TRANSVERSE HYDRODYNAMIC FORCES ON SLENDER BODIES IN FREE - SURFACE FLOWS AT LOW SPEED

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ABSTRACT

The forces and moments on a moving body partially immersed in the surface of a deep ocean of heavy fluid are considered in the limit of small Froude number, F. Asymptotic expressions for velocity potential and free surface elevation are developed. The choice of the first terms of the asymptotic sequence is indicated by the behavior, at small F, of the classical results of "small disturbance theory" - analysis starting from the linearized free-surface boundary conditions. It is found that the leading terms depend on the local disturbance, which can be expanded as a power series in F. The wave pattern contributes higher-order terms which are not analytic about $F = 0$; only estimates of the order of these terms are obtained. Consequently the present work does not estimate drag but is confined to consideration of transverse forces and moments.

Once the asymptotic sequence is assumed, perturbation of the exact equations and boundary conditions about $F = 0$ is straightforward. The zero-order potential is that of the "reflection-plane" model of Davidson. For a restricted class of shapes, the slender body theory is applied to the zero-order and first-order problems. A general method is developed using conformal mapping to solve the first-order problem for sufficiently slender shapes of arbitrary cross-section. This method is applied to two particular shapes, viz. a wing of zero thickness and a half-submerged body of revolution, both in sideslip. The correction to the reflection plane model is found to be generally quite small in the range of F for which this theory is expected to apply.

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I. INTRODUCTION

The determination of the forces and moments on a partially immersed body is important for the prediction of performance, control and stability in the design of surface ships and other interface vehicles. Historically, by far the greatest concentration of attention has been on the drag problem. The interest in drag is certainly justified by the fact that commercial and military vessels spend most of their time in steady, rectilinear motion at zero sideslip, and by the direct relation of drag to the economically and militarily important factors of speed and fuel consumption. However, for some types of operation more general motions must be considered.

Interest in sideslipping and yawing motion of ships has been motivated by considerations of maneuvering and turning characteristics. Davidson and Schiff (Ref. 1, 1946) is a discussion of maneuvering problems and a summary of earlier work. It was apparently Davidson who first suggested the "reflection plane" model, which appears in the current work as the zero-order theory: the free surface is regarded as a rigid flat wall, in which the submerged part of hull may be reflected. Advances in the aerodynamics of low-aspect-ratio wings and slender bodies have been applied by Tsakonas (Ref. 2, 1959) to calculate hydrodynamic coefficients of ships, still based on the reflection-plane model. To date no theory has appeared on yawing and sideslipping which takes into account the changes in elevation of the free surface in the vicinity of the ship, and no theoretical justification has been presented for the reflection-plane model, or its range of validity. It is the purpose of the current work to make some contribution to a clarification of these matters. The present author's initial motivation came not from any love of modern ships - in fact he has good personal reasons for a definite aversion to them - but rather from problems arising in the design of sailing yachts, considered in a previous paper (Ref. 3, 1965). The yacht hull normally operates in side-slip to develop a side force equal to that of the sails; control problems demand a knowledge of the yawing moment involved.

The significant dimensionless parameter for motion of geometrically similar shapes near the free surface of a semi-infinite inviscid fluid is the Froude number $F \equiv c^2/g\ell$ where c is a characteristic speed of the motion, g is acceleration due to gravity, and ℓ is a characteristic length of the body, taken here as the length measured at the water-line. Typical upper limits of F are 0.15 for a fast steamer, 0.10 for a fast yacht close-hauled. Plausible qualitative arguments have been presented by Davidson for the validity of the reflection model in the limit of vanishing Froude number. (It is most useful to think of this limit as g becoming very large while c, ℓ, and geometry are held fixed.) It was felt, then, that an expansion of all flow quantities as power series in the small parameter F might lead to significant results. When this perturbation analysis is applied to the exact equations and the boundary conditions on the free surface, on the solid surface, and in the distant field, the zero-order set of equations and boundary conditions are found to be exactly those of the reflection model. Order F yields a mixed boundary value problem for Laplace's equation in the portion of the lower half-space outside the body. Normal first-order velocity on the plane of the undisturbed surface is prescribed by the zero-order pressure on that surface.

In the case of sufficiently slender shapes and small angles of sideslip, two simplifications emerge: the slender-body theory provides an analysis for the zero-order pressures; and the first order problem can be solved in the cross-flow plane. This work deals primarily with some general results and some specific solutions of the second-order problem for slender geometry.

II. EQUATIONS AND BOUNDARY CONDITIONS

The governing relations will be formulated first in terms of a rectangular Cartesian co-ordinate system fixed in the distant fluid. The X, Z - plane coincides with the undisturbed surface far away, and the Y-axis is vertically upward; T is the time measured from some arbitrary reference. The following assumptions are made:

1) The flow is incompressible, irrotational, and inviscid. The existence of a velocity potential Φ is thus assured, such that the components of velocity in the X, Y, Z-directions are $\frac{\partial \Phi}{\partial X}, \frac{\partial \Phi}{\partial Y}, \frac{\partial \Phi}{\partial Z}$ respectively.

2) The free surface is given by $Y = H(X, Z, T)$, a single-valued function over the region of the X, Z-plane exterior to the body. This assumption is uniformly valid at sufficiently small values of F; it may break down in the vicinity of the bow for larger F, depending on the geometry (a breaking bow wave). Such cases are beyond the scope of this treatment.

3. The surface of the rigid body is given by $E(X,Y,Z,T)=0$,

with the exterior $E > 0$.

4) No wave trains are incident, coming from far away. In other words, the only disturbance present is that caused by the body's motion.

5) Velocity is finite in the vicinity of a sharp trailing edge. This is the well-known Kutta condition of airfoil theory, the only manifestation of viscosity considered here.

6) The Weber number $\rho c^2 \ell/(\text{surface tension})$ is very large, so that surface tension effects are negligible.

Then the following relations govern the flow: the field equation, which expresses continuity:

$$\nabla^2 \Phi = 0 \qquad \text{in} \quad Y \leqslant H, \qquad E \geqslant 0 ; \qquad (2\text{-}1)$$

the boundary conditions expressing tangency of the flow on the solid surface:

$$DE/DT = E_T + \Phi_X E_X + \Phi_Y E_Y + \Phi_Z E_Z = 0 \quad \text{on} \quad E = 0; \qquad (2\text{-}2)$$

and on the free surface:

$$D(H-Y)/DT = H_T + \Phi_X H_X - \Phi_Y + \Phi_Z H_Z = 0 \quad \text{on} \quad Y = H; \qquad (2\text{-}3)$$

the dynamic condition requiring constant pressure (taken zero) on the free surface, using Bernoulli's equation for unsteady flow:

$$\frac{P}{\rho} = -\Phi_T - \tfrac{1}{2} (\nabla\Phi)^2 - gY = 0 \quad \text{on} \quad Y = H; \qquad (2\text{-}4)$$

a distant boundary condition far below:

$$\nabla \Phi \rightarrow 0 \qquad \text{as} \qquad Y \rightarrow -\infty \tag{2-5}$$

Several difficulties are apparent:

1) Nonlinearity of boundary conditions (2-3) and (2-4)

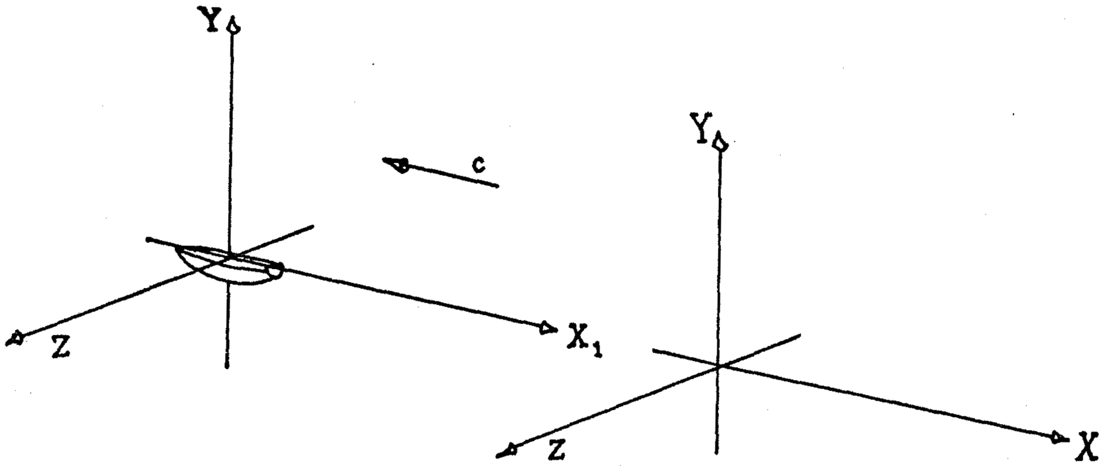
2) Boundary conditions (2-3) and (2-4) are applied on the surface $Y = H(X, Z, T)$ which is not known at the outset and must appear as part of the solution. Indeed, existence and uniqueness proofs are apparently lacking for such unknown boundary problems in more than one dimension.

3) A further distant boundary condition is necessary to satisfy condition 4) above (no incident wave trains) while still permitting persistent waves to be generated by the body's motion. For steady motion in the $-X$ direction,

$$\left.\begin{array}{l} \nabla \Phi \rightarrow 0 \\ H \rightarrow 0 \end{array}\right\} \qquad \text{as } X \rightarrow -\infty \tag{2-6}$$

is appropriate.

This case of steady motion in the $-X$ direction is of special interest, and it is useful to introduce a moving Cartesian frame (X_1, Y, Z, T) where $X_1 = X + cT$. The moving body is at the origin of the moving frame.



The perturbation potential and free-surface elevation are

$$\Phi_1(X_1, Y, Z, T) \quad = \quad \Phi(X, Y, Z, T) \qquad (2\text{-}7)$$

$$H_1(X_1, Y, Z, T) \quad = \quad H(X, Y, Z, T) \qquad (2\text{-}8)$$

The function representing the solid surface is independent of time for steady motions:

$$E_1(X_1, Y, Z) \quad = \quad E(X, Y, Z, T) \qquad (2\text{-}9)$$

Under this transformation (2-1) to (2-6) become

$$\nabla_1^2 \Phi_1 = 0 \quad \text{in} \quad Y \leqslant H_1 \;, \quad E_1 \geqslant 0 \qquad (2\text{-}10)$$

$$\left(c + \frac{\partial \Phi_1}{\partial X_1}\right) \frac{\partial E_1}{\partial X_1} + \frac{\partial \Phi_1}{\partial Y}\frac{\partial E_1}{\partial Y} + \frac{\partial \Phi_1}{\partial Z}\frac{\partial E_1}{\partial Z} = 0 \quad \text{on } E_1 = 0 \qquad (2\text{-}11)$$

$$\frac{\partial H_1}{\partial T} + \left(c + \frac{\partial \Phi_1}{\partial X_1}\right) \frac{\partial H_1}{\partial X_1} - \frac{\partial \Phi_1}{\partial Y} + \frac{\partial \Phi_1}{\partial Z}\frac{\partial H_1}{\partial Z} = 0 \quad \text{on } Y = H_1 \qquad (2\text{-}12)$$

$$\frac{\partial \Phi_1}{\partial T} + c\,\frac{\partial \Phi_1}{\partial X_1} + \tfrac{1}{2}(\nabla_1 \Phi_1)^2 + gY = 0 \quad \text{on } Y = H_1 \qquad (2\text{-}13)$$

$$\nabla_1 \Phi_1 \to 0 \qquad Y \to -\infty \ \text{or} \ X_1 \to -\infty \tag{2-14}$$

$$H_1 \to 0 \qquad X_1 \to -\infty \tag{2-15}$$

III. THE FORM OF SOLUTIONS

The problem described in the previous section, (2-1) to (2-6), involving a free boundary, is notoriously intractable; not a single exact solution is available.

The usual approximation, leading to what may be called <u>small-disturbance theory</u>, assumes vanishingly small displacement of the free surface and vanishingly small velocity perturbations, so that the free-surface boundary conditions (2-3) and (2-4) are linearized to

$$H_T = \Phi_Y, \qquad \Phi_T + gH = 0 \tag{3-1}$$

applied on $Y = 0$. Small-disturbance theory can be derived as a rational power series expansion (Wehausen, Ref. 4) in a small geometric parameter, e.g. ratio of amplitude to length for surface waves; or thickness ratio in Michell's thin ship theory, which treats a symmetric planar wing penetrating the free surface. The small-disturbance solutions are characterized by two different kinds of terms: (1) a local disturbance, which dies out rapidly with distance from the body, and (2) a superposition of surface waves of the form

$$\Phi = A \sin(\kappa_1 X + \omega_1 T) \sin(\kappa_2 Z + \omega_2 T) \exp(\kappa_1^2 + \kappa_2^2)^{1/2} Y \tag{3-2}$$

which follow a traveling disturbance; presumably similar waves would be present in solutions of the exact equations. The energy radiated by the waves is provided by work done against "wave resistance", the calculation of which is the object of most small-disturbance problems.

In the present analysis quite a different approximation is made. The parameter which is assumed to be small is the Froude number $F = c^2/g\ell$. It is most useful and convenient to think of the limit $F \to 0$ as g becoming very large while c, ℓ, and geometry are held fixed. Then the free surface is clamped very tightly to $Y=0$; a finite H over a finite region of the X, Z-plane would require infinite energy. Another way to regard the limit $F \to 0$ is to recognize c^2/g as the wavelength of the surface waves which keep up with the disturbance at speed c, which becomes much smaller than ℓ. But since the wavelength appears in the exponent in (3-2), wave effects become very small at depths below a wavelength. For very small F, the waves affect only a very thin layer of fluid near the free surface.

The only appearance of the parameter F in the governing relations is in the pressure boundary condition (2-4), which can be written

$$H = -F \frac{\ell}{c^2} \left[\Phi_T + \frac{1}{2} (\nabla \Phi)^2 \right]_{Y=H} \tag{3-3}$$

F appears as the ratio of typical inertial forces to the gravitational force, or the ratio of typical fluid accelerations to gravitational acceleration. In the limit $F \to 0$ gravity dominates. If this is true uniformly throughout the flow, the appropriate first approximation is to neglect the right-hand side of (3-3), which now becomes $H=0$. Consequently (2-3) becomes $\Phi_Y = 0$ on $Y=0$ and the field equation

(2-1) is valid in $Y \leqslant 0$, $E \geqslant 0$. This is the reflection-plane model, on which there will be some further discussion in later sections. A further approximation could be calculated by using the reflection-plane potential to evaluate the right-hand side of (3-3), arriving at a first approximation for H; using this H the full set of equations becomes linear and can be solved for a second approximate Φ. This iteration process repeated will generate an asymptotic expansion of Φ.

In this analysis the equivalent and more systematic perturbation procedure is used. The choice of an asymptotic sequence for the expansion is guided by results of small-disturbance theory, expanded for small F. The simplest choice - a power series in F -

$$\Phi \sim \Phi^{(0)} + F\,\Phi^{(1)} + F^2\,\Phi^{(2)} + \cdots \tag{3-4}$$

is at first discouraged by the frequent appearance of nonanalytic terms, e.g. $e^{-2\beta/F}/F^3$ in the wave resistance of a submerged cylinder as given by Lamb, Art. 249 (Ref. 5); or $F^{5/2}\sin 1/F$ in Havelock's results for resistance in thin ship theory (Ref. 6, 1923). In fact, it quickly becomes apparent in carrying out the expansion in powers of F that no wavelike behavior appears at any order and, consequently, no wave resistance shows up at all.

The distinction between local disturbance and wave pattern is important here, and another aspect of it will be pointed out by consideration of the fundamental source-like solution, for a source of strength m at depth f, as given by Havelock (Ref. 7, 1951).

$$\Phi_1(X_1, Y, Z) = \frac{m}{R_1} - \frac{m}{R_2}$$

$$- \frac{2}{\pi} \kappa_0 m \int_{-\pi/2}^{\pi/2} \sec^2\theta \int_0^\infty \frac{e^{\kappa(Y-f)}\cos(\kappa X_1 \cos\theta)\cos(\kappa Z \sin\theta)}{\kappa - \kappa_0 \sec^2\theta}\, d\kappa\, d\theta$$

$$+ 2\kappa_0 m \int_{-\pi/2}^{\pi/2} e^{\kappa_0(Y-f)\sec^2\theta} \sin(\kappa_0 X_1 \sec\theta)\cos(\kappa_0 Z \sin\theta \sec^2\theta)\sec^2\theta\, d\theta \qquad (3\text{-}5)$$

where $\kappa_0 = g/c^2$, $R_1^2 = X_1^2 + (Y+f)^2 + Z^2$, and

$R_2^2 = X_1^2 + (Y-f)^2 + Z^2$.

The first integral, representing the local disturbance, is readily expanded in powers of $F = 1/\kappa_0 \ell$:

$$\frac{2}{\pi} m \int_{-\pi/2}^{\pi/2} \int_0^\infty \left[1 + F\kappa\ell\cos\theta + F^2\kappa^2\ell^2\cos^2\theta + O(F^3) \right]$$

$$e^{\kappa(Y-f)}\cos(\kappa X_1 \cos\theta)\cos(\kappa Z \sin\theta)\, d\kappa\, d\theta ,$$

while the second integral, representing a superposition of plane surface waves, cannot be placed in this form. The same conclusion holds for more complex flows resulting from superpositions of sources and other singularities derived from the source by differentiation and integration: while waves do not appear in an expansion in powers of F, such an expansion is a valid representation of the local disturbance.

It is interesting to see where the waves do appear. If (3-4) were a uniformly valid asymptotic representation of Φ, and wave-like terms did not appear in (3-4), then wavelike terms, and hence wave resistance, would have to be smaller than any power of F;

this conclusion, however, is at variance with the results of thin ship theory (for example, Ref. 6) where terms in the powers of F do appear. The nonuniformity occurs in the vicinity of stagnation points on the free surface - for instance, at the bow and stern of a ship - where acceleration is no longer small compared with that of gravity, and the right-hand side of (3-3) is no longer negligible. In the expansion (3-4) successive terms become larger and more singular.

By using expanded co-ordinates it is possible to make a closer investigation of the region of nonuniformity. If, in a steady motion, the choice of inner variables is $x=X_1/F\ell$, $y=Y/F\ell$, $z=Z/F\ell$, $h=H/F\ell$, $\varphi=\Phi_1/F\ell c$, the first-order set of equations and boundary conditions from (2-10) to (2-15) are

$$\nabla^2\varphi = 0 \qquad y<0 \tag{3-6}$$

$$(c+\varphi_x)e_x + \varphi_y e_y + \varphi_z e_z = 0 \qquad \text{on } e = E = 0 \tag{3-7}$$

$$ch_x - \varphi_y = 0 \qquad \text{on } y = 0 \tag{3-8}$$

$$\varphi_x + h = 0 \qquad \text{on } y = 0 \tag{3-9}$$

$$\nabla\varphi \rightarrow 0 \qquad y \rightarrow -\infty \text{ or } X \rightarrow -\infty \tag{3-10}$$

$$h \rightarrow 0 \qquad X \rightarrow -\infty \tag{3-11}$$

which are the small-disturbance equations. The solutions depend strongly on the details of the shape of the body described by the function e near the singular point, but always involve the typical free wave pattern downstream, far outside the region of nonuniformity. This picture of waves being generated at the singular point and propagating into the rest of the solution is in good agreement with a general result of thin-ship theory, due to Inui (Ref. 8, 1962). By considering the wave pattern due to a continuous distribution of sources, Inui has demonstrated that for small F the waves all originate at the ends of the distribution, and that the strength of the waves is strongly dependent on the details of the distribution near the ends. It should also be pointed out that in those cases of wave motion that have been carried out for completely submerged bodies, where no singular stagnation points occur, (particularly, the entirely general case of Lamb, Art. 25a (Ref. 5)) the wave resistance is exponentially small.

