachine theory in analogy with the corresponding classical problems in the theory of finite wings.

The inverse problem, which is definitely the less difficult of the two, derives this advantage from the fact that it may be separated into two independent problems. The first of these consists in determining the velocity field corresponding to the prescribed blade loading by replacing this blade loading with an equivalent force system distributed over some definite surface or region. In the second step the two-dimensional theory of airfoils or airfoil lattices is applied to find the radial distribution of blade shape and orientation providing the prescribed load distribution when placed in this particular velocity field. Recent investigations into the theory of airfoil lattices (e.g. Garrick[1], Lighthill[2], and A. Goldstein and Jerison[3]) have reduced most cases of the second step to one of numerical calculation. The present paper is concerned with a quantitative description of the three-dimensional velocity field that prescribes the mean flow in which each blade element is situated and indicates the accuracy with which the two-dimensional airfoil theory may be applied to each element of the blade.

For a perfect fluid the solution to the three-dimensional problem is simple only when the distribution of tangential velocity in any plane normal to the turbomachine axis is that of a vortex situated on the axis.[4] Under this condition no radial or axial disturbances are induced, the circulation about each blade element is constant along its length, and the blade behaves very much like an infinite wing. In spite of the increased complexity of the flow under more general conditions of varying circulation along the blade, the approximate difference between the axial velocity profiles far upstream and far downstream of the blade row is easily calculated (e.g. Traupel[5], Sinnette[6], and Eckert and Korbacher[7]) by neglecting the radial transport of vorticity shed from the blades. From considerations of the simplified vortex system (e.g. Ruden[8]) it follows that one half of this axial velocity change has taken place by the time the fluid reaches the blade row. The velocity distribution in the rest of the flow field, which is of importance in answering such practical questions as the interference of adjacent blade rows in a multistage turbomachine, can not be found so simply.

For the discussion of the three-dimensional flow in the present paper, the physical problem is simplified by considering a non-viscous and incompressible fluid, inner and outer boundaries consisting of concentric cylinders, and by assuming an infinite number of blades in each blade row so that the flow possesses axial symmetry. Therefore the vorticity shed from each blade row is no longer concentrated in sheets but is continuously distributed over the region downstream of the blade row. The difficulty of this problem lies in the non-linear partial differential equations which describe the rotational fluid motion. By the approximation that the vorticity is transported by the mean velocity and not by its own induced velocity, the problem reduces to the solution of a well known non-homogeneous linear partial differential equation. The resulting solutions provide the linear approximation to the radial, tangential, and axial velocities associated with any loading of a blade of finite chord or of an infinitely thin blade row corresponding to a discontinuous change of tangential velocity.

2. Formulation of the Mathematical Problem: - The flow is described (figure IV.1) in a cylindrical coordinate system r, ϑ, z, by the velocity components u, v, w, respectively. The corresponding radial, tangential, and axial vorticity components are

$$\xi = -\frac{\partial v}{\partial z} \qquad 2.1$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \qquad 2.2$$

$$\zeta = \frac{\partial}{r\partial r}(vr) \qquad 2.3$$

Because of the axial symmetry only the tangential vorticity is associated with the radial and axial velocities while the radial and axial vorticity components are associated with the tangential velocity. Consequently information concerning the radial and axial velocities is obtained by considering (cf. R. E. Meyer[9]) an annular vortex ring of small cross section (figure IV.2) which consists always of the same fluid elements. In particular the circulation Γ about the ring is given by $\Gamma = \eta \times$ cross section area and remains constant throughout all deformations in a conservative force field. The only deformation of this ring consistent with the assumption of axial symmetry is a radial stretching and, inasmuch as the volume of the ring must remain constant, the cross sectional area varies as the ratio $\frac{r}{r_0}$ where r_0 and r are the initial and final radii. Under such a deformation the constancy of circulation demands that the initial and final vorticities η_0 and η, satisfy the relation

$$\frac{\eta}{r} = \frac{\eta_0}{r_0} = \text{constant} \qquad 2.4$$

Hence as the ring moves along the stream surface in a conservative force field, the rate of change of $\frac{\eta}{r}$ vanishes.

The circulation about the cross section of the annular vortex will vary, however, in the presence of a non-conservative force field. Since the pressure is a scalar quantity and consequently generates a conservative force field, the circulation may be changed only by the action of the centrifugal force and the forces applied by the blades in the radial and axial directions. The rate of change of circulation is easily found from the equations of motion

$$v\zeta - w\eta = -F_r + \frac{\partial}{\partial r}\left(\frac{P}{\rho}\right) \qquad 2.5$$

$$w\xi - u\zeta = -F_\vartheta \qquad 2.6$$

$$u\eta - v\xi = -F_z + \frac{\partial}{\partial z}\left(\frac{P}{\rho}\right) \qquad 2.7$$

where P is the local stagnation pressure of the fluid and F_r , F_ϑ , and F_z are the applied forces per unit mass in the radial, tangential, and axial direction respectively, By subtracting the partial derivative by r of equation 2.7 from the partial derivative by z of equation 2.5 and simplifying by means of the continuity equation

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \qquad 2.8$$

it follows that

$$u\frac{\partial \eta}{\partial r} - \frac{u\eta}{r} + w\frac{\partial \eta}{\partial z} = \frac{\partial}{\partial r} v\xi + \frac{\partial}{\partial z}(v\zeta) + \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}$$

Finally by dividing by r , simplifying the term on the left and applying the definition of the vorticity components to the term on the right, it follows that

$$\frac{\partial}{\partial z}\left(\frac{\eta}{r}\right)\frac{ds}{dt} = \left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\left(\frac{\eta}{r}\right) = \frac{1}{r}\left(\frac{\partial}{\partial z}\left(\frac{v^2}{r}\right)+\frac{\partial F_r}{\partial z}+\frac{\partial F_z}{\partial r}\right) \qquad 2.9$$

where s is the distance measured along a stream surface.

The operator $u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}$ occuring on the left side of equation 9 represents the time rate of variation taken while moving with the stream. If $|u| \ll w$ so that the stream surfaces are very nearly co-axial cylinders and if the tangential vorticity has a smooth radial distribution, then $\left|u\frac{\partial \eta}{\partial r}\right| \ll \left|w\frac{\partial \eta}{\partial z}\right|$. If furthermore the radial variation of axial velocity is small with respect to the mean value w_o, i.e. $\left|\frac{w-w_o}{w_o}\right| \ll 1$, then this operator may be taken with good approximation to be $w_o\frac{\partial}{\partial z}$. From definition of the tangential vorticity and by applying the equation of continuity (equation 2.8)

$$\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\left(\frac{\eta}{r}\right) \approx \frac{w_o}{r}\frac{\partial \eta}{\partial z} = \frac{w_o}{r}\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{u}{r}\right)+\frac{\partial^2 u}{\partial z^2}\right) \qquad 2.10$$

For calculation of the centrifugal forces which occur in equation 2.9 it will be sufficient to approximate the tangential velocity in the following manner. The second equation of motion, equation 2.6, may be rewritten

$$\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)vr = rF_\theta \qquad 2.11$$

in which from it states that the rate of change, along a stream surface, of the moment of momentum about the turbomachine axis is equal to the moment of the tangential force. To the same approximation as above

$$F_\vartheta \approx w \frac{\partial v}{\partial z} \approx w_0 \frac{\partial v}{\partial z} \qquad 2.12$$
$$= - w_0 \xi$$

where ξ corresponds to the density of "bound" radial vorticity associated with a particular point of the blade row. From equation 2.12 the approximate tangential velocity becomes

$$v_0(r,z) - v_{-\infty}(r) = - \int_{-\infty}^{z} r\,\xi(r,\beta)\,d\beta = \int_{-\infty}^{z} \frac{r F_\vartheta(r,\beta)}{w_0}\,d\beta \qquad 2.13$$

where $v_{-\infty}(r)$ is the initial tangential velocity far ahead of the first blade row.

The forces applied to the fluid by the blades are related through the condition that their resultant must act normal to the blade surface. Furthermore the blade surface is parallel to the relative velocity of the fluid and consequently the forces exerted by the blades are normal to the relative velocity of the fluid. If the blades move about the axis with an angular velocity ω, this condition is expressed through the relation

$$u F_r + (v - \omega r) F_\vartheta + w F_z = 0 \qquad 2.14$$

But since the radial force is usually of smaller order than the tangential and axial forces, this becomes very nearly

$$(v_0 - \omega r) F_\vartheta + w_0 F_z = 0 \qquad 2.15$$

inasmuch as $\left|\frac{u}{w_0}\right| \ll 1$, $\left|\frac{w - w_0}{w_0}\right| \ll 1$, $\frac{v - v_0}{w_0} \ll 1$.

Now equations 2.12, and 2.15 may be used to express the right hand member of equation 2.9 in terms of known functions

$$\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{v^2}{r}\right) + \frac{1}{r}\left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right) \approx \frac{1}{r}\frac{\partial}{\partial z}\left\{(v_0 - \omega r)\frac{1}{r}\frac{\partial}{\partial r}(v_0 r) + F_r\right\} \qquad 2.16$$

Thus equation 2.9 is reduced, within the approximations stated, to the non-homogeneous linear partial differential equation

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{\bar{u}}{r}\right) + \frac{\partial^2 \bar{u}}{\partial z^2} = \frac{\partial}{\partial z}\left\{\left(\bar{v}_0 - \frac{\omega r}{w_0}\right)\frac{1}{r}\frac{\partial}{\partial r}(\bar{v}_0 r) + \frac{F_r}{w_0^2}\right\} \equiv \frac{\partial}{\partial z} f(r, z) \qquad 2.17$$

where non-dimensional velocities $\frac{u}{w_0}$, $\frac{v_0}{w_0}$, are denoted $\bar{u}$, $\bar{v}_0$, and when the blade forces are prescribed, $\bar{v}_0$ is known from equation 2.13.

The axial and tangential velocities corresponding to this radial velocity follow from the equation of continuity and the second equation of motion respectively. Through integration of equation 2.8 the axial velocity is

$$\frac{w}{w_0} - 1 \equiv \bar{w} - 1 = \int_{-\infty}^{z} \frac{1}{r}\frac{\partial}{\partial r}(ru)\, d\beta \qquad 2.18$$

while by the appropriate approximation that $|\bar{v} - \bar{v}_0| \ll 1$, the tangential velocity is, from equation 2.11

$$\bar{v} - \bar{v}_0 = -\int_{-\infty}^{z} \frac{\bar{u}}{r}\frac{\partial}{\partial r}(\bar{v}_0 r)\, d\beta \qquad 2.19$$

The complete mathematical problem of calculating the linearized radial, tangential, and axial velocity components in a turbomachine with prescribed blade loading is given by equations 2.13, 2.17, 2.18 2.19, together with the following boundary conditions

$$\bar{u} = 0 \; ; \quad \left\{ \begin{array}{l} z = \pm\infty \\ r = r_1, r_2 \end{array} \right\} \qquad \begin{array}{ll} \bar{w} = 1 & ; \quad z = -\infty \\ \bar{v} = \bar{v}_\infty(r) & ; \quad z = -\infty \end{array} \tag{2.20}$$

and either $F_\theta(r, z)$, $F_r(r, z)$ or $\bar{v}_o(r, z)$, $F_r(r, z)$ 2.21

prescribed throughout the annular space occupied by the fluid.

3. <u>Linearized Solutions for the Velocity Components</u>: - The linearized solution for the radial velocity component will be obtained by finding the appropriate Green's function; that is, a solution $G(r,z;\alpha,\beta)$ which gives the radial velocity, consistent with the boundary conditions, induced at any point of a circle r, z by a unit change in tangential vorticity of a vortex ring of radius α at an axial coordinate β. The complete radial velocity is simply the sum of such solutions corresponding to all changes of tangential vorticity. From the manner in which the problem was linearized it is clear that the solution $G(r,z;\alpha,\beta)$ may also be interpreted as the velocity induced at a point r, z by a cylindrical surface made up of annular vorticies with radius α extending from the axial coordinate β to ∞. It will be of particular interest to find the Green's function by considering the case (figure IV.3) where all forces are concentrated in a plane $z = \beta$ so that the mean tangential velocity changes discontinuously from $v_0^{(1)}$ to $v_0^{(2)}$. This plane corresponds to a sheet of radial and tangential vorticity normal to the turbomachine axis. Clearly $\frac{\partial}{\partial z} f(r,z) = 0$ when $z \neq \beta$; equation 2.17 becomes homogeneous and may be solved directly, using the given boundary conditions, for independent solutions on either side of the discontinuity. Now it is clear from the left side of equation 2.17 and the symmetry of the boundary conditions that any solution of equation 2.17 corresponding to a unit disturbance at $z = \beta$ must be symmetrical in z about β. Therefore both radial and axial velocity components are continuous at $z = \beta$. If the nota-

tion $\left[f(r,z)\right]_\beta$ is used to denote $\lim_{\varepsilon\to 0}\left(f(r,\beta+\varepsilon)-f(r,\beta-\varepsilon)\right)$, the jump in the value of $f(r,z)$ across the discontinuity at $z=\beta$, then the jump in tangential vorticity may be written as

$$\left[\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}\right]_\beta \equiv \left[\frac{\partial u}{\partial z}\right]_\beta = \omega_0\left[f(r,z)\right]_\beta \qquad 3.1$$

$$= \frac{1}{\omega_0}\left[(v_0-\omega r)\frac{\partial}{r\partial r}(v_0 r)+F_r\right]_\beta$$

inasmuch as the axial velocity is continuous at $z=\beta$. Thus the continuity of the radial velocity together with the prescribed discontinuity in tangential vorticity at $z=\beta$ (equation 3.1) are sufficient to join the independent solutions at $z=\beta$.

