I. TIME-DEPENDENT ACCRETION DISKS AROUND COMPACT OBJECTS
and
II. THEORETICAL FRAMEWORKS FOR ANALYZING AND TESTING GRAVITATION THEORIES

thesis by

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NOTE BY THE AUTHOR

Because of a logistics problem, the "theoretical frameworks for analyzing relativistic theories of gravity" portion of the thesis occurs in Part II. while the "analysis (Newtonian) of time-dependent accretion disks" portion occurs in Part I. In actuality, the research for Part II. was undertaken and completed first. The explanation is as follows: After beating his head against the wall in the process of probing the inner recesses of modern-day gravitation theories (in an effort to isolate the correct theory of gravity), the author inadvertently stumbled upon Newton's theory. Although Newtonian theory has a few slight disagreements with solar-system experiments, it lends itself to calculation somewhat more than most "modern" theories of gravitation (e.g., General Relativity) and was therefore cheerfully and immediately snatched up by the author in his research, eventually resulting in Part I. of the thesis.

Omnia Tristia

Post Eggplantum Sunt

ABSTRACT

Part I.

The theory of time-independent accretion disks around compact objects is developed, generalizing the stationary models of various authors to allow time dependence on the radial-flow time scale. Equations are derived for the time evolution of matter surface density Σ and for implicit expressions of relevant disk variables in terms of Σ. Analytic and numerical studies of these equations yield numerical models of mass accretion from a disk onto a compact object and a discovery of the unstable nature of the "inner region" of the disk, causing a breakdown of current accretion disk models.

Part II.

Theoretical frameworks for analyzing and testing gravitation theories are developed for both nonmetric and metric theories. Highly precise experimental confirmation of the Weak Equivalence Principle is shown to be deadly if not fatal evidence for ruling out all nonmetric theories of gravity. For the class of metric theories we demonstrate the necessity for going beyond current frameworks of analysis (e.g., the PPN framework) by constructing a new theory of gravity identical to GRT in the Post-Newtonian limit. As a first step in transcending current frameworks, we develop a formalism for delineating and testing all metric theories of gravity on the basis of their gravitational-wave properties and thereby emphasize gravitational-wave observations as a future tool for testing gravitation theories. We also investigate conservation laws and some common properties of Lagrangian-based metric theories of gravity.

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INTRODUCTION

INTRODUCTION TO PART I.

Accretion disks are currently thought to play important roles in various binary star systems, either powering emission or regulating mass flow onto a central compact object (e.g. a white dwarf in U-Gem systems, a neutron star in Her X-1, and probably a black hole in Cyg X-1).

The detailed accretion disk models of Pringle and Rees (1972), Shakura and Sunyaev (1973) and Novikov and Thorne (1973) are time-independent; they assume a constant (in time and radius) mass flux through the disk, deposited near its outer edge by the companion star. It is of considerable interest to also study the properties of time-dependent disks. Such a study can investigate, among other problems, the stability of time-independent disks and variability of mass deposition rate as a possible explanation for various phenomena.

In this Part of the thesis we develop the theory of time-dependent disks (allowing variables to change on the radial flow time scale), and we construct numerical models, with speculative applications to astrophysical systems. In Section A (Paper I) we give a brief overview of the most important results: implications for black holes of an instability in the inner region of the disk. Basic equations governing the structure and evolution of the disk are set forth in Section B (Paper II). The time evolution equation for disk surface density (the sole independent variable characterizing the disk) is solved numerically in Section C (Paper III). Also discussed in this Section is an instability of the "inner region" of the disk -- an instability causing a breakdown of current accretion disk models.

INTRODUCTION TO PART II.

Several years ago Thorne's research group initiated (Thorne and Will 1971) a project of constructing theoretical foundations for experimental tests of gravitation theories. In the first couple of years of that project (with principal contributions made by Will and Ni at Caltech and Nordtvedt at Montana State) it was found very fruitful to restrict attention to metric theories of gravity, for which it was possible to develop a simple and elegant formalism (PPN) delineating the significant differences of such theories for purposes of typical solar-system experiments.

At first it was hoped that each metric theory of gravity was uniquely characterized by its form in the PPN limit -- solar system experiments could then home in on the "correct" theory (if that theory were in the class of metric theories). Counter examples, however, soon began to arise (one of which is in Section C) -- different theories with identical forms in the Post-Newtonian limit. It therefore became apparent that theoretical formalisms and experimental tests extending beyond the Post-Newtonian regime had to be developed to probe more deeply into those theories which agreed with each other and with the data in conventional solar-system experiments.

New investigations also opened upon another front -- the murky morass of nonmetric theories. Nonmetric theories of gravity, in the literature for more years and with greater fecundity than their metric counterparts, did and do not have the simplifying feature of metric theories: identical equations of motion for matter in a given gravitational field (represented by the metric). It appeared that each nonmetric theory had to be treated on an individual basis, with no possibility of a systematic formalism such as PPN to encompass the entire class of theories. Then a crucial idea, first conjectured by Leonard Schiff around 1960 and then vigorously reasserted by Thorne a decade later -- that nonmetric theories might violate the experimentally verified (Dicke et al. 1964, Braginsky and Panov 1971) Weak Equivalence Principle (WEP) -- offered a new foothold and column of attack into nonmetric theories as a class of theories. Along those lines, we began our investigations.

The two fronts of investigations referred to above -- a formalism beyond PPN for comparing metric theories of gravity, and a consideration of WEP as a tool for ruling out nonmetric theories -- constitute the principal subject matter of this Part of the thesis.

Since our investigations have led us through the realms of both metric and nonmetric theories, we require some foundations and common ground of definitions, basic concepts, and viewpoints from which to discuss gravitation theories in general. Such a foundation is the subject of Section A (Paper IV). In this Section we also present a theorem delineating the overlap between Lagrangian-based theories and metric theories, a theorem whose results are exploited in Paper X of Section C.

In Section B we give a treatment of the Equivalence Principles (Weak and Einstein) and their implications for gravitation theories. More specifically our treatment consists of a partial proof that the Weak Equivalence Principle is consistent exclusively with metric theories of gravity (Paper V), and a detailed analysis (Paper VI) of a particular nonmetric theory, the Belinfante-Swihart (1957) theory, which is a prototype of theories ruled out by experimental verification of WEP. In actuality we first studied the Belinfante-Swihart theory in order to build up our intuition and understanding of nonmetric theories. With some knowledge of the possible structure of such theories, we then studied other similar theories and slowly catalyzed the approach and methodology to be used in our partial proof of the "Schiff Conjecture." Beginning with complete ignorance in the complex class of nonmetric theories, our program was largely one of trial and error.

In Section C we turn to metric theories and our second front of investigation. As an outgrowth of our study of the Belinfante-Swihart theory and in an attempt to construct a theory with the same Post-Newtonian form as General Relativity, we devised a new Lagrangian-based metric theory of gravity (Paper VII). The aims of such a construction were to determine how contrived such a theory would necessarily be and to identify possible new theoretical and experimental tools for testing relativistic gravity. In the course of our analysis of the new theory, two key ideas emerged: (1) the presence of cosmological effects on local scales in theories with "prior geometry" and (2) the richness of gravitational-wave structure in metric theories of gravity. (Our new theory turned out to have the most general gravitational wave possible in metric theories of gravity.)

Further investigation of (2) above resulted in a formalism for delineating all possible polarizations of gravitational waves in metric theories and a systematic procedure for cataloguing theories on the basis of their gravitational-wave structure (Papers VIII and IX). Central to the analysis is a discussion of experimentally testing such theories on the basis of their gravitational-wave properties.

All of the theories of gravity which are now viable competitors with General Relativity are Lagrangian-based metric theories. In Paper X we analyze some of the common properties of such theories and, in particular, focus upon the conservation laws which are usually found in their structure.

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I. TIME-DEPENDENT ACCRETION DISKS AROUND COMPACT OBJECTS

A. Overview of Most Important Results -- Black Holes in Binary Systems: Instability of Disk Accretion (Paper I; collaboration with D.M. Eardley, published in Astrophys. J. Lett., **187**, L1, 1974)

BLACK HOLES IN BINARY SYSTEMS: INSTABILITY OF DISK ACCRETION*

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ABSTRACT

We have tested the stability of a thin, orbiting accretion disk near a black hole. Under conditions appropriate for a binary X-ray source, with the usual (ad hoc) assumptions about viscosity, the disk is always secularly unstable on timescales of a few seconds or less. Therefore current thin-disk models for such X-ray sources are self-inconsistent. We mention possibilities for alternative models; perhaps the secular instability explains chaotic time-variations in Cygnus X-1.

Current models [Pringle and Rees (1972), Shakura and Syunyaev (1973), Novikov and Thorne (1973)] for binary X-ray sources powered by accretion onto a black-hole companion envisage the gas flow near the hole as either a thin, orbiting disk or a thick, perhaps chaotic cloud. If the X-ray luminosity L exceeds the Eddington limit, $L^{ED} \sim (10^{38}$ ergs/sec$)$ $(M_{BH}/M_{\odot})$, where $M_{BH} \equiv$ mass of black hole, then the cloud picture is more likely. Moreover, even at luminosities somewhat lower than the Eddington limit, say $L \gtrsim 10^{-2} L^{ED}$ (all figures quoted will be for typical parameters of accretion models), thermal instabilities caused by optical thinness (Pringle et al. 1973) may disrupt the inner region of a thin disk, transforming it into a thick cloud. We wish to point out in this Letter that, with the usual (ad hoc) assumption about the viscosity, detailed thin-disk models are always secularly unstable over the whole "inner region" (that region where radiation pressure dominates gas pressure, $P_R > P_G$, and the dominant opacity is electron scattering). Such an inner region exists near the hole when $L \gtrsim 10^{-4} L^{ED}$. Therefore these models are inconsistent. The observational consequences are great since most of the X-ray luminosity originates in the inner region.

The current thin-disk models [Pringle and Rees (1972), Shakura and Syunyaev (1973), Novikov and Thorne (1973)] are stationary and include two key assumptions:

(a) Accreting matter forms a thin, orbiting, non-selfgravitating disk drifting inward on a slow timescale t_{drift} (slow compared to thermal and Kepler timescales). The drift is caused by viscous stress removing angular momentum.

(b) Although the viscous stress $t_{\hat{\varphi}\hat{r}}$ arises from intricate processes (e.g., turbulent motions on fast timescales, or magnetic fields), it may be approximated on slow timescales $\sim t_{drift}$ and longer by

$$t_{\hat{\varphi}\hat{r}} = \alpha P_{tot} , \tag{1}$$

where $P_{tot} = P_R + P_G$ and α is a number believed to lie between 10^{-3} and 1.

To investigate stability of the above models we generalize them to allow time-dependence in the radial disk structure on the slow timescale t_{drift} (a few seconds at the outer edge of the inner region; a few milliseconds at the inner edge). We shall sketch the development here. For a complete discussion of the stationary models, see Novikov and Thorne (1973), Shakura and Syunyaev (1973). For a complete discussion of the time-dependent generalization, see Lightman (1974).

Variables describing the local, instantaneous state of the disk are surface density $\Sigma(r,t)$ (g cm^{-2}), total inward mass flux $\dot{M}(r,t)$ (g sec^{-1}), mean half-thickness $h(r,t)$ (cm), mean pressure $P(r,t)$, mean temperature $T(r,t)$, radiative flux $F(r,t)$ (erg cm^{-2} sec^{-1}) from top of disk (= same from bottom), and vertically-integrated viscous stress $W(r,t) \approx 2h\, t_{\hat{\varphi}\hat{r}}$ (dyne cm^{-1}) (means are vertical averages). The structural equations relating these variables (ignoring relativistic corrections) are:

Equations of radial structure:

$$2\pi r \frac{\partial \Sigma}{\partial t} = \frac{\partial \dot{M}}{\partial r} \qquad \text{(conservation of mass)}, \tag{2a}$$

$$\frac{d(\Omega r^2)}{dr} \dot{M} = \frac{\partial}{\partial r} (2\pi r^2 W) \quad \text{(conservation of angular momentum)}; \tag{2b}$$

here $\Omega \equiv (GM_{BH}/r^3)^{\frac{1}{2}}$. Equations (2) are exact.

Equations of vertical structure (specialized to inner region):

$$F = \frac{3}{4} \Omega W \quad \text{(conservation of dissipated energy)}, \tag{3a}$$

$$F = \frac{2}{3}\, ac\, T^4/(\kappa_{Compt}\, \Sigma) \text{ (vertical radiative diffusion, Compton opacity)}, \tag{3b}$$

$$P = \frac{1}{2}\, h\, \Omega^2 \Sigma \text{ (vertical pressure balance against out-of-plane gravitational forces of black hole)}, \tag{3c}$$

$$P = P_R \equiv \frac{1}{3}\, aT^4 \text{ (equation of state, } P_R \gg P_G\text{)}, \tag{3d}$$

$$W = 2\alpha\, h\, P \text{ (source of viscosity, equation (1))}. \tag{3e}$$

Equations (3) are only approximate, because of uncertainties in averaging over vertical structure.

The stationary models are obtained by setting $\partial\Sigma/\partial t \equiv 0$ in equations (2), (3).

For time-dependent models, it is best to choose $\Sigma(r,t)$ as the sole independent variable characterizing the local, instantaneous state of the disk. Then, at each (r,t), one solves equations (3) algebraically for h, P, T, F, and W as functions of (Σ,r). It is essential to determine $W(r,t)$ self-consistently in this way, rather than to fix W through equation (2b) from a given $\dot{M}$, as one does in the stationary case. Equations (2) yield, as the evolution equation of $\Sigma(r,t)$,

$$\frac{\partial\Sigma}{\partial t} = \frac{\partial}{\partial r}\left[\frac{d(\Omega r^2)}{dr}\right]^{-1} \frac{\partial}{\partial r}\left[r^2 W(\Sigma,r)\right] . \tag{4}$$

The instability arises in the inner region for the following reason: Equations (3) give

$$W(\Sigma,r) = \text{const.}/\Sigma . \tag{5}$$

(To justify this paradoxical result: Since $P_R \gg P_G$, P is not determined directly by ρ ($\rho \equiv \Sigma/2h$), but only by T; and in fact T and P turn out to be independent of Σ. Equation (3c) implies $h \propto \Sigma^{-1}$; then equation (3e) shows $W \propto \Sigma^{-1}$.) The integrated stress W is here a <u>decreasing</u> function of Σ; hence the nonlinear diffusion equation for Σ, equation (4), has a <u>negative</u> effective diffusion coefficient. As a result an initially stationary disk tends to break up into rings $\Delta r \gtrsim h$, on timescales $\sim (\Delta r/r)^2\, t_{drift}$; alternate rings have high-Σ/low-W and low-Σ/high-W. The density contrast grows because matter is pushed into regions of minimum viscous stress W. Eventually the low-Σ regions become optically thin and hence thermally unstable (Pringle <u>et al.</u> 1973). As Σ grows in the high-Σ regions, eventually a regime is reached in which the disk cannot radiate as much energy as it is generating and the vertical structure equations fail to admit a solution. Therefore the growing instability causes a complete breakdown in the thin-disk picture, assumption (a). These conclusions are supported by detailed analytic and numerical calculations which one of us (APL) will report elsewhere (Lightman 1974).

Definitive models must therefore await a better understanding of viscosity: we mention two quite distinct possible alternatives to current models:

(1) Assumption (a) fails because (b) is roughly correct. Around the hole forms a cloud, which is 10 to 100 times larger than the hole. If dissipation is efficient (expected, since accreting matter must still lose its angular momentum), the cloud may emit X-rays as a hot, thin plasma with Comptonization probably important [Felten and Rees (1972), Illarionov and Syunyaev (1972)]. Alternatively, synchrotron cooling may be important.

Gross time variations, both in intensity and in spectrum, are expected on the hydrodynamical timescale of the cloud ~ tens to hundreds of milliseconds and longer. If the cloud is optically thick to Compton scattering, time variations on timescales shorter than the random walk time of a photon through the cloud $\sim \tau r/c$ (τ = optical depth) may be lost (F. K. Lamb, private communication). In particular, submillisecond time variations in signal, originating very near the hole (Syunyaev 1972), might be hopelessly smeared out by scattering in the translucent cloud.

(2) Assumption (b) is seriously wrong. With α a function of Σ rather than a constant in the time-dependent case [eq. (1)], a stable, stationary, thin disk is possible if α falls at least as fast as Σ^{-1} in the inner region (less efficient viscosity). Such an α leads in turn to a $\Sigma(r)$ that increases steeply towards the hole. For example (Cunningham 1973), equation (1) might be replaced by

$$t_{\hat{\varphi}\hat{r}} = \beta\, P_G, \ \beta = \text{const.} \ , \tag{6}$$

even when $P_R \gg P_G$. (Perhaps this relation is preferable for a self-limiting magnetic viscosity, since gas is frozen to the B-field while radiation is not.) The stationary, thin-disk model resulting from equation (6) is stable and is much like current models except that Σ is much greater in the inner region (typically 25 times greater at $r = 10\ GM_{BH}/c^2$). The thickness, 2h, is still $\lesssim 2 \cdot 10^5$ cm. This dense disk is quite optically thick and is probably immune to thermal or magnetic disturbances on length scales ~ h; hence, chaotic variations in the X-ray signal are likely to be negligible.

Observations (Schreier et al. 1971) of Cygnus X-1 (and similar sources which have been advanced as black-hole candidates) favor alternative (1), since the observed signal is chaotic on all timescales from tens of seconds to $\sim$ 50 milliseconds (instrumental limit). For either alternative, we believe that the prospects of seeing characteristic ($\lesssim$ msec.) time variations originating very near the hole are poorer than has been generally supposed on the basis of current models (Syunyaev 1972).

The same instability arises in a disk around an unmagnetized neutron star. For a magnetized neutron star, a disk does not extend inside the magnetosphere (Pringle and Rees 1972); there is no inner region, hence there is no instability.

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B. Complete Details -- I. Theory and Basic Equations
(Paper II; submitted to Astrophys. J., 1974)

TIME-DEPENDENT ACCRETION DISKS AROUND COMPACT OBJECTS

I. THEORY AND BASIC EQUATIONS*

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ABSTRACT

We consider accretion disks around compact stars and black holes, generalizing the stationary disk models of various authors to allow time dependence on the radial-flow time scale. The structure and evolution of the disk are governed by an "evolution equation" for matter surface density $\Sigma(r,t)$, plus a set of implicit algebraic equations determining various thermodynamic and radiation variables in terms of Σ. The analytic structure of these equations is studied. It is shown that there exists a maximum permissible value of the surface density $\Sigma_{crit}(r)$ -- at which gas and radiation pressure are approximately equal, and beyond which viscosity generates more energy than radiative transport can remove.

I. INTRODUCTION

When a star or black hole accretes matter with high relative angular momentum, the matter probably goes into an approximately circular orbit about the central star. With gravitational attraction providing the centripetal acceleration in the plane of rotation, a thin "accretion disk" is formed. If viscous stresses are present in the disk to transport angular momentum outwards, the matter in the disk can spiral inwards.

Accretion disks are currently thought [Pringle and Rees 1972, Shakura and Sunyaev ("SS") 1973, Novikov and Thorne ("NT") 1973] to play important roles in various binary X-ray sources — e.g., in Cyg X-1 (HDE 226868), where the observed X-rays are probably emitted by a disk, and in Her X-1 (HZ-Her), where a disk presumably produces much optical emission and X-ray absorption. Disks may also be important in the regulation of accretion in U-Gem systems (Smak 1971).

All of the accretion disk models of the above authors are time-independent. A constant mass flux from the normal star into the disk is assumed. It is of considerable interest to also study the properties of time-dependent accretion models. Such a study can investigate, among other things: (1) the stability of time-dependent disks and (2) variability of the mass deposition rate as a possible explanation for the observed 35-day cycle of Her X-1 (Pringle 1973, McCray 1973), or as an explanation for the "extended lows" and "offs" in the X-ray heating of HZ Herc (Jones et al. 1973) or in the X-ray emissions of SMC X-1 (Schreier et al. 1972).

In this paper we develop the theory of time-dependent disks. The underlying physics is essentially the same as that of the stationary models above, except that we allow variables to evolve in time on the "drift" (radial flow) time scale [see (v), §II]. We follow the general format of NT (1973). Our principal results are a nonlinear "evolution equation" (of the form of a diffusion equation) for surface density of the disk and a complete set of auxiliary equations for determining all of the other disk structure variables in terms of surface density (§VII), and an analytic and physical investigation of the solutions of the dependent variables for given surface density and radius (§VIII).

In a companion paper (Lightman 1974), hereafter referred to as Paper II, we solve our evolution equation and auxiliary equations numerically, and apply our results to the investigations (1) and (2) mentioned above. Some of our conclusions have been published previously in abbreviated form (Lightman and Eardley 1974).

II ASSUMPTIONS AND APPROXIMATIONS

(i) The analysis does not attempt to treat the outermost regions of the disk, where the gravitational pull of the normal star and interaction with gas streaming off the normal star are important. Instead, the analysis is confined to the inner portions of the disk ($r \lesssim 1/10\ r_{outer\ edge}$), where the influence of the normal star and streaming matter are negligible.

(ii) Relativistic effects can be neglected. This assumption is valid everywhere for disks around neutron stars and white dwarfs and everywhere except at $r \lesssim 3r_{inner\ edge}$ for disks around black holes. The relations required are

$$v/c \ll 1 \quad \text{and} \quad \frac{GM}{rc^2} \ll 1,$$

where v and M are a typical gas element velocity and mass of the central compact object respectively. (See NT 1973 for a completely relativistic treatment of time-independent disks.) All of the results reported in this paper and Paper II would be qualitatively unchanged in going from our Newtonian calculations to fully relativistic ones.

(iii) The disk is thin. If h is the half thickness of the disk and r the distance to the compact object,

$$\frac{h}{r} \ll 1.$$

(iv) The gas of the disk moves in nearly circular Keplerian orbits, on which is superimposed a small radial flow. This assumption requires the gravitational force of the central object to be much greater than internal stress and pressure gradients inside the disk, and the gravitational energy of the disk gas to be much greater than its internal energy. It therefore requires (cf. §5.5 of NT)

$$\frac{V^r}{V_K} \ll 1 , \qquad \frac{C_s}{V_K} \ll 1 ,$$

where V^r, V_K, and C_s are the radial, Kepler (orbital), and sound velocities of the gas in the disk.

(v) The time scale for gas to drift radially inwards is long compared to the time scales for energy (heat) and sound waves to travel vertically through the disk:

$$\begin{pmatrix}\text{drift}\\ \text{time scale}\end{pmatrix} \equiv t_D \equiv \frac{r}{V^r} \gg t_T \equiv \begin{pmatrix}\text{thermal}\\ \text{time scale}\end{pmatrix} ,$$

$$t_D \gg t_H \equiv \frac{h}{C_s} = \begin{pmatrix}\text{hydrodynamical}\\ \text{time scale}\end{pmatrix} .$$

(vi) The principal sources of opacity are free-free and Compton scattering. For typical temperatures (10^6 ^oK), densities (10^{-3} g/cm^3), and magnetic fields (10^5 G) in the gas, free-free emissivity dominates cyclotron emissivity. Furthermore, for a fully ionized plasma [see (viii) below] bound-free and bound-bound processes are negligible.

(vii) The optical depth is large everywhere in the disk. Stated more precisely: If τ_{ff} and τ_{es} are the free-free and electron scattering optical depths respectively, then (cf. Felten and Rees 1972)

$$\tau_{ff} \gg 1 \text{ when } \tau_{ff} \gg \tau_{es},$$

$$(\tau_{ff}\tau_{es})^{\frac{1}{2}} \equiv (\text{root mean optical depth}) \gg 1$$

$$\text{when } \tau_{es} \gg \tau_{ff} .$$

Rees has pointed out (see Pringle et al. 1973) that failure of the above requirement leads to a thermal instability: If the root mean optical depth becomes small, the disk has difficulty radiating, so its temperature rises. The higher temperature increases the disk thickness and lowers its density. Both these effects result in further decreasing τ_{ff} (see eq. [17]) and thus further lowering the root mean optical depth, further raising the temperature..... The disk quickly "blows up."

(viii) The gas is a fully ionized plasma. This assumption is true for the typical disk temperatures of 10^6 ^oK in hydrogen gas. It can be relaxed easily if necessary. For simplicity we assume a specific value of the mean molecular weight: 0.5 (gas almost all hydrogen).

(ix) Radiation emitted from either the disk or the compact object does not reimpinge on the disk. This assumption, which may be relaxed in future work, certainly has only limited validity. (Narrowly beamed radiation from the poles of a neutron star may not influence an equatorial disk at all. Radiation exceeding the Eddington limit in luminosity, $L_{ED} = 10^{38}$ ergs/sec $(M_c/M_\odot)$, from an unmagnetized neutron star might disrupt the disk completely.)

(x) Energy generated in the disk is transported to the surface by radiation rather than by convection. At a future date, if one has a more detailed understanding of viscosity in the disk (cf. §IV), convective transport of energy due to turbulent mixing might replace the above assumption. See §VIII,b) for further discussion.

In §VI, after the equations of disk structure have been formulated, we will examine the consistency and validity of the various assumptions of this section.

III. FUNDAMENTAL EQUATIONS OF DISK STRUCTURE

The fundamental variables we will use are[1]:

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[1]We are working in standard cylindrical coordinates.

$\rho(r,z,t)$	= density of mass,	(1a)
$p(r,z,t)$	= vertical pressure,	(1b)
$t_{\varphi r}(r,z,t)$	= shear stress (coordinate component),	(1c)
$T(r,z,t)$	= temperature,	(1d)
$q(r,z,t)$	= vertical flux of energy,	(1e)
$F(r,t)$	= energy flux at surface of disk,	(1f)

$V^r(r,z,t)$	= radial velocity of gas,	(1g)
$V^z(r,z,t)$	= vertical velocity of gas,	(1h)
$h(r,t)$	= half-thickness of disk,	(1i)
$\overline{\kappa}(r,z,t)$	= Rosseland mean opacity,	(1j)
M	= mass of central compact object.	(1k)

The laws governing these variables are delineated below:

a) Local Equations

Conservation of mass

$$0 = \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho V^r) + \frac{\partial}{\partial z}(\rho V^z) \tag{2a}$$

Conservation of angular momentum

From assumption (iv) of §II, the angular momentum per unit mass of a gas element at radius r is approximately $(GMr)^{\frac{1}{2}}$. Thus, equating the total time derivative of the angular momentum density to the torque per unit volume, one obtains

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\rho V^r (GMr)^{\frac{1}{2}}\right] + \frac{\partial}{\partial z}\left[\rho V^z (GMr)^{\frac{1}{2}}\right] + (GMr)^{\frac{1}{2}}\frac{\partial \rho}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}(r t_{\varphi r}).$$

If the equation of mass conservation (2a) is used, the above equation simplifies to

$$\rho(GMr)^{\frac{1}{2}} V^r = -2\frac{\partial}{\partial r}(r t_{\varphi r}). \tag{2b}$$

Conservation of energy

Energy is generated locally by viscous heating through action of the shear stress $t_{\varphi r}$. If σ_{ij} and t_{ij} are the components of fluid shear and shear stress, the rate of generation of energy density is

$$- t_{ij}\sigma^{ij} = - 2\, t_{\varphi r}\sigma^{\varphi r} .$$

At this point we do not need any details of the shear stress (e.g., nature of magnetic viscosity or nature of turbulent viscosity). The fluid shear is simply derived from the assumed Kepler (orbital) velocity of the gas

$$\sigma^{\varphi r} = \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{V_K}{r}\right) = \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{GM}{r^3}\right)^{\frac{1}{2}} = -\frac{3}{4}\left(\frac{GM}{r^5}\right)^{\frac{1}{2}} .$$

Because we assume internal energy to be small compared to released gravitational energy [assumption (iv) of §II], and because we assume thermal time scales are much shorter than gas drift time scales [(v) of §II], we may equate the energy generation rate to the divergence of the energy flux (assumed to be in vertical direction because of disk thinness)

$$\frac{\partial q}{\partial z} = \frac{3}{2}\left(\frac{GM}{r^5}\right)^{\frac{1}{2}} t_{\varphi r} . \qquad (2c)$$

Equation of state

$$p = p_{gas} + p_{radiation} , \qquad (2d)$$

$$p_{gas} = \frac{2\rho kT}{m} , \qquad p_{rad} = \frac{1}{3}\, bT^4 . \qquad (2e)$$

In equation (2e), m, k and b are the proton mass, Boltzmann constant, and familiar radiation constant respectively. The factor of 2 in the expression for the gas pressure comes from (viii) of §II.

Equation of energy transport

From our assumption of radiative transport [(x) of §II] as the energy transport mechanism and assumptions (vi) and (vii), the radiative transport is described by the radiative diffusion equation

$$q = -\frac{1}{\bar{\kappa}\rho}\frac{\partial}{\partial z}\left(\frac{1}{3}\ cbT^4\right)\ , \tag{2f}$$

where c is the speed of light,

$$\bar{\kappa} = \bar{\kappa}_{ff} + \bar{\kappa}_{es}\ , \tag{2g}$$

($\bar{\kappa}_{ff}$ = free-free opacity, $\bar{\kappa}_{es}$ = electron scattering opacity) and

$$\bar{\kappa}_{es} = 0.40\ \text{cm}^2/\text{g}\quad ,\quad \bar{\kappa}_{ff} = 0.64\times 10^{23}\left(\frac{\rho}{\text{g/cm}^3}\right)\left(\frac{T}{^{\circ}\text{K}}\right)^{-\frac{7}{2}}\ \text{cm}^2/\text{g}\ , \tag{2h}$$

(see, e.g., NT 1973, p. 378, for a standard derivation of these opacities).

Vertical pressure balance

Because of assumption (v) §II, we may assume that the vertical structure adjusts itself instantaneously to slow changes in radial structure and is always in hydrostatic equilibrium:

$$\frac{\partial p}{\partial z} = -\frac{\rho GMz}{r^3}\ . \tag{2i}$$

b) Vertically Averaged Equations

We now average all equations vertically by integrating them over the thickness of the disk. Define the following new variables:

$$\Sigma(r,t) = \text{surface density} \equiv \int_{-h}^{h} \rho\ dz\ , \tag{3}$$

$$W(r,t) = \text{integrated shear stress} \equiv \int_{-h}^{h} r^{-1}\ t_{\varphi r}\ dz\ . \tag{4}$$

We will continue to use the same symbols V^r, T, p, $\bar{\kappa}$ for the vertically averaged velocity, temperature, pressure, and Rosseland mean opacity, although these variables are now functions of only r and t. All of our vertically averaged equations below will be approximate up to factors of order unity because of uncertainty in vertical averaging. A much greater uncertainty, however, is an explicit form for W (cf. §IV) so that a more precise vertical averaging procedure is not justified. Using the facts that $\rho(z = \pm h) = p(z = \pm h) = 0$ and $F(r,t) = q(z=h,r,t) = q(z=-h,r,t)$, equations (2) and (3) may be integrated over the disk thickness to yield

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r \Sigma V^r) = 0, \tag{5a}$$

$$\Sigma V^r (GMr)^{\frac{1}{2}} = -2 \frac{\partial}{\partial r}(r^2 W), \tag{5b}$$

$$F = \frac{3}{4}\left(\frac{GM}{r^3}\right)^{\frac{1}{2}} W, \tag{5c}$$

$$p = \frac{\Sigma k T}{hm} + \frac{1}{3} b T^4, \tag{5d}$$

$$F = \frac{2/3 \; c b T^4}{\Sigma \bar{\kappa}} \quad , \tag{5e}$$

where

$$\bar{\kappa} = 0.4 \text{ cm}^2/\text{g} + 0.32 \times 10^{23} \left(\frac{\Sigma/h}{\text{g/cm}^3}\right)\left(\frac{T}{{}^oK}\right)^{-\frac{7}{2}} \text{cm}^2/\text{g} \tag{5f}$$

$$p = \frac{\Sigma h}{2}\left(\frac{GM}{r^3}\right) \quad . \tag{5g}$$

Equations (5a)-(5c) are "radial structure equations" and equations (5d)-(5g) are "vertical structure equations." It should be noted that the radial structure equations are exact — in particular, equations (5a) and (5b) allow us to define a physically meaningful conserved mass and angular momentum of the disk, which are useful in calculations (see Paper II).

Equations (5) represent six equations for seven unknowns Σ, W, F, V^r, p, T, and h. The final equation comes from a model for viscosity, i.e., the viscous stress which defines the variable $t_{\varphi r}$ and subsequently the variable W.

Note in equations (5) that the only variable on which a time derivative acts is the surface density Σ. The time variation of all other variables is enforced implicitly, through their functional dependence upon Σ. Thus Σ is a natural choice for the sole independent variable characterizing the state of the disk at each instant of time.

IV. MODEL FOR VISCOSITY AND RESULTANT EQUATIONS

The dominant sources of viscosity are probably turbulence in the gas and chaotic magnetic fields (see SS). Letting B, V_t, C_s be the strength of magnetic field in the gas, turbulent velocity, and sound velocity respectively, SS and NT estimate

$$(r^{-1}\ t_{\varphi r}) = \alpha p, \tag{6a}$$

where

$$\alpha \lesssim \frac{V_t}{C_s} + \frac{B^2}{4\pi\rho C_s^2} \quad . \tag{6b}$$

The first contribution to the dimensionless parameter α is from turbulence; the second is from magnetic stress. As pointed out by the above authors, α is almost certainly less than unity: supersonic turbulent velocities generate shocks which heat and expand the gas, thus raising the sound speed; magnetic stress exceeding the thermal pressure causes magnetic field lines to bulge out of the disk, reconnect, and escape.

In all previous calculations involving stationary disks (e.g., SS and NT) the dimensionless viscosity parameter α has been assumed to be constant. This assumption probably becomes increasingly tenuous as an increasing degree of time dependence is allowed in the problem and as one looks at shorter and shorter time scales. We shall make the assumption that α is constant in radius and time on the drift time scale, $t_D = r/V^r$ [see (iv) of §II and §VI below]. On the thermal and hydrodynamical time scales, t_T and t_H, α is almost certainly not a constant. (Situations can develop in which energy generated by the disk must be transported vertically by sonic convective elements, requiring α to vary in time and space. See §VIII,b) for further discussion.) We assume (cf. eqs. [6a] and [4])

$$W \sim 2h\alpha p \ , \tag{7a}$$

$$\alpha < 1 \ , \tag{7b}$$

and α = const. for $\Delta t \gtrsim t_D$.

If equation (7a) is now substituted into equations (5), the resulting equations, in dimensionless form, are

$$\frac{\partial \Sigma_4}{\partial t} + \frac{10^{-3}}{r_6} \frac{\partial}{\partial r_6} (r_6 \Sigma_4 V_3{}^r) = 0, \tag{8a}$$

$$\Sigma_4 V_3{}^r (M_* r_6)^{\frac{1}{2}} = -0.2 \frac{\partial}{\partial r_6} (\alpha r_6^2 h_6 p_{10}), \tag{8b}$$

$$F_{21} = 0.3 \ h_6 \alpha p_{10} \left(\frac{M_*}{r_6^3}\right)^{\frac{1}{2}} \tag{8c}$$

$$p_{10} = 8.3 \frac{\Sigma_4 \, T_5}{h_6} + (2.5 \times 10^{-5}) \, T_5^{\,4} \, , \tag{8d}$$

$$F_{21} = (1.5 \times 10^{-9}) \, T_5^{\,4} \, (\Sigma_4 \, \bar{\kappa}_1)^{-1} \, , \tag{8e}$$

$$p_{10} = (2.0 \times 10^{8}) \, h_6 \, \Sigma_4 \left(\frac{M_*}{r_6^{\,3}} \right) \, , \tag{8f}$$

$$\bar{\kappa}_1 = 0.4 + (1.0 \times 10^{3}) \, \Sigma_4 \, T_5^{-\frac{7}{2}} \, h_6^{-1} \, , \tag{8g}$$

where we have used the notation: $\Sigma_4 \equiv (\Sigma/10^4 \text{ g cm}^{-2})$, $r_6 \equiv (r/10^6 \text{ cm})$, $V_3^{\,r} \equiv (V^r/10^3 \text{ cm sec}^{-1})$, $M_* \equiv (M/3 \, M_\odot)$, $h_6 \equiv (h/10^6 \text{ cm})$, $p_{10} \equiv (p/10^{10} \text{ dynes cm}^{-2})$, $F_{21} \equiv (F/10^{21} \text{ ergs cm}^{-2} \text{ sec}^{-1})$, $T_5 \equiv (T/10^5 \, {}^{\circ}\text{K})$, $\bar{\kappa}_1 \equiv (\bar{\kappa}/\text{cm}^2 \text{ g}^{-1})$.

If the time derivative of Σ in equation (8a) is set to zero, equations (8) reproduce the results of NT (1973) for their stationary disk.

V. APPROXIMATE REGIONAL SOLUTIONS FOR DEPENDENT VARIABLES

Equations (8c)-(8g) represent four algebraic relations for the four dependent variables F, T, h, p; as such, they can be solved (in principle, although the equations are very implicit) in terms of the independent variable Σ. It is useful to consider their approximate solution in three physical regimes (these regimes defined in the stationary disk models of SS and NT):

Outer Region $(\bar{\kappa}_{ff} \gg \bar{\kappa}_{es}, \; p_g \gg p_r)$

Using §III and equations (8c)-(8g)

$$T_5 = (1.9 \times 10^{2}) \, \Omega^{2/7} \, \Sigma_4^{\,3/7} \, \alpha^{1/7} \, , \tag{9a}$$

$$p_{10} = (5.6 \times 10^{5}) \, \Omega^{8/7} \, \Sigma_4^{\,17/14} \, \alpha^{1/14} \, , \tag{9b}$$

$$h_6 = (2.8 \times 10^{-3})\ \Omega^{-6/7}\ \Sigma_4^{3/14}\ \alpha^{1/14}\ , \tag{9c}$$

$$F_{21} = (4.7 \times 10^{2})\ \Omega^{9/7}\ \Sigma_4^{10/7}\ \alpha^{8/7}\ , \tag{9d}$$

Middle Region $(\bar{\kappa}_{es} \gg \bar{\kappa}_{ff}\ ,\ p_g \gg p_r)$

$$T_5 = (8.7 \times 10^{2})\ \Omega^{1/3}\ \Sigma_4^{2/3}\ \alpha^{1/3}\ , \tag{10a}$$

$$p_{10} = (1.2 \times 10^{6})\ \Omega^{7/6}\ \Sigma_4^{4/3}\ \alpha^{1/6}\ , \tag{10b}$$

$$h_6 = (6.0 \times 10^{-3})\ \Omega^{-5/6}\ \Sigma_4^{1/3}\ \alpha^{1/6}\ , \tag{10c}$$

$$F_{21} = (2.2 \times 10^{3})\ \Omega^{4/3}\ \Sigma_4^{5/3}\ \alpha^{4/3}\ , \tag{10d}$$

Inner Region $(\bar{\kappa}_{es} \gg \bar{\kappa}_{ff}\ ,\ p_r \gg p_g)$

$$T_5 = (2.5 \times 10^{2})\ \Omega^{1/4}\ \alpha^{-1/4}\ , \tag{11a}$$

$$p_{10} = (9.8 \times 10^{4})\ \Omega\ \alpha^{-1}\ , \tag{11b}$$

$$h_6 = (4.9 \times 10^{-4})\ \Omega^{-1}\ \Sigma_4^{-1}\ \alpha^{-1}\ , \tag{11c}$$

$$F_{21} = (1.4 \times 10)\ \Omega\ \Sigma_4^{-1}\ \alpha^{-1}\ . \tag{11d}$$

Note that T and p are independent of Σ in the inner region, and that h decreases with increasing Σ. The units in which the above equations are written are given in and below equations (8) and we have used the notation

$$\Omega \equiv \left(\frac{M_*}{r_6^{\,3}}\right)^{\frac{1}{2}} \approx (2.0 \times 10^4\ \mathrm{sec}^{-1})^{-1} \left(\frac{GM}{r^3}\right)^{\frac{1}{2}} = \frac{\Omega_k}{2 \times 10^4\ \mathrm{sec}^{-1}}\ . \tag{12}$$

VI. DISCUSSION OF CONSISTENCY AND VALIDITY OF APPROXIMATION

a) Sound Velocity Versus Kepler Velocity

From the equation of vertical pressure balance (cf. eq. [5g])

$$\frac{h}{r} \sim \left(\frac{p}{\rho}\right)^{\frac{1}{2}} \left(\frac{r}{GM}\right)^{\frac{1}{2}} \sim \frac{C_s}{V_K} \quad . \tag{13}$$

Thus the sound velocity is much smaller than the Kepler velocity, $C_s/V_K \ll 1$, part of the assumption (iv) §II, as long as the disk is thin, $h/r \ll 1$, assumption (iii) of §II.

b) Hydrodynamical and Drift Time Scales

The hydrodynamical time scale, t_H, is

$$t_H \sim \frac{h}{C_s} \sim \frac{r}{V_K} \quad . \tag{14a}$$

The drift time scale, t_D, is (cf. eqs. [5b] and [8b])

$$t_D \sim \frac{r}{V^r} \sim \frac{(\Sigma/h)}{\alpha p} (GMr)^{\frac{1}{2}} \sim \left(\frac{r}{h}\right)^2 \alpha^{-1} \; t_H \; , \tag{14b}$$

where we have used equations (13) and (14a). Thus

$$\frac{t_D}{t_H} \sim \frac{V_K}{V^r} \gg 1 \qquad \text{if } \left(\frac{r}{h}\right)^2 \gg \alpha \quad . \tag{14c}$$

Equation (14c) establishes consistency between assumptions (iii), (iv), and (v) of §II.

c) Optical Thickness

The free-free and electron scattering optical depths, τ_{ff} and τ_{es}

respectively, are

$$\tau_{ff} = \Sigma \, \overline{\kappa}_{ff} \qquad , \qquad \tau_{es} = \Sigma \, \overline{\kappa}_{es} . \tag{15}$$

(The expressions given in eq. [15] are for the frequency averaged optical depths, since we are using the Rosseland mean opacities. Thus our relations are only approximately true. For a complete discussion of frequency dependent optical depths, see Felten and Rees (1972) or Illarionov and Sunyaev (1972).)

The root mean optical depth is then (cf. eq. [5f])

$$\tau_* \equiv (\tau_{ff} \, \tau_{es})^{\frac{1}{2}} = (2.0 \times 10^5) \, \Sigma_4^{3/2} \, (h_6 \, T_5^{7/2})^{-\frac{1}{2}} \; , \tag{16}$$

while the optical depth due to free-free opacity alone is

$$\tau_{ff} = (1.0 \times 10^7) \, \Sigma_4^{\,2} \, (h_6 \, T_5^{7/2})^{-1} \qquad . \tag{17}$$

For our radiative diffusion equation (2f) to be valid, we must require (see, e.g., Felten and Rees 1972)

$$\tau_{ff} \gg 1 \quad \text{in outer region of disk,} \tag{18a}$$

$$\tau_* \gg 1 \quad \text{in middle region of disk,} \tag{19a}$$

$$\tau_* \gg 1 \quad \text{in inner region of disk .} \tag{20a}$$

Using equations (9), (10), (11,), (16), and (17), we then must require

$$(4 \times 10) \, (\Sigma_4^{\,2} \, \Omega^{-1} \, \alpha^{-4})^{1/7} \gg 1 \quad \text{in outer region,} \tag{18b}$$

$$(2 \times 10) \, (\Omega^{-1} \, \Sigma_4 \, \alpha^{-4})^{1/6} \gg 1 \quad \text{in middle region,} \tag{19b}$$

$$(6 \times 10^2) \, \Sigma_4^{\,2} \Omega^{1/16} \, \alpha^{15/16} \gg 1 \quad \text{in inner region .} \tag{20b}$$

In our explicit models (see Paper II), equations (18), (19), and (20) will turn out to be satisfied, except in the evolution of the perturbations of certain stationary disk models; there, however, the inner region of the disk goes unstable and nearly all of our assumptions become invalid.

d) Thermal Time Scale

In the middle and inner regions, where electron scattering is the dominant source of opacity, photons generated in the interior of the disk must random walk their way to the surface of the disk. The time it takes for a photon to random walk from $z = 0$ to $z = h$ is an upper limit to the thermal time scale t_T. If N is the number of scatterings such a photon undergoes, and λ_{es} the mean free path between scatterings, then

$$t_T \lesssim \frac{N \lambda_{es}}{c} \quad . \qquad (21a)$$

But since the process is a random walk, N is related to the optical depth τ_{es} in the standard way

$$N = \tau_{es}^2 \quad . \qquad (21b)$$

If we substitute equation (21b) into equation (21a) and use the fact that $\lambda_{es} = h/\tau_{es}$, and equations (2h) and (15), then equation (21a) becomes

$$t_T \lesssim (0.13 \text{ sec}) \Sigma_4 \, h_6 \quad . \qquad (21c)$$

Thus, using equations (10), (11), equation (21c) implies the relations

$$t_T \lesssim (7.8 \times 10^{-4} \text{ sec}) \, \Omega^{-5/6} \, \Sigma_4^{4/3} \, \alpha^{1/6} \qquad \text{in middle region,} \qquad (22a)$$

$$t_T \lesssim (6.4 \times 10^{-5} \text{ sec}) \, \Omega^{-1} \, \alpha^{-1} \qquad \text{in inner region .} \qquad (22b)$$

Note that equation (22b), together with equations (12) and (14a), shows that in the inner region

$$t_T \sim \alpha^{-1} t_H . \qquad (23)$$

Note also that t_T is independent of Σ in the inner region.

In summary, we can conclude (cf. eqs. [12], [14b], [22], [23]) that as long as the disk is thin, $h/r \ll 1$, the three time scales t_T, t_D, t_H satisfy

$$t_D \gg t_H , \qquad (24a)$$

$$t_D \gg t_T , \qquad (24b)$$

where the last relation holds as long as Σ_4 is not many orders of magnitude larger than unity.

e) Thinness of Disk

As can be seen from the above discussion, most of the assumptions of §II are satisfied as long as the disk is thin. By using equations (9c), (10c), and (11c) we can obtain approximate criteria for disk thinness in the three regimes of §V.

$$\frac{h}{r} = (2.8 \times 10^{-3})\, \Sigma_4^{3/14}\, \alpha^{1/14} \left(\frac{r_6^{\,2}}{M_*^{\,3}}\right)^{1/7} \ll 1 \quad \text{in outer region,} \qquad (25a)$$

$$\frac{h}{r} = (6.0 \times 10^{-3})\, \Sigma_4^{1/3}\, \alpha^{1/6}\, M_*^{-5/12}\, r_6^{1/4} \ll 1 \quad \text{in middle region,} \qquad (25b)$$

$$\frac{h}{r} = (4.9 \times 10^{-4})\, \Sigma_4^{-1}\, \alpha^{-1} \left(\frac{r_6}{M_*}\right)^{\frac{1}{2}} \ll 1 \quad \text{in inner region.} \qquad (25c)$$

For typical values of the parameters involved, i.e., $10^{-3} \le \alpha \le 1$, $M \sim 1$, $10^{-3} \le \Sigma \le 10$, $r \lesssim 50$ in inner region, $r < 5 \times 10^{4}$ in middle and outer regions, equations (25) are all satisfied — except when Σ becomes too small in the inner region. There will be more discussion of this last possibility in Paper II, where Σ is evolved dynamically in time. Equations (18b), (19b), (20b), (22a), (25a), (25b), and (25c) are a minimal set of criteria for validity of the model.

VII. REDUCTION OF EQUATIONS

In §V we gave approximate solutions, in various regimes, for the variables F, p, h, and T in terms of the single independent variable Σ. In this section we give exact, although implicit, solutions for these variables in terms of Σ, which are valid in all regimes and which will be used in future numerical work (see Paper II). We also put the dynamical equation for Σ into a new, more useful form and thereby complete our specification of the equations governing disk structure.

In this section and henceforth, except where indicated otherwise, we drop the units subscripts on variables, with the units shown in and below equations (8) to be understood.

a) Equations for Dependent Variables

If equations (8c) and (8e) are equated, then equations (8c)-(8g) can be reduced to 3 equations for the three unknowns p, h, T. (Remember, we are solving for dependent variables in terms of the independent variable Σ; Σ is not considered an unknown here.):

$$h = C_1 p \qquad (26a)$$

$$p = C_2 \frac{T}{h} + C_3 T^4 , \qquad (26b)$$

$$T^4 = ph \left(C_4 \frac{T^{-7/2}}{h} + C_5\right) \quad , \tag{26c}$$

where

$$C_1 \equiv (5.0 \times 10^{-9}) \, \Sigma^{-1} \, \Omega^{-2} \quad , \tag{27a}$$

$$C_2 \equiv 8.3 \, \Sigma \quad , \tag{27b}$$

$$C_3 \equiv (2.5 \times 10^{-5}) \quad , \tag{27c}$$

$$C_4 \equiv (2.0 \times 10^{11}) \, \alpha \Omega \Sigma^2 \quad , \tag{27d}$$

$$C_5 \equiv (8.0 \times 10^{7}) \, \alpha \Sigma \Omega \quad . \tag{27e}$$

Equations (26a), (26b) can be solved for h and p in terms of T (quadratic equation in p):

$$p = \frac{1}{2} \left(C_3 \, T^4 + X\right) \quad , \tag{28a}$$

where

$$X = \left(C_3^2 \, T^8 + \frac{4C_2}{C_1} \, T\right)^{\frac{1}{2}} \quad . \tag{28b}$$

Note that we have chosen the plus sign in front of the radical in our solution to the quadratic equation for p so that p will be positive.

If equations (28a), (28b), and (26a) are now substituted into equation (26c), a single equation containing only the dependent variable T is obtained:

$$H(T) \equiv T^4 - \tfrac{1}{2} \, C_4 T^{-7/2} \left(C_3 T^4 + X\right) - \tfrac{1}{2} \, C_1 C_5 \left(C_3^2 \, T^8 + \frac{2C_2}{C_1} \, T + C_3 T^4 X\right) = 0 \tag{29}$$

Equation (29) gives implicitly the temperature $T(r,\Sigma)$. From equations (28a) and (26a) one then can determine $h(r,\Sigma)$ and $p(r,\Sigma)$, and from equation (8c), for example, one can then determine $F(r,\Sigma)$.

b) Nonlinear Diffusion Equation for Surface Density

If equation (8b) is used to eliminate ΣV^r on the RHS of equation (8a), one obtains the equation

$$\frac{\partial\Sigma}{\partial t} = \frac{(2.0 \times 10^{-4})}{r} \frac{\partial}{\partial r} \left\{ \left(\frac{r}{M}\right)^{\frac{1}{2}} \frac{\partial}{\partial r} \left[\alpha r^2 h(r,\Sigma) p(r,\Sigma) \right] \right\} \quad . \qquad (30)$$

Mathematically speaking, equation (30) is a nonlinear diffusion equation for the surface density $\Sigma(r,t)$; when proper boundary conditions have been specified (see Paper II), it, together with equations (29), (28a), and (26a), constitute a complete set of equations for the dynamical evolution of the disk variables.