The question that must now be considered is the extent of effects of the wavelike part of Φ on the transverse force distribution on a body. Within small-disturbance theory, the only component of force on a body due to its waves is a drag, since infinitesimal waves can carry away energy but not momentum. This consideration does not, however, rule out pure couples due to waves; a pitching moment is present in general and has been treated by Lunde (Ref. 9). The simplest case that might involve yawing moments is that of a planar wing of zero thickness penetrating the free surface. (The thickness case, which can be superimposed, is called "thin ship theory.") The lifting surface is replaced by a distribution of elementary horseshoe vortices, whose potential is calculated in Appendix I, having density proportional to the wing loading.

The co-ordinate system and variables set up for steady motion are used - see (2-7) to (2-15). The lifting wing is represented by a vortex distribution over the plane $Z = 0$, of strength $\gamma(X_1, Y)$, so the pressure difference between the two sides of the wing is $\rho c \gamma$. The potential associated with such a vortex system is

$$\Phi_1(X_1, Y, Z) = \int_{-\infty}^{0}\int_{-\infty}^{\infty} \gamma(\xi, \eta) V(X_1, Y, Z; \xi, \eta, 0; \pi/2) d\xi d\eta \qquad (3\text{-}12)$$

where V is the elementary vortex potential derived in Appendix I. If $w(X_1, Y)$ is the downwash velocity on the plane $Z = 0$, defined by

$$w(X_1, Y) = \left[\frac{\partial \Phi_1}{\partial Z}\right]_{Z=0} \qquad (3\text{-}13)$$

then

$$w(X_1, Y) = \int_{-\infty}^{0}\int_{-\infty}^{\infty} \gamma(\xi, \eta)\left[\frac{\partial}{\partial Z} V(X_1, Y, Z; \xi, \eta, 0; \pi/2)\right]_{Z=0} d\xi\, d\eta\, . \qquad (3\text{-}14)$$

Defining the kernel function

$$K(X_1, Y; \xi, \eta) = \left[\frac{\partial}{\partial Z} V\right]_{Z=0} \quad , \qquad (3\text{-}15)$$

we write (3-14) as

$$w(X_1, Y) = \int_{-\infty}^{0}\int_{-\infty}^{\infty} \gamma(\xi, \eta) K(X_1, Y; \xi, \eta)\, d\xi\, d\eta \qquad (3\text{-}16)$$

which is in the form of the fundamental integral equation of lifting-surface theory, except that the kernel is more complex in this case. The kernel can be written as the sum of two parts: K_ℓ arising from the first three terms of V, representing the local disturbance, and K_w arising from the last term, the wave disturbance. For the present purpose of investigating the magnitude of the wave effects, the interest is in K_w:

$$K_w(X_1, Y; \xi, \eta) = 2\kappa_0^2 \int_{-\pi/2}^{\pi/2} \sin^2\theta \sec^5\theta e^{\kappa_0(Y+\eta)\sec^2\theta} \cos[\kappa_0(X_1-\xi)\sec\theta]\, d\theta \tag{3-17}$$

and in the associated component of the downwash

$$w_w(X_1, Y) = \int_{-\infty}^{0} \int_{-\infty}^{\infty} \gamma(\xi, \eta) K_w(X_1, Y; \xi, \eta) d\xi\, d\eta \tag{3-18}$$

We investigate the behavior of K_w for large values of $\kappa_0 = g/c^2$ by the method of stationary phase. Since $\cos[K_0(X_1-\xi)\sec\theta]$ is a rapidly oscillating function, the principal contribution to the integral (3-17) comes from the vicinity of points for which the phase $\kappa_0(X_1-\xi)\sec\theta$ is stationary. The only stationary point in the range of integration is $\theta = 0$; however, the point $\theta = 0$ is a zero for the rest of the integrand, so the ordinary method requires modification. The required treatment is carried out in Appendix II. In this case we have $f(0) = \kappa_0(X_1-\xi)$, $f''(0) = \kappa_0(X_1-\xi)$, $f'''(0) = 0, \phi'(0) = 0$, $\phi''(0) = 2e^{\kappa_0(Y+\eta)}$ and (3-17) becomes approximately

$$K_w(X_1, Y; \xi, \eta) \sim 2\sqrt{\pi}\kappa_0^{1/2} \frac{e^{\kappa_0(Y+\eta)}}{|X_1-\xi|^{3/2}} \{\cos\kappa_0(X_1-\xi) \pm \sin\kappa_0(X_1-\xi)\} \tag{3-19}$$

where the $\pm$ sign is the sign of $(X_1-\xi)$. This expression has a nonintegrable singularity at $X_1 = \xi$; however, the application of stationary phase is valid only if $\kappa_0(X_1-\xi)$ is large. At $X_1 = \xi$, (3-17) gives

$$K_w(X_1, Y; X_1, \eta) = 2\kappa_0^2 \int_{-\pi/2}^{\pi/2} \sin^2\theta \sec^5\theta\, e^{\kappa_0(Y+\eta)\sec^2\theta}\, d\theta \tag{3-20}$$

For large κ_0 this integral can be evaluated approximately by the method of steepest descent, done in a fashion quite analogous to the method of Appendix II. Again the principal contribution is from values of θ near the stationary point $\theta = 0$; the result is

$$K_w(X_1, Y; X_1, \eta) \sim 2\kappa_0^2 \, e^{\kappa_0(Y+\eta)} \int_{-\infty}^{\infty} \theta^2 e^{\kappa_0(Y+\eta)\theta^2} d\theta = \sqrt{\pi}\, \kappa_0^{1/2} \, \frac{e^{\kappa_0(Y+\eta)}}{|Y+\eta|^{3/2}} \quad (3\text{-}21)$$

so the kernel is finite and integrable.

Assume $|\gamma(X_1, Y)| \leq M$ for $O \leq X_1 \leq \ell$ and $\gamma(X_1, Y) = 0$ outside that region. Then from (3-18)

$$w_w \leq \int_{-\infty}^{0} \int_{0}^{\ell} M \, \kappa_0^{1/2} \, e^{\kappa_0(Y+\eta)} \, f(X_1-\xi) \, d\xi) \, d\eta \qquad (3\text{-}22)$$

where $f(X_1-\xi)$ is finite and integrable. So

$$w_w \leq \frac{M}{\kappa_0^{1/2}} \, e^{\kappa_0 Y} \int_{0}^{\ell} f(X_1-\xi) \, d\xi \qquad (3\text{-}23)$$

and using $F = \kappa_0 \ell$,

$$w_w \leq (F\ell)^{1/2} \, e^{Y/F\ell} M \int_{0}^{\ell} f(X_1-\xi) \, d\xi \, . \qquad (3\text{-}24)$$

The contribution to the downwash from the wave part of the kernel in (3-16) is exponentially small at any finite depth Y below the free surface. Any integrated effect of this downwash from the free surface downward is of order $F^{3/2}$. The oscillatory behavior of $f(X_1-\xi)$, displayed in (3-19), indicates that integrated forces and moments connected with the wave part of the kernel are even smaller than $F^{3/2}$.

This result of small-distribution theory is used to justify neglecting the wave term in the kernel while calculating forces and moments of order unity and order F.

IV. PERTURBATION EQUATIONS

To obtain the perturbation expansion, the potential and the free-surface elevation are expanded in power-series form:

$$\Phi(X,Y,Z,T) = \Phi^{(0)}(X,Y,Z,T) + F\,\Phi^{(1)}(X,Y,Z,T) + F^2\Phi^{(2)}(X,Y,Z,T) + \cdots \quad (4\text{-}1)$$

$$H(X,Z,T) = H^{(0)}(X,Z,T) + F\,H^{(1)}(X,Z,T) + F^2 H^{(2)}(X,Z,T) + \cdots \quad (4\text{-}2)$$

Throughout this paper the notation of superscripts in parentheses is used to denote the coefficients of powers of F in similar series expansions for various quantities.

When (4-1) and (4-2) are substituted into (2-1) to (2-5) and the coefficients of the various powers of F are collected, there result sets of equations and boundary conditions for the $\Phi^{(i)}$ and $H^{(i)}$. The conditions on the surface $Y = H$ must be transferred to the $Y = 0$ plane, by expanding Φ in the form of (4-1), and its derivatives, in Taylor series about $Y = 0$:

$$\begin{aligned}\Phi(X,H,Z,T) &= \Phi^{(0)}(X,O,Z,T) + F\,\{\Phi^{(1)}(X,O,Z,T) + \Phi_y^{(0)}(X,O,Z,T)\,H^{(1)}(X,Z,T)\}\\ &+ F^2\,\{\Phi^{(2)}(X,O,Z,T) + \Phi_y^{(0)}(X,O,Z,T)\,H^{(2)}(X,Z,T)\\ &+ \Phi_y^{(1)}(X,O,Z,T)\,H^{(1)}(X,Z,T) + \tfrac{1}{2}\,\Phi_{yy}^{(0)}(X,O,Z,T)\,[\,H^{(1)}(X,Z,T)\,]^2\,\}\\ &+ O(F^3) \quad \text{etc}\end{aligned} \quad (4\text{-}3)$$

4.1 Zero Order

$$\nabla^2 \Phi^{(0)} = 0 \qquad \text{in } Y \leqslant 0, \quad E \geqslant 0 \tag{4-4}$$

$$E_T + \nabla\Phi^{(0)} \cdot \nabla E = 0 \qquad \text{on } E = 0 \tag{4-5}$$

$$\Phi^{(0)}_y = 0 \qquad \text{on } Y = 0, \quad E \geqslant 0 \tag{4-6}$$

$$H^{(0)} = 0 \tag{4-7}$$

$$\nabla \Phi^{(0)} \rightarrow 0 \qquad Y \rightarrow -\infty \tag{4-8}$$

Together with a further distant boundary condition on $\nabla\Phi^{(0)}$ near $Y = 0$, and the Kutta condition on trailing edges, these relations determine $\Phi^{(0)}$. They are precisely the relations governing the motion in an unbounded fluid, of a body (called the "reflected body") whose surface is given by $E(X, Y, Z, T) = 0$, $Y < 0$, and its reflection in the $Y = 0$ plane, constrained to move on that plane. So the reflection-plane model of Davidson emerges, as expected, as the limit for vanishingly small Froude number.

Heaving, rolling and pitching motions produce changes in the geometry of the reflected body, while for translations in the $Y = 0$ plane (forward motion and sideslip) and yawing motions the body's shape is time-invariant. The zero-order problem with fixed geometry is a familiar and fundamental problem in aerodynamics, and much attention has been given to its solution. The zero-order problem with variable geometry is not so well developed; however, most of the techniques developed for unsteady motion of bodies having fixed geometry are suitable for extension to time-variable geometry.

Equations (4-4) to (4-8) with an additional distant boundary condition have a unique solution if the potential is assumed single-valued. That solution, however, in general involves infinite velocities around the trailing edges, if any are present. Satisfaction of the Kutta condition requires the presence in the fluid of free vortex sheets, surfaces across which $\Phi^{(0)}$ is discontinuous - the familiar trailing vortex sheets of wing theory, essential to the development of lift.

The zero-order forces and moments are calculated by integrating pressures derived from the zero-order approximation to Bernoulli's equation (2-4):

$$\frac{P^{(0)}}{\rho} = -\Phi_T^{(0)} - \tfrac{1}{2}\left(\nabla\Phi^{(0)}\right)^2 \quad . \tag{4-9}$$

The term $-gY$ is omitted, since it contributes only the uninteresting hydrostatic force and moments.

4.2 First Order

$$\nabla^2 \Phi^{(1)} = 0 \qquad \text{in} \quad Y \leqslant 0, \quad E \geqslant 0 \tag{4-10}$$

$$\nabla \Phi^{(1)} \cdot \nabla E = 0 \qquad \text{on} \quad E = 0 \tag{4-11}$$

$$\left.\begin{aligned} \Phi_Y^{(1)} &= H_T^{(1)} + \Phi_X^{(0)} H_X^{(1)} + \Phi_Z^{(0)} H_Z^{(1)} - \Phi_{YY}^{(0)} H^{(1)} && (4\text{-}12) \\ H^{(1)} &= -\frac{\ell}{c^2}\left[\Phi_T^{(0)} + \tfrac{1}{2}\left(\nabla\Phi^{(0)}\right)^2\right] && (4\text{-}13) \end{aligned}\right\} \text{on } Y = 0$$

$$\nabla \Phi^{(i)} \rightarrow 0 \qquad \text{as} \quad Y \rightarrow -\infty \tag{4-14}$$

$H^{(1)}$ can be eliminated between (4-12) and (4-13) to yield a single condition on $\Phi_Y^{(1)}$ in terms of $\Phi^{(0)}$. From (4-9) it is apparent that the quantity $H^{(1)}$ is directly related to $P^{(0)}$: free-surface height, to order F, is precisely the head of fluid supported by the zero-order pressure occurring on the reflection plane.

So $\Phi^{(1)}$ is a harmonic function of the space variables, with normal derivatives given on the solid boundary and on the plane $Y = 0$ and derivatives vanishing far ahead. Again the continuous solution fails in general to satisfy the Kutta condition, and vortex sheets must be present in a lifting problem. Now in the exact problem there is only one vortex sheet from each trailing edge; its position should respond to all orders of perturbation velocities. Strictly, it should be treated as another free surface which is slightly perturbed from its strength and position in the zero-order solution. However, the available wing theories for solution of the zero-order problem neglect even the inductions of the bound vortices in locating the trailing vortex sheet; compared to these the velocity contributions from $\Phi^{(1)}$ are $O(F)$. It is consistent with the accuracy of wing-theory solutions for $\Phi^{(0)}$ to allow the vortex sheet for all orders to coincide with the zero-order sheet.

In the most general case this first-order problem does not admit of easily computed solutions, on account of the complex shape of the boundary on which the normal derivatives are specified; however, for a certain class of sufficiently slender shapes the problem becomes a two-dimensional one and can be approached by complex analysis. When $\Phi^{(1)}$ has been found, first-order forces are computed from the pressures given by the first-order Bernoulli's equation:

$$\frac{P^{(1)}}{\rho} = -\Phi_T^{(1)} - \nabla\Phi^{(0)} \cdot \nabla\Phi^{(1)} \qquad (4\text{-}15)$$

4.3 Second and Higher Orders

The first order problem is typical of all the higher-order ones. All are governed by Laplace's equation $\nabla^2\Phi^{(i)} = 0$ in the lower half-space outside the body, with the same condition $\nabla E \cdot \nabla\Phi^{(i)} = 0$ on $E=0$ and $\nabla\Phi^{(i)}$ vanishing far ahead. The normal velocity over the $Y=0$ plane is prescribed by a functional of the $\Phi^{(j)}$ and $H^{(j)}$, $j<i$; for example

$$\Phi_Y^{(2)} = \Big\{ H_T^{(2)} + \Phi_X^{(0)} H_X^{(2)} + \Phi_X^{(1)} H_X^{(1)} + \Phi_{XY}^{(0)} H_X^{(1)} H^{(1)} - \Phi_{YY}^{(0)} H^{(2)}$$
$$- \tfrac{1}{2}\Phi_{YYY}^{(0)} (H^{(1)})^2 - \Phi_{YY}^{(1)} H^{(1)} + \Phi_Z^{(1)} H_Z^{(1)} + \Phi_{ZY}^{(0)} H_Z^{(1)} H^{(1)} \Big\}, \qquad (4\text{-}16)$$

where

$$H^{(2)} = -\frac{\ell}{c^2}\Big\{ \Phi_T^{(1)} + \Phi_{TY}^{(0)} H^{(1)} + H^{(1)} \nabla\Phi^{(0)} \cdot \nabla\Phi_Y^{(0)} + \nabla\Phi^{(0)} \cdot \nabla\Phi^{(1)} \Big\}, \qquad (4\text{-}17)$$

on the plane $Y=0$. The methods developed for the first-order problem are equally applicable to higher orders, since the problems fall into the same form; the computations are apparently more complicated.

V. SUBMERGED CYLINDER

The perturbation equations in Section IV have been especially developed to deal with transverse forces. It would be reassuring at this point to find them in agreement with the results of small-disturbance theory under circumstances where both are applicable.

One case in which transverse forces have been calculated by small-disturbance theory is that of the lift on a submerged cylinder normal to the stream (or more accurately, a submerged line doublet pointed upstream) treated by Havelock (Ref. 10, 1928 and Ref. 11, 1936). The result is valid for any speed provided the depth of submergence is sufficiently great, compared to the radius.

Using the present method it is possible to compute the lift on a cylinder at any submergence, total or partial, provided the speed is low enough. To do so would only be an uninteresting exercise; for purposes of comparison we want to calculate the combination of deep submergence and low speed.

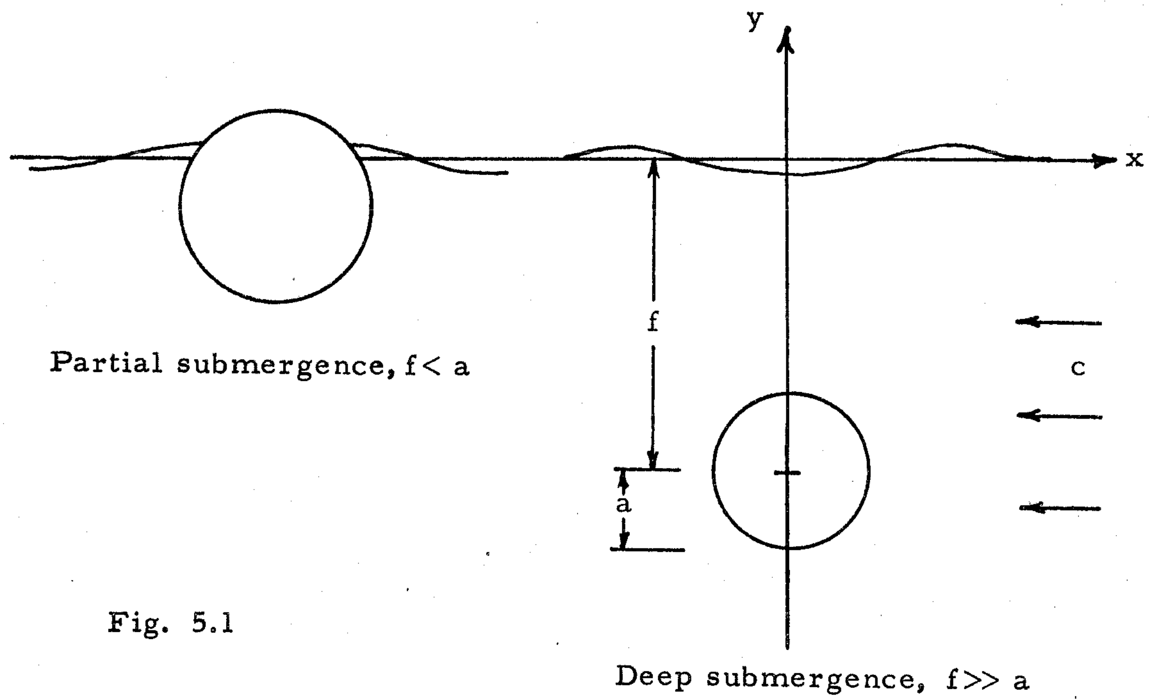


Fig. 5.1

In this case the cylinder of radius a is represented by a doublet of strength $M = ca^2$. The co-ordinates used in this section only are shown in Fig. 5.1. F is referred to the typical dimension of the body, a: $F \equiv c^2/ga$.