If the applied forces, and hence the discontinuity in tangential vorticity, be limited to the neighborhood of a circle $r=\alpha$ and vanish for other radii the solution obtained corresponds to the desired Green's function or unit solution. If, on the other hand, there exists a finite jump of tangential vorticity for all radii, the solution bears the same relation to that for a continuous axial distribution of forces as the lifting line solution for finite wings does to the solution for continuous chordwise load distribution.

If $\bar{u}^{(1)}$ and $\bar{u}^{(2)}$ are the non-dimensional radial velocities upstream and downstream of the discontinuity at $z=\beta$, this new problem may be formulated as

$$\frac{\partial^2\bar{u}^{(k)}}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{\bar{u}^{(k)}}{r}\right)+\frac{\partial^2\bar{u}^{(k)}}{\partial z^2}=0 \qquad k=1,2 \qquad 3.2$$

with the boundary conditions

$$\bar{u}^{(1)} = 0 \quad ; \quad z = -\infty$$
$$\bar{u}^{(2)} = 0 \quad ; \quad z = +\infty$$
$$\bar{u}^{(k)} = 0 \quad ; \quad r = r_1, r_2 \qquad k = 1,2 \tag{3.3}$$

and the auxiliary conditions at the discontinuity

$$\bar{u}^{(1)}(r,\beta) = \bar{u}^{(2)}(r,\beta)$$
$$\left.\frac{\partial \bar{u}^{(2)}}{\partial z}\right|_{z=\beta} - \left.\frac{\partial \bar{u}^{(1)}}{\partial z}\right|_{z=\beta} = \left[(\bar{v}_o - \frac{\omega r}{\omega_o})\frac{\partial}{r\partial r}(\bar{v}_o r) + \frac{F_r}{\omega_o^2}\right] = \left[f(r,z)\right]_\beta \tag{3.4}$$

A solution to equation 3.2 of the form

$$\bar{u}^{(k)} = \sum_{n=1}^{\infty} U_1(\varepsilon_n r)\left(A_n^{(k)} e^{\varepsilon_n z} + B_n^{(k)} e^{-\varepsilon_n z}\right) \tag{3.5}$$

holds on each side of the discontinuity where $U_1(\varepsilon_n r)$ is the linear combination of Bessel functions of order one

$$U_1(\varepsilon_n r) = J_1(\varepsilon_n r)\, Y_1(\varepsilon_n r_1) - J_1(\varepsilon_n r_1)\, Y_1(\varepsilon_n r) \tag{3.6}$$

and the characteristic values ε_n are the roots of the transcendental equation

$$U_1(\varepsilon_n r_2) = J_1(\varepsilon_n r_2)\, Y_1(\varepsilon_n r_1) - J_1(\varepsilon_n r_1)\, Y_1(\varepsilon_n r_2) \tag{3.7}$$

The conditions 3.4 relate the solutions $\bar{u}^{(1)}$ and $\bar{u}^{(2)}$ and consequently $A_n^{(1)}$, $A_n^{(2)}$, $B_n^{(1)}$, and $B_n^{(2)}$.

$$\sum_{n=1}^{\infty} U_1(\varepsilon_n r)\,\varepsilon_n e^{\varepsilon_n \beta}\left(A_n^{(2)} - A_n^{(1)}\right) = \tfrac{1}{2}\left[f(r,z)\right]_\beta$$
$$\sum_{n=1}^{\infty} U_1(\varepsilon_n r)\,\varepsilon_n e^{-\varepsilon_n \beta}\left(B_n^{(1)} - B_n^{(2)}\right) = \tfrac{1}{2}\left[f(r,z)\right]_\beta \tag{3.8}$$

In order to satisfy the conditions that the radial velocity vanish at large distances upstream and downstream of the discontinuity (boundary condition 3.4 at $\pm\infty$) $A_n^{(2)} = B_n^{(1)} = 0$. Hence by applying the orthogonality relations of the Bessel functions to equation 3.8 the coefficients are

$$A_n^{(1)} = -e^{-\varepsilon_n \beta} \int_{r_1}^{r_2} \frac{\alpha U_1(\varepsilon_n \alpha)}{2 \nu_n^2 \varepsilon_n} \left[f(\alpha, z) \right]_\beta d\alpha$$

$$B_n^{(2)} = -e^{\varepsilon_n \beta} \int_{r_1}^{r_2} \frac{\alpha U_1(\varepsilon_n \alpha)}{2 \nu_n^2 \varepsilon_n} \left[f(\alpha, z) \right]_\beta d\alpha \qquad 3.9$$

where α is introduced as the variable of integration and ν_n is the norm of the function $U_1(\varepsilon_n r)$ over the interval r_1, r_2

$$\nu_n^2 = \int_{r_1}^{r_2} \alpha U_1^2(\varepsilon_n \alpha)\, d\alpha = \frac{r_2^2 U_0^2(\varepsilon_n r_2) - r_1^2 U_0^2(\varepsilon_n r_1)}{2} \qquad 3.10$$

The corresponding solution for the axial velocity follows from equation 2.18

$$\bar{w}^{(k)} - 1 = -\sum_{n=1}^{\infty} U_0(\varepsilon_n r) \left(A_n^{(k)} e^{\varepsilon_n z} - B_n^{(k)} e^{-\varepsilon_n z} \right) + \gamma^{(k)}(r) \qquad 3.11$$

where

$$U_0(\varepsilon_n r) = J_0(\varepsilon_n r) Y_1(\varepsilon_n r_1) - J_1(\varepsilon_n r_1) Y_0(\varepsilon_n r) \qquad 3.12$$

and the functions $\gamma^{(k)}(r)$ are determined by the initial distribution of axial velocity and the continuity of axial velocity at $z = \beta$. Thus

$$\bar{w}^{(1)} = 1 \quad ; \quad z = -\infty$$

$$\bar{w}^{(1)}(r, \beta) = \bar{w}^{(2)}(r, \beta) \qquad 3.13$$

and applying these relations

$$\gamma^{(1)}(r) = 0$$

$$\gamma^{(2)}(r) = \sum_{n=1}^{\infty} U_0(\varepsilon_n r) \int_{r_1}^{r_2} \frac{\alpha U_1(\varepsilon_n \alpha)}{2 \nu_n^2 \varepsilon_n} \left[f(\alpha, z) \right]_\beta d\alpha \qquad 3.14$$

The linearized solution to the tangential velocity distribution is found through integrating the known value of $\bar{u}^{(k)}$ in equation 2.19. Substituting the known values for the constants and interchanging the order of summation and integration, the complete solution for the single discontinuity becomes

$$\bar{u} = \int_{r_1}^{r_2} [f(\alpha,z)]_\beta \left\{ \sum_{n=1}^{\infty} \frac{\alpha U_1(\varepsilon_n r)\, U_1(\varepsilon_n \alpha)}{2\varepsilon_n \vartheta_n^2} e^{-\varepsilon_n |z-\beta|} \right\} d\alpha \qquad 3.15$$

$$\bar{w} - 1 = \int_{r_1}^{r_2} [f(\alpha,z)]_\beta \left\{ \sum_{n=1}^{\infty} \frac{\alpha U_0(\varepsilon_n r)\, U_1(\varepsilon_n \alpha)}{2\varepsilon_n \vartheta_n^2} e^{-\varepsilon_n |z-\beta|} \right\} d\alpha \qquad 3.16$$

$$+ p \int_{r_1}^{r_2} [f(\alpha,\beta)]_\beta \left\{ 2 \sum_{n=1}^{\infty} \frac{\alpha U_0(\varepsilon_n r)\, U_1(\varepsilon_n \alpha)}{2\varepsilon_n \vartheta_n^2} \left(1 - e^{-\varepsilon_n |z-\beta|}\right) \right\} d\alpha$$

$$v \cdot \bar{v}_0^{(2)} = -\int_{\beta}^{z} \frac{\partial}{r\partial r}\left(v_0^{(2)} r\right) \int_{r_1}^{r_2} [f(\alpha,z)]_\beta \left\{ \sum_{n=1}^{\infty} \frac{\alpha U_1(\varepsilon_n r)\, U_1(\varepsilon_n \alpha)}{2\varepsilon_n \vartheta_n^2} e^{-\varepsilon_n |z-\beta|} \right\} d\alpha\, dz \qquad 3.17$$

where

$$p = \begin{cases} 0 \; ; & z < \beta \\ 1 \; ; & z > \beta \end{cases}$$

An examination of these solutions shows that they consist essentially in unit solutions of weight $[f(\alpha,z)]_\beta$ summed over the radius from r_1 to r_2 ; the expressions in curly brackets represent solutions which give the velocities consistent with the boundary conditions induced at a circle r, z by a unit disturbance on the circle of radius α at the coordinate $z = \beta$. These functions are denoted

$$G(r,z;\alpha,\beta) = \sum_{n=1}^{\infty} \frac{\alpha\, U_1(\varepsilon_n r)\, U_1(\varepsilon_n \alpha)}{2\,\varepsilon_n\, \gamma_n^2}\, e^{-\varepsilon_n |z-\beta|}$$

$$H(r,z;\alpha,\beta) = \sum_{n=1}^{\infty} \frac{\alpha\, U_0(\varepsilon_n r)\, U_1(\varepsilon_n \alpha)}{2\,\varepsilon_n\, \gamma_n^2}\, e^{-\varepsilon_n |z-\beta|} \qquad 3.18$$

$$K(r,z;\alpha,\beta) = 2\sum_{n=1}^{\infty} \frac{\alpha\, U_0(\varepsilon_n r)\, U_1(\varepsilon_n \alpha)}{2\,\varepsilon_n\, \gamma_n^2}\left(1 - e^{-\varepsilon_n |z-\beta|}\right)$$

The discontinuous approximations (equations 3.15, 3.16 and 3.17) are solutions of a linear problem and may be superposed with the convention that $[f(\alpha,z)]_\beta$ at each value of β be interpreted not as the sum of the superposed discontinuities but as the discontinuity corresponding to the sum of the individual tangential velocities (see equation 3.1). This is necessary inasmuch as the approximate tangential velocity enters non-linearly in evaluating the jump across the discontinuity. The solution for a continuous blade loading follows either by considering the limit of such a superposed sum of discontinuous jumps in which $[f(\alpha,z)]_\beta$ becomes $\frac{\partial}{\partial\beta} f(\alpha,\beta)\, d\beta$, or by considering the original equations 2.17, 2.18, 2.19 in relation to the unit solutions, equations 3.18, which have been found. Hence for any continuous loading

$$\bar{u} = \int_{-\infty}^{\infty} \frac{\partial}{\partial\beta} f(\alpha,\beta)\, G(r,z;\alpha,\beta)\, d\alpha\, d\beta \qquad 3.19$$

$$\bar{\omega} - 1 = \int_{-\infty}^{\infty} \int_{r_1}^{r_2} \frac{\partial}{\partial \beta} f(\alpha, \beta) \; H(r, z; \alpha, \beta) \, d\alpha \, d\beta$$

$$+ \int_{-\infty}^{z} \int_{r_1}^{r_2} \frac{\partial}{\partial \beta} f(\alpha, \beta) \; K(r, z; \alpha, \beta) \, d\alpha \, d\beta \qquad 3.20$$

$$v \cdot \bar{v}_0 = \int_{-\infty}^{z} \frac{\partial}{r \partial r} (\bar{v}_0 \, r) \int_{-\infty}^{\infty} \int_{r_1}^{r_2} \frac{\partial}{\partial \beta} f(\alpha, \beta) \; G(r, z; \alpha, \beta) \, d\alpha \, d\beta \, dz \qquad 3.21$$

4. <u>Special Forms of the Blade Loading</u>: - The case where the blade loading is similar at all radii and the effect of the radial force component may be neglected allows simplification of the expressions for the velocity components. Let R and Z be non-dimensional functions of r and z alone and $\xi_0 c$ be a parameter indicative of the total circulation about a blade of chord c. The radial and axial vorticities and the tangential velocity may be expressed in the form

$$\begin{aligned} \xi &= -\xi_0 c\, R Z' \\ \zeta &= \xi_0 c \left(R' + \frac{R}{r}\right) Z \\ v_\theta &= \xi_0 c\, R Z \end{aligned} \qquad 4.1$$

where the prime denotes differentiation with respect to the appropriate variable. Then according to equation 2.17

$$\frac{\partial}{\partial z} f(r,z) = \left(\frac{\xi_0 c}{w_0}\right)^2 \left\{ 2R\left(R' + \frac{R}{r}\right) Z Z' - \frac{\omega r_2}{\xi_0 c}\left(\frac{R}{r_2} + \frac{r}{r_2} R'\right) Z' \right\} \qquad 4.2$$

Using this relation in the solutions (equations 3.19, 3.20, 3.21) and interchanging the order of integration and summation, the following four fundamental integrals appear

$$T_n^{(1)} = \int_{r_1}^{r_2} R\,(R'\alpha + R)\, U_1(\varepsilon_n \alpha)\, d\alpha \quad ; \quad T_n^{(2)} = \int_{r_1}^{r_2} \frac{\alpha}{r_2}\,(R'\alpha + R)\, U_1(\varepsilon_n \alpha)\, d\alpha$$

$$V_n^{(1)}(\delta) = \int_{-\infty}^{\delta} Z Z' e^{-\varepsilon_n |z - \beta|}\, d\beta \quad ; \quad V_n^{(2)}(\delta) \int_{-\infty}^{\delta} Z' e^{-\varepsilon_n |\delta - \beta|}\, d\beta \qquad 4.3$$