One may think of the system of equations in the following way: For given r and t, there is a given $\Sigma(r,t)$; equations (29), (28a), and (26a) may then be used to compute $T(r,\Sigma)$, $h(r,\Sigma)$, and $p(r,\Sigma)$; h and p may then be substituted into equation (30) to determine $\Sigma\,(r, t+\Delta t)$, and so on.

VIII. THE TEMPERATURE FUNCTION AND SOLUTION REGIMES

Without considering the dynamical equation (30) at all, one can learn a considerable amount about the nature of the physical problem by investigating the solutions of equation (29) for fixed r and Σ.

At large T, H(T) goes as $-T^8$ and at small T, as $-T^{-3}$ (cf. eqs. [28b] and [29]). Further investigation reveals that for typical values of the parameters involved (α and M for each model, Σ and r for each solution of eq. [29] in a given model) H(T) can have two, one, or no zeros.

a) One and Two Solution Regimes

Let us restrict our attention to a given model (fixed α and M). For values of (r,Σ) such that H(T) has two zeros, the high-T zero always has radiation pressure providing the dominant contribution to the total pressure (second term on RHS of eq. [26b]) and the low-T zero always has gas pressure providing the dominant contribution to the total pressure (first term on RHS of eq. [26b]). These two roots are illustrated, for typical values of (α, M, r, Σ), in figure 1a; we shall refer to the high temperature solution as the radiation root and the low temperature solution as the gas root. The gas root yields (from our experience in computing models) an h satisfying $h/r << 1$, but the radiation root may yield an h such that $h/r > 1$ if Σ is too small or r too large, providing an inconsistent [see (iii) §II and §VI] and therefore unacceptable solution. This behavior of the radiation root is somewhat expected, since the inner region (see §V) of stationary disks (see, e.g., NT) if it exists at all, occurs only at small radii as its name implies.

If one kept fixed the radius r of figure 1a, but plotted H(T) for increasingly greater Σ, the peak would continually decrease and the gas and radiation roots would move closer together, until a Σ was reached at which the curve would become tangent to the T axis and the two roots would coalesce. Further increase of Σ lowers the peak of H(T) below the axis and there are no solutions.

b) No Solution Regime

As was mentioned above, for a given r, if Σ exceeds a critical value, $\Sigma_{crit}(r)$, there are no zeros of H(T) and thus no temperature at which the vertical structure equations (26) have a solution. In order to understand physically what this situation corresponds to, one must return to equations (8c) and (8e), expressions for the energy generation rate and energy transport rate respectively. Equating H(T) to zero is equivalent to equating the energy generation rate to the energy transport rate, satisfying the equation of pressure balance, and satisfying the equation of state (cf. eqs. [26]). If equations (26a) and (28a) are substituted into equations (8c) and (8e), then one finds, for fixed r and Σ, that the generation rate G(T) and the radiative transport rate R(T) satisfy

$$\left.\begin{aligned} G &\propto T \qquad (31a)\\ R &\propto T^8 \qquad (31b)\end{aligned}\right\} \text{ small T,}$$

$$\left.\begin{aligned} G &\propto T^8 \qquad (32a)\\ R &\propto T^4 \qquad (32b)\end{aligned}\right\} \text{ large T.}$$

Since G(T) has both a shallower slope at small T and a steeper slope at large T, G(T) > R(T) for all T when there is no intersection of the G(T) and R(T) curves. The situation is illustrated in figure 1b. The solid-line graphs correspond to the graph of H(T) in figure 1a, while the dotted-line graphs correspond to a situation in which $\Sigma > \Sigma_{crit}$ and there are no solutions.

In conclusion, for $\Sigma > \Sigma_{crit}(r)$ the model breaks down: the viscosity in the disk generates energy more rapidly than radiative diffusion can transport it away.

The subsequent evolution of the disk cannot be analyzed within the framework of our assumptions. We can only speculate, on physical grounds, that the disk's internal temperature will rise rapidly (perhaps violating assumption (iv) of §II), the disk will expand vertically (perhaps violating assumption (iii) of §II), and the temperature gradient (cf. eq. [2f]) will become sufficiently superadiabatic to drive convective energy transport at the rate necessary to counterbalance the energy generation (violation of assumption (x) of §II). In the inner region, where radiation pressure dominates, the convective elements are forced to travel with sonic velocities [in contrast with the usual assumption of subsonic velocities in convection — see e.g., Cox and Giuli (1968) or Aller (1954)] in order to effectively transport energy, and the problem becomes quite complicated. (Subsonic velocities would allow pressure equilibration and hence temperature equilibration, between the convective element and its surrounding medium.) A better understanding of viscosity (and the associated variation of α) on short time scales and as a function of height is almost certainly required for the proper treatment of the above problem — until such an understanding is developed, we can deal only with radiation as the mechanism for energy transport.

c) Boundary of the No Solution Regime

We can find an approximate analytic expression for the boundary of the "no solution regime," $\Sigma_{crit}(r)$, by noting that as Σ is increasing the radiation and gas roots coalesce just as the peak of H(T) dips below the T axis. Since for models of X-ray sources the radiation root always yields solutions such that $\bar{\kappa}_{es} \gg \bar{\kappa}_{ff}$, such a situation must also hold when the radiation and gas roots are equal. Thus the boundary is given approximately by equating any of the variables of the inner region to the same variable in the middle region. Equating the two temperatures, for example, yields

$$\Sigma_{crit}(r) \approx 0.15 \left(\frac{r^3}{M}\right)^{1/16} \alpha^{-7/8} .$$

By considering the full H(T), rather than its limiting form in the two regimes, one can get a better determination of the overall numerical factor in the previous equation:

$$\Sigma_{crit}(r) \approx 0.07 \left(\frac{r^3}{M}\right)^{1/16} \alpha^{-7/8} . \tag{33}$$

Thus $\Sigma_{crit}(r)$ changes by less than a factor of ten as r ranges over five orders of magnitude, whereas it is much more sensitive to the viscosity parameter α. Models with more efficient viscosity (higher values of α) have a smaller range of allowed surface density.

IX. COMPARISON WITH STATIONARY MODELS

The purpose of this section is principally to reconcile the one, two, and no root phenomena of the previous section with the structure of the stationary disk models of SS and NT, since the latter models always have a unique temperature which is consistent with all of the equations. We should be able to understand such time-independent models within our formalism since they are a special case of time-dependent models.

In the stationary disk models equation (30) is replaced by the statement that the mass flux be constant,

$$\dot{M} = 4\pi r h \Sigma V^r = \text{const.}, \qquad \text{(conventional units)} \tag{34}$$

in both radius and time throughout the disk. With such an equation, the disk variables, including Σ, become functions of only radius and the constant $\dot{M}$. In our language, we may regard the stationary disks as corresponding to a situation in which: (1) our independent variable Σ assumes a specified functional form

$$\Sigma(r,t) \rightarrow \tilde{\Sigma}(r),$$

and (2) using $\tilde{\Sigma}(r)$, roots of equation (29) are chosen that yield values for the disk variables h, p, etc., satisfying constant mass flux (cf. eq. [32]).

The boundary radius, r_B, between the middle and inner region of the disk (cf. §V) occurs at (cf. NT or SS)

$$(r_6)_B \approx 18\, \alpha^{2/21}\, M_*^{1/3} \left(\frac{\dot{M}}{10^{17}\ \mathrm{g\ sec^{-1}}}\right)^{4/5} , \tag{35}$$

and exists only if the inner edge of the disk can extend within this radius. (The inner region typically exists for stationary disks around black holes, but not for disks around magnetized neutron stars where the Alfvén surface occurs outside of r_B.)

Stationary models with no inner region correspond to situations in which the radiation zero of H(T) (cf. §VII and §VIII) always violates $h/r \ll 1$ and is therefore unacceptable. Hence, in such situations, the gas zero of H(T) is always chosen as the solution of equation (29). Stationary models in which there is an inner region correspond to situations in which, for $r \leq r_B$, the radiation root of equation (29) not only yields $h/r \ll 1$, but also is required for satisfaction of equation (34). In such situations $\tilde{\Sigma}(r)/\Sigma_{crit}(r)$ increases with decreasing radius [peak of H(T) falling closer to T axis; gas root chosen] until the transition from middle to inner regions at r_B [peak of H(T) just tangent to T axis; gas and radiation roots coalesce] at which point $\tilde{\Sigma}(r)/\Sigma_{crit}(r)$ begins decreasing with decreasing radius [peak of H(T) rising after tangency at r_B; radiation root chosen]. The transition from gas root to radiation root is thus a smooth one and the peak of H(T) never dips below the T axis, i.e., the no solution region is never reached. Whether such a situation is stable under perturbations will be investigated in Paper II.

In the actual literature on stationary disks, the function H(T) and consequently the multiple and no root phenomena are not encountered because the equations are solved under the imposed restriction of equation (34) and done so not exactly, but only approximately, by dividing the solutions into outer, middle, and inner regions as in §V.

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FIGURE CAPTIONS

Fig. 1a. A plot of the temperature function $H(T)$ for the model $M = 1$, $\alpha = 0.01$ at $r = 5 \times 10^{3}$, $\Sigma = 10$. The two roots are indicated by A and B; at root A, $p_g \gg p_r$ and $h/r \ll 1$ while at root B $p_r \gg p_g$ and $h/r > 1$.

Fig. 1b. Plots of the generation and radiation functions $G(T)$ and $R(T)$. The solid line graph corresponds to the same graph as fig. 1a with the two solutions indicated. The dotted line graph corresponds to different values of r and Σ such that $\Sigma > \Sigma_{crit}(r)$, (see eq. [33] of text), and there are no intersections. The scale in the y direction has been compressed by a factor of 3 for clarity.

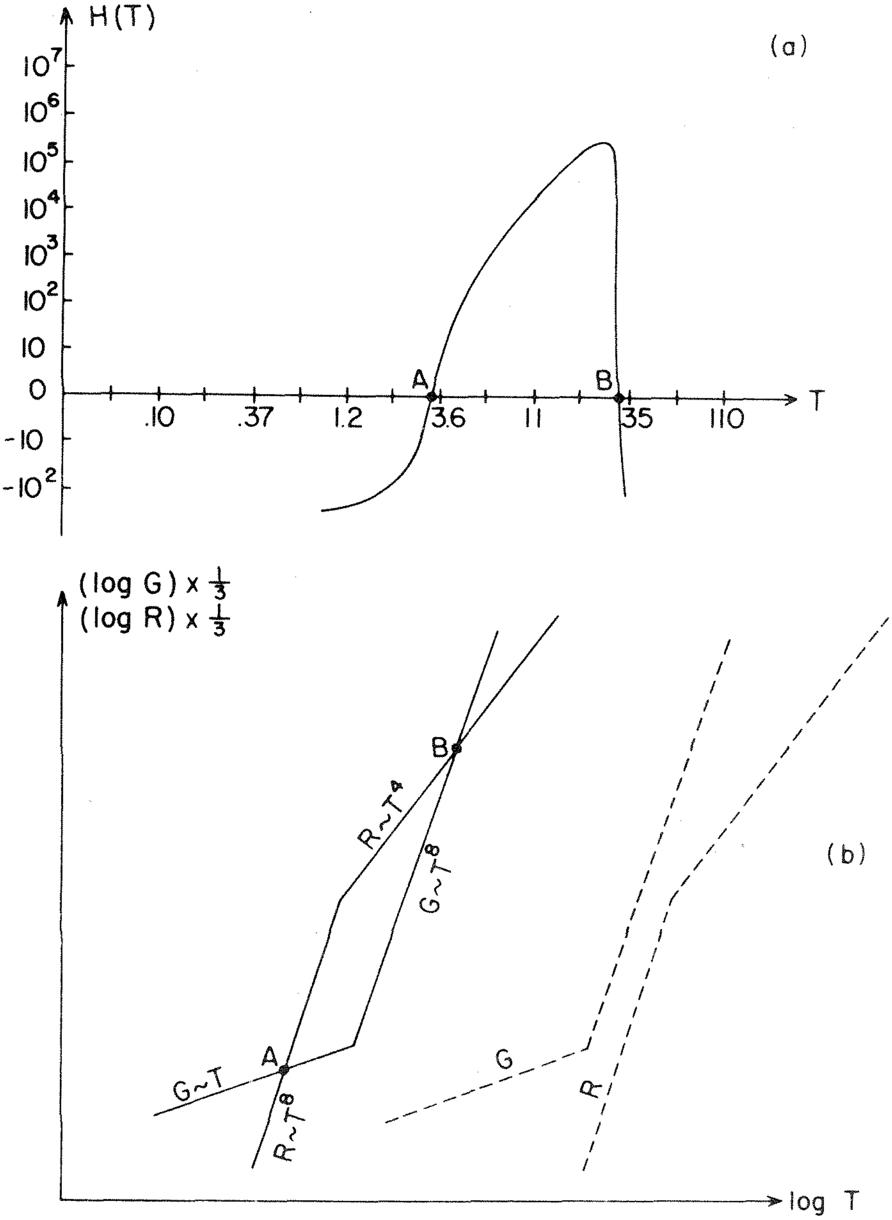
H(T)
(a)
10^7
10^6
10^5
10^4
10^3
10^2
10
0
-10
-10^2
A
B
.10
.37
1.2
3.6
11
35
110
T
(log G) x $\frac{1}{3}$
(log R) x $\frac{1}{3}$
B
$R \sim T^4$
$G \sim T^8$
(b)
A
$G \sim T$
$R \sim T^8$
G
R
log T

Fig. 1

C. Complete Details -- II. Numerical Models and Instability of Inner Region (Paper III; submitted to Astrophys. J., 1974)

TIME-DEPENDENT ACCRETION DISKS AROUND COMPACT OBJECTS
II. NUMERICAL MODELS AND INSTABILITY OF INNER REGION*

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ABSTRACT

Using the theory of time-dependent accretion disks developed in a previous paper, we numerically solve, for various examples, the time evolution equation for disk surface density Σ; from Σ all other variables of the disk may be calculated by algebraic equations. From both analytic considerations and numerical results we find that the "inner region" of a disk around a black hole is secularly unstable against clumping of the gas into rings. This invalidates the standard steady-state theory of such disks. Possible revisions of the theory are suggested. For accretion onto a magnetized neutron star, we study how the evolution of the disk is affected by time varying deposition of gas into the disk, and by changes in location of the Alfvén surface (magnetosphere-disk interface). Finally, we describe briefly some possible implications of our results for models of the binary X-ray sources Cyg X-1, Herc X-1, and SMC X-1.

I. INTRODUCTION AND SUMMARY

This is the second of a sequence of papers which develop and apply the theory of time-dependent accretion disks. In Paper I (Lightman 1974) we indicated the astrophysical setting in which accretion disks around compact objects probably play an important role and then proceeded to develop the basic theory and equations of disks that are nonstationary on the radial flow time scale. In this paper, using the results of Paper I, we solve the time-dependent equations, investigate stability of stationary disks, and numerically build disk models. It would be very helpful to the reader to be familiar with Paper I, as we shall frequently borrow results from there.

In §II we show, first analytically with approximate solutions and then numerically, that the inner region of a disk (region where radiation pressure is much greater than gas pressure and the dominant opacity is due to electron scattering) is secularly unstable. Because of this result, the standard models of disks around black holes [e.g., Pringle and Rees (1972), Novikov and Thorne ("NT") (1973), or Shakura and Sunyaev ("SS") (1973)] must be reformulated.

In §III, we numerically construct models in which the "inner region" does not exist, so there is no problem with the instability discussed in §II. Such models would be typical, for example, of magnetized neutron stars, in which the external magnetic field prevents the inner edge of the disk from extending to a radius so small that radiation pressure dominates gas pressure.

Accretion disks may play crucial roles in binary X-ray sources. The extra-disk physics of such systems, however, can be quite complicated and is as yet poorly understood, e.g., precession of the compact object (Brecher

1972) or regulation of accretion rate at the Alfvén surface by a magnetic gate (Pines 1973) or regulation of accretion rate by X-rays impinging onto the companion star [McCray (1973), Arons (1973), Alme and Wilson (1973)] or forced precession of the disk itself (Katz 1973). Consequently, our point of view — rather than attempting to fit the detailed observed properties of observed X-ray sources with a time-dependent disk model — will be to do a parameter study of disks that are time dependent on the radial flow time scale and to investigate their properties. In short, we would like to find out what such disks can do and what they cannot do.

In §V we discuss implications of various of our results and indulge in some speculation as to future model building. Boundary conditions appropriate for the different physical situations considered are discussed in §IV, and a brief treatment of the numerical methods used is given in the Appendix.

II. INSTABILITY OF INNER REGION

In this section we consider only the inner region of the disk (radiation pressure dominant over gas pressure, electron scattering opacity dominant over free-free opacity), which may or may not exist, depending upon physical conditions. (For accretion disks around black holes, the inner region typically exists; for disks around magnetized neutron stars it typically does not.)

a) Analytic Analysis

The time evolution of the disk is governed by the nonlinear diffusion equation for surface density Σ (cf., eqs. [I-30], [I-7a])[1]

---

[1]Equations from Paper I will be referred to as I-equation number.

$$\frac{\partial\Sigma_4}{\partial t} = \frac{10^{-4}}{r_6} \frac{\partial}{\partial r_6} \left\{ \left(\frac{r_6}{M_*}\right)^{1/2} \frac{\partial}{\partial r_6} [r_6{}^2 W_{16}(r,\Sigma)] \right\} , \tag{1a}$$

$$W_{16} \equiv 2\,\alpha\, h_6(r,\Sigma)\; p_{10}(r,\Sigma) , \tag{1b}$$

together with implicit algebraic expressions for disk half thickness, h, and pressure, p, in terms of Σ and r (cf., eqs. [I-26-29]). [In equations (1) we have used the notation Σ_4, r_6, M_*, h_6, p_6, W_{16} to denote Σ (disk surface density)/10^4 g cm^{-2}, r(radius)/10^6 cm, M(mass of compact object)/3 $M_\odot$, h(disk half thickness)/10^6 cm, p(pressure)/10^{10} dynes cm^{-2}, and W(integrated shear stress)/10^{16} dynes cm^{-1}, respectively. Here unlike Paper I we will always use subscripts to indicate the above units.] In the inner region of the disk, one has approximately (cf., eqs. [I-11b] and [I-11c]) for the integrated shear stress W

$$W \propto (\alpha\Sigma)^{-1} . \tag{2}$$

[The parameter α appearing in eqs. (1b) and (2) is a dimensionless measure of the viscosity — see Paper I.] Hence, from equation (2),

$$\partial W/\partial r = (\partial W/\partial\Sigma)\,(\partial\Sigma/\partial r) \propto -\,\alpha^{-1}\Sigma^{-2}(\partial\Sigma/\partial r) , \tag{3}$$

and the effective diffusion coefficient in equation (1a) becomes negative. (Compare it with the linear equation

$$\partial\Sigma/\partial t \quad \mu\partial^2\Sigma/\partial r^2 , \tag{4}$$

where μ is the diffusion coefficient.) The result of a negative diffusion coefficient is for material to clump rather than to smooth out as happens for "normal" diffusion equations with positive diffusion coefficients. High density zones get higher in density; low density zones get lower, and the material clumps into rings. Since the clumping is secular and occurs at all wavelengths (see discussion below), there is nothing to prevent the high Σ zones from eventually achieving $\Sigma > \Sigma_{crit}$ (see §VIII of Paper I) and thus causing a breakdown of the model. At the same time, the low Σ zones eventually become optically thin and the model again breaks down.

We can estimate the time scale of the instability in the following simple way. For modes of wavelength λ solutions to the linear diffusion equation (4) have form

$$\Sigma \sim \cos(r/\lambda)\; e^{mt} \quad ,$$

where $m = -\mu/\lambda^2$ (positive for negative μ). On the other hand, the drift time scale, t_D, is the time it takes Σ to diffuse a distance r and is thus $t_D \sim r^2/\mu$. Therefore, the time scale for growth of an instability of wavelength λ is

$$t_{inst.} \equiv m^{-1} \approx (\lambda/r)^2\; t_D \quad . \qquad (5)$$

Both the existence of the instability discussed above and the time scale on which it occurs have been arrived at by considering approximations to the equations and by analogy with the linear diffusion equation. Validation of our results is given in §II.c, where equations (1a) and (1b) are solved numerically under conditions in which the inner region exists; there we graphically observe the clumping phenomenon and consequent breakdown of the model.

b) Physical Explanation

To understand why the instability of §II.a exists, we first must understand the puzzling result of equation (2): that integrated stress decreases as surface density increases. From equations (I-8c) and (I-8e) we have that the energy generation rate and the energy transport rate (radiative diffusion toward disk surface) are, respectively,

$$G \propto hp \quad ,$$

$$R \propto T^4 (\Sigma\bar{\kappa})^{-1} \quad .$$

But in the inner region, where radiation pressure dominates and electron scattering opacity dominates, $P \propto T^4$, $\bar{\kappa}$ const., so equating G to R gives

$$h \propto \Sigma^{-1} \quad .$$

Since $p \propto \Sigma h$ for hydrostatic equilibrium (cf., eq. [I-8f]), we have the result that pressure is independent of surface density and thus $W \propto hp \propto \Sigma^{-1}$. Physically, low stress in high-Σ regions and high stress in low-Σ regions means that matter is pushed into regions of low stress and thus the density contrast grows, forming rings of gas.

Let us turn for the moment to the time scale of the clumping phenomenon. Equation (5) suggests that infinitely short wavelength modes grow infinitely fast. However, there is a physical lower cutoff on the size of the wavelength. Because of the likelihood of turbulent mixing (one of the contributions to viscous stress) on length scales of the thickness of the disk ~ 2h, any structure in the disk must have wavelengths approximately satisfying $\lambda \gtrsim h$. Therefore, equation (5) should read

$$t_{inst.} \gtrsim (h/r)^2 \; t_D \quad . \qquad (6)$$

Since $t_D \sim \alpha^{-1}(r/h)^2 \; t_H \sim (r/h)^2 \; t_T$, where t_H and t_T are the hydrodynamical and thermal time scales (see §VI of Paper I), equation (6) can be rewritten in the form

$$t_{inst.} \gtrsim \alpha^{-1} t_H \approx t_T \quad . \qquad (7)$$

Again, we should point out that estimates such as equation (7) must be investigated and refined by actual numerical calculations, as is done in the next subsection.

It is worthwhile to point out that, according to the above analysis (and in actual numerical calculations), the middle and outer regions of the disk are stable against the clumping phenomenon discussed in this section (W increases with Σ in both middle and outer regions — see eqs. [I-9], [I-10].)

c) Numerical Analysis

We now illustrate the clumping instability of the previous subsection by considering perturbations of the inner region of the stationary disk model. In our model we take a 3 $M_\odot$ central object and an initial density profile of

$$\Sigma \propto r^{3/2} \quad , \tag{8}$$

in accordance with the functional dependence of Σ in the inner region of the stationary models (cf., SS or NT). We assume that the central object is a nonrotating black hole so that the inner edge of the disk extends to the innermost stable circular orbit of 3 Schwarzschild radii ($r_6 \approx 2.6$ for $M_* = 1$). The appropriate inner boundary condition on the diffusion equation is that surface density, Σ, vanish (see §IV for discussion). The outer boundary condition in the inner region (where it joins the middle region of the disk; see Paper I) is not crucial for purposes of investigating the clumping instability since the time scale of the instability is shorter than the drift time scale. For simplicity, we therefore use $\Sigma = 0$ at the outer boundary also.

In accordance with the discussion of the previous subsection we choose radial zone widths to be the thickness of the disk. (By eqs. [8] and [I-11c] the disk thickness and hence radial zone widths are approximately independent of radius in the inner region.) This necessarily prevents structure on length scales smaller than is physically allowed by turbulent mixing. It is not necessary to explicitly introduce a perturbation onto the initial density distribution, because of the automatic round-off errors of the computer.

The results are depicted in Figures 1a-1d for $\alpha = 10^{-2}$ as a typical example. As can be seen, on a time scale $t_{inst.} \sim 0.1$ sec the density distribution becomes irregular, the density contrast grows, radial zones become optically thin in the low-Σ regions, eventually one of the high-Σ zones reaches the "no solution region" $[\Sigma > \Sigma_{crit}(r)]$ and the computation cannot proceed any farther. Actually, the model partially breaks down as soon as a radial zone goes optically thin, since the diffusion equation for radiative transport [see assumption (vii) and eq. (I-2f) of Paper I] is no longer strictly applicable. In order to continue the time evolution as far as possible, however, we continue to use the radiative diffusion equation even in optically thin zones.

Figures 1a-1d numerically validate the "clumping instability" discussed previously. The time scale of the instability agrees with that estimated in the previous subsection, since

$$t_H \approx 10^{-3} \text{ sec} \qquad \text{for } r_6 \approx 8 \, , \quad M_* \quad 1 \, ,$$

and

$$t_{inst} \approx t_T \approx \alpha^{-1} t_H \approx 10^2 \cdot 10^{-3} \quad 0.1 \text{ sec} \quad .$$

We should point out that superimposed upon the time evolution of Σ as depicted in Figures 1a-1d are local fluctuations orginating from thermal processes, occurring on approximately the same time scale, $t_{inst} \simeq t_T$ — we have neglected such local fluctuations because our initial assumptions and equations restrict our attention to time scales $\gg t_T$. The overall instability, however, cannot be affected by the thermal fluctuations because it occurs on wavelengths $\lambda \gg h$ as well as $\lambda \sim h$ (cf., eqs. [5], [6], and [7]) and thus on time scales much longer than t_T. The time scale t_T only estimates the time of growth of the shortest allowed wavelength perturbations. Thus the clumping instability persists even after a time average over many thermal time scales.

III. NEUTRON-STAR-LIKE MODELS

When the inner edge of the disk occurs at a large enough radius, typically the case for magnetized neutron stars and white dwarfs, the inner region of the disk does not exist (gas pressure always greater than radiation pressure) and the clumping instability discussed in the previous section never occurs. In such cases, the density distribution may be evolved to completion, i.e., until all of the mass in the disk has flowed inwards onto the compact object or outwards to infinity or until the disk has reached a steady state in the case of a constant mass deposition rate into the disk.

In order to allow mass deposition into the disk as well as mass flux out of the disk at its inner and outer edges, we generalize the density diffusion equation, equations (1a), (1b) to allow a source function s(r,t):

$$\frac{\partial \Sigma_4}{\partial t} = \frac{10^{-4}}{r_6} \frac{\partial}{\partial r_6}\left[\left(\frac{r_6}{M_*}\right)^{1/2} \frac{\partial}{\partial r_6} (r_6{}^2 W_{16})\right] + s(r,t) \quad . \qquad (9)$$

The source function s(r,t) may, for example, be the result of variable mass deposition into the disk from a normal companion star. For simplicity we consider only the cases

i) $s(r,t) = 0$

ii) $s(r,t) = \begin{cases} \text{constant} \cdot \delta_{r,r_o}, & 0 < t \leq \Delta t \\ 0, \quad t > \Delta t, & \end{cases}$

where r_o is a radial zone near the outer edge of the disk ($r_o \approx 2/3\ r_{max}$). The constant multiplying the Kronecker delta in ii) is chosen so that in time Δt, 10^{24} grams are deposited in the disk. Thus in case ii) $s(r,t)$ is determined completely by r_o and Δt. In case i) the initial distribution of Σ at $t = 0$ is chosen so that there are 10^{24} grams in the disk and all of the mass is in a narrow annulus.

Figures 2 and 3 are typical illustrations of the time evolution of Σ for the cases $\eta = 1.0$ and $\eta = 0.1$, where

$$\eta \equiv \frac{\text{Alfvén radius}}{\text{corotation radius}} \equiv \frac{r_A}{r_c} \quad . \tag{10}$$

Once $\Sigma(r,t)$ is known, all other variables may be easily calculated by algebraic equations (see Paper I). We will now define the parameter η.

The corotation radius, r_c, is defined as that radius at which the Kepler period equals the period of rotation of the compact object. The Alfvén radius, r_A, is that radius at which the magnetic pressures originating from the magnetic field of the compact object exceed the internal pressures in the disk and thus force gas to leave the disk and follow magnetic field lines onto the compact object. The inner edge of the disk thus occurs at $r = r_A$. For $r_A > r_c$ the compact object spins up the disk (transfers mechanical energy and angular momentum into the disk). For $r_A < r_c$ the disk produces a mechanical torque on the compact object. It is likely that accretion from the disk onto the compact object can occur only if $r_A < r_c$ (Lamb, Pethick, Pines

1973), so we consider only $\eta \leq 1.0$.

The principal impact of the value of η is to determine the boundary condition at the inner edge of the disk r_A (see §IV). Since that boundary condition is actually a function of $\sigma \equiv (1 - \eta^{3/2})$, (eq. [17a] below), we can explore essentially the full range of possible cases by considering for η values of 1.0 and 0.1.

In reality r_A and hence η do not remain constant in the evolution of a nonstationary disk. Rough estimates (see, e.g., Lamb, Pethick, Pines 1973) show that η depends upon such time changing quantities as mass flux and disk thickness at the inner edge of the disk. To simplify a very complex situation, we fix the mass and period of the compact object ($M = 1/3$, $T = 1.2$ sec in Figs. 2 and 3), thus fixing r_c, and we assume that η remains constant during the evolution of the disk (assume r_A remains constant). This simplification, although not quite correct, does allow us to investigate the effect of different inner boundary conditions (different η) on the time evolution of Σ and on the mass flux $\dot{M}$ onto the compact object.

As can be seen from Figures 2 and 3, the effect of different values of η is to change the curvature of the Σ distribution at the inner edge of the disk — values of η increasingly smaller than unity require an increasingly steeply peaked Σ at r_A in order to transport into the disk the angular momentum required for accretion at r_A. See §IV.b for a further discussion of this point.

Figures 4 and 5 give the mass flux $\dot{M}$ from the disk onto the compact object (see §IV.c for a calculation of mass fluxes) for various values of α and Δt for the two cases $\eta = 1.0$ and $\eta = 0.1$. The intrinsic effect of different values of η on $\dot{M}(t)$ can be seen by comparing the B curves of Figures 4 and 5 [where $\Delta t = 0$; source law (i) above]. For $\eta = 1.0$, the mass flux peaks sooner, has a higher maximum, and a faster fall off than for $\eta = 0.1$. This can be understood in terms of the different boundary conditions (and associated Σ distributions) at the inner edge. (See Figs. 2 and 3 and discussion in previous paragraph.) The positive slope of the $\eta = 0.1$ Σ distribution at r_A reflects the effective resistance to accretion due to the requirement that stress transport angular momentum outwards at r_A — consequently $\dot{M}$ is not as sharply peaked a function (and consequently has a smaller maximum for the same total mass) as the corresponding function for $\eta = 1.0$, where the Σ distribution has <u>negative</u> slope at r_A and angular momentum does not have to be transported outwards at r_A for accretion to take place.

The mass fluxes of Figures 4 and 5 may be converted to luminosities L of the compact object by consideration of the released gravitational energy. In order of magnitude, the relations are

$$L \sim 10^{-4}\, \dot{M}c^2 \quad \text{for white dwarf} \quad , \tag{11a}$$

$$L \sim 10^{-1}\, \dot{M}c^2 \quad \text{for neutron star} \quad . \tag{11b}$$

In Figures 2-5, the time evolution is carried nearly to completion — i.e., in Figures 2 and 3 curves F and E represent late time shapes and in Figures 4 and 5 curves are continued (with the exception of the A curves) until at least 99.9% of the mass put into the disk has escaped onto the compact object or outwards through the outer edge of the disk. The outer edge of the disk in Figures 2-5 is located at $r_6 = 30{,}000$, a typical value for models of compact X-ray sources.

It is interesting to note that curves F and E of Figures 2 and 3, respectively, which depict the density distribution after it has equilibrated, fall off with power laws consistent with those calculated in the stationary models. For example, curve E of Figure 3 has the form

$$\Sigma \sim r^{-0.742} \quad ,$$

and a mass flux so low that the entire disk is in the "outer region" (e.g., free-free opacity dominant over electron scattering, gas pressure dominant over radiation pressure — see Paper I). The calculated functional form for Σ in the stationary models, in the outer region of the disk is (SS, NT)

$$\Sigma \propto r^{-3/4} \quad .$$

Therefore it is fairly clear that, for parameters such that the "inner region" of the disk does not exist, stationary disks of the form currently in the literature can be built up from nonstationary disks.

An important feature of the $\dot{M}$ curves (Figures 4 and 5) is the spiral time, t_s, defined as the time required for an initial ring of matter near the outer edge of the disk to diffuse inwards and yield an appreciable (e.g., $\dot{M} \sim 0.01$ of maximum subsequent value) accretion rate onto the compact object. The spiral time depends strongly on α and less strongly on the average value of Σ in the outer regions of the disk during the initial stages of diffusion — which we denote by $\bar{\Sigma}$. A fairly good analytic fit to the numerical results for an initial ring at radius $r \approx 2 \times 10^{10}$ cm and for $\Delta t = 0$ [case (i) above; no matter fed to disk after $t = 0$] is

$$t_s \approx (2.3 \times 10^3 \text{ sec.})\ \alpha^{-1.2}\left(\frac{\bar{\Sigma}}{5 \times 10^3 \text{ g cm}^{-2}}\right)^{-0.5} \quad . \qquad (12)$$

There is good theoretical basis for the form of equation (12). The spiral time t_s should be closely related to the drift time scale, t_D (see §VI of Paper I for definition of t_D). From Paper I,

$$t_D \sim (r/h)^2\ \alpha^{-1}(r^3/GM)^{1/2} \quad , \qquad (13a)$$

and, in the outer region of the disk, where time scales are longest, we have the approximate relation

$$h \propto \Sigma^{3/14} \alpha^{1/14} (r^3/GM)^{3/7} \quad . \tag{13b}$$

So, neglecting the r and M dependence, which are fixed in the examples of Figures 4 and 5, we should have approximately

$$t_s \propto t_D \propto \alpha^{-8/7} \Sigma^{-3/7} \quad . \tag{14}$$

Within the degree of approximation, equation (14) agrees well with the empirically determined equation (12). We will return to a brief discussion of the spiral time in the next section.

Note that the case $\alpha = 1.0$, $\eta = 0.1$ is not depicted (cf., Fig. 5). This is because regardless of how slowly mass is deposited into the disk (regardless of Δt), the surface density always builds up until $\Sigma > \Sigma_{crit}$ at the inner boundary of the disk, for $\alpha \approx 1.0$, $\eta \approx 0.1$. Thus for such a case the inner radial zones of the disk eventually create an inner region of the disk, the clumping instability sets in, and the computation can proceed no further. Such a situation also arises for $\alpha \approx 0.1$, $\eta \approx 0.1$ unless $\Delta t \gtrsim 10^6$ sec. We will return to a brief discussion of these results in §V.

IV. BOUNDARY CONDITIONS AND MASS FLUX

In this section we discuss the appropriate boundary conditions for our diffusion equation in surface density (cf., eqs. [1], [9]) and the calculation of the mass flux through the inner and outer boundaries.

a) Outer Boundary Condition

In a realistic accretion problem, the location of the outer edge of the disk, r_{Max}, is determined by a balance of the gravitational pull of the compact object and the centripetal acceleration of the inflowing gas. Gas flowing by the disk at $r > r_{Max}$ has too much energy and angular momentum to be captured into orbit and siphons off the angular momentum of gas moving outwards through r_{Max}. Since the evolution of the interesting small-r regions of the disk is relatively insensitive to the outer boundary condition on Σ, we set $\Sigma = 0$ at $r = r_{Max}$ for simplicity. (The principal effect of variations of this boundary condition is to change by a factor of order unity the ratio of mass escaping outwards through the outer edge to mass accreting inwards through the inner edge of the disk.)

b) Inner Boundary Condition

The inner boundary condition on Σ at the inner edge of the disk, r_{Min} ($= r_A$ for magnetized compact objects), is more important than the outer boundary condition, since it is near the regions of largest energy production and has a direct influence on mass accretion from the disk onto the compact object.

For accretion onto a black hole, the inner edge of the disk is at the innermost stable circular orbit; after gas reaches r_{Min}, it falls rather directly into the hole, providing little contact with gas just outside of r_{Min}. Thus the transferred stress at r_{Min} in this case vanishes and the proper boundary condition is $\Sigma = 0$ at $r = r_{Min}$ (insures that stress, W, vanishes at $r = r_{Min}$).

For accretion onto a magnetized compact object the shear stress, W, may or may not vanish at the inner edge of the disk, depending on whether the magnetic field lines are rotating at the same rate as the gas at $r = r_A$. Although the detailed interaction of the magnetic field and the accreting gas may be quite complicated at the Alfvén radius r_A, the stress at r_A may be calculated just from the basic conservation laws. In addition to the quantities M, $\dot{M}$, r_A, and r_c which have been previously defined, let us define the following quantities:

ω_B = angular velocity of compact object and it magnetic field,

ℓ_B = specific angular momentum of gas after it has accreted onto magnetic field lines at r_A,

ℓ_D = specific angular momentum of gas in disk at r_A.

Then, equating the torque to the time rate of change of angular momentum across r_A, we get

$$(2\pi r^2 W)_{r=r_A} = [\dot{M}(\ell_D - \ell_B)]_{r=r_A} \quad . \tag{15}$$

We may use further relations among the quantities above

$$\ell_D = (GMr_A)^{1/2} \quad , \tag{16a}$$

$$\ell_B = \omega_B r_A^2 \quad , \tag{16b}$$

$$\omega_B = (GM/r_c^3)^{1/2} \quad , \tag{16c}$$

to rewrite equation (15) as

$$(2\pi r^2 W)_{r_A} = [\dot{M}(GMr_A)^{1/2}(1 - \eta^{3/2})]_{r_A} \quad , \tag{17a}$$

where

$$\eta \equiv r_A/r_c \quad . \tag{17b}$$

Since W is a function of Σ, and $\dot{M}$ is a function of Σ and its spatial derivative (see §IV.c below), the boundary condition of equations (17) is a linear lation between Σ and its first derivative with respect to r at r_A. Note at if $r_A = r_c$, we have the same inner boundary condition, i.e., $\Sigma = 0$ at r_A, used for a disk around a black hole.

c) Mass Flux at Boundaries

The diffusion equation for surface density has the form (cf., eqs. [9], [10])

$$\dot{\Sigma} = r^{-1}(\partial F/\partial r) + s \quad , \tag{18a}$$

where

$$F \propto \left(\frac{r}{M}\right)^{1/2} \frac{\partial}{\partial r} \left[r^2 W(r,\Sigma) \right] \quad . \tag{18b}$$

The mass of the disk, M_D, is

$$M_D = \int 2\pi r \Sigma dr \quad . \tag{19}$$

Combining equations (18) and (19), we get

$$\begin{aligned} \dot{M}_D &= 2\pi \int_{r_{Min}}^{r_{Max}} \left(\frac{\partial F}{\partial r} + rs\right) dr \\ &= 2\pi (F_{Max} - F_{Min}) + 2\pi \int rsdr \quad . \end{aligned} \tag{20}$$

The first and second terms in equation (20) are the negatives of the mass fluxes out of the disk at the outer (to "infinity") and inner (onto compact object) edges. It is the second of these expressions which is used in calculating $\dot{M}$ in Figures 4 and 5. For the initial mass distribution given in Figures 2 and 3 and for mass fed into the disk near its outer radius, typically half of the disk mass escapes to infinity and half accretes onto the compact object.

The third term in equation (20), which may be zero in some applications (e.g., Fig. 1 and curve A in Fig. 4), represents the mass flux into the disk.

Since this mass, in our computations, is deposited a few radial zones in from the outer edge of the disk (see discussion in §III), it can be easily distinguished from mass leaving the disk at its outer edge.

V. DISCUSSION OF RESULTS

a) Implications of Instability of Inner Region of Disk

The "inner region" of the disk typically exists for disks around black holes and, under certain conditions (see, e.g., discussion at end of §III), for disks around neutron stars. Because of the instability of the inner region, discussed in §II, a revision of current disk models must be made. In this subsection we discuss the three most likely revisions: (1) departure from thinness in the inner region, (2) reformulation of model of viscosity, and/or (3) reformulation of model of energy transport. Items (2) and (3) may not be completely independent.

One of the principal assumptions going into the model is that the disk is everywhere thin. This may not be true in the inner region of the disk; the secular instability of §II may produce around the compact object a cloud tens to hundreds of times its size. If dissipation is efficient (expected, since accreting matter must still lose its angular momentum), the cloud will emit X-rays as a hot, thin plasma with Comptonization probably important [Felten and Rees (1972), Illarinov and Sunyaev (1972)]. Gross time variations, both in intensity and in spectrum, are expected on the hydrodynamical time scale of the cloud [~ tens to hundreds of milliseconds (for black hole) and longer]. If the cloud is optically thick to Compton scattering, time variations on time scales shorter than the random walk time of a photon through the cloud may be lost. Recent results obtained by Rothschild et al. (1973)

suggest, however, Cyg X-1 has time variations on time scales $\lesssim$ 1 msec., significantly shorter than the shortest expected time scale if Cyg X-1 is surrounded by an optically thick cloud. The inner region of the disk around Cyg X-1 may still be thin, therefore, but with a different model of viscosity and/or energy transport than is currently used (see discussion below). The role of a disk and the nature of its inner region for the exciting case (probably a black hole!) of Cygnus X-1 must await further taking and analysis of data.

Another likely possibility is that the disk is everywhere thin, but that the standard simplified model of viscosity (e.g., eqs. [I-6], [I-7]) is not correct. For example, the viscous stress might not be proportional to the total pressure. With the viscosity parameter α a function of Σ rather than a constant, equation (2) shows that if α falls at least as fast as Σ^{-1} in the inner region (less efficient viscosity) the effective diffusion coefficient for Σ (cf., eqs. [3], [4] and accompanying discussion) will be nonnegative. Such a functional dependence would therefore prevent the clumping instability and consequent breakdown of the model. For example (Cunningham 1973), equation (I-6) might be replaced by

$$(r^{-1}t_{\varphi r}) = \beta P_G \ , \qquad \beta = \text{const.} \ , \tag{21}$$

even when radiation pressure, P_R, greatly exceeds gas pressure, P_G. Using equations (I-5), one can easily show that equation (21) corresponds to an effective α satisfying, in the inner region,

$$\alpha \propto \Sigma^{-8/3} \ .$$

The thin-disk model resulting from equation (21) is stable and much like current models except that Σ is much greater in the inner region of a stationary disk.

A third related possibility (which also requires a better understanding of viscosity for proper treatment) is that the method of energy transport is different from the simplified model of radiative transport (cf., §II and eq. [I-2f] of Paper I). Convection probably sets in at $\Sigma \lesssim \Sigma_{crit}$, since the radiative temperature gradient becomes superadiabatic there (see §VIII.b of Paper I) and may be effective enough as an energy transport mechanism to prevent the inner region from entering the "no solution" region (§VIII of Paper I) — the disk may then remain thin. There may actually not even be a need for energy transport, if the energy generation is near the surface of the disk.

In summary, the existence of the inner-region instability discussed in §II depends upon the assumption that the energy transport is radiative, that the energy generation occurs in an optically thick region and must therefore be transported, that the standard model of viscosity is valid, and that the disk is everywhere thin. One or more of the above assumptions may be invalid.

b) Implications for Extended Lows in Compact X-Ray Sources

As can be seen by comparing Figure 2 with curve A in Figure 4 (representing same values of parameters), substantial mass does not accrete from the inner edge of the disk onto the compact object until the peak of the Σ distribution has moved from the outer to the inner edge of the disk. Both the spiral time scale t_s and the "turn-on" time (rise time of $\dot{M}$ in Figs. 4 and 5) are sharply decreasing functions of α. A fairly common feature for disks which are not fed with mass after $t = 0$ is that the time scale for fall off of mass flux onto the compact object is about 100 times the spiral time scale. This fact may be important in explaining phenomena which are regulated by quasiperiodic loading and unloading of mass in an accretion disk.

As an example, one of the various explanations for the peculiar 35-day periodicity in the X-ray flux of Her X-1 is that the 24 day off time occurs while gas is spiralling in from the outer edge of an accretion disk, and the 11 day on time occurs while the accumulated mass of the disk is unloading onto the neutron star (McCray 1973). In this model, while the disk is unloading, radiation pressure from the emission of the neutron star prevents further deposition of gas by the primary into the outer edge of the disk. From the results indicated in Figures 4 and 5 it is clear that, in order to yield a spiral time of the order of 24 days, α must be less than or of the order of 10^{-2}. For such a value of α, however, the decay time of the mass flux is far longer than 11 days. Thus, it seems unlikely that the 35-day periodicity of Her X-1 is regulated by this type of loading and unloading of an accretion disk. [Actually, the evidence (Forman et al., 1972) for almost constant X-ray heating of HZ Herc during the 35-day cycle is more crucial in ruling out McCray's model.]

Another example in which disk dynamics may be important is in the extended highs and lows of some X-ray sources, e.g., Cen X-3 (Schreier et al., 1972b), HZ Herc (Jones et al., 1973), or SMC X-1 (Schreier et al., 1972a). It is conceivable that the $\approx$ 20 year quasiperiodicity in HZ Herc could be explained by a very slow ($\alpha \approx 10^{-4}$, see A curves in Figs. 4 and 5) unloading of an accretion disk.

Finally, the existence of an "inner region" in neutron-star-like models for large values of the viscosity parameter α and with $\eta > 0.1$ (see discussion in §III) indicates that either such values of the parameters are not realized, or that some reformulation of the model must be made, as discussed in §V.a.

c) Future Work

Future work on accretion disks should be along the lines discussed in §V.a. A more detailed model of viscosity is needed particularly strongly. Such a model must await a better understanding of turbulence and chaotic magnetic fields.

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APPENDIX I

NUMERICAL METHODS

In this section we give a brief discussion of our numerical techniques for solving the time evolution equations of the disk. The equations reduce to an implicit algebraic equation for $T(r,\Sigma)$ and a nonlinear diffusion equation for $\Sigma(r,t)$, (see Paper I, §VII).

a) Algebraic Equation for T

After each time step, one has a $\Sigma(r)$. With this Σ, one then solves in each radial zone

$$H(T,\Sigma,r) = 0 \quad , \tag{A-1}$$

where H is an implicit nonlinear equation in T (see eq. [I-29]). Equation (A-1) is solved for both the gas and radiation roots T_G and T_R (see §VIII of Paper I) by Newton's method, using as initial guesses for T_G and T_R their values at the previous time step. The iteration procedure is continued until convergence of about 0.01 percent is achieved. From T all other disk variables may be calculated by explicit algebraic equations.

b) Nonlinear Diffusion Equation for Σ

In problems with widely differing time scales (e.g., drift time scale t_D in different radial zones) it is advantageous to always take time steps in accordance with the ongoing dynamical processes rather than with the shortest time scale in the problem (which may describe processes which have already equilibrated). To do so, one must use implicit differing methods (e.g., Isaacson and Keller 1966) in order to avoid numerical instabilities.

In nonlinear equations, some linearization procedure must be used in order to solve the implicitly differenced equations, and such a procedure must not introduce enough "explicitness" to cause numerical instabilities. In our case the implicitly differenced equation, before linearization, is (Here we let n denote the time step label and suppress spatial zone indices.)

$$\frac{\Sigma_n - \Sigma_{n-1}}{\Delta t} = \frac{10^{-4}}{r} \frac{\partial}{\partial r} \left\{ \left(\frac{r}{M}\right)^{1/2} \frac{\partial}{\partial r} \left[r^2 W_n(r, \Sigma_n) \right] \right\} + s_n \quad , \tag{A-2a}$$

and we Taylor expand W_n about the previous time step

$$W_n \approx W_{n-1} + \frac{W_{n-1} - W_{n-2}}{\Sigma_{n-1} - \Sigma_{n-2}} (\Sigma_n - \Sigma_{n-1}) \quad , \tag{A-2b}$$

where

$$W_n = 2\alpha h_n p_n \quad . \tag{A-2c}$$

When spatial indices are put in, equations (A-2) can be solved for $\Sigma_n(r)$ by standard matrix techniques (see, e.g., Isaacson and Keller 1966).

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FIGURE CAPTIONS

Figs. 1a-1d. Time evolution of surface density in the inner region of a disk around a black hole beginning, at $t = 0$, with the distribution of a stationary disk. Parameters of this model are $M_* = 1$, $\alpha = 10^{-2}$. Circles indicate optically thin zones and the square in Fig. 1d indicates a zone having gone into the "no-solution region." The peak value of Σ in each plot is indicated by Σ_{Max}.

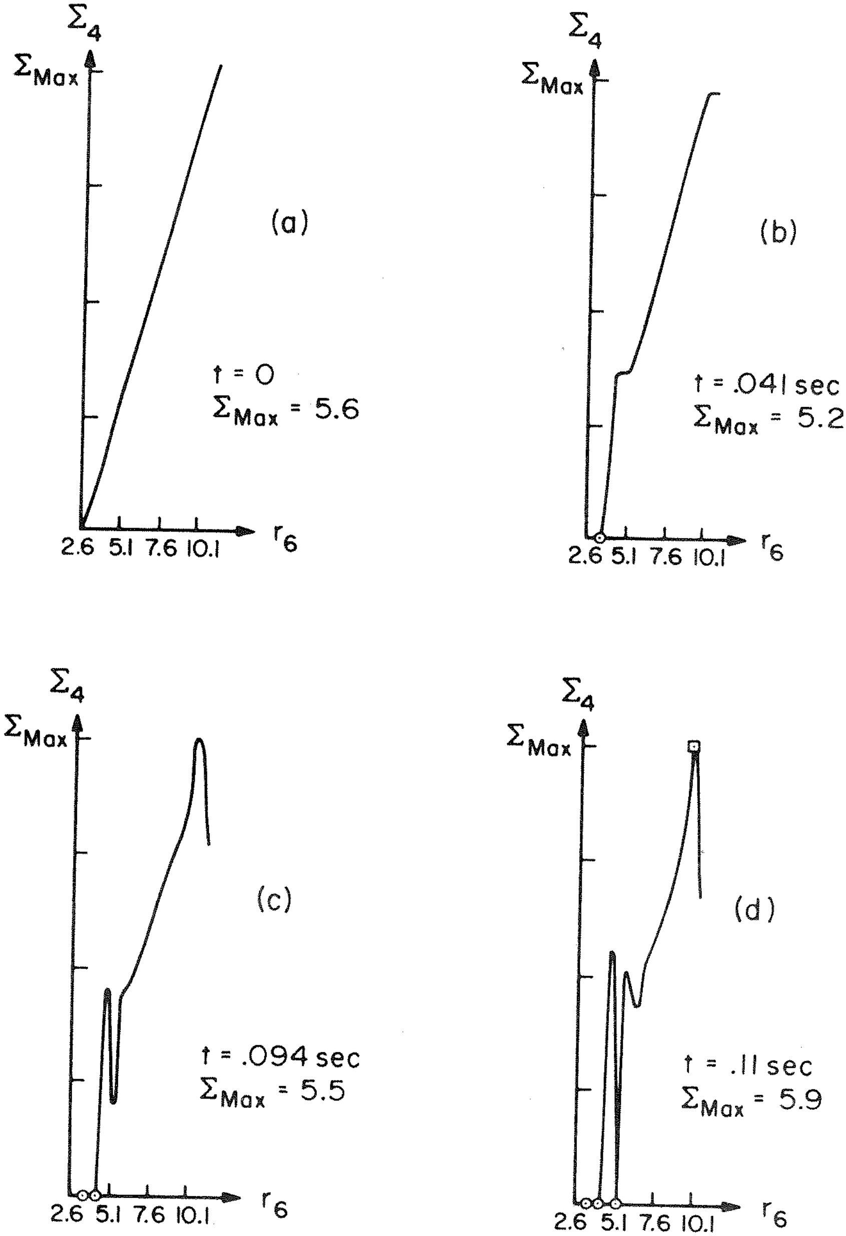
Fig. 2. Time evolution of surface density of disk for "neutron-star-like" models. In this model $M_* = 1/3$, $\alpha \approx 0.01$, $\eta = 1.0$. The various curves are A: $t = 0$, $\Sigma_{Max} = 0.76$; B: $t = 3.5 \times 10^5$, $\Sigma_{Max} = 0.19$; C: $t = 5.8 \times 10^5$, $\Sigma_{Max} = 0.20$; D: $t = 6.6 \times 10^5$, $\Sigma_{Max} = 0.22$; E: $t = 8.0 \times 10^5$, $\Sigma_{Max} = 0.37$; F: $t = 4.3 \times 10^7$, $\Sigma_{Max} = 0.03$. (Time is measured in seconds.)