First, the zero-order force is calculated by the reflection-plane model. In this approximation an image doublet of strength ca^2 is located at $(0, + f)$ and it is necessary to calculate the force on one doublet due to the other. The force on a doublet is given by Milne-Thompson (Ref. 12) as:

$$X + iY = 2\pi\rho M e^{i\alpha} f'(z_0) \tag{5-1}$$

where α is the inclination of the doublet's axis (here zero) and $f(z)$ is the velocity with the doublet at z_0 removed. In the present application this becomes

$$Y = 2\pi\rho M \frac{\partial u}{\partial y} . \tag{5-2}$$

The velocity gradient $\partial u/\partial y$ is calculated from the potential of a doublet, with the result

$$Y^{(0)} = \frac{1}{2} \pi\rho c^2 a \left(\frac{a}{f}\right)^3 . \tag{5-3}$$

The first order is calculated from the perturbation equations (4-10) to (4-14), using for the zero-order potential

$$\phi^{(0)}(x, y) = M \frac{x}{x^2 + (y+f)^2} + M \frac{x}{x^2 + (y-f)^2} \tag{5-4}$$

From (4-13) it is found

$$h^{(1)} = -\frac{2Ma}{c}\left[\frac{f^2 - x^2}{(f^2 + x^2)^2}\right] \tag{5-5}$$

then from (4-12),

$$\phi_y^{(1)}(x,0) = 4\,Max\left[\frac{x^2 - 3f^2}{(f^2 + x^2)^3}\right]. \tag{5-6}$$

This condition, prescribing the normal velocity over $y = 0$, is satisfied by a source distribution along the x-axis with strength $m(x) = 2\phi_y^{(1)}(x, \ )$, or

$$m(x) = 8Max\left[\frac{x^2 - 3f^2}{(x^2 + f^2)^3}\right] \tag{5-7}$$

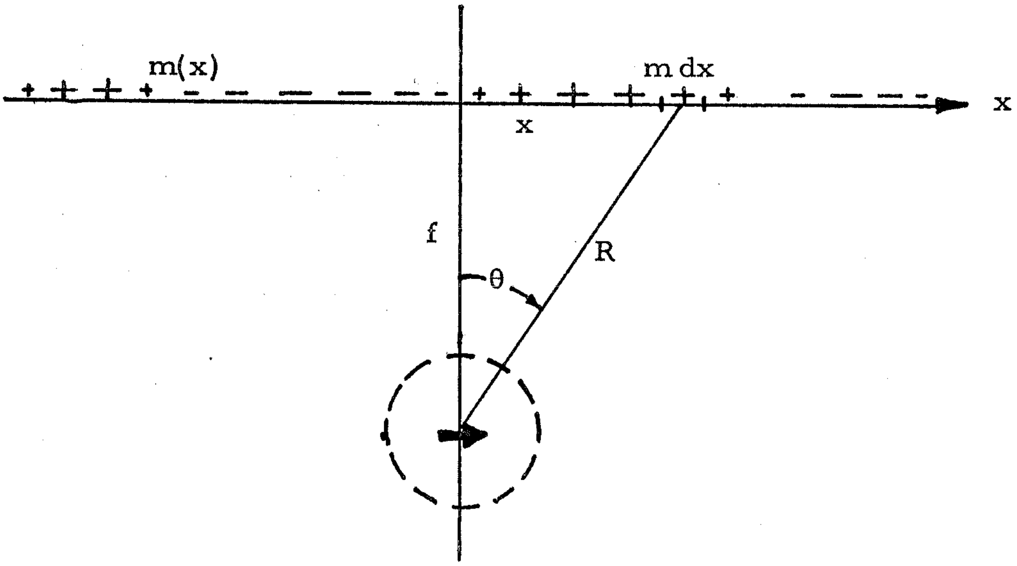


Fig. 5.2 First-order Problem

Now it is necessary to find the velocity gradient $\partial u^{(1)}/\partial y$ at $(0, -f)$ due to the source distribution $m(x)$. The contribution to $\partial u/\partial y$ from the part of m between x and $x + dx$ is

$$d\left(\frac{\partial u^{(1)}}{\partial y}\right) = -\frac{\sin\theta\cos\theta}{\pi R^2}\quad mdx \tag{5-8}$$

Consequently $\partial u^{(1)}/\partial y$ is found by integrating (5-8) from $-\infty$ to $+\infty$; or, using $x = f\tan\theta$,

$$\frac{\partial u^{(1)}}{\partial y} = -\frac{8Ma}{\pi}\frac{1}{f^4}\int_{-\pi/2}^{\pi/2} \sin^2\theta\cos^6\theta(\tan^2\theta - 3)d\theta \qquad (5\text{-}9)$$

The integration is done with the help of Dwight (Ref.13), No. 858.514, with the result

$$\frac{\partial u^{(1)}}{\partial y} = \frac{3}{4}\frac{Ma}{f^4}\ . \qquad (5\text{-}10)$$

Putting this in (5-2), we find

$$Y^{(1)} = \frac{3}{2}\pi\rho c^2 a\left(\frac{a}{f}\right)^4 . \qquad (5\text{-}11)$$

Combining the results (5-3) and (5-11),

$$Y \sim Y^{(0)} + FY^{(1)} = \frac{1}{2}\pi\rho c^2 a\left[\left(\frac{a}{f}\right)^3 + 3F\left(\frac{a}{f}\right)^4\right] \qquad (5\text{-}12)$$

Havelock's result is given in the form (Ref.10)

$$Y = -\frac{\pi\rho c^2 a^4}{2f^3}\left\{1 + 2\kappa f + 4\kappa^2 f^2 - 8\kappa^3 f^3 e^{-2\kappa f}\,\ell i(e^{2\kappa f})\right\} \qquad (5\text{-}13)$$

where $\kappa \equiv g/c^2$ and ℓi is the logarithmic integral. Using the identity between the logarithmic integral and the exponential integral, and using the asymptotic expansion of the latter given by Jahnke and Emde (Ref.14):

$$\ell i(e^{2\kappa f}) = \mathcal{E}i(2\kappa f) \sim \frac{e^{2\kappa f}}{2\kappa f}\left(1 + \frac{1!}{2\kappa f} + \frac{2!}{(2\kappa f)} + \frac{3!}{(2\kappa f)^3} + \cdots\right) \qquad (5\text{-}14)$$

(5-13) can be represented for large κf by

$$Y \sim -\frac{\pi\rho c^2 a^4}{2f^3}\left\{-1 - \frac{3}{\kappa f}\right\} = \frac{\pi}{2}\rho c^2 a\cdot\frac{a^3}{f^3}\left[1 + 3F\frac{a}{f}\right] \qquad (5\text{-}15)$$

which is gratifyingly identical with the present result (5-12).

No sufficiently simple solution involving a trailing vortex sheet is available for comparison.

VI. SLENDER BODY APPROXIMATION

In the slender-body theory (see, for example, Adams and Sears (Ref.15, 1953) and Sacks (Ref.16, 1954) we have an analytic approximate solution for the zero-order problem for $\Phi^{(0)}$, for a general class of bodies, denoted as "slender", and motions restricted to slow maneuvers and small angles of attack. To fit into the ordinary slender-body theory, a shape must have its lateral dimensions small compared with distance from the nose, slopes of its surface small compared with unity, and curvatures in the streamwise direction small compared with the reciprocal of distance from the nose. Under these circumstances, conditions change so slowly along the length that in each "crossflow plane" normal to the body the flow is essentially the two-dimensional flow past a cylinder having the same cross-section as the body.

For a further restricted subclass of slender bodies the same reasoning leads to a treatment of the second-order problem in the cross flow plane. The restriction is that the zero-order pressure distribution $P^{(0)}(X, O, Z, T)$ be also effectively "slender"; that is,

1) the lateral extent of significant pressure disturbances must be small compared with distance from the nose, and of the same order as the lateral dimensions of the body; and

2) pressure disturbances must change slowly along the length. If these conditions obtain, then the boundary condition (4-13) changes

slowly along the length; appealing to the continuous dependence of $\Phi^{(1)}$ on its boundary conditions, we conclude that the second-order flow changes slowly along the length and so is essentially the two-dimensional flow in the cross-flow plane.

The geometrical restrictions that will guarantee satisfaction of restrictions 1) and 2) on pressure are not immediately clear, and their investigation is postponed until later in the paper, when some analysis relating to slender configurations has been carried out.

VII. STEADY SIDESLIP - BODY-FIXED COORDINATES

For the special case of steady forward motion and sideslip, two factors result in a simplified treatment:

1) Geometry of the reflected body is fixed.

2) The free-surface boundary conditions assume simple forms in body-fixed co-ordinates.

A rectangular Cartesian co-ordinate system (x,y,z,t) fixed in the body and translating with respect to the (X,Y,Z,T) system is introduced (Fig. 7.1). At $T = t = 0$, the two systems are co-incident. The origin of the (x,y,z,t) system is chosen to be the foremost point of the intersection of the body surface with $Y = 0$; i.e., the nose of the reflected body.

The x-axis is chosen more or less along the length of the body. In cases where the body has a vertical plane of symmetry, the x-axis will always be taken in that plane. The origin has the steady velocities $-c\cos\alpha$, $-c\sin\alpha$ in the X - and Z - directions respectively. Then the transformation between moving and stationary frames is

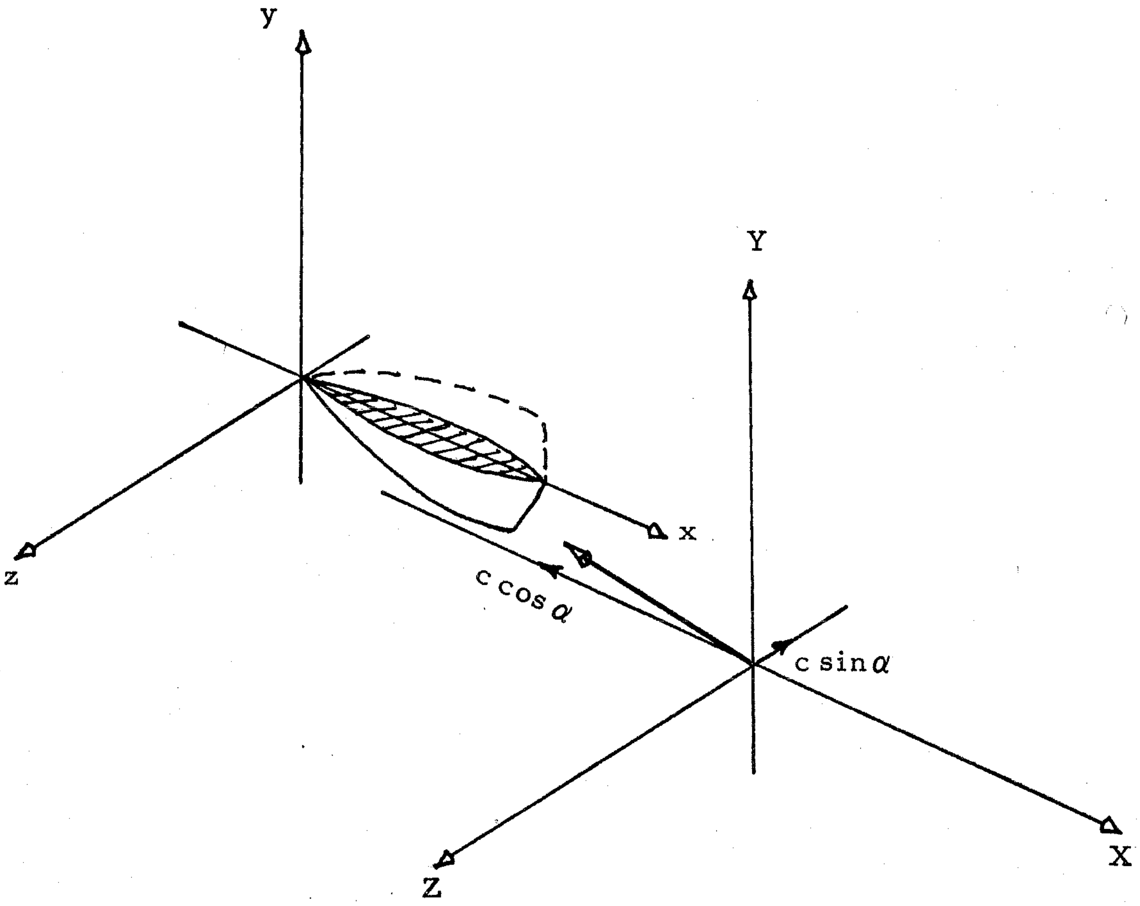


Fig. 7.1 Moving Co-ordinate System

$$\begin{cases} x &= X + c\cos\alpha\, T \\ y &= Y \\ z &= Z + c\sin\alpha\, T \\ t &= T \end{cases} \qquad (7\text{-}1)$$

The following functions are introduced for the perturbation potential, free-surface elevation, and pressure:

$$\varphi(x,y,z,t) \equiv \Phi(X,Y,Z,T) \qquad (7\text{-}2)$$

$$h(x,z,t) \equiv H(X,Z,T) \qquad (7\text{-}3)$$

$$p(x,y,z,t) \equiv P(X,Y,Z,T) \qquad (7\text{-}4)$$

under the transformation (7-1). By this definition φ is the "perturbation potential"; its derivatives are the "perturbation velocities" - the fluid velocity components at a point (x,y,z,t), minus the free-stream components. φ and h are assumed for the present to be functions of t; whereas the function representing the solid surface

$$e(x,y,z) \equiv E(X,Y,Z,T) \qquad (7\text{-}5)$$

is independent of t for the motions considered.

7.1 Zero Order

The zero-order relations (4-4) to (4-8) now become

$$\nabla^2\varphi^{(0)} = 0 \quad \text{in} \quad y \leqslant 0, \quad e \geqslant 0. \qquad (7\text{-}6)$$

$$\nabla\varphi^{(0)} \cdot \nabla e = -c\cos\alpha\, e_x - c\sin\alpha\, e_z \quad \text{on} \quad e = 0 \qquad (7\text{-}7)$$

$$\varphi_y^{(0)} = 0 \quad \text{on} \quad y = 0, \quad e \geqslant 0 \qquad (7\text{-}8)$$

$$\varphi^{(0)} \to 0 \quad \text{as} \quad x \to -\infty \quad \text{or} \quad y \to -\infty \tag{7-9}$$

$$\nabla\varphi^{(0)} \text{finite,} \quad \text{trailing edge} \tag{7-10}$$

It is to be observed by differentiating (7-6) - (7-9) with respect to time, that the problem for $\varphi_t^{(0)}$:

$$\nabla^2 \varphi_t^{(0)} = 0 \quad \text{in } y \leqslant 0, \quad e \geqslant 0 \tag{7-6'}$$

$$\nabla\varphi_t^{(0)} \cdot \nabla e = 0 \quad \text{on} \quad e = 0 \tag{7-7'}$$

$$\varphi_{ty}^{(0)} = 0 \quad \text{on} \quad y = 0 \tag{7-8'}$$

$$\varphi_t^{(0)} \to 0 \quad x \to -\infty \quad \text{or} \quad y \to -\infty \tag{7-9'}$$

is homogeneous, and has only the solution $\varphi_t^{(0)} = 0$ in $y \leqslant 0$, $e \geqslant 0$. Thus the zero-order solution $\varphi^{(0)}$ is independent of time, and so the variable t is omitted in the following.

Now for slender geometry $\varphi^{(0)}$ is approximated by

$$\varphi^{(0)}(x,y,z) = \phi^{(0)}(z + iy; x)$$

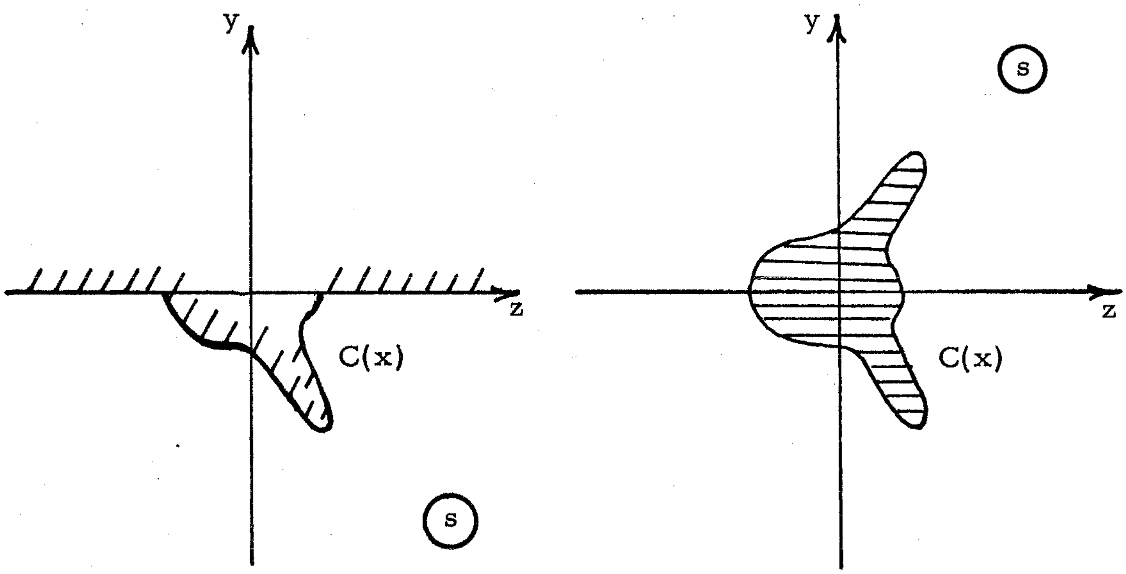
$$= \phi^{(0)}(s; x) \tag{7-11}$$

where the complex variable $s \equiv z + iy$ has been introduced. $\phi^{(0)}$ is a real function of s, with x appearing as a parameter labeling the cross-flow planes. In the approximation of slender-body theory, (7-6) becomes

$$\varphi^{(0)}_{zz} + \varphi^{(0)}_{yy} = 0 \tag{7-12}$$

which is satisfied by requiring $\phi^{(0)}$ to be the real part of an analytic function of s, called the complex potential $f^{(0)}(s;x) = \phi^{(0)}(s;x) + i\psi^{(0)}(s;x)$, where $\psi^{(0)}$ is also real.