The linearized approximation to the velocity components may then be separated into the velocity associated with a stationary blade row and the additional velocity resulting from rotating a blade row of the same loading with an angular velocity ω. Hence

$$\begin{aligned} \bar{u} &= \bar{u}_1 + \frac{\omega r_2}{\xi_o c}\, \bar{u}_2 \\ \bar{w}-1 &= \bar{w}_1 - 1 + \frac{\omega r_2}{\xi_o c}\, \bar{w}_2 \\ \bar{v}-\bar{v}_o &= \bar{v}_1 - \bar{v}_o + \frac{\omega r_2}{\xi_o c}\, \bar{v}_2 \end{aligned} \qquad 4.4$$

where

$$\begin{aligned} \frac{\bar{u}_1}{\left(\frac{\xi_o c}{w_o}\right)^2} &= 2\sum_{n=1}^{\infty} \frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2}\, T_n^{(1)} V_n^{(1)}(\infty) \\ \frac{\bar{u}_2}{\left(\frac{\xi_o c}{w_o}\right)^2} &= -\sum_{n=1}^{\infty} \frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2}\, T_n^{(2)} V_n^{(2)}(\infty) \\ \frac{\bar{w}_1 - 1}{\left(\frac{\xi_o c}{w_o}\right)^2} &= 2\sum_{n=1}^{\infty} \frac{U_o(\varepsilon_n r)}{2\varepsilon_n \nu_n^2}\, T_n^{(1)} \left\{ V_n^{(1)}(\infty) - 2V_n^{(1)}(z) + z^2 \right\} \\ \frac{\bar{w}_2}{\left(\frac{\xi_o c}{w_o}\right)^2} &= -\sum_{n=1}^{\infty} \frac{U_o(\varepsilon_n r)}{2\varepsilon_n \nu_n^2}\, T_n^{(2)} \left\{ V_n^{(2)}(\infty) - 2V_n^{(2)}(z) + 2z \right\} \\ \frac{\bar{v}_1 - \bar{v}_o}{\left(\frac{\xi_o c}{w_o}\right)^2} &= 2\sum_{n=1}^{\infty} \frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2}\, \frac{(Rr)'}{r}\, T_n^{(1)} \int_{-\infty}^{z} z\, V_n^{(1)}(\infty)\, dz \\ \frac{\bar{v}_2}{\left(\frac{\xi_o c}{w_o}\right)^2} &= -\sum_{n=1}^{\infty} \frac{U_1(\varepsilon_n r)}{2\varepsilon_n \nu_n^2}\, \frac{(Rr)'}{r}\, T_n^{(2)} \int_{-\infty}^{z} z\, V_n^{(2)}(\infty)\, dz \end{aligned} \qquad 4.5$$

Therefore when the series 4.5 have been evaluated for a given type of blade loading, the velocities induced by any stationary or rotating blade row of similar loading follow directly from equations 4.4. It should be noted that as the angular velocity is changed, the blade loading and not the blade shape is preserved.

This flow may be approximated by a discontinuous solution or, more conveniently by two discontinuous solutions, one associated with the velocities induced by the stationary blade row, the other with the effect of rotation. If these two discontinuities are located at $z = \beta_1$ and $z = \beta_2$ respectively, the solutions follow from equations 4.4 and 4.5 by the correspondence

$$V_n^{(1)}(\infty) \longrightarrow \frac{(Z^{(2)})^2 - (Z^{(1)})^2}{2}\, e^{-\varepsilon_n |z-\beta_1|}$$

$$V_n^{(2)}(\infty) \longrightarrow \frac{Z^{(2)} - Z^{(1)}}{2}\, e^{-\varepsilon_n |z-\beta_2|}$$

$$V_n^{(1)}(z) \longrightarrow \begin{cases} 0 & ; \quad z < \beta_1 \\ \dfrac{(Z^{(2)})^2 - (Z^{(1)})^2}{2}\, e^{-\varepsilon_n |z-\beta_1|} & ; \quad z > \beta_1 \end{cases}$$

$$V_n^{(2)}(z) \longrightarrow \begin{cases} 0 & ; \quad z < \beta_2 \\ \dfrac{Z^{(2)} - Z^{(1)}}{2}\, e^{-\varepsilon_n |z-\beta_2|} & ; \quad z > \beta_2 \end{cases} \qquad 4.6$$

$$Z^2 \longrightarrow \begin{cases} 0 & ; \quad z < \beta_1 \\ (Z^{(2)})^2 - (Z^{(1)})^2 & ; \quad z > \beta_1 \end{cases}$$

$$Z \longrightarrow \begin{cases} 0 & ; \quad z < \beta_2 \\ Z^{(2)} - Z^{(1)} & ; \quad z > \beta_2 \end{cases}$$

where $Z^{(1)}$ and $Z^{(2)}$ are defined by the approximate tangential velocity upstream and downstream of the blade row

$$v_\theta^{(1)} = \xi_{oc} R Z^{(1)}$$

$$v_\theta^{(2)} = \xi_{oc} R Z^{(2)} \qquad 4.7$$

and $\xi_0 c$ the circulation parameter remains constant as $c \to 0$.

The values of β_1 and β_2 are chosen such that each of the discontinuities acts at the centroid of the loading they replace. Thus β_1 and β_2 are solutions of the equations

$$\int_{-c/2}^{c/2} Z Z'(z-\beta_1)\,dz = 0$$

$$\int_{-c/2}^{c/2} Z'(z-\beta_2)\,dz = 0 \qquad 4.8$$

5. Single Row of Stationary or Rotating Blades: - In the following example the velocities induced by a single stationary or rotating blade row are calculated, assuming that the radial force vanishes, using both a particular continuous chordwise loading and the corresponding discontinuous approximation. In addition to illustrating the general procedure of application, the numerical results provide a direct comparison of the continuous distribution and the discontinuous approximation.

Consider a single row of blade chord c and prescribe the vorticity distribution $\xi(r,z)$ as shown in figure IV.4. This distribution is similar at all radii and increases linearly from the blade root to tip. If the corresponding physical blade row is supposed to have n blades, the linear vorticity distribution of the blade at any radius r is given by

$$\bar{\xi}(r,z) = \frac{2\pi r}{n}\,\xi(r,z) \qquad 5.1$$

and the corresponding total circulation is

$$\Gamma(r) = \frac{3\pi}{2n}\,\xi_0 c\left(\frac{r}{r_1}\right) \qquad 5.2$$

The tangential velocity becomes, from equation 2.13

$$\begin{aligned}
v_\theta &= 0 && -\tfrac{c}{2} \geq z \\
v_\theta &= \xi_0 c\left(\tfrac{z}{c}+\tfrac{1}{2}\right)\tfrac{r}{r_1} && -\tfrac{c}{2} \leq z \leq 0 \\
v_\theta &= \xi_0 c\left(\tfrac{3}{4}-\left(\tfrac{1}{2}-\tfrac{z}{c}\right)^2\right)\tfrac{r}{r_1} && 0 \leq z \leq \tfrac{c}{2} \\
v_\theta &= \tfrac{3}{4}\,\xi_0 c\left(\tfrac{r}{r_1}\right) && z \geq \tfrac{c}{2}
\end{aligned} \qquad 5.3$$

and the parameters R and Z are

$$R = r/r_1 \qquad 5.4$$

$$\begin{aligned} Z &= 0 & -\tfrac{c}{2} &\geq \zeta \\ Z &= \zeta/c + 1/2 & -\tfrac{c}{2} &\leq \zeta \leq 0 \\ Z &= \tfrac{3}{4} - \left(\tfrac{1}{2} - \tfrac{\zeta}{c}\right)^2 & 0 &\leq \zeta \leq \tfrac{c}{2} \\ Z &= \tfrac{3}{4} & \zeta &\geq c/2 \end{aligned} \qquad 5.5$$

The basic integrals, equations 4.3 are then evaluated by straightforward calculation and the results are tabulated in the appendix.

Using the tabulated relations the velocity perturbations associated with the stationary or rotating blade row follow directly from equations 4.5. For the case where $\frac{r_1}{r_2} = 0.6$, $\frac{r_2 - r_1}{c} = 2$, and the vorticity distribution is of the form given in figure IV.4, the induced velocity distribution has been calculated for various distances from the center of the blade row. The results are presented in figure IV.5

Figures IV.5a and IV.5b indicate that, for the blade loading under consideration, an appreciable change in the axial velocity profile takes place both before the fluid enters and after it leaves the blade row, the coordinates $\frac{\zeta}{c} = \pm 0.5$ marking the termini of the blade row. These axial velocity changes and the corresponding ones for the tangential velocity (figures IV.5c and IV.5d) are those involved in considering the interference of adjacent blade rows and for the sake of simplicity it is convenient if they can be estimated from the lifting line of approximation. The radial velocity disturbances (figures

IV.5e and IV.5f), which are of practical importance only since they produce the changes in tangential and axial velocity, exhibit a non-symmetry about the center line ($\frac{z}{c} = 0$) of the blade row which results from the unsymmetrical blade loading. The disturbance associated with the stationary blade attains its maximum value behind the center line while the additional disturbance associated with the rotation attains its maximum value ahead of the center line. This difference is a direct result of the load distribution producing the flow in each case and is clearly reflected in the position of the discontinuities appropriate for approximating each. Substituting the known load distribution into equations 4.8 it follows that

$$\beta_1 = 0.026c$$
$$\beta_2 = -0.111c \qquad 5.6$$

which agrees with the observed non-symmetry of the flow.

The discontinuous approximation to the axial velocity variation associated with the stationary blade row follows directly from the correspondence given by equations 4.6 where the discontinuity is located at $\frac{\beta_1}{c} = 0.026$. The axial velocity variation given by the discontinuous approximation is compared with the continuous solution in figure IV.6 and clearly provides a very reasonable approximation except, of course, in the immediate vicinity of the lifting line. This discrepancy near the lifting line is to be expected because of the excessively high radial velocity induced in this region which causes a correspondingly high axial gradient of the axial velocity perturbation.

6. <u>Approximation to the Velocity Components:</u> - The distribution of velocity far downstream of the blade rows, which is analogous to the downwash in the Trefftz plane for the theory of finite wings, may be calculated with comparative ease. For upon integration with respect to z , equations 2.10 and 2.17 give far downstream

$$\frac{\partial}{\partial r}\left(\bar{w}_{\infty} - \bar{w}_{-\infty}\right) = f(r,-\infty) - f(r,\infty) \qquad 6.1$$

where $\bar{w}_{-\infty}$ and $\bar{w}_{\infty}$ correspond to the distributions of axial velocity far upstream and downstream respectively of the blade row. Thus

$$\bar{w}_{\infty} - \bar{w}_{-\infty} = \int\left(f(r,-\infty) - f(r,\infty)\right) dr + C$$

so that is the approximate tangential velocity distribution is given by $v_0 = \xi_0 c \; g(r,z)$, it follows that

$$\frac{(\bar{w}_1 - 1)_{\infty}}{\left(\frac{\xi_0 c}{w_0}\right)^2} - \frac{(\bar{w}_1 - 1)_{-\infty}}{\left(\frac{\xi_0 c}{w_0}\right)^2} = \int\left(\frac{g(r,-\infty)}{r} \frac{\partial}{\partial r} r g(r,-\infty) - \frac{g(r,\infty)}{r} \frac{\partial}{\partial r} r g(r,-\infty) dr + c_1 \right.$$

$$= h_1(r) \qquad 6.2$$

$$\frac{(\bar{w}_2)_{\infty}}{\left(\frac{\xi_0 c}{w_0}\right)^2} - \frac{(\bar{w}_2)_{-\infty}}{\left(\frac{\xi_0 c}{w_0}\right)^2} = \int\left(\frac{\partial}{r_2 \partial r} r g(r,-\infty) - \frac{\partial}{r_2 \partial r} g(r,\infty)\right) dr + c_2$$

$$= h_2(r)$$

where the constants c_1 and c_2 are evaluated such that

$$\int_{r_1}^{r_2} (\bar{w}_1 - 1)_{\infty} \, r \, dr = 0$$

$$\int_{r_1}^{r_2} (\bar{w}_2)_{\infty} \, r \, dr = 0 \qquad 6.3$$

Now for the case in which the velocity far upstream is uniform it follows from the general discontinuous approximation to the axial velocity (equations 4.4, 4.5, 4.6) that

$$\lim_{z/c \to \infty} \left\{ 2 \sum_{n=1}^{\infty} \frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(1)}\left((Z^{(2)})^2 - (Z^{(1)})^2\right)\left(1 - \tfrac{1}{2} e^{-\varepsilon_n(z-\beta_1)}\right) \right.$$

$$\left. - \frac{\omega r_2}{\xi_0 c} \sum_{n=1}^{\infty} \frac{U_0(\varepsilon_n r)}{2\varepsilon_n \nu_n^2} T_n^{(2)}\left(Z^{(2)} - Z^{(1)}\right)\left(1 - \tfrac{1}{2} e^{-\varepsilon_n(z-\beta_2)}\right) \right\} \qquad 6.4$$

$$= h_1(r) + \frac{\omega r_2}{\xi_0 c} h_2(r)$$

This suggests the approximation

$$\frac{\overline{w}_1 - 1}{\left(\frac{\xi_0 c}{w_0}\right)^2} = \begin{cases} \dfrac{h_1(r)}{2} e^{-\frac{\lambda_1}{R}\frac{z-\beta_1}{c}} & z < \beta_1 \\ h_1(r)\left(1 - \tfrac{1}{2} e^{-\frac{\lambda_1}{R}\frac{z-\beta_1}{c}}\right) & z > \beta_1 \end{cases}$$