Fig. 3. Time evolution of surface density of disk for "neutron-star-like" models. In this model $M_* = 1/3$, $\alpha = 0.01$, $\eta = 0.1$. The various curves are A: $t = 0$, $\Sigma_{Max} = 0.1$; B: $t = 8.1 \times 10^5$, $\Sigma_{Max} = 0.089$; C: $t = 9.0 \times 10^5$, $\Sigma_{Max} = 0.09$; D: $t \approx 9.6 \times 10^5$, $\Sigma_{Max} = 0.18$; E: $t \approx 2.3 \times 10^7$, $\Sigma_{Max} \approx 1.36$.

Fig. 4. Mass flux from disk onto compact object for various values of α and of time interval, Δt, over which mass is deposited into outer extremities of disk. In this model $M_* = 1/3$, $\eta = 1.0$. The various curves are A: $\alpha = 0.0001$, $\Delta t = 0$; B: $\alpha = 0.01$, $\Delta t = 0$; C: $\alpha = 1.0$, $\Delta t = 10^6$ sec; D: $\alpha = 0.1$, $\Delta t \approx 0$; E: $\alpha \approx 1.0$, $\Delta t \approx 0$. (Recall: for $\Delta t \approx 0$ the disk has mass in it, distributed over several zones, at $t \approx 0$. For $\Delta t \neq 0$ there

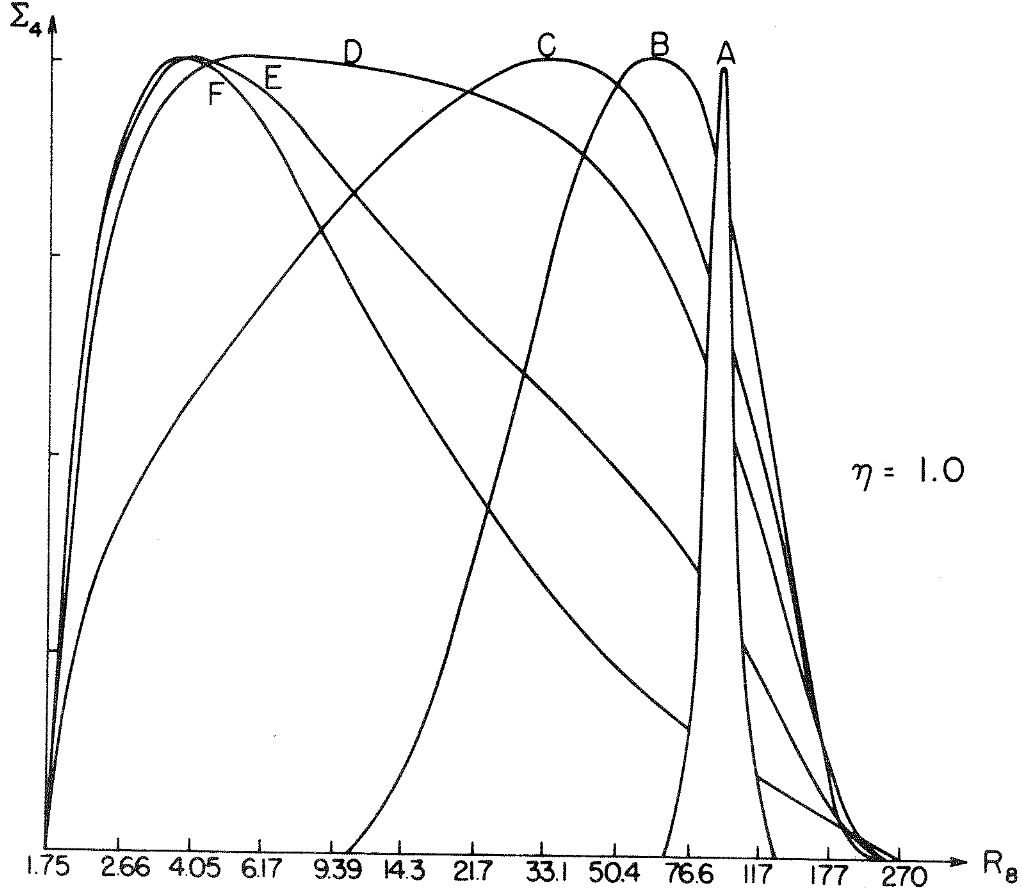
is no initial mass content.)

Fig. 5. Mass flux from disk onto compact object for various values of α and time interval, Δt, over which mass is deposited into outer extremities of disk. In this model $M_* = 1/3$, $\eta = 0.1$. The curves are A: $\alpha = 0.0001$, $\Delta t = 0$; B: $\alpha = 0.01$, $\Delta t = 0$; C: $\alpha = 0.1$, $\Delta t = 10^6$ sec.



Σ4
ΣMax
(a)
t = 0
ΣMax = 5.6
2.6 5.1 7.6 10.1
r6
Σ4
ΣMax
(b)
t = .041 sec
ΣMax = 5.2
2.6 5.1 7.6 10.1
r6
Σ4
ΣMax
(c)
t = .094 sec
ΣMax = 5.5
2.6 5.1 7.6 10.1
r6
Σ4
ΣMax
(d)
t = .11 sec
ΣMax = 5.9
2.6 5.1 7.6 10.1
r6

Fig. 1



Σ_4
A
B
C
D
E
F
η = 1.0
1.75
2.66
4.05
6.17
9.39
14.3
21.7
33.1
50.4
76.6
117
177
270
R_8

Fig. 2

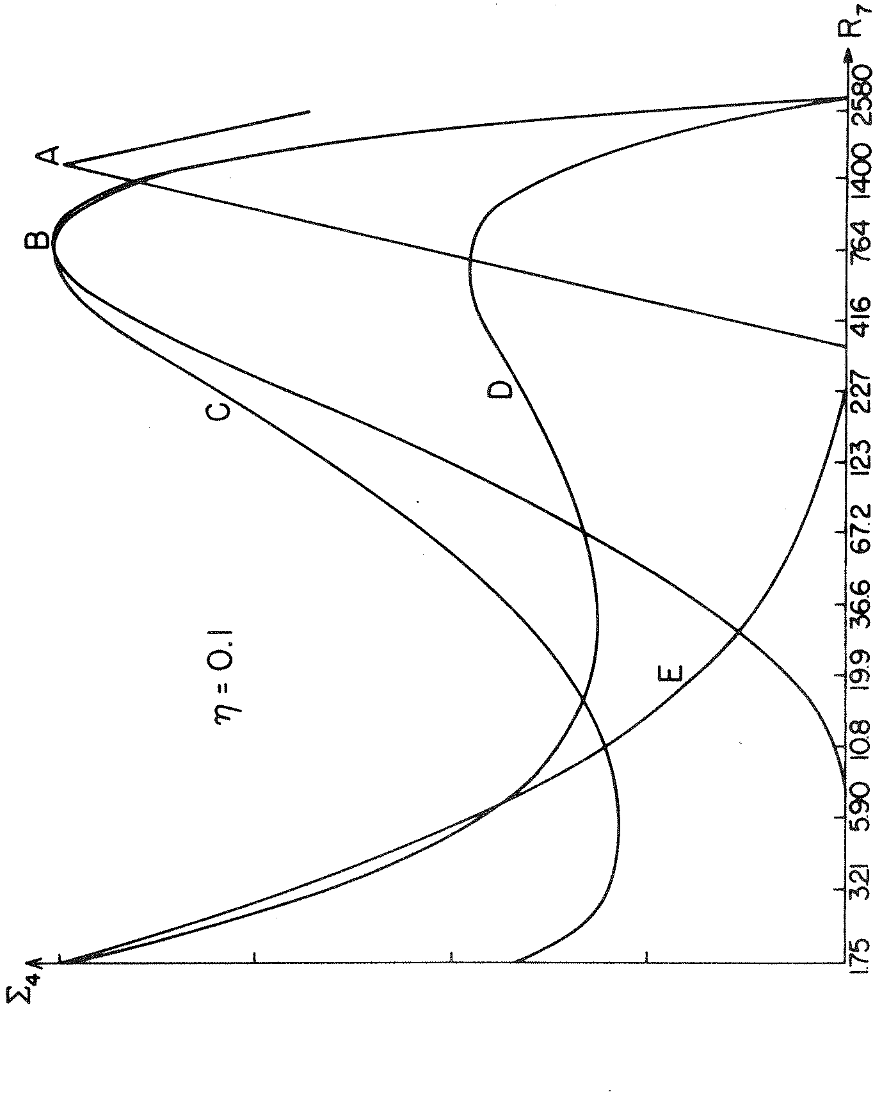
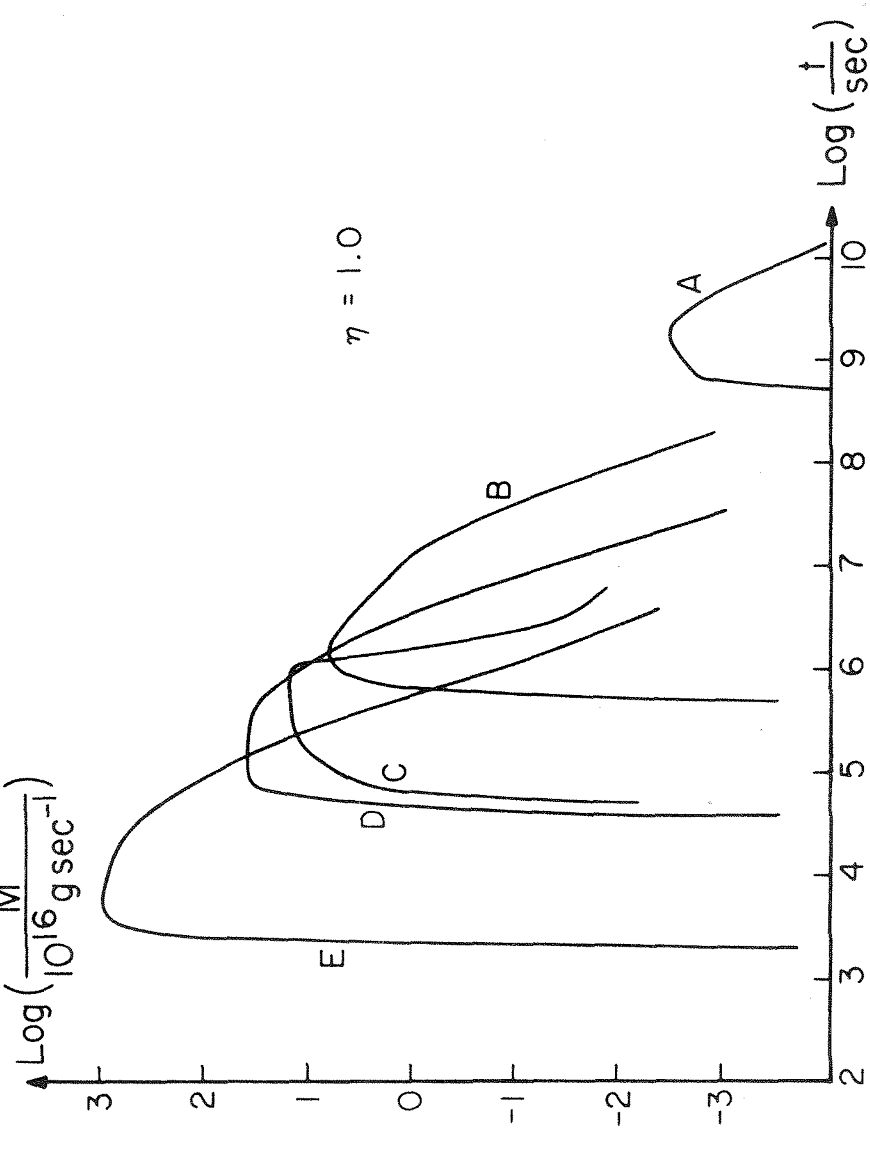

$\eta = 0.1$
A
B
C
D
E
Σ_4
R_7
1.75
3.21
5.90
10.8
19.9
36.6
67.2
123
227
416
764
1400
2580

Fig. 3



$\eta = 1.0$
$\mathrm{Log}\left(\frac{\dot{M}}{10^{16}\,\mathrm{g\,sec^{-1}}}\right)$
$\mathrm{Log}\left(\frac{t}{\mathrm{sec}}\right)$
A
B
C
D
E
3
2
1
0
-1
-2
-3
2
3
4
5
6
7
8
9
10

Fig. 4

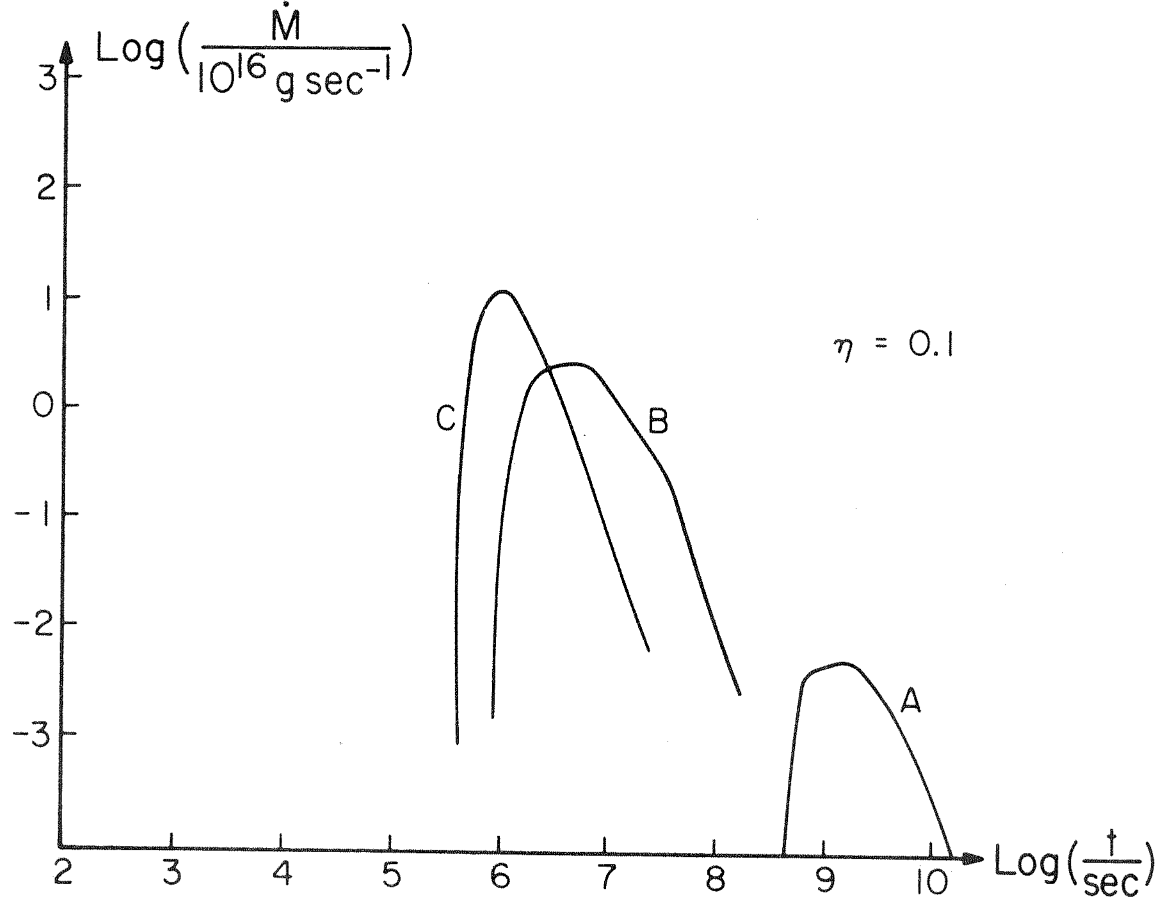


Fig. 5

II. THEORETICAL FRAMEWORKS FOR ANALYZING AND TESTING GRAVITATION THEORIES

A. DEFINITIONS AND BASIC CONCEPTS

a) Foundations for a Theory of Gravitation Theories (Paper IV; collaboration with K.S. Thorne and D.L. Lee, published in Phys. Rev. D, **7**,3563,1973.)

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Foundations for a Theory of Gravitation Theories*

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A foundation is laid for future analyses of gravitation theories. This foundation is applicable to any theory formulated in terms of geometric objects defined on a 4-dimensional spacetime manifold. The foundation consists of (i) a glossary of fundamental concepts, (ii) a theorem that delineates the overlap between Lagrangian-based theories and metric theories; (iii) a conjecture (due to Schiff) that the weak equivalence principle implies the Einstein equivalence principle; and (iv) a plausibility argument supporting this conjecture for the special case of relativistic, Lagrangian-based theories.

I. INTRODUCTION

Several years ago our group initiated[1] a project of constructing theoretical foundations for experimental tests of gravitation theories. The results of that project to date (largely due to Will and Ni) and the results of a similar project being carried out by the group of Nordtvedt at Montana State University are summarized in several recent review articles.[2-4] Those results have focused almost entirely on "metric theories of gravity" (relativistic theories that embody the Einstein equivalence principle; see Sec. III below).

By January 1972, metric theories were sufficiently well understood that we began to broaden our horizons to include nonmetric theories. The most difficult aspect of this venture has been communication. The basic concepts used in discussing nonmetric theories in the past have been defined so vaguely that discussions and "cross-theory analyses" have been rather difficult. To remedy this situation we have been forced, during these last eleven months, to make more precise a number of old concepts and to introduce many new ones. By trial and error, we have gradually built up a glossary of concepts that looks promising as a foundation for analyzing nonmetric theories.

Undoubtedly we shall want to change some of our concepts, and make others more precise, as we proceed further. But by now our glossary is sufficiently stabilized, and we have derived enough interesting results using it, that we feel compelled to start publishing.

This paper presents the current version of our glossary (Secs. II–IV), and uses it to outline some key ideas and results about gravitation theories, both nonmetric and metric (Secs. V and VI). Subsequent papers will explore some of those ideas and results in greater depth.

Central to our current viewpoint on gravitation theories is the following empirical fact. Only two ways have ever been found to mesh a set of gravitational laws with all the classical, special relativistic laws of physics. One way is the route of the Einstein equivalence principle (EEP) – (i) Describe gravity by one or more gravitational fields, including a metric tensor $g_{\alpha\beta}$; and (ii) insist that in the local Lorentz frames of $g_{\alpha\beta}$ all the nongravitational laws take on their standard special relativistic forms. The second way of meshing is the route of the Lagrangian – (i) Take a special relativistic Lagrangian for particles and nongravitational fields, and (ii) insert gravitational fields into that Lagrangian in a manner that retains general covariance. The equivalence-principle route always leads to a metric theory. (Example: general relativity.) The Lagrangian route always leads to a "Lagrangian-based theory." [Example: Belinfante-Swihart theory (Table IV, later in this paper).] Thus, in the future we expect most of our attention to focus on metric theories and on Lagrangian-based theories; and in the nonmetric case we might be able to confine attention to theories with Lagrangians.

Since metric theories are so well understood,[4] it would be wonderful if one could prove that all nonmetric, Lagrangian-based theories are defective in some sense. A conjecture due to Schiff[5] points to a possible defect. Schiff's conjecture says[5] that *any complete and self-consistent theory that obeys the weak equivalence principle (WEP) must also, unavoidably, obey the Einstein equivalence principle (EEP).* (See Sec. III for precise definitions.) Since any relativistic, Lagrangian-based theory that obeys EEP is a metric theory, this conjecture suggests that nonmetric, relativistic, Lagrangian-based theories should always violate WEP.

The experiments of Eötvös *et al.*[6] and Dicke *et al.*,[7] with modifications by Braginsky *et al.*[8] (ED experiments), are high-precision tools for testing WEP. Hence, the Schiff conjecture suggests that, if one has a nonmetric Lagrangian-based theory, one should test whether it violates the ED experiments. (Such tests for the Belinfante-Swihart and Naida-Capella theories reveal violations of ED and WEP.[9])

In this paper, after presenting our glossary of concepts (Secs. II–IV), we shall (i) derive a criterion for determining whether a Lagrangian-based theory is a metric theory ("principle of universal coupling," Sec. V), and (ii) discuss and make plausible Schiff's conjecture (Sec. VI).

II. CONCEPTS RELEVANT TO SPACETIME THEORIES

This section, together with Secs. III and IV, presents our glossary of concepts. To understand these concepts fully, the reader should be familiar with the foundations of differential geometry as laid out, for example, by Trautman.[10] He should also be familiar with Chap. 4 of Anderson's textbook[11] (cited henceforth as JLA), from which we have borrowed many concepts. However, he should notice that we have modified slightly some of JLA's concepts, and we have reexpressed some of them in the more precise notation and terminology of Trautman[10] and of Misner, Thorne, and Wheeler (MTW).[12]

The concepts introduced in this section apply to any "spacetime theory" (see below for definition). In Secs. III and IV we shall specialize to "gravitation theories," which are a particular type of spacetime theory. To make our concepts clear, we shall illustrate them using four particular gravitation theories: the Newton-Cartan theory (Table I), general relativity (Table II), Ni's theory (Table III),[13] and the Belinfante-Swihart theory (Table IV).[14,15] Of these theories, general relativity and Ni's theory are metric; the Newton-Cartan and Belinfante-Swihart theories are nonmetric.

Mathematical representations of a theory. Two different mathematical formalisms will be called "different representations of the same theory" if they produce identical predictions for the outcome of every experiment or observation. Here by "outcome of an experiment or observation" we mean

TABLE I. Newton-Cartan theory.

1. Reference for this version of the theory:

   Chapter 12, and especially Box 12.4 of MTW[12]

2. Gravitational fields.

   a. Symmetric covariant derivative (affine connection) .... $\underline{\nabla}$
   b. Spatial metric .... $\underline{\gamma}$
   c. Universal time .... t

3. Gravitational field equations:

   a. $\underline{\nabla}\,\underline{\boldsymbol{d}t} = 0$.
   b. $\mathcal{R}(\underline{u}, \underline{n})\underline{w} = 0$,

      where $\mathcal{R}$ is the curvature operator formed from $\underline{\nabla}$; $\underline{u}$ and $\underline{n}$ are arbitrary vectors; $\underline{w}$ is any spatial vector ($\langle \underline{\boldsymbol{d}t}, \underline{w}\rangle = 0$).
   c. $\mathcal{R}(\underline{v}, \underline{w}) = 0$
      for every pair of spatial vectors, $\underline{v}, \underline{w}$. [*Note:* a, b, c guarantee the existence of the metric, $\underline{\gamma}$ or "•", defined on spatial vectors only, such that

      $$\underline{\nabla}_{\underline{u}}(\underline{w}\cdot\underline{v}) = (\underline{\nabla}_{\underline{u}}\underline{w})\cdot\underline{v} + \underline{w}\cdot(\underline{\nabla}_{\underline{u}}\underline{v})$$

      for any $\underline{u}$ and for any spatial $\underline{w}, \underline{v}$.]
   d. $v\cdot[\mathcal{J}(\underline{u}, \underline{n})\underline{w}] = w\cdot[\mathcal{J}(\underline{u}, \underline{n})\underline{v}]$
      for all spatial $\underline{v}, \underline{w}$ and for any $\underline{u}, \underline{n}$, where

      $$\mathcal{J}(\underline{u}, \underline{n})\underline{p} \equiv \tfrac{1}{2}[\mathcal{R}(\underline{p}, \underline{n})\underline{u} + \mathcal{R}(\underline{p}, \underline{u})\underline{n}] .$$

   e. $\underline{\text{Ricci}} = 4\pi\rho\,\underline{\boldsymbol{d}t}\otimes\underline{\boldsymbol{d}t}$,
      where $\underline{\text{Ricci}}$ is the Ricci tensor formed from $\underline{\nabla}$, and ρ is mass density.

4. Influence of gravity on matter:

   a. Test particles move along geodesics of $\underline{\nabla}$, with t an affine parameter.
   b. Each test particle carries a local *inertial frame* with orthonormal, parallel-transported spatial basis vectors ($\underline{e}_{\hat{j}}\cdot\underline{e}_{\hat{k}} = \delta_{jk}$, $\underline{\nabla}_{\underline{u}}\underline{e}_{\hat{j}} = 0$) and with $\underline{e}_{\hat{0}} = d/dt$ = (tangent to geodesic world line).
   c. All the nongravitational laws of physics take on their standard, Newtonian forms in every local inertial frame.

the raw numerical data, before interpretation in terms of theory. Any theory can be given a variety of different mathematical representations. [Example – The Dicke-Brans-Jordan theory has two "standard representations: (i) the original representation,[16,17] in which test particles move on geodesics but the field equations differ significantly from those of Einstein; and (ii) the conformally transformed representation,[18] in which the scalar field produces deviations from geodesic motion but the field equations are nearly the same as Einstein's.] A theory can be regarded as the

TABLE II. General relativity theory.

1. Reference: Standard textbooks, e.g., MTW.[12]

2. Gravitational field:

   The metric of spacetime .... $\underline{g}$

3. Gravitational field equations:

   $\underline{G} = 8\pi\underline{T}$,

   where $\underline{G}$ is the Einstein tensor formed from $\underline{g}$, and $\underline{T}$ is the stress-energy tensor.

4. Influence of gravity on matter:

   a. Test particles move along geodesics of $\underline{g}$, with proper time τ an affine parameter.
   b. Each test particle carries a local inertial ("local Lorentz") frame with parallel-transported, orthonormal basis vectors $\underline{e}_{\hat{\alpha}}$, and with $\underline{e}_{\hat{0}} = d/d\tau$ = (tangent to geodesic world line).
   c. All the nongravitational laws of physics take on their standard, special-relativistic forms in every local inertial frame (aside from delicate points associated with "curvature coupling"; see Chap. 16 of MTW[12]).

TABLE III. Ni's "New Theory."

1. Reference: Ni[13]
2. Gravitational fields:
   - a. Background metric (signature +2) .... $\underline{\eta}$
   - b. Universal time .... t
   - c. Scalar field .... φ
   - d. One-form field .... $\underline{\psi}$
   - e. Physical metric .... $\underline{g}$
3. Gravitational field equations:
   - a. Background metric is flat,

     $$\underline{\mathrm{Riemann}}(\underline{\eta}) = 0 .$$

   - b. "Meshing" of $\underline{\eta}$, t, $\underline{\psi}$:

     $$t_{|\alpha\beta} = 0 ,$$

     $$t_{|\alpha} t_{|\beta} \eta^{\alpha\beta} = -1 ,$$

     $$t_{|\alpha} \psi_{\beta} \eta^{\alpha\beta} = 0 ,$$

     where "$_|$" denotes covariant derivative with respect to $\underline{\eta}$, and $\| \eta^{ij} \|$ is the inverse of $\| \eta_{ij} \|$.
   - c. $\underline{g} = f_2(\varphi)\underline{\eta} + [f_2(\varphi) - f_1(\varphi)]\, \underline{d}t \otimes \underline{d}t - \underline{\psi} \otimes \underline{d}t - \underline{d}t \otimes \underline{\psi}$.
     Here $f_1(\varphi)$ and $f_2(\varphi)$ are arbitrary functions to be determined finally by experiment.
   - d. Field equations for φ and $\underline{\psi}$ follow from the action principle

     $$\delta \int \mathfrak{L}\, d^4x = 0 , \text{ where } \mathfrak{L} = \mathfrak{L}_{NG} + \mathfrak{L}_G ,$$

     $$\mathfrak{L}_G = -\frac{1}{8\pi}\left\{ \frac{1}{e} \psi_{\alpha|\gamma} \psi_{\beta|\delta} \eta^{\alpha\beta} \eta^{\gamma\delta} - \varphi_{|\alpha} \varphi_{|\beta} \eta^{\alpha\beta} + [f_3(\varphi) + 1][\varphi_{|\alpha} t_{|\beta} \eta^{\alpha\beta}]^2 \right\} \sqrt{-\eta} ;$$

     e is a constant to be determined by experiment, $\mathfrak{L}_{NG} = L_{NG}\sqrt{-g}$, and L_{NG} is the standard Lagrangian density of special relativity with the metric of special relativity replaced by $\underline{g}$.
4. Influence of gravity on matter:

   Governed by action principle

   $$\delta \int \mathfrak{L}_M\, d^4x = 0 ,$$

   where particle world lines and nongravitational fields are varied.

equivalence class of all its representations. Tables I–IV present particular representations for the theories described there.

Spacetime theory. A "spacetime theory" is any theory that possesses a mathematical representation constructed from a 4-dimensional spacetime manifold and from geometric objects defined on that manifold. (For the definition of "geometric object," see Sec. 4.13 of Trautman.[10]) Henceforth we shall restrict ourselves to spacetime theories and to the above type of mathematical representations. The geometric objects of a particular representation will be called its *variables*; the equations which the variables must satisfy will be called the *physical laws* of the representation. [Example – general relativity (Table II): The physical laws are the Einstein field equations, Maxwell's equations, the Lorentz force law, etc.] [Example – Belinfante-Swihart theory (Table IV): The physical laws are $\underline{\mathrm{Riemann}}(\underline{\eta}) = 0$, and the Euler-Lagrange equations that follow from $\delta \int \mathfrak{L} d^4x = 0$.]

Manifold mapping group (MMG). The MMG is the group of all diffeomorphisms of the spacetime manifold onto itself. Each diffeomorphism h, together with an initial coordinate system $x^{\alpha}(\mathcal{P})$, produces a new coordinate system

$$x^{\alpha'}(\mathcal{P}) = x^{\alpha}(h^{-1}\mathcal{P}) . \tag{1}$$

(Events are denoted by capital script letters.)

Kinematically possible trajectory (kpt). Consider a given mathematical representation of a given

TABLE IV. Belinfante-Swihart theory.

1. References: Summary and analysis of the theory by Lee and Lightman[14]; original paper by Belinfante and Swihart.[15]
2. Gravitational fields:
   a. Metric .... $\underline{\eta}$
   b. Symmetric second-rank tensor .... $\underline{h}$
3. Nongravitational variables:
   a. Electromagnetic vector potential .... $\underline{A}$
   b. Electromagnetic field tensor (second-rank, antisymmetric) .... $\underline{H}$
   c. World line of particle J, parametrized in an arbitrary manner .... $z_J^\alpha(\lambda_J)$ [in a given coordinate system, world line is $x^\alpha = z_J^\alpha(\lambda_J)$].
   d. Velocity vector of particle J (defined along world line) .... $\underline{a}_J(\lambda_J)$
   e. Momentum vector of particle J (defined along world line) .... $\underline{\pi}_J(\lambda_J)$
4. Gravitational field equations:
   a. Metric is flat: $\underline{\text{Riemann}}(\eta) = 0$.
   b. Field equation for $\underline{h}$ follows from varying $h_{\alpha\beta}$ in $\delta \int \mathfrak{L} d^4x = 0$, where $\mathfrak{L}$ is given below.
5. Influence of gravity on matter:

   Equations for $\underline{A}$, $\underline{H}$, $\underline{z}_J$, $\underline{a}_J$, $\underline{\pi}_J$ follow from varying these quantities in $\int \mathfrak{L} d^4x = 0$.
6. Lagrangian density:
   a. $\mathfrak{L} = \mathfrak{L}_G + \mathfrak{L}_{NG}$.
   b. $\mathfrak{L}_G = -(1/16\pi)\eta^{\alpha\beta}\eta^{\lambda\mu}\eta^{\rho\sigma}(a h_{\lambda\rho|\alpha} h_{\mu\sigma|\beta} + f h_{\lambda\mu|\alpha} h_{\rho\sigma|\beta})(-\eta)^{1/2}$, where "$|$" denotes covariant derivative with respect to $\underline{\eta}$; a and f are constants to be determined by experiment, and $\eta \equiv \det \| \eta_{ij} \|$.
   c. $\mathfrak{L}_{NG} = (1/4\pi)(\tfrac{1}{4} H^{\nu\mu} H_{\nu\mu} - H^{\nu\mu} A_{\mu|\nu})(-\eta)^{1/2}$

$$+\sum_J \int_{-\infty}^{+\infty} [-m_J b_J + (\pi_{J\mu} - e_J A_\mu)\dot{z}_J^\mu - \pi_{J\mu} a_J^\mu]\delta^4[\underline{x} - \underline{z}_J(\lambda_J)] d\lambda_J$$

$$+\tfrac{1}{2} T^{\mu\nu} h_{\mu\nu} + K \sum_J \int_{-\infty}^{+\infty} m_J b_J \eta^{\alpha\beta} h_{\alpha\beta} \delta^4[\underline{x} - \underline{z}_J(\lambda_J)] d\lambda_J \ .$$

   d. Here e_J and m_J are the charge and rest mass of particle J; $\dot{z}_J^\mu \equiv dz_J^\mu/d\lambda_J$; $b_J \equiv (-a_J^\alpha a_{J\alpha})^{1/2}$; K is a constant to be determined by experiment; indices are raised and lowered with $\eta_{\alpha\beta}$; and

$$T^{\mu\nu} \equiv (1/4\pi)(H^{\lambda\mu} H_\lambda{}^\nu - \tfrac{1}{4}\eta^{\mu\nu} H^{\alpha\beta} H_{\alpha\beta})$$

$$+\sum_J \int_{-\infty}^{+\infty} a_J^\mu \pi_J^\nu \delta^4[\underline{x} - \underline{z}_J(\lambda_J)] d\lambda_J \ .$$

   e. In the action principle one varies $h_{\mu\nu}$, A_μ, $H_{\mu\nu}$, $z_J^\alpha(\lambda_J)$, $a_J(\lambda_J)$, $\pi_J(\lambda_J)$ independently; but one holds $\eta_{\mu\nu}$ fixed.

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spacetime theory. A kpt of that representation is any set of values for the components of all the variables in any coordinate system. A kpt need not satisfy the physical laws of the representation. (Example – general relativity (Table II): A kpt is any set of functions $\{g_{\alpha\beta}(x) = g_{\beta\alpha}(x);\ F_{\alpha\beta}(x) = -F_{\beta\alpha}(x);\ z_k^\alpha(\tau_k); \ldots\}$ in any coordinate system, which – if they were to satisfy the physical laws – would represent metric, electromagnetic field, particle world lines, etc.) (Example – Belinfante-Swihart theory (Table IV): A kpt is any set of functions $\{\eta_{\alpha\beta}(x) = \eta_{\beta\alpha}(x),\ h_{\alpha\beta}(x) = h_{\beta\alpha}(x),\ A_\alpha(x),\ H_{\alpha\beta}(x) = -H_{\beta\alpha}(x),\ z_J^\alpha(\lambda_J),\ a_J^\alpha(\lambda_J),\ \pi_J^\alpha(\lambda_J)\}$ in any coordinate system.)

Dynamically possible trajectory (dpt). A dpt is any kpt that satisfies all the physical laws of the representation.

Covariance group of a representation. A group $\mathcal{G}$ is a covariance group of a representation if (i) $\mathcal{G}$ maps kpt of that representation into kpt; (ii) the kpt constitute "the basis of a faithful realization of $\mathcal{G}$" (i.e., no two elements of $\mathcal{G}$ produce identical mappings of the kpt)[19]; (iii) $\mathcal{G}$ maps dpt into dpt. (Example – MMG is a covariance group of each of the representations of theories in Tables I–IV.) (Example – Electromagnetic gauge transformations, $A_\mu \to A_\mu + \varphi_{,\mu}$, are a covariance group of the representation of Belinfante-Swihart theory given in Table IV.) By *complete covariance group* we shall mean the largest covariance group of the representation. By *generally covariant representation* of a theory we shall mean any representation for which MMG is a covariance group. (An argument due to Kretschmann[20] shows that every spacetime theory possesses generally covariant representations.) By *internal covariance group* we shall mean a covariance group that involves *no* diffeomorphisms of spacetime onto itself. (Example – Electromagnetic gauge transformations are an internal covariance group.) By *external covariance group* we shall mean a covariance group that is a subgroup of MMG. The complete covariance group of a representation need not be the direct product of its complete (i.e., largest) internal covariance group with its complete external covariance group. It may also include transformations that are "partially internal" and "partially external" and cannot be split up. [Example – When one formulates Newton-Cartan theory in a Galilean coordinate representation (see the Appendix, which should not be read until one has finished this entire section), one obtains a complete covariance group described by Eqs. (A5). The complete external covariance group consists of (A5a) and (A5b). There is no internal covariance group. The transformations (A5c) are mixed internal-external transformations that belong to the complete covariance group.]

We shall use the following notation to describe a particular element G of the covariance group, and its effect. G consists of a diffeomorphism h [Eq. (1), above] and an internal transformation H:

$$G = (h, H). \qquad (2)$$

If G is an external transformation (element of MMG), then H must be the identity operation; if G is an internal transformation, then h is the identity mapping; if G is a mixed internal-external transformation, then neither h nor H is an identity. Denote the variables of the representation (geometric objects) by y, and their components at a point $\mathcal{P}$ in a coordinate system $\{x^\alpha\}$ by $y_A(\mathcal{P}, \{x^\alpha\})$. The set of functions

$$y_A(\mathcal{P}, \{x^\alpha\}), \quad \mathcal{P} \text{ varying and } \{x^\alpha\} \text{ fixed} \qquad (3)$$

constitute a kpt. The diffeomorphism h maps this kpt into $y_A(\mathcal{P}, \{x^{\alpha'}\})$, where $\{x^{\alpha'}\}$ is the coordinate system of Eq. (1). The internal transformation H converts y into a new geometric object,

$$y' \equiv Hy. \qquad (4)$$

The net effect of G on the kpt (3) is

$$G: \; y_A(\mathcal{P}, \{x^\alpha\}) \rightarrow y'_A(\mathcal{P}, \{x^{\alpha'}\}). \qquad (5)$$

It is often useful to characterize G by the functions

$$\bar{\delta} y_A(\mathcal{P}, \{x^{\alpha'}\}) \equiv y'_A(\mathcal{P}, \{x^{\alpha'}\}) - y_A(h^{-1}\mathcal{P}, \{x^\alpha\})$$
$$= y'_A \big|_{\text{evaluated at } x^{\alpha'}(\mathcal{P})} - y_A \big|_{\text{evaluated at } x^\alpha = x^{\alpha'}(\mathcal{P})}. \qquad (6)$$

Note that these "changes in y" satisfy the relation

$$\bar{\delta}(y_{A,\mu})(\mathcal{P}, \{x^{\alpha'}\}) = [\bar{\delta} y_A(\mathcal{P}, \{x^{\alpha'}\})]_{,\mu'}, \qquad (7)$$

where a comma denotes partial derivative, and also the relation

$$\bar{\delta} y_A = (Hy)_A(\mathcal{P}, \{x^{\alpha'}\}) - (hy)_A(\mathcal{P}, \{x^{\alpha'}\}), \qquad (8)$$

where hy is the geometric object obtained by "dragging along with h" (see p. 86 of Trautman[10]).

Of particular interest are the infinitesimal elements of a covariance group. [From them one can generate that topologically connected component[21] of the group which contains the identity. The other connected components, if any, are typically obtained by bringing into play a discrete set of group elements (space reflections, time inversions, etc.).] Let $G_\epsilon = (h_\epsilon, H_\epsilon)$ be a one-parameter family of elements (curve in group space parametrized by ϵ), with G_0 the identity. Denote by $\underline{\xi}$ the infinitesimal generator of the diffeomorphism h_ϵ:

$$\underline{\xi} \equiv [d(h_\epsilon \mathcal{P})/d\epsilon]_{\epsilon=0}. \qquad (9)$$

Then, to first order in ϵ, Eq. (8) reduces to

$$\bar{\delta} y_A(\mathcal{P}, \{x^\alpha\}) = \epsilon \left\{ (\pounds_{\underline{\xi}}\, y)_A(\mathcal{P}, \{x^\alpha\}) + \left[\frac{d}{d\epsilon}(H_\epsilon y)_A(\mathcal{P}, \{x^\alpha\}) \right]_{\epsilon=0} \right\}, \qquad (10)$$

where $\pounds_{\underline{\xi}}$ is the Lie derivative along $\underline{\xi}$ (Sec. 4.15 of Trautman[10]).

Equivalence classes of dpt. Two dpt are members of the same equivalence class if one of them is mapped into the other by some element of the complete covariance group. (Example – When MMG is a covariance group, all dpt that are obtained from each other by coordinate transformations belong to the same equivalence class.) If a generally covariant representation possesses no internal covariance groups, then there is a one-to-one correspondence between equivalence classes of dpt and the geometric, coordinate-independent solutions of its geometric, coordinate-independent physical laws.

Confined, absolute, and dynamical variables. The variables of a generally covariant representation split up into three groups: "confined variables," "absolute variables," and "dynamical variables." The confined variables are those which do *not* constitute the basis of a faithful realization of MMG. (Examples – All universal constants, such as the charge of the electron, are confined variables. The world line of a particle is not a confined variable, as one sees by this procedure: (i) Characterize the world line by the scalar field

$$\tau(\mathcal{P}) = \begin{cases} 0, & \text{if } \mathcal{P} \text{ is } \textit{not} \text{ on world line;} \\ \text{proper time of particle,} & \\ \quad \text{if } \mathcal{P} \textit{ is} \text{ on world line.} & \end{cases} \qquad (11)$$

(ii) Verify that an element of MMG can be characterized uniquely by the manner in which it maps the set of all kinematically possible world lines [all functions $\tau(x^\alpha)$ that are zero everywhere except along a curve, and are monotonic along that curve] into each other. (iii) Thereby conclude that a particle world line does constitute the basis for a faithful realization of MMG, and therefore that it is not a confined variable.) To determine whether an unconfined variable B is absolute or dynamical, perform the following test: Pick out an arbitrary dpt, and let $\bar{B}_A(x^\alpha)$ be the functions which describe the components of B for that dpt. Then examine each equivalence class of dpt to see whether these same functions $\bar{B}_A$ appear somewhere in it. If they do, for every equivalence class and for every choice of the arbitrary initial dpt, then B is an *absolute variable*. If they do not, for some particular choice of the initial dpt and for some particular equivalence class, then B is a *dynamical variable*. Some dynamical variables contain absolute parts, and some dynamical and absolute variables contain confined parts. [Example – Belinfante-Swihart theory (Table IV): $\eta_{\alpha\beta}$ is an absolute variable; $h_{\alpha\beta}$ and all the nongravitational variables are dynamical.] [Example – Ni's theory (Table III): $\underline{\eta}$ and t are absolute variables; $\underline{\psi}$, φ, and $\underline{g}$ are dynamical. Although $\underline{\psi}$ is dynamical, it contains an absolute part – the projection of $\underline{\psi}$ on $\underline{dt}$ (i.e., $\psi_\alpha t_{|\beta}\eta^{\alpha\beta}$). The remaining, "spatial" part of $\underline{\psi}(\underline{\psi} + \psi_\alpha t_{|\beta}\eta^{\alpha\beta}\underline{dt})$ is fully dynamical. Although t is absolute, it contains a confined part – its "origin," or equivalently, its value at some fixed fiducial event $\mathcal{P}_0$. One can remove this confined part from t by passing from t to the 1-form field $\underline{dt}$.] [Example – general relativity (Table II): All the unconfined variables are dynamical, and they contain no absolute parts. It is this feature that distinguishes general relativity from almost all other theories of gravity (see JLA[11]; also Chap. 17 of MTW, where absolute variables are called "prior geometry").] (Example – Newton-Cartan theory: In the representation of Table I, t and $\underline{\gamma}$ are absolute variables; $\underline{\nabla}$ is dynamical. As in Ni's theory, the origin of t is a confined variable and can be split off by passing from t to $\underline{dt}$. Although the covariant derivative $\underline{\nabla}$ is dynamical, it contains absolute parts. A decomposition of $\underline{\nabla}$ into its absolute and dynamical parts is performed in the Appendix [Eq. (A1e)]. After that decomposition the theory takes on a new mathematical representation with absolute variables $\underline{\beta}$, $\underline{\gamma}$, $\underline{D}$, and dynamical variables Φ and $\underline{\nabla}$.)

Irrelevant variables. A set of variables of a generally covariant representation is called irrelevant if (i) its variables are not coupled by the physical laws to the remaining variables of the representation, and (ii) its variables can be eliminated from the representation without altering the structure of the equivalence classes of dpt and without destroying general covariance. A variable that is not irrelevant is called "relevant." Some variables contain both relevant and irrelevant parts. (Example – The gauge of the electromagnetic vector potential is irrelevant. So is any other variable that can be forced to take on any desired set of values by imposing an appropriate internal covariance transformation.) [Example – In Ni's theory (Table IV) and the Newton-Cartan theory (Table I) the origin of universal time t is an irrelevant variable.]

Fully reduced, generally covariant representation. A generally covariant representation is called "fully reduced" if (i) it contains no irrelevant variables, (ii) its dynamical variables contain no absolute parts, and (iii) its dynamical and absolute variables contain no confined parts. [Example – Newton-Cartan theory: The representation of Table I is generally covariant, but not fully reduced. To reduce it one must follow the procedure of the Appendix: (i) Remove the irrelevant origin of t by passing from t to $\underline{\beta} = \underline{dt}$; (ii) split $\underline{\nabla}$ into its absolute and dynamical parts. The resulting representation is not quite fully reduced because it possesses the internal covariance transformation (A3'a) with an associated, irrelevant "gauge arbitrariness" in $\underline{D}$ and Φ. When one removes that irrelevance by fixing the "gauge" once and for all (e.g., by requiring, for an island universe, that $\left\{ {}^{\alpha}_{\beta\gamma} \right\} = 0$ in any Galilean frame where the total 3-momentum vanishes), then one obtains a fully reduced representation.]

Boundary conditions, prior geometric constraints, decomposition equations, and dynamical laws. In a given mathematical representation of a given theory, the physical laws break up into four sets: (i) boundary conditions – those laws which involve only confined variables; (ii) prior geometric constraints[22] – those which involve absolute variables and possibly also confined variables, but not dynamical variables; (iii) decomposition equations – those which express a dynamical variable algebraically in terms of other variables; (iv) dynamical laws – all others. [Example – Ni's theory (Table III): Equations (3a) and (3b) are prior geometric constraints; Eq. (3c) is a decomposition equation; and the equations that follow from the variational principle are all dynamical. If one augments the theory by cosmological demands that ψ and φ go to zero at spatial infinity, those demands are boundary conditions.] [Example – general relativity (Table II): All physical laws are dynamical.] [Example – Belinfante-Swihart theory (Table IV): $\underline{\text{Riemann}}\,(\underline{\eta})=0$ is a prior geometric constraint; the equations obtained from the variational principle are dynamical.] [Example – Newton-Cartan theory (Table I): In the mathematical formulation of Table I, Eqs. (3a)–(3d) are all dynamical laws. One has the feeling, however, that they ought not to be dynamical, because they involve only gravitational fields; they make no reference to any source of gravity. Only (3e) contains a source, so only it "ought to be" dynamical. The failure of one's "ought-to" intuition results from one's failure to split $\underline{\nabla}$ up into its absolute and dynamical pieces. Such a split (see Appendix) results in a new mathematical formulation of the theory, with just one dynamical gravitational law: (A1f), which is equivalent to (3e) of Table I. Of the other gravitational equations in the new formulation, (A1a)–(A1d) are prior geometric constraints, and (A1e) is a decomposition equation.]

Symmetry group. Let G be an element of the complete covariance group of a representation. Examine the change produced by G in every variable B that (i) is absolute, and (ii) has had all irrelevant, confined parts removed from itself. If

$$\bar{\delta} B_A(\mathcal{P}, \{x^{\alpha'}\})=0 \text{ at all } \mathcal{P} \text{ and for all coordinate systems } \{x^{\alpha'}\} \tag{12}$$

for every such B, then G is called a *symmetry transformation*. Any group of symmetry transformations is called a *symmetry group*; the largest group of symmetry transformations is called the *complete symmetry group* of the representation. [Note: That component of the complete symmetry group which is topologically connected to the identity is generated by infinitesimal transformations. One can find all the infinitesimal generators by solving Eqs. (10) and (12) for $\underline{\xi}$, and for $(dH_c/d\epsilon)_{\epsilon=0}$.] [Another note: If the absolute variables B are all tensor or affine-connection fields, then $\bar{\delta} B$ are all tensor fields, so

$$(\bar{\delta} B_A = 0 \text{ for all } \mathcal{P} \text{ in one coordinate system}) \Rightarrow (\bar{\delta} B_A = 0 \text{ for all } \mathcal{P} \text{ in every coordinate system}). \tag{13}$$

Hence, in this case one can confine attention to any desired, special coordinate system when testing for symmetry transformations.] [Example – Belinfante-Swihart theory (Table IV): The complete symmetry group consists of the Poincaré group (inhomogeneous Lorentz transformations) together with the electromagnetic gauge transformations. One proves this most easily in a global Lorentz frame of $\underline{\eta}$; one can restrict calculations to this frame because the absolute variable $\underline{\eta}$ is a tensor.] [Example – Ni's theory (Table III): Symmetry transformations are analyzed most easily in a coordinate system where $x^0 = t$ (universal time), and $\eta_{\alpha\beta}$ has the Minkowski form. Any symmetry transformation must leave $\bar{\delta}\eta_{\alpha\beta} = \bar{\delta} t_{,\alpha} = \bar{\delta}(\eta^{\alpha\beta} t_{,\alpha} \psi_\beta) = 0$. Thus, the symmetry transformations are (i) electromagnetic gauge transformations; (ii) spacetime translations, $x^{\alpha'} = x^\alpha + a^\alpha$ with a^α a constant; (iii) time-independent spatial rotations, $x^{0'} = x^0$ and $x^{j'} = R^{jk} x^k$ with $\|R^{jk}\|$ a rotation matrix; (iv) spatial reflections.] [Example – general relativity (Table II): There are no absolute variables, so the complete covariance group and the complete symmetry group are identical; they are the MMG plus electromagnetic gauge transformations.] (Example – Newton-Cartan theory: See Appendix.) An *external symmetry group* is a symmetry group that is a subgroup of MMG. An *internal symmetry group* is a symmetry group that involves no diffeomorphisms of spacetime onto itself. The complete symmetry group need not be the direct product of the external symmetries and the internal symmetries; it may also include symmetries that are partially internal and partially external and cannot be split up. [Example – Newton-Cartan theory in the representation of the Appendix: Transformations (A5c) are partially internal and partially external.]