The problem for $\varphi^{(0)}$ then reduces to finding the harmonic real function $\phi^{(0)}$ in each cross-flow plane, subject to the conditions:



$$\frac{\partial \phi^{(0)}}{\partial n} = \phi^{(0)}_y = 0 \quad \text{on} \quad y = 0 \tag{7-13}$$

$$\frac{\partial \phi^{(0)}}{\partial n} = -n_z c \sin\alpha \quad \text{on the body cross-section } C(x) \tag{7-14}$$

$$\phi^{(0)} \to 0 \qquad z + iy \to \infty \tag{7-15}$$

$n = n_z + in_y$ being the outward normal to $C(x)$. This is a well-posed mixed boundary value problem for Laplace's equation in a simply connected region of the plane.

It will be useful at times to consider the problem in the whole s-plane exterior to the reflected body cross-section. Noting the Cauchy-Riemann conditions expressing the analyticity of $f^{(0)}$:

$$\phi_z^{(0)} = \psi_y^{(0)} \tag{7-16}$$

$$\phi_y^{(0)} = -\psi_z^{(0)} \tag{7-17}$$

we observe that (7-17) with (7-13) requires that $\psi_z^{(0)}$ vanish along the real axis; hence $\psi^{(0)}$ = constant on $z = 0$ and we take that constant to be zero. Then $f^{(0)}$ is purely real on the real axis and may be continued into the upper half-plane by $f^{(0)}(\bar{s}) = \bar{f}^{(0)}(s)$. The region is now doubly connected; however the circulation about the interior boundary vanishes by virtue of (7-13) and the symmetry of (7-14) about the real axis.

The form of Bernoulli's equation (4-9) appropriate to slender body theory, in the moving co-ordinates, is

$$\frac{p^{(0)}}{\rho} = -c\varphi_x^{(0)} - \alpha c\, \varphi_z^{(0)} - \tfrac{1}{2}\left[(\varphi_z^{(0)})^2 + (\varphi_y^{(0)})^2 \right] \tag{7-18}$$

where $p^{(0)} = p^{(0)}(s;x)$ is now a function of s, and $\cos\alpha$, $\sin\alpha$ have been approximated by $1, \alpha$ respectively. On the plane $y = 0$, (7-13) reduces (7-18) to

$$\frac{p^{(0)}(x, 0, z)}{\rho} = -c\,\varphi_x^{(0)} - \alpha c \varphi_z^{(0)} - \tfrac{1}{2}(\varphi_z^{(0)})^2 . \tag{7-18'}$$

7.2 First Order

The following relations are found by expressing (4-10)-(4-14) in terms of the new variable:

$$\nabla^2 \varphi_1^{(1)} = 0 \quad \text{in} \quad y \leqslant 0\,, \quad e \geqslant 0 \tag{7-19}$$

$$\nabla \varphi^{(1)} \cdot \nabla e = 0 \quad \text{on} \quad e = 0 \tag{7-20}$$

$$\varphi_y^{(1)} = c h_x^{(1)} + (\alpha c + \varphi_z^{(0)}) h_z^{(1)} - \varphi_{yy}^{(0)} h^{(1)} \quad \text{on} \quad y = 0 \tag{7-21}$$

$$h^{(1)} = -\frac{\ell}{c^2} \left\{ c\varphi_x^{(0)} + \alpha c \varphi_z^{(0)} + \tfrac{1}{2}(\varphi_z^{(0)})^2 \right\} \quad \text{on } y = 0 \tag{7-22}$$

$$\nabla \varphi^{(1)} \to 0 \qquad x \to -\infty, \quad y \to -\infty \tag{7-23}$$

The $h^{(1)}$ of (7-22) can be substituted into (7-21) to give $\varphi_y^{(1)}$ directly in terms of $\varphi^{(0)}$ (using (7-12) to replace $\varphi_{yy}^{(0)}$ with $-\varphi_{zz}^{(0)}$):

$$\varphi_y^{(1)} = -\frac{\ell}{c^2} \Big[c^2 \varphi_{xx}^{(0)} + 2\alpha c^2 \varphi_{xz}^{(0)} + \alpha^2 c^2 \varphi_{zz}^{(0)} + c\,\varphi_x^{(0)} \varphi_{zz}^{(0)}$$

$$+ 3\alpha c\, \varphi_z^{(0)} \varphi_{zz}^{(0)} + 2c\, \varphi_z^{(0)} \varphi_{xz}^{(0)} + \frac{3}{2} (\varphi_z^{(0)})^2 \varphi_{zz}^{(0)} \Big]_{y=0} \tag{7-24}$$

Since $\varphi^{(0)}$ was found to be independent of time, we may differentiate (7-19)-(7-23) with respect to time and arrive at the same homogeneous problem as (7-6')-(7-9') for $\varphi_t^{(1)}$; hence it is concluded that $\varphi^{(1)}$ is likewise independent of time.

If the cross flow analysis can be applied to the second-order problem, as discussed in Section VI, then it will be useful to introduce a notation similar to that used in the zero-order analysis. We define

$$\phi^{(1)}(s; x) = \phi^{(1)}(z + iy; x) = \varphi^{(1)}(x, y, z) \tag{7-25}$$

in analogy to (7-11), while (7-19) requires that $\phi^{(1)}$ be the real part of an analytic function $f^{(1)}(s; x)$.

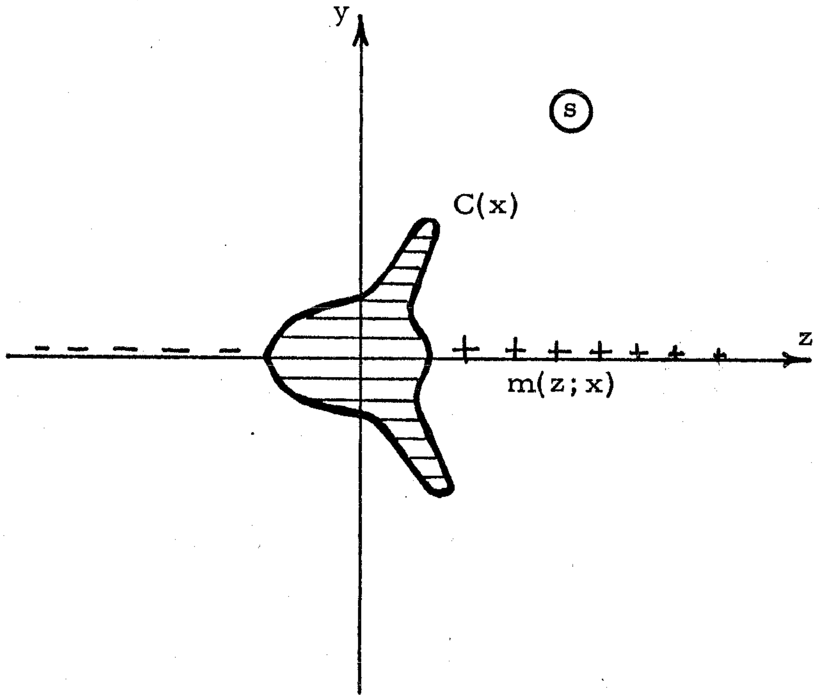
The boundary conditions on the harmonic $\phi^{(1)}$ in the whole s-plane are

$$\frac{\partial \phi^{(1)}}{\partial n} = 0 \quad \text{on the reflected body cross-section} \quad C(x) \tag{7-26}$$

$$\frac{\partial \phi^{(1)}}{\partial n} = \phi_y^{(1)} = -\tfrac{1}{2} m(z;x) \quad \text{on} \quad y = 0 \tag{7-27}$$

$$\phi^{(1)} \to 0 \qquad \text{far away} \tag{7-28}$$

Here the line-source distribution $m(z;x)$ has been introduced to satisfy the normal velocity requirement on the z-axis. The strength of m is minus twice the normal velocity $\phi_y^{(1)}$ as given by (7-24)



The first-order Bernoulli equation (4-15) expressed in the new variables, and simplified consistent with slender-body theory, is

$$\frac{p^{(1)}(s;x)}{\rho} = -c\varphi_x^{(1)} - \alpha c\,\varphi_z^{(1)} - \varphi_z^{(0)}\varphi_z^{(1)} - \varphi_y^{(0)}\varphi_y^{(1)} \tag{7-29}$$

VIII. SYMMETRY CONSIDERATIONS

The calculation of the pressures $p^{(0)}$ and $p^{(1)}$ is further simplified, in fact to the point where some analytic results can be obtained for specific shapes, if the body under consideration is symmetric about the $z = 0$ plane; this means the reflected body has two planes of symmetry. Then any term in the pressure equations which is an even function of z does not contribute to the side force; also the even part of the source strength $m(z;x)$ will produce the even part of $\varphi^{(1)}$, while the odd part of m accounts for the odd part of $\varphi^{(1)}$. By an "even function of z" is meant a function $f(z)$ or $\underline{f}(s)$ with the property

$$f(-z) = f(z)$$

$$\text{or} \quad \underline{f}(-\bar{s}) = \underline{f}(-z + iy) = \underline{f}(z + iy) = \underline{f}(s)$$

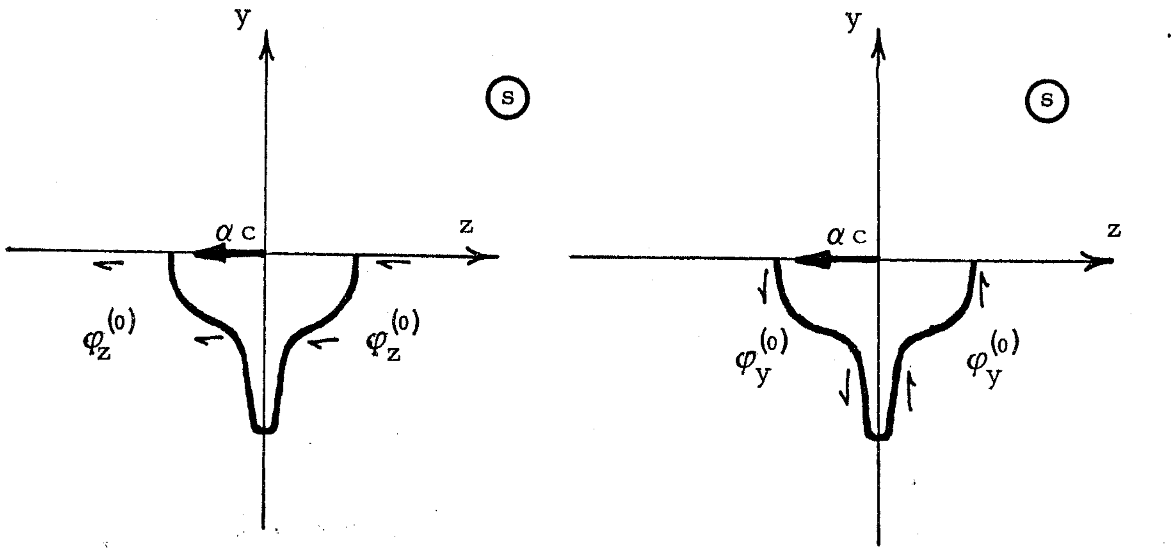
whereas for an "odd function of z", $g(z)$ or $\underline{g}(s)$,

$$g(-z) = -g(z)$$

$$\text{or} \qquad \underline{g}(-\bar{s}) = -\underline{g}(s)\,.$$

Now consider the symmetry properties of the various terms of the zero-order Bernoulli equation:

$$\frac{p^{(0)}}{\rho} = - c\varphi_x^{(0)} - \alpha c \varphi_z^{(0)} - \frac{1}{2}\left[(\varphi_z^{(0)})^2 + (\varphi_y^{(0)})^2 \right] . \qquad (7\text{-}18)$$



It is clear that $\varphi_z^{(0)}$ has the same value at corresponding points $(\pm z + iy)$ on the two sides, so the second term of (7-18) is even in z. Immediately, then, $(\varphi_z^{(0)})^2$ must also be even. Furthermore $\varphi_y^{(0)}$ is odd, so $(\varphi_y^{(0)})^2$ is even. So the only term that contributes to zero-order side force is

$$p_{odd}^{(0)} (x,y,z) = p_{odd}^{(0)} (s;x) = - \rho c \phi_x^{(0)} . \qquad (8\text{-}1)$$

Now the source strength distribution entering into the second-order problem is determined by (7-21). For this purpose $h^{(1)}$ (7-22) is broken into odd and even parts:

$$h_{odd}^{(1)} = - \frac{\ell}{c} \varphi_x^{(0)} (x, 0, z) \qquad (8\text{-}2)$$

$$h^{(1)}_{\text{even}} = -\frac{\ell}{c^2}\left\{\alpha c\,\varphi^{(0)}_z(x,0,z) + \frac{1}{2}\left[\varphi^{(0)}_z(x,0,z)\right]^2\right\} \quad (8\text{-}3)$$

and the odd and even source distributions are derived from them:

$$m_{\text{odd}}(z;x) = -2\left[c(h^{(1)}_{\text{odd}})_x + (\alpha c + \varphi^{(0)}_z)(h^{(1)}_{\text{even}})_z + \varphi^{(0)}_{zz}\,h^{(1)}_{\text{even}}\right]_{y=0} \quad (8\text{-}4)$$

$$m_{\text{even}}(z;x) = -2\left[c(h^{(1)}_{\text{even}})_x + (\alpha c + \varphi^{(0)}_z)(h^{(1)}_{\text{odd}})_z + \varphi^{(0)}_{zz}\,h^{(1)}_{\text{odd}}\right]_{y=0} \quad (8\text{-}5)$$

Now the first-order Bernoulli equation (7-29) is investigated, with $\varphi^{(1)} = \varphi^{(1)}_{\text{even}} + \varphi^{(1)}_{\text{odd}}$. We separate out the odd pressure terms:

$$\frac{p^{(1)}_{\text{odd}}}{\rho} = -\,c(\varphi^{(1)}_{\text{odd}})_x - \alpha c(\varphi^{(1)}_{\text{even}})_z - \varphi^{(0)}_z(\varphi^{(1)}_{\text{even}})_z - \varphi^{(0)}_y(\varphi^{(1)}_{\text{even}})_y , \quad (8\text{-}6)$$

which are the only ones that contribute to first-order side force.

IX. SOLUTIONS BY CONFORMAL MAPPING.

Solutions of the two potential problems with boundary conditions (7-13)-(7-15) and (7-26)-(7-28) may be obtained by conformal transformation of the s-plane into another complex plane in which the boundaries assume shapes suited to the necessary computations. Since we are interested in $\phi^{(1)}$ and its derivatives only on the interior boundary, it seems natural to consider a transformation that maps the interior boundary and the real axis, where the sources are located, onto a single line (say the real axis) in the mapped plane. The transformation of this type that leaves the plane unchanged at infinity will be chosen. The mapped plane is called the σ-plane $(\sigma = \zeta + i\eta)$, and the transformation is represented by

$$\sigma = \gamma(s;x) = s + \sum_{n=1}^{\infty} \frac{A_n(x)}{s^n} \tag{9-1}$$

with the inverse

$$s = g(\sigma;x) = \sigma + \sum_{n=1}^{\infty} \frac{B_n(x)}{\sigma^n} \tag{9-2}$$

The complex potential is the same at corresponding points in s- and σ-planes; so a potential $\tilde{f}(\sigma;x) = \tilde{\phi}(\sigma;x) + i\psi(\sigma;x)$ is introduced in the σ-plane, defined by

$$\tilde{f}(\sigma;x) = \tilde{f}[\gamma(s;x);x] = f(s;x). \tag{9-3}$$

If the mapping (9-1)-(9-2) is conformal then $\tilde{f}$ is an analytic function of σ, and so $\tilde{\phi}$ is harmonic.

9.1 Zero Order

It is useful to consider a slightly different zero-order problem from the one posed in (7-13)-(7-15). Rather than deal with the rigid boundary moving in the negative z-direction with speed αc, it is preferred to solve the altogether equivalent problem of a fixed boundary in a stream which far away has velocity αc in the positive z-direction. Call the potential in the latter problem $f_1^{(0)} = \phi_1^{(0)} + i\psi_1^{(0)}$. Its boundary conditions are

$$\frac{\partial \phi_1^{(0)}}{\partial n} = 0 \quad \text{on} \quad y = 0 \tag{9-4}$$

$$\frac{\partial \phi_1^{(0)}}{\partial n} = 0 \quad \text{on} \quad C(x) \tag{9-5}$$

$$\phi_1^{(0)} \to \alpha c z \qquad z + iy \to \infty \tag{9-6}$$

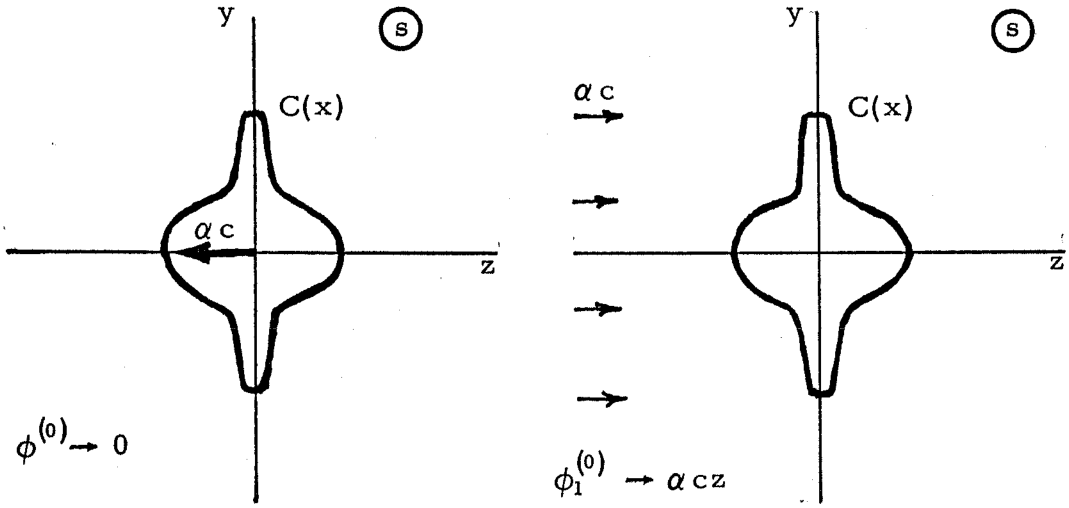


Fig. 9.1. Equivalent zero-order problems.