and 6.5

$$\frac{\overline{w}_2}{\left(\frac{\xi_0 c}{w_0}\right)^2} = \begin{cases} \dfrac{h_2(r)}{2} e^{-\frac{\lambda_2}{R}\frac{z-\beta_2}{c}} & z < \beta_2 \\ h_2(r)\left(1 - \tfrac{1}{2} e^{-\frac{\lambda_2}{R}\frac{z-\beta_2}{c}}\right) & z > \beta_2 \end{cases}$$

where R is the blade aspect ratio $\frac{r_2 - r_1}{c}$ and the λ are undetermined constants. Similarly using the continuity relations equation 2.8 and from equation 2.19 the corresponding radial and tangential velocity components are

$$\frac{\overline{u}_1}{\left(\frac{\xi_0 c}{w_0}\right)^2} = -\frac{\lambda_1}{r_2 - r_1} k_1(r)\, e^{-\frac{\lambda_1}{R}\left|\frac{z-\beta_1}{c}\right|}$$

$$\frac{\overline{u}_2}{\left(\frac{\xi_0 c}{w_0}\right)^2} = -\frac{\lambda_2}{r_2 - r_1} k_2(r)\, e^{-\frac{\lambda_2}{R}\left|\frac{z-\beta_1}{c}\right|} \qquad 6.6$$

and

$$\frac{\bar{v}_1 - \bar{v}_0}{\left(\frac{\xi_0 c}{\omega_0}\right)^2} = \frac{\xi_0 c}{\omega_0} k_2(r) \frac{(r\,g(r,\infty))'}{r} \left(e^{-\frac{\lambda_1}{R}\left(\frac{z-\beta_1}{c}\right)} - 1 \right); \quad z > \beta_1$$

$$\frac{\bar{v}_2}{\left(\frac{\xi_0 c}{\omega_0}\right)^2} = \frac{\xi_0 c}{\omega_0} k_2(r) \frac{(r\,g(r,\infty))'}{r} \left(e^{-\frac{\lambda_2}{R}\left(\frac{z-\beta_2}{c}\right)} - 1 \right); \quad z > \beta_2 \qquad 6.7$$

where

$$r k_{1,2} = \frac{1}{2} \int r h_{1,2}(r)\, dr + C \qquad 6.8$$

and the constant of integration is evaluated to satisfy the boundary conditions on the radial velocity.

If the radial velocities (equation 6.6) satisfied the homogeneous differential equation 3.2, the velocity components equations 6.5-6.7 would be true solutions of the linearized problem. Hence the values of λ_1 and λ_2 will be chosen so that the radial velocity satisfies equation 3.2 except for a minimum weighted mean square error over the entire field of flow. From the first of equations 6.6 and equation 3.2

$$\frac{1}{\left(\frac{\xi_0 c}{\omega_0}\right)^2} \left(\frac{\partial^2 \bar{u}_1}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{\bar{u}_1}{r}\right) + \frac{\partial^2 \bar{u}_1}{\partial z^2} \right) = -\frac{\lambda_1}{r_2 - r_1} \left\{ k_1''(r) + \left(\frac{k_1(r)}{r}\right)' + \left(\frac{\lambda_1}{r_2 - r_1}\right)^2 k_1(r) \right\} e^{-\frac{\lambda_1}{R}\left(\frac{z-\beta_1}{c}\right)}$$

and the value of λ_1 is found by minimizing the integral

$$\int_{r_1}^{r_2} \int_{\beta_1}^{\infty} \left(\frac{\lambda_1}{r_2 - r_1}\right)^2 \left(k_1'' + \left(\frac{k_1}{r}\right)' + \left(\frac{\lambda_1}{r_2 - r_1}\right)^2 k_1 \right)^2 e^{\frac{-2\lambda_1}{R}\left(\frac{z-\beta_1}{c}\right)} (z-\beta_1)\, r\, dr\, dz \qquad 6.9$$

By integrating with respect to z, minimizing with respect to λ_1, and simplifying the result through integration by parts, the parameter λ_1 is

$$\lambda_1 = \pm \left(\frac{\int_{r_1}^{r_2} \left(\frac{k_1'}{r_2 - r_1}\right)^2 r\,dr + \int_{r_1}^{r_2} \left(\frac{k_1}{r_2 - r_1}\right)^2 \frac{dr}{r}}{\int_{r_1}^{r_2} \left(\frac{k_1}{r_2 - r_1}\right)^2 \frac{r\,dr}{(r_2 - r_1)^2}} \right)^{\frac{1}{2}} \qquad 6.10$$

where each of these integrals is positive and hence the values of the parameter are real. Similarly for the solutions of the additional velocity associated with a rotating blade row

$$\lambda_2 = \pm \left(\frac{\int_{r_1}^{r_2} \left(\frac{k_2'}{r_2 - r_1}\right)^2 r\,dr + \int_{r_1}^{r_2} \left(\frac{k_2}{r_2 - r_1}\right)^2 \frac{dr}{r}}{\int_{r_1}^{r_2} \left(\frac{k_2}{r_2 - r_1}\right)^2 \frac{r\,dr}{(r_2 - r_1)^2}} \right)^{\frac{1}{2}} \qquad 6.11$$

In the example for which detailed calculations were carried out in the previous section, $v_0^{(1)} = 0$; $v_0^{(2)} = \frac{3}{4} \xi_0 c \left(\frac{r}{r_1}\right)$. It follows directly that

$$h_1 = \frac{9}{32} \left(\left(\frac{r_2}{r_1}\right)^2 + 1 - 2\left(\frac{r}{r_1}\right)^2 \right)$$

$$h_2 = \frac{3}{8} \frac{r_1}{r_2} \left(\left(\frac{r_2}{r_1}\right)^2 + 1 - 2\left(\frac{r}{r_1}\right)^2 \right) \qquad 6.12$$

where

$$k_1 = \frac{9}{128} \frac{1}{r} \left(\left(\frac{r_2}{r_1}\right)^2 - \left(\frac{r}{r_1}\right)^2 \right) \left(r^2 - r_1^2 \right)$$

$$k_2 = \frac{3}{32} \frac{1}{r} \left(\frac{r_1}{r_2}\right) \left(\left(\frac{r_2}{r_1}\right)^2 - \left(\frac{r}{r_1}\right)^2 \right) \left(r^2 - r_1^2 \right) \qquad 6.13$$

where $\frac{r_1}{r_2} = 0.6$. The appropriate values of the exponents λ_1 and λ_2 are then

$$\lambda_1 = \lambda_2 = 2.58 \qquad 6.14$$

The exponential approximation to the velocity distribution may be found very simply by applying equations 6.5-6.7. In particular the axial velocity perturbation associated with the stationary blade row is

$$\frac{\bar{w}_1 - 1}{\left(\frac{\xi_0 c}{w_0}\right)^2} = \begin{cases} \frac{9}{64}\left(\left(\frac{r_2}{r_1}\right)^2 + 1 - 2\left(\frac{r}{r_1}\right)^2\right) e^{1.29\left(\frac{z-\beta_1}{c}\right)} & z < \beta_1 \\ \frac{9}{32}\left(\left(\frac{r_2}{r_1}\right)^2 + 1 - 2\left(\frac{r}{r_1}\right)^2\right)\left(1 - \frac{1}{2} e^{-1.29\left(\frac{z-\beta_1}{c}\right)}\right) & z > \beta_1 \end{cases} \qquad 6.15$$

and is compared in figure IV.7 with the general linearized solution for finite blade chord. This exponential approximation appears, at least in this case to be quite sufficient for estimating the general order of the velocity components. Their simplicity makes these relations quite useful in estimating the interference of adjacent blade rows.

A further simplification, useful in the neighborhood of the blade row, follows by approximating the exponential function in equations 6.9-6.11 in the form

$$e^{\pm \frac{\lambda}{\mathcal{R}}\left(\frac{z-\beta}{c}\right)} \approx \frac{1}{1 \mp \frac{\lambda}{\mathcal{R}}\left(\frac{z-\beta}{c}\right)} \qquad 6.16$$

The axial velocity upstream of the blade row is given, for example, by

$$\frac{\bar{w}_1 - 1}{\left(\frac{\xi_0 c}{w_0}\right)^2} \approx \frac{h_1(r)}{2\left(1 + \frac{\lambda_1}{\mathcal{R}}\,\frac{z-\beta}{c}\right)} \qquad 6.17$$

and similar relations may be easily obtained for the other velocity components.

APPENDIX: TABULATED VALUES OF THE BASIC INTEGRALS

$$T_n^{(1)} = \frac{2}{\varepsilon_n}\left(U_0(\varepsilon_n r_1) - \left(\frac{r_2}{r_1}\right)^2 U_0(\varepsilon_n r_2) \right)$$

$$T_n^{(2)} = \frac{r_1}{r_2}\, T_n^{(1)}$$

The functions V_n must be evaluated in four separate cases corresponding to the four different representations of Z.

Case I $\quad z \leq -\frac{c}{2}$

$$V_n^{(1)}(z) = 0$$

$$V_n^{(1)}(\infty) = \left(\frac{1}{\varepsilon_n c}\right)^2 \left(\frac{\varepsilon_n c}{4} - \frac{5}{2} + e^{\frac{\varepsilon_n c}{2}} + \frac{3}{2} e^{-\frac{\varepsilon_n c}{2}} \right) e^{\varepsilon_n z}$$
$$- \frac{2}{(\varepsilon_n c)^4} \left(\left(\frac{\varepsilon_n c}{2}\right)^3 - 3\left(\frac{\varepsilon_n c}{2}\right)^2 + 6\left(\frac{\varepsilon_n c}{2}\right) - 6 + 6e^{-\frac{\varepsilon_n c}{2}} \right) e^{\varepsilon_n z}$$

$$V_n^{(2)}(z) = 0$$

$$V_n^{(2)}(\infty) = \left(\frac{2}{(\varepsilon_n c)^2}\left(1 - e^{-\frac{\varepsilon_n c}{2}}\right) - \frac{1}{\varepsilon_n c}\, e^{\frac{\varepsilon_n c}{2}} \right) e^{\varepsilon_n z}$$

Case II $\quad -\frac{c}{2} \leq z \leq 0$

$$V_n^{(1)}(z) = \frac{1}{(\varepsilon_n c)^2} \left(\varepsilon_n c\left(\frac{z}{c} + \frac{1}{2}\right) - 1 + e^{-\varepsilon_n c\left(\frac{z}{c} + \frac{1}{2}\right)} \right)$$

$$V_n^{(1)}(\infty) = \frac{1}{(\varepsilon_n c)^2} \left(e^{\varepsilon_n z}\left(\frac{\varepsilon_n c}{4} - \frac{5}{2}\right) + \frac{3}{2} e^{-\frac{\varepsilon_n c}{2}} + e^{-\varepsilon_n c\left(\frac{z}{c} + \frac{1}{2}\right)} + 2\varepsilon_n c\left(\frac{z}{c} + \frac{1}{2}\right) \right)$$
$$- \frac{2}{(\varepsilon_n c)^4} \left(6e^{-\frac{\varepsilon_n c}{2}} + \left(\frac{\varepsilon_n c}{2}\right)^3 - 3\left(\frac{\varepsilon_n c}{2}\right)^2 + 6\,\frac{\varepsilon_n c}{2} - 6 \right) e^{\varepsilon_n z}$$

$$V_n^{(2)}(z) = \frac{1}{\varepsilon_n c} \left(e^{-\varepsilon_n c\left(\frac{z}{c} + \frac{1}{2}\right)} - 1 \right)$$

$$V_n^{(2)}(\infty) = \frac{1}{\varepsilon_n c} \left(1 + e^{-\frac{\varepsilon_n c}{2}} - 2e^{-\varepsilon_n z} \right) e^{\varepsilon_n z} - \frac{2}{(\varepsilon_n c)^2} \left(\frac{\varepsilon_n c}{2} - 1 + e^{-\frac{\varepsilon_n c}{2}} \right) e^{\varepsilon_n z}$$