III. GRAVITATION THEORIES AND EQUIVALENCE PRINCIPLES

We now turn from general spacetime theories to the special case of gravitation theories. We cannot discuss gravitation theories without making somewhat precise the distinction between gravitational phenomena and nongravitational phenomena. There seem to be a variety of ways in which one might make this distinction. Somewhat arbitrarily, but after considerable thought, we have chosen to regard as "gravitational" those phenomena which either are absolute or "go away" as the amount of mass-energy in the experimental laboratory decreases. In other words, gravitational phenomena are either prior geometric effects or effects generated by mass-energy. This means that the flat background metric $\underline{\eta}$ of Belinfante-Swihart theory is a gravitational field; the metric of general relativity is a gravitational field; but the torsion of Cartan's modified general relativity,[23] which is generated by spin rather than by mass-energy, is not a gravitational field.

We try to make the above statements more precise by introducing the following concepts.

Local test experiment. A "local test experiment" is any experiment, performed anywhere in spacetime, in the following manner. A shield is set up around the experimental laboratory. When analyzed using the concepts and experiments of special relativity, this shield must have arbitrarily small mass-energy and must be impermeable to electromagnetic fields, to neutrino fields, and to real (as opposed to virtual) particles. The experiment is performed, with freely falling apparatus, in the center of the shielded laboratory, in a region so small that inhomogeneities in all external fields are unimportant. One makes sure that external inhomogeneities are unimportant by performing a sequence of experiments of successively smaller size (with size of shield and external conditions unchanged), until the experimental result approaches a constant value asymptotically. (Examples – The experiment might be a local measurement of the electromagnetic fine-structure constant, or a Cavendish experiment with two lead spheres, or a series of Cavendish experiments involving lead spheres and small black holes.)

Local, nongravitational, test experiment. A "local, nongravitational test experiment" is a local test experiment with these properties: (i) When analyzed in the center-of-mass Galilean frame, using the Newtonian theory of gravity, and using all forms of special relativistic mass-energy as sources for the Newtonian potential Φ, the matter and fields inside the shield must produce a Φ with

$$|\Phi \text{ (at any point inside shield)} - \Phi \text{ (at any point on shield)}| \ll 1 .$$

(ii) When the experiment is repeated, with successively smaller mass-energies inside the shield (as deduced using special relativity theory) – but leaving unchanged the characteristic sizes, intrinsic angular momenta, velocities, and charges (electric, baryonic, leptonic, etc.) of its various parts – the experimental result does not change. (Examples: A measurement of the electromagnetic fine-structure constant *is* a local, nongravitational test experiment; a Cavendish experiment is not.)

Gravitation theory. A "gravitation theory," or "theory of gravity," is any space-time theory which correctly predicts Kepler's laws for a binary star system that (i) is isolated in interstellar space ("local test experiment"); (ii) consists of two "normal stars" (stars with $|\Phi| \ll 1$ throughout their interiors); and (iii) has periastron p large compared to the stellar radii, $p \gg R$. The theory's predictions must not deviate from Kepler's laws by fractional amounts exceeding the larger of $|\Phi|_{max}$ and p/R. (Note: To agree with experiment in the solar system, the theory will have to reproduce Kepler much more accurately than this.) (Examples – The theories in Tables I–IV are all gravitation theories.)

In the absence of gravity. The phrase "in the absence of gravity" means "when analyzing any local, nongravitational test experiment for which the shield is spherical, has arbitrarily large radius, and is surrounded by a spherically symmetric sea of matter." "*To turn off gravity*" means "to pass from a generic situation to a situation where gravity is absent." "*To turn on gravity*" means "to pass from a situation where gravity is absent to a generic situation."

Gravitational field. In a given representation of a given gravitation theory, any unconfined, relevant variable B is a "gravitational field" if, in the absence of gravity, it reduces to a constant, or to an absolute variable, or to an irrelevant variable. In particular, every absolute, relevant variable is a gravitational field. [Example – general relativity (Table II): For local, nongravitational test experiments, analyzed using Fermi-normal coordinates, one gets the same result whether one uses the correct $\underline{g}$ or one replaces it by a flat Minkowski metric $\underline{\eta}$ (absolute variable). Thus $\underline{g}$ is a gravitational field.] [Example – Newton-Cartan theory (Table I): t and $\underline{\gamma}$ are already absolute, so they are gravitational fields; $\underline{\nabla}$ can be replaced by the Riemann-flat $\underline{D}$ of the Appendix without affecting local, nongravitational experiments, so it is also a gravitational field.] (Example – Cartan's modification of general relativity, with torsion[23]: The torsion is generated by spin. Therefore, it must remain a dynamical variable in analyses of local, nongravitational test experiments. It is not a gravitational field.)

Dicke's[24] weak equivalence principle (WEP).[25] The weak equivalence principle states: *If an uncharged test body is placed at an initial event in spacetime, and is given an initial velocity there, then its subsequent world line will be independent of its internal structure and composition.* Here by "uncharged test body" is meant an object (i) that is shielded, in the sense used above in defining "local test experiments"; (ii) that has negligible self-gravitational energy, when analyzed using Newtonian theory; (iii) that is small enough in size so its coupling (via spin and multipole moments) to inhomogeneities of external fields can be ignored. These constraints guarantee that any test of WEP is a local, nongravitational test experiment.

WEP is called "universality of free fall" by MTW,[12] and is called "equality of passive and inertial masses" by Bondi.[26]

The experiments of Eötvös *et al.*,[6] Dicke *et al.*,[7] and Braginsky *et al.*[8] are direct tests of WEP. Braginsky, whose experiment is the most recent, reports that the relative acceleration of an aluminum test body and a platinum test body placed in the sun's gravitational field at the location of the earth's orbit is

$$\text{(relative acceleration)} < 0.9\times 10^{-12}(GM_{\odot}/r_{\text{orbit}}^2)$$
$$= 0.5\times 10^{-12}\ \text{cm/sec}^2$$
$$(95\%\ \text{confidence}).$$

If WEP is correct, then the world lines of test bodies are a preferred family of curves (without parametrization) filling spacetime – with a single unique curve passing in each given direction through each given event. But WEP does *not* guarantee that these curves can be regarded as geodesics of the spacetime manifold; only if these curves have certain special properties can they be geodesics.[27]

Einstein equivalence principle (EEP). The Einstein equivalence principle states that (i) WEP is valid, and (ii) the outcome of any local, nongravitational test experiment is independent of where and when in the universe it is performed, and independent of the velocity of the (freely falling) apparatus. (Example – Dimensionless ratios of nongravitational physical constants must be independent of location, time, and velocity.) The experimental evidence supporting EEP is reviewed in Secs. 38.5 and 38.6 of MTW.[12]

Dicke's[24] *strong equivalence principle (SEP).* SEP states that (i) WEP is valid, and (ii) the outcome of any local test experiment – gravitational or nongravitational – is independent of where and when in the universe it is performed, and independent of the velocity of the (freely falling) apparatus. (Example – The Dicke-Brans-Jordan theory, with its variable "gravitational constant" as measured by Cavendish experiments, satisfies EEP but violates SEP.)

Two types of effects can lead to a breakdown of SEP: "preferred-location effects" and "preferred-frame effects." Perform a local test experiment, gravitational or nongravitational. If the experimental result depends on the location of the freely falling experimenter, but not on his velocity there, the phenomenon being measured is called a *preferred-location effect.* If it depends on the velocity of the experimenter, it is called a *preferred-frame effect.*[28] [Examples – A cosmological time variation in the "gravitational constant" (as measured by Cavendish experiments) is a preferred-location effect. Anomalies in the earth's tides and rotation rate due to the orbital motion of the earth around the sun and the sun through the galaxy[28] are preferred-frame effects.]

A theory of gravity obeys SEP if and only if it obeys EEP, and it possesses no preferred-frame or preferred-location effects.

Any theory for which the complete external symmetry group excludes boosts will presumably exhibit preferred-frame effects. But preferred-frame effects can also show up when boosts are in the symmetry group. (Example – The vector-tensor theory of Nordtvedt, Hellings, and Will[29] exhibits preferred-frame effects but possesses MMG as a symmetry group.) For further discussion see "*metric theory of gravity*," below.

IV. PROPERTIES AND CLASSES OF GRAVITATION THEORIES

Completeness of a theory. A gravitation theory is "complete" if it makes a definite prediction (not necessarily the correct prediction) for the outcome of any experiment that current technology is capable of performing. (Standard quantum-mechanical limitations on the definiteness of the prediction are allowed.) To be complete, the theory must predict results for nongravitational experiments as well as for gravitational experiments. Of course, it can do so only if it meshes with and incorporates (perhaps in modified form) all the nongravitational laws of physics. If a theory is complete so far as all "classical" experiments are concerned, but has not yet been meshed with the quantum-mechanical laws of physics, we shall call it *classically complete.*

Self-consistency of a theory. A gravitation theory is "self-consistent" if its prediction for the outcome of every experiment is unique – i.e., if, when one calculates the prediction by different methods, one always gets the same result.

Reference 2 discusses completeness and self-consistency in greater detail, and gives examples of incomplete theories and self-inconsistent theories.

Relativistic theory of gravity. A theory of gravity is "relativistic" if it possesses a representation ("relativistic representation") in which, in the absence of gravity, the physical laws reduce to the standard laws of special relativity. (Examples – General relativity, Ni's theory, and the Belinfante-Swihart theory are relativistic; the Newton-Cartan theory is not, nor is Cartan's torsion-endowed modification of general relativity.[23])

Metric theory of gravity. By "metric theory" we mean any theory that possesses a mathematical representation ("metric representation") in which (i) spacetime is endowed with a metric; (ii) the world lines of test bodies are the geodesics of that metric; and (iii) EEP is satisfied, with the nongravitational laws in any freely falling frame reducing to the laws of special relativity.[29] Any theory or representation that is not metric will be called *nonmetric*. [Examples – General relativity and Ni's theory are metric theories, and the representations given in Tables II and III are metric; the Belinfante-Swihart theory is nonmetric,[14] but can be made metric by suitable modifications.[14,30] The Newton-Cartan theory is nonmetric. The Dicke-Brans-Jordan theory is metric; the representation of Ref. 16 is a metric representation; the representation of Ref. 18 ("conformally transformed representation"; "rubber meter sticks") is nonmetric.]

In any metric theory, the metric that enters into EEP is called the "*physical metric*." All other gravitational fields are called "*auxiliary gravitational fields*." Relevant auxiliary scalar fields typically produce preferred-location effects; other relevant auxiliary gravitational fields (vector, tensor, etc.) typically produce preferred-frame effects. This is true independently of whether or not the auxiliary fields are absolute variables or are dynamical – i.e., independently of whether the complete external symmetry group is MMG or is more restrictive.

Clearly, every metric theory is relativistic, but relativistic theories need not be metric [example: the Belinfante-Swihart theory]. Ni[31] has given a partial catalog of metric theories. Will and Nordtvedt[32] have developed a "parametrized post-Newtonian formalism" for comparing metric theories with each other and with experiment.

Prior geometric theories. Any gravitation theory will be called a "prior geometric theory" if it possesses a fully reduced, generally covariant representation that contains absolute variables. (Examples – The Newton-Cartan theory, Ni's theory, and the Belinfante-Swihart theory are prior geometric; general relativity and the Dicke-Brans-Jordan theory are not.)

Lorentz-symmetric representations and theories. A generally covariant representation is called "Lorentz symmetric" if its complete external symmetry group is the Poincaré group – with or without inversions and time reversal. We suspect that, for any theory, all fully reduced, generally covariant representations must have the same complete external symmetry group. Assuming so, we define a theory to be "Lorentz symmetric" if its fully reduced, generally covariant representations are Lorentz symmetric. (Example – General relativity is not Lorentz symmetric; the complete external symmetry group of its fully reduced, standard representation is too big – it is MMG rather than Poincaré.) (Example – Ni's theory is not Lorentz symmetric; as with the Newton-Cartan theory, the complete external symmetry group is too small.) (Example – Belinfante-Swihart theory is Lorentz symmetric.)

Elsewhere in the literature one sometimes finds Lorentz-symmetric theories called "Lorentz-invariant theories" or "flat-space theories."

Lagrangian-based representations and theories. A generally covariant representation of a spacetime theory is called Lagrangian-based if (i) there exists an action principle that is extremized with respect to variations of all dynamical variables but *not* with respect to variations of absolute or confined variables, and (ii) from the action principle follow all the dynamical laws but none of the other physical laws. The issue of whether the other physical laws (boundary conditions, decomposition equations, and prior geometric constraints) are imposed before the variation or afterwards does not affect the issue of whether the representation is Lagrangian-based. A theory is called Lagrangian-based if it possesses a generally covariant, Lagrangian-based representation. (Examples – General relativity, Ni's theory, and the Belinfante-Swihart theory are all Lagrangian-based.)

The Lagrangian density $\mathfrak{L}$ of a Lagrangian-based representation (which appears in the action principle in the form $\delta\int\mathfrak{L}d^4x=0$) can be split up into two parts: $\mathfrak{L}=\mathfrak{L}_G+\mathfrak{L}_{NG}$. The *gravitational part* $\mathfrak{L}_G$ is the largest part that contains only gravitational fields. The *nongravitational part* $\mathfrak{L}_{NG}$ is the rest.

V. UNIVERSAL COUPLING

We turn attention, now, from our glossary of concepts to some applications. We begin in this section by analyzing the overlap between metric theories and relativistic, Lagrangian-based theories.

As motivation for the analysis, consider any relativistic representation of a relativistic theory of gravity. In the absence of gravity that representation reduces to special relativity – so, in particular, it possesses a flat Minkowski metric $\eta_{\alpha\beta}$. By continuity one expects the representation to possess, in the presence of gravity, at least one second-rank, symmetric tensor gravitational field $\psi_{\alpha\beta}$ that reduces to $\eta_{\alpha\beta}$ as gravity is turned off. Indeed, this is the case for all relativistic theories with which we are familiar. (Example –

general relativity: The curved-space metric $g_{\alpha\beta}$ reduces to $\eta_{\alpha\beta}$ when gravity is turned off.) (Example – Ni's theory: There are a variety of second-rank, symmetric tensor gravitational fields that reduce to $\eta_{\alpha\beta}$. They include the flat background metric $\eta_{\alpha\beta}$, the physical metric $g_{\alpha\beta}$, any tensor field of the form $[1+f(\varphi)]\eta_{\alpha\beta}$, where $f(\varphi)$ is an arbitrary function with $f(0)=0$, etc.) [Example – Belinfante-Swihart theory: $\eta_{\alpha\beta}$, $\eta_{\alpha\beta}+h_{\alpha\beta}$, $\eta_{\alpha\beta}(1+3h_\mu{}^\mu)-17h_\alpha{}^\mu h_\mu{}^\beta$ all reduce to $\eta_{\alpha\beta}$ when gravity is turned off.]

Next consider any *Lagrangian-based, relativistic theory*. Being relativistic, it must possess a generally covariant, Lagrangian-based representation in which, as gravity is turned off, the nongravitational part of the Lagrangian $\mathfrak{L}_{NG}$ approaches the total Lagrangian of special relativity. Adopt that representation. Then, in the presence of gravity $\mathfrak{L}_{NG}$ will presumably contain at least one second-rank, symmetric, tensor gravitational field $\psi_{\alpha\beta}$ that reduces to $\eta_{\alpha\beta}$ as gravity is turned off. Roughly speaking, if $\mathfrak{L}_{NG}$ contains precisely one such $\psi_{\alpha\beta}$ and contains no other gravitational fields, then the theory is said to be "universally coupled."[33]

More precisely, we say that a Lagrangian-based, relativistic theory is *universally coupled* if it possesses a representation ("universally coupled representation") with the following properties: (i) The representation is generally covariant and Lagrangian-based. (ii) $\mathfrak{L}_{NG}$ contains precisely one gravitational field, and that field is a second-rank, symmetric tensor $\psi_{\alpha\beta}$ with signature +2 throughout spacetime. (iii) In the limit as gravity is turned off $\psi_{\alpha\beta}$ becomes a Riemann-flat second-rank, symmetric tensor field $\eta_{\alpha\beta}$; and whenever $\psi_{\alpha\beta}$ is replaced by such an $\eta_{\alpha\beta}$, $\mathfrak{L}_{NG}$ becomes the total Lagrangian of special relativity. (iv) The prediction for the result of any local, nongravitational experiment anywhere in the universe is unchanged when, throughout the laboratory, one replaces $\psi_{\alpha\beta}$ by a Riemann-flat second-rank, symmetric tensor.

The following theorem reveals the key role of universal coupling as a link between Lagrangian-based theories and metric theories: *Consider all Lagrangian-based, relativistic theories of gravity. Every such theory that is universally coupled is a metric theory; and, conversely, every metric theory in this class is universally coupled.*

Proof: Let $\mathfrak{A}$ be a Lagrangian-based, relativistic, universally coupled theory. Adopt a universally coupled representation. Use that representation to analyze any local, nongravitational test experiment anywhere in spacetime. Use the mathematical tools of Riemannian geometry, treating the unique gravitational field $\psi_{\alpha\beta}$ that appears in $\mathfrak{L}_{NG}$ as a metric tensor. In particular, introduce a Fermi-normal coordinate system ($\psi_{\alpha\beta}=\eta_{\alpha\beta}$, $\Gamma^\alpha{}_{\beta\gamma}=0$ at the center of mass of the laboratory). Condition (iv) for universal coupling guarantees that the predictions of the representation will be unchanged if we replace $\psi_{\alpha\beta}$ by $\eta_{\alpha\beta}$ throughout the laboratory. Do so. Then condition (iii) for universal coupling guarantees that $\mathfrak{L}_M$ is the total Lagrangian of special relativity. The dynamical laws that follow from

$$\delta\int(\mathfrak{L}_G+\mathfrak{L}_{NG})d^4x=0$$

by varying all nongravitational variables also follow from

$$\delta\int\mathfrak{L}_{NG}\,d^4x=0\,;$$

in this representation and coordinate system they are the laws of special relativity. Thus, the outcome of the local, nongravitational test experiment is governed by the standard laws of special relativity, irrespective of the location and velocity of the apparatus. This guarantees that theory $\mathfrak{A}$ is a metric theory.

Proof of converse: Let $\mathfrak{H}$ be a Lagrangian-based, metric theory. Adopt a Lagrangian-based, metric representation. Since all unconfined, nongravitational variables are dynamical, they must all be varied in $\delta\int\mathfrak{L}d^4x=0$. Moreover, since they appear in $\mathfrak{L}_{NG}$ but not in $\mathfrak{L}_G$, their Euler-Lagrange equations are obtained equally well from

$$\delta\int\mathfrak{L}_{NG}\,d^4x=0\,.$$

Call those Euler-Lagrange equations (obtained by varying all unconfined, nongravitational variables in $\delta\int\mathfrak{L}_{NG}d^4x=0$) the "nongravitational laws." Let a freely falling observer anywhere in spacetime, with any velocity, perform a local, nongravitational test experiment. Analyze that experiment in a local Lorentz frame of the physical metric $g_{\alpha\beta}$ using the above nongravitational laws. Because the theory is metric, the predictions must be the same as those of special relativity. Hence, the nongravitational laws – in *any* local Lorentz frame of $g_{\alpha\beta}$ *anywhere* in the universe – must reduce to the laws of special relativity. This is possible only if (i) those laws – and hence also $\mathfrak{L}_{NG}$ – contain no reference to any gravitational field except $g_{\alpha\beta}$,[34] and (ii) $\mathfrak{L}_{NG}$ is some version of the total special relativistic Lagrangian, with $\eta_{\alpha\beta}$ replaced by $g_{\alpha\beta}$. These properties of $\mathfrak{L}_{NG}$, plus the definition of "metric theory," guarantee directly that the four conditions for universal coupling are satisfied. Hence, theory $\mathfrak{H}$ is universally coupled. QED.

VI. SCHIFF'S CONJECTURE

Schiff's conjecture[5] *states that any complete and self-consistent gravitation theory that obeys WEP must also, unavoidably, obey EEP.*

General relativity is an example. It endows spacetime with a metric; it obeys WEP by predicting that all uncharged test bodies fall along geodesics of that metric, with each geodesic world line determined uniquely by an initial event and an initial velocity; it achieves completeness by demanding that in every local, freely falling frame the nongravitational laws of physics take on their standard special relativistic forms; and by this method of achieving completeness, it obeys EEP.

The Newton-Cartan theory is another example. It was complete and self-consistent within the framework of nineteenth century technology. It endows spacetime with an affine connection; it obeys WEP by predicting that all uncharged test bodies fall along geodesics of that affine connection, with each geodesic world line determined uniquely by an initial event and an initial velocity; it achieves completeness by demanding that in every local, freely falling frame the laws of physics take on their standard nongravitational Newtonian form; and by this method of achieving completeness, it obeys EEP.

Before accepting Schiff's conjecture as plausible, one should search the literature for a counterexample – i.e., for a theory of gravity which somehow achieves completeness, and somehow obeys WEP, but fails to obey EEP. Several Lagrangian-based theories which one finds in the literature might conceivably be counterexamples, but they have not been analyzed with sufficient care to allow any firm conclusion. Subsequent papers[9,14] will show that the most likely counterexample, Belinfante-Swihart theory, actually fails to satisfy WEP, violates the ED experimental results, and is thus not a counterexample at all.

One can make Schiff's conjecture seem very plausible within the framework of relativistic, Lagrangian-based theories (the case of greatest interest; see Sec. I) by the following line of argument.[35]

Consider a Lagrangian-based, relativistic theory, and ask what constraints WEP places on the Lagrangian. WEP probably forces $\mathcal{L}_{NG}$ to involve one and only one gravitational field (and that field must, of course, be a second-rank symmetric tensor $g_{\alpha\beta}$ which reduces to $\eta_{\alpha\beta}$ far from all gravitating bodies). If $\mathcal{L}_{NG}$ were to involve, in addition, some other gravitational field φ, then to satisfy WEP $g_{\alpha\beta}$ and φ would have to conspire to produce *identically* the same gravitational accelerations on a test body made largely of rest mass, as on a body made largely of electromagnetic energy, as on a body made largely of internal kinetic energy, as on a body made largely of nuclear binding energy, as on a body made largely of .... This seems implausible, unless $g_{\alpha\beta}$ and φ appear everywhere in $\mathcal{L}_{NG}$ in the same "mutually coupled" form $f(\varphi)g_{\alpha\beta}$ – in which case one can absorb $f(\varphi)$ into $g_{\alpha\beta}$ and end up with just one gravitational field in $\mathcal{L}_{NG}$. Thus, it seems likely that WEP forces $\mathcal{L}_{NG}$ to involve only $g_{\alpha\beta}$. This means that the theory is universally coupled – and, hence, by the theorem of Sec. V, it is a metric theory.

This argument convinces us that Schiff's conjecture is probably correct, when one restricts attention to Lagrangian-based, relativistic theories. And it is hard to see how the conjecture could fail in other types of theories.

A formal proof of Schiff's conjecture for a more limited class of theories will be given in a subsequent paper.[9]

APPENDIX: ABSOLUTE AND DYNAMICAL FIELDS IN NEWTON-CARTAN THEORY

In order to separate the absolute gravitational fields of Newton-Cartan theory from the dynamical fields, one must change mathematical representations. In place of the representation given in Table I, one can adopt the following.

1. Gravitational fields.

a. Symmetric covariant derivatives (two of them): $\underline{D}$ and $\underline{\nabla}$.

b. Scalar gravitational field: Φ.

c. Spatial metric [defined on vectors $\underline{w}$ such that $\langle \underline{\beta}, \underline{w}\rangle = 0$]: $\underline{\gamma}$.

d. Universal 1-form: $\underline{\beta}$.

(Note: t has been replaced by $\underline{\beta}$ in order to remove from the theory the "irrelevant" choice of origin of universal time; see "irrelevant variables" in Sec. II A. $\underline{D}$ and Φ will turn out to be absolute and dynamical parts of $\underline{\nabla}$; see below.)

2. Gravitational field equations.

a. $\underline{\beta}$ is perfect: $\underline{d}\underline{\beta} = 0$. (A1a)

b. $\underline{\beta}$ is covariantly constant: $\underline{D}\underline{\beta} = 0$. (A1b)

c. $\underline{D}$ is flat: $\underline{\text{Riemann}}(\underline{D}) = 0$. (A1c)

d. Compatibility of $\underline{D}$ and $\underline{\gamma}$:

$\underline{D}_{\underline{n}}(\underline{v}\cdot\underline{w}) = (\underline{D}_{\underline{n}}\underline{v})\cdot\underline{w} + \underline{v}\cdot(\underline{D}_{\underline{n}}\underline{w})$ for any vector $\underline{n}$, and for any spatial vectors $\underline{v}$, $\underline{w}$. (A1d)

e. Decomposition of $\underline{\nabla}$:

$\underline{\nabla} = \underline{D} + \underline{A}\otimes\underline{\beta}\otimes\underline{\beta}$, where $\underline{A}$ is the spatial vector "dual" to $\underline{d}\Phi$: $\langle \underline{d}\Phi, \underline{w}\rangle = \underline{A}\cdot\underline{w}$ for all spatial $\underline{w}$.

(A1e)

f. Field equation for Φ:

$$\underline{D} \cdot \underline{A} \equiv (\text{divergence of } \underline{A}) = 4\pi\rho. \qquad \text{(A1f)}$$

3. *Influence of gravity on matter*. Same as in part 4 of Table I where t is *any* scalar field such that $\underline{\beta} = \underline{d}t$.

To prove that this and the formalism given in Table I are different mathematical representations of the same theory, we can show that they become identical in Galilean coordinate frames. The reduction of the formalism of Table I to a Galilean frame is performed in Exercise 12.6 of MTW.[12] The reduction of the above formalism proceeds as follows: (i) Let t be any particular scalar field such that $\underline{\beta} = \underline{d}t$. (ii) At some particular event in spacetime pick a set of basis vectors $\{\underline{e}_\alpha\}$ such that (a) $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ are spatial, $\langle \underline{\beta}, \underline{e}_j \rangle = 0$, and orthonormal, $\underline{e}_j \cdot \underline{e}_k = \delta_{jk}$; (b) $\underline{e}_0$ is not spatial, $\langle \underline{\beta}, \underline{e}_0 \rangle \neq 0$. (iii) From each vector $\underline{e}_\alpha$ construct a vector field on all of spacetime by parallel transport with $\underline{D}$. The resulting field is unique because $\underline{D}$ is flat; and it has $\underline{D}\underline{e}_\alpha = 0$. Hence, the commutators vanish:

$$[\underline{e}_\alpha, \underline{e}_\beta] = \underline{D}_\alpha \underline{e}_\beta - \underline{D}_\beta \underline{e}_\alpha = 0 .$$

This guarantees the existence of a coordinate system $\{x^\alpha\}$ in which $\underline{e}_\alpha = \partial/\partial x^\alpha$. (iv) The condition (valid in *any* coordinate frame) $\langle \underline{d}x^0, \underline{e}_j \rangle = 0$, when compared with $\langle \underline{d}t, \underline{e}_j \rangle = 0$, guarantees that the surfaces of constant x^0 and constant t are identical, i.e., $t = f(x^0)$. Moreover, because the connection coefficients of $\underline{D}$ vanish in this coordinate frame,

$$\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} \equiv \langle \underline{d}x^\alpha, \underline{D}_\gamma \underline{e}_\beta \rangle = 0 , \qquad \text{(A2a)}$$

the condition $\underline{D}\underline{d}t = 0$ becomes $\partial^2 t/\partial x^\alpha \partial x^\beta = 0$; in particular, $\partial^2 t/\partial x^0 \partial x^0 = 0$, so $t = ax^0 + b$ for some constants a and b. Renormalize x^0 so $t = x^0$. (v) In the resulting coordinate frame $\underline{\beta}$, $\underline{\gamma}$, and $\underline{A}$ have components

$$\beta_0 = 1 , \quad \beta_j = 0 , \quad \gamma_{jk} = \delta_{jk} ,$$
$$A^0 = 0 , \quad A^j = \partial\Phi/\partial x^j ; \qquad \text{(A2b)}$$

so the field equation for Φ is Poisson's equation

$$\frac{\partial^2 \Phi}{\partial x^j \partial x^j} = 4\pi\rho ; \qquad \text{(A2c)}$$

and the connection coefficients of $\underline{\nabla}$ are $\Gamma^\alpha{}_{\beta\gamma} = A^\alpha t_{,\beta} t_{,\gamma}$, i.e.,

$$\Gamma^j{}_{00} = \frac{\partial\Phi}{\partial x^j} , \quad \text{all other } \Gamma^\alpha{}_{\beta\gamma} \text{ vanish}. \qquad \text{(A2d)}$$

This Galilean coordinate version of the above formalism is identical to the Galilean coordinate version of the formalism of Table I, as given in Chap. 12 of MTW.[12] Thus, the two formalisms are different mathematical representations of the same theory.

In the above formalism it is easy to verify that $\underline{D}$, $\underline{\beta}$, and $\underline{\gamma}$ are absolute gravitational fields, while Φ is a dynamical gravitational field. In fact, $\underline{D}$, $\underline{\beta}$, and $\underline{\gamma}$ are the absolute parts of $\underline{\nabla}$; Φ is its dynamical part; Eqs. (A1a)–(A1d) are the prior geometric constraints of the theory; Eq. (A1e) is the decomposition of $\underline{\nabla}$ into its absolute and dynamical parts; and Eq. (A1f) is the dynamical field equation for Φ.

The covariance group for the above mathematical representation of Newton-Cartan theory is slightly larger than that for the representation of Table I. For Table I the covariance group is MMG. For the above representation it is the direct product of MMG with a group of *internal covariance transformations*. In a Galilean frame the internal transformations are

$$\left\{ \begin{matrix} j \\ 00 \end{matrix} \right\} \to \left\{ \begin{matrix} j \\ 00 \end{matrix} \right\}' = \left\{ \begin{matrix} j \\ 00 \end{matrix} \right\} + a^j(t) = a^j(t) ,$$

$$\Phi \to \Phi' = \Phi - a_j(t) x^j + \text{constant} , \qquad \text{(A3)}$$

all other variables, including $\Gamma^\alpha{}_{\beta\gamma}$,

left unchanged.

In coordinate-free form the internal transformations are

$$\underline{D} \to \underline{D}' = \underline{D} + \underline{a} \otimes \underline{\beta} \otimes \underline{\beta} ,$$
$$\Phi \to \Phi' = \Phi - b , \qquad \text{(A3'a)}$$

where $\underline{a}$ is any vector field which is covariantly constant in the surfaces of $\underline{\beta}$,

$$\underline{D}_{\underline{w}} \underline{a} = \underline{\nabla}_{\underline{w}} \underline{a} = 0 \quad \text{for all spatial vectors } \underline{w} ; \qquad \text{(A3'b)}$$

and where b is any scalar field such that

$$\langle \underline{d}b, \underline{w} \rangle = \underline{a} \cdot \underline{w} \quad \text{for all spatial vectors } \underline{w} . \qquad \text{(A3'c)}$$

The complete symmetry group for the above mathematical representation of Newton-Cartan theory is best analyzed in a Galilean coordinate system. [Because the absolute objects are all tensors or affine connections, one can restrict attention to a single coordinate system; see Eq. (13) and associated discussion in the text.] The symmetry transformations are those which leave

$$\bar{\delta}\gamma_{jk} = \bar{\delta}\beta_\alpha = \bar{\delta} \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} = 0 . \qquad \text{(A4)}$$

Clearly, the symmetry transformations include (i) spacetime translations

$$x^{\alpha} \to x^{\alpha'} = x^{\alpha} + c^{\alpha}, \tag{A5a}$$

where c^{α} are constants, and (ii) spatial rotations

$$x^{j} \to x^{j'} = R^{jk} x^{k}, \tag{A5b}$$

$\|R^{jk}\|$ a constant rotation matrix. They also include (iii) the combination of an arbitrary time-dependent spatial translation with a carefully matched internal covariance transformation

$x^{j} \to x^{j'} = x^{j} + c^{j}(t)$, where c^{j} are arbitrary functions of t,

$$\left.\begin{aligned} \left\{ {j \atop 0\,0} \right\} \to \left\{ {j \atop 0\,0} \right\}^{\dagger} &= \left\{ {j \atop 0\,0} \right\} + \ddot{c}^{j}(t) \\ \Phi \to \Phi^{\dagger} &= \Phi - \ddot{c}^{j}(t) x^{j} \end{aligned}\right\} \text{ where } \ddot{c}^{j} = \frac{d^2 c^j}{dt^2}. \tag{A5c}$$

Note that these symmetry transformations are precisely the transformations that lead from one Galilean coordinate system to another (cf. Sec. 12.3 of MTW[12]).

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[1]K. S. Thorne and C. M. Will, Astrophys. J. 163, 595 (1971).

[2]K. S. Thorne, W.-T. Ni, and C. M. Will, in Proceedings of the Conference on Experimental Tests of Gravitation Theories, edited by R. W. Davies, NASA-JPL Tech. Memo 33–499, 1971 (unpublished).

[3]C. M. Will, Physics Today 25 (No. 10), 23 (1972).

[4]C. M. Will, Proceedings of Course 56 of the International School of Physics "Enrico Fermi", edited by B. Bertotti (Academic, New York, to be published) [also distributed as Caltech Report No. OAP-289, 1972 (unpublished)].

[5]Our form of Schiff's conjecture is a classical analog of Schiff's original quantum mechanical conjecture. Schiff briefly outlined his version of the conjecture on page 343 of his article in Am. J. Phys. 28, 340 (1960). So far as we know, he never pursued it in any detail until November 1970, when his interest in the issue was rekindled by a vigorous argument with one of us (KST) at the Caltech-JPL Conference on Experimental Tests of Gravitation Theories. Unfortunately, his sudden death 2 months later took him from us before he had a chance to bring his analysis of the conjecture to fruition.

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[10]A. Trautman, lectures in A. Trautman, F. A. E. Pirani, H. Bondi, *Lectures on General Relativity* (Prentice-Hall, Englewood Cliffs, N. J., 1965).

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[15]F. J. Belinfante and J. C. Swihart, Ann. Phys. (N.Y.) 1, 168 (1957); 1, 196 (1957); 2, 81 (1957).

[16]C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).

[17]P. Jordan, Z. Physik 157, 112 (1959).

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[19]This definition of "faithful realization" differs from that given on page 26 of JLA; we think that this is what JLA intended to say and should have said.

[20]E. Kretschmann, Ann. Phys. (Leipzig) 53, 575 (1917).

[21]For the topological properties of groups see, e.g., L. S. Pontrjagin, *Topological Groups* (Oxford Univ. Press, Oxford, 1946).

[22]This concept is due to C. W. Misner; see Chap. 17 of MTW (Ref. 12).

[23]*Note added in proof.* A. Trautman (private communication) argues that we should define as "gravitational" those phenomena which are absolute or "go away" as the amounts of mass-energy *and spin* in the experimental laboratory decrease. Such a definition, he argues, would be "in accordance with our knowledge, based on special relativity, where mass and spin are equally fundamental." He goes on to say, "I feel that it is a little misleading to arrange the definitions (by omitting the phrase 'and spin' above) so as to exclude the Einstein-Cartan theory from the framework of relativistic theories of gravity (Sec. IV of this paper). Clearly, the Einstein-Cartan theory reduces to special relativity when $G \to 0$. Moreover, in the real world, spin, charge, baryon number, etc. are always accompanied by energy and momentum. Therefore, if the amount of mass-energy in the laboratory decreases, then not only gravitational phenomena go away: so do electromagnetic and nuclear effects as well as the (so far hypothetical) effects of torsion."

We (the authors) find Trautman's argument cogent. Nevertheless, we adhere to our set of definitions — for a very pragmatic reason. By refusing to allow torsion into "relativistic theories of gravity" we make it possible for ourselves to prove certain theorems about relativistic theories which otherwise we would not know how to prove and which might well be false. An example is the theorem on universal coupling proved in Sec. V of this paper.

For details and references on Cartan's theory with torsion, see A. Trautman, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. 20, 185 (1972), and references cited therein.

[24]R. H. Dicke, lectures in *Relativity, Groups, and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).

[25]For a more detailed discussion of WEP see Chap. 38 of MTW – where WEP goes under the name "universality of free fall".

[26]H. Bondi, Rev. Mod. Phys. 29, 423 (1957).

[27]See, e.g., Chap. 10 of MTW.

[28]K. Nordtvedt, Jr. and C. M. Will, Astrophys. J. 177, 775 (1972).

[29]This definition of "metric theory" originated in Chap. 39 of MTW. Here and henceforth we shall adhere to it, even though earlier work by our group (e.g., Ref. 1) used a slightly less restrictive definition. (Any theory that is "metric" according to the present definition is also "metric" according to the old definition.)

[30]A. Lightman and D. Lee (unpublished).

[31]W.-T. Ni, Astrophys. J. 176, 769 (1972).

[32]C. M. Will and K. Nordtvedt, Jr., Astrophys. J. 177, 757 (1972); also earlier references by Nordtvedt and by Will cited therein. For reviews see Refs. 2, 3, 4, and 12.

[33]We have adapted the principle of universal coupling from R. V. Wagoner, Phys. Rev. D 1, 3209 (1970). Wagoner enunciated this principle only for the special case of scalar-tensor theories, and he gave it the more restrictive name "principle of mutual coupling." However, our concept is a straightforward generalization of his.

[34]To see more clearly why no gravitational fields other than $g_{\alpha\beta}$ can enter the Euler-Lagrange equations, argue as follows: The only gravitational effects which vanish as the size of the frame vanishes arise from terms of a Taylor series type expansion of some gravitational field(s) B. But if there are any B other than $g_{\alpha\beta}$, then somewhere in spacetime there will be local Lorentz frames of $g_{\alpha\beta}$ in which the lowest order Taylor series term of some of the B does not vanish, thus violating the local validity of special relativity.

[35]This is a classical version of Schiff's original quantum-mechanical line of reasoning (Ref. 5).

B. THE EQUIVALENCE PRINCIPLE AND ITS SEVERE CONSTRAINT UPON GRAVITATION THEORIES

a) Restricted Proof that the Weak Equivalence Principle Implies the Einstein Equivalence Principle (Paper V; collaboration with D.L. Lee, published in Phys. Rev.D, **8**,364,1973)

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Restricted Proof that the Weak Equivalence Principle Implies the Einstein Equivalence Principle*

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Schiff has conjectured that the weak equivalence principle (WEP: free-fall trajectories independent of test-body composition) implies the Einstein equivalence principle (EEP: all nongravitational laws of physics the same in every freely falling frame). This paper presents a proof of Schiff's conjecture, restricted to (i) test bodies, made of electromagnetically interacting point particles, that fall from rest in a static, spherically symmetric gravitational field; and (ii) theories of gravity within a certain broad class—a class that includes almost all complete relativistic theories that we have found in the literature, but with each theory truncated to contain only point particles plus electromagnetic and gravitational fields. The proof shows that every "nonmetric" theory in the class (every theory that violates EEP) must violate WEP. A formula is derived for the magnitude of the violation. Comparison with the results of Eötvös-Dicke-type experiments rules out various nonmetric theories, including those of Belinfante and Swihart and of Naida and Capella—theories that previously were believed to agree with all current experiments. It is shown that WEP is a powerful theoretical and experimental tool for constraining the manner in which gravity couples to electromagnetism in gravitation theories.

I. INTRODUCTION

In a previous paper[1] we have discussed the content and significance of Schiff's conjecture. In brief, the conjecture states that all theories of gravity which satisfy the weak equivalence principle[1] (WEP), i.e., predict a unique composition-independent trajectory for any test body at a given point of spacetime and with a given initial velocity through that point, must satisfy the Einstein equivalence principle (EEP), i.e., must show that the nongravitational[1] laws of physics are the same in every freely falling frame. When specialized to "relativistic theories of gravity"[1] (as will be done throughout this paper), Schiff's conjecture says that every theory satisfying WEP is necessarily a "metric theory."[1] Plausibility arguments (e.g., Refs. 1 and 2) have frequently been given for the conjecture, but there have been few detailed calculations that bear upon its validity or invalidity. Indeed, the conjecture is so sweeping that it will probably never be proved with complete generality. (Such a proof would require a moderately deep understanding of all gravitation theories that satisfy WEP—including theories not yet invented, and never destined to be invented. Such understanding is well beyond one's grasp in 1973.)

On the other hand, one can gain useful insight by proving restricted versions of the conjecture, and by searching for the most general versions that are provable. For example, one might first analyze test bodies with purely electromagnetic internal interactions and thereby attempt to show that particles and electromagnetism must interact with gravity in the manner of metric theories (EEP) in order that WEP be satisfied; next analyze purely nuclear systems and attempt to show that nuclear fields must couple to gravity metrically; etc. Unfortunately, for our purposes, nuclear interactions have not been given an adequate mathematical representation even in the absence of gravity; and the nonmetric theories known to us make no attempt to write down nuclear force laws. Hence our present program must end one way or another after the first stage. Even a general proof of the first stage (Schiff conjecture for bodies with internal electromagnetic interactions) is too much to expect. To make it manageable, one must assume some restricted (but hopefully quite general) form for the interactions. This we shall do in the present paper—with an interaction form general enough to include all metric theories plus almost all nonmetric theories we have found in the published literature. As a byproduct of our proof, we can rule out several nonmetric theories in the literature.

In order not to prejudice ourselves, the language and concepts used in the calculation will be those employed in standard classical field theory with gravity treated as just another ordinary field. In particular, we will not use such phrases as "curved spacetime" and will not make any coordinate transformations to real or pseudo- "freely falling frames." The concept of gravity as a metric phenomenon should be forced upon us by WEP.

As spelled out in Sec. II, we shall take a nonquantum-mechanical approach and shall use a particle rather than a fluid picture for the test body. Since the gravitation theories with which we attempt to tie in are largely classical theories, we feel that a classical approach is completely justified and perhaps essential. There are two reasons why a particle approach has been taken: first, more often than not, classical field theories formulate the interaction of gravity with matter in the form of point particles; second, a charged-particle approach allows one to deal with the exact "gravitationally modified Maxwell equations" of a given theory, rather than with their smeared-out averages.

Our calculation is not the first of its type. For several particular theories, and at lower orders of approximation, the acceleration of electromagnetic test bodies in a gravitational field has been previously calculated. Nordtvedt[3] and Belinfante and Swihart[4] have both done calculations, to first order in the gravitational field potential and squared particle velocities; Nordtvedt for general metric theories, and Belinfante and Swihart for their theory of gravity. In addition, Post[5] has done a calculation, at post-Newtonian order, of the acceleration of a confined quantity of electromagnetic energy in a gravitational field. Had his calculation been carried to higher order it is conceivable he could have obtained part of our result: that $\epsilon = \mu$ [cf. Eq. (21)].

Section II of this paper gives an outline of the assumptions, procedure, and techniques of our calculation, including the results; Sec. III presents the details. Section IV compares the predictions for WEP violation with the results of Eötvös-Dicke-type experiments, and thereby rules out the nonmetric theories of Belinfante and Swihart,[4,6] Capella,[7] Naida,[8] and Whitehead.[9] Also discussed is the manner—both quantitative and qualitative—in which WEP is an experimental probe of the "gravitational-Maxwell equations," as contrasted to previously recognized experimental tests of those equations.

II. GENERAL FRAMEWORK AND RESULTS

In calculating the center-of-mass acceleration of an electromagnetic test body, we would like to set up a formalism which includes as many types of gravitation theories as possible, but which is not too complicated. In particular, our formalism should be able to deal with scalar, vector, tensor, scalar-tensor, etc. theories.

We have found that all of these different types of theories can be put into a somewhat universal form when describing a static, spherically symmetric (SSS) gravitational field—providing their dynamical law[1] for particle motion is derivable from a Lagrangian. (The restriction to SSS fields is certainly a limitation in principle, but it allows us to handle many different theories at once; and, as discussed in Sec. IV, is not a limitation in practice.) The quasiuniversal description of particles and electromagnetism in an SSS field is as follows:

The motion of charged particles under the joint action of gravity and the electromagnetic field A_μ can be derived from the Lagrangian[10]

$$\bar{L} = \sum_k \int [-m_{0k}(T - H\vec{v}_k^{\,2})^{1/2} + e_k A_\mu v_k^\mu] dt, \qquad (1)$$

where we have used the bar above the L to indicate that $\bar{L}$ may be only a part of the total Lagrangian, and where the various symbols will be defined below. The "gravitationally modified Maxwell equations" (GMM: Maxwell's equations in the presence of a gravitational field) are of the form

$$\vec{\nabla} \cdot (\epsilon \vec{E}) = 4\pi\rho\,, \qquad (2)$$

$$\vec{\nabla} \times (\mu^{-1}\vec{B}) = 4\pi\vec{J} + \frac{\partial}{\partial t}(\epsilon\vec{E})\,. \qquad (3)$$

Definitions of the quantities in Eqs. (1) - (3) and of other quantities that will be used in the calculation are given below:

$x^i \equiv$ spatial coordinates; they are nearly Cartesian when gravity is weak,

$t \equiv$ a time coordinate associated with the static nature of the SSS field, nearly equal to proper time for slowly moving particles when gravity is weak,

$m_{0k} \equiv$ rest mass of particle k, a constant,

$e_k \equiv$ charge of particle k, a constant,

$x_k^\mu(t) \equiv$ world line of particle k,

$v_k^\mu \equiv dx_k^\mu/dt$,

$x^0 \equiv t$,

$\vec{v}_k^{\,2} \equiv \delta_{ij} v_k^i v_k^j$ with δ_{ij} the 3-Kronecker δ,

$U(r) \equiv$ a gravitational potential equal to M_s/r, where M_s is a constant ("active gravitational mass") characterizing the source of the SSS field, and r is coordinate distance, $[(x - x_s)^2 + (y - y_s)^2$

$+(z-z_s)]^{1/2}$, from source of field point,

$\vec{\nabla}, \vec{\nabla}\cdot \equiv$ the usual differential operators of gravity free Euclidean space,

$\vec{g} \equiv \vec{\nabla} U \equiv$ the gravitational acceleration to be expected if the theory in question were Newtonian theory,

$T, H, \epsilon, \mu \equiv$ functions of the gravitational potential U; functions that are arbitrary in this calculation but that have a specific form in each theory of gravity when the coordinate system has been suitably specified,

$A^\mu \equiv$ components of an electromagnetic vector potential, a four-vector,

$(\vec{A})^i \equiv A_i \equiv$ spatial part of vector potential,

$$\varphi \equiv -A_0,$$

$$\vec{J} \equiv \sum_k e_k \vec{v}_k \delta^3(\vec{x} - \vec{x}_k(t)), \tag{4a}$$

$$\rho \equiv \sum_k e_k \delta^3(\vec{x} - \vec{x}_k(t)), \tag{4b}$$

$$\vec{E} \equiv \vec{\nabla} A_0 - \partial \vec{A}/\partial t, \tag{4c}$$

$$\vec{B} \equiv \vec{\nabla} \times \vec{A}. \tag{4d}$$

Although in most theories the form of $\bar{L}$ in Eq. (1) is typical only of SSS fields, it turns out that all of the results we shall obtain hold even if U is an arbitrary, but time-independent function of position.

For an SSS field in a given theory, T, H, ϵ, and μ will be particular functions of U (and hence of position). *Here we assume that T, H, ϵ, and μ have been given and we seek the relations among them, if any, that are required for compliance with* WEP. It is clear from Eq. (1) that we have sacrificed general covariance of the particle Lagrangian in order to encompass a wide range of theories.

Note that Eqs. (2)–(3) can be reinterpreted (different physics; same mathematical representation) as the usual Maxwell equations for a permeable medium in which the free sources originate from charged particles labeled by k. Thus ϵ and μ play the role of "gravitationally induced dielectric and permeability parameters," respectively. We require that T, H, ϵ, μ all approach unity as U vanishes so that the special relativistic limit is maintained.

Given the SSS restriction, one may ask how general are Eqs. (1)–(3). Except in the most general (nonmetric) case of Jordan's theory,[11] which is incomplete[1] in the sense that it involves unspecified processes of particle creation, all theories we know of which are complete enough to formulate the interaction of the electromagnetic field with gravity have GMM equations of the form of Eqs. (2)–(3).[12] In fact, the "ϵ-μ formulation" of the sourceless Maxwell equations in metric theories has sometimes been used in calculations.[13] The particle Lagrangian $\bar{L}$ [cf. Eq. (1)] also appears to be fairly general, except for a class of theories discussed by Naida[8] which includes the theory of Capella.[7] We treat the Capella-Naida theory on an individual basis in Sec. IV, using the methods developed in this section. We point out that it is sometimes necessary to perform a reformulation (same theory; new "mathematical representation") of a theory in order to put it into the form of Eqs. (1)–(3) (see, for example, the Belinfante-Swihart theory as analyzed in Ref. 14). Finally, we should emphasize that, even more important than the generality of Eqs. (1)–(3), are the techniques and methods developed in this section, since they can also be applied on an individual basis to that handful of theories which is not included in Eqs. (1)–(3). We now proceed with an outline of our calculations.

Variation of Eq. (1) yields an expression for the acceleration of the kth particle, which, together with Eqs. (2) and (3) constitutes three coupled equations. We seek a perturbation solution. There are two obvious, small dimensionless quantities in which one could expand: the gravitational potential U and the squared particle velocities $\vec{v}_k^{\,2}$. Since we prefer a result correct to all orders in the gravitational potential, we expand only in $\vec{v}_k^{\,2}$ and leave T, H, ϵ, and μ as arbitrary functions of U. We do, however, expand these latter functions in a Taylor series about the instantaneous center of mass of the test body (defined below), i.e.,

$$T = T_0 + (\vec{g}\cdot\vec{x})T_0' + \cdots, \tag{5}$$

where

$$T' \equiv dT/dU \quad \text{and} \quad T_0' \equiv (dT/dU)_{\vec{x}=0}. \tag{6}$$

We shall assume that the body is small enough so that second derivatives of U make negligible contributions. Indeed, this is part of the definition of "test body" (Ref. 1) and is a necessary and integral qualification in Schiff's conjecture.

We define the center of mass for the test body by the following sequence of equations:

$$m_k = m_{0k}\{1 + F[U(\vec{x}_k)]\} + \tfrac{1}{2} m_{0k}\vec{v}_k^{\,2}\{1 + G[U(\vec{x}_k)]\} + \tfrac{1}{2} e_k \sum_i e_i |\vec{x}_{ik}|^{-1}\{1 + K[U(\vec{x}_i)] + S[U(\vec{x}_k)]\} + O(m_0 v^4), \tag{7}$$

$$\vec{x}_{ik} \equiv \vec{x}_i - \vec{x}_k, \qquad M \equiv \sum_k m_k, \tag{8}$$

$$\vec{X}_{\text{c.m.}} \equiv M^{-1} \sum_k m_k \vec{x}_k. \tag{9}$$

Here F, G, K, S are again arbitrary functions of the potential U. (Whenever two indices, e.g., i and k, occur in terms, in double or single sums, it is always assumed that $i \neq k$ in the sum.) Any credible result should be independent of the particular definition of the center of mass as long as it remains inside of the body, that is, the result should not depend on the specific forms of the functions F, G, K, and S.