The potentials for the two problems are related by

$$f^{(0)}(s;x) = f_1^{(0)}(s;x) - \alpha c s \; . \tag{9-7}$$

Now the problem for $\phi_1^{(0)}$ is mapped into the σ-plane, where the solution is trivial:

$$\tilde{f}_1^{(0)}(\sigma;x) = \alpha c \sigma \; . \tag{9-8}$$

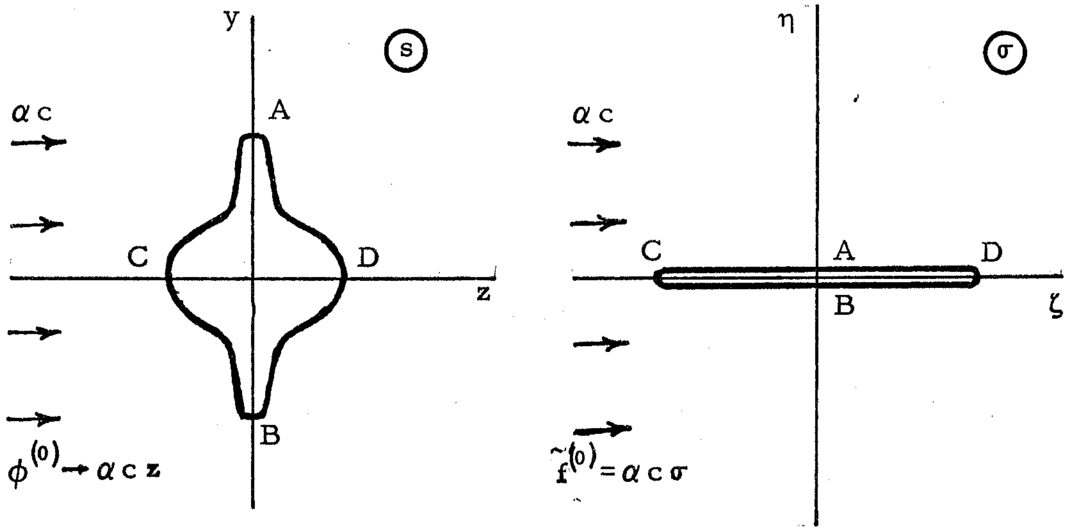


Fig. 9.2. Mapping of zero-order problem.

By (9-1),

$$f_1^{(0)}(s;x) = \alpha c \gamma(s;x) \tag{9-9}$$

and

$$f^{(0)}(s;x) = \alpha c [\gamma(s;x) - s] . \tag{9-10}$$

Let $s_0 = z_0 + iy_0$ represent a value of s on the interior boundary, where $\psi^{(0)}$ has been taken equal to zero. Then

$$\phi^{(0)}(s_0;x) = \alpha c [\gamma(s_0;x) - z_0] \tag{9-11}$$

and, by (8-1),

$$p_{odd}^{(0)}(s_0;x) = -\rho c^2 \alpha \gamma_x(s_0;x) . \tag{9-12}$$

The force on a differential length dx of the body is obtained by an integration:

$$dS + idL = \frac{1}{2} dx \oint_{C(x)} p_{odd}(-dy + i dz) = \frac{i}{2} dx \oint_{C(x)} p_{odd}(s_0;x) d s_0 \tag{9-13}$$

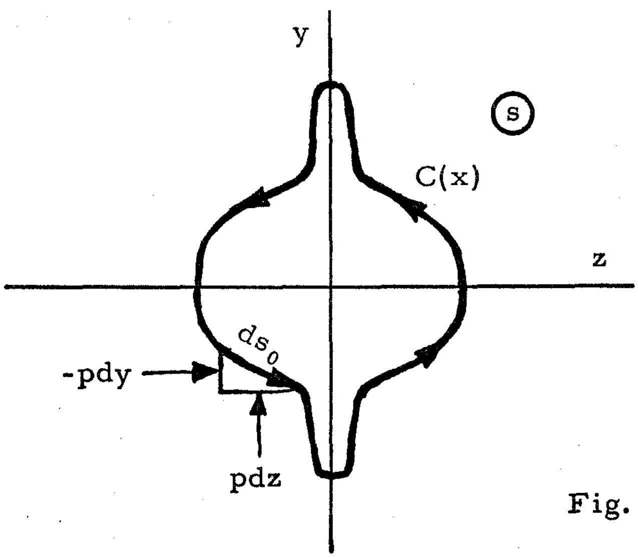


Fig. 9.3 Force resolution

Here dL is the force along the y-axis due to p_{odd}, which integrates to zero by symmetry about the real axis. The fraction $\frac{1}{2}$ is to make dS represent the side force on the actual body, that is, on the lower half of the reflected body. Using (9-12),

$$dS^{(0)} = -\frac{i}{2}\,dx \cdot \rho c^2 \alpha \oint_{C(x)} \gamma_x(s_0\,;x)\,ds_0 \tag{9-14}$$

Alternatively, $dS^{(0)}$ can be obtained from the virtual momentum considerations of ordinary slender-body theory, or from integrations in a complex plane other than the σ-plane.

Next the conformal mapping is used to investigate the distribution of $p^{(0)}$ over the surface $y=0$ - to see whether in general this distribution is effectively "slender". The imaginary part $\psi^{(0)}$ of $f^{(0)}$ has been taken equal to zero here, so, from (9-10) and (9-1)

$$\varphi^{(0)}(x,0,z) = \alpha c \sum_{n=1}^{\infty} \frac{A_n(x)}{z^n} \tag{9-15}$$

Differentiating and substituting into (7-18) we find that $p^{(0)}(x,0,z)$ has the form of a power series in inverse powers of z, starting with $-\rho c^2 \alpha \frac{dA_1}{dx} \frac{1}{z}$. The pressure disturbance dies off rapidly with increasing z; and so we may expect that the lateral extent of the major pressure disturbance is small, of the order of dA_1/dx.

9.2 First Order

The mapping to the σ-plane was especially chosen to simplify the first-order problem. Under the conformal transformation (9-2), the source distribution $m(z;x)$ maps into a source distribution

$$\mu(\zeta; x) = m\,[\,g(\zeta; x)\,;\,x]\;g_\sigma(\zeta; x) \qquad (9\text{-}16)$$

along the ζ-axis (shown in Appendix III).

In the σ-plane now there is the distribution

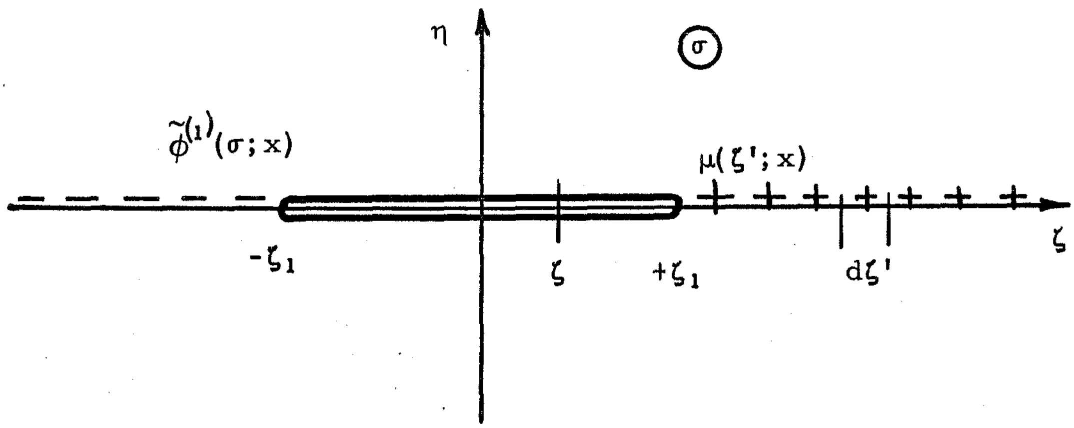


Fig. 9.4. Mapping of second-order problem.

$\mu(\zeta; x)$ of sources along the ζ-axis with $|\zeta| \geqslant \zeta_1$, and the potential $\tilde{\phi}^{(1)}$ need be calculated only on the ζ-axis with $|\zeta| < \zeta_1$. Poisson's integral reduces to a one-dimensional integration over ζ' outside $(-\zeta_1, \zeta_1)$:

$$2\pi\tilde{\phi}^{(1)}(\zeta; x) = \int_{-\infty}^{-\zeta_1} \mu(\zeta'; x)\log(\zeta - \zeta')d\zeta' + \int_{\zeta_1}^{\infty} \mu(\zeta'; x)\log(\zeta' - \zeta)\,d\zeta' \qquad (9\text{-}17)$$

$$= \int_{\zeta_1}^{\infty} \Big[\mu(-\zeta'; x)\log(\zeta' + \zeta) + \mu(\zeta'; x)\log(\zeta' - \zeta)\Big]d\zeta' \qquad (9\text{-}18)$$

where the latter form has been obtained by substituting $-\zeta'$ for ζ' in the first integral of (9-17). When μ is expressed as $\mu = \mu_{even} + \mu_{odd}$, there results

$$\tilde{\phi}^{(1)}_{odd}(\zeta;x) = \frac{1}{2\pi}\int_{\zeta_1}^{\infty} \mu_{odd}(\zeta';x)\log\left(\frac{\zeta'-\zeta}{\zeta'+\zeta}\right) d\zeta' \tag{9-19}$$

$$\phi_{even}(\zeta;x) = \frac{1}{2\pi}\int_{\zeta_1}^{\infty} \mu_{even}(\zeta';x)\log(\zeta'^2-\zeta^2)\,d\zeta' \tag{9-20}$$

Here μ_{odd} and μ_{even} are given by (9-16) applied to m_{odd} and m_{even}, equations (8-4) and (8-5) respectively.

Since the integrals containing logarithms are hard to dealwith, it is useful to obtain algebraic forms by differentiating (9-19) and (9-20) with respect to ζ:

$$(\tilde{\phi}^{(1)}_{odd})_{\zeta} = -\frac{1}{\pi}\int_{\zeta_1}^{\infty} \mu_{odd}(\zeta';x)\,\frac{\zeta' d\zeta'}{\zeta'^2-\zeta^2} \tag{9-21}$$

$$(\tilde{\phi}^{(1)}_{even})_{\zeta} = -\frac{\zeta}{\pi}\int_{\zeta_1}^{\infty} \mu_{even}(\zeta';x)\,\frac{d\zeta'}{\zeta'^2-\zeta^2} \tag{9-22}$$

The results of (9-21) and (9-22) can be integrated to find $\tilde{\phi}^{(1)}_{even}(\zeta;x)$ and $\tilde{\phi}^{(1)}_{odd}(\zeta;x)$; the constants of integration are immaterial constant potentials.

$\phi^{(1)}$ on the boundary is obtained by the mapping (9-1) :

$$\phi^{(1)}(s_0;x) = \tilde{\phi}^{(1)}[\gamma(s_0;x);x] \tag{9-23}$$

so that the first-order side force is obtained through the first-order Bernoulli equation (8-6) integrated by (9-13) :

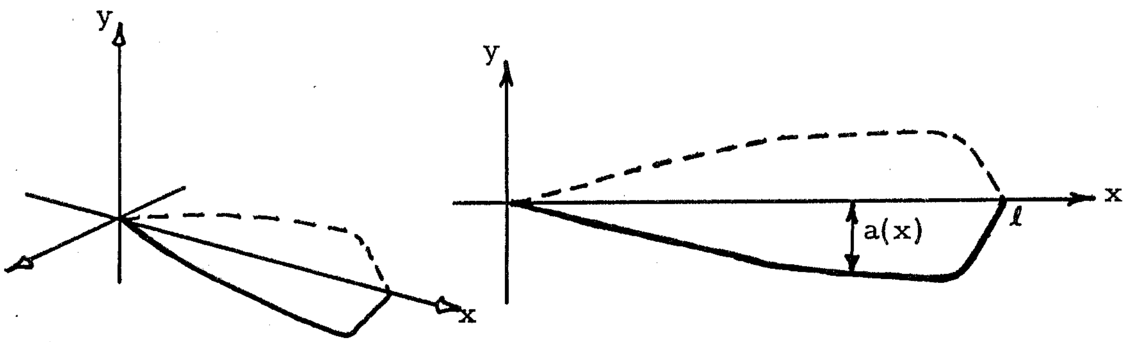
$$\frac{1}{\rho}\frac{dS^{(1)}}{dx} = -\frac{i}{2}c\oint_{C(x)}(\phi^{(1)}_{odd})_x\,ds_0 - \frac{i}{2}\alpha c\oint_{C(x)}(\phi^{(1)}_{even})_z\,ds_0$$

$$-\frac{i}{2}\int_{C(x)}\left[\phi^{(0)}_z(\phi^{(1)}_{even})_z + \phi^{(0)}_y(\phi^{(1)}_{even})_y\right]ds_0\,. \qquad (9\text{-}24)$$

Since $\phi^{(1)}$ was calculated only on the boundary, it may appear that the z- and y- derivatives in (6-24) are not determined; however knowledge of $\phi^{(1)}$ on the boundary plus the vanishing there of $\partial\phi^{(1)}/\partial n$ (7-26) suffice to determine $\phi^{(1)}_z$ and $\phi^{(1)}_y$.

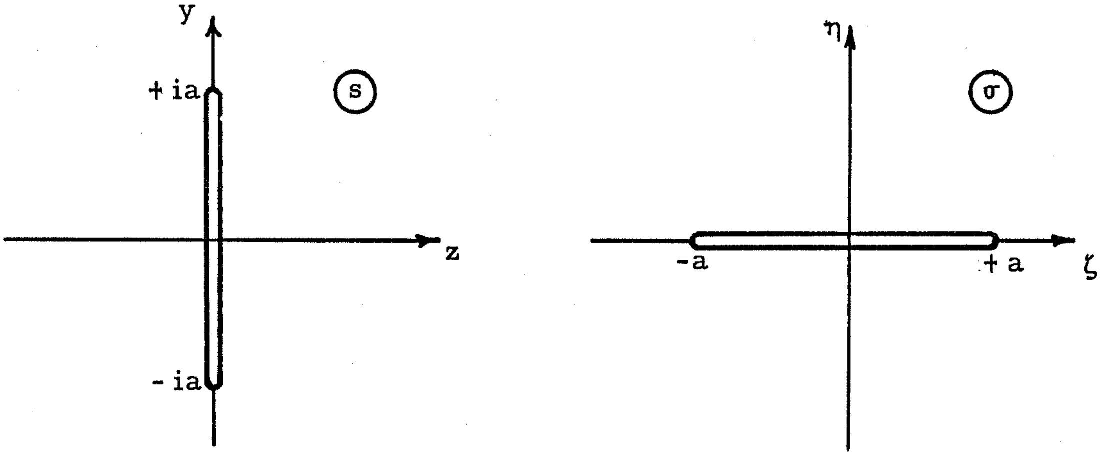
X. SLENDER WING OF ZERO THICKNESS

The first specific geometry considered is a flat plate wing of arbitrary slender plan form is given by $y = -a(x)$.



The reflected body is the flat plate wing with span $2a(x)$.

The interior boundary $C(x)$ in the cross-flow plane is the cut from $-ia(x)$ to $+ia(x)$. Under the



transformation

$$\sigma = \gamma(s;x) = \pm \{s^2 + [a(x)]^2\}^{\frac{1}{2}} = s\left[1 + \frac{a^2}{s^2}\right]^{\frac{1}{2}} \qquad (10\text{-}1)$$

with the inverse

$$s = g(\sigma;x) = \pm \{\sigma^2 - [a(x)]^2\}^{\frac{1}{2}} = \sigma\left[1 - \frac{a^2}{\sigma^2}\right]^{\frac{1}{2}} \qquad (10\text{-}2)$$

the boundary $C(x)$ maps into the cut from $-a(x)$ to $+a(x)$ in the σ-plane, the z-axis maps into the ζ-axis with $|\zeta| > a$, and the plane is unchanged at infinity. The sign of the square root is chosen so that σ is in the same quadrant as s.

The complex potential is, by (9-10),

$$f^{(0)}(s;x) = \alpha c \{\pm (s^2 + a^2)^{\frac{1}{2}} - s\} \qquad (10\text{-}3)$$

and on the boundary, where $s_0 = iy$, the real potential is, by (9-11),

$$\phi^{(0)}(iy;x) = \pm \alpha c (a^2 - y^2)^{\frac{1}{2}} \qquad (10\text{-}4)$$

The zero-order side force distribution is given by (9-14)

$$\frac{dS^{(0)}}{dx} = -\frac{i}{2}\rho c^2 \alpha \cdot \oint \frac{aa_x}{(s_0^2 + a^2)^{\frac{1}{2}}}\, ds_0 = -\frac{i}{2}\rho c^2 \alpha \cdot aa_x \cdot 2\int_{-a}^{a} \frac{i\,dy}{(a^2 - y^2)^{\frac{1}{2}}} = \pi\rho c^2 \alpha\, aa_x. \tag{10-5}$$

This is the familiar result of the Jones slender wing theory (Ref. 17, 1946). The section of maximum span is treated as the trailing edge, and $x = \ell$ is its location.

The next step is to calculate $h^{(1)}$ from (8-2) and (8-3). The real potential on the surface $y = 0$ is, from (10-3), (with $\pm$ = sign z)

$$\varphi^{(0)}(x, 0, z) = \alpha c \{\pm(z^2 + a^2)^{\frac{1}{2}} - z\} \tag{10-6}$$

Consequently,

$$h^{(1)}_{odd}(x,z) = -\alpha\ell \frac{aa_x}{\pm(z^2+a^2)^{\frac{1}{2}}} \tag{10-7}$$

$$h^{(1)}_{even}(x,z) = \frac{\alpha^2}{2}\ell \frac{a^2}{z^2 + a^2} \tag{10-8}$$

These expressions are used in (8-4) and (8-5) to determine the source distributions in the physical plane:

$$m_{odd} = \pm 2\ell c\alpha \left[(a_x^2 + aa_{xx})\frac{1}{(z^2+a^2)^{\frac{1}{2}}} - (a_x^2 - \alpha^2)\frac{a^2}{(z^2+a^2)^{3/2}} - \frac{3}{2}\alpha^2 \frac{a^4}{(z^2+a^2)^{5/2}}\right] \tag{10-9}$$

$$m_{even} = 2\ell c\alpha^2 a_x \left[\frac{3a^3}{(z^2+a^2)^2} - \frac{2a}{z^2+a^2}\right]. \tag{10-10}$$

"Slenderness" of this source distribution, as required for a crossflow analysis of the first-order problem, demands that a_{xx} not change too rapidly in x: $a_{xxx} << 1/\ell^2$.