Case III $\quad 0 \le z \le \frac{c}{2}$

$$V_n^{(1)}(z) = \frac{1}{(\varepsilon_n c)^2}\left(\left(\frac{\varepsilon_n c}{2} - \frac{1}{2}\right)e^{-\frac{\varepsilon_n c}{2}} + e^{-\varepsilon_n c} + \frac{3}{2}\left(\varepsilon_n c\left(\frac{1}{2} - \frac{z}{c}\right)+1\right)e^{\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right)} - \frac{3}{2}\left(\frac{\varepsilon_n c}{2}+1\right)e^{-\frac{\varepsilon_n c}{2}}\right)e^{-\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right)}$$

$$\frac{-2}{(\varepsilon_n c)^4}\left(\left(\varepsilon_n c\left(\frac{1}{2} - \frac{z}{c}\right)\right)^3 + 3\left(\varepsilon_n c\left(\tfrac{1}{2} - \frac{z}{c}\right)\right)^2 + 6\left(\varepsilon_n c\left(\frac{1}{2} - \frac{z}{c}\right)\right) + 6\right) + \frac{2}{(\varepsilon_n c)^4}\left(\left(\frac{\varepsilon_n c}{2}\right)^3 + 3\left(\frac{\varepsilon_n c}{2}\right)^2 + 6\left(\frac{\varepsilon_n c}{2}\right) + 6\right)e^{-\varepsilon_n z}$$

$$V_n^{(1)}(\infty) = V_n^{(1)}(z) + \frac{3}{2}\frac{1}{(\varepsilon_n c)^2}\left(1 - \left(\varepsilon_n c\left(\frac{z}{c} - \frac{1}{2}\right)+1\right)e^{-\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right)}\right)e^{\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right)}$$

$$-\frac{2}{(\varepsilon_n c)^4}\left(6e^{\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right)} - \left(\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right)\right)^3 + 3\left(\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right)\right)^2 + 6\left(\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right)\right) + 6\right)$$

$$V_n^{(2)}(z) = -\frac{1}{\varepsilon_n c}\left(1 - e^{-\frac{\varepsilon_n c}{2}}\right)e^{-\varepsilon_n z} + \frac{2}{(\varepsilon_n c)^2}\left(\left(\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right)-1\right)e^{\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right)} + \left(\frac{\varepsilon_n c}{2}+1\right)e^{-\frac{\varepsilon_n c}{2}}\right)e^{-\frac{\varepsilon_n c}{2}\left(\frac{z}{c}-\frac{1}{2}\right)}$$

$$V_n^{(2)}(\infty) = -\frac{1}{\varepsilon_n c}\left(1 - e^{-\frac{\varepsilon_n c}{2}}\right)e^{-\varepsilon_n z} + \frac{2}{(\varepsilon_n c)^2}\left(2\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right) + \left(\frac{\varepsilon_n c}{2}+1\right)e^{-\varepsilon_n z} - e^{\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right)}\right)$$

Case IV $\quad \frac{c}{2} \le z$

$$V_n^{(1)}(z) = V_n^{(1)}(\infty)$$

$$V_n^{(1)}(\infty) = \frac{1}{(\varepsilon_n c)^2}\left(\left(\frac{\varepsilon_n c}{2} - 1\right)e^{\frac{\varepsilon_n c}{2}} + 1 - \frac{3}{2}\left(\left(\frac{\varepsilon_n c}{2}+1\right)e^{\frac{\varepsilon_n c}{2}} - e^{\varepsilon_n c}\right)\right)e^{-\varepsilon_n c\left(\frac{z}{c}+\frac{1}{2}\right)}$$

$$+\frac{2}{(\varepsilon_n c)^4}\left(\left(\frac{\varepsilon_n c}{2}\right)^3 + 3\left(\frac{\varepsilon_n c}{2}\right)^2 + 6\left(\frac{\varepsilon_n c}{2}\right) + 6 - 6e^{\frac{\varepsilon_n c}{2}}\right)e^{-\varepsilon_n z}$$

$$V_n^{(2)}(z) = V_n^{(2)}(\infty)$$

$$V_n^{(2)}(\infty) = -\frac{1}{\varepsilon_n c}\left(1 - e^{-\frac{\varepsilon_n c}{2}}\right)e^{-\varepsilon_n z} + \frac{2}{(\varepsilon_n c)^2}e^{-\varepsilon_n c\left(\frac{z}{c}-\frac{1}{2}\right)}\left(\left(\frac{\varepsilon_n c}{2}+1\right)e^{-\frac{\varepsilon_n c}{2}} - 1\right)$$

REFERENCES

1. Garrick, I. E.: On the Plane Potential Flow Past a Lattice of Arbitrary Airfoils. NACA Technical Report No. 788. 1944.

2. Lighthill, M. J.: A Mathematical Method of Cascade Design. Aeronautical Research Council R. and M. No. 2104. June 1945.

3. Goldstein, Arthur W. and Jerison, Meyer: Isolated and Cascade Airfoils with Prescribed Velocity Distribution. NACA Technical Note No. 1038. May 1947.

4. Keller, Curt: The Theory and Performance of Axial Flow Fans. McGraw-Hill Book Co., New York. 1937. Chapter 1. pp. 1-11.

5. Traupel, W: Neue Allgemeine Theorie der Mehrstufigen Axialen Turbomaschinen. Leeman and Col. Zurich, 1942.

6. Sinnette, John T. Jr.: Increasing the Range of Axial Flow Compressors by Use of Adjustable Stator Blades. Journal of Aero. Sci. Vol. 14, No. 5, May 1947. pp. 269-282.

7. Eckert and Korbacher: The Flow Through Axial Turbine Stages of Large Radial Blade Length. NACA Technical Memorandum No. 1118. April 1947.

8. Ruden, F.: Investigation of Single Stage Axial Fans. NACA Technical Memorandum No. 1062. April 1944.

9. Meyer, Richard E.: Beitrag zur Theorie feststehender Schaufelgitter. Mit. Inst. Aerodynamik E.T.H. Zurich, 1946.

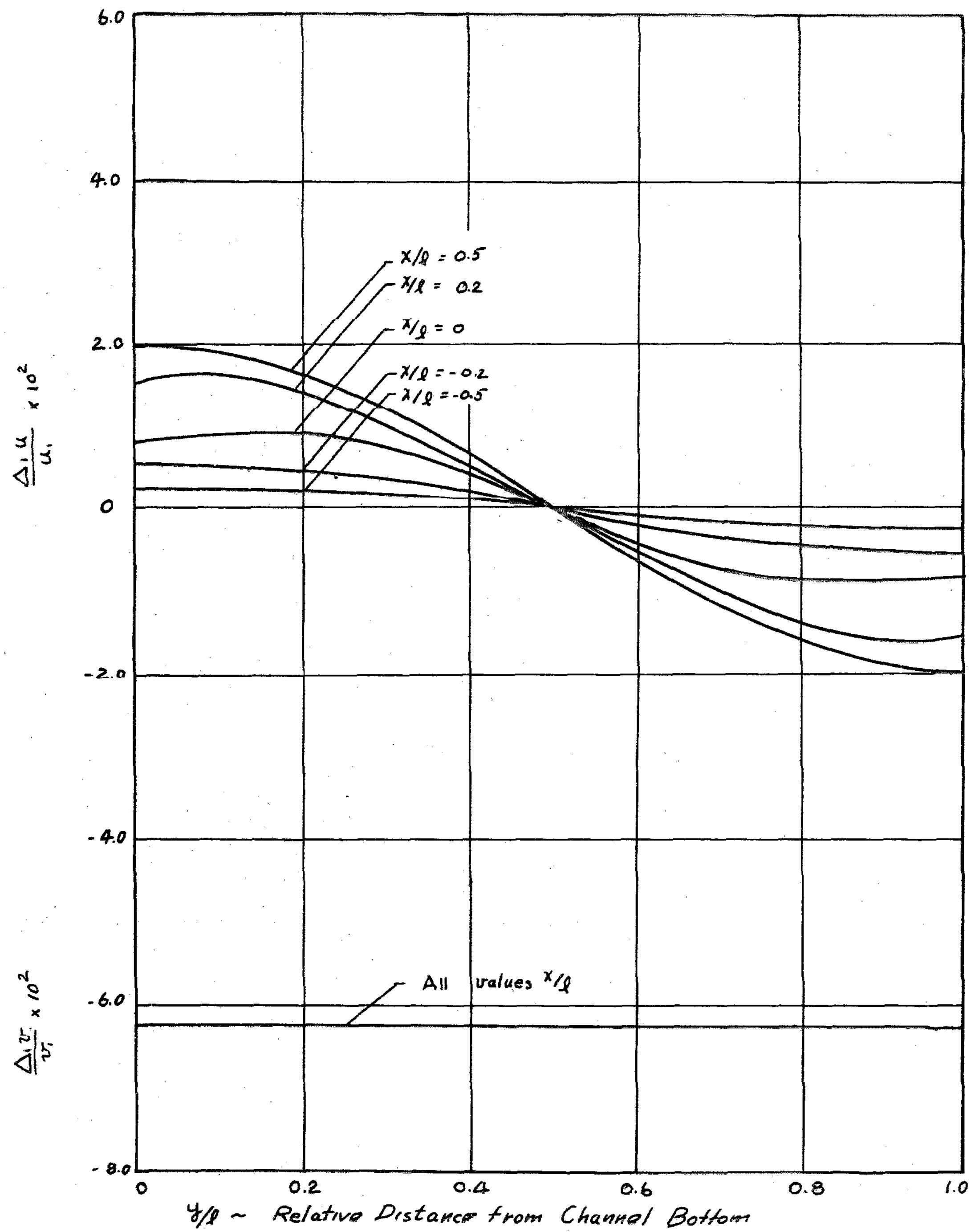

Figure II. 2. - Relative Correction Caused by Transport of Second Order Vorticity Increment Along Initial Streamlines

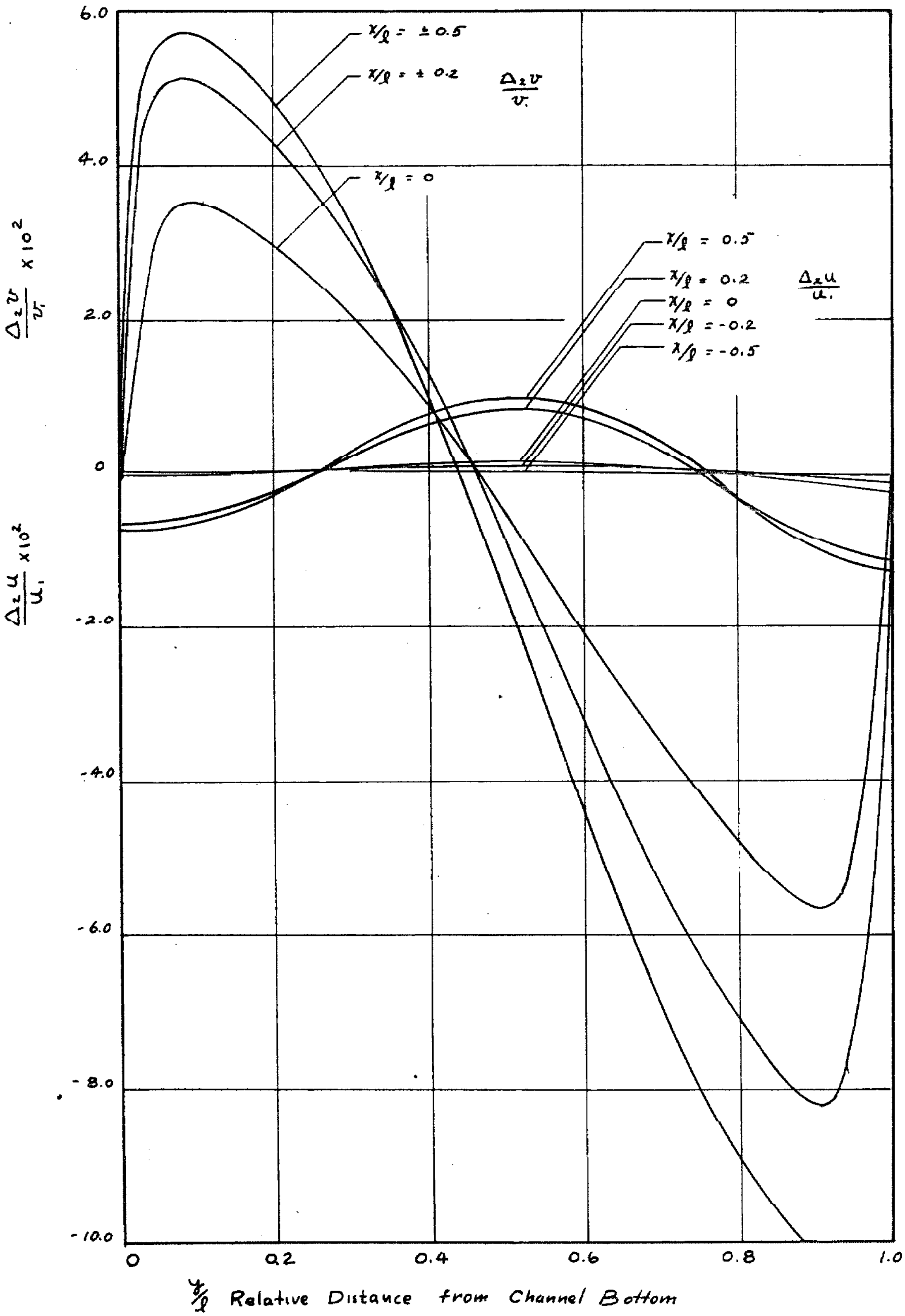

Figure II. 3. - Relative Correction Caused by Transport of Initial Vorticity Along First Order Streamlines