We now assume that at $t=0$, the center of mass of the test body is momentarily at rest, at the origin of the coordinate system,

$$(\vec{X}_{\text{c.m.}})_{t=0} = (d\vec{X}_{\text{c.m.}}/dt)_{t=0} = 0\,. \tag{10}$$

By differentiating Eq. (9) twice and combining with Eqs. (10), we obtain for the instantaneous center-of-mass acceleration

$$\vec{A}_{\text{c.m.}} = M^{-1}\Big(\sum_k \ddot{m}_k \vec{x}_k + 2\sum_k \dot{m}_k \vec{v}_k + \sum_k m_k \vec{a}_k\Big), \tag{11}$$

where

$$\vec{A}_{\text{c.m.}} \equiv d^2\vec{X}_{\text{c.m.}}/dt^2\,,$$

$$\vec{a}_k \equiv d\vec{v}_k/dt\,,$$

$$\dot{m}_k \equiv dm_k/dt\,, \quad \text{etc.}$$

Return for a moment to the details of the expansion scheme. Our expansion is in the quantity

$$v^2 \equiv (\text{typical squared particle velocity}) \lesssim v_k^2\,. \tag{12a}$$

The virial theorem guarantees that

$$v^2 \approx \frac{(\text{typical charge of a particle})^2}{(\text{typical mass})(\text{typical separation of neighboring particles})} \lesssim \frac{e_k^2}{m_k|\vec{x}_{ik}|}\,. \tag{12b}$$

Thus, without serious error, we may treat both terms on the right-hand sides of Eqs. (12a) and (12b) as $O(v^2)$ when ordering the terms in the expansion.

Besides the dimensionless quantity v^2 in which we *do* expand, and the dimensionless quantity U in which we *do not* expand, there is a third, less obvious dimensionless quantity:

$$gs \equiv |\vec{g}|(\text{size of test body}) \lesssim |\vec{g}|\,|\vec{x}_k|\,. \tag{13}$$

We shall expand in this quantity—independently of the v^2 expansion—but, in practice, by examining powers of g rather than gs.

Now, if $\vec{A}_{\text{c.m.}}$ is to be body-independent in general, it must be so for each order in v^2 and each order in g, independently. Surprisingly, perhaps, it will be sufficient to work to first order in v^2 and to first order in g. The imposition of WEP at this order will force the dynamical equations (1)—(3) to take on metric form, thereby guaranteeing that EEP *(and hence* WEP *a fortiori)* is satisfied at all orders.

To first order in v^2 and g, after solving Eqs. (1)-(3) for $\vec{a}_k$ and substitution into Eq. (11), we find (details given in Sec. III)

$$\vec{A}_{\text{c.m.}} = -\tfrac{1}{2}\vec{g}(T_0' H_0^{-1}) + \vec{g} M_0^{-1}\Big[\tfrac{1}{2}(H_0' H_0^{-1})\sum_i m_{0i} v_i^2 + \bar{\eta}\sum_{i,k}\eta_{ik}\Big] + M_0^{-1}\bar{\omega}\sum_{i,k}\vec{\omega}_{ik} + M_0^{-1}\theta\sum_i m_{0i}(\vec{g}\cdot\vec{v}_i)\vec{v}_i\,, \tag{14}$$

where

$$M_0 \equiv \sum_i m_{0i}\,, \tag{15a}$$

$$\bar{\eta} \equiv (T_0^{1/2} H_0^{-1})(\tfrac{1}{2}\epsilon_0'\epsilon_0^{-2} + \tfrac{1}{4}T_0'\mu_0 H_0^{-1})\,, \tag{15b}$$

$$\bar{\omega} \equiv \tfrac{1}{2}(T_0^{1/2} H_0^{-1})(\tfrac{1}{2}T_0' H_0^{-1}\mu_0 + \tfrac{1}{2}T_0' T_0^{-1}\epsilon_0^{-1} - H_0' H_0^{-1}\epsilon_0^{-1})$$
$$+ (1+F_0)^{-1}[F_0' T_0^{1/2} H_0^{-1}\epsilon_0^{-1} - \tfrac{1}{2}(1+G_0)T_0' T_0^{1/2} H_0^{-2}\epsilon_0^{-1}]\,, \tag{15c}$$

$$\theta = T_0' T_0^{-1} - H_0' H_0^{-1} + 2(1+F_0)^{-1}[F_0' - \tfrac{1}{2}(1+G_0)T_0' H_0^{-1}]\,, \tag{15d}$$

$$\eta_{ik} \equiv e_i e_k |\vec{x}_{ik}|^{-1}\,, \tag{15e}$$

$$\vec{\omega}_{ik} \equiv e_i e_k (\vec{g}\cdot\vec{x}_{ik})|\vec{x}_{ik}|^{-3}\vec{x}_{ik}\,. \tag{15f}$$

Equation (14) becomes much simplified when we use some gravitationally modified virial relations (see Sec. III C for details):

$$\Big\langle \sum_i m_{0i} v_i^m v_i^p + \tfrac{1}{2}(T_0^{1/2} H_0^{-1}\epsilon_0^{-1})\sum_{i,k} e_i e_k x_{ik}^m x_{ik}^p |x_{ik}|^{-3}\Big\rangle = O(M_0 v^2 gs)\,, \tag{16}$$

where m, p refer to to components of the appropriate vectors and $\langle\,\rangle$ denotes the usual time average. Using Eq. (16), Eq. (14) becomes

$$\langle\vec{A}_{c.m.}\rangle = -\tfrac{1}{2}\vec{g}(T_0' H_0^{-1}) - \tfrac{1}{4}\vec{g}M_0^{-1}(T_0^{1/2}H_0^{-1}\epsilon_0^{-1})(H_0' H_0^{-1} - 2\epsilon_0'\epsilon_0^{-1} - T_0'\epsilon_0\mu_0 H_0^{-1})\left\langle\sum_{ik}\eta_{ik}\right\rangle$$
$$-\tfrac{1}{4}M_0^{-1}(T_0' T_0^{1/2}H_0^{-2}\epsilon_0^{-1})(H_0 T_0^{-1} - \epsilon_0\mu_0)\left\langle\sum_{i,k}\vec{\omega}_{ik}\right\rangle . \tag{17}$$

The first term of this acceleration is body-independent (satisfies WEP); the second term depends on the body's self-electromagnetic energy; the third term depends on the electromagnetic energy, the shape of the body, and the orientation of the body with respect to the gravitational field gradient. Thus $\langle\vec{A}_{c.m.}\rangle$ will always be body-independent only if the second and third terms always vanish, i.e.,

$$H_0'/H_0 - 2\epsilon_0'/\epsilon_0 - T_0'\epsilon_0\mu_0/H_0 = 0 , \tag{18a}$$

$$H_0/T_0 - \epsilon_0\mu_0 = 0 \tag{18b}$$

(the other factors in the body-dependent terms must be nonzero for correct Newtonian and special relativistic limits), or equivalently,

$$\epsilon_0'/\epsilon_0 = \tfrac{1}{2}(H_0'/H_0 - T_0'/T_0) , \tag{19a}$$

$$\mu_0 = H_0/(T_0\epsilon_0) . \tag{19b}$$

Since we have not specified the initial location of our test body with respect to the external gravitating source, and Eqs. (19) should be satisfied at any point we choose to deposit the body, the naught subscript can be removed from quantities in those equations, yielding, upon integration,

$$\epsilon = C(H/T)^{1/2} , \tag{20a}$$

$$\mu = C^{-1}(H/T)^{1/2} , \tag{20b}$$

where C is a constant. Since, "in the absence of gravity,"[1] we must have $\epsilon = H = T = 1$, C must also be unity. Therefore we finally obtain, as a necessary condition for our electromagnetic test body to fall with a composition-independent acceleration:

$$\epsilon = \mu = (H/T)^{1/2} . \tag{21}$$

It is worth noting that, using heuristic arguments (see, e.g., Ref. 15) about the electromagnetic energy content of atoms and the expression for the fine-structure "constant" α in a dielectric medium

$$\alpha = (\epsilon\,\mu)^{1/2}e^2/(\epsilon\hbar)$$

one can see why WEP should require constancy of the ratio (ϵ/μ).

Comparison of Eqs. (21) and (1)–(3) with the discussion in Sec. III E reveals that Eq. (21) is a necessary and sufficient condition for the dynamical equations (1)–(3) to take on the familiar metric form

$$\overline{L} = \sum_k \int -m_{0k}\,ds_k + e_k A_\mu dx_k^\mu , \tag{22}$$

$$F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha . \tag{23}$$

In this metric form

$$ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta , \tag{24a}$$

$$g_{00} = T , \tag{24b}$$

$$g_{ij} = -\delta_{ij}H \text{ (spherical coordinates turn out to be "isotropic")} , \tag{24c}$$

; denotes the covariant derivative with respect to $g_{\alpha\beta}$,

$$F^{\alpha\beta} = g^{\alpha\tau}g^{\beta\mu}(A_{\mu,\tau} - A_{\tau,\mu}) , \tag{24d}$$

$$J^\alpha \equiv \sum_k \int e_k\delta^4(\underline{x} - \underline{z}(s))(dx_k^\alpha/ds)(-g)^{-1/2}ds_k . \tag{24e}$$

Note that all dependence on the arbitrary functions used in the center-of-mass definition, Eq. (7), has vanished by the time one reaches Eq. (17).

Higher-order calculations [v^4 or $(gs)^2$, for example] could only yield results consistent with Eq. (21), since WEP *at first order implies that gravity has a metric-theory description (automatically satisfying* WEP*) to all orders.*

Our theoretical results can be summarized by the following statement: *Consider the class of gravitation theories that possesses a mathematical representation of the form of Eqs. (1)–(3). For that class, with each theory written in that representation,*

(WEP) ⟺ [Eq. (21)] ⟺ [the theory is metric with the metric given by Eqs. (24b)–(24c)].

III. DETAILS OF THE CALCULATION

A. Single-Particle Equations of Motion

Variation of Eq. (1) with respect to the coordinates of particle k yields

$$(HW^{-1})\vec{a}_k + \vec{v}_k\,\frac{d(HW^{-1})}{dt} + \tfrac{1}{2}W^{-1}\vec{\nabla}(T - Hv_k^2) = \vec{A}_L(\vec{x}_k) , \tag{25}$$

where

$$W \equiv (T - Hv_k^2)^{1/2} , \tag{26a}$$

$$\vec{A}_L(\vec{x}_k) \equiv \text{Lorentz acceleration of particle } k$$
$$= (e_k/m_{0k})\left\{-\vec{\nabla}\varphi(\vec{x}_k) + \vec{\nabla}[\vec{v}_k\cdot\vec{A}(\vec{x}_k)] - \frac{d}{dt}\vec{A}(\vec{x}_k)\right\}, \tag{26b}$$

and all functions of U are evaluated on the particle's world line, e.g., $H \equiv H(U[\vec{x}_k(t)])$. Using Eqs. (5)–(6) and the discussion following Eqs. (13), we can write, to the order of our calculation,

$$\vec{\nabla}H = H_0'\vec{g}, \text{ etc}. \tag{27}$$

We shall regard $\vec{g}$ as spatially constant [see discussion following Eq. (6)]. Equation (25) can then be written as

$$\vec{a}_k = \tfrac{1}{2}\vec{g}(H_0'v_k^2 - T_0')H_0^{-1} - \vec{v}_k(\vec{v}_k\cdot\vec{g})[H_0'H_0^{-1} - \tfrac{1}{2}(T_0' - v_k^2H_0')W^{-2}] - \vec{v}_k(\vec{v}_k\cdot\vec{a}_k)HW^{-2} + (WH^{-1})\vec{A}_L. \tag{28}$$

Note that whenever functions like H, T, ϵ, etc. occur in terms multiplied by $\vec{g}$, we may evaluate them at naught, i.e.,

$$H\vec{g} \to H_0\vec{g},$$

because we work only to first order in g.

We further expand W in a power series in v^2 and, since we are only working to $O(v^2)$, we can set $W = T^{1/2}$ in Eq. (28). This follows from the fact that $\vec{A}_L \sim O(v^2)$ and from the explicit velocity dependence of other terms in Eq. (28). [It should be mentioned that when a term is considered $O(v^2)$, it is not necessarily intended that the term is dimensionless, but only that v^2 (or the expression in Eq. (12b)) is a multiplicative factor in the term. The same applies to the notation $O(g)$.]

By dotting $\vec{v}_k$ into both sides of Eq. (28), solving for $(\vec{a}_k\cdot\vec{v}_k)$, and substituting the result back into Eq. (28), we obtain

$$\vec{a}_k = \tfrac{1}{2}\vec{g}(H_0'v_k^2 - T_0')H_0^{-1} + \vec{v}_k(\vec{v}_k\cdot\vec{g})(T_0'T_0^{-1} - H_0'H_0^{-1}) + (T^{1/2}H^{-1})\vec{A}_L + O(v^4) + O(g^2). \tag{29}$$

B. The Gravitationally Modified Maxwell Equations

We must now solve Maxwell's equations and compute the quantity $\vec{A}_L$ which occurs in Eq. (29). If Eqs. (4c) and (4d) are substituted into Eqs. (2) and (3) and one uses the gauge

$$(\epsilon\mu)\frac{\partial\varphi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0, \tag{30}$$

the result is

$$\nabla^2\varphi = \epsilon\mu\frac{\partial^2\varphi}{\partial t^2} - 4\pi\rho\epsilon^{-1} - \epsilon^{-1}\vec{\nabla}\epsilon\cdot\left(\vec{\nabla}\varphi + \frac{\partial\vec{A}}{\partial t}\right), \tag{31a}$$

$$\nabla^2\vec{A} = \epsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} - 4\pi\mu\vec{J} + (\epsilon\mu)^{-1}(\vec{\nabla}\cdot\vec{A})\vec{\nabla}(\epsilon\mu) + \mu^{-1}(\vec{\nabla}\times\vec{A})\times\vec{\nabla}\mu. \tag{31b}$$

We can now do a perturbation solution of these equations by expanding simultaneously in powers of v^2 and g, treating *formally* $v^2 \sim g$:

$$\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \cdots, \tag{32a}$$

$$\vec{A} = \vec{A}_0 + \vec{A}_1 + \vec{A}_2 + \cdots, \tag{32b}$$

$$\nabla^2\varphi_0 = -4\pi\epsilon^{-1}\rho, \tag{33a}$$

$$\nabla^2\varphi_1 = \epsilon\mu(\partial^2\varphi_0/\partial t^2) - \epsilon_0^{-1}\epsilon_0'[\vec{g}\cdot(\vec{\nabla}\varphi_0 + \partial\vec{A}_0/\partial t)], \tag{33b}$$

$$\nabla^2\varphi_2 = \epsilon\mu(\partial^2\varphi_1/\partial t^2) - \epsilon_0^{-1}\epsilon_0'[\vec{g}\cdot(\vec{\nabla}\varphi_1 + \partial\vec{A}_1/\partial t)], \quad \text{etc.}, \tag{33c}$$

$$\nabla^2\vec{A}_0 = -4\pi\mu\vec{J}, \tag{34a}$$

$$\nabla^2\vec{A}_1 = \epsilon\mu(\partial^2\vec{A}_0/\partial t^2) + (\epsilon\mu)_0^{-1}(\vec{\nabla}\cdot\vec{A}_0)\vec{\nabla}(\epsilon\mu) + \mu_0^{-1}\mu_0'(\vec{\nabla}\times\vec{A}_0)\times\vec{g}, \quad \text{etc.} \tag{34b}$$

(One should not confuse the perturbation order of $\vec{A}$, $\vec{A}_k$, with the kth component of the vector A_k.)

The solution of these equations is far simpler if we remember from the beginning that since the particle acceleration is required only to $O(v^2)$ and $O(g)$ we need $\vec{A}_L$ only to the same order. Remember also that $\vec{a}_k = O(v^2) + O(g)$ whenever the solution of Eqs. (33)–(34) requires a particle acceleration as a source term (right-hand side of equations).

We solve the equations for $\vec{A}$ first. Clearly, from the expression for $\vec{J}$ [cf. Eq. (4a)],

$$\vec{A}_0(\vec{x}_k) = \sum_i e_i\vec{v}_i\mu(\vec{x}_i)|\vec{x}_{ki}|^{-1}. \tag{35}$$

Equation (35) gives the lowest-order vector potential at particle k due to all other particles $(i \neq k)$. Note that $\mu(\vec{x}_i)$ is considered to be a constant with respect to the d'Alembertian operator acting on functions of $\vec{x}_k$. The above $\vec{A}_0$ can produce terms of the desired order in $\vec{A}_L$. For example,

$$e_k\frac{d}{dt}\vec{A}_0(x_k) = \sum_i e_ie_k\vec{a}_i\mu(\vec{x}_i)|\vec{x}_{ki}|^{-1} + \cdots \tag{36a}$$

$$= \sum_i e_ie_k\vec{g}\mu(\vec{x}_i)|\vec{x}_{ki}|^{-1} + \cdots, \tag{36b}$$

where we have substituted $\vec{a}_i = \vec{g} + O(v^2) + O(g^2)$. The indicated term in Eq. (36b) is bilinear in v^2 and g and is therefore acceptable. However it can be shown that no higher orders of $\vec{A}$ after $\vec{A}_0$ can contribute. For example, the second source term on the right-hand side of Eq. (34b) makes the contribution

$$\vec{A}_1 \sim O(g)\vec{A}_0 \sim O(g)O(v^3),$$

$$\vec{A}_L \sim \frac{d\vec{A}_1}{dt} + \vec{\nabla}(\vec{v}\cdot\vec{A}_1) = O(gv^4) + O(g^2v^2).$$

From the expression for ρ [cf. Eq. (4b)], we can write down the lowest-order solution for the scalar electromagnetic potential:

$$\varphi_0(\vec{x}_k) = \sum_i e_i \epsilon^{-1}(\vec{x}_i)|\vec{x}_{ki}|^{-1}. \tag{37}$$

The source term proportional to $\partial\vec{A}_0/\partial t$ in Eq. (33b) doesn't contribute to our order of calculation. Now, define a "superpotential" χ by the equation

$$\nabla^2\chi = \varphi_0. \tag{38}$$

Using χ we can write Eq. (33b) as, to appropriate order,

$$\nabla^2\varphi_1 = \nabla^2\left(\epsilon\mu\frac{\partial^2\chi}{\partial t^2}\right) - 2\vec{\nabla}(\epsilon\mu)\cdot\vec{\nabla}\left(\frac{\partial^2\chi}{\partial t^2}\right) - \nabla^2[\epsilon_0^{-1}\epsilon_0'(\vec{g}\cdot\vec{\nabla}\chi)]. \tag{39}$$

Using Eqs. (37) and (38), we obtain

$$\chi(\vec{x}_k) = \tfrac{1}{2}\sum_i e_i\epsilon^{-1}(\vec{x}_i)|\vec{x}_{ki}|, \tag{40a}$$

$$\frac{\partial\chi}{\partial t} = -\tfrac{1}{2}\sum_i e_i(\vec{v}_i\cdot\vec{x}_{ki})\epsilon^{-1}(\vec{x}_i)|\vec{x}_{ki}|^{-1} - \tfrac{1}{2}\sum_i e_i(\vec{g}\cdot\vec{v}_i)\epsilon^{-2}(\vec{x}_i)\epsilon_0'|\vec{x}_{ki}|, \tag{40b}$$

$$\frac{\partial^2\chi}{\partial t^2} = -\tfrac{1}{2}\sum_i e_i(\vec{a}_i\cdot\vec{x}_{ki})\epsilon^{-1}(\vec{x}_i)|\vec{x}_{ki}|^{-1} + O(v^4), \tag{40c}$$

where we have carefully interpreted the partial time derivative on functions of $\vec{x}_k$ as acting on coordinates of particles labeled i with $i \neq k$. From Eq. (40c) it is clear that the second source term in Eq. (39) does not contribute and the remaining equation is trivially integrated to yield

$$\varphi_1 = \epsilon\mu\frac{\partial^2\chi}{\partial t^2} - \epsilon_0^{-1}\epsilon_0'(\vec{g}\cdot\vec{\nabla}\chi). \tag{41}$$

Using Eqs. (40a) and (40c), Eq. (41) becomes

$$\varphi_1 = -\tfrac{1}{2}\mu_0\sum_i e_i(\vec{a}_i\cdot\vec{x}_{ki})|\vec{x}_{ki}|^{-1} - \tfrac{1}{2}\epsilon_0^{-2}\epsilon_0'\sum_i e_i(\vec{g}\cdot\vec{x}_{ki})|\vec{x}_{ki}|^{-1} \tag{42}$$

and, using Eq. (29) for $\vec{a}_i$,

$$\varphi_1 = (\tfrac{1}{4}T_0'\mu_0H_0^{-1} - \tfrac{1}{2}\epsilon_0^{-2}\epsilon_0')\sum_i e_i(\vec{g}\cdot\vec{x}_{ki})|\vec{x}_{ki}|^{-1}. \tag{43}$$

In the same manner as with the vector potential, one can show that φ_2, φ_3, etc. do not contribute to the Lorentz acceleration at the desired order. Using Eqs. (26b), (35), (37), (43), one obtains

$$\vec{A}_L(\vec{x}_k) = (e_k/m_{0k})\sum_i\{[\vec{x}_{ki}\epsilon^{-1}(\vec{x}_i)|\vec{x}_{ki}|^{-3} - \vec{a}_i\mu(\vec{x}_i)|\vec{x}_{ki}|^{-1}]e_i\} + \tfrac{1}{2}(\tfrac{1}{2}T_0'\mu_0H_0^{-1} - \epsilon_0^{-2}\epsilon_0')\sum_i[\vec{\omega}_{ki} - (e_k/m_{0k})e_i\vec{g}|\vec{x}_{ki}|^{-1}], \tag{44}$$

where $\vec{\omega}_{ki}$ is as defined in Eq. (15f). From Eqs. (29) and (6) we obtain the relations

$$\vec{a}_i = -\tfrac{1}{2}(T_0'H_0^{-1})\vec{g} + O(v^2), \tag{45a}$$

$$\epsilon(\vec{x}_i) = \epsilon_0 + (\vec{g}\cdot\vec{x}_i)\epsilon_0', \tag{45b}$$

which, when substituted into Eq. (44), yield

$$\vec{A}_L(\vec{x}_k) = \sum_i(e_ie_k/m_{0k})\left[\frac{\vec{x}_{ki}\epsilon_0^{-1}}{|\vec{x}_{ki}|^3} - \frac{\epsilon_0'\epsilon_0^{-2}(\vec{g}\cdot\vec{x}_i)\vec{x}_{ki}}{|\vec{x}_{ki}|^3} + \frac{1}{2}\frac{(\epsilon_0^{-2}\epsilon_0' + \frac{1}{2}T_0'\mu_0H_0^{-1})}{|\vec{x}_{ki}|}\vec{g}\right] + \tfrac{1}{2}(\tfrac{1}{2}T_0'\mu_0H_0^{-1} - \epsilon_0^{-2}\epsilon_0')\sum_i\vec{\omega}_{ki}. \tag{46}$$

C. Virial Conditions

We now have enough information to derive some useful virial conditions. Substitution of the expression for $\vec{A}_L$ [cf. Eq. (46)] into Eq. (29) reveals

$$m_{0k}(a_k)^p = T_0^{1/2}H_0^{-1}\epsilon_0^{-1}\sum_i e_ie_k(x_{ki})^p|\vec{x}_{ki}|^{-3} + O(g), \tag{47}$$

where p denotes a particular vector component. Multiplication of both sides of Eq. (47) with $(x_k)^l$ yields

$$m_{0k}x_k^la_k^p = m_{0k}\frac{d(x_k^lv_k^p)}{dt} - m_{0k}v_k^lv_k^p = T_0^{1/2}H_0^{-1}\epsilon_0^{-1}\sum_i e_ie_k(x_{ki})^px_k^l|x_{ki}|^{-3}. \tag{48}$$

If we sum Eq. (48) over the index k, use the antisymmetry of $\vec{x}_{ki}$, and take a time average, the result is Eq. (16). Summing Eq. (16) on l and p produces another useful virial relation:

$$\left\langle \sum_k m_{0k}v_k^2 + \tfrac{1}{2}T_0^{1/2}H_0^{-1}\epsilon_0^{-1}\sum_{i,k} e_i e_k |\vec{x}_{ik}|^{-1} \right\rangle = 0 + O(g) . \tag{49}$$

D. Center-of-Mass Acceleration

We now have all of the necessary tools at our disposal for calculating the test-body acceleration. We begin with Eq. (7). To the required order

$$\dot{m}_k = m_{0k}[F_0'(\vec{g}\cdot\vec{v}_k) + (1+G)(\vec{a}_k\cdot\vec{v}_k) + \tfrac{1}{2}v_k^2 G_0'(\vec{g}\cdot\vec{v}_k)]$$
$$+ \tfrac{1}{2}\sum_i e_i e_k \{K_0'(\vec{g}\cdot\vec{v}_i) + S_0'(\vec{g}\cdot\vec{v}_k) - [K(\vec{x}_i) + S(\vec{x}_k)](\vec{x}_{ik}\cdot\vec{v}_{ik})|\vec{x}_{ik}|^{-2}\}|\vec{x}_{ik}|^{-1} , \tag{50}$$

$$\ddot{m}_k = m_{0k}[F_0'(\vec{g}\cdot\vec{a}_k) + (\vec{a}_k\cdot\vec{a}_k)(1+G)] . \tag{51}$$

In obtaining Eqs. (50)–(51) we have, as before, used the fact that $\vec{a}_k \sim O(g) + O(v^2)$. To be exact, Eqs. (29) and (46) show that

$$\vec{a}_k = -\tfrac{1}{2}\vec{g}(T_0'H_0^{-1}) + T_0^{1/2}H_0^{-1}\epsilon_0^{-1}\sum_i (e_i e_k/m_{0k})\vec{x}_{ki}|\vec{x}_{ki}|^{-3} + O(gv^2) . \tag{52}$$

Using Eqs. (50)–(52), the first two terms in the expression for $\vec{A}_{c.m.}$ [cf. Eq. (11)] become

$$M^{-1}\sum_k \ddot{m}_k\vec{x}_k = \tfrac{1}{2}M^{-1}H_0^{-1}\epsilon_0^{-1}T_0^{1/2}[F_0' - (1+G_0)T_0'H_0^{-1}]\sum_{i,k}\vec{\omega}_{ki} , \tag{53a}$$

$$2M^{-1}\sum_k \dot{m}_k\dot{\vec{x}}_k = 2M^{-1}[F_0' - \tfrac{1}{2}(1+G_0)T_0'H_0^{-1}]\sum_k m_{0k}(\vec{v}_k\cdot\vec{g})\vec{v}_k . \tag{53b}$$

Again using Eqs. (29) and (46) to get the $O(gv^2)$ contribution to $\vec{a}_k$ [cf. Eq. (52)], the third and last term contributing to $\vec{A}_{c.m.}$ is

$$M^{-1}\sum_k m_k\vec{a}_k = M^{-1}\vec{g}\left\{-\tfrac{1}{2}M_0T_0'(1+F_0)H_0^{-1} + \tfrac{1}{2}H_0^{-1}[H_0'(1+F_0) - \tfrac{1}{2}T_0'(1+G_0)]\sum_k m_{0k}v_k^2 + \tfrac{1}{2}\tau_1\sum_{i,k}\eta_{ik}\right\}$$
$$+ (1+F_0)(T_0'T_0^{-1} - H_0'H_0^{-1})M^{-1}\sum_k m_{0k}(\vec{v}_k\cdot\vec{g})\vec{v}_k + \tfrac{1}{2}\tau_2 M^{-1}\sum_{i,k}\vec{\omega}_{ik} , \tag{54}$$

where

$$\tau_1 \equiv T_0^{1/2}H_0^{-1}(1+F_0)(\epsilon_0^{-2}\epsilon_0' + \tfrac{1}{2}T_0'\mu_0H_0^{-1}) - \tfrac{1}{2}T_0'H_0^{-1}(1+K_0+S_0) , \tag{55a}$$

$$\tau_2 \equiv T_0^{1/2}H_0^{-1}[(1+F_0)H_0^{-1}(\tfrac{1}{2}T_0'\mu_0 - \epsilon_0^{-1}H_0') + \epsilon_0^{-1}F_0' + \tfrac{1}{2}(1+F_0)\epsilon_0^{-1}T_0^{-1}T_0'] , \tag{55b}$$

with M_0, η_{ik}, $\vec{\omega}_{ik}$ defined in Eqs. (15).

Now, expand the expression for M^{-1} using Eqs. (7) and (8):

$$M^{-1} = M_0^{-1}(1+F_0)^{-1}\left[1 - \frac{1}{2}\frac{(1+G_0)}{M_0(1+F_0)}\sum_k m_{0k}v_k^2 - \frac{1}{2}\frac{1+K_0+S_0}{M_0(1+F_0)}\sum_{i,k}\eta_{ik}\right] + O(v^4) + O(g) . \tag{56}$$

With Eqs. (53)–(56), the expression for $\vec{A}_{c.m.}$, Eq. (11), becomes that given in Eq. (14). Use of Eqs. (16) and (49) then yields Eq. (17), and subsequently Eq. (21).

E. The "ϵ-μ" Formulation for Metric Theories

In any static, spherically symmetric, locally Lorentz manifold with metric, one can introduce "spatially isotropic coordinates," for which

$$g_{00} = g_{00}(r) , \tag{57a}$$

$$g_{0k} = 0 , \tag{57b}$$

$$g_{ij} = -\delta_{ij}f(r) ,$$
$$r \equiv [(x^1 - x_s^1)^2 + (x^2 - x_s^2) + (x^3 - x_s^3)^2]^{1/2} . \tag{57c}$$

(For proof, see any standard textbook on general relativity.) For the problem at hand we can regard g_{00} and f as functions of $U = M_s/r$ rather than as functions of r. In such a coordinate system, the standard metric-theory Lagrangian for the motion of charged particles reduces to

$$L = \sum_k\left[-m_{0k}\int (g_{\alpha\beta}dx_k^\alpha dx_k^\beta)^{1/2} + e_k\int A_\mu dx_k^\mu\right]$$
$$= \sum_k \int [-m_{0k}(g_{00} - f\vec{v}_k^2)^{1/2} + e_k A_\mu v_k^\mu]dt , \tag{58}$$

and the metric-theory Maxwell equations read

$$F^{\alpha\beta}{}_{;\beta} = (-g)^{-1/2}[F^{\alpha\beta}(-g)^{1/2}]_{,\beta} = 4\pi J^{\alpha}, \qquad (59a)$$

where

$$J^{\alpha} \equiv \sum_k e_k \int (dx_k^{\alpha}/ds_k)\delta^4(\underline{x} - \underline{x}_k)(-g)^{-1/2} ds_k$$

$$= \sum_k e_k(-g)^{-1/2}\delta^3(\vec{x} - \vec{x}_k)(dx_k^{\alpha}/dt). \qquad (59b)$$

Here $g \equiv$ determinant of $g_{\alpha\beta}$, and commas and semicolons denote partial and covariant differentiation, respectively. Combining Eqs. (59) gives

$$[g^{\alpha\tau}g^{\beta\mu}F_{\tau\mu}(-g)^{1/2}]_{,\beta} = 4\pi\sum_k e_k\delta^3(\vec{x} - \vec{x}_k)(dx_k^{\alpha}/dt). \qquad (60)$$

Equation (60), when written out for the diagonal, spatially isotropic metric of Eq. (57), has the "ϵ-μ" form of Eqs. (2) and (3), with

$E_i = F_{0i}$, etc.

and

$$\epsilon = \mu = (f/g_{00})^{1/2}. \qquad (61)$$

Conversely, for a theory with GMM equations of the form of Eqs. (2) and (3) and with

$$\epsilon = \mu \qquad (62)$$

one can define an "effective electromagnetic metric" by

$$g_{00} = \Psi, \qquad (63a)$$

$$g_{ij} = -\epsilon^2\Psi\delta_{ij}; \qquad (63b)$$

then the GMM equations will take on metric-theory form. In Eqs. (63) Ψ is an arbitrary function and reflects the well-known conformal invariance of Maxwell's equations. If, in addition to satisfying Eq. (62), the effective metric determined by Eqs. (63) is correctly related to the functions appearing in the particle Lagrangian [cf. Eqs. (57)–(58)], then the entire theory of particles and electromagnetic fields can be consistently put into metric form.

IV. CONCLUSIONS AND APPLICATIONS

A. Theoretical Implications of the Results

We have shown that, in a spherically symmetric gravitational field, a theory of gravity described by Eqs. (1)–(4) can be put into metric form (with respect to the dynamical equations for particles and electromagnetic fields) if and only if it satisfies the weak equivalence principle.[16] Equivalently, if such a theory is nonmetric then Eq. (21) will not be satisfied, the acceleration of test bodies will have body-dependent contributions [cf. Eq. (17)], and WEP will be violated. The result has far-reaching consequences if one accepts WEP as a valid principle: Having proved, from WEP, the metric nature of the GMM equations inside of an electromagnetic test body, one knows how to describe all gravitational-electromagnetic phenomena—e.g., the bending of light by the sun, electromagnetic radiation in a gravitational field, etc.

There are two potential weaknesses of our calculation. First we have assumed a spherically symmetric gravitational field. Now, it is conceivable that a theory could be of "metric form" for spherically symmetric gravitational fields, but nonmetric in other cases. Such theories would have to be analyzed on an individual basis, to see whether their non-SSS fields violated WEP. However, we feel that such a theory would be difficult to formulate and, in fact, have seen no examples in the literature. In practical applications, one considers a particular nonmetric theory, solves the spherically symmetric problem, and finds that Eq. (21) is not satisfied, thus constituting a violation of WEP at some order. Examples will be given below.

A second possible weakness, discussed previously, is the limitation to the types of equations discussed in the beginning of Sec. II. However, except for the Naida-Capella nonmetric theory, discussed below, Eqs. (1)–(4) appear to be quite general among "complete" theories. (There are many theories which are not explicit as to the formulation of the GMM equations, and we must require that such theories be completed before given further consideration.)

Finally, we point out that WEP and Eq. (21) demand that the center-of-mass acceleration be body-independent at each order in the external gravitational potential U. As will be seen below, a given theory violating the WEP will do so at some order of U. To be more explicit, suppose that one expands the functions H, T, μ, ϵ appearing in Eq. (17) in a power series in U, i.e.,

$$H = 1 + 2\gamma U + \tfrac{3}{2}\delta U^2 + \cdots, \qquad (64a)$$

$$T = 1 - 2\alpha U + 2\beta U^2 + \cdots, \qquad (64b)$$

$$\epsilon = 1 + \epsilon_1 U + \epsilon_2 U^2 + \cdots, \qquad (64c)$$

$$\mu = 1 + \mu_1 U + \mu_2 U^2 + \cdots. \qquad (64d)$$

Then, Eq. (17) can be written in the form

$$\langle \vec{A}_{\text{c.m.}} \rangle = -\tfrac{1}{2}\vec{g}(T_0'H_0^{-1}) - \tfrac{1}{2}\vec{g}M_0^{-1}\Big\langle \sum_{i,k} \eta_{ik} \Big\rangle \times (\Gamma_0 + \Gamma_1 U_0 + \Gamma_2 U_0^2 + \cdots)$$

$$+ \frac{\alpha}{2} M_0^{-1}\Big\langle \sum_{i,k} \vec{\omega}_{ik} \Big\rangle \times (\Upsilon_0 + \Upsilon_1 U_0 + \Upsilon_2 U_0^2 + \cdots), \qquad (65)$$

where

$$\Gamma_0 \equiv \gamma - \epsilon_1 + \alpha\,, \qquad (66a)$$

$$\Upsilon_0 = 0\,, \qquad (66b)$$

$$\Gamma_1 \equiv 2(\tfrac{3}{4}\delta - 2\gamma^2 - \epsilon_2 - \beta + \epsilon_1{}^2) + \gamma\epsilon_1 + \alpha(\mu_1 - 5\gamma + \epsilon_1 - \alpha)\,, \qquad (66c)$$

$$\Upsilon_1 \equiv 2\gamma + 2\alpha - \epsilon_1 - \mu_1\,, \qquad (66d)$$

etc.

(For the correct Newtonian limit, one must require that $\alpha = 1$, but we leave α arbitrary here.) Each theory will yield certain values for the Γ's and Υ's. We have shown that nonmetric theories must have some of the Γ's or Υ's nonzero—the first nonzero Γ or Υ determines the order at which the theory violates WEP.

B. Experimental Verification of WEP and Applications of Our Calculations

Thus far, our results have been completely within a theoretical context. We now investigate the experimental and practical applications.

Experimental support for WEP comes from the type of experiment developed by Eötvös in the late nineteenth century, and redesigned extensively by Dicke in the 1960's.[17] The particular Eötvös-Dicke (ED) experiments of highest reported precision are the Princeton experiment of Roll, Krotkov, and Dicke,[17] and the Moscow experiment of Braginsky and Panov.[18] These experiments measure the relative acceleration toward the sun of two different substances (gold and aluminum in the Princeton experiment; platinum and aluminum in the Moscow experiment). The reported results are

$$\frac{|\langle\vec{A}_{c.m.}\rangle_{Al} - \langle\vec{A}_{c.m.}\rangle_{Au}|}{|\langle\vec{A}_{c.m.}\rangle|} \approx \frac{|\langle\vec{A}_{c.m.}\rangle_{Al} - \langle\vec{A}_{c.m.}\rangle_{Au}|}{|\vec{g}|} < 10^{-11}\,, \qquad (67a)$$

$$\frac{|\langle\vec{A}_{c.m.}\rangle_{Al} - \langle\vec{A}_{c.m.}\rangle_{Pt}|}{|\langle\vec{A}_{c.m.}\rangle|} < 10^{-12}\,. \qquad (67b)$$

Our calculation involved a test body dropped in a static field. The following argument justifies direct comparison of our calculation with the results of the above experiments:

(i) The 24-hour component of the acceleration can easily be isolated so that the sun can really be considered as the sole external source of gravitation (see page 173 of Ref. 17). To make this more clear, if one uses the 24-hour period variation to select out $\vec{g}_{sun}$ from $\vec{g}_{sun} + \vec{g}_{earth}$, then Eq. (17) has body-dependent terms of the form

$$\langle\vec{A}_{c.m.}\rangle \approx \vec{g}_{sun} M_0^{-1}\Big\langle \sum_{i,k} \eta_{ik}\Big\rangle \times [\Gamma_0 + \Gamma_1(U_{sun} + U_{earth}) + \cdots] \approx \vec{g}_{sun} M_0^{-1}\Big\langle \sum_{i,k} \eta_{ik}\Big\rangle [\Gamma_0 + \Gamma_1 U_{sun} + \cdots]$$

since $U_{sun} \approx 10 U_{earth}$.

(ii) The fact that the earth is rotating rather than at rest can only contribute *inertial* accelerations; in particular no *relative* accelerations between the two test bodies can be introduced in this manner.

(iii) We have considered only electromagnetic test bodies; but we wish to apply our results to the actual atoms used in the experiments, atoms which have nuclear as well as electromagnetic interactions. Thus the complete equation for $\langle\vec{A}_{c.m.}\rangle$ for realistic atoms has, in addition to the terms shown in Eq. (17), terms which involve nuclear energies. Is it possible that the nuclear and electromagnetic terms would cancel each other? The only mechanism by which the terms could be combined and related is through the virial relations; yet an examination of Eq. (17) reveals that μ_0 does not even occur in the electromagnetic portion of the virial relations. In particular, *given* the combined virial relations for both electromagnetic and nuclear interactions one could construct an infinity of different theories merely by changing μ (and thus changing the body-dependent terms in $\langle\vec{A}_{c.m.}\rangle$). Thus there is no credible mechanism by which nuclear and electromagnetic body dependent terms could conspire to cancel each other. The "electromagnetic violation" of WEP thus constitutes a lower limit to the total violation (allowing for possible nuclear violations).

We can now ask to what order does Eq. (67) test the GMM equations of a theory. Equation (17) has the form

$$\langle\vec{A}_{c.m.}\rangle \sim \vec{g}\left[\frac{\text{electromagnetic energy}}{\text{total mass}}\right] \times F(H_0, T_0, \epsilon_0, \mu_0, H_0', T_0', \epsilon_0') + \text{body-independent term}\,, \qquad (68)$$

where F is a function of the indicated variables. Now, the largest contribution to the electromagnetic energy of the total atom certainly comes from the nuclear protons and for platinum or gold this amounts to, using the semiempirical mass formula,[19]

$$\left[\frac{\text{electromagnetic energy}}{\text{total mass}}\right]_{\text{Pt or Au}} \approx 5 \times 10^{-3}\,. \qquad (69a)$$

For aluminum, the corresponding quantity is

$$\left[\frac{\text{em energy}}{\text{total mass}}\right]_{Al} \sim \frac{(Z^2A^{-4/3})_{Al}}{(Z^2A^{-4/3})_{Pt\ or\ Au}}\left[\frac{\text{em energy}}{\text{total mass}}\right]_{Pt} \approx 2\times10^{-3}. \tag{69b}$$

Noting that U_0 has the magnitude

$U_0 \equiv$ potential of sun at earth $\sim 10^{-8}$

and using Eqs. (65) and (67), we see that current experimental accuracy bears upon the Γ_k and Υ_k only for $k \leq 1$. *The accuracy*[20] *of the experiment must go up by a factor of* 10^7 *to require that* Γ_2 *and* Υ_2 *vanish.* Equations (66) show that the experiment thus measures H, T, and ϵ to $O(U^2)$, but μ only to $O(U)$. We expect that almost all theories will do well enough to have $\Gamma_0 \equiv 0$.

Before continuing with direct applications to theories of the current experimental verification of WEP, let us return to Eq. (17) and analyze the specific way in which it constrains the GMM equations of a gravitation theory. The second body-dependent term in Eq. (17)—the "directional Coulomb energy" term—involves the GMM equations only through the product $\epsilon\mu$. This particular product is also equal to the square of the index of refraction, n^2, and is tested by light-bending and time-delay experiments (see, e.g., Ref. 21 for a discussion of these experiments—although in the context of metric theories). In fact, exploiting the "ϵ-μ" analogy for the GMM equations and taking the geometrical optics limit, one sees that the current experimental tests, with the exception of WEP, are sensitive *only* to the product $\epsilon\mu$—and only to first order in U of that quantity. On the other hand, the *first* body-independent term in Eq. (17)—the "nondirectional Coulomb energy" term—samples the GMM equations in a deeper manner, both qualitatively and quantitatively. Not only is ϵ distinguished from μ (magnetic and electric effects distinguished) but also is ϵ explored to *second* order in U (cf. the ϵ'_0) for the current experimental verification of WEP. Thus WEP is revealed as a powerful tool for probing the GMM equations—the most sensitive probe of those equations existing in 1973.

On purely theoretical grounds one can require, as we have previously remarked, that the Γ's and Υ's vanish independently. However, in practical experimental applications, the second body-dependent vector in Eq. (65) has some particular relation to the first for any given experiment. Since the nuclei of the atoms in the ED experiment are approximately spherical,

$$\left\langle \sum_{i,k} \vec{\omega}_{ik}\right\rangle \approx \tfrac{1}{3}\vec{g}\left\langle\sum_{i,k}\eta_{ik}\right\rangle . \tag{70}$$

Usings Eqs. (65) – (70), one finally obtains, for $\alpha = 1$ (correct Newtonian limit)

$$\frac{\langle\vec{A}_{c.m.}\rangle_{Pt,\,Au} - \langle\vec{A}_{c.m.}\rangle_{Al}}{\vec{g}} \approx -3\times10^{-3}[\Gamma_0 + 10^{-8}(\Gamma_1 - \tfrac{1}{3}\Upsilon_1)] . \tag{71}$$

C. Applications to Specific Nonmetric Theories

In this section we discuss WEP for three particular nonmetric theories. The Belinfante-Swihart and Whitehead theories have equations of the form of Eqs. (1)–(3). As an illustration of the formalism of Sec. IV A and IV B, the WEP violation is calculated explicitly in the case of the Belinfante-Swihart theory. The Naida-Capella theory, which is an apparently rare example of a theory *not* having a particle Lagrangian of the form of Eq. (1) in the SSS limit, is treated on an individual basis, using the techniques developed in Secs. II and III.

1. *Belinfante-Swihart Theory*[4,5]

An analysis of the Belinfante-Swihart theory in Ref. 14 reveals that its *particle* Lagrangian can be put into metric form with

$$g_{\alpha\beta} = (1 - Kh)^2[\eta_{\alpha\beta} + h_{\alpha\beta} + \tfrac{3}{4}h_{\alpha\tau}h_{\beta\mu}\eta^{\mu\tau} + O(h^3)], \tag{72}$$

where K is an arbitrary constant, $h \equiv \eta^{\alpha\beta}h_{\alpha\beta}$, and $\eta_{\alpha\beta}$ is the Minkowski metric. The GMM equations are of "ϵ-μ" form [i.e., have the form of Eqs. (2)–(3)], with, in the SSS limit,

$$\epsilon = [1 - \tfrac{1}{2}(h_{00} + h_{11})]^{-1}, \tag{73a}$$

$$\mu = [1 + \tfrac{1}{2}(h_{00} + h_{11})]. \tag{73b}$$

In the SSS limit, $h_{\mu\nu}$ has the form

$$h_{00} = C_0 U, \tag{74a}$$

$$h_{ij} = \delta_{ij}C_1 U, \tag{74b}$$

$$h_{0k} = 0, \tag{74c}$$

where C_0, C_1 are arbitrary constants, but with the implicit relation

$$2K(3C_1 - C_0) + C_0 - 2 = 0 \tag{75}$$

in order to satisfy the Newtonian limit ($g_{00} = -1 + 2U + \cdots$). Defining T and H by comparison of Eqs. (72), (74) with Eqs. (24) and then evaluating the various Γ_k and Υ_k [cf. Eqs. (64) and (66)], one finds

$$\Gamma_0 = 0, \tag{76a}$$

$$\Gamma_1 - \tfrac{1}{3}\Upsilon_1 \doteq -\tfrac{1}{2}C_0(C_0+C_1) \neq 0 . \qquad (76b)$$

In order to predict an amount of light bending and perihelion shift compatible with experiment, one must require that C_0 and C_1 satisfy

$$0.9 \leq \tfrac{1}{2}(C_0+C_1-2) \leq 1.1 , \qquad (77a)$$

$$0.8 \leq \tfrac{1}{2}(C_0+1) \leq 1.3 . \qquad (77b)$$

The combinations of C_0 and C_1 occurring in Eqs. (77a) and (77b) correspond to the γ and β parameters, respectively, of the "PPN formalism"[21] and the experimental limits indicated above are discussed in Ref. 21.

Using Eqs. (71) and (77), we find that the nonmetric theory of Belinfante and Swihart predicts

$$4\times10^{-11} \leq \left|\frac{\langle\vec{A}_{\rm c.m.}\rangle_{\rm Pt\ or\ Au} - \langle\vec{A}_{\rm c.m.}\rangle_{\rm Al}}{\langle\vec{A}_{\rm c.m.}\rangle}\right| \leq 1\times10^{-10} . \qquad (78)$$

If one requires the light-bending and perihelion-shift predictions of the Belinfante-Swihart theory to be same as in general relativity, Eq. (78) becomes

$$\left|\frac{\langle\vec{A}_{\rm c.m.}\rangle_{\rm Au\ or\ Pt} - \langle\vec{A}_{\rm c.m.}\rangle_{\rm Al}}{\langle\vec{A}_{\rm c.m.}\rangle}\right| \approx 6\times10^{-11} . \qquad (79)$$

Thus, the Belinfante-Swihart theory violates seriously both the Princeton and the Moscow versions of the ED experiment.

2. *Whitehead's Theory*[9]

Synge analyzes only the motion of uncharged particles and the sourceless GMM equations in Whitehead's theory:

$$\delta\int (g_{\alpha\beta}\,dx^\alpha dx^\beta)^{1/2} = 0 \quad \text{[Eq. (1.7) of Ref. 8]} \qquad (80a)$$

$$(g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu})_{,\beta} = 0 \quad \text{[Eq. (1.9) of Ref. 8]} \qquad (80b)$$

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \quad \text{[Eq. (1.9) of Ref. 8]}. \qquad (80c)$$

A straightforward generalization of these equations to include sources shows that the GMM equations have "ϵ-μ" form in the SSS limit, with

$$\epsilon = (-g_{00}f)^{-1} , \qquad (81a)$$

$$\mu = f^2 \qquad (81b)$$

[in the notation of Eqs. (57)]. Using Eqs. (17), (57), and (81), one can then show that

$$\left|\frac{\langle\vec{A}_{\rm c.m.}\rangle_{\rm Au\ or\ Pt} - \langle\vec{A}_{\rm c.m.}\rangle_{\rm Al}}{\langle\vec{A}_{\rm c.m.}\rangle}\right| \approx 10^{-3}\frac{d}{dU}[\ln(-g_{00}f^3)] , \qquad (82)$$

so that, for experimentally acceptable values of g_{00} and f^3, this version of Whitehead's theory violates WEP at the order of 10^{-3}. [Note that in Whitehead's theory the product $\epsilon\mu$ is the same as in metric theories, so that the coefficient of the second body-dependent term in Eq. (17) vanishes identically. In some sense one can say that, with respect to the light bending and radar time-delay experiments, Whitehead's theory is a metric theory.]

3. *Naida-Capella Theory*

The nonmetric theory of Capella[7] as completed by Naida[8] has the following Lagrangian [cf. Eq. (2.1) of Ref. 7]:

$$L = m_0\int ds[-(\eta_{\alpha\beta}u^\alpha u^\beta)^{1/2} + \chi h_{\alpha\beta}u^\alpha u^\beta(\eta_{\rho\sigma}u^\rho u^\sigma)^{-1/2}]$$
$$-e\int A_\mu dx^\mu , \qquad (83)$$

where $\eta_{\alpha\beta}$ is the Minkowski metric and

$$ds \equiv (\eta_{\alpha\beta}\,dx^\alpha dx^\beta)^{1/2} ,$$

$$\chi \equiv (7\pi)^{1/2} ,$$

$$u^\alpha \equiv (dx^\alpha/ds) .$$

The GMM equations are of "ϵ-μ" form [cf. Eq. (3.7) of Ref. 7] with

$$\epsilon = 1 + \chi(h_{00}+h_{11}) , \qquad (84a)$$

$$\mu = [1 - \chi(h_{00}+h_{11})]^{-1} . \qquad (84b)$$

Solutions to the SSS gravitational field equations yield

$$h_{00} = C_0\chi^{-1}U , \qquad (85a)$$

$$h_{ij} = C_1\chi^{-1}U\delta_{ij} , \qquad (85b)$$

where C_0 and C_1 are arbitrary constants. Variation of Eq. (83) and use of Eqs. (85) gives the particle equation of motion [analog of Eq. (29)]

$$\begin{aligned}\vec{a}_k = \vec{g}[&C_0 - C_0(C_0+2C_1)U_0 + C_1 v_k^2 \\ &- U_0 v_k^2(2C_0C_1 + C_0^2 + 2C_1^2)] \\ &-2\vec{v}_k(\vec{v}_k\cdot\vec{g})[C_0 + C_1 - 2C_1(C_0+C_1)U_0] \\ &+\vec{A}_L[1 - U(C_0+2C_1)] . \end{aligned} \qquad (86)$$

Using Eqs. (84)–(86), the GMM equations give

$$\begin{aligned}\vec{A}_L(\vec{x}_i) = &(m_{0i})^{-1}(1 - CU_0 + C^2U_0^2)\sum_k e_ie_k|\vec{x}_{ik}|^{-3}\vec{x}_{ik} \\ &+\tfrac{1}{2}(m_{0i})^{-1}[C_1 - U_0(2C^2 - C_0C_1)]\vec{g}\sum_k \eta_{ik} \\ &-\tfrac{1}{2}(m_{0i})^{-1}[C_0 + C - U_0(2C^2 + C_0C_1)]\sum_k \vec{\omega}_{ik} \\ &-(m_{0i})^{-1}C(1 - 2CU_0)\sum_k e_ie_k(\vec{g}\cdot\vec{x}_k)|\vec{x}_{ik}|^{-3}\vec{x}_{ik} , \end{aligned} \qquad (87)$$

with $C \equiv C_0 + C_1$.