The source distribution in the σ-plane is found by (9-16) to be

$$\mu_{odd}(\zeta;x) = \pm 2\ell c\alpha\left[\frac{a_x^2+aa_{xx})}{(\zeta^2-a^2)^{\frac{1}{2}}} - \frac{(a_x^2-\alpha^2)a^2}{\zeta^2(\zeta^2-a^2)^{\frac{1}{2}}} - \frac{3}{2}\,\frac{\alpha^2 a^4}{\zeta^4(\zeta^2-a^2)^{\frac{1}{2}}}\right] \quad (10\text{-}11)$$

$$\mu_{even}(\zeta;x) = 2\ell c\alpha^2 a_x\left[\frac{3a^3}{\zeta^3(\zeta^2-a^2)^{\frac{1}{2}}} - \frac{2a}{\zeta(\zeta^2-a^2)^{\frac{1}{2}}}\right] \quad (10\text{-}12)$$

The integrations of (9-19) and (9-20), using these source distributions, are carried out in Appendix IV, with the results

$$\tilde{\phi}^{(1)}_{odd} = \ell c\left\{-\alpha(a_x^2+aa_{xx})\sin^{-1}\frac{\zeta}{a} + \alpha a_x^2\,\frac{a-(a^2-\zeta^2)^{\frac{1}{2}}}{\zeta} + \frac{\alpha^3}{4}\,\frac{a}{\zeta}\left[\frac{a-(a^2-\zeta^2)^{\frac{1}{2}}}{\zeta}\right]^2\right\}$$

(10-13)

$$\tilde{\phi}^{(1)}_{even} = -\frac{\ell c\alpha^2 a_x}{2}\quad \log\left[\frac{a+(a^2-\zeta^2)^{\frac{1}{2}}}{a}\right] + \frac{3}{2}\left[\frac{a-(a^2-\zeta^2)^{\frac{1}{2}}}{\zeta}\right]^2 \quad (10\text{-}14)$$

with a constant term ignored in the latter. Transforming back to the s-plane by (10-1):

$$\varphi^{(1)}_{odd}(x,y,0) = -\tfrac{1}{2}\,\ell c\alpha^2 a_x\left\{\log\frac{a+|y|}{a} + \frac{3}{2}\,\frac{a-|y|}{a+|y|}\right\} \quad (10\text{-}16)$$

The first-order Bernoulli equation (8-6) gives the spanwise loading:

$$\frac{p^{(1)}_{odd}}{\rho c^2\ell} = \alpha\, a_x a_{xx}\left[3\cos^{-1}\frac{y}{a} + \frac{3y-2a}{(a^2-y^2)^{\frac{1}{2}}}\right] + \alpha\, aa_{xxx}\left[\cos^{-1}\frac{y}{a}\right]$$

$$+\,\alpha a_x^3\left[\frac{y^2}{a(a+y)(a^2-y^2)^{\frac{1}{2}}}\right] - \tfrac{1}{4}\,\alpha^3 a_x\left[\frac{y^2-2ay}{(a+y)^2(a^2-y^2)^{\frac{1}{2}}}\right] \quad (10\text{-}17)$$

The side force is given by the integration (9-13)

$$\frac{dS^{(1)}}{dx} = 2\int_0^a p^{(1)}_{odd}\,dy\,. \quad (10\text{-}18)$$

The necessary integrals are worked out in Appendix V, with the result

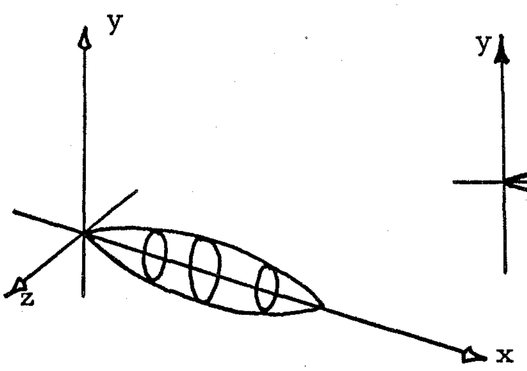
$$\frac{dS^{(1)}}{dx} = \rho c^2 \ell \alpha \left\{ 2(6-\pi)aa_x a_{xx} + 2a^2 a_{xxx} + (4-\pi)a_x^3 + (1-\pi/4)\alpha^2 a_x \right\} \quad (10\text{-}19)$$

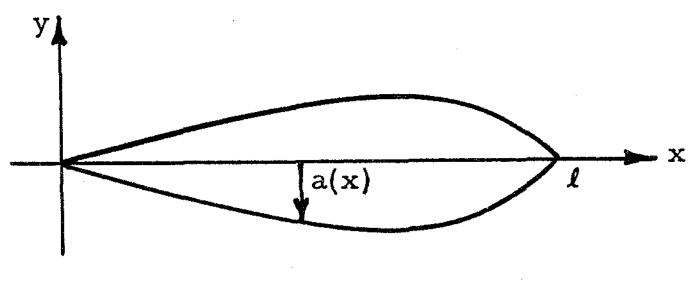
$$= \rho c^2 \ell \left\{ \alpha\left[(4-\pi)aa_x^2 + 2a^2 a_{xx}\right]_x + \frac{4-\pi}{4}\alpha^3 a_x \right\} . \quad (10\text{-}20)$$

Further comment on this result and its application to specific plan forms is reserved for the concluding Chapter XII.

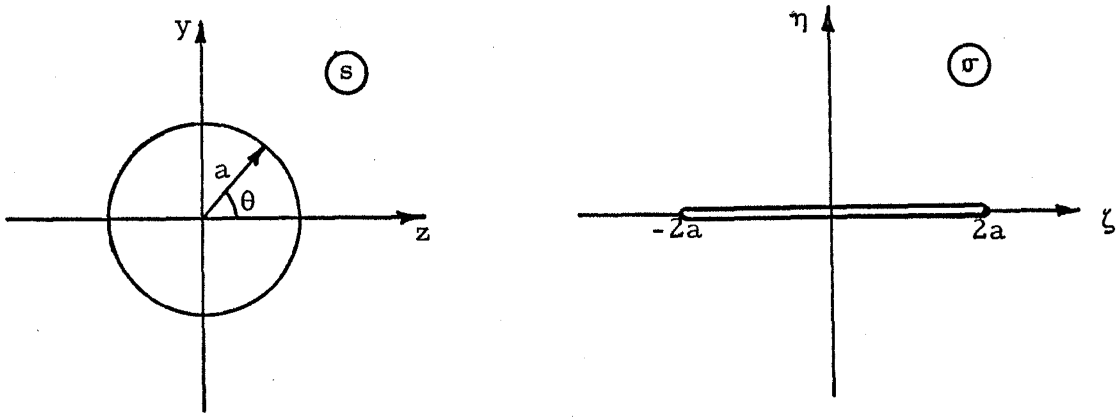
XI. HALF-SUBMERGED BODY OF REVOLUTION

The second specific geometry considered is a slender body whose submerged portion is half of a body of revolution, so that the reflected body is a slender body of revolution. The x-axis is along the line of centers of the body cross-sections, whose radii are given by a(x). If the body has a base, it must be approximately normal to the x-axis.





The interior boundary $C(x)$ in the cross-flow plane is the circle $|s| = a(x)$



. Under the transformation

$$\sigma = \gamma(s;x) = s + \frac{[a(x)]^2}{s} \tag{11-1}$$

with the inverse

$$s = g(\sigma;x) = \frac{\sigma}{2} \pm \left\{ \left(\frac{\sigma}{2}\right)^2 - \left[a(x)\right]^2 \right\}^{\frac{1}{2}} \tag{11-2}$$

the boundary $C(x)$ maps into the cut from $-2a(x)$ to $2a(x)$ in the σ-plane, the z-axis maps into the ζ-axis with $|\zeta| > 2a$, and the plane is unchanged at infinity. The sign of the radical is chosen so s and σ are in the same quadrant.

The complex potential is, by (9-10)

$$f^{(0)}(s;x) = \alpha c \frac{a^2}{s} \tag{11-3}$$

and on the boundary, where $s_0 = ae^{i\theta}$, the real potential is, by (9-11)

$$\phi^{(0)}(s_0;x) = \alpha ca \cos\theta . \tag{11-4}$$

The zero-order side force distribution is given by (9-14)

$$\frac{dS^{(0)}}{dx} = -\frac{i}{2}\rho c^2\alpha \oint \frac{2aa_x}{s_0}\, ds_0 = \pi\rho c^2\alpha a a_x, \tag{11-5}$$

which is the famous result of Munk in the original work on slender body theory (Ref.18, 1924).

The real potential on the surface $y=0$ is, from (10-3),

$$\varphi^{(0)}(x;0,z) = \alpha c \frac{a^2}{z} \tag{11-6}$$

Putting this into (8-2) and (8-3):

$$h^{(1)}_{odd}(x,z) = -2\ell\alpha a_x \frac{a}{z} \tag{11-7}$$

$$h^{(1)}_{even}(x,z) = \ell\alpha^2\left[\frac{a^2}{z^2} - \frac{1}{2}\frac{a^4}{z^4}\right]. \tag{11-8}$$

These expressions are used in (8-4) and (8-5) to determine the source distributions in the physical plane:

$$m_{odd}(z;x) = 2\ell c\left[2\alpha(a_x^2 + aa_{xx})\frac{1}{z} + 2\alpha^3\frac{a^2}{z^3} - 6\alpha^3\frac{a^4}{z^5} + a\alpha^3\frac{a^6}{z^7}\right] \tag{11-9}$$

$$m_{even}(z;x) = 8\ell c\alpha^2 a_x\left[-\frac{a}{z^2} + 2\frac{a^3}{z^4}\right] \tag{11-10}$$

Slenderness of the source distribution requires $a_{xxx} \ll 1/\ell^2$.

By differentiation of (11-12) it is found

$$g_\sigma(\zeta;x) = \frac{1}{2}\frac{\zeta + (\zeta^2 - 4a^2)^{\frac{1}{2}}}{(\zeta^2 - 4a^2)^{\frac{1}{2}}} \tag{11-11}$$

$$z = g(\zeta;x) = \frac{1}{2}\left[\zeta + (\zeta^2 - 4a^2)^{\frac{1}{2}}\right] \tag{11-12}$$

Then, by (9-16)

$$\mu_{odd}(\zeta;x) = 2\ell c\alpha \left\{ 2(a_x^2 + aa_{xx}) \frac{1}{(\zeta^2-4a^2)^{\frac{1}{2}}} + 8\alpha^2 \frac{a^2}{(\zeta^2-4a^2)^{\frac{1}{2}}} Z^{-2} \right.$$

$$\left. -96\alpha^2 \frac{a^4}{(\zeta^2-4a^2)^{\frac{1}{2}}} Z^{-4} + 192\alpha^2 \frac{a^6}{(\zeta^2-4a^2)^{\frac{1}{2}}} Z^{-6} \right\} \qquad (11\text{-}13)$$

The last three terms are eventually found to make no contribution to the pressure; so they are carried along in the form:

$$\mu_{odd}(\zeta;x) = 2\ell c\alpha \left\{ 2(a_x^2 + aa_{xx}) \frac{1}{(\zeta^2-4a^2)^{\frac{1}{2}}} + \frac{\alpha^2}{a} f\left(\frac{\zeta}{2a}\right) \right\} . \qquad (11\text{-}14)$$

Also, by (9-16)

$$\mu_{even}(\zeta;x) = 16\ell c\alpha^2 a_x \left\{ -\frac{a}{(\zeta^2-4a^2)^{\frac{1}{2}}} Z^{-1} + 8\frac{a^3}{(\zeta^2-4a^2)^{\frac{1}{2}}} Z^{-3} \right\} , \qquad (11\text{-}15)$$

where Z stands for $\zeta + (\zeta^2-4a^2)^{\frac{1}{2}}$.

The integrations over sources are carried out by (9-19) and (9-22). The necessary integrals are worked out in Appendix VI.

$$\tilde{\phi}^{(1)}_{odd}(\zeta;x) = -2\ell c\alpha \left\{ (a_x^2 + aa_{xx}) \sin^{-1}\frac{\zeta}{2a} + \alpha^2 f_1\left(\frac{\zeta}{2a}\right) \right\} \qquad (11\text{-}16)$$

$$\left(\tilde{\phi}^{(1)}_{even}\right)_\zeta = -\frac{16\ell c}{\pi a}\alpha^2 a_x \left\{ \frac{\zeta}{a} - \frac{\pi}{8}\frac{\zeta(7a^2-2\zeta^2}{a^2(\zeta^2-4a^2)^{\frac{1}{2}}} + \frac{3a^2-2\zeta^2}{8a^2}\log\frac{2a+\zeta}{2a-\zeta} \right\} \qquad (11\text{-}17)$$

The first-order Bernoulli equation (8-6) gives the pressure around the circumference. It is convenient to write this pressure as a function of the angular position θ , since on the boundary $\zeta = 2a\cos\theta$ and $\partial\phi/\partial\theta = -2a\sin\theta\, \partial\tilde{\phi}/\partial\zeta$. Also on the boundary we have $a\partial\phi/\partial y = \cos\theta\, \partial\phi/\partial\theta$ and $a\,\partial\phi/\partial z = -\sin\theta\, \partial\phi/\partial\theta$.

Thus,

$$(\phi^{(1)}_{even})_z = 2\sin^2\theta\, (\tilde{\phi}^{(1)}_{even})_\zeta , \qquad (11\text{-}18)$$

$$(\phi^{(1)}_{\text{even}})_y = -2\cos\theta\sin\theta(\tilde{\phi}^{(1)}_{\text{even}})_\zeta , \tag{11-19}$$

and using these results and (11-4) in (8-6) we have

$$\begin{aligned}\frac{p^{(1)}_{\text{odd}}}{\rho} = 2\ell c^2\alpha \Bigg\{ &(3a_x a_{xx} + a a_{xxx})\left(\frac{\pi}{2} - \theta\right) \\ &+ 4\alpha^2 a_x \frac{\sin^2\theta}{a}\left[\frac{16}{\pi}\cos\theta - (7-8\cos^2\theta)\frac{\cos\theta}{\sin\theta}\right. \\ &\left.+ \frac{1}{\pi}(3-8\cos^2\theta)\log\frac{1+\cos\theta}{1-\cos\theta}\right]\Bigg\}\end{aligned} \tag{11-20}$$

The side force is given by the integration (9-13) :

$$\frac{dS^{(1)}}{dx} = 2\int_0^{\pi/2} p^{(1)}_{\text{odd}}\cos\theta\ a\ d\theta . \tag{11-21}$$

The necessary integrals are worked out in Appendix VII, with the result

$$\frac{dS^{(1)}}{dx} = 4\rho c^2 \ell\alpha \left\{ (3aa_x a_{xx} + a^2 a_{xxx}) + 2\alpha^2 a_x \right\} . \tag{11-22}$$

Further developments are treated in the following Chapter XII.

XII. CONCLUSIONS

A principal conclusion of this study is that the local part of the disturbance caused by a moving body is representable as a power series in Froude number F, except in small regions of nonuniformity near stagnation points on the free surface. Equations are presented for the calculation of the coefficients of the power series, to arrive at in asymptotic expansion, for small F, for any flow quantity. The magnitude of wave effects is investigated with the conclusion that they may be neglected in calculating at least the first two terms of the expansions.

A method of solution is presented for partially submerged, sufficiently slender, bodies of arbitrary cross-section. The method is applied to calculate the first two terms in the expansions for side force distribution for two specific cross-sections: the planar wing of zero thickness, for which the result is

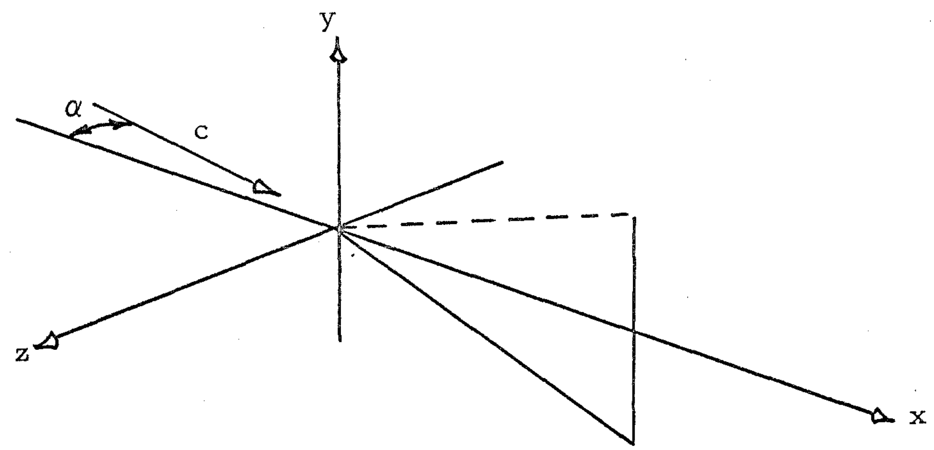
$$\frac{1}{\frac{1}{2}\rho c^2}\frac{dS}{dx} \sim 2\pi\alpha a a_x + 2F\ell\left\{\alpha\frac{d}{dx}\left[(4-\pi)a a_x^2 + 2a^2 a_{xx}\right] + \frac{4-\pi}{4}\alpha^3 a_x\right\}; \qquad (12\text{-}1)$$

and the half-submerged body of revolution, for which the result is

$$\frac{1}{\frac{1}{2}\rho c^2}\frac{dS}{dx} \sim 2\pi\alpha a a_x + 8F\ell\left[\alpha(3a a_x a_{xx} + a^2 a_{xxx}) + 2\alpha^3 a_x\right]. \qquad (12\text{-}2)$$

These formulas are now applied to a few specific shapes.

1. Delta Wing. This is the case of a flat wing



of zero thickness having triangular planform: $a(x) = a_x x$, where a_x is constant, the tangent of the half-angle at the nose. The side force distribution is

$$\frac{1}{\frac{1}{2}\rho c^2}\frac{dS}{dx} \sim 2\pi\alpha a_x^2 x + \frac{4-\pi}{2}F\alpha a_x(4a_x^2 + \alpha^2)\ell \qquad (12\text{-}3)$$

The net side force is

$$S \sim \tfrac{1}{2}\rho c^2 \ell^2 \left[\pi \alpha a_x^2 + \tfrac{1}{2}(4-\pi) F \alpha a_x (4a_x^2 + \alpha^2) \right] \quad . \tag{12-4}$$

The yawing moment about the y-axis is

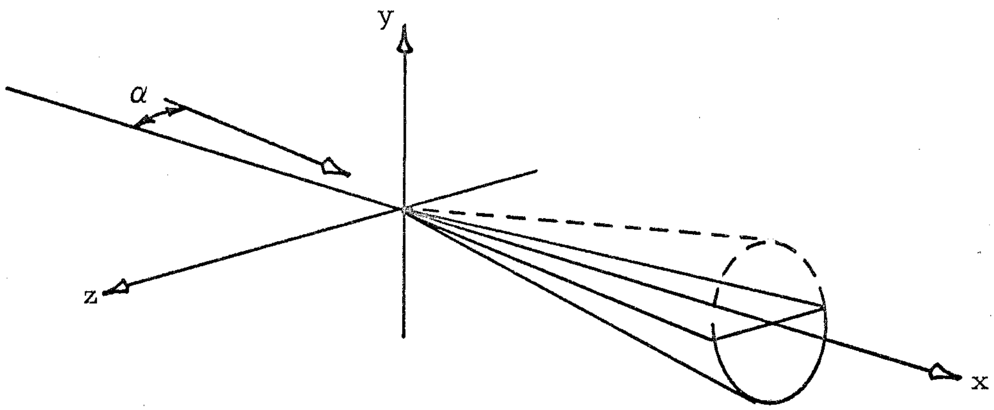
$$M \sim \tfrac{1}{2}\rho c^2 \ell^2 \left[\tfrac{2}{3} \pi \alpha a_x^2 + \frac{4-\pi}{4} F \alpha a_x (4a_x^2 + \alpha^2) \right] \tag{12-5}$$

and the center of pressure is at

$$\frac{\bar{x}}{\ell} \sim \frac{\frac{2}{3} + F \cdot \frac{4-\pi}{4\pi} \frac{4a_x^2 + \alpha^2}{a_x}}{1 + 2F \cdot \frac{4-\pi}{4\pi} \frac{4a_x^2 + \alpha^2}{a_x}} \tag{12-6}$$

so that if $a_x \to 0$ with α and F fixed, $\bar{x} \to \ell/2$. The first-order side force is uniformly distributed along the length. As a_x becomes small $[O(F\alpha^2)]$ the entire wing is very close to the free surface, and free-surface effects are felt strongly; the coefficient $S^{(1)}$ can then be large compared with $S^{(0)}$.