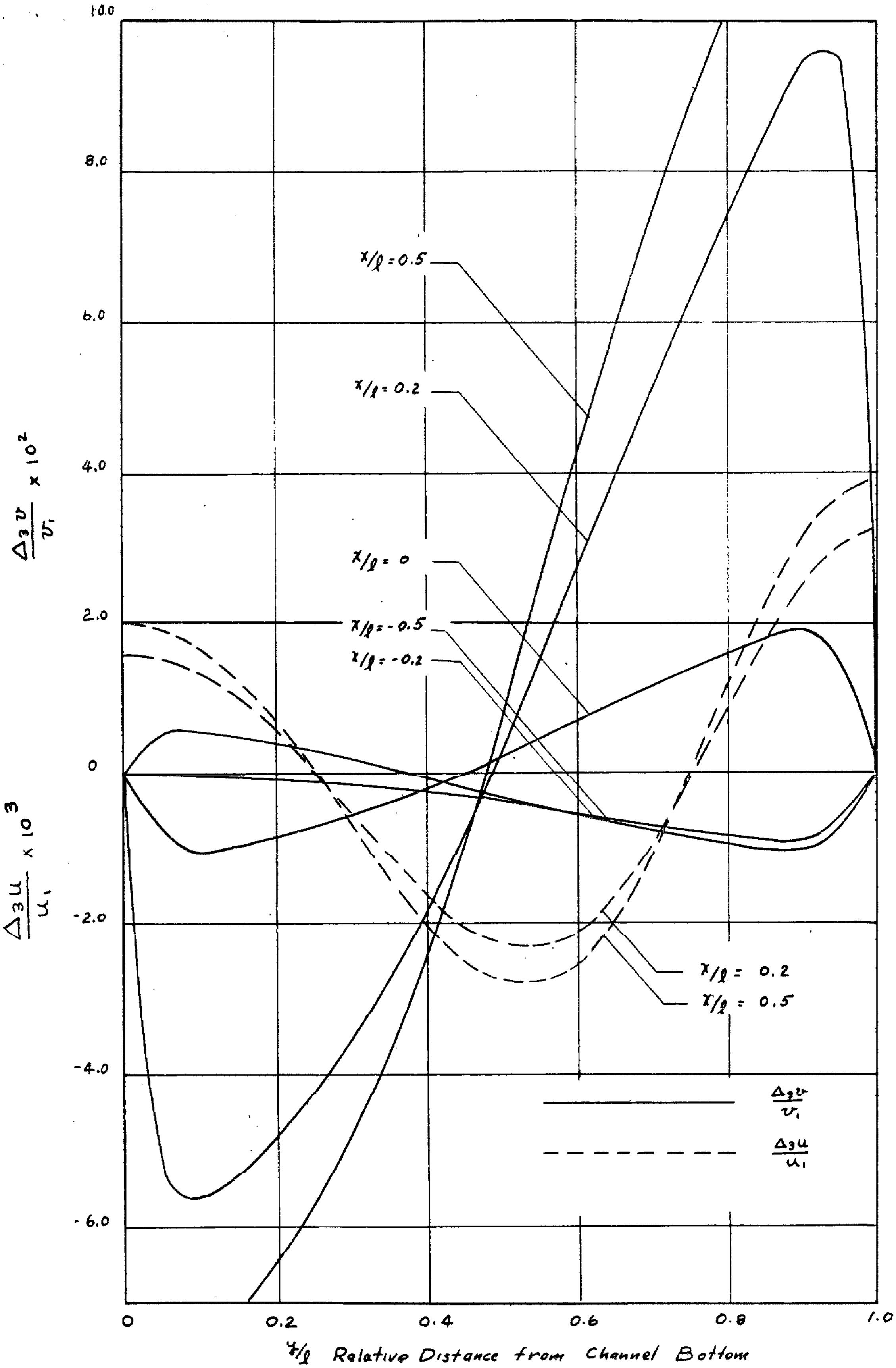

Figure II. 4. - Transport of Second Order Vorticity Increment Along First Order Streamlines

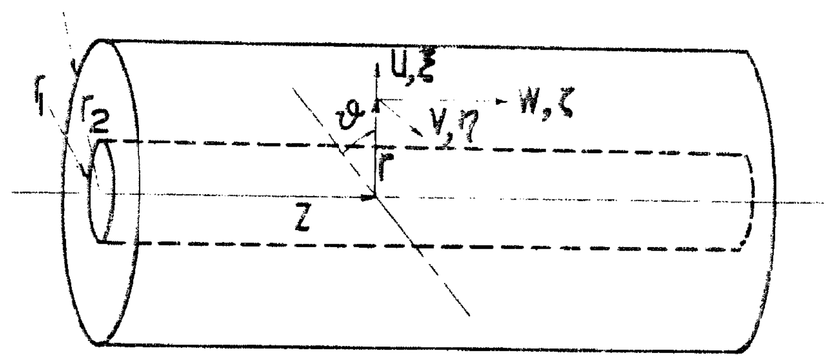

Figure IV.1. - Coordinate System and Designation of Velocity and Vorticity Components.

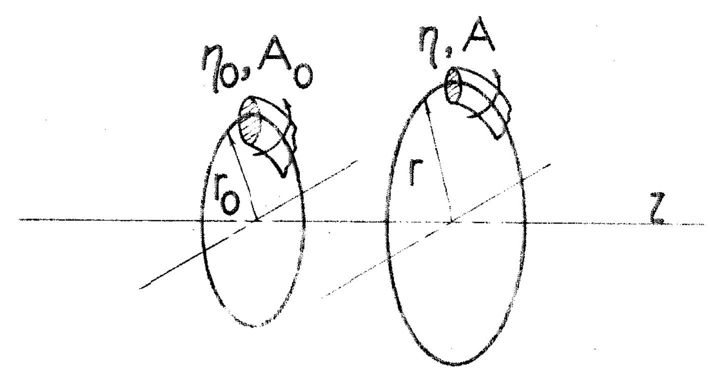

Figure IV.2. - Deformation of a Vortex Ring.

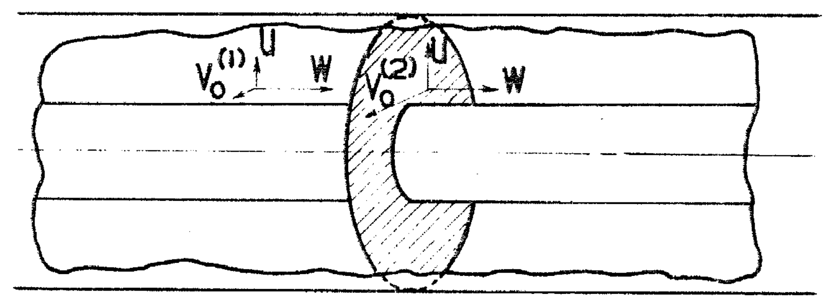

Figure IV.3. - Jump of Tangential Velocity across a Discontinuity.

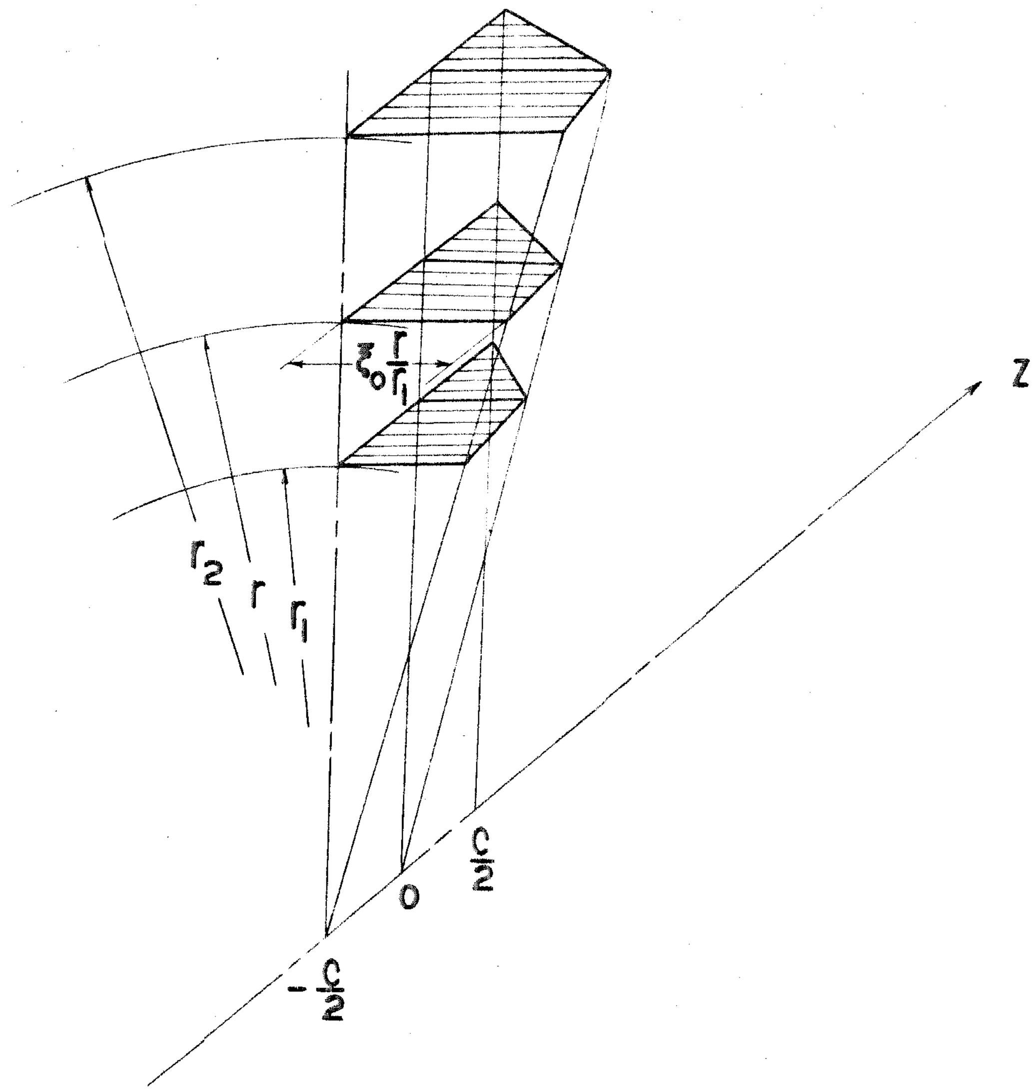

Figure IV.4. - Distribution of Load on the Blade Row.

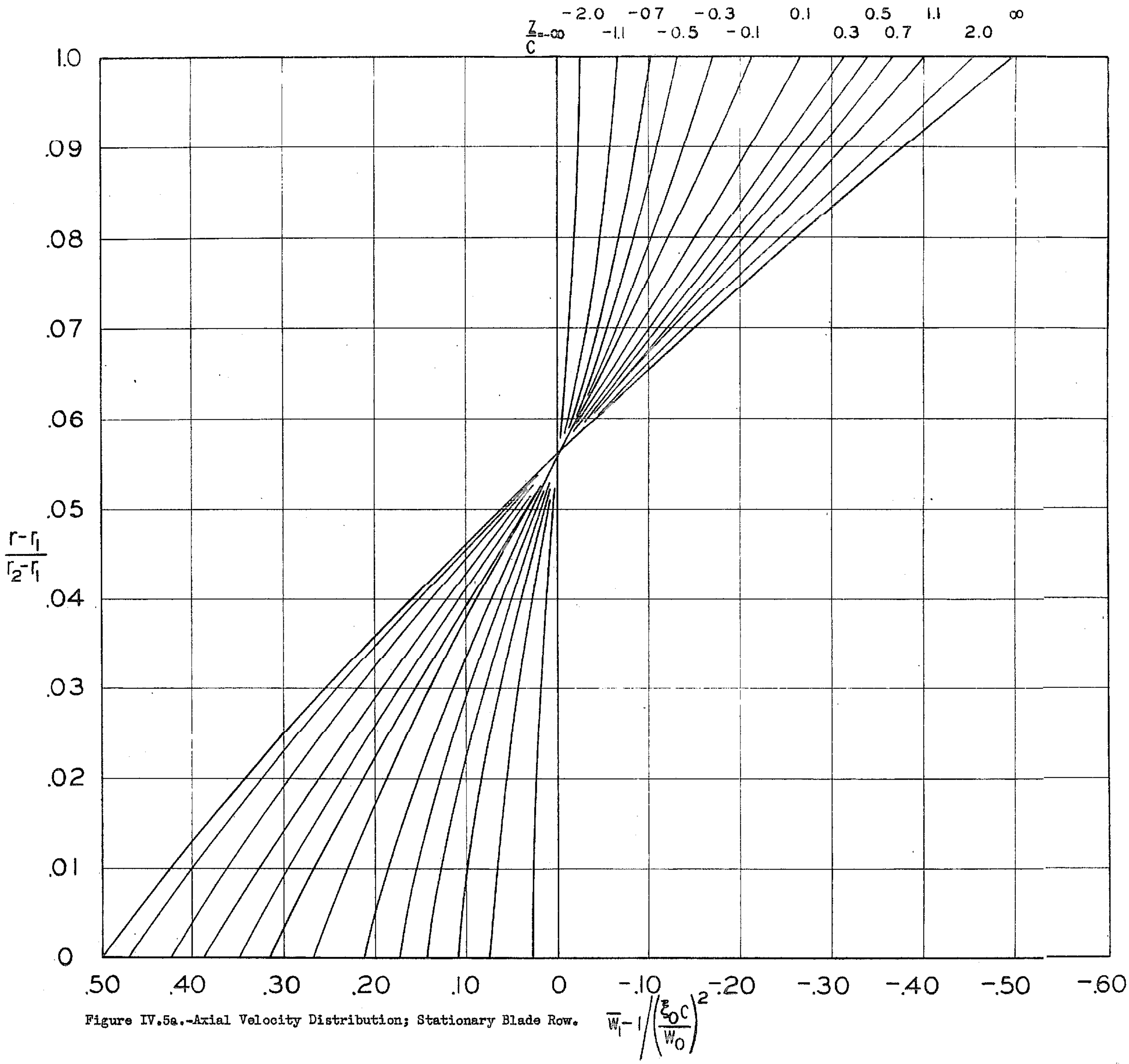

Figure IV.5a.-Axial Velocity Distribution; Stationary Blade Row.