Using the same center-of-mass formulas as given in Eqs. (7)–(9) and the virial theorem

$$\left\langle \sum_i m_{0i}(v_i)^\alpha (v_i)^\beta + \tfrac{1}{2}[1 - U_0(3C_1 + 2C_0)] \sum_{i,k} e_i e_k (x_{ik})^\alpha (x_{ik})^\beta |x_{ik}|^{-3} \right\rangle = 0 + O(g) \tag{88}$$

one finally obtains

$$\langle \vec{A}_{c.m.} \rangle = \vec{g} C_0 [1 + U_0(-2C_1 + C_0)] - \tfrac{1}{2} M_0^{-1} (C_0^2 + 3C_1^2) U_0 \vec{g} \left\langle \sum_{i,k} \eta_{ik} \right\rangle + M_0^{-1} (\tfrac{1}{2} + \tfrac{3}{2} C_1 - 5C_1^2 - C_0^2 - 4C_0 C_1) U_0 \left\langle \sum_{i,k} \vec{\omega}_{ik} \right\rangle . \tag{89}$$

Now, with Eqs. (69)–(71) we get

$$\frac{|\langle \vec{A}_{c.m.} \rangle_{\text{Pt or Au}} - \langle \vec{A}_{c.m.} \rangle_{\text{Al}}|}{|\vec{g}|} \approx 10^{-11} (1 + 3C_1 - 19C_1^2 - 5C_0^2 - 8C_0 C_1) . \tag{90}$$

The correct Newtonian and light-bending results require, respectively,

$$C_0 = 1 , \tag{91a}$$

$$0.9 \le \tfrac{1}{2}(C_1 + 1) \le 1.1 . \tag{91b}$$

Equations (90) and (91) indicate then the relation

$$2 \times 10^{-10} \le \left| \frac{\langle \vec{A}_{c.m.} \rangle_{\text{Au or Pt}} - \langle \vec{A}_{c.m.} \rangle_{\text{Al}}}{\langle \vec{A}_{c.m.} \rangle} \right| \le 4 \times 10^{-10} . \tag{92}$$

Thus the Naida-Capella nonmetric theory seriously violates both the Princeton and Moscow versions of the ED experiment.

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b) Analysis of the Belinfante-Swihart Theory of Gravity (Paper VI; collaboration with D.L. Lee, published in Phys. Rev. D, **7**, 3578, 1973)

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Analysis of the Belinfante-Swihart Theory of Gravity*

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We show that the Belinfante-Swihart (BS) theory can be reformulated in a representation in which uncharged matter responds to gravity in the same way as in metric theories. The BS gravitationally modified Maxwell equations can also be put into metric form to first order in the deviations of the physical metric from flat space, but not to second order; consequently, the theory is nonmetric except in first order. We also show that the theory violates the high-precision Eötvös-Dicke experiment, but cannot be ruled out by the gravitational precession of gyroscopes.

I. INTRODUCTION AND SUMMARY

This paper analyzes the most complete and extensively developed nonmetric theory that exists: the 1957 theory of Belinfante and Swihart.[1-3] Belinfante and Swihart (BS) constructed their theory as a Lorentz-symmetric[4] linear field theory which would be easily quantized. However, as we shall show, in terms of measurable quantities the theory has all the nonlinearities of typical "curved-spacetime" theories. Moreover, it is nearly a metric[4] theory: We construct a new mathematical representation which has metric form to first order in deviations of the physical metric from flatness, but does not have metric form to higher orders.

Section II gives a brief summary of the original BS representation. Included are discussions of nonlinearities and the behavior of rods and clocks. Section III presents our new mathematical representation of the theory. Section IV gives a prescription for obtaining the post-Newtonian limit[5,6] of the theory, and Sec. V considers various experimental tests. Contrary to previous calculations[7] it is found that both the geodetic and the Lens-Thirring precessions of gyroscopes[8] cannot distinguish the BS theory from general relativity (for a particular choice of adjustable parameters). However, using results of another paper,[9] we show that the failure of the theory to be metric at second order causes a violation of the Eötvös-Dicke[10,11] experimental results. Our calculations confirm the the Belinfante-Swihart conclusion that their theory agrees with the three classical tests of gravitation theories (perihelion shift of Mercury, bending of light by the sun, and red shift of light), and also agrees with the weak equivalence principle[4] (WEP) to *first* order.

II. THE BELINFANTE-SWIHART REPRESENTATION OF THEIR THEORY

A. Lagrangian and Equations of Motion

The original representation of the BS theory is Lagrangian-based,[4] but is not in generally covariant form.[4] In this section we generalize, in a trivial manner, the original representation so that it is generally covariant. The dynamical equations are obtained by extremization of the following action:

$$I = \int \mathfrak{L}_G d^4x + \int \mathfrak{L}_M d^4x + \int \mathfrak{L}_I d^4x \, . \tag{1}$$

where

$$\mathfrak{L}_G = -(16\pi)^{-1}\eta^{\alpha\beta}\eta^{\lambda\mu}\eta^{\rho\sigma}(a h_{\lambda\rho|\alpha} h_{\mu\sigma|\beta} + f h_{\lambda\mu|\alpha} h_{\rho\sigma|\beta})(-\eta)^{1/2}, \tag{2}$$

$$\mathfrak{L}_M = \sum_A \int \left[-m_A b_A + (\Pi_\mu + e_A A_\mu)\frac{dx^\mu}{d\lambda_A} - \Pi_\mu a^\mu \right] \delta^4(\underline{x} - \underline{z}_A(\lambda_A))\, d\lambda_A + (4\pi)^{-1}(\tfrac{1}{4} H^{\nu\mu} H_{\nu\mu} - H^{\nu\mu} A_{\mu|\nu})(-\eta)^{1/2}, \tag{3}$$

$$\mathfrak{L}_I = \tfrac{1}{2}\overline{T}^{\mu\nu} h_{\mu\nu} + \sum_A \int K m_A b_A h_\lambda{}^\lambda \, \delta^4(\underline{x} - \underline{z}_A(\lambda_A))\, d\lambda_A \, . \tag{4}$$

$$\overline{T}^{\mu\nu} = (4\pi)^{-1}(H^{\lambda\mu} H_\lambda{}^\nu - \tfrac{1}{4}\eta^{\mu\nu} H^{\alpha\beta} H_{\alpha\beta})(-\eta)^{1/2} + \sum_A \int a^\mu_A \Pi^\nu_A \, \delta^4(\underline{x} - \underline{z}_A(\lambda_A))\, d\lambda_A \, . \tag{5a}$$

$$b_A \equiv a_{A\mu} a^\mu_A \, . \tag{5b}$$

Equations (1)–(5) describe the interactions of a collection of charged particles (labeled by A) with the electromagnetic and gravitational fields. Conventions and definitions for the above are the following:

(i) We use units such that $c = G = 1$.

(ii) $\eta_{\alpha\beta}$ is a Riemann flat background metric (absolute gravitational field[4]). In some coordinate system, it therefore takes on Minkowski values, $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. *All tensorial indices* occurring in Eqs. (1)–(5) are raised and lowered with $\eta_{\alpha\beta}$.

(iii) Greek and Latin indices run through 0–3 and 1–3, respectively.

(iv) a, f, K are adjustable parameters.

(v) $h_{\mu\nu} = h_{\nu\mu}$ is a symmetric second-rank dynamical gravitational field.[4]

(vi) The world line of particle A is parametrized by an arbitrary, monotonic parameter λ_A which varies from $-\infty$ to $+\infty$. Particle A is described by its *coordinate* z^μ_A and its "velocity and momentum variables" a^μ_A and Π^μ_A, which are all functions of λ_A.

(vii) The electromagnetic field is described by the tensor fields A_μ and $H_{\mu\nu} = -H_{\nu\mu}$.

(viii) $\overline{T}^{\mu\nu}$ is a "stress-energy tensor" for particles and electromagnetic fields. (The bar above is used to distinguish it from a different "stress-energy tensor" defined in Sec. IV.)

(ix) Slashes denote covariant derivatives with respect to the flat background metric $\eta_{\alpha\beta}$.

(x) $\eta \equiv$ determinant of $\eta_{\alpha\beta}$.

Equations (5a) and (5b) are decomposition equations[4] for $\overline{T}^{\mu\nu}$ and b_A. The dynamical variables which one varies independently in the action are $h_{\mu\nu}(x)$, $z^\mu_A(\lambda_A)$, $a^\mu_A(\lambda_A)$, $\Pi^\mu_A(\lambda_A)$, $A_\mu(x)$, and $H_{\mu\nu}(x)$. *Variation of the matter variables yields the following* dynamical laws[12]:

$$m a^\mu (1 - Kh) = b(\Pi^\mu - \tfrac{1}{2} h_\nu{}^\mu \Pi^\nu) \quad [\text{BS, I, (29)}], \tag{6}$$

$$dz^\mu_A / d\lambda_A = a^\mu_A - \tfrac{1}{2} h_\nu{}^\mu (z_A) a^\nu_A \quad [\text{BS, I, (30)}] \cdot \tag{7}$$

$$F_{\mu\nu} \equiv A_{\nu|\mu} - A_{\mu|\nu}$$
$$= H_{\mu\nu}(1 - \tfrac{1}{2}h) + H_{\mu\lambda} h_\nu{}^\lambda - H_{\nu\lambda} h_\mu{}^\lambda$$
$$[\text{BS, II, (11)}], \tag{8}$$

$$H^{\mu\nu}{}_{|\nu} = 4\pi \sum_A e_A \int \frac{dz^\mu_A}{d\lambda_A} \delta^4(\underline{x} - \underline{z}_A)\, d\lambda_A (-\eta)^{-1/2}$$
$$[\text{BS, II, (10)}]. \tag{9}$$

$$\frac{d\Pi_A{}^\mu}{d\lambda_A} = e_A F_{\mu\nu}\frac{dz_A^\nu}{d\lambda_A} + \tfrac{1}{2} a_A^\rho \Pi_A^\sigma h_{\rho\sigma|\mu} + K m_A b_A h_{|\mu}$$

[BS, II, (5)], (10)

where $h \equiv h_\alpha{}^\alpha$.

Variation of $h_{\mu\nu}$ yields

$$a\Box h_{\alpha\beta} + f\eta_{\alpha\beta}\Box h = -4\pi \overline{T}_{\alpha\beta} - 8\pi K \eta_{\alpha\beta}\sum_A \int m_A b_A \delta^4(\underline{x} - \underline{z}_A) d\lambda_A . \quad (11)$$

Here we have used the symbol $\Box h_{\mu\nu} \equiv \eta^{\alpha\beta} h_{\mu\nu|\alpha|\beta}$.

B. Nonlinearities in the Theory

Linear gravitational field equations do not preclude a nonlinear form for the response of particles to gravity. The BS theory is an example: Equations (6) and (7) endow the canonical variables a_A^μ and Π_A^μ with gravitational contributions. Consequently, the equation of motion for a particle, Eq. (10), is nonlinear in the gravitational field $h_{\mu\nu}$. Indeed, although the BS theory is often called a "linear" theory, its linear first-order matter Lagrangian produces qualitatively many of the nonlinear effects of general relativity (GRT), for example (see Secs. III and IV). Hence one should be cautious in the labeling of theories as linear or nonlinear on the mere basis of the linear forms of their gravitational equations.

C. Behavior of Rods and Clocks

In the third paper of their series,[3] Belinfante and Swihart quantize the theory and obtain a gravitationally modified Dirac theory. We remind the reader that all nonmetric theories must exhibit explicitly the manner in which all the laws of physics are changed in the presence of gravity. Belifante and Swihart find that, in the case of a static spherically symmetric (SSS) gravitational source, the standard solutions to the unmodified Dirac equation are related to those in the presence of gravity in the following way:

$$\varphi_0(\vec{x}_0, t_0) = N\varphi(\vec{x}, t), \quad (12)$$

$$\vec{x}_0 = C\vec{x} \quad [\text{BS, III, (78)}], \quad (13)$$

$$t_0 = (1-U)t, \quad (14)$$

$$N \equiv C^{-3/2} \equiv \left(\frac{1-U}{1-U/2a}\right)^{-3/2} . \quad (15)$$

Here the subscripted quantities are those in the absence of gravity, φ is the electron wave function, U is the Newtonian gravitational potential for an SSS source, and a is the previously mentioned adjustable parameter. The coordinate system is one in which $\eta_{\alpha\beta} = \text{diag}\ (-1, 1, 1, 1)$. The energy eigenvalues, i.e., E in $\varphi(\vec{x}, t) = \varphi(\vec{x}) \exp(-iEt/\hbar)$, are shifted in the presence of gravity:

$$E_0 = (1+U)E \quad [\text{BS, III, (82)}] \quad (16)$$

–a result following essentially from Eq. (14). It is Eq. (16) which produces qualitatively the correct red shift. Equations (12) and (13) also indicate the effect of gravity on the coordinate sizes of atoms. Consider the expectation value of the coordinate size of an atom:

$$\langle r\rangle = \int |\varphi(\vec{x}, t)|^2 r\, d^3x . \quad (17)$$

Using Eqs. (12) and (13) we obtain

$$\langle r\rangle = \int N^{-2}|\varphi_0(\vec{x}_0, t_0)|^2 C^{-1} r_0 C^{-3} d^3x_0 = C^{-1}\langle r_0\rangle = \frac{1-U/2a}{1-U}\langle r_0\rangle \approx [1 - U(\tfrac{1}{2}a^{-1} - 1)]\langle r_0\rangle \quad (18)$$

According to Eq. (16), the coordinate ticking rate of an atomic clock decreases in a gravitational field:

$$\omega = (1-U)\,\omega_0 .$$

According to Eq. (18) the coordinate size of a rod made of atoms decreases in a gravitational field:

$$l = [1 - U(\tfrac{1}{2}a^{-1} - 1)]\, l_0 .$$

Since $a \approx \frac{1}{4}$ to agree with the light bending experiment (see later sections), the above results are the same, to first order in U, as one obtains in GRT, using an "isotropic, post-Newtonian" coordinate system.[5]

III. ATTEMPTS TO PUT THE THEORY INTO METRIC FORM

The BS theory is a Lagrangian-based relativistic theory of gravity.[4] Therefore, according to a theorem proved in Ref. 4, it is a metric theory if and only if the "nongravitational part" of its Lagrangian,

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$$\mathcal{L}_{NG} \equiv \mathcal{L}_M + \mathcal{L}_I ,$$

can be put into universally coupled form.[4] Let us try to achieve universal coupling by a change of variables, i.e., by introducing a new mathematical representation of the theory.

A. Particle Part of Lagrangian

Begin with the terms in $\mathcal{L}_{NG}$ that refer only to particles and define the following tensors:

$$\bar{\Delta}_\nu{}^\mu \equiv \delta_\nu{}^\mu - \tfrac{1}{2} h_\nu{}^\mu , \tag{19}$$

$$\Delta_\alpha{}^\beta \equiv (\bar{\Delta}_\alpha{}^\beta)^{-1}, \ \text{i.e.,} \ \bar{\Delta}_\alpha{}^\beta \Delta_\beta{}^\tau = \delta_\alpha{}^\tau . \tag{20}$$

Then, from Eq. (7), obtain the relation

$$a^\nu = \Delta_\mu{}^\nu dz^\mu/d\lambda . \tag{21}$$

Equation (21), which is obtained after variation of the Lagrangian, suggests that one define a new variable v^μ to replace a^μ in the Lagrangian:

$$a^\nu \equiv \Delta_\mu{}^\nu v^\mu . \tag{22}$$

Then, the relation $v^\mu = dz^\mu/d\lambda$ will presumably turn out to be an Euler-Lagrange equation. Using Eqs. (19)–(21), bring the particle portion of the Lagrangian into the form

$$I_{\text{part}} \equiv \left[\int \mathcal{L}_M d^4x + \int \mathcal{L}_I d^4x \right]_{\text{part}} \tag{23}$$

$$= \sum_A \int \left[-(1-Kh) m_A b_A + e_A A_\mu \frac{dz^\mu_A}{d\lambda_A} + \Pi_{A\mu} \left(\frac{dz^\mu_A}{d\lambda_A} - a^\mu_A + \tfrac{1}{2} h_\nu{}^\mu a^\nu_A \right) \right] d\lambda_A$$

$$= \sum_A \int \left\{ -m_A [-(1-Kh)^2 \Delta_\mu{}^\alpha \Delta_{\alpha\nu} v^\mu_A v^\nu_A]^{1/2} + e_A A_\mu \frac{dz^\mu_A}{d\lambda_A} + \Pi_{A\mu} \left(\frac{dz^\mu_A}{d\lambda_A} - v^\mu_A \right) \right\} d\lambda_A . \tag{24}$$

In obtaining Eq. (24) from Eq. (23) we have performed the integrations over d^4x and, thus, all of the space-time functions should be evaluated at the particle position z^μ_A.

If we now define an "effective metric,"

$$g_{\alpha\beta} \equiv (1-Kh)^2 \Delta_\alpha{}^\mu \Delta_{\mu\beta} = \eta_{\alpha\beta}(1-2Kh) + h_{\alpha\beta} + O(h^2) , \tag{25}$$

Eq. (24) takes the universally coupled form, with $g_{\alpha\beta}$ being the only gravitational field occurring in I_{part}. Variation of Π_μ then yields the desired relation

$$v^\mu = \frac{dz^\mu}{d\lambda} . \tag{26}$$

To make our results look simpler, we explicitly introduce Eq. (26) into Eq. (24), thus eliminating Π_μ completely and obtaining

$$I_{\text{part}} = \sum_A \int \left[-m_A \left(-g_{\alpha\beta} \frac{dz^\alpha_A}{d\lambda_A} \frac{dz^\beta_A}{d\lambda_A} \right)^{1/2} + e_A A_\mu \frac{dz^\mu}{d\lambda_A} \right] d\lambda_A . \tag{27}$$

Variation of Eq. (27) yields equations of motion which, by the use of Eqs. (6), and (19)–(21), can be shown to be identical to the BS equations of motion, Eqs. (10). Equation (27) is the familiar "metric theory" action principle describing the interaction of charged particles with the gravitational field $g_{\mu\nu}$ and the electromagnetic field A_μ.

B. Electromagnetic Part of Lagrangian

It will now be shown that, to first order in $h_{\mu\nu}$, the electromagnetic Lagrangian can also be put into metric form. Change variables from $H_{\mu\nu}$ to an antisymmetric tensor $F_{\mu\nu}$ by

$$H_{\mu\nu} = F_{\mu\nu} (1 + \tfrac{1}{2}h + \tfrac{1}{4}h^2) + 2F_{\lambda[\mu}h_{\nu]}{}^\lambda (1+h) - 2F_{\alpha[\mu}h_{\nu]}{}^\lambda h^\alpha{}_\lambda - 2F_{\lambda\alpha}h^\alpha{}_{[\mu}h_{\nu]}{}^\lambda + O(Fh^3) . \tag{28}$$

Equation (28) is simply the result of an inversion of Eq. (8). Square brackets around indices denote anti-symmetrization of indices (with the usual normalization of a factor of $\tfrac{1}{2}$). Variation of $F_{\mu\nu}$ in the new Lagrangian presumably will yield the relation

$$F_{\mu\nu} = A_{\nu|\mu} - A_{\mu|\nu} \ . \tag{29}$$

Substitution of Eq. (28) into the electromagnetic portion of the action yields

$$\begin{aligned} L_{em} &\equiv (4\pi)^{-1} \int \{ \tfrac{1}{2} H_{\alpha\beta} H_{\nu\mu} [\tfrac{1}{2}\eta^{\beta\mu}(1-\tfrac{1}{2}h) + h^{\beta\mu}]\eta^{\alpha\nu} - A_{\mu|\nu} H_{\alpha\beta}\eta^{\alpha\nu}\eta^{\beta\mu}\} (-\eta)^{1/2} d^4x \\ &= (4\pi)^{-1} \int \{ \tfrac{1}{2}[F_{\alpha\beta}(1+\tfrac{1}{2}h) + 2F_{\lambda[\alpha}h_{\beta]}{}^{\lambda}] [F_{\nu\mu}(1+\tfrac{1}{2}h) + 2F_{\lambda[\nu}h_{\mu]}{}^{\lambda}] \eta^{\alpha\nu}\Gamma^{\beta\mu} \\ &\qquad - A_{\mu|\nu}\eta^{\alpha\nu}\eta^{\beta\mu}[F_{\alpha\beta}(1+\tfrac{1}{2}h) + 2F_{\lambda[\alpha}h_{\beta]}{}^{\lambda}]\} (-\eta)^{1/2} d^4x \qquad (30) \\ &= (4\pi)^{-1} \int \left(A_{[\mu|\nu]} + \tfrac{1}{4}F_{\mu\nu}\right) F_{\alpha\beta}[\eta^{\alpha\mu}\eta^{\beta\nu}(1+\tfrac{1}{2}h) - h^{\alpha\mu}\eta^{\beta\nu} - h^{\beta\nu}\eta^{\alpha\mu}] (-\eta)^{1/2} d^4x + O(h^2) \ , \qquad (31) \end{aligned}$$

where

$$\Gamma^{\beta\mu} \equiv \tfrac{1}{2}\eta^{\beta\mu}(1-\tfrac{1}{2}h) + h^{\beta\mu} \ .$$

If one now uses the inverse of Eq. (25), i.e.,

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} + 2Kh\eta^{\alpha\beta} + O(h^2)$$

and

$$(-g)^{1/2} = (-\eta)^{1/2}[1 + h(\tfrac{1}{2} - 4K)] + O(h^2) \ ,$$

one finds Eq. (31) can be written as

$$L_{em} = (4\pi)^{-1} \int (A_{[\mu|\nu]} + \tfrac{1}{4}F_{\mu\nu}) \times F_{\alpha\beta} g^{\alpha\mu} g^{\beta\nu} (-g)^{1/2} d^4x + L_{corr} \ , \tag{32}$$

where

$$L_{corr} \equiv (4\pi)^{-1} \int F_{\mu\nu} (\tfrac{1}{4}F_{\alpha\beta} - A_{[\alpha|\beta]}) \Gamma^{\mu\alpha\nu\beta} d^4x = O(h^2) \tag{33a}$$

and

$$\Gamma^{\mu\alpha\nu\beta} \equiv \tfrac{1}{8} h^2 \eta^{\mu\alpha}\eta^{\nu\beta} - h\eta^{\mu\alpha}h^{\nu\beta} + \tfrac{3}{2}\eta^{\mu\alpha}h^{\nu\sigma}h_{\sigma}{}^{\beta} + 3h^{\mu\alpha}h^{\nu\beta}. \tag{33b}$$

Thus L_{em} has universally coupled form at $O(h)$; at $O(h^2)$ deviations occur, arising from the L_{corr} term in Eq. (32). Variation of $F_{\mu\nu}$ in Eq. (32) yields the desired relation between $F_{\mu\nu}$ and A_μ, i.e., Eq. (29). Completely equivalent equations are obtained if Eq. (29) is now substituted into Eq. (32), yielding

$$L_{em} = -(16\pi)^{-1} \int F_{\alpha\beta} F_{\mu\nu} g^{\alpha\mu} g^{\beta\nu} (-g)^{1/2} d^4x + L_{corr} \tag{34a}$$

$$\equiv -(16\pi)^{-1} \int F_{\alpha\beta} F^{\alpha\beta} (-g)^{1/2} d^4x + L_{corr} \ . \tag{34b}$$

The relation given in Eq. (29) is now understood to hold in Eqs. (34). Since we now have constructed a second metric $g_{\alpha\beta}$ (the "physical metric"), indices on all quantities except the constituents of $g_{\alpha\beta}$ ($\eta_{\alpha\beta}, h_{\alpha\beta}, \Delta_{\alpha\beta}$) henceforth will be raised and lowered with $g_{\alpha\beta}$. Equation (34), aside from the $O(h^2)$ correction term, is recognized as the electromagnetic Lagrangian for metric theories. Thus the BS theory is a metric theory at first order, but nonmetric at all higher orders (in h).

C. Summary of Our New Representation

Our new representation of the BS theory is summarized succinctly in Table I. In particular, one sees that for uncharged particles the theory is metric to all orders in h, with $g_{\alpha\beta}$ playing the role of the "physical" metric.[4] When electromagnetic phenomena are included, and when one goes beyond first order in h, the theory is nonmetric (cf $\mathcal{L}_{corr}$ in Table I).

IV. THE POST-NEWTONIAN LIMIT OF THE THEORY

We now proceed to calculate the post-Newtonian (PN) limit of the theory. The PN limit is a perturbation solution of the gravitational field equations – expanding in the small quantities occurring in the solar system, e.g.,

$$v^2 \equiv (\text{macroscopic velocities of bodies})^2 = O(\epsilon^2),$$

$$U \equiv \text{Newtonian gravitational potential} = O(\epsilon^2),$$

$$p/\rho \equiv \frac{\text{pressure}}{\text{proper density of rest mass}} = O(\epsilon^2),$$

$$\Pi \equiv \frac{\text{internal energy density}}{\text{rest-mass density}} = O(\epsilon^2) \ .$$

We refer the reader to Ref. 5 for further details of the expansion scheme.

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A. The Metric-Theory Approximation

Using Table I, we write the field equations as

$$\frac{\delta \mathfrak{L}_G}{\delta h_{\mu\nu}} = -\frac{\delta \mathfrak{L}_{NG}}{\delta h_{\mu\nu}}$$
$$= -\left(\frac{\delta \mathfrak{L}_{metric}}{\delta h_{\mu\nu}} + \frac{\delta \mathfrak{L}_{corr}}{\delta h_{\mu\nu}}\right)$$
$$= -\left(\frac{\delta \mathfrak{L}_{metric}}{\delta g_{\mu\nu}}\frac{\partial g_{\mu\nu}}{\partial h_{\mu\nu}} + \frac{\delta \mathfrak{L}_{corr}}{\delta h_{\mu\nu}}\right)$$
$$= -\left(\frac{(-g)^{1/2}}{2} T^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial h_{\mu\nu}} + \frac{\delta \mathfrak{L}_{corr}}{\delta h_{\mu\nu}}\right), \qquad (35)$$

where we have used the usual definition (as in metric theories)

$$T^{\mu\nu} = \frac{2}{(-g)^{1/2}}\frac{\delta \mathfrak{L}_{metric}}{\delta g_{\mu\nu}} \quad . \qquad (36)$$

To PN order, the first term on the right-hand side of Eq. (35) is of order

$$\text{first term} \approx (\text{total energy density}) \times \epsilon^2 ,$$

while the second term is of order (see Table I)

$$\text{second term} \approx (\text{electromagnetic energy density}) \times \epsilon^2 .$$

Since the electromagnetic energy of a substance is typically smaller than the total mass-energy by a factor $\lesssim 10^{-3}$, the second source term in Eq. (35) can be neglected at PN order, by comparison with the first. Similarly, one can make a metric-theory approximation for the response of matter to gravity. For metric-theory (i.e., universally coupled[4]) Lagrangians, one always has

$$T^{\mu\nu}{}_{;\nu} = 0 \qquad (37)$$

when the matter field equations are satisfied, where the semicolon denotes covariant differentiation with respect to the physical metric $g_{\alpha\beta}$. In the BS case

$$T^{\mu\nu}{}_{;\nu} = O\left(\frac{\delta \mathfrak{L}_{corr}}{\delta h_{\mu\nu}} h_{,\nu}\right); \qquad (38)$$

TABLE I. A new mathematical representation of the Belinfante-Swihart theory.

1. Gravitational fields:
   a. Absolute field .... $\underline{\eta}$
   b. Dynamical symmetric second-rank tensor .... $\underline{h}$
   c. "Physical" metric .... $\underline{g}$
2. Nongravitational variables:
   a. Particle coordinates .... $\underline{z}_A$
   b. Electromagnetic vector potential .... $\underline{A}$
   c. Affine parameter of particle world lines .... λ_A
3. Gravitational field equations:
   a. Flatness of $\underline{\eta}$: $\underline{\text{Riemann}}\,(\underline{\eta}) = 0$
   b. Field equations for $\underline{h}$ obtained by variation of $h_{\alpha\beta}$ in Lagrangian below
   c. Decomposition equation for g: $g_{\alpha\beta} = (1-Kh)^2\Delta_\alpha{}^\mu\Delta_{\mu\beta}$ where we have defined $\Delta_\alpha{}^\beta(\delta_\beta{}^\tau - \tfrac{1}{2}h_\beta{}^\tau) \equiv \delta_\alpha{}^\tau$, K is an arbitrary constant, $h = \eta^{\alpha\beta}h_{\alpha\beta}$, and indices are raised and lowered on $h_{\alpha\beta}, \Delta_{\alpha\beta}$ with $\eta_{\alpha\beta}$.
4. Influence of gravity on matter:

   Equations for $\underline{A}, \underline{z}_A$, obtained by variation of those quantities in Lagrangian
5. Lagrangian density:
   a. $\mathfrak{L} = \mathfrak{L}_G + \mathfrak{L}_{NG}$
   b. $\mathfrak{L}_G = -(16\pi)^{-1}(a h^{\mu\sigma}{}_{|\alpha} h_{\mu\sigma}{}^{|\alpha} + f h_{|\alpha} h^{|\alpha})(-\eta)^{1/2}$
   c. $\mathfrak{L}_{NG} = \sum_A \int \left[-m_A\left(-g_{\alpha\beta}\frac{dz_A^\alpha}{d\lambda_A}\frac{dz_A^\beta}{d\lambda_A}\right)^{1/2} + e_A A_\mu \frac{dz_A^\mu}{d\lambda_A}\right] d\lambda_A\, \delta^4(\underline{x} - \underline{z}_A) - (16\pi)^{-1}F_{\alpha\beta}F^{\alpha\beta}(-g)^{1/2} + \mathfrak{L}_{corr}$

      $= \mathfrak{L}_{metric} + \mathfrak{L}_{corr}$,

   where $\mathfrak{L}_{corr}$, the "correction term" in the Lagrangian, which represents the amount by which the purely electromagnetic portion of the Lagrangian fails to have metric form, satisfies

   $\mathfrak{L}_{corr} = O(F^2h^2)$ [see Eqs. (33)] .

so again one can conclude that effects resulting from the deviation in the matter response equation from Eq. (37) will be $\lesssim 10^{-3}$ of PN effects. Thus for all PN phenomena we can neglect $\mathcal{L}_{\rm corr}$ and treat the BS theory as a metric theory.

B. From Point Particles to Perfect Fluid

In one of their original papers[1] Belinfante and Swihart, when solving their gravitational field equations with the sun as the external source, use an *ad hoc* perfect-fluid stress-energy tensor for $\bar{T}^{\mu\nu}$, rather than the expression given in Eq. (5). Their $\bar{T}_{\mu\nu}$ is precise enough to yield an adequate treatment of the "three classical gravitation tests" but is not precise enough to adequately handle such effects as the effective gravitational mass of gravitational energy (cf. "Nordtvedt effect" in Ref. 5). To avoid such problems, and to ensure self-consistency of the theory when dealing with gravitating sources in the solar system, we will build up the fluid BS stress-energy tensor $T^{\mu\nu}$ as an average over charged point particles and their electromagnetic fields [cf. Eq. (27) and Table I].

The kinetic-theory procedure for constructing a perfect fluid out of interacting particles is the same in any metric theory as in general relativity, and the same in general relativity as in special relativity ("equivalence principle").[13] By following that standard procedure and by neglecting the resulting nonperfect fluid terms, we obtain the standard stress-energy tensor:

$$T^{\mu\nu} = (\epsilon + p)\, u^\mu u^\nu + p g^{\mu\nu} \ . \tag{39}$$

Here u^μ is a suitable macroscopic average of the microscopic particle 4-velocities, ϵ is the density of total mass-energy (rest mass plus kinetic energy of particles plus electromagnetic energy) as measured in the macroscopic rest frame, and p is the similarly measured averaged pressure.

C. The Parametrized Post-Newtonian (PPN) Formalism

References 5 and 6 present a "parametrized post-Newtonian formalism" in which the PN limit of every metric theory is summarized by the coefficients of various integral functions in its metric. These coefficients, the so-called PPN parameters, are obtained by the previously mentioned perturbation solution (PN limit) of the gravitational field equations. We have constructed such a solution for our new mathematical representation of the BS theory, using Eqs. (35) and (39) and Table I. The details are spelled out in Ref. 14. (Actually Ref. 14 is the presentation of an exact gravitation theory closely related to the metric-theory approximation of the BS theory.) We refer the reader to Ref. 14 and here quote only the PPN parameters of the BS theory:

$$\begin{aligned} &\gamma = \bar{\gamma} + O(w), \quad \zeta_2 = 0, \quad \alpha_1 = O(w)\ , \\ &\beta = \bar{\beta} + O(w), \quad \zeta_3 = 0, \quad \alpha_2 = O(w)\ , \\ &\zeta_1 = 0, \qquad\qquad \zeta_4 = 0, \quad \alpha_3 = 0 \ . \end{aligned} \tag{40}$$

Here $\bar{\gamma}$ and $\bar{\beta}$ are given implicitly in terms of a and f by

$$a = 1/(2\bar{\gamma} + 2), \tag{41}$$

$$f = \frac{10\bar{\beta} + 6\bar{\gamma}\bar{\beta} - 7\bar{\gamma}^2 - 8\bar{\gamma} - 6}{2(\bar{\gamma}+1)(3\bar{\gamma} + 5 - 4\bar{\beta})^2}\ , \tag{42}$$

and to obtain the correct Newtonian limit one must require

$$\frac{16K^2 a - 4aK + a + 3f}{a(a+4f)} = 2 \ . \tag{43}$$

By $O(w)$, we denote terms involving the cosmological boundary values of $h_{\mu\nu}$ (see Ref. 14 for further details). Imposing Eq. (43) reduces the number of arbitrary parameters to two (a and f, for example); so we may regard $\bar{\gamma}$ and $\bar{\beta}$ as being arbitrary. For comparison, general relativity has no arbitrary parameters and its only nonzero parameters are $\gamma = \beta = 1$.

V. EXPERIMENTAL CONSEQUENCES AND TESTS OF THE THEORY

In his 1972 Varenna Lectures, Will[8] summarizes, within the PPN framework, the constraints which may be placed on a metric theory's parameters by current solar system gravitation experiments. As has been indicated in Sec. IV, the difference between the BS theory and a metric theory for PPN-type experiments is less than one part in 10^3. For most experiments the microscopic internal energies play a minor role; e.g., it is the macroscopic rotation of the earth which produces the macroscopic Lens-Thirring precession of gyroscopes. For such experiments the BS theory is effectively a metric theory to a much higher accuracy than indicated above. In summary, so far as PN experiments are concerned, to the precision of the technology of the 1970's the BS theory is accurately summarized by the values of its PPN parameters, Eqs. (40). We refer the reader to Ref. 8 for the experimental consequences of those values. Here we merely point out a few salient features.

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Perhaps the most important feature is this: If the $O(w)$ terms in the parameters are sufficiently small, and if the arbitrary parameters are chosen so that $\bar{\gamma}=\bar{\beta}=1$, then the PN predictions of the metric-theory approximation to BS are the same as the PN predictions of general relativity. In particular, the predictions for the "three classical tests" are the same as Belinfante and Swihart[1] themselves deduced by complicated calculations.

A. Preferred-Frame Effects

For the coordinate system in which $\underline{\eta}$ is Minkowskian, it is natural to set the boundary values of $\underline{h}$ to zero when treating the solar system, as was done originally by Belinfante and Swihart. However, the correct way to determine the boundary values of $\underline{h}$ is through the solution of the cosmological problem. If the solution produces nonzero cosmological boundary values of $\underline{h}$, then those values will effect certain of the PPN parameters [cf. $O(w)$ terms in Eqs. (40)]. In the case of the BS theory the presence of such terms is a direct consequence of the presence of the "absolute gravitational field" $\underline{\eta}^4$ (cf. Table I), and leads to various preferred-frame effects[8] such as anomalous solid earth tides and contributions to the perihelion shift of mercury. We refer the reader to Ref. 14 for a more complete discussion of the derivation of such effects in the BS theory.

B. Precession of Gyroscopes

We specifically mention this experimental test only because there seems to be some confusion[15] as to the prediction of the BS theory. Using formulas from Ref. 8 and the BS PPN parameters, Eq. (40), one obtains for the precession of the spin $\vec{S}$ of a gyroscope orbiting the earth

$$\frac{d\vec{S}}{ds}=\vec{\Omega}\times\vec{S} \ , \tag{44}$$

where

$$\vec{\Omega}=\vec{\Omega}_{\text{Lens-Thirring}}+\vec{\Omega}_{\text{geodetic}} \ , \tag{45a}$$

$$\Omega_{\text{L-T}}=\tfrac{1}{8}[4\bar{\gamma}+4+O(w)] \ (0.05'' \text{ of arc/year}) \ , \tag{45b}$$

$$\Omega_{G}=\tfrac{1}{3}[1+2\bar{\gamma}+O(w)] \ (7'' \text{ of arc/year}) \ . \tag{45c}$$

Thus the results of the upcoming Everitt-Fairbank[16] gyroscope experiment (to be launched before 1977) can only place upper limits on the cosmological boundary values of $h_{\mu\nu}$ [cf. $O(w)$ terms in Eqs. (45)] for a given choice of $\bar{\gamma}$.

C. The Weak Equivalence Principle and Eötvös-Dicke-Type Experiments

We conclude by considering the Eötvös-Dicke (ED) type experiments,[10,11] which test gravity so precisely that they fall outside of the PN realm of precision. Braginsky,[11] in his recent version of the ED experiment, reports that the difference in accelerations of test bodies of aluminum and platinum in the gravitational field of the sun is smaller than one part in 10^{12}. Such a result represents a strong validation of the weak equivalence principle[4] (WEP). Consider the contribution of electromagnetic energy at order F^2h^2 (see bottom of Table I) to the gravitational mass and acceleration $\underline{a}$ of a test body:

$$\left|\frac{\vec{a}}{\vec{g}}\right|\sim\left|\frac{1}{\vec{g}}\vec{\nabla}\left[\left(\frac{\text{electromagnetic energy}}{\text{total energy}}\right)h^2\right]\right| \approx \frac{\text{electromagnetic energy}}{\text{total mass}}U \ , \tag{46}$$

where $h^2\approx U^2$ and $\vec{g}\equiv\vec{\nabla}U$. For platinum, the following relation holds:

$$\frac{\text{electromagnetic energy}}{\text{total mass}}\approx 10^{-3},$$

and the Newtonian potential due to the sun at the earth is

$$U\approx 10^{-8} \ .$$

Equation (46) and the above numerical estimates indicate that the ED experiment can distinguish between the BS theory and its metric-theory approximation (cf. $\mathcal{L}_{\text{corr}}$ in Table I). All metric theories satisfy WEP identically. The BS theory, however, as is shown in Ref. 9, predicts

$$\left|\frac{\langle\vec{a}\rangle_{\text{Pt}}-\langle\vec{a}\rangle_{\text{Al}}}{\vec{g}}\right|\approx 6\times10^{-11} \approx \left(\frac{\text{electromagnetic energy}}{\text{total mass}}\right)U \tag{47}$$

in clear violation of the Dicke[10] and Braginsky[11] versions of the experiment. The reader is referred to Ref. 9 for complete details as to the derivation of Eq. (47) from considerations of particles interacting with gravity and electromagnetism.

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[15]V. I. Pustovoĭt and A. V. Bautin (see Ref. 7) obtain the gyroscopic precession by integrating the BS point-particle equation of motion. It is clear from Eq. (27) of our paper that such a calculation, if carried out without error, *must* give the standard metric-theory result [cf. Eqs. (44)–(45)]. Pustovoĭt and Bautin get the wrong result because they omit several terms from their Eq. (20).

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C. METRIC THEORIES: BEYOND THE POST-NEWTONIAN LIMIT

a) A New Two-Metric Theory of Gravity with Prior Geometry (Paper VII; collaboration with D.L. Lee, published in Phys. Rev. D, **8**, 3293,1973)

A New Two-Metric Theory of Gravity with Prior Geometry*

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ABSTRACT

We present a Lagrangian-based metric theory of gravity with three adjustable constants and two tensor fields, one of which is a nondynamical "flat-space metric" $\underset{\sim}{\eta}$. With a suitable cosmological model and a particular choice of the constants, the "Post-Newtonian limit" of the theory agrees, in the current epoch, with that of General Relativity (GRT); consequently our theory is consistent with current gravitation experiments. Because of the role of $\underset{\sim}{\eta}$, the gravitational "constant" G is time dependent and gravitational waves travel null geodesics of $\underset{\sim}{\eta}$ rather than the physical metric $\underset{\sim}{g}$. Gravitational waves possess six degrees of freedom. The general exact static spherically symmetric solution is a four parameter family and one of these solutions is investigated in detail. Future experimental tests of the theory are discussed.

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I. INTRODUCTION AND SUMMARY

Within the past few years an elegant theoretical formalism, the "Parametrized Post-Newtonian" (PPN) framework, has been developed[1] to analyze metric[2] theories of gravity. The PPN framework is structured around the "weak gravitational fields" and low velocities of the gravitational matter which characterize typical solar-system tests of gravity. It classifies each gravitation theory as to its form "in the Post-Newtonian (PN) limit." At first it was hoped, and indeed seemed to be true, that the PN limit of each theory of gravity is unique — thus by solar-system experiments alone, one could, in principle, determine the "correct PN limit," which would then correspond to one and only one "correct theory of gravity." In addition, it was hoped and is hoped, that the "correct PN limit" is that of General Relativity (GRT) (although we try not to let this fact prejudice our investigations). To play devil's advocate, a program was initiated to attempt to formulate theories of gravity with the same PN limit (and hence PPN parameters[1]) as GRT. The aims of such a program are two-fold, as one can ask the following questions: (i) If such theories exist, how complex and contrived are their formulations? (ii) Do such theories have anything in common and in what respect do they differ from GRT outside of the PN limit? The first question is primarily only of aesthetic interest. But the second has the possibility of identifying powerful new theoretical and experimental tools for testing relativistic gravity — indeed that has been the case (see Sec. VI and Refs. 3 and 4).

In this paper we present and analyze a new theory of gravity — one which has the same PN limit (for the current epoch) as GRT, given a suitable cosmological model and a particular choice of the adjustable constants.

Analysis of our new theory provides partial answers to questions (i) and (ii) above.

A further motivation for study of this particular theory is to analyze in detail the role of prior geometry[2] in gravitation theories, a role which will be investigated in more general terms in another paper.[5]

To date the authors are aware of three other new metric theories which are candidates for sharing the property of having the same PN limit as GRT (candidates in the sense of contingency upon the existence of special but acceptable cosmological solutions and certain choices of the available adjustable constants). These theories are the Hellings-Nordtvedt theory,[6] Ni's theory,[7] and the Will-Nordtvedt theory.[8] Of these three, Ni's theory contains prior geometric elements like our own.

A. The Lagrangian Formulation

The equations of the theory are obtained, in the usual way, by varying the dynamical variables in the Lagrangian:

$$L = \int \mathcal{L}_G(\eta, h)\, d^4x + \int \mathcal{L}_{NG}(g, q_\lambda)\, d^4x \quad , \tag{1a}$$

$$g = g(\eta, h) \quad , \tag{1b}$$

$$\mathrm{Riem}(\eta) = 0 \quad , \tag{1c}$$

where η, h, g are second-rank symmetric tensor fields: η is an absolute variable[2] (not varied in L), h is dynamical, and g is constructed algebraically from η and h. The Riemann tensor constructed out of η is denoted by $\mathrm{Riem}(\eta)$, and consequently Eq. (1c) states that η is a "flat-space metric." It is Eq. (1c), the "field equation" for η, that introduces geometrical structure into the theory which is independent of the matter distribution

— thus the "prior geometry." The gravitational Lagrangian density is denoted by $\mathcal{L}_G$ while the nongravitational Lagrangian density,[2] $\mathcal{L}_{NG}$, is the same as the corresponding quantity in other metric theories, (e.g., GRT), with q_λ representing the matter fields. The "physical metric," governing the response of matter to gravity, is denoted by $\underset{\sim}{g}$.

Explicitly, $\mathcal{L}_G$ and $\underset{\sim}{g}$ are defined by the following:

$$\mathcal{L}_G = -(16\pi)^{-1}\, \eta^{\alpha\beta}\eta^{\lambda\mu}\eta^{\rho\sigma}(a h_{\lambda\rho|\alpha} h_{\mu\sigma|\beta} + f h_{\lambda\mu|\alpha} h_{\rho\sigma|\beta})(-\eta)^{1/2}, \tag{2}$$

$$g_{\mu\nu} = (1 - Kh)^2 \Delta_\mu{}^\tau \Delta_{\tau\nu}, \tag{3a}$$

$$\Delta^\mu{}_\nu(\delta_\mu{}^\alpha - \frac{1}{2} h_\mu{}^\alpha) = \delta^\alpha{}_\nu \quad . \tag{3b}$$

Conventions and definitions for the above are the following:

(i) Greek indices run 0-3, Latin 1-3.

(ii) units chosen such that G = c = 1 (gravitational constant today and speed of light) (see Sec. VI).

(iii) slashes "|" and semicolons ";" denote covariant differentiation with respect to the flat space-metric $\eta_{\alpha\beta}$ and the curved-space metric $g_{\alpha\beta}$ respectively. Comma "," denotes a partial coordinate derivative.

(iv) η is the determinant of $\eta_{\alpha\beta}$.

(v) $\delta^\alpha{}_\nu$ is the Kronecker delta.

(vi) $\Delta_\mu{}^\nu$ is defined by Eq. (3b).

(vii) indices on $\Delta_{\alpha\beta}$ and $h_{\alpha\beta}$ <u>only</u> are raised and lowered with $\eta_{\mu\nu}$, i.e., $h^\alpha{}_\alpha = h^{\alpha\beta}\eta_{\alpha\beta} \equiv h$, and $\eta^{\alpha\beta}\eta_{\beta\gamma} = \delta^\alpha{}_\gamma$; indices on all other tensors will be raised and lowered with $g_{\alpha\beta}$.

(viii) signatures of $\underset{\sim}{\eta}$ and $\underset{\sim}{g}$ are + 2.

(ix) a, f, K are adjustable constants.

Motivation for the rather ungainly expression for the metric [Eqs. (3)] comes from an analysis[9] of the Belinfante-Swihart theory of gravity[10] — a theory which can be reformulated, at lowest order, into a metric theory with "effective metric" of the form of Eqs. (3). From that suggested algebraic form for the metric we have constructed the present full metric theory.

B. Summary

Section II includes a discussion of the field equations and a calculation of the PN limit of the theory. It is shown that there are mathematically ten degrees of freedom in the initial value problem for $h_{\mu\nu}$ (compared with two for $g_{\mu\nu}$ in GRT). In the PN limit there are, in general, "preferred frame effects";[1] such effects are, however, functions of only the cosmological boundary values of $h_{\mu\nu}$. By a certain choice of the cosmological model one can make these effects vanish for the current epoch. We suspect that such time-dependent preferred-frame effects are a common property of prior geometric gravitation theories. At any rate, the observed absence of preferred-frame effects can only place upper limits on the cosmological boundary values of $h_{\mu\nu}$.

Section III derives and discusses the equations of stellar structure for static, spherically symmetric stars. The equations are much more complicated than the corresponding ones in GRT (see Table I) and there is probably no analytic solution even for a star of constant density. In addition, a stellar model is not uniquely specified by giving its equation of state and central pressure, as is the case in most other theories. The exact exterior, static spherically symmetric solution is obtained and is found to be a 4-parameter family.

Section IV includes an analysis of a special exterior spherically symmetric solution. For this special solution, the effective potentials for particles and photons are similar to the corresponding quantities in the Schwarzschild geometry of GRT, outside of a couple of "Schwarzschild radii." However, the physical manifold extends only to $\rho = 1.5$ m,[11] which is a "point at infinity" (not reachable in finite affine parameter by any geodesic).

There are no singularities or horizons (i.e., no black hole) in the physical manifold in this exact solution, but a peculiar geometrical effect in which the proper surface areas of concentric spheres centered on $\rho = 0$ pass through a minimum and then increase as one moves radially inward (decreasing ρ and increasing proper time for radially falling observer). The minimum of areas is approximately $97\pi m^2$ and occurs near $\rho = 2.7$ m. The areas then increase to infinity at $\rho = 1.5$ m, although space is not flat there.

It is also found that one cannot embed the entire constant time, equatorial geometry in a Euclidean 3-space, but that a pseudo-Euclidean space is necessary for $1.5\ m < \rho \lesssim 2.1$ m.

Section V discusses time-dependent solutions, conservation laws, and gravitational waves. Birkhoff's theorem[12] does not hold in this theory, i.e., the exterior geometry of a spherically symmetric and asymptotically flat spacetime need not be static — collapsing stars can radiate monopole gravitational waves. The general plane gravitational wave has six physical degrees of freedom, the maximum number possible in a metric theory of gravity.[3,4]

As the theory is Lagrangian-based, conservation laws follow and one can construct a gravitational stress-energy complex. Appropriately defined, the stress energy-density of this object is positive definite for all possible polarizations of plane waves. In addition there is a purely gravitational quantity conserved all by itself, probably of only mathematical interest.

Section VI discusses the time dependence of the gravitational "constant" and further possible experimental tests of the theory. In particular, a search for time delays between reception of gravitational and electromagnetic bursts and a search for "non-GRT" type polarizations of gravitational waves promise to be important future experimental tests of the theory. Such tests would also be crucial in the theories of Refs. 6, 7, 8; and their identification represents an important success in our program of "devil's advocate."

II. FIELD EQUATIONS AND POST-NEWTONIAN LIMIT

Variation of Eq. (1) with respect to the dynamical field variable $h_{\mu\nu}$ yields the following gravitational field equations:

$$(-\eta)^{1/2}(a\Box h^{\nu\mu} + f\,\eta^{\mu\nu}\Box h) = -4\pi T^{\alpha\beta}(-g)^{1/2}(\partial g_{\alpha\beta}/\partial h_{\mu\nu}) \ , \qquad (4a)$$

where

$$\Box h^{\mu\nu} \equiv \eta^{\alpha\beta}\, h^{\mu\nu}{}_{|\alpha|\beta} \ , \qquad (4b)$$

$$T^{\alpha\beta} \equiv 2(-g)^{-1/2}\,(\delta \mathscr{L}_{NG}/\delta g_{\alpha\beta}) \ , \qquad (4c)$$

and δ is the variational derivative.