2. Cone. The body is a circular cone having



radius $a(x) = a_x x$, where a_x is a constant, the tangent of the half-angle. The side force distribution is

$$\frac{1}{\frac{1}{2}\rho c^2}\ \frac{dS}{dx} \sim 2\pi\alpha a_x^2 x + 16F\ell\alpha^3 a_x, \tag{12-7}$$

so the net side force is

$$S \sim \tfrac{1}{2}\rho c^2 \ell^2 \left[\pi\alpha\, a_x^2 + 16F\alpha^3 a_x\right], \tag{12-8}$$

The yawing moment is

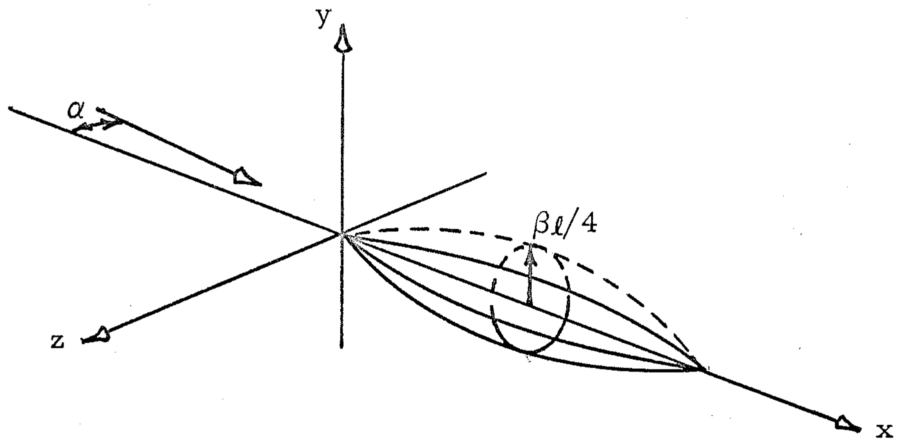
$$M \sim \tfrac{1}{2}\rho c^2 \ell^2 \left[\tfrac{2}{3}\pi\alpha a_x^2 + 8F\alpha^3 a_x\right] \tag{12-9}$$

so the center of pressure is at

$$\frac{\overline{x}}{\ell} = \frac{\frac{2}{3} + \frac{8}{\pi} F \frac{\alpha^2}{a_x}}{1 + \frac{16}{\pi} F \frac{\alpha^2}{a_x}} . \tag{12-10}$$

Again, the first-order side force is uniformly distributed and for very small $a_x [\, O(F\alpha^2)\,]$ the first order term may be comparable to the zero-order term.

3. Parabolic meridian. The body is a figure of revolution with radius $a(x) = \beta x(\ell - x)/\ell$, so $\beta\ell$ is four times the maximum radius. Then $a_x = \beta(\ell - 2x)/\ell$, $a_{xx} = -2\beta/\ell$, $a_{xxx} = 0$.



$$\frac{1}{\frac{1}{2}\rho c^2}\frac{dS}{dx} \sim 2\pi\alpha\beta^2 x(\ell-x)(\ell-2x)/\ell^2 + 8F\left[-6\alpha\beta^3 x(\ell-x)(\ell-2x)/\ell^2 + 2\alpha^3\beta(\ell-2x)\right], \tag{12-11}$$

so the net side force integrates to zero as expected for a shape with no base or trailing edge. The moment is

$$M \sim -\tfrac{1}{2}\rho c^2 \ell^3 \left[\frac{\pi}{30}\alpha\beta^2 - F\left(\frac{4}{5}\beta^2 - \frac{8}{3}\alpha^2\right)\alpha\beta\right]. \tag{12-12}$$

The small size of the first-order correction for moderate values of the parameters is noteworthy. For $\alpha = 0.1$, $a_x = 0.1$ and the fairly high Froude number of 0.1, the correction to side force is less than 2% for the delta wing and about 5% for the cone; the correction to center of pressure is about 0.2% ℓ and 1.0% ℓ respectively. In the case of the closed, nonlifting body of revolution treated, however, the correction to the moment is considerable; with $\beta = \frac{1}{2}$ and $\alpha = 0.1$, the correction amounts to a reduction in moment of about 33% at $F = 0.1$.

The shapes calculated here do not have any practical value in naval architecture; however, they do have qualitative similarities to practical shapes, and so they might provide qualitative indications of the Froude number effects for practical shapes.

Perhaps fortunately, no experimental data is available for comparison. Tsakonas (Ref. 2) gives data for towing-tank tests of model hulls and flat wings; however, all his data are at one Froude number, $F = 0.02$, and he concludes that the zero-order results are adequate. A major value of the present theory is its suggestion of certain simple shapes for which the calculations are easy, as the three special shapes considered above. Towing-tank tests of these simple shapes would provide an interesting test of the theory.

The calculations of this theory could be applied to other slender shapes, either by direct but tedious analysis starting from the mapping functions and following the same course as Chaps. X and XI, or by numerical methods. For shapes that do not qualify as slender in the sense of this theory, the perturbation equations of Chap. IV can be applied by numerical methods.

APPENDIX I

THE POTENTIAL OF AN ELEMENTARY SUBMERGED HORSESHOE VORTEX[1]

The fundamental lifting solution for free-surface flows is derived from the fundamental source-like solution. If y is vertically upward and the stream velocity is c in the positive x-direction, the potential of a source of strength m located at (0, -f, 0) is, in the form given by Havelock (Ref. 7):

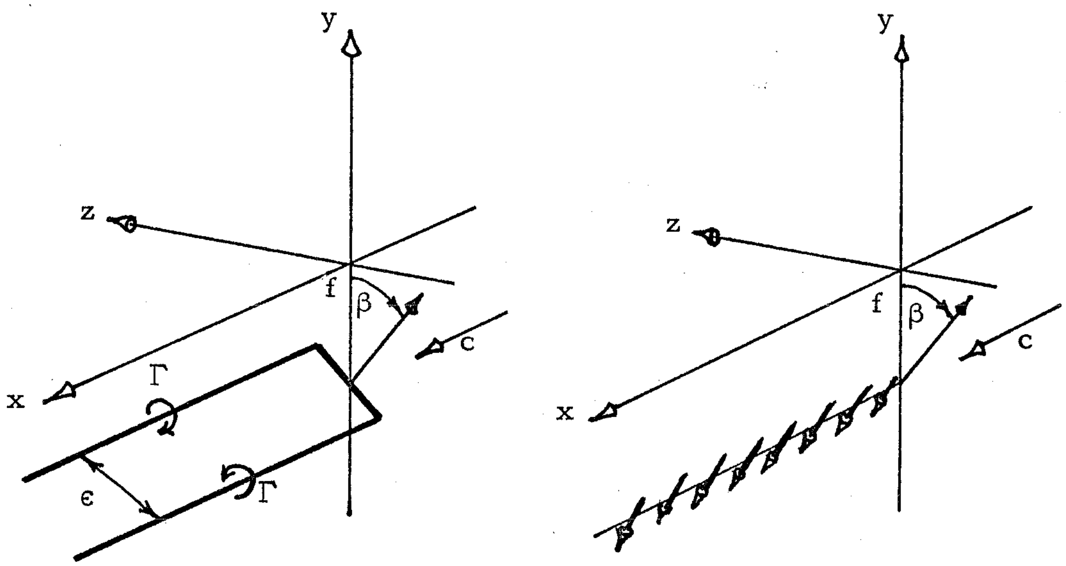
$$mH(x, y, z; 0, -f, 0) = \frac{m}{r_1} - \frac{m}{r_2}$$
$$- \frac{2}{\pi}\varkappa_o m \int_{-\pi/2}^{\pi/2} \sec^2\theta \int_0^\infty \frac{e^{\varkappa(y-f)}\cos(\varkappa x\cos\theta)\cos(\varkappa z\sin\theta)}{\varkappa - \varkappa_o\sec^2\theta}\, d\varkappa\, d\theta$$
$$+ 2\varkappa_o m \int_{-\pi/2}^{\pi/2} e^{\varkappa_o(y-f)\sec^2\theta}\sin(\varkappa_o x\sec\theta)\cos(\varkappa_o z\sin\theta\sec^2\theta)\sec^2\theta\, d\theta \qquad \text{(I-1)}$$

where $\varkappa_o = g/c^2$, $r_1{}^2 = x^2+(y+f)^2+z^2$, and $r_2{}^2 = x^2+(y-f)^2+z^2$. The first term gives the sourcelike behavior at (0, -f, 0); the second term and the first integral (of which the principal value is to be taken) are a local symmetric disturbance; and the second integral is the superposition of free surface waves that make up the wake. The entire expression satisfies the linearized free-surface boundary condition, $\varphi_{xx} + \varkappa_o\varphi_y = 0$, on y = 0. Consequently derivatives and integrals of

[1] I has been pointed out to the writer that the submerged lifting singularity has been presented previously, first by Wu (Ref. 21, 1954) for the case of $\beta = 0$ in the present notation, and later by other authors for general β. The appendix is included because the approach is quite different from Wu's and because the present writer has not been able to locate another derivation.

H also satisfy this condition and can be used to construct solutions corresponding to different singularities at (0, -f, 0).

The fundamental lifting solution is a bound vortex element of infinitesimal span ϵ and infinite strength Γ such that $\Gamma\epsilon$ is constant and equal to the lift force L divided by ρc, the product of density and speed, according to the Kutta-Joukowsky lift theorem. To satisfy Helmholtz' theorems the bound vortex filament is continued into a



pair of free trailing vortices of strength Γ and separation ϵ, which extend parallel to the x-axis to infinity.

The horseshoe vortex flow is identical with the flow induced by a line doublet extending from (0, -f, 0) to infinity parallel to the x-axis, the doublet axis pointing opposite to the lift vector. This can be thought of as a pair of line sources of strength Γ and separation ϵ, or as a distribution of doublets of strength/unit length $\Gamma\epsilon$. The potential of one doublet at (0, -f, 0) having strength $m\epsilon$ and axis pointing opposite the lift direction is

$$m\epsilon D(x, y, z; 0, -f, 0; \beta) = m\epsilon \left(\sin\beta \frac{\partial}{\partial z} - \cos\beta \frac{\partial}{\partial y}\right) H(x, y, z; 0, -f, 0) \qquad \text{(I-2)}$$

Using the expression for H above (I-1),

$$\epsilon m D(x, y, z; 0, -f, 0; \beta) = -\frac{\epsilon m[z\sin\beta-(y+f)\cos\beta]}{r_1^3} + \frac{\epsilon m[z\sin\beta-(y-f)\cos\beta}{r_2^3}$$

$$+ \frac{2}{\pi}\varkappa_o \epsilon m \int_{-\pi/2}^{\pi/2} \sec^2\theta \int_0^\infty \frac{e^{\varkappa(y-f)}\cos(\varkappa x\cos\theta)}{\varkappa-\varkappa_o\sec^2\theta}$$

$$[\sin\beta\sin\theta\sin(\varkappa z\sin\theta) - \cos\beta\cos(\varkappa z\sin\theta)]\varkappa d\theta$$

$$- 2\varkappa_o^2\epsilon m \int_{-\pi/2}^{\pi/2} [\sin\beta\sin\theta\sin(\varkappa_o z\sin\theta\sec^2\theta)-\cos\beta\cos(\varkappa_o z\sin\theta\sec^2\theta)]$$

$$e^{\varkappa_o(y-f)\sec^2\theta}\sin(\varkappa_o x\sec\theta)\sec^4\theta d\theta. \qquad \text{(I-3)}$$

By changing coordinates it is easily established that

$$D(x, y, z; \xi, -f, 0; \beta) = D(x-\xi, y, z; 0, -f, 0; \beta) \qquad \text{(I-4)}$$

is the potential of a similar doublet at $(\xi, -4, 0)$.

The next step is to integrate over a distribution of such doublets along the line $y = -f$, $z = 0$. In order to obtain convergent integrals it is necessary to consider the desired uniform distribution as the limiting case of a broader family of distributions. A family of potentials is defined by

$$V_a(x, y, z; 0, -f, 0; \beta) \equiv \int_0^\infty e^{-a\xi} D(x, y, z; \xi, -f, 0) d\xi \qquad \text{(I-5)}$$

Then the desired vortex potential is

$$\frac{L}{\rho c} V(x, y, z; 0, -f, 0; \beta) = \frac{L}{\rho c} \lim_{a \to o} V_a(x, y, z; 0, -f, 0; \beta)$$

$$= -\frac{L}{\rho c} \frac{z \sin\beta - (y+f)\cos\beta}{(y+f)^2 + z^2} \left\{ 1 + \frac{x}{[x^2+(y+f)^2+z^2]^{\frac{1}{2}}} \right\}$$

$$+ \frac{L}{\rho c} \frac{z \sin\beta - (y-f)\cos\beta}{(y-f)^2 + z^2} \left\{ 1 + \frac{x}{[x^2+(y-f)^2+z^2]^{\frac{1}{2}}} \right\}$$

$$+ \frac{2}{\pi} \varkappa_o \frac{L}{\rho c} \int_{-\pi/2}^{\pi/2} \sec^2\theta \int_0^\infty \frac{e^{\varkappa(y-f)} \sin(\varkappa x \cos\theta) \sec\theta}{\varkappa - \varkappa_o \sec^2\theta}$$

$$[\sin\beta \sin\theta \sin(\varkappa z \sin\theta) - \cos\beta \cos(\varkappa z \sin\theta)]\, d\varkappa\, d\theta$$

$$+ 2\varkappa_o \frac{L}{\rho c} \int_{-\pi/2}^{\pi/2} [\sin\beta \sin\theta \sin(\varkappa_o z \sin\theta \sec^2\theta) - \cos\beta \cos(\varkappa_o z \sin\theta \sec^2\theta)]$$

$$e^{\varkappa_o(y-f)\sec^2\theta} \cos(\varkappa_o x \sec\theta) \sec^3\theta\, d\theta \qquad \text{(I-6)}$$

where the strength has been identified with $L/\rho c$. Again, this expression is to be regarded as composed of the submerged vortex represented in the first term, the local disturbance of the second and third terms, and the free wave pattern of the last term. The free wave pattern at a distance downstream can be calculated from the last term according to

$$h = -\frac{c}{g} \varphi_x \Big|_{y=o}$$

$$= -2\varkappa_o \frac{L}{\rho c^2} \sin\beta \int_{-\pi/2}^{\pi/2} \sin\theta \sec^4\theta\, e^{-\varkappa_o f \sec^2\theta}$$

$$\cos\{\varkappa_o \sec^2\theta (x\cos\theta + z\sin\theta)\}\, d\theta$$

$$-2\varkappa_o \frac{L}{\rho c^2} \cos\beta \int_{-\pi/2}^{\pi/2} \sec^4\theta\, e^{-\varkappa_o f \sec^2\theta}$$

$$\sin\{\varkappa_o \sec^2\theta (x\cos\theta + z\sin\theta)\}\, d\theta \qquad \text{(I-7)}$$

Using this result the wave resistance can be calculated by the methods of Havelock (Ref. 19). It is interesting to note that for the case $\beta = 0$ (lift vector vertical) the wave pattern and resistance are identical with those of a submerged sphere whose volume is $2L/3\kappa_0 \rho c^2$.

APPENDIX II

MODIFIED METHOD OF STATIONARY PHASE

This appendix deals with the approximate evaluation of integrals of the type

$$u = \int_a^b \varphi(x)\, e^{if(x)}\, dx \tag{II-1}$$

with special attention to cases in which $\varphi(x)$ has a zero at a stationary point of $f(x)$. The notation and treatment are similar to Lamb's (Ref. 5, Art. 241). $\varphi(x)$ and $f(x)$ are required to be analytic at a stationary point α such that $f'(\alpha) = 0$. Then, writing $\xi = x - \alpha$, we have

$$f(x) = f(\alpha) + \frac{1}{2}\xi^2 f''(\alpha) + \frac{1}{6}\xi^3 f'''(\alpha) + \cdots \tag{II-2}$$

$$\varphi(x) = \xi\varphi'(\alpha) + \frac{1}{2}\xi^2\varphi''(\alpha) + \frac{1}{6}\xi^3\varphi'''(\alpha) + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \left.\frac{d^n\varphi}{dx^n}\right|_{x=\alpha} \xi^n \tag{II-3}$$

since $\varphi(\alpha)$ is assumed zero.

Provided the quotient $f'''(\alpha)/|f''(\alpha)|^{\frac{3}{2}}$ is small, so that the third term in (II-2) can be neglected, the important part of the integral, coming from the neighborhood of α, is approximately

$$e^{if(\alpha)} \sum_{n=1}^{\infty} \frac{1}{n!}\varphi^{(n)}(\alpha) \int_{-\infty}^{\infty} \xi^n e^{\frac{1}{2}if''(\alpha).\,\xi^2}\, d\xi \tag{II-4}$$

For odd n the integral in (II-4) vanishes by virtue of the integrand being an odd function of ξ. For even n we have the integral, for a a positive integer,

$$\int_{-\infty}^{\infty} \xi^{2a}\, e^{\pm i m^2 \xi^2}\, d\xi = \left(\pm \frac{1}{2i}\right)^a \sqrt{\pi}\, e^{\pm i\pi/4}\, \frac{1\cdot 3\cdot 5 \cdot \cdot \cdot (2a-1)}{m^{2a+1}} \qquad \text{(II-5)}$$

which is established by repeated differentiations of

$$\int_{-\infty}^{\infty} e^{\pm i m^2 \xi^2}\, d\xi = \frac{\sqrt{\pi}}{m}\, e^{\pm i\pi/4} \qquad \text{(II-6)}$$

with respect to m. The principal contribution to (II-4), then, is from the lowest even derivative of φ that does not vanish at α. If this is the derivative of order 2a, we have by (II-5)

$$u \sim \frac{1}{(\pm 2i)^a a!}\, \frac{\sqrt{\pi}}{|f''(\alpha)|^{a+\frac{1}{2}}}\, e^{i[f(\alpha) \pm \pi/4]}\, \varphi^{(2a)}(\alpha) \qquad \text{(II-7)}$$

where the ± sign is taken according to the sign of $f''(\alpha)$.