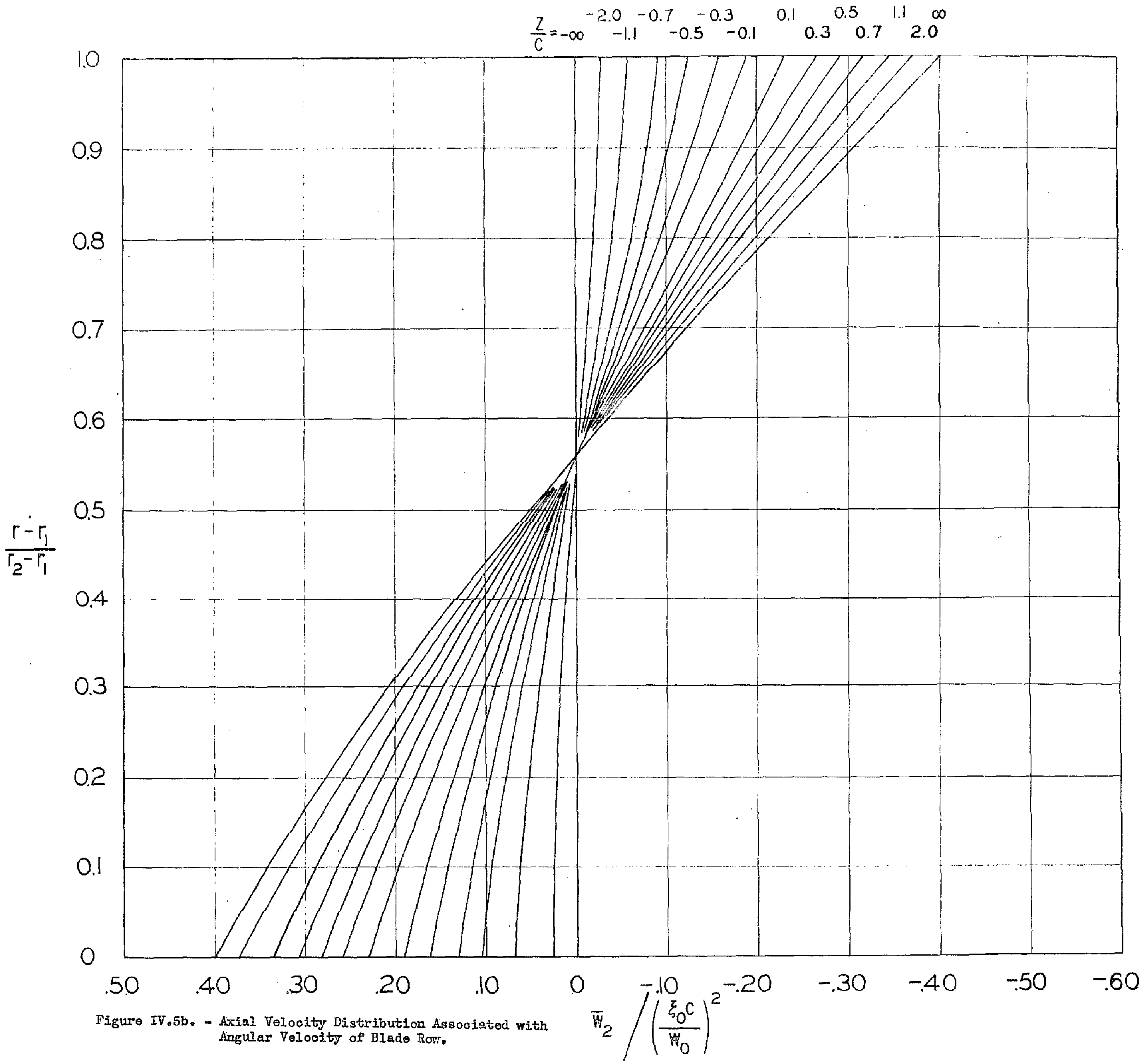

Figure IV.5b. - Axial Velocity Distribution Associated with Angular Velocity of Blade Row.

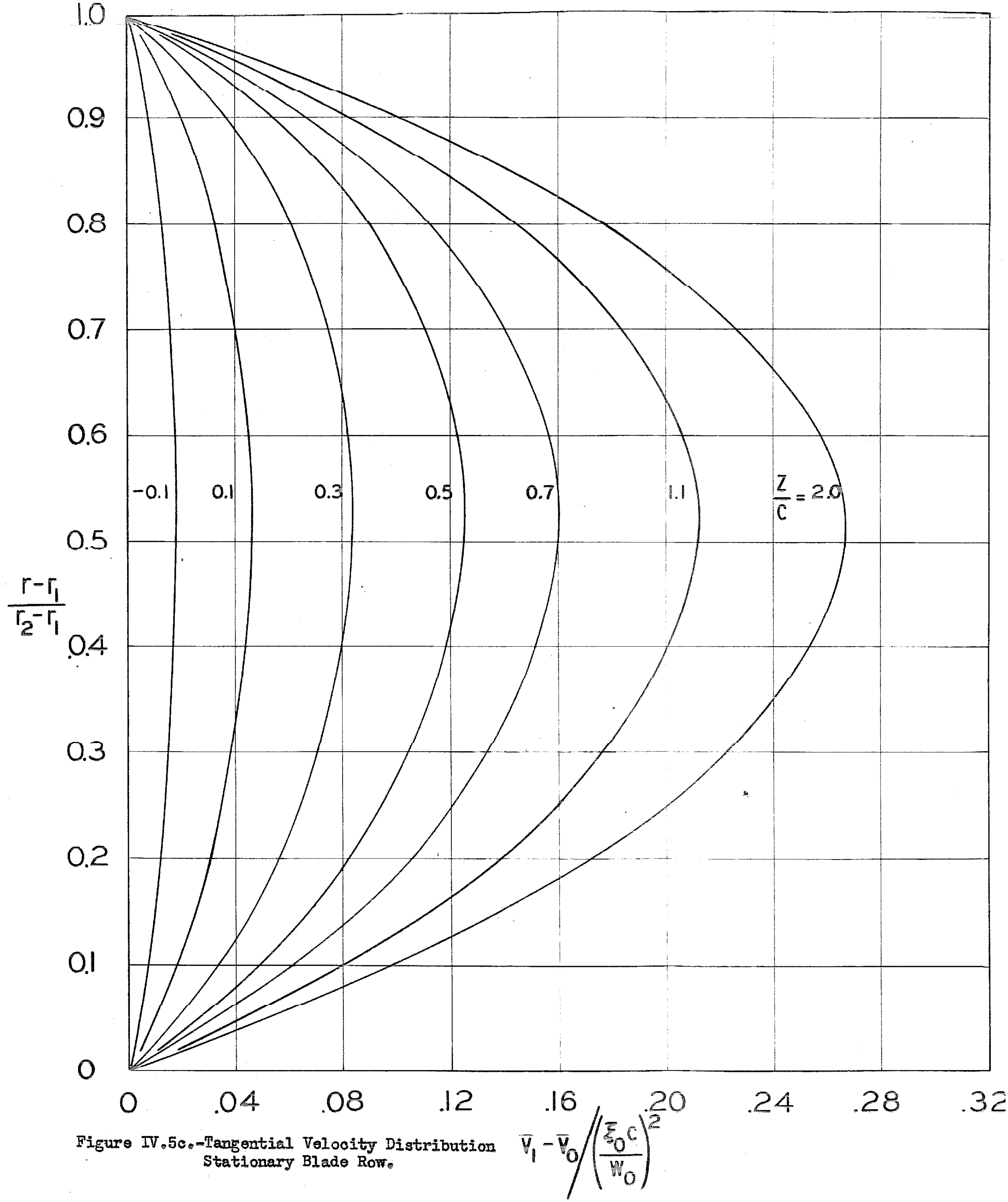

Figure IV.5c.-Tangential Velocity Distribution Stationary Blade Row.

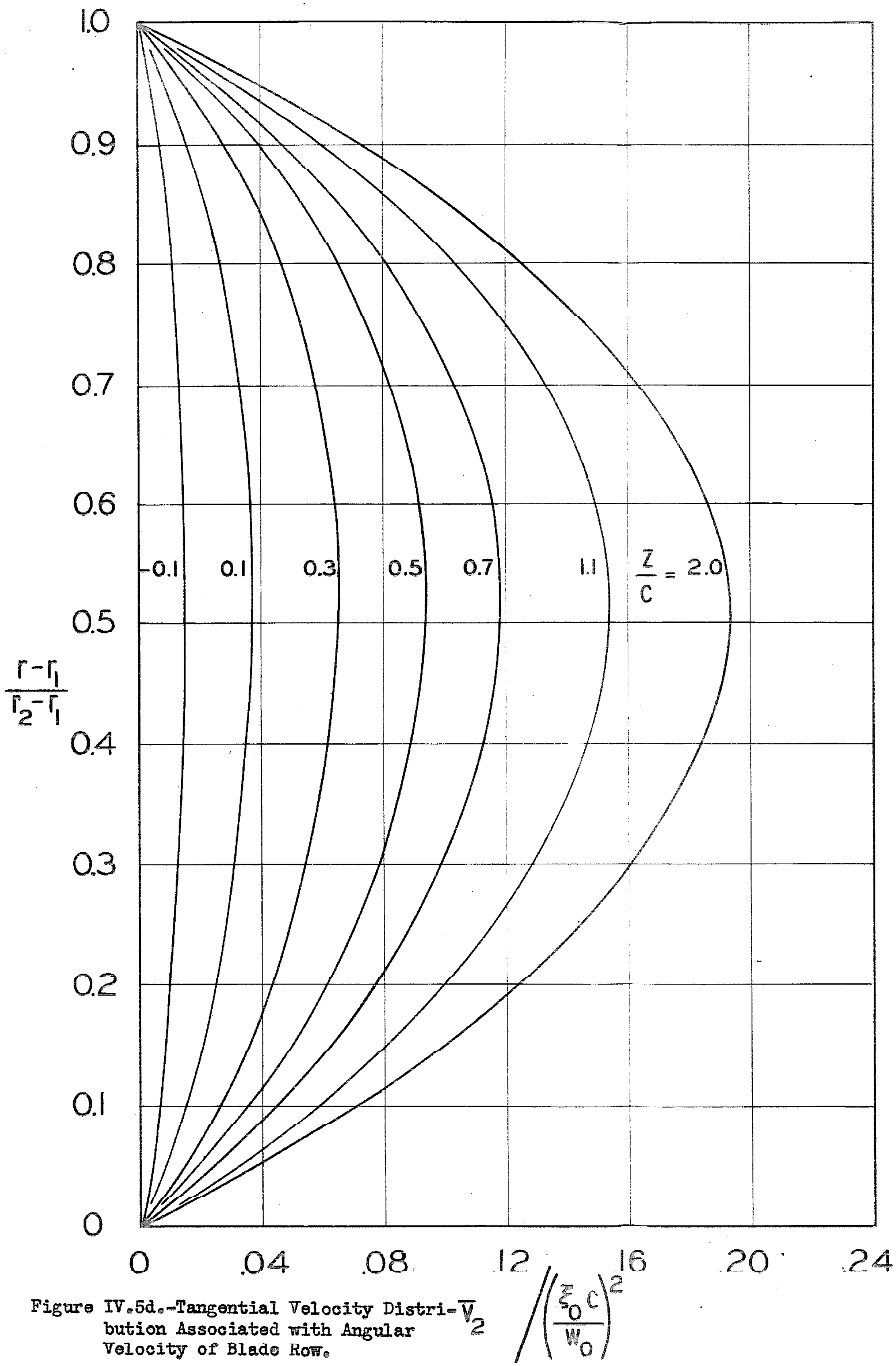

Figure IV.5d.-Tangential Velocity Distribution Associated with Angular Velocity of Blade Row.

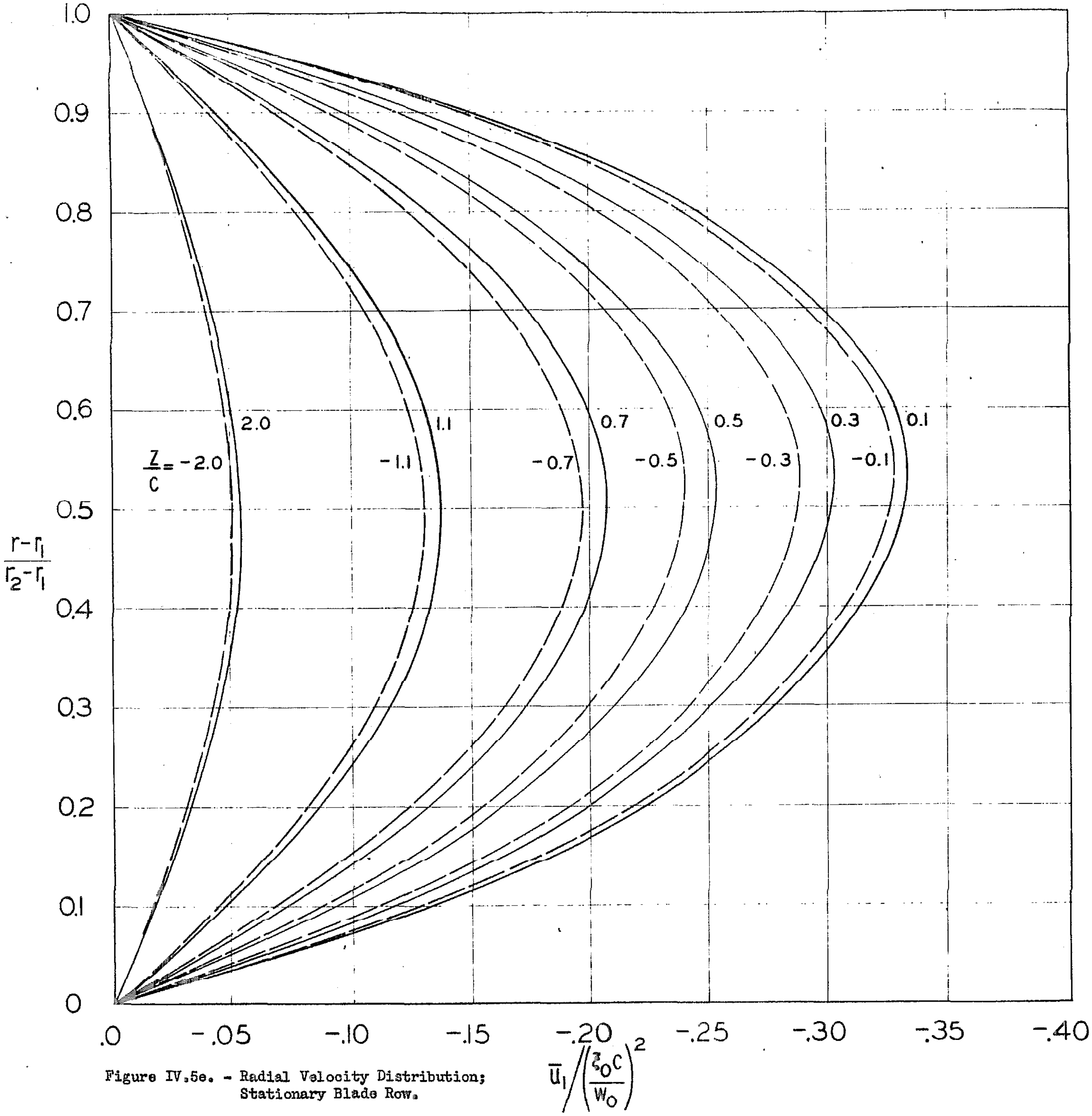

Figure IV,5e. - Radial Velocity Distribution; Stationary Blade Row.