From the matter equations, obtained by variation of q_λ in Eq. (1), one can show in the usual manner (see, e.g., Ref. 13)

$$T^{\alpha\beta}{}_{;\beta} = 0 \ . \qquad (5)$$

Equation (5) is the typical "matter response equation" in metric theories.

Contraction of Eq. (4a) with $\eta_{\mu\nu}$ yields an equation for h alone, which can be substituted back into Eq. (4a) to yield

$$\Box h^{\mu\nu} = -(4\pi/a)(-g)^{1/2}(-\eta)^{-1/2}T^{\alpha\beta}[\Theta^{\mu\nu}_{\alpha\beta} - f(a+4f)^{-1}\Theta^{\gamma\tau}_{\alpha\beta}\eta_{\gamma\tau}\eta^{\mu\nu}] \quad , \tag{6a}$$

where

$$\Theta^{\mu\nu}_{\alpha\beta} \equiv \partial g_{\alpha\beta}/\partial h_{\mu\nu} \quad . \tag{6b}$$

The linearized limit of Eq. (6a) is

$$\Box h^{\mu\nu} = -(4\pi/a)\ T^{\alpha\beta}[\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \eta_{\alpha\beta}\eta^{\mu\nu}(f+2Ka)(a+4f)^{-1}] \quad . \tag{7}$$

Unlike metric theories without prior geometry, the four Eqs. (5) do not follow from the gravitational field equations; they are additional equations.[5] However, there is no problem of overdetermination because all of the 10 components of $h^{\mu\nu}$ are now dynamical variables; i.e., if all of the essential coordinate freedom is used up in choosing a frame in which $\eta_{\alpha\beta}$ has a particular set of components, [usually diag(-1,1,1,1)], then there is no coordinate freedom left to adjust the components of $h_{\mu\nu}$.

For example, for a perfect fluid $T^{\alpha\beta}$ is described by four matter variables once an equation of state is given (3 components of four velocity and energy density, for example). Thus Eqs. (5) and (6a) comprise a system of fourteen independent equations for the fourteen unknowns.

We also note that all of the ten Eqs. (6a) involve second time derivatives of $h_{\mu\nu}$. Thus in the Cauchy problem all of the $h_{\mu\nu}$ are to be regarded as dynamical variables and there are ten degrees of freedom. Once $g_{\alpha\beta}$ has been constructed from $\eta_{\alpha\beta}$ and $h_{\alpha\beta}$, however, coordinate transformations can be performed and so there can only be six "physical" degrees of freedom. This is to be contrasted with GRT in which not only can four of the $g_{\alpha\beta}$ be chosen arbitrarily by coordinate conditions, but also four of the field equations involve only first time derivatives. Thus in the corresponding Cauchy problem, the Einstein gravitational field has only two physical degrees of freedom.

The PPN framework of Nordtvedt, Will, and others can be used to analyze the predictions of all metric theories with respect to solar-system experiments (e.g., light bending, perihelion shift, gravimeter data, earth-moon separation, etc.). The reader is referred to Ref. 1 for a complete summary of the PPN framework. Briefly, this formalism involves expanding the metric, in the manner of Chandrasekhar,[14] in the small dimensionless quantities which occur in the solar system stress energy tensor, e.g.,

$$v^2 \sim U \sim (P/\rho) \sim \Pi \sim O(\epsilon^2) \approx 10^{-7} \quad , \tag{8}$$

where v^2 is the squared velocity of a typical fluid element, U is the Newtonian potential, P/ρ is the pressure divided by energy density (specific pressure) and Π is the specific internal energy. It is found that, in a particular coordinate gauge, and for most metric theories — including ours — there are only nine different functionals which can occur in the metric at PN order and only nine independent parameters multiplying these functionals. Almost all twentieth century gravitation experiments to date can be summarized by their constraints on these nine parameters, the "PPN parameters."

We now calculate in our theory the PN limit, which will involve a perturbation solution of Eq. (6a). For calculational ease we assume a coordinate system in which $\eta_{\alpha\beta}$ takes on Minkowski values. Before we begin, a crucial point must be recognized.[15] The metric $g_{\alpha\beta}$ has the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + O(h) \quad ,$$

and we know that far away from the solar system there is some coordinate system in which $g_{\alpha\beta}$ takes on Minkowski values. However, this coordinate system will, in general, not be the same frame in which $\eta_{\alpha\beta}$ takes on

Minkowski values; there is no a priori reason why the boundary values of $h_{\mu\nu}$ should be zero in this coordinate system. Thus in solving Eq. (6a) we are not at liberty to set equal to zero for all time the "arbitrary constant" which may be added to $h_{\mu\nu}$; this complicates considerably the construction of the PN limit of our theory. However, we feel that this complication and its origin are of sufficient educational value to warrant a detailed discussion.

Denote the nearly constant boundary values of $h_{\mu\nu}$ by $\omega_{\mu\nu}$ ($\omega_{\mu\nu}$ can only change on a cosmological time scale by definition) and the part tied directly to the solar system by $h^*_{\mu\nu}$; i.e.,

$$h_{\mu\nu} = h^*_{\mu\nu} + \omega_{\mu\nu} \quad . \tag{9}$$

Now use the six-parameter invariance group of the Minkowski metric to pick a coordinate system in which $\omega_{\mu\nu}$ is diagonal, reducing $\omega_{\mu\nu}$ to four components. Without justification, but for simplicity, we now assume that the three spatial components of $\omega_{\mu\nu}$ are equal. Such an assumption does not affect the qualitative conclusions of this section. Further assume that

$$|\omega_{\mu\nu}| \ll 1 \quad , \tag{10}$$

although $\omega_{\mu\nu}$ does not have to be as small as the $O(\epsilon)$ indicated in Eq. (8). Equation (10) will turn out to be an assumption consistent with the ultimate experimental limits on the $\omega_{\mu\nu}$.

Next expand Eqs. (3a) and (3b) in a power series in $h_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} - 2Kh\eta_{\mu\nu} + h_{\mu\nu} + K^2h^2\eta_{\mu\nu} - 2Khh_{\mu\nu} + \frac{3}{4}\, h_{\mu\tau}h^{\tau}{}_{\nu} + \dots \quad . \tag{11}$$

When Eq. (9) is substituted into Eq. (11) one obtains

$$g_{00} = - D_0 + E_0\, h_{00}^{*} - F_0 h^{*} - K^2 h^2 - 2K h^{*}_{00} h^{*} - \frac{3}{4}\, h^{*\,2}_{00} \quad , \tag{12a}$$

$$g_{ij} = D\delta_{ij} + E h^{*}_{ij} + F\delta_{ij} h^{*} - 2K h^{*} h^{*}_{ij} + K^2 h^{*2} \delta_{ij} + \frac{3}{4}\, h^{*\,2}_{ij} \quad , \tag{12b}$$

$$g_{0k} = H h^{*}_{0k} \quad , \tag{12c}$$

where all of the constants appearing in Eqs. (12) have the form: $D_0 = 1 + O(\omega)$, etc., and are given explicitly to $O(\omega^2)$ in Appendix A, along with other constants appearing below. Using Eqs. (12) and a perfect fluid for the matter stress-energy tensor, one obtains from Eq. (6a)

$$\begin{aligned} \Box h^{*\mu\nu} = - (4\pi/a)\, I^{-1} \rho v^{\alpha} v^{\beta} (1 + I_1 h^{*}_{00} + I_2 h^{*} + I_3 v^2) \Big[&(1 - 2K\omega)\, \delta^{\mu}{}_{\alpha} \delta^{\nu}{}_{\beta} \\ &+ L\eta_{\alpha\beta}\eta^{\mu\nu} + \frac{3}{2}\, \delta^{\mu}{}_{\alpha} \omega^{\nu}{}_{\beta} + M\eta^{\mu\nu}\omega_{\alpha\beta} + N\eta^{\mu\nu}\eta_{\alpha\beta} h^{*} \\ &+ \frac{3}{2}\, \delta^{\mu}{}_{\alpha} h^{*\nu}{}_{\beta} - 2K h^{*} \delta^{\mu}{}_{\alpha} \delta^{\nu}{}_{\beta} + M\eta^{\mu\nu} h^{*}{}_{\alpha\beta} \Big] \quad . \end{aligned} \tag{13}$$

In Eq. (13) I, I_1, I_2, I_3, M, N are all functions of a, f, K, $\omega_{\mu\nu}$ (see Appendix A) and

$$\omega \equiv 3\omega_{11} - \omega_{00} \quad , \tag{14a}$$

$$v^{\alpha} \equiv dx^{\alpha}/dt \quad , \tag{14b}$$

$\rho \equiv$ proper mass-energy density measured in the rest-frame of the fluid. (14c)

To simplify an already complex presentation, we have omitted the pressure from the perfect fluid stress energy tensor and included the internal energy in the total proper energy density ρ. (Such terms are not omitted in quoting the final PPN parameters.) We now write

$$h^{*\mu\nu} = {}^{(1)}h^{*\mu\nu} + {}^{(2)}h^{*\mu\nu} + \ldots \quad , \tag{15}$$

in a perturbation expansion and obtain (see Appendix A for notation)

$$\nabla^2\ {}^{(1)}h^{*00} = -4\pi\tau\rho\left[(1-2K\omega)+L-\omega_0\left(\tfrac{3}{2}+M\right)\right] \equiv -4\pi\rho\ C_0\ , \qquad (16a)$$

$$\nabla^2\ {}^{(1)}h^{*ij} = -4\pi\rho\tau(M\omega_0 - L)\ \delta^{ij} \equiv -4\pi\rho\ C_1\delta^{ij}\ , \qquad (16b)$$

$$\nabla^2\ {}^{(1)}h^{*0k} = -4\pi\tau\rho\left[v^k(1-2K\omega)+\tfrac{3}{2}\ \omega_1 v^k\right] \equiv -4\pi\rho\ C_2 v^k\ , \qquad (16c)$$

$$\nabla^2\ {}^{(2)}h^{*00} = -4\pi\tau\rho(S_0\ {}^{(1)}h^{*00} + S_1\ {}^{(1)}h^{*} + B_0 v^2) + {}^{(1)}h^{*00}_{\ \ ,00}\ , \qquad (16d)$$

$$\nabla^2\ {}^{(2)}h^{*ij} = -4\pi\tau\rho\left[R_0 v^i v^j + \delta^{ij}(R_1\ {}^{(1)}h^{*00} + R_2\ {}^{(1)}h^{*} + B_1 v^2)\right] + {}^{(1)}h^{*ij}_{\ \ ,00}\ , \qquad (16e)$$

where

$$\tau \equiv (aI)^{-1}\ . \qquad (17)$$

Solutions of the equations are

$${}^{(1)}h^{*00} = C_0 U\ , \qquad (18a)$$

$${}^{(1)}h^{*ij} = \delta^{ij} C_1 U\ , \qquad (18b)$$

$${}^{(1)}h^{*0k} = C_2 V_k\ , \qquad (18c)$$

$${}^{(2)}h^{*00} = \tau\left[S_0 C_0 + S_1(3C_1 - C_0)\right]\Phi_2 + \tau B_0 \Phi_1 + C_0 \chi_{,00}\ , \qquad (18d)$$

$$\begin{aligned}{}^{(2)}h^{*ij} = {} & \tau R_0 \mathcal{J}_3^{ij} + \tau\delta^{ij}\left[R_1 C_0 + R_2(3C_1 - C_0)\right]\Phi_2 \\ & + \tau B_1 \delta^{ij}\Phi_1 + C_1 \delta^{ij}\chi_{,00}\ ,\end{aligned} \qquad (18e)$$

where we have defined the five "potentials" U, V_k, Φ_1, Φ_2, $\mathcal{J}_3^{ij}$, and the "superpotential" χ aa follows:

$$U(\underset{\sim}{x},t) \equiv \int \rho(\underset{\sim}{x}',t)\,|\underset{\sim}{x}-\underset{\sim}{x}'|^{-1}\ d^3x'\ , \qquad (19a)$$

$$V_k(\underset{\sim}{x},t) \equiv \int \rho(\underset{\sim}{x}',t)\,|\underset{\sim}{x}-\underset{\sim}{x}'|^{-1}\ v^k d^3x'\ , \qquad (19b)$$

$$\Phi_1(\underset{\sim}{x},t) \equiv \int \rho(\underset{\sim}{x}',t)\ v^2\ |\underset{\sim}{x}-\underset{\sim}{x}'|^{-1}\ d^3x'\ , \qquad (19c)$$

$$\Phi_2(\underset{\sim}{x}, t) = \int \rho(\underset{\sim}{x}', t)|\underset{\sim}{x} - \underset{\sim}{x}'|^{-1} U(\underset{\sim}{x}', t)\, d^3x' \quad , \tag{19d}$$

$$\mathcal{I}_3^{ij}(\underset{\sim}{x}, t) = \int \rho(\underset{\sim}{x}', t)|\underset{\sim}{x} - \underset{\sim}{x}'|^{-1} v^i v^j\, d^3x' \quad , \tag{19e}$$

$$\nabla^2 \chi = U \quad . \tag{19f}$$

Using Eqs. (12) and our solutions, Eqs. (19), we now compute the metric:

$$g_{00} = - D_0 + K_1 U + K_2 U^2 + K_3 \Phi_2 + K_4 \Phi_1 + K_1 \chi_{,00} \quad . \tag{20a}$$

$$g_{ij} = \delta_{ij}(D + K_5 U) \quad , \tag{20b}$$

$$g_{0k} = - HC_2 V_k \quad . \tag{20c}$$

Notice that the metric does not approach the standard Minkowski tensor far away from the solar system (when the potentials U, Φ_1, Φ_2, V_k, $\chi \to 0$) because of the leading constants D_0 and D_1. We must therefore make a "scaling" transformation:

$$t = D_0^{-1/2}\, \bar{t} \quad , \tag{21a}$$

$$\underset{\sim}{x} = D^{-1/2}\, \bar{\underset{\sim}{x}} \quad . \tag{21b}$$

In the tensor transformation law for the metric

$$\bar{g}_{\mu\nu}(\bar{x}) = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} = g_{\alpha\beta}[U(\underset{\sim}{x}, t),\ \Phi_1(\underset{\sim}{x}, t),\ \ldots] \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \quad , \tag{22}$$

we also need to express the potentials as functions of the new (barred) coordinates. An example of the procedure is the following: since ρ is a scalar

$$\bar{\rho}(\bar{\underset{\sim}{x}}, t) = \rho(\underset{\sim}{x}, t) \quad , \tag{23a}$$

$$U(\underset{\sim}{x}, t) = \int \rho(\underset{\sim}{x}', t)|\underset{\sim}{x} - \underset{\sim}{x}'|^{-1} d^3x' = \int \bar{\rho}(\bar{\underset{\sim}{x}}', \bar{t})|\underset{\sim}{x} - \underset{\sim}{x}'|^{-1} d^3x' \quad ,$$

$$= D^{-1} \int \bar{\rho}(\bar{\underset{\sim}{x}}', \bar{t})|\bar{\underset{\sim}{x}} - \bar{\underset{\sim}{x}}'|^{-1} d^3\bar{x}' \equiv D^{-1}\, \bar{U}(\bar{\underset{\sim}{x}}, \bar{t}) \quad . \tag{23b}$$

In a similar manner one finds

$$\Phi_2(\underset{\sim}{x}, t) = D^{-2}\, \bar{\Phi}_2(\underset{\sim}{\bar{x}}, \bar{t}) \ , \qquad (23c)$$

$$\Phi_1(\underset{\sim}{x}, t) = D_0 D^{-2}\, \bar{\Phi}_1(\underset{\sim}{\bar{x}}, \bar{t}) \ , \qquad (23d)$$

$$V_k(\underset{\sim}{x}, t) = D_0^{\ 1/2} D^{-3/2}\, \bar{V}_k(\underset{\sim}{\bar{x}}, \bar{t}) \ , \qquad (23e)$$

$$\chi_{,00} = D^{-2} D_0 \bar{\chi}_{,\bar{0}\bar{0}} \ . \qquad (23f)$$

Making the transformation indicated in Eqs. (22) and (23) and then dropping the bars, $g_{\mu\nu}$ becomes

$$g_{00} = -1 + D_0^{\ -1}D^{-1}K_1U + D_0^{\ -1}D^{-2}K_2U^2 + D_0^{\ -1}D^{-2}K_3\Phi_2 + D^{-2}K_4\Phi_1 + D^{-2}K_1\chi_{,00} \ , \qquad (24a)$$

$$g_{ij} = \delta_{ij}(1 + D^{-2}K_5U) \ , \qquad (24b)$$

$$g_{0k} = -\, HC_2D^{-2}V_k \ . \qquad (24c)$$

A final coordinate transformation must be made to remove the $\chi_{,00}$ term from g_{00} and reduce the metric to "standard PPN form." However, additional transformations of the form of Eqs. (23) are now negligible corrections and no distinction need be made between functions of new and old coordinates. The result of the final transformation, $t \to t + 1/2\, D^{-2}K_1\chi_{,0}$, is

$$g_{00} \to g_{00} - K_1D^{-2}\chi_{,00} \ , \qquad (25a)$$

$$g_{ij} \to g_{ij} \ , \qquad (25b)$$

$$g_{0k} \to g_{0k} + \frac{1}{4} K_1D^{-2}(V_k - W_k) \ , \qquad (25c)$$

where W_k is a new potential defined by

$$W_k \equiv \int \rho[\underset{\sim}{v} \cdot (\underset{\sim}{x} - \underset{\sim}{x}')]\, |\underset{\sim}{x} - \underset{\sim}{x}'|^{-1}\, (\underset{\sim}{x} - \underset{\sim}{x}')_k d^3x' \ . \qquad (26)$$

We now demand the proper Newtonian limit, i.e.,

$$g_{00} \approx 1 - 2U + \ldots \quad ,$$

which requires

$$K_1 D_0^{-1} D^{-1} = 2 \quad \underline{\text{today}} \quad : \tag{27}$$

(a consequence of our choosing units in which the gravitational constant is unity today). Equation (27) expresses a constraint between the three adjustable constants a, f, and K for a given set of $\omega_{\mu\nu}$. Comparing Eqs. (24)-(25) with the definitions of the PPN parameters[1] and using Eq. (27) to simplify. one finds

$$\gamma = \frac{1}{2} D^{-2} K_5 \equiv \bar{\gamma}(a, f, K) + O(\omega) \quad , \tag{28a}$$

$$\beta = -\frac{1}{2} D_0^{-1} D^{-2} K_2 \equiv \bar{\beta}(a, f, K) + O(\omega) \quad , \tag{28b}$$

$$\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \alpha_3 = 0 \quad , \tag{28c}$$

$$\alpha_1 = 2HC_2 D^{-2} - 4\gamma - 4 = O(\omega) \quad , \tag{28d}$$

$$\alpha_2 = D_0 D^{-1} - 1 = O(\omega) \quad . \tag{28e}$$

where $\bar{\gamma}$ and $\bar{\beta}$ are defined implicitly by the relations

$$a = (2\bar{\gamma} + 2)^{-1} \quad , \tag{29a}$$

$$f = (10\bar{\beta} + 6\bar{\gamma}\,\bar{\beta} - 7\bar{\gamma}^2 - 8\bar{\gamma} - 6)[2(\gamma + 1)(3\bar{\gamma} - 5 - 4\bar{\beta})^2]^{-1} \, . \tag{29b}$$

In GRT, $\gamma = \beta = 1$ and the other seven parameters vanish. In our theory it is clear that the two adjustable constants, a and f, may be so chosen to give any value to γ and β. For example, if the $\omega_{\mu\nu}$ are all zero, one can satisfy Eq. (27) and have $\gamma = \beta = 1$ with the choice

$$(a, f, K) = (\tfrac{1}{4}, -\tfrac{5}{64}, \tfrac{1}{16}) \quad . \tag{30}$$

It has been shown[16] that the nonvanishing of α_1, α_2, or α_3 leads to noninvariance of the functional form of the metric of Eqs. (24)-(25) under post-Galilean transformations[17] (curved-space versions of Lorentz transformations). New terms, involving the velocity of the Lorentz boost with respect to the current "preferred frame" and multiplied by combinations of α_1, α_2, α_3, appear in the metric. Nordtvedt and Will[18] have calculated the experimental consequences of the resulting "preferred-frame effects" and find that they lead to periodic anomalies in such phenomena as the solid earth tides, secular perihelion shifts, etc. The reader is referred to their paper for further details and we quote here only the current experimental limits on α_1 and α_2:

$$\alpha_1 \leq 0.1 \quad , \tag{31a}$$

$$\alpha_2 \leq 0.02 \quad . \tag{31b}$$

We have calculated explicitly the quite complicated functions $\alpha_1(\omega_{\mu\nu})$, $\alpha_2(\omega_{\mu\nu})$ and have examined their numerical values over a large range of constants a and f (consistent with the experimental limits on γ and β). We find that the experimental constraints indicated in Eqs. (31) require approximately

$$|\omega_0| + |\omega_1| \lesssim .015 \quad . \tag{32}$$

Even if we had not made the simplifying assumptions about the form of $\omega_{\mu\nu}$, its individual elements presumably would still be required to satisfy roughly the constraint of Eq. (32).

Since the $\omega_{\mu\nu}$ are cosmological boundary values of $h_{\mu\nu}$, one must solve the cosmological problem for a particular cosmological model to obtain the theoretical values of the $\omega_{\mu\nu}$. Because of the absolute nature of $\eta_{\alpha\beta}$, it should be possible to construct cosmologies such that, during the current epoch, the curved and flat-space metrics approach Minkowski form, far from the solar system, in the same coordinate system. Such a cosmology would guarantee that the $\omega_{\mu\nu}$ vanish at present, although a time dependent cosmology would certainly cause nonzero values of $\omega_{\mu\nu}$ to occur over cosmological time scales. Indeed, preliminary results from a cosmological solution[19] indicate that it is possible to make all of the $\omega_{\mu\nu}$ arbitrarily small for the current epoch — and still have a reasonable cosmological model. Thus, a consistent solution exists for which the PN limit of our theory is arbitrarily close to that of GRT in the current epoch.

Further details regarding the time dependence of the $\omega_{\mu\nu}$ are given in Sec. VI.

III. THE GENERAL STATIC SPHERICALLY SYMMETRIC SOLUTION AND EQUATIONS OF STELLAR STRUCTURE

A. The General Exterior Static Spherically Symmetric Solution

Before writing down the equations of stellar structure for a static spherically symmetric star, let us construct the general static spherically symmetric exterior solution (which must then be joined onto the solution inside the star).

First of all, choose a coordinate system in which

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} . \qquad (33)$$

The most general form of $h_{\mu\nu}$ in this coordinate system which satisfies the symmetry requirements is[20]

$$h_{\mu\nu} = \begin{pmatrix} \varphi(r) & \mu(r) & 0 & 0 \\ \mu(r) & \psi(r) & 0 & 0 \\ 0 & 0 & r^2\lambda(r) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta\,\lambda(r) \end{pmatrix} . \qquad (34)$$

The homogeneous field equations for $h_{\mu\nu}$ are simply

$$\eta^{\alpha\beta} h^{\mu\nu}{}_{|\alpha|\beta} = 0 \quad . \qquad (35)$$

The solutions to Eqs. (35) which are well behaved at infinity are[21]

$$h_{\mu\nu} = \begin{pmatrix} a_1/r & -2a_4/r^2 & 0 & 0 \\ -2a_4/r^2 & a_2/r - 2a_3/r^3 & 0 & 0 \\ 0 & 0 & r^2(a_2/r + a_3/r^3) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta(a_2/r + a_3/r^3) \end{pmatrix} , \qquad (36)$$

where a_1, a_2, a_3, and a_4 are arbitrary constants. We remind the reader that the r coordinate in Eq. (36) has, at this point, no interpretation other than its relation to the group — theoretically defined assumption of spherical symmetry. Construction of $g_{\mu\nu}$ from $h_{\mu\nu}$ is purely algebraic [see Eqs. (3)], and the details will not be given here. Since $h_{\mu\nu}$ has off-diagonal terms, so will $g_{\mu\nu}$. However, having obtained $g_{\mu\nu}$, we can make the coordinate transformation

$$t \to t + \int \frac{g_{0r}}{g_{00}}\, dr \quad , \qquad (37)$$

which then diagonalizes the metric, and we finally obtain

$$g_{00} = (1 - Kh)^2 \gamma^2 \left[\frac{a_4{}^2}{r^4} - \left(1 - \frac{1}{2}\,\frac{a_2}{r} + \frac{a_3}{r^3}\right)^2 \right] \quad , \qquad (38a)$$

$$g_{rr} = (1 - kh)^2 \gamma^2 \left\{ \left(1 + \frac{1}{2} \frac{a_1}{r}\right)^2 - \frac{a_4^{\,2}}{r^4} + \frac{\left(\frac{a_4^{\,2}}{r^4}\right)\left[2 + \frac{1}{2}(a_1 - a_2)\, r^{-1} + a_3 r^{-3}\right]^2}{\frac{a_4^{\,2}}{r^4} - \left(1 - \frac{1}{2} \frac{a_2}{r} + \frac{a_3}{r^3}\right)^2} \right\}, \quad (38b)$$

$$g_{\Theta\Theta} = (1 - kh)^2 \, r^2 \left(1 - \frac{1}{2} \frac{a_2}{r} - \frac{1}{2} \frac{a_3}{r^3}\right)^{-2} , \quad (38c)$$

$$g_{\varphi\varphi} = \sin^2\Theta \; g_{\Theta\Theta} , \quad (38d)$$

$$h \equiv r^{-1}(3a_2 - a_1) , \quad (38e)$$

$$\gamma \equiv \left[1 + \frac{1}{2}(a_1 - a_2)\, r^{-1} - \frac{1}{4} a_1 a_2 r^{-2} + a_3 r^{-3} + \left(a_4^{\,2} + \frac{1}{2} a_1 a_3\right) r^{-4}\right]^{-1} , \quad (38f)$$

$$ds^2 = g_{00} dt^2 + g_{rr} dr^2 + g_{\Theta\Theta} d\Theta^2 + g_{\varphi\varphi} d\varphi^2 . \quad (39)$$

Equations (38) for the metric indicate a 4-parameter family of solutions for the general static spherically symmetric exterior metric. One can convince himself that all four of the parameters are physical (not removable by coordinate transformations) by transforming to curvature coordinates and verifying that four arbitrary parameters remain.[22] In Sec. IV we will investigate more closely a particular member of the 4-parameter family.

B. Stellar Models

We idealize a star as a spherically symmetric, static mass of perfect fluid and assume a temperature-independent equation of state

$$p = p(\rho) , \quad (40)$$

where p is the pressure and ρ the energy density. We work in the coordinate system in which $\eta_{\mu\nu}$ has the form of Eq. (33).[23] For mathematical simplicity we seek solutions for $h_{\mu\nu}$ which are diagonal, i.e., with $\mu(r) = 0$ in Eq. (34).

Such solutions represent a subclass of all possible solutions and correspond to the condition $a_4 = 0$ in the exterior metric [cf. Eqs. (36) and (38)].[24]

The metric now has the form

$$g_{\mu\nu} = (1 - Kh)^2 \begin{pmatrix} -(1+\frac{1}{2}\varphi)^{-2} & & & \\ & (1-\frac{1}{2}\psi)^{-2} & & \\ & & r^2(1-\frac{1}{2}\lambda)^{-2} & \\ & & & r^2\sin^2\theta(1-\frac{1}{2}\lambda)^{-2} \end{pmatrix}, \quad (41a)$$

where

$$h \equiv -\varphi + \psi + 2\lambda \quad . \qquad (41b)$$

Equations (5) and (6a) together with Eq. (40) are the necessary set for computing the structure of our stellar model. With the usual fluid stress energy tensor

$$T^{\alpha\beta} = (\rho + p)\, u^\alpha u^\beta + p g^{\alpha\beta} \quad , \qquad (42)$$

one finds that the only nonvacuous equation resulting from Eqs. (5) is

$$dp/dr = \frac{1}{2}(\rho + p)\left[2K(1 - Kh)^{-1}\, dh/dr + (1 + \frac{1}{2}\varphi)^{-1}\, d\varphi/dr\right] . \qquad (43)$$

Using the Christoffel symbols for $\underset{\sim}{\eta}$, one finds that Eqs. (6a) yield the following:

$$\nabla^2 h = -4\pi\Gamma(a + 4f)^{-1}\left\{8K(\rho - 3p) + (1 - Kh)\left[p(1 - \frac{1}{2}\psi)^{-1} + 2p(1 - \frac{1}{2}\lambda)^{-1} - \rho(1 + \frac{1}{2}\varphi)^{-1}\right]\right\}, \qquad (44a)$$

$$\nabla^2\varphi = (f/a)\nabla^2 h - (4\pi/a)\Gamma\left[2K(3p - \rho) + (1 + \frac{1}{2}\varphi)^{-1}(1 - Kh)\rho\right] , \qquad (44b)$$

$$\nabla^2\psi = 4(\psi - \lambda)\, r^{-2} - (f/a)\,\nabla^2 h - (4\pi/a)\Gamma\left[2K(\rho - 3p) + (1 - \frac{1}{2}\psi)^{-1}(1 - Kh)p\right] , \qquad (44c)$$

where

$$\nabla^2 \equiv d^2/dr^2 + 2r^{-1}\, d/dr \quad , \tag{45a}$$

$$\Gamma \equiv (1 - Kh)^3 \, (1 + \tfrac{1}{2}\varphi)^{-1} \, (1 - \tfrac{1}{2}\psi)^{-1} \, (1 - \tfrac{1}{2}\lambda)^{-2} \quad . \tag{45b}$$

Equation (44a) follows from taking the trace (with respect to $\underset{\sim}{\eta}$) of Eq. (6a). Equations (44b) and (44c) are the 0-0 and r-r components respectively of Eq. (6a). Altogether, Eqs. (40), (43)-(44) are five highly nonlinear coupled equations for the five unknowns p, ρ, φ, λ, and ψ. Linear combinations of Eqs. (44) can be taken to yield

$$\nabla^2(\psi - \lambda) = \left[\tfrac{1}{2}(\Gamma/a)(1 - Kh)(1 - \tfrac{1}{2}\psi)^{-1} \, (1 - \tfrac{1}{2}\lambda)^{-1} + 6r^{-2}\right](\psi - \lambda) \quad , \tag{46}$$

which is an equation we will later discuss.

Outside of the star the physically acceptable solutions to the homogeneous forms of Eqs. (44) are [cf. Eq. (36)]

$$\varphi = a_1/r \quad , \tag{47a}$$

$$\psi = a_2/r - 2a_3/r^3 \quad , \tag{47b}$$

$$\lambda = a_2/r + a_3/r^3 \quad . \tag{47c}$$

The constants a_1, a_2, and a_3 are to be determined by matching conditions at the surface of the star. The general procedure in constructing stellar models is to choose various central values for the variables, integrate the equations outward from the center until the pressure vanishes, and thus establish the surface of the star. Various boundary conditions must typically be satisfied, but in the case of GRT, for example, the conditions can be satisfied in a trivial manner without multiple trial integrations. The situation here, as we shall see, is vastly more complicated.

As long as the denominators do not vanish (see discussion below), Eqs. (44) are regular at the stellar surface and hence require that $r\varphi$, $r\psi$, $r\lambda$ and their first derivatives be continuous across the surface. Using Eqs. (47) and denoting quantities evaluated at the surface by a subscript s, one obtains the six matching conditions:

$$\varphi_s = a_1/R, \quad (\varphi_{,r})_s = -a_1/R^2 \quad , \tag{48a}$$

$$\psi_s = a_2/R - 2a_3/R^3, \quad (\psi_{,r})_s = -a_2/R^2 + 6a_3/R^4 \quad , \tag{48b}$$

$$\lambda_s = a_2/R + a_3/R^3, \quad (\lambda_{,r})_s = -a_2/R^2 - 3a_3/R^4 \quad , \tag{48c}$$

where $r = R$ is the surface of the star.

What are the appropriate central quantities to be specified? Suppose we regard $(\psi - \lambda)$, and φ as the three independent gravitational potentials. Then a possible but nonunique solution to Eq. (46) is $\psi - \lambda = 0$ everywhere, corresponding to considering r an isotropic radial coordinate. However, forgetting this special case for the moment, the regular solution of Eq. (46) near the origin is

$$\psi - \lambda \sim \text{const.}\ r^2 \quad .$$

Thus one central condition to be specified is

$$[(\psi - \lambda)/r^2]_c \quad ,$$

where we denote by c quantities at the center, analogously to the quantities at the surface discussed above. The equations for h and φ are regular at the origin as long as the potentials are sufficiently small and therefore, in analogy with the corresponding electrostatic equations, the derivatives, at the center, of φ and λ must vanish. However, the central values of the potentials themselves must be specified, and hence the two other central parameters are φ_c and λ_c. Thus in general we have six parameters to adjust, e.g., a_1, a_2, a_3, φ_c, λ_c, $[(\psi - \lambda)/r^2]_c$ in order to satisfy the six matching constraints given in Eqs. (48), for a given equation of state and central pressure. One way of viewing the boundary conditions is that φ_c, λ_c, $[(\psi - \lambda)/r^2]_c$ must be so chosen as to match onto a regular exterior solution at the star's surface — such a two-point boundary value problem in general has a discrete set of solutions, i.e., for a given p_c and equation of state there may be no $\left|\varphi_c, \lambda_c, [(\psi - \lambda)/r^2]_c\right|$ such that there is a solution, or there may be many different sets. Thus the central pressure and equation of state do not uniquely specify the stellar model in general. However, we do know that for a weakly gravitating star $(p_c/\langle\rho\rangle \ll 1,\ \varphi,\ \lambda,\ \psi \ll 1)$. Equations (44) become linear and do indeed have unique and well behaved solutions for each central pressure (Newtonian, and post-Newtonian regimes, see Sec. II). However, we can expect that as the models become more and more relativistic, a point is reached where each p_c and equation of state branches into a discrete spectrum of stellar models.

If one trys as a solution to Eq. (46) $\psi = \lambda$, then a more convenient form of the boundary condition is

$$[\varphi/(r\varphi_{,r})]_s = -1 \quad , \tag{49a}$$

$$[(3\lambda + r\lambda_{,r})/3\lambda]_s = \frac{2}{3} \quad . \tag{49b}$$

One then adjusts λ_c and φ_c to satisfy Eqs. (49) and defines a_1 and a_2 $(a_3 = 0)$ by

$$a_1 = R\varphi_s \quad , \tag{50a}$$

$$a_2 = \frac{1}{2} R(3\lambda + R\lambda_{,r})_s \quad . \tag{50b}$$

If $\psi \neq \lambda$, then the proper constraints are

$$[\varphi/r\varphi_{,r}]_s = -1 \quad , \tag{51a}$$

$$[(3\lambda + r\lambda_{,r})/(2\lambda + \psi)]_s = \frac{2}{3} \quad , \tag{51b}$$

$$R[(\psi_{,r} - \lambda_{,r})/(\lambda - \psi)]_s = 3 \quad , \tag{51c}$$

and one adjusts λ_c, φ_c and $[(\psi - \lambda)/r^2]_c$ to satisfy these three constraints; defining a_1 and a_2 as in Eqs. (50), and

$$a_3 = \frac{1}{3} R^3(\lambda_s - \psi_s) \quad . \tag{52}$$

As far as the exterior metric is concerned, all of the information about the stellar model is contained in the parameters a_1, a_2, and a_3 (and a_4 in the general case). Each different set of values for these constants corresponds to a different mass and radius of the star. Indeed, the total mass-energy of the star ("gravitating mass") as determined by g_{00} and using Eqs. (41) and (47) is

$$m = \frac{1}{2} a_1 + K(3a_2 - a_1) \quad , \tag{53}$$

(a_1 and a_2 determined by matching conditions at the surface). It is difficult to say what each parameter corresponds to physically (in terms of integrals over the source, etc.) because of the complexity of the inhomogeneous equations [cf. Eqs. (44)]. The only definite statement is that the particular combination of a_1 and a_2 given in Eq. (53) corresponds to the total mass.

A further interesting fact is that, for a given choice of a, f, K, the PPN parameters γ and β — as determined by a 1/r expansion of the isotropic version of the metric — are functions of a_1 and a_2 and in general are not equal to their values as determined in the PN limit. (This situation is also true in the Dicke-Brans-Jordan theory.)[25] Only in the case of a weakly gravitating star can one be sure that the two different determinations of γ and β will agree approximately (to within PN precision). In GRT, on the other hand, expansion of the Schwarzschild metric gives $\gamma = \beta = 1$ regardless of stellar model, and in agreement with the γ and β as determined in the PN limit of the theory.

Table I gives a comparison between our stellar-structure equations and those of GRT.

IV. ANALYSIS OF AN EXACT EXTERIOR SOLUTION

A. The Metric

As pointed out in the last section, the general exterior metric of a static spherically symmetric spacetime is a 4-parameter family [cf. Eq. (38)]. Let us analyze a member of that family. First of all, for simplification, we choose $a_3 = a_4 = 0$, which puts the metric of Eq. (38) in isotropic form. Next, using Eq. (53) as a definition of the mass m, we choose a_1, a_2, and K such that a 1/r expansion of the metric indicates that the PPN parameters γ and β are both unity (see Sec. II). In other words, choose a_1 a_2, and K such that[11]

$$g_{00} = - 1 + 2m/\rho - 2(m/\rho)^2 + O(\rho^{-3}) \quad , \tag{54a}$$

$$g_{ij} = - \delta_{ij}(1 + 2m/\rho) + O(\rho^{-2}) \quad , \tag{54b}$$

which requires

$$a_1/m = 1 \quad , \tag{55a}$$

$$a_2/m = 3 \quad , \tag{55b}$$

$$K = 1/16 \quad . \tag{55c}$$

It is interesting to note that the value for K given in Eq. (55c) is the same value required for $\gamma = \beta = 1$ in the weak-field PN expansion [cf. Sec. II and Eq. (30)]. Using Eqs. (55) and Eq. (38), one can now write the line element as

$$ds^2 = -\frac{(1-\frac{1}{2}m/\rho)^2}{(1+\frac{1}{2}m/\rho)^2}dt^2 + \frac{(1-\frac{1}{2}m/\rho)^2}{(1-\frac{3}{2}m/\rho)^2}(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\varphi^2) \quad . \quad (56)$$

The line element given in Eq. (56) is the simplest static spherically symmetric metric which yields the same light bending and perihelion shift (viz., $\gamma = \beta = 1$) as in GRT. (Note that the value of g_{00} is identical to the corresponding term in the isotropic form[26] of the GRT Schwarzschild geometry.)

B. Geodesic Completeness and Radial Geodesics

A glance at Eq. (56) reveals that $\rho = 1.5$ m is an infinite proper radial distance away from any $\rho > 1.5$ m. To investigate whether this point is removed from the physical manifold we need to look at null and timelike geodesics. Consider equatorial orbits (no loss of generality with spherical symmetry) and consider the first integrals of the motion for particles and photons:

$$u^\alpha u_\alpha = -1 = (u_0)^2 g^{00} + g_{\rho\rho}(u^\rho)^2 + (u_\varphi)^2 g^{\varphi\varphi} \quad , \quad (57a)$$

$$g_{\rho\rho}(P_\rho)^2 + g^{00}(P_0)^2 + g^{\varphi\varphi}(P_\varphi)^2 = 0 \quad , \quad (57b)$$

where $u^\alpha = dx^\alpha/d\tau$ for particles and $P^\alpha = dx^\alpha/d\lambda$ (with λ the affine parameter) for photons. It is well known (see, e.g., Ref. 27) that for a metric of the form of Eq. (26), u_0, u_φ, and (P_φ/P_0) are all constants of the motion, which we shall denote by $-\tilde{E}$, $\tilde{L}$, and ℓ, respectively. Physically, these constants are energy per unit rest mass, angular momentum per unit rest mass, and impact parameter respectively.

Using the above, Eq. (57a) can be written as

$$u^\rho = d\rho/d\tau = (\bar{\rho} - \tfrac{1}{2})^{-2} (\bar{\rho} + \tfrac{1}{2})(\bar{\rho} - \tfrac{3}{2})[\tilde{E}^2 - \Gamma^2(\tilde{\bar{L}},\bar{\rho})] \quad , \tag{58a}$$

where

$$\bar{\rho} \equiv \rho/m, \quad \tilde{\bar{L}} \equiv \tilde{L}/m, \text{ etc.} \quad , \tag{58b}$$

and

$$\Gamma(\tilde{\bar{L}},\bar{\rho}) \equiv (\bar{\rho} + \tfrac{1}{2})^{-1}[(\bar{\rho} - \tfrac{1}{2})^2 + (\tilde{\bar{L}}/\bar{\rho})^2(\bar{\rho} - \tfrac{3}{2})^2]^{1/2} \quad . \tag{58c}$$

The function Γ plays the role of an effective potential, which we shall discuss later. Equation (57b) can be written as

$$(d\rho/dt)^2 = \frac{1}{\bar{\rho}^2}\left[\frac{\bar{\rho} - \frac{3}{2}}{\bar{\rho} + \frac{1}{2}}\right]^4 (\gamma^2 - \ell^2) \quad , \tag{59a}$$

where

$$\gamma \equiv \bar{\rho}(\bar{\rho} + \tfrac{1}{2})(\bar{\rho} - \tfrac{3}{2})^{-1} \quad . \tag{59b}$$

Consider first radial geodesics ($\tilde{L} = \ell = 0$). Then Eq. (58a) indicates clearly that $\bar{\rho} = 3/2$ is an infinite proper time away from timelike geodesics. If one then uses the fact that $P_0 = g_{00}\, dt/d\lambda$ = constant for the null radial geodesics together with Eq. (59a), then it is also easy to show that $\bar{\rho} = 3/2$ is an infinite affine parameter distance away for null radial geodesics. Equations (58) and (59) indicate that nonradial geodesics between any two values of $\bar{\rho}$ take even longer proper time and affine parameter than do radial geodesics. Thus we have shown that $\bar{\rho} = 3/2$ is really unreachable by particles and photons; in particular, the manifold covered by our coordinate system is maximal.[28] Since one can also show that there are no singularities for $\bar{\rho} \geq 3/2$, our manifold is geodesically complete.[28]

For the special case of radial geodesics, we integrate Eq. (58a) to yield

$$\tau = \pm\left[- b^{-1}\ell n\left|\frac{2b(X + 1) + 2c(\bar{\rho} - \frac{3}{2})}{\bar{\rho} - \frac{3}{2}}\right| - d^{-1}X + d^{-3/2}\sin^{-1}\left(\frac{2d\bar{\rho} - 1 - \tilde{E}^2}{2\tilde{E}}\right)\right] \quad (60a)$$

$$+ \text{const. for } \frac{1}{2} < \tilde{E} < 1 \quad ,$$

where

$$b \equiv (4\tilde{E}^2 - 1)^{1/2} \quad , \quad (60b)$$

$$c \equiv 2\tilde{E}^2 - 1 \quad , \quad (60c)$$

$$d \equiv 1 - \tilde{E}^2 \quad , \quad (60d)$$

$$X \equiv [(1 + \tilde{E}^2)\,\bar{\rho} - c(\bar{\rho}^2 + \frac{1}{4})]^{1/2} \quad . \quad (60e)$$

We will not be interested in analytic solutions for values of $\tilde{E}$ other than those indicated in Eq. (60). To obtain the functional relationship between coordinate time t and ρ for $1/2 < \tilde{E} < 1$, add to Eq. (60a) a factor of 4 multiplying the log term and a factor of $(3 - 2\tilde{E}^2)$ multiplying the inverse sine term.

For radial photon geodesics, Eq. (59a) can be integrated to yield

$$\bar{t} = \pm\,(\bar{\rho} + 2\ell n|\bar{\rho} - \frac{3}{2}|) + \text{const.} \quad . \quad (61)$$

Figure 1 illustrates a few of the radial geodesics for photons and particles, the latter released from rest at $\bar{\rho} = 10$ and $\bar{\rho} = 5$. It is interesting to note that the analogous metric in GRT is geodesically incomplete: $\bar{\rho} = 1/2$ can be reached in finite proper time, but requires infinite coordinate time.

It can be shown, from analysis of the metric, that another complete universe exists for $1/2 \le \bar{\rho} \le 3/2$. However, if we assume the geometry to be produced by a star which originated in our universe, then its surface lies outside $\bar{\rho} = 3/2$. In the following we consider only the region $\bar{\rho} > 3/2$.

B. Proper Surface Areas and Embedding Diagrams

There are some curious geometrical effects in our manifold, not to be found in the Schwarzschild geometry of GRT. The proper surface area of a sphere described by ρ = const. is

$$A = 4\pi m^2 \,\bar{\rho}^2(\bar{\rho} - \tfrac{1}{2})^2\,(\bar{\rho} - \tfrac{3}{2})^{-2} \quad . \tag{62}$$

A plot of this area is given in Fig. 2, in which the abcissa is marked off not only by ρ but also by the proper time as measured by a radially falling observer. As can be seen in the figure, the observer sees the sequence of surface areas pass through a minimum, $A_{MIN} = 4\pi m^2\,(49/4 + 5\sqrt{6})$ at $\bar{\rho} = 3/2 + 1/2\sqrt{6}$, and then increase without bound as $\bar{\rho} = 3/2$ is approached.

Another interesting feature arises when we examine the intrinsic geometry of the 2-surface: t = const., $\Theta = \pi/2$ by the use of an embedding diagram. By equating the two-dimensional metric

$$ds^2 = (\bar{\rho} - \tfrac{1}{2})^2\,(\bar{\rho} - \tfrac{3}{2})^{-2}\,(d\rho^2 + \rho^2 d\Theta^2) \tag{63a}$$

to the metric of a surface of revolution in a Euclidean 3-space

$$ds^2 = dz^2 + dr^2 + r^2 d\varphi^2 = [(dz/dr)^2 + 1]\,dr^2 + r^2 d\varphi^2 \quad , \tag{63b}$$

one can visualize the geometry of Eq. (63a). If we can find $z(r)$, or more easily $z(\rho)$ and $r(\rho)$, then the line element of Eq. (63b) can be drawn. Clearly

$$\bar{r} = \bar{\rho}(\bar{\rho} - \tfrac{1}{2})(\bar{\rho} - \tfrac{3}{2})^{-1} \quad . \tag{64}$$

The function $z(\rho)$ is the solution of the equation

$$\left(\frac{dz}{d\rho}\right)^2 = \left(\frac{\partial s}{\partial \rho}\right)^2 - \left(\frac{dr}{d\rho}\right)^2 = \frac{(\bar{\rho} - \frac{1}{2})^2}{(\bar{\rho} - \frac{3}{2})^2} - \left[\frac{\bar{\rho}^2 - 3\bar{\rho} + \frac{3}{4}}{(\bar{\rho} - \frac{3}{2})^2}\right]^2 \quad , \tag{65a}$$

or

$$d\bar{z}/d\bar{\rho} = \bar{\rho}^{\,1/2}(2\bar{\rho}^2 - 5\bar{\rho} + \tfrac{3}{2})^{1/2}(\bar{\rho} - \tfrac{3}{2})^{-2} \quad . \tag{65b}$$

The right-hand side of Eq. (65b) becomes complex at

$$\bar{\rho} = \tfrac{1}{4}(5 + \sqrt{13}) \approx 2.1 \quad \text{or} \quad \bar{r} \approx 4.8 \quad . \tag{66}$$

This indicates that for $1.5 < \bar{\rho} \lesssim 2.1$ we will have to embed in a pseudo-Euclidean space, i.e.,

$$ds^2 = - dz^2 + dr^2 + r^2 d\varphi^2 \quad . \tag{67}$$

The embedding diagram is given in Fig. 3 and includes both the Euclidean part and the pseudo-Euclidean part. The surface is obtained by rotating the curve about the z or iz axis.

C. Particle and Photon Orbits

Analysis of orbits is facilitated by use of the effective potential. Equations (58c) and (59b) give the effective potentials for massive particles and photons. For a given value of $\tilde{L}$, the particle is allowed only in those regions for which $\Gamma(\tilde{L},\bar{\rho}) \leq \tilde{E}$. For photons, γ^2 acts as an "inverse" effective potential; photons are allowed only in regions for which $\gamma \gtrsim \ell$. Figures 4 and 5 illustrate the effective potentials for particles and photons, respectively, with the dots in Fig. 4 indicating extrema of the potential (circular orbits). The closest stable circular orbit for particles occurs for $\bar{\tilde{L}} \sim 3.88$ at $\bar{\rho} \sim 7$. For particles with larger $\bar{\tilde{L}}$, the circular orbits with $\bar{\rho} < 7$ are unstable and those with $\bar{\rho} > 7$ are stable. The circular photon orbit occurs at $\bar{\rho} = 1.5 + \sqrt{3}$ or $\bar{r} \sim 5$. This can be compared with the corresponding value of $\bar{r} = 3$ in GRT.

V. GRAVITATIONAL WAVES AND CONSERVATION LAWS

A. Monopole Waves

In the full theory (no linearized approximation) the homogeneous field equations are, as indicated previously,

$$\eta^{\alpha\beta} h^{\mu\nu}{}_{|\alpha|\beta} = 0 \quad , \tag{68}$$

and gravitational waves travel geodesics of $\underset{\sim}{\eta}$ rather than $\underset{\sim}{g}$. The implication of this last fact will be explored later. The simplicity of the vacuum field equations [cf. Eq. (68)] is of great help in constructing solutions.

Consider a time-dependent spherically symmetric solution to Eq. (68), for example

$$h_{00} = r^{-1} e^{i\omega(r-t)} \quad , \tag{69a}$$

$$h_{ij} = \delta_{ij} \, r^{-1} e^{i\omega(r-t)} \quad . \tag{69b}$$

The Riemann tensor constructed from the resulting time-dependent spherically symmetric metric is itself time dependent. From this we conclude the presence of physical monopole waves; thus there is no analogue of Birkhoff's theorem[12] in this theory. The existence of such solutions in our theory and the accompanying monopole radiation complicate the problem of the spherical collapse of a star. As will be shown below, there are other "non-GRT" type gravitation-wave modes in addition to the monopole waves.