If α coincides with one of the limits of integration in (II-1), the limits in (II-4) must be 0 to ∞ or $-\infty$ to 0. In that case we use the formula for odd n

$$\int_0^{\infty} \xi^{2a+1}\, e^{\pm i m^2 \xi^2}\, d\xi = \frac{1}{2}\int_0^{\infty} u^a\, e^{\pm i m^2 u}\, du = \frac{1}{2}\left(\pm \frac{1}{i}\right)^{a+1} \frac{a!}{m^{2a+2}} \qquad \text{(II-8)}$$

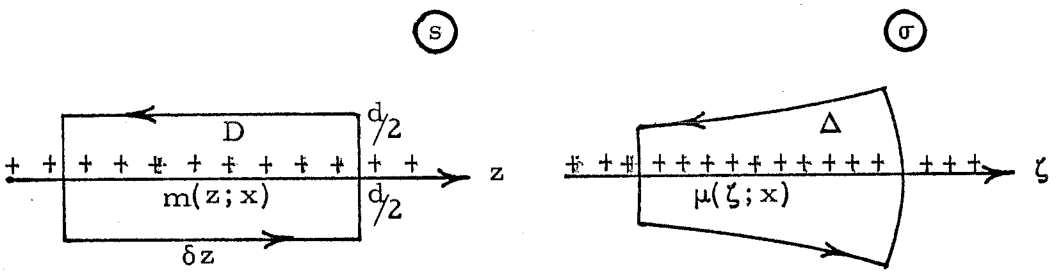
The principal contribution to (II-4) is now from the lowest derivative of φ that does not vanish at α. If this derivative is <u>even</u> the result is half of (II-7). If the lowest nonvanishing derivative is odd, the result is

$$u \sim \left(\pm \frac{1}{i}\right)^{a+1} \frac{2^a\, a!}{(2a+1)!}\, \frac{e^{if(\alpha)}}{|f''(\alpha)|^{a+1}}\, \varphi^{(2a+1)}(\alpha). \qquad \text{(II-9)}$$

APPENDIX III

MAPPING OF A SOURCE DISTRIBUTION

Under a conformal transformation a source of finite strength located at a conformal point of the mapping goes into an equal source at the mapped point (Milne-Thompson, Ref. 12). To see how a continuous source distribution along the real axis, of strength m(z;x), is mapped, consider a small rectangular region near the z-axis -- a two-dimensional "pillbox" of height d and length δz.



D maps into the closed region Δ in the σ-plane, symmetric about the ζ-axis. The net efflux through ∂D is $\oint_{\partial D} d\psi = m(z;x)\,\delta z + O(\delta z^2)$. Since the stream-function ψ has the same value at corresponding points in the two planes, the integral $\oint_{\partial \Delta} d\psi$ must be also equal to $m(z;x)\,\delta z + O(\delta z^2)$. Now the height d can be made arbitrarily small, in which case the boundary $\partial\Delta$ converges toward the segment $\delta\zeta$ while the integral $d\psi$ is constant. This shows there are sources along the ζ-axis, say with the strength $\mu(\zeta;x)$ where $\zeta = \gamma(z;x)$, such that

$$m(z;x)\delta z = \mu[\gamma(z;x);x]\delta\zeta + O(\delta z^2)$$

Now taking the limit as $\delta z \to 0$,

$$m(z;x) = \mu[\gamma(z;x);x]\,\gamma_s(z;x) \tag{III-1}$$

Here the subscript partial derivative notation is extended to derivatives of the mapping functions with respect to their complex arguments, just as in (6-12) it was used for a derivative with respect to x. By substitution of $z = g(\zeta;x)$ and use of the identity $\sigma = \gamma[g(\sigma;x);x]$ and its derivative with respect to σ, $1 = \gamma_s[g(\sigma;x);x]g_\sigma(\sigma;x)$, (III-1) is put in the form

$$\mu(\zeta;x) = m[g(\zeta;x);x]\,g_\sigma(\zeta;x). \tag{III-2}$$

APPENDIX IV

PLANAR WING: INTEGRATION OVER SOURCES.

The following integrals are required:

$$I_1(\zeta) = \int_a^\infty \frac{1}{(\xi^2 - a^2)^{\frac{1}{2}}} \log\left(\frac{\xi-\zeta}{\xi+\zeta}\right) d\xi$$

$$I_2(\zeta) = \int_a^\infty \frac{a^2}{\xi^2(\xi^2 - a^2)^{\frac{1}{2}}} \log\left(\frac{\xi-\zeta}{\xi+\zeta}\right) d\xi$$

$$I_3(\zeta) = \int_a^\infty \frac{a^4}{\xi^4(\xi^2 - a^2)^{\frac{1}{2}}} \log\left(\frac{\xi-\zeta}{\xi+\zeta}\right) d\xi$$

$$I_4(\zeta) = \int_a^\infty \frac{a^3}{\xi^3(\xi^2 - a^2)^{\frac{1}{2}}} \log(\xi^2 - \zeta^2)\, d\xi$$

$$I_5(\zeta) = \int_a^\infty \frac{a}{\xi(\xi^2 - a^2)^{\frac{1}{2}}} \log(\xi^2 - \zeta^2)\, d\xi$$

A useful transformation is:

$$\xi = a \sec\theta \quad , \quad d\xi = a\sec\theta\tan\theta\, d\theta$$

$$(\xi^2 - a^2)^{\frac{1}{2}} = a\tan\theta \qquad \left[\begin{array}{l} a = a\sec 0 \\ \infty = a\sec\frac{\pi}{2} \end{array}\right.$$

We also use $\zeta = qa \qquad |q| < 1$.

Then $I_1 - I_5$ become:

$$I_1(qa) = \int_0^{\pi/2} \frac{1}{\cos\theta} \log\left(\frac{1 - q\cos\theta}{1 + q\cos\theta}\right) d\theta$$

$$I_2\ (qa) = \int_0^{\pi/2} \cos\theta \log\left(\frac{1-q\cos\theta}{1+q\cos\theta}\right) d\theta$$

$$I_3\ (qa) = \int_0^{\pi/2} \cos^3\theta \log\left(\frac{1-q\cos\theta}{1+q\cos\theta}\right) d\theta$$

$$I_4\ (qa) = \int_0^{\pi/2} \cos^2\theta \log a^2 (\sec^2\theta - q^2)\, d\theta$$

$$I_5\ (qa) = \int_0^{\pi/2} \log a^2(\sec^2\theta - q^2)\, d\theta$$

I_1 is tabulated in Dwight (Ref. 13) No. 865.37:

$$I_1(qa) = -\pi \sin^{-1} q\ .$$

I_2 is first written

$$I_2 = \int_0^{\pi/2} \cos\theta \log(1-q\cos\theta)\, d\theta - \int_0^{\pi/2} \cos\theta \log(1+q\cos\theta)\, d\theta\ ,$$

then in the second integral $\pi = \theta$ is substituted for θ :

$$I_2 = \int_0^{\pi} \cos\theta \log(1-q\cos\theta)\, d\theta\ .$$

Now substituting $q = \frac{2k}{1+k^2}$ ($k < 1$ since $q < 1$)

$$I_2 = -\log(1+k^2) \int_0^{\pi} \cos\theta\, d\theta + \int_0^{\pi} \log(1-2k\cos\theta + k^2)\cos\theta\, d\theta.$$

The first integral vanishes, and the second is tabulated in Dwight, No. 865.74 with $m = 1$:

$$I_2(qa) = -\pi k = -\pi \frac{1-\sqrt{1-q^2}}{q}$$

I_3 is similarly put in the form

$$I_3 = \int_0^{\pi} \cos^3\theta \log(1-q\cos\theta)\, d\theta$$

$$= \int_0^{\pi} \log(1-2k\cos\theta + k^2) \cos^3\theta\, d\theta$$

and, using $\cos^3\theta = \frac{1}{4}\cos^3\theta + \frac{3}{4}\cos\theta$,

$$I_3 = \frac{3}{4}\int_0^{\pi} \log(1-2k\cos\theta + k^2)\cos\theta\, d\theta + \frac{1}{4}\int_0^{\pi} \log(1-2k\cos\theta + k^2)\cos^3\theta\, d\theta.$$

Again Dwight, No. 865.74 with $m = 1, 3$:

$$I_3(qa) = \frac{-\pi k}{12}(9 + k^2) = -\pi \frac{2 + 3q^2 - 2(1+2q^2)\sqrt{1-q^2}}{6q^3}.$$

I_5 is broken up as

$$I_5 = \int_0^{\pi/2} 2\log a\, d\theta + \int_0^{\pi/2} \log(1-q^2\cos^2\theta)\, d\theta - \int_0^{\pi/2} 2\log\cos\theta\, d\theta .$$

Using Dwight No. 865.34 (with $p = -q^2$) and No. 865.11 :

$$I_5 = \pi\log a + \pi\log(1 + \sqrt{1-q^2}) .$$

I_4 is split up similarly :

$$I_4 = 2\log a \int_0^{\pi/2} \cos^2\theta\, d\theta - 2\int_0^{\pi/2} \cos^2\theta \log(\cos\theta)d\theta + \int_0^{\pi/2} \cos^2\theta \log(1-q^2\cos^2\theta)d\theta$$

Using 865.25 on the second,

$$I_4 = \frac{\pi}{2}\log a - \frac{\pi}{4}(1-2\log 2) + \int_0^{\pi/2} \cos^2\theta \log(1-q^2\cos^2\theta)d\theta\,.$$

The last integral is called

$$I_{4a} = \frac{1}{2}\int_0^{\pi} \cos^2\theta \log(1-q^2\cos^2\theta)\, d\theta$$

Now using $\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$,

$$I_{4a} = \frac{1}{4}\int_0^{\pi} \log(1-q^2\cos^2\theta)d\theta + \frac{1}{4}\int_0^{\pi}\cos 2\theta \log(1-q\cos\theta)d\theta + \frac{1}{4}\int_0^{\pi}\cos 2\theta \log(1+q\cos\theta)\, d\theta$$

The first term is evaluated by Dwight No. 865.34:

$$\frac{\pi}{2}\log\frac{1+\sqrt{1-q^2}}{2}\,.$$

The second and third are put in tabulated form by substituting $q = \frac{2k}{1+k^2}$, $q = -\frac{2k}{1+k^2}$ respectively:

$$\frac{1}{4}\int_0^{\pi} \log(1-2k\cos\theta + k^2)\cos 2\theta\, d\theta + \frac{1}{4}\int_0^{\pi} \log(1-2k\cos\theta + k^2)\cos 2\theta\, d\theta$$

which are in the form of Dwight No. 865.74 .

$$-\frac{\pi}{4}k^2 = -\pi\,\frac{2-q^2-2\sqrt{1-q^2}}{4q^2}$$

So finally, collecting terms,

$$I_4(qa) = \frac{\pi}{2}\log a - \frac{\pi}{4} + \frac{\pi}{2}\log\left(1+\sqrt{1-q^2}\right) - \frac{\pi}{4}\;\frac{2-q^2-2\sqrt{1-q^2}}{q^2} .$$

APPENDIX V

PLANAR WING: SPANWISE INTEGRATION

Two of the integrals are tabulated: Dwight (Ref.13) Nos. 520., 320.01, 321.01

$$\int_0^a \cos^{-1}\frac{y}{a}\,dy = a$$

$$\int_0^a \frac{2a-3y}{\sqrt{a^2-y^2}}\,dy = (\pi-3)\;a$$

The remaining two are

$$J_1 = \int_0^a \frac{y^2\,dy}{(a+y)\sqrt{a^2-y^2}} = a\int_0^{\pi/2} \frac{\sin^2\theta\,d\theta}{1+\sin\theta}$$

$$J_2 = \int_0^a \frac{(2ay-y^2)\,dy}{(a+y)^2\sqrt{a^2-y^2}} = \int_0^{\pi/2} \frac{(2\sin\theta - \sin^2\theta)\,d\theta}{(1+\sin\theta)^2}$$

where the latter forms have been obtained by substituting $y = a\sin\theta$.

J_1 is multiplied inside, top and bottom, by $1 - \sin\theta$:

$$J_1 = a\int_0^{\pi/2} \frac{(\sin^2\theta - \sin^3\theta)\,d\theta}{\cos^2\theta} .$$

With the help of Dwight Nos. 452.22, 452.32

$$J_1 = a\left[\tan\theta - \theta - \cos\theta - \sec\theta\right]_0^{\pi/2} = (2 - \frac{\pi}{2})\, a$$

since $\tan(\frac{\pi}{2} - \epsilon) - \sec(\frac{\pi}{2} - \epsilon) = \cot\epsilon - \csc\epsilon = O(\epsilon)$.

J_2 is treated similarly, being multiplied inside, top and bottom, by $(1-\sin\theta)^2$:

$$J_2 = \int_0^{\pi/2} \frac{2\sin\theta - 5\sin^2\theta + 4\sin^3\theta - \sin^4\theta}{\cos^4\theta}\, d\theta$$

Then, using Dwight Nos. 452.14, 452.24, 452.34, 480.4:

$$J_2 = \left[\tan\theta - 4\sec\theta - 2\tan^3\theta + 2\sec^3\theta\right]_0^{\pi/2} - \frac{\pi}{2} = 2 - \frac{\pi}{2}$$

Since $\cot^3\epsilon = \frac{1}{\epsilon^3} - \frac{1}{\epsilon} + O(\epsilon)$, $\csc^3\epsilon = \frac{1}{\epsilon^3} + \frac{1}{2\epsilon} + O(\epsilon)$,

$\cot\epsilon = \frac{1}{\epsilon} + O(\epsilon)$ and $\csc\epsilon = \frac{1}{\epsilon} + O(\epsilon)$, the quantity in brackets is $O(\epsilon)$ at $\frac{\pi}{2} - \epsilon$, and the integral is finite.

APPENDIX VI

BODY OF REVOLUTION: INTEGRATION OVER SOURCES

The following integrals are required:

$$I_1 = \int_{2a}^{\infty} \frac{1}{(\xi^2 - 4a^2)^{\frac{1}{2}}} \log \frac{\xi - \zeta}{\xi + \zeta}\, d\xi\ .$$

$$I_2 = \int_{2a}^{\infty} \frac{a^3\, d\xi}{(\xi^2 - 4a^2)^{\frac{1}{2}} \left[\xi + (\xi^2 - 4a^2)^{\frac{1}{2}}\right](\xi^2 - \zeta^2)}$$

$$I_3 = \int_{2a}^{\infty} \frac{a^5 \, d\xi}{(\xi^2-4a^2)^{\frac{1}{2}} \left[\xi^2-4a^2)^{\frac{1}{2}}\right]^3 (\xi^2-\zeta^2)}$$

I_1 is identical with the I_1 of Appendix , except for $2a$ replacing a. Consequently,

$$I_1 = -\pi \sin^{-1} \frac{\zeta}{2a} .$$

I_2 and I_3 are transformed by

$$\xi = 2a \csc \theta , \qquad d\xi = -2a \csc \theta \cot \theta \, d\theta$$

$$(\xi^2-4a^2)^{\frac{1}{2}} = 2a \cot \theta$$

We also use $q \equiv \zeta/2a$. $|q| < 1$.

Then the integrals become

$$I_2 = \frac{1}{8} \int_0^{\pi/2} \frac{(1-\cos\theta)\, d\theta}{1-q^2 \sin^2\theta}$$

$$I_3 = \frac{1}{32} \int_0^{\pi/2} \frac{(1-\cos\theta)^3 \, d\theta}{\sin^2\theta (1-q^2 \sin^2\theta)}$$

I_2 is written in two parts, and in the second the substitution $u = \sin\theta$ is made:

$$I_2 = \frac{1}{8} \int_0^{\pi/2} \frac{d\theta}{1-q^2 \sin^2\theta} - \frac{1}{8} \int_0^{1} \frac{du}{1-q^2 u^2}$$

Using Dwight (Ref. 13) Nos. 858.541, 140.02

$$I_2 = \frac{\pi a}{8\sqrt{4a^2 - \zeta^2}} - \frac{a}{8\zeta} \log \frac{2a + \zeta}{2a - \zeta}$$

I_3 is expanded as follows:

$$I_3 = \frac{1}{32} \int_0^{\pi/2} \left[\frac{4}{\sin^2 \theta} + \frac{-3+4q^2}{1-q^2 \sin^2 \theta} + \frac{(-4+\sin^2 \theta)\cos \theta}{\sin^2 \theta (1-q^2 \sin^2 \theta)} \right] d\theta$$

$$= \frac{-3+4q^2}{32} \int_0^{\pi/2} \frac{d\theta}{1-q^2 \sin^2 \theta} + \frac{1}{8} \int_0^{\pi/2} \frac{d\theta}{\sin^2 \theta} + \frac{1}{32} \int_0^1 \frac{(-4+u^2)du}{u^2(1-q^2u^2)}$$

where $u = \sin \theta$ has been used in the last. The last two terms have infinities at the lower limit which cancel each other off, as they must; since the integrand of I_3 is certainly regular even at $\theta = 0$. Hence, using Dwight Nos. 140.02, 152.1, and 432.20 :

$$I_3 = \frac{1}{8} - \frac{3a^2 - \zeta^2}{32a} \frac{\pi}{\sqrt{4a^2 - \zeta^2}} - \frac{1}{32} \frac{\zeta^2 - a^2}{a\zeta} \log \frac{2a + \zeta}{2a - \zeta}$$

APPENDIX VII

BODY OF REVOLUTION: CIRCUMFERENTIAL INTEGRALS

All the integrals are straightforward, excepting

$$J \equiv \int_0^{\pi/2} \sin^2 \theta \cos\theta \,(3 - 8\cos^2 \theta) \log \frac{1+\cos\theta}{1-\cos\theta} \, d\theta \,.$$

First the $\sin^2 \theta$ is replaced by $1 - \cos^2 \theta$:

$$J = \int_0^{\pi/2} (3\cos\theta - 11\cos^3\theta + 8\cos^5\theta)\log\frac{1+\cos\theta}{1-\cos\theta}\,d\theta$$

$$= \int_0^{\pi/2} (3\cos\theta - 11\cos^3\theta + 8\cos^5\theta)\log(1+\cos\theta)\,d\theta$$

$$- \int_0^{\pi/2} (3\cos\theta - 11\cos^3\theta + 8\cos^5\theta)\log(1-\cos\theta)\,d\theta$$

Now in the second integral θ is replaced by $\pi - \theta$, and the integral assumes the form

$$J = \int_0^{\pi} (3\cos\theta - 11\cos^3\theta + 8\cos^5\theta)\log(1+\cos\theta)\,d\theta\,.$$

$$= \int_0^{\pi} (3\cos\theta - 11\cos^3\theta + 8\cos^5\theta)\log(2+2\cos\theta)\,d\theta$$

$$- \log 2 \int_0^{\pi} (3\cos\theta - 11\cos^3\theta + 8\cos^5\theta)\,d\theta$$

The second integral is zero. By writing thepowers of $\cos\theta$ in terms of the cosines of multiple angles:

$$J = \int_0^{\pi} \left(-\tfrac{1}{4}\cos\theta - \tfrac{1}{4}\cos 3\theta + \tfrac{1}{2}\cos 5\theta\right)\log(2+2\cos\theta)\,d\theta$$

the integral is put in the form of Gröbner and Hofreiter No. 338.13a (ref. 20) with $r = 1$:

$$J = -\tfrac{1}{4}(-\pi) - \tfrac{1}{4}\left(-\frac{\pi}{3}\right) + \tfrac{1}{2}\left(-\frac{\pi}{5}\right) = \frac{7\pi}{30}\,.$$

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