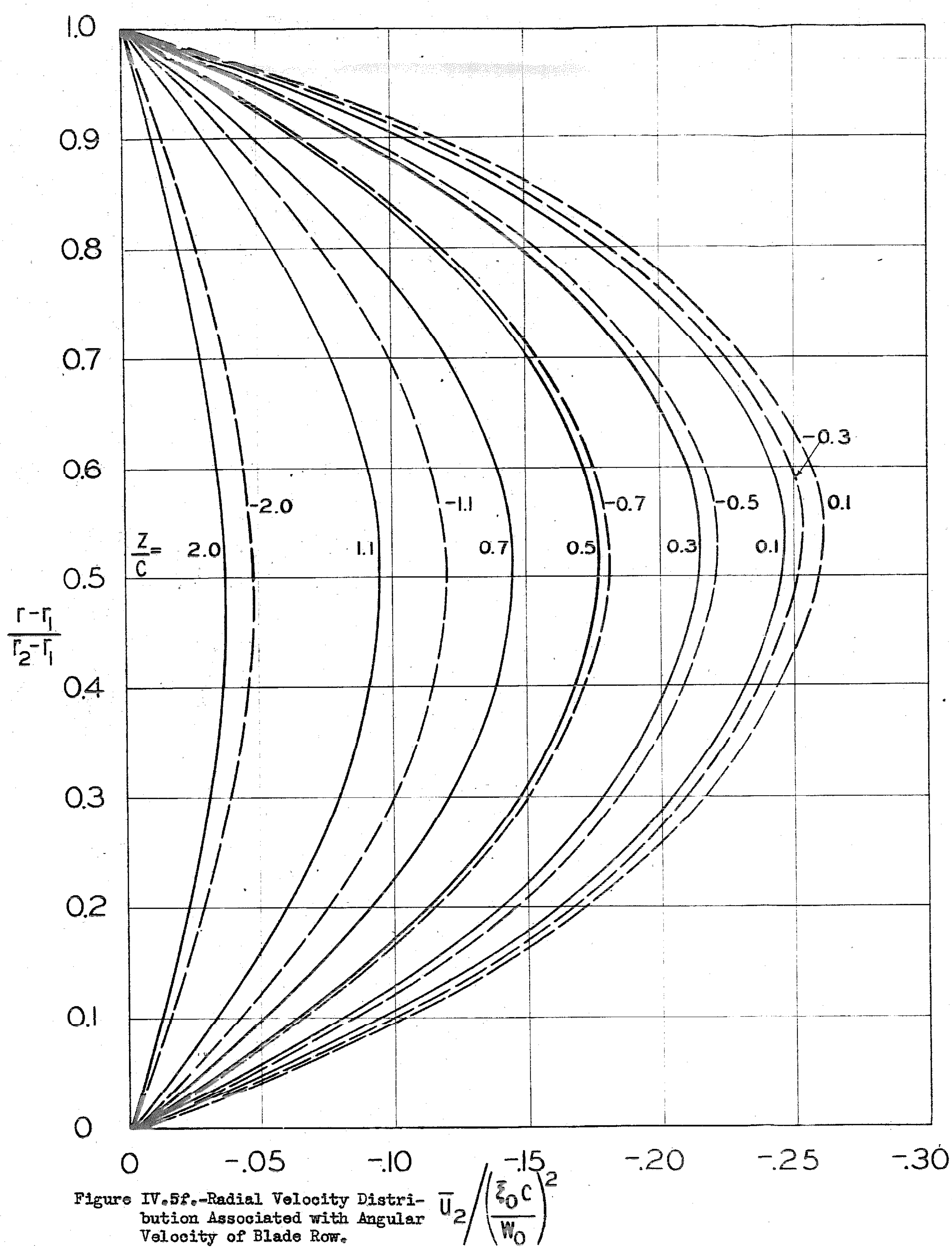

Figure IV.5f.-Radial Velocity Distribution Associated with Angular Velocity of Blade Row.

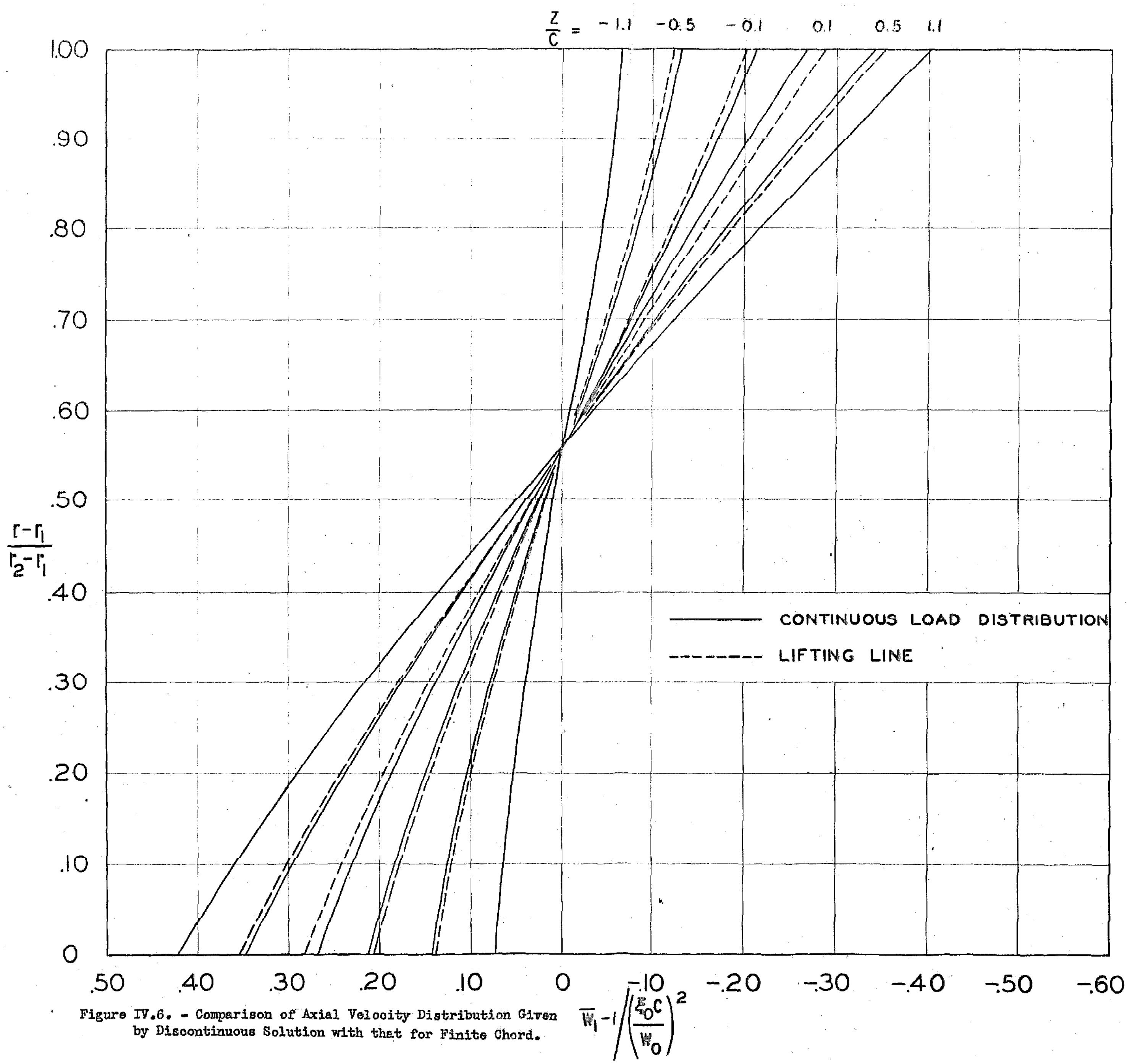

Figure IV.6. - Comparison of Axial Velocity Distribution Given by Discontinuous Solution with that for Finite Chord.

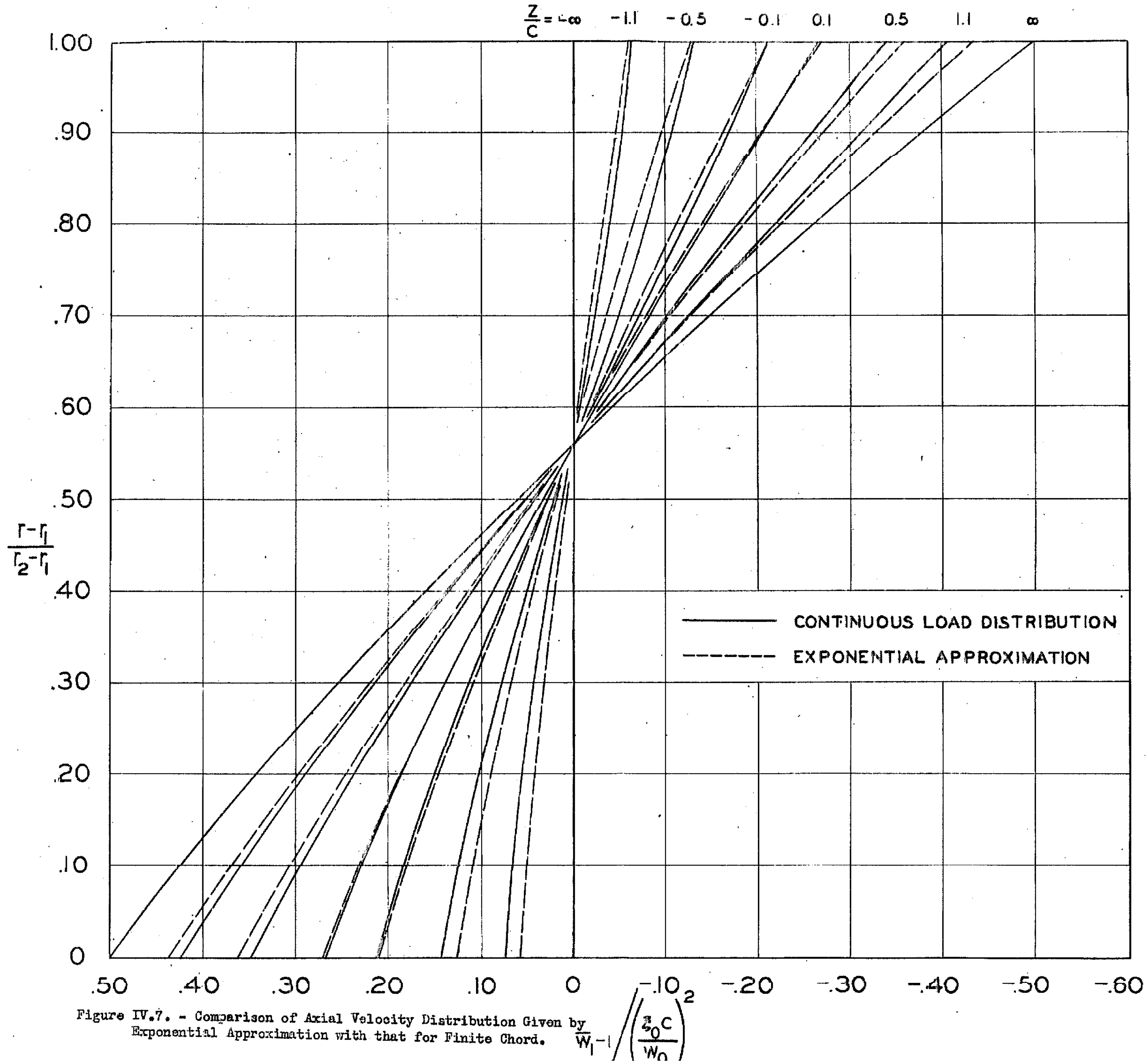

Figure IV.7. - Comparison of Axial Velocity Distribution Given by Exponential Approximation with that for Finite Chord.