B. Linearized Theory and Plane Gravitational Waves

In analyzing weak gravitational waves, one should restrict one's attention to the form and behavior of the Riemann tensor, not only because it is gauge invariant (under infinitesimal coordinate transformations) but also because it is that feature of the gravitational wave which interacts directly with test bodies. Work in a coordinate system in which $\eta_{\mu\nu}$ is Minkowskian and $h_{\mu\nu}$ is small (small deviations from flat space). Then

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - 2Kh\eta_{\mu\nu} + O(h^2) \equiv \eta_{\mu\nu} + h'_{\mu\nu} + O(h^2) \quad , \tag{70}$$

and

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(h'_{\alpha\delta,\beta\gamma} + h'_{\beta\gamma,\alpha\delta} - h'_{\alpha\gamma,\beta\delta} - h'_{\beta\delta,\alpha\gamma}) \quad . \tag{71}$$

Furthermore, restrict one's attention to those solutions of Eq. (68) which represent plane waves travelling in the z direction, i.e.,

$$h'_{\mu\nu} = A_{\mu\nu} e^{ik(z-t)} \quad , \tag{72}$$

where $A_{\mu\nu}$ is a constant amplitude and k a wave number. To analyze the decomposition of $R_{\alpha\beta\gamma\delta}$ into independent "wave modes" in as invariant a manner as possible, one should investigate the transformation properties of $R_{\alpha\beta\gamma\delta}$ under those Lorentz transformations which leave the wave direction fixed. With such transformations in mind one selects a new basis in which the components of $R_{\alpha\beta\gamma\delta}$ are to be computed — the quasi-orthonormal tetrad basis (see, e.g., Ref. 29 for a complete discussion of the "tetrad formalism").

$$\underset{\sim}{k} = 2^{-1/2}(-1,0,0,1) \quad , \tag{73a}$$

$$\underset{\sim}{\ell} = 2^{-1/2}(1,0,0,1) \quad , \tag{73b}$$

$$\underset{\sim}{m} = 2^{-1/2}(0,1,i,0) \quad , \tag{73c}$$

$$\underset{\sim}{\overline{m}} = 2^{-1/2}(0,1,-i,0) \quad . \tag{73d}$$

Note that one of the "tetrad legs" points along the direction of the wave. In such a basis the components of the Riemann tensor are

$$R_{nkm\ell} = R_{\alpha\beta\gamma\delta} n^{\alpha} k^{\beta} m^{\gamma} \ell^{\delta} \text{ , etc. } \quad . \tag{74}$$

Using Eqs. (71)-(74) one finds that the only nonvanishing components of the Riemann tensor are those with two ℓ's — thus there are six possible degrees of freedom. Since there are no restrictions on the Riemann tensor once Eqs. (68) are satisfied, all six tetrad components will in general be nonvanishing and our theory thus has six independent gravitational wave modes. In GRT, as a contrast, the field equations $R_{\alpha\beta} = 0$ imply vanishing of $R_{\ell k\ell k}$, $R_{\ell k\ell m}$, $R_{\ell k\ell\bar{m}}$, and $R_{\ell m\ell\bar{m}}$ so that there are only two degrees of freedom — those represented by $R_{\ell m\ell m}$ and its complex conjugate $R_{\ell\bar{m}\ell\bar{m}}$.

The reader is referred to Refs. 3 and 4 for details of the transformation properties of the objects indicated in Eq. (74). Here we quote only the results: We denote the six wave modes by Ψ_2, Ψ_3, $\bar{\Psi}_3$, Ψ_4, $\bar{\Psi}_4$, Φ_{22} and in terms of the tetrad components of the Riemann tensor and "electric" coordinate components of the Riemann tensor (those which are directly physically measurable) these are

$$\Psi_2 \equiv -\frac{1}{6} R_{\ell k\ell k} = -\frac{1}{6} R_{tztz} \quad , \tag{75a}$$

$$\Psi_3 \equiv -\frac{1}{2} R_{\ell k\ell\bar{m}} = \frac{1}{2}(R_{txtz} - iR_{tytz}) \quad , \tag{75b}$$

$$\bar{\Psi}_3 \equiv -\frac{1}{2} R_{\ell k\ell m} = \frac{1}{2}(R_{txtz} + iR_{tytz}) \quad , \tag{75c}$$

$$\Psi_4 \equiv -R_{\ell\bar{m}\ell\bar{m}} = R_{tyty} - R_{tztz} + 2iR_{txty} \quad , \tag{75d}$$

$$\bar{\Psi}_4 \equiv -R_{\ell m\ell m} = R_{tyty} - R_{tztz} - 2iR_{txty} \quad , \tag{75e}$$

$$\Phi_{22} \equiv \frac{1}{2} R_{\ell m\ell\bar{m}} = -R_{txtx} - R_{tyty} \quad . \tag{75f}$$

The presence or absence of a Ψ_2 component in a gravitational wave is Lorentz invariant. If Ψ_2 is absent in a particular wave, the presence or absence of Ψ_3 (or $\bar{\Psi}_3$) in that wave is also Lorentz invariant. As outlined in Refs.

3 and 4, if either Ψ_2 or Ψ_3 is present in a wave (in many theories they are always absent, but not ours), then it is impossible to decompose the wave into states of definite helicity (spin) in a Lorentz invariant manner: what one observer identifies as "pure spin 0" another observer will identify as "pure spin 0" plus "pure spin 1," etc. . Only waves containing only Φ_{22}, Ψ_4, and $\overline{\Psi}_4$ can be decomposed into pure spins: spin 0 and spin 2. In general, then, there is no unique spin decomposition of waves in our theory and it is of class II_6 (see Refs. 3 and 4 for a complete discussion of the "classification scheme"). The physical imprints of the various modes will be discussed in Sec. VI.

B. The Stress-Energy Pseudo Tensor for Gravitational Waves

For all Lagrangian-based theories a very general method, with roots going back to Noether,[30] exists for constructing conserved quantities (see Ref. 5 and the references quoted therein for a more complete discussion). Invariance of the gravitational Lagrangian under coordinate transformations leads to the following identities:

$$(t'_{\mu}{}^{\nu} - U_{\mu A}{}^{\nu}\mathcal{L}_G{}^{A})_{,\nu} \equiv 0 \quad , \tag{76}$$

where $\mathcal{L}_G$ is the gravitational Lagrangian density, $\mathcal{L}_G{}^A$ is the variational derivative of $\mathcal{L}_G$ with respect to field y_A occurring in $\mathcal{L}_G$,

$$t'_{\mu}{}^{\nu} = -\delta_{\mu}{}^{\nu}\mathcal{L}_G + \frac{\partial\mathcal{L}_G}{\partial y_{A,\nu}}\, y_{A,\mu} \quad , \tag{77a}$$

and $U_{\mu A}{}^{\nu}$ is defined by the functional changes of the y_A, δy_A, under infinitesimal coordinate transformations, i.e.,

$$\overline{x}^{\mu} = x^{\mu} + \xi^{\mu} \quad , \tag{77b}$$

$$\delta y_A = U_{\mu A}{}^{\nu}\xi^{\mu}{}_{,\nu} - y_{A,\mu}\xi^{\mu} \quad . \tag{77c}$$

We have assumed $\mathcal{L}_G$ contains no higher than first derivatives of the y_A; generalization to higher derivatives is straightforward. Equations (76) are of the form of conservation laws and our object is to identify in a physically meaningful way the gravitational portion of the conserved quantity. To facilitate the computation, we assume $\mathcal{L}_G$ has been rewritten in terms of $\eta_{\alpha\beta}$ and $g_{\alpha\beta}$ [which can be done in principle by solving for $h_{\alpha\beta}(g_{\mu\nu}, \eta_{\mu\nu})$]. Using the tensor transformation law for $g_{\alpha\beta}$ and $\eta_{\alpha\beta}$, one easily shows

$$U_{\mu A}{}^{\nu} = - 2y_{\mu(\alpha}\delta_{\beta)}{}^{\nu} \quad \text{for} \quad y_A = y_{\alpha\beta} \quad , \tag{78}$$

where parentheses denote symmetrization of indices. Using Eq. (78) we find the relation

$$U_{\mu A}{}^{\nu}\mathcal{L}_G'{}^{,A} = - 2\eta_{\mu(\alpha}\delta_{\beta)}{}^{\nu}(\delta\mathcal{L}_G'/\delta\eta_{\alpha\beta}) - 2g_{\mu(\alpha}\delta_{\beta)}{}^{\nu}(\delta\mathcal{L}_G'/\delta g_{\alpha\beta}) \quad , \tag{79}$$

where

$$\mathcal{L}_G'(\underset{\sim}{\eta}, \underset{\sim}{g}) = \mathcal{L}_G(\underset{\sim}{h}, \underset{\sim}{\eta}) \quad .$$

If we now use the field equations

$$\delta\mathcal{L}_G'/\delta g_{\alpha\beta} = - \delta\mathcal{L}_{NG}/\delta g_{\alpha\beta} \quad , \tag{80}$$

and Eq. (4c), Eq. (79) becomes

$$U_{\mu A}{}^{\nu}\mathcal{L}_G'{}^{,A} = - 2\eta_{\mu(\alpha}\delta_{\beta)}{}^{\nu}(\delta\mathcal{L}_G'/\delta\eta_{\alpha\beta}) + (- g)^{1/2}\, T_{\mu}{}^{\nu} \equiv \Lambda_{\mu}{}^{\nu} + (- g)^{1/2} T_{\mu}{}^{\nu} \quad . \tag{81}$$

We point out that although Eqs. (76) are "strong conservation laws"[31] (identities), one must use Eqs. (80) to get out a physically useful result. Substitution of Eq. (81) into Eq. (76) yields

$$\left(t_{\mu}{}^{\nu} - (- g)^{1/2} T_{\mu}{}^{\nu}\right)_{,\nu} = 0 \quad , \tag{82a}$$

where

$$t_\mu{}^\nu \equiv t'_\mu{}^\nu - \Lambda_\mu{}^\nu \quad . \tag{82b}$$

The conserved energy momentum vector is then

$$P_\mu = \int \left(t_\mu{}^0 - (-g)^{1/2} T_\mu{}^0 \right) d^3x \quad . \tag{83}$$

Since P_μ in Eq. (83) contains a contribution from the matter stress energy tensor, we know we are on the right track. Problems arise when we notice that the quantity defined in Eq. (82b) is in general not positive definite, as a result of contributions from $\Lambda_\mu{}^\nu$. However, it can be shown from the generalized Bianchi identities of this theory (see Appendix B) that $\Lambda_\mu{}^\nu$ obeys the equation

$$\Lambda_\mu{}^\nu{}_{|\nu} = 0 \quad . \tag{84}$$

Actually, Eddington[32] was the first to point out that conservation laws of the form of Eq. (84) follow from theories with absolute objects.[2] If we now choose to work in the coordinate system in which $\eta_{\alpha\beta}$ is the globally constant Minkowski metric, Eq. (84) becomes

$$\Lambda_\mu{}^\nu{}_{,\nu} = 0 \quad , \tag{85}$$

and we see that $\Lambda_\mu{}^\nu$ is conserved by itself, independently of energy gain or loss from matter ($T_\mu{}^\nu$). Since our usual idea of total energy conservation involves interactions, it is perhaps more useful to omit the separately conserved $\Lambda_\mu{}^\nu$ from consideration and to define, in this frame, the gravitational stress-energy tensor as

$$t_\mu{}^\nu = t_\mu{}'^\nu \quad . \tag{86}$$

Thus $\Lambda_\mu{}^\nu$ represents the energy density of a quantity associated with the absolute field $\eta_{\mu\nu}$; at present we must regard it as a purely mathematical quantity whose noninteraction with matter mirrors the absolute nature of $\eta_{\mu\nu}$. (As an aside, there always exists a $t_\mu^{\ \nu}$ which is a real tensor and not a pseudo-tensor in prior-geometric theories of gravity.[5])

We point out that in the linearized approximation Eq. (85) is always the expression of Eq. (84) in all frames related to the global Minkowski frame by infinitesimal coordinate (gauge) transformations. We proceed by explicitly calculating $t_\mu^{\ \nu}$ for the linearized theory. From Eq. (77a)

$$t_\mu^{\ \nu} = -\delta_\mu^{\ \nu}\mathscr{L}'_G + \frac{\partial\mathscr{L}'_G}{\partial g_{\alpha\beta,\nu}}\, g_{\alpha\beta,\mu} = -\delta_\mu^{\ \nu}\mathscr{L}_G + \frac{\partial\mathscr{L}_G}{\partial h_{\gamma\delta,\omega}}\,\frac{\partial h_{\gamma\delta,\omega}}{\partial g_{\alpha\beta,\nu}}\, g_{\alpha\beta,\mu} \ . \tag{87}$$

Inverting the linearized relation between $g_{\alpha\beta}$ and $h_{\alpha\beta}$ [cf. Eq. (70)] and taking the required partial derivatives, we find

$$\partial h_{\gamma\delta,\omega}/\partial g_{\alpha\beta,\nu} = \delta_\gamma^{(\alpha}\delta_\delta^{\beta)}\delta^\nu_{\ \omega} + 2K(1-8K)^{-1}\,\eta_{\gamma\delta}\eta^{\alpha\beta}\delta^\nu_{\ \omega} \ . \tag{88}$$

Using Eqs. (87), (88), and Eq. (2) for $\mathscr{L}_G$, we finally obtain

$$t_\mu^{\ \nu} = (16\pi)^{-1}\,[\delta_\mu^{\ \nu}(ah^{\gamma\sigma,\beta}h_{\gamma\sigma,\beta} + fh^{,\alpha}h_{,\alpha}) - 2(ah^{\alpha\beta,\nu}h_{\alpha\beta,\mu} + fh_{,\mu}h^{,\nu})] . \tag{89}$$

Since $h_{\mu\nu}$ transforms as a tensor, the above expression is gauge invariant. Equation (89) expresses a naturally defined stress-energy complex for the gravitational field.

Consider the energy density in a plane gravitational wave

$$h^{\gamma\sigma} = A^{\gamma\sigma}e^{ik_\alpha x^\alpha} \ ; \quad k_\alpha k^\alpha = 0 \ . \tag{90}$$

Then the first two terms in Eq. (89) do not contribute to $t_\mu^{\ \nu}$ and one obtains

$$t_\mu^{\ \nu} \propto k_\mu k^\nu \ , \tag{91a}$$

with

$$t_0^{\ 0} = (8\pi)^{-1}[a(h_{\alpha\beta,0})^2 + f(h_{,0})^2] \quad . \tag{91b}$$

With the suggested values for a and f [cf. Eq. (30)], Eq. (91b) indicates a positive definite energy density. It is encouraging to note that for pure spin 2 waves (only ψ_4 present), Eq. (91b) becomes, for a = 1/4 [cf. Eq. (30)],

$$t_{0\,\mathrm{spin}\,2}^{\ 0} = a(4\pi)^{-1}(h_{xx,0})^2 = (16\pi)^{-1}(h_{xx,0})^2 \quad , \tag{92}$$

which is identical to the corresponding expression in GRT.

VI. THE GRAVITATIONAL CONSTANT AND FURTHER EXPERIMENTAL TESTS

A. A Time-Dependent Gravitational Constant

As discussed in Sec. II, a number of existing solar system experiments place upper limits on the cosmological boundary values of $h_{\mu\nu}$ [cf. Eqs. (31)-(32)]. These constraints can always be satisfied in a given epoch. A more relevant point is the time dependence of the $\omega_{\mu\nu}$, which is directly related to the time dependence of the gravitational constant G. With the choice of adjustable constants given in Eq. (30), and using the explicit functional forms for K_1 D_0, D, one finds from Eq. (27) and Appendix A that

$$1 - \frac{1}{16}\,(19\omega_1 + 7\omega_0) + O(\omega^2) = G \quad . \tag{93a}$$

Thus

$$(\frac{1}{G})\ dG/dt \approx -\frac{1}{16}\,(19\omega_1/dt + 7d\omega_0/dt) \quad . \tag{93b}$$

Shapiro et al.[33] have placed limits on the time dependence of the gravitational constant by comparing the periods of planets with the ticking rates of atomic clocks. They find

$$|(\frac{1}{G})\ (dG/dt)| < 4 \times 10^{-10}/\text{year} \quad . \tag{94}$$

This constitutes an experimental constraint on the magnitude of the time derivatives of $\omega_{\mu\nu}$ occurring in Eq. (93b). Preliminary results from our cosmological solution[19] indicate that the time dependences of ω_0 and ω_1 satisfy Eq. (94), but an improved Shapiro experiment might still prove to be a crucial experimental test of our theory.

B. Gravitational-Wave Experiments

The analysis of the preceding section reveals two crucial new experimental tests of our theory involving gravitational waves — two tests which have blossomed from our current program (see introductory remarks in Sec. I) — two tests which emphasize gravitational wave detection as a powerful new tool for probing metric theories of gravity.[3,4] The two tests are (i) time delay between simultaneously emitted gravitational and electromagnetic waves and (ii) polarizations of gravitational waves.

Since gravitational waves travel along geodesics of the "fast metric" $\eta_{\alpha\beta}$ and electromagnetic waves travel along geodesics of the "slow metric" $g_{\alpha\beta}$, there should be a time delay in reception of the two waves — emitted, for example, in simultaneous bursts by a supernova explosion. For waves emitted at the center of the galaxy, an order of magnitude estimate indicates

$$\begin{aligned}\text{Time Delay} &\sim (m/r)_{\text{galaxy}} \cdot (\text{light travel time}) \\ &\sim (5 \times 10^{-7}) \cdot (3 \times 10^{4}\ \text{light years}) \approx 5\ \text{days} \quad .\end{aligned} \tag{95}$$

Much longer delay times would hold for the Virgo Cluster.

Polarization information is also a crucial experimental test. Equations

(75) indicate a purely longitudinal mode (ψ_2), mixed longitudinal-transverse quadrupole type modes (ψ_3, $\overline{\psi}_3$), a purely transverse "breathing" mode (Φ_{22}), and the familiar transverse quadrupole modes of GRT (ψ_4, $\overline{\psi}_4$). If an observer knows the direction of the wave, he can use Eqs. (75) to unambiguously catalogue the modes. If he does not know the direction of the source, he can still draw some conclusions. For example, if displacements do occur in more than one plane, then either the longitudinal-transverse modes (ψ_3, $\overline{\psi}_3$) are present, or the purely longitudinal mode (ψ_2) is mixed in with one of the purely transverse modes (ψ_4, $\overline{\psi}_4$, Φ_{22}).

It is important to note that until the problem of the generation of the various types of waves by particular sources is solved, our theory can only be verified by the presence of — but not ruled out by the absence of — the various possible modes indicated in Eqs. (75). This is unfortunate. But new doorways have been opened in the area of experimental tests and it is clear that gravitational tests outside of the PPN formalism must be contemplated in the future.

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APPENDIX A

CONSTANTS APPEARING IN PN LIMIT (Sec. II)

$$\omega_0 \equiv \omega_{00}$$

$$\omega_1 \equiv \omega_{11}$$

$$\omega \equiv 3\omega_1 - \omega_0$$

Eq. (12a): $D_0 \equiv 1 - 2K\omega + K^2\omega^2 + 2K\omega\omega_0 + \frac{3}{4}\omega_0^2 - \omega_0$

$E_0 \equiv 1 - 2K\omega - \frac{3}{2}\omega_0$

$F_0 \equiv -2K + 2K^2\omega + 2K\omega_0$

Eq. (12b): $D \equiv 1 - 2K\omega + \omega_1 + K^2\omega^2 - 2K\omega\omega_1 + \frac{3}{4}\omega_1^2$

$E \equiv 1 - 2K\omega + \frac{3}{2}\omega_1$

$F \equiv -2K(1 + \omega_1) + 2K^2\omega$

Eq. (12c): $H \equiv 1 - 2K\omega - \frac{3}{4}\omega_0 + \frac{3}{4}\omega_1$

Eq. (13): $I \equiv D_0^{1/2} D^{-3/2}$

$I_1 \equiv \frac{1}{2}\left(\frac{E}{D} + \frac{E_0}{D_0}\right)$

$I_2 \equiv \frac{1}{2}\left(\frac{3F}{D} - \frac{F_0}{D_0} + \frac{E}{D}\right)$

$I_3 \equiv \frac{D}{D_0}$

$L \equiv -(a + 4f)^{-1}\left[f(1 - 2K\omega) + 2Ka(1 - K\omega)\right]$

$M \equiv -(a + 4f)^{-1}\left(2Ka + \frac{3}{2}f\right)$

$N \equiv 2k(f + Ka)(a + 4f)^{-1}$

Eq. (16d): $S_0 \equiv I_1(1 - 2K\omega + L - \frac{3}{2}\omega_0 - M\omega_0) - \frac{3}{2} - M$

$S_1 \equiv I_2(1 - 2K\omega + L - \frac{3}{2}\omega_0 - M\omega_0) + N - 2K$

$B_0 \equiv I_3(1 - 2K\omega + L - \frac{3}{2}\omega_0 - M\omega_0) - L - M\omega_1$

Eq. (16e): $R_0 \equiv 1 - 2K\omega + \frac{3}{2}\omega_1$

$R_1 \equiv I_1(M\omega_0 - L) + M$

$R_2 \equiv I_2(M\omega_0 - L) - N$

$B_1 \equiv I_3(M\omega_0 - L) + L + M\omega_1$

Eq. (20a): $K_1 \equiv E_0C_0 - F_0(3C_1 - C_0)$

$K_2 \equiv -[K^2(3C_1 - C_0)^2 + 2KC_0(3C_1 - C_0) + \frac{3}{4}C_0^2]$

$K_3 \equiv \tau[S_0C_0 + S_1(3C_1 - C_0)](E_0 + F_0) - 3\tau F_0[RC_0 + R_2(3C_1 - C_0)]$

$K_4 \equiv \tau[E_0B_0 - F_0(R_o + 3B_1 - B_0)]$

Eq. (20b): $K_5 \equiv EC_1 + F(3C_1 - C_0)$

APPENDIX B

RELATIONS FOLLOWING FROM GENERALIZED BIANCHI IDENTITIES

Assume that L_G has been rewritten as a function of $\eta_{\mu\nu}$ and $g_{\mu\nu}$. Since L_G is a scalar, its variation under infinitesimal coordinate transformations must vanish, i.e.,

$$\delta L_G = \int \left(\frac{\delta \mathscr{L}'_G}{\delta \eta_{\alpha\beta}} \delta \eta_{\alpha\beta} + \frac{\delta \mathscr{L}'_G}{\delta g_{\alpha\beta}} \delta g_{\alpha\beta} \right) d^4x = 0 \quad . \tag{B1}$$

Under the coordinate transformation

$$\bar{x}^\alpha = x^\alpha + \xi^\alpha \quad , \tag{B2}$$

the functional changes in the tensors $\underset{\sim}{\eta}$ and $\underset{\sim}{g}$ are

$$\begin{aligned} \delta \eta_{\alpha\beta} &= - \eta_{\alpha\beta,\mu} \xi^\nu - \eta_{\alpha\gamma} \xi^\gamma{}_{,\beta} - \eta_{\beta\gamma} \xi^\gamma{}_{,\alpha} \quad , \\ &= - 2\tau_{(\alpha|\beta)}, \quad \text{where} \quad \tau_\alpha = \eta_{\alpha\beta} \xi^\beta \quad , \end{aligned} \tag{B3}$$

$$\delta g_{\alpha\beta} = - 2\xi_{(\alpha;\beta)} \quad . \tag{B4}$$

Now define

$$\delta \mathscr{L}'_G / \delta \eta_{\alpha\beta} = (-\eta)^{1/2} \gamma^{\alpha\beta} \quad , \tag{B5}$$

and use the field equations to write

$$\delta \mathscr{L}'_G / \delta g_{\alpha\beta} = - \frac{1}{2} (-g)^{1/2} T^{\alpha\beta} \quad . \tag{B6}$$

Using Eqs. (B2)-(B6), Eq. (B1) can be written in the form

$$\begin{aligned} 0 = \int \Big[& (-\eta)^{1/2} (\gamma^{\alpha\beta} \tau_\alpha)_{|\beta} - \frac{1}{2} (-g)^{1/2} (T^{\alpha\beta} \xi_\alpha)_{;\beta} + \frac{1}{2} (-g)^{1/2} T^{\alpha\beta}{}_{;\beta} \xi_\alpha \\ & - (-\eta)^{1/2} \gamma^{\alpha\beta}{}_{|\beta} \tau_\alpha \Big] d^4x \quad . \end{aligned} \tag{B7}$$

Now if we remember that

$$(\gamma^{\alpha\beta}\tau_\alpha)_{|\beta} = (-\eta)^{-1/2}\left[(-\eta)^{1/2}\gamma^{\alpha\beta}\tau_\alpha\right]_{,\beta} \quad , \tag{B8}$$

and also the corresponding equation for the covariant derivative with respect to $g_{\alpha\beta}$, the first two terms in (B7) vanish with proper boundary conditions on ξ^α. Now use the matter equations, Eqs. (5), and the arbitrariness of ξ^α (and hence τ_α) to get from Eq. (B7)

$$\gamma^{\alpha\beta}{}_{|\beta} = 0 \quad . \tag{B9}$$

Equation (B9) is not an identity; we had to use both the matter and gravitational field equations to obtain it. [We would have obtained an identity in the place of Eq. (B9) had we not enforced the dynamical equations.] Since $\underset{\sim}{\eta}$ is covariantly constant with respect to "slash," Eqs. (B5) and (B9) imply the desired relation

$$\Lambda_\mu{}^\nu{}_{|\nu} \equiv [\,-\,2\eta_{\mu(\alpha}\delta^\nu{}_{\beta)}(\delta\mathscr{L}'_G/\delta\eta_{\alpha\beta})]_{|\nu} = 0 \quad . \tag{B10}$$

TABLE I

Comparison of Construction of Stellar Models

	GRT	Two-Metric Theory
Number of coupled differential equations which must be integrated to find star's surface	2	4
Type of differential equations used in determining metric functions	First-order linear	Second order nonlinear
Number of quantities whose central values must be chosen to satisfy boundary conditions	1	4
Analytic Solutions	Yes	Probably not
Uniqueness of solution for given central pressure and equation of state	Yes	No
Number of parameters in exterior metric	1	4

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21. In this section and for the rest of the paper, except Sec. VI, we assume that the cosmological boundary values of $h_{\mu\nu}$ are arbitrarily small for the current epoch. See Sec. II for a discussion and justification of this point.

22. One can argue as follows: Let A be a coordinate system which contains the minimum number of arbitrary parameters. A transformation from A to curvature coordinates C cannot decrease the number of arbitrary parameters, by definition, and cannot increase the number since the transformation is only a function of the parameters occurring in A. Hence C has the same number of arbitrary parameters as A, i.e., the minimum possible number.

23. If $\eta_{\mu\nu}$ were not of the form of Eq. (34), aside from transformations of the type given in Eq. (37), in the same frame in which we assume the star to be static, then the resultant $g_{\mu\nu}$ would have such features as gravitational waves, anisotropies, etc. and thus be physically

inconsistent with the assumed condition of the star.

24. Note that only in such solutions (with $a_4 = 0$) are the gravitational wave cones (based upon $\eta_{\mu\nu}$ — Sec. V) and light cones (based upon $g_{\mu\nu}$) mutually symmetric. When one metric is diagonal and the other not, the two cones are "tilted" with respect to each other.

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FIGURE CAPTIONS

Fig. 1. Radial Geodesics of Particles and Photons. The numbers along the curves indicate proper time values for massive particles released from rest at $\bar{\rho} = 10, 5$. One of the curves is a photon geodesic and all curves have $\bar{t} \to \infty$ and affine parameter $\to \infty$ as $\bar{\rho} \to 1.5$.

Fig. 2. Proper Surface Area of Sphere $\bar{\rho}$ = Const. The upper abcissa gives the proper time of an observer released from $\bar{\rho} = 10$ as a local coordinate marker.

Fig. 3. Embedding Diagram for Equatorial Geometry. Solid line indicates Euclidean embedding (refers to z ordinate) and dashed line indicates pseudo-Euclidean embedding (refers to iz ordirate). Numbers along curve indicate values of $\bar{\rho}$.

Fig. 4. Effective Potential for Massive Objects. Dots indicate circular orbits.

Fig. 5. Effective Potential for Photons.

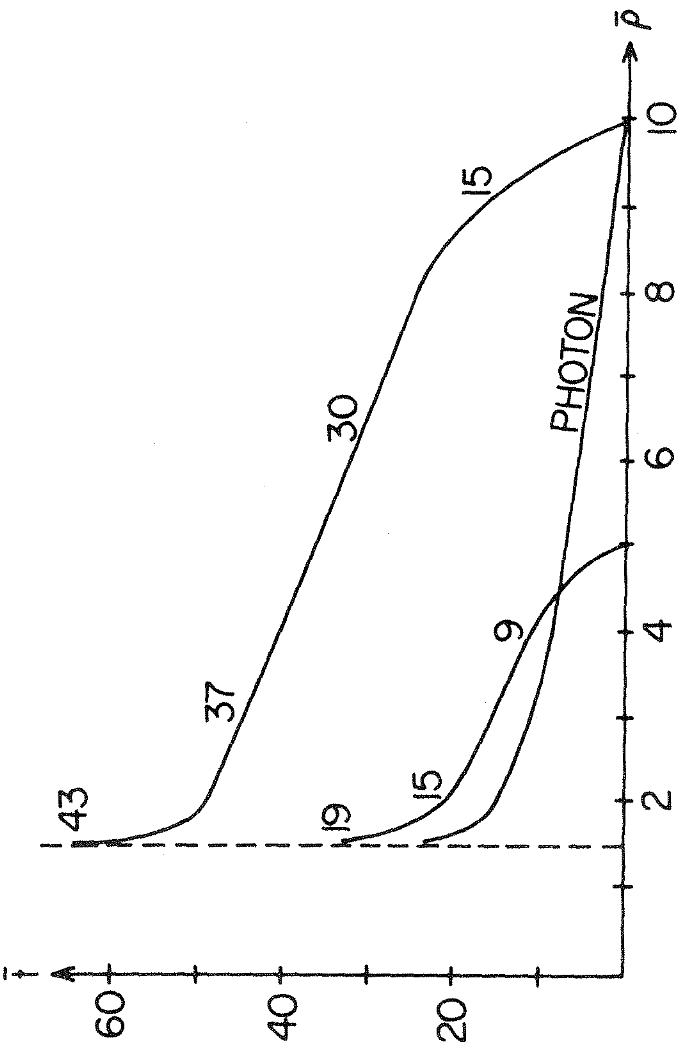

Fig. 1

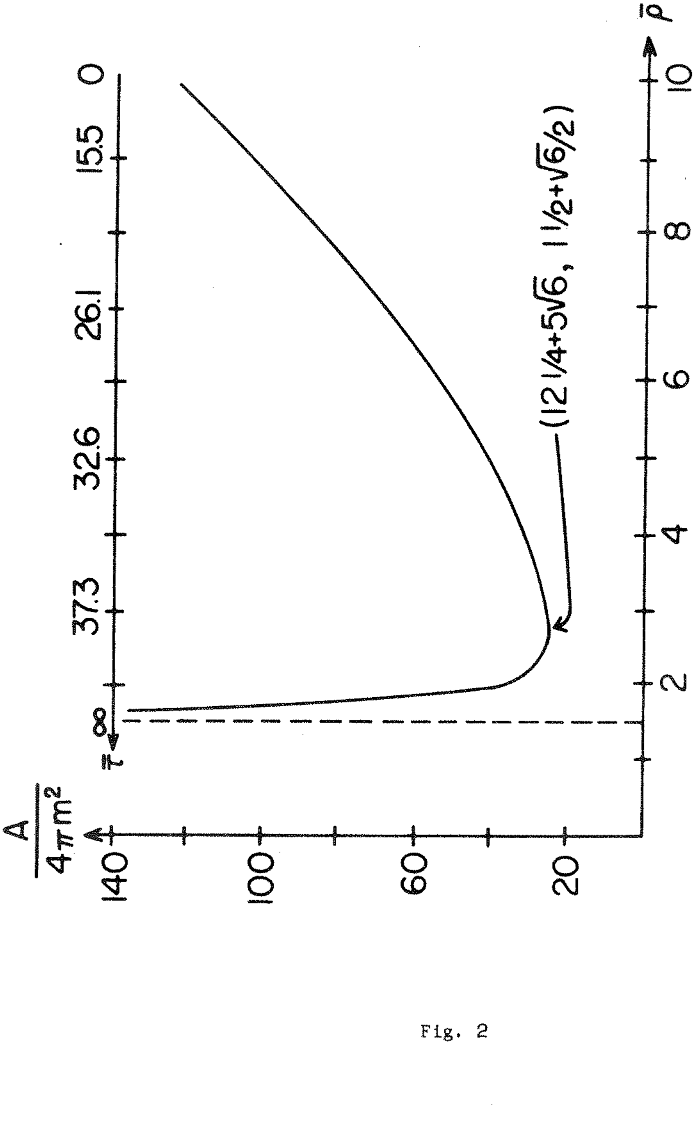

Fig. 2

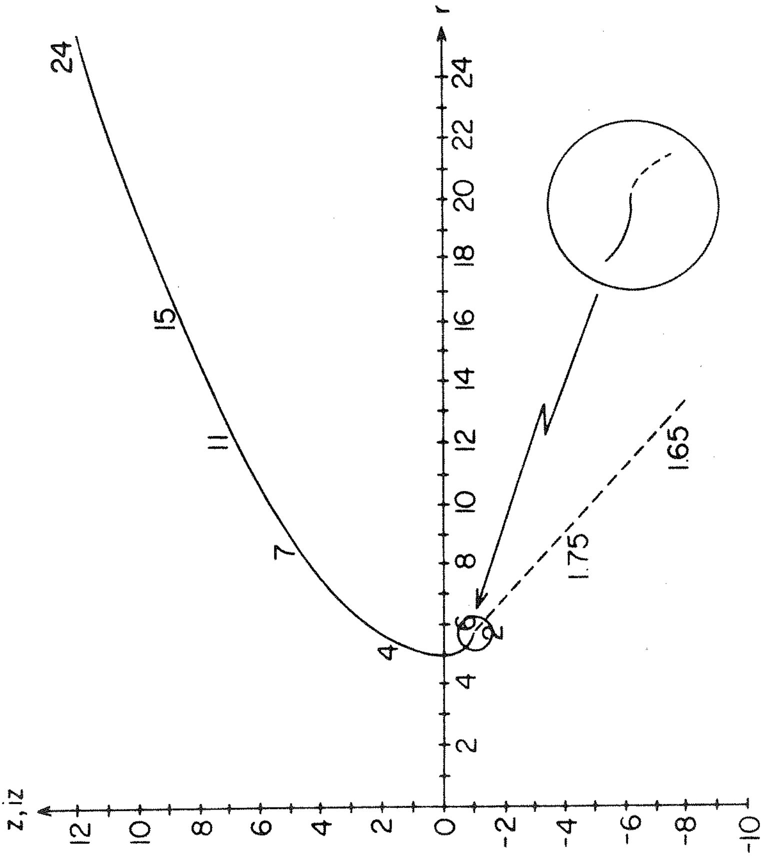

Fig. 3

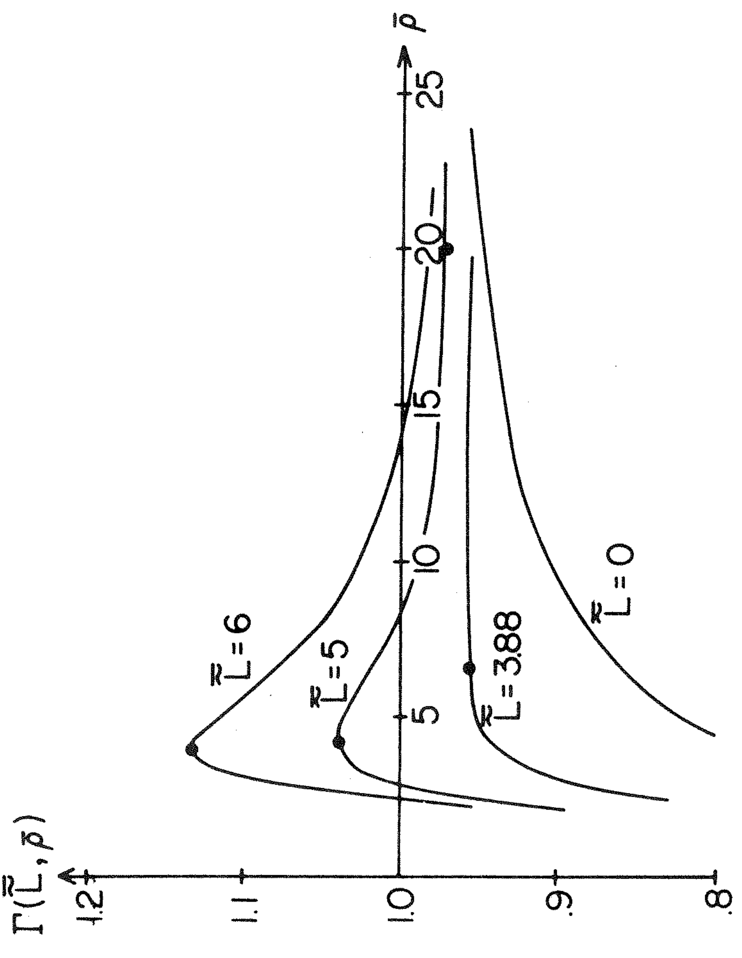

Fig. 4

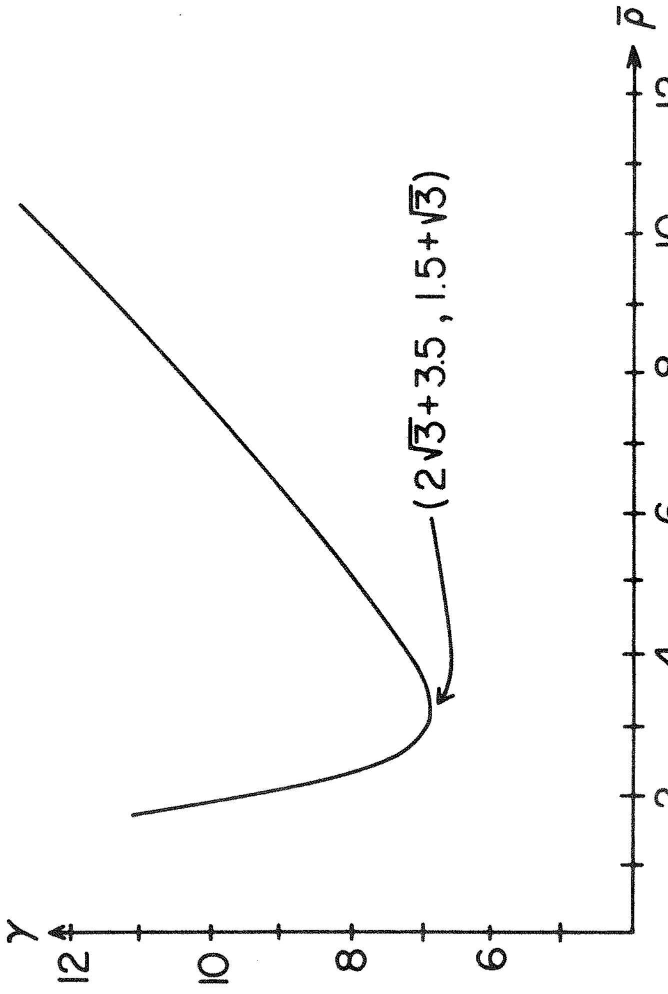

Fig. 5

b) Gravitational-Wave Observations as a Tool for Testing Relativistic Gravity

i) Brief Overview (Paper VIII; collaboration with D.M. Eardley, D.L. Lee, R. V. Wagoner and C.M. Will, published in *Phys. Rev. Lett.*, **30**, 884, 1973)

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Gravitational-Wave Observations as a Tool for Testing Relativistic Gravity*

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Gravitational-wave observations can be powerful tools in the testing of relativistic theories of gravity. Future experiments should be designed to search for *six* different types of polarization, and for anomalies in the propagation speed of the waves: $|c_{\text{grav waves}} - c_{\text{em waves}}| \gtrsim 10^{-7} c_{\text{em waves}}$. This Letter outlines the nature and implications of such measurements.

Several viable gravitation theories now exist that differ radically when describing strong gravitational fields, but that can be made to be identical to each other and to general relativity in the "post-Newtonian limit." During the next twenty years, one will probably not be able to distinguish these theories from general relativity or from each other by means of "solar-system experiments" (gravitational redshift, perihelion shift, light deflection, time delay, gyroscope precession, lunar-laser ranging, gravimetry, Earth rotation, ...). However, gravitational-wave experiments offer hope: These theories differ in their predictions of (i) propagation speed and (ii) polarization properties of gravitational waves.

(i) Some of the competing theories[1-4] predict the same propagation speed for gravitational waves (c_g) as for light (c_{em}). But others[5-7] predict a difference that, in weak gravitational fields, is typically

$$(c_g - c_{em})/c \sim (1/c^2) \times |\text{Newtonian potential}|$$

$\sim 10^{-7}$, for waves traveling in our region of the Galaxy or in the field of the Virgo cluster. An experimental limit of $\lesssim 10^{-8}$ would disprove most such theories and would stringently constrain future theory building. Perhaps the most promising way to obtain such a limit is by comparing arrival times for gravitational waves and for light that come from the onset of a supernova, or from some other discrete event. If current experimental efforts continue unabated, by 1980 one may detect gravitational-wave bursts from supernovae in the Virgo cluster (~three supernovae per year, 11 Mpc from Earth). Then a limit of

$$|c_g - c_{em}|/c \lesssim 10^{-9} \times (\text{time-lag precision})/(1\text{ week})$$

will be possible.

(ii) All of the currently viable theories fall into a class called "metric theories of gravity."[8,9] Recently, we have completed an analysis of the polarization properties of the most general weak, plane, null wave permitted by any metric theory. In general, the wave involves the metric field $g_{\mu\nu}$ and also auxiliary gravitational fields, such as the scalar field ϕ in Dicke-Brans-Jordan[2] theory. We include all these contributions by basing our analysis on the resultant Riemann tensor, the only directly measurable field. Our analysis also applies to waves that are approximately, rather than exactly, null.[7,10] Details will be published elsewhere.[11]

Our main result is that the Riemann tensor of the most general wave is composed of *six* modes of polarization, which are expressible in terms of the six "electric" components R_{i0j0} (i, j spatial) that govern driving forces in a detector.[12] Consequently, *currently feasible detectors can obtain all measurable information contained in the most general wave permitted by any metric theory of gravity.* It is important that future experiments be designed to measure all six "electric" components.

The amplitudes of the six polarization modes are related to the "electric" components R_{i0j0} in the following manner: Use coordinates $txyz$; let the wave propagate in the $+z$ direction. The six amplitudes are, in the notation of Newman and Penrose,[13] two real functions $\Psi_2(u)$, $\Phi_{22}(u)$ and the real and imaginary parts of two complex functions $\Psi_3(u)$, $\Psi_4(u)$, where $u \equiv t - z/c$ is the "retarded time." Then

$$\Psi_2 = -\tfrac{1}{6}R_{z0z0},$$

$$\Psi_3 = \tfrac{1}{2}(-R_{x0z0} + iR_{y0z0}),$$

$$\Psi_4 = R_{y0y0} - R_{x0x0} + 2iR_{x0y0},$$

$$\Phi_{22} = -(R_{x0x0} + R_{y0y0}).$$

Figure 1 shows the action of each mode on a sphere of test bodies. Ψ_4 and Φ_{22} are purely transverse, Ψ_2 is purely longitudinal, and Ψ_3 is mixed. General relativity permits only the two Ψ_4 modes.

The entire Riemann tensor of any observed wave can be reconstructed from these amplitudes.

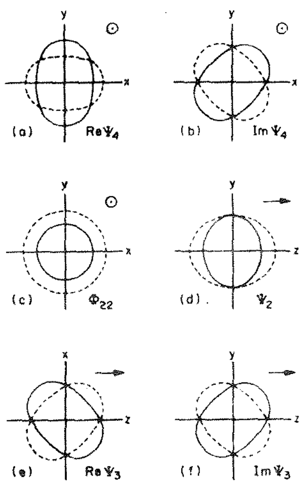

FIG. 1. The six polarization modes of a weak, plane, null gravitational wave permitted in the generic metric theory of gravity. Shown is the displacement that each mode induces on a sphere of test particles. The wave propagates in the $+z$ direction (arrow at upper right) and has time dependence $\cos(\omega t)$. Solid line, snapshot at $\omega t = 0$; the broken line, one at $\omega t = \pi$. There is no displacement perpendicular to the plane of the figure.

Comparison with waves permitted by various metric theories of gravity then allows one to rule out some theories. To facilitate this comparison, we have set up a classification scheme for waves based on the properties of the six amplitudes under certain Lorentz transformations. We choose[14] a restricted set of "standard observers" such that (a) each observer sees the wave traveling in the $+z$ direction, and (b) each observer sees the same Doppler shift, e.g., each measures the same frequency for a monochromatic wave. These standard observers are related by the subgroup of Lorentz transformations that leaves the wave vector $\vec{k}$, $\vec{k} \equiv \nabla u$, invariant ("little group"). The six amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ are generally observer dependent. However, there are certain "invariant" statements about them that are true for all standard observers if they are true for one. These statements characterize *invariant classes of waves*:

Class II_6: $\Psi_2 \neq 0$. All standard observers measure the same nonzero amplitude in the Ψ_2 mode. (But the presence or absence of all other modes is observer dependent.)

Class III_5: $\Psi_2 \equiv 0 \neq \Psi_3$. All standard observers measure the absence of Ψ_2 and the presence of Ψ_3. (But the presence or absence of Ψ_4 and Φ_{22} is observer dependent.)

Class N_3: $\Psi_2 \equiv 0 \equiv \Psi_3$, $\Psi_4 \neq 0 \neq \Phi_{22}$. Presence or absence of all modes is independent of observer.

Class N_2: $\Psi_2 \equiv 0 \equiv \Psi_3$, $\Psi_4 \neq 0 \equiv \Phi_{22}$. Independent of observer.

Class O_1: $\Psi_2 \equiv 0 \equiv \Psi_3$, $\Psi_4 \equiv 0 \neq \Phi_{22}$. Independent of observer. Class II_6 is the most general; as one demands that successive amplitudes vanish identically, one descends to less and less general classes. The class of the most general permitted wave in some currently viable metric theories is, for general relativity,[1] N_2; Dicke-Brans-Jordan,[2] N_3; Will-Nordtvedt,[3] III_5; Hellings-Nordtvedt,[4] N_3; Ni's new theory,[5] II_6; and Lightman-Lee,[6] II_6. All these but Dicke-Brans-Jordan theory can be adjusted to have the same post-Newtonian limit as general relativity, for certain choices of possible cosmological models and arbitrary theory parameters.

We see that measuring the polarization of gravitational waves provides a sharp experimental test of theories of gravity. The class of the "correct" theory is at least as general as that of any observed wave. The observation of a wave more general than N_2 would contradict general relativity but would be consistent with other viable theories.[2-6] Weber[15] has initiated such experiments by searching for the Φ_{22} mode, with negative results.

To test theories, an experimenter must classify the waves that he detects. If he knows the direction of a wave *a priori* (e.g., from a particular supernova), he can directly extract the amplitude of each mode from his data and determine the class. If he does not know the direction, he *cannot* extract the amplitudes or determine the direction without applying some further assumption to his data (e.g., that the wave is no more general than N_3 and is therefore purely transverse). But he can always place limitations on what the class may be (e.g., if driving forces in his detector do not remain in one plane, the wave must be more general than N_3, i.e., II_6 or III_5).

We now sketch the arguments that lead to these results. Consider a weak, plane, null wave described by a linearized Riemann tensor $R_{\alpha\beta\gamma\delta}(u)$, with $\nabla u \cdot \nabla u = 0$. Work in an approximately constant quasiorthonormal null tetrad[13] $(\vec{k}, \vec{l}, \vec{m}, \vec{m}^*)$, where $\vec{k} = \nabla u$. The Bianchi identities imply that there are six functionally independent real components of the Riemann tensor; take them to be $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$, as above. (The other components are $\Phi_{21} = \Psi_3$, $-2\Lambda = \frac{2}{3}\Phi_{11} = \Psi_3$, $\Phi_{00} = \Phi_{01} = \Phi_{02} = \Psi_0 = \Psi_1 = 0$.) Consider the "little group"[16] $E(2)$ of Lorentz transformations of the tetrad which fix $\vec{k}$: $\vec{k}' = \vec{k}$, $\vec{m}' = e^{i\varphi}(\vec{m} + \alpha\vec{k})$, $\vec{l}' = \vec{l} + \alpha^*\vec{m} + \alpha\vec{m}^* + \alpha\alpha^*\vec{k}$, where α is complex and φ is a real phase. The action of $E(2)$ on the amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ is

$$\begin{aligned} &\Psi_2' = \Psi_2, \quad \Psi_3' = e^{-i\varphi}(\Psi_3 + 3\alpha^*\Psi_2), \\ &\Psi_4' = e^{-2i\varphi}(\Psi_4 + 4\alpha^*\Psi_3 + 6\alpha^{*2}\Psi_2), \\ &\Phi_{22}' = \Phi_{22} + 2\alpha\Psi_3 + 2\alpha^*\Psi_3^* + 6\alpha^*\alpha\Psi_2. \end{aligned} \tag{1}$$

The invariant classes of waves that are defined above correspond precisely to the different representations of $E(2)$ that can arise through Eqs. (1).

The helicity (spin) decomposition of a wave is $E(2)$ invariant *only* for classes N_3, N_2, and O_1. Theories in classes N_3, N_2, and O_1 provide a unitary representation of $E(2)$ which is a direct sum of one-dimensional massless-particle representations,[16-18] containing at most spins $0, \pm 2$. Theories in classes II_6 and III_5 provide a reducible representation of $E(2)$ which is not completely reducible and is therefore nonunitary[18]; it is likely that such theories cannot be quantized. No other representation of $E(2)$ (such as one with "continuous spin"[18]) can occur.

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ii) Details (Paper IX; collaboration with D.M. Eardley and D.L. Lee, Published in <u>Phys. Rev. D</u>, **8**, 3308, 1973)

Gravitational-Wave Observations as a Tool for Testing Relativistic Gravity*

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ABSTRACT

Gravitational-wave observations can be powerful tools in the testing of relativistic theories of gravity — perhaps the only tools for distinguishing between certain extant theories in the foreseeable future. In this paper we examine gravitational radiation in the far field using a formalism that encompasses all "metric theories of gravity." There are six possible modes of polarization, which can be completely resolved by feasible experiments. We set forth a theoretical framework for classification of waves and theories, based on the Lorentz transformation properties of the six modes. We also show in detail how the six modes may be experimentally identified and to what extent such information limits the "correct" theory of gravity.

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†Imperial Oil Predoctoral Fellow.

I. INTRODUCTION

Within the past few years, as experimental tests of gravity have been analyzed and refined, and as gravitation theories have been systematically compared,[1] most extant theories have been ruled out. Indeed, analysis of data from existing "solar system" experiments promises to distinguish more and more clearly between the theories that today remain viable. [For example, within the next two years, a search for the Nordtvedt effect[2] in lunar laser ranging data[3] should either rule out general relativity (GRT),[4] or place a limit of $\omega > 30$ on the Dicke coupling constant of Dicke-Brans-Jordan theory.[5]] An elegant theoretical formalism, the "Parametrized Post-Newtonian" (PPN) framework,[6] exists for analysis of metric theories[7] in the limit of weak gravitation and slow motion. All gravitation experiments that have played key roles in ruling out theories, except the Eötvös-Dicke experiment,[8] fall within the PPN framework. The Eötvös-Dicke experiment itself probably forces the "correct" theory of gravity to be a metric theory[7,9] and, in fact, there are no known complete[7] nonmetric theories which do not violate the Eötvös-Dicke experiment.

In the last year or so, it has become evident that the PPN framework has fundamental limitations. New metric theories of gravity,[10-13] with widely varying structures, have been invented which are virtually indistinguishable from one another and from GRT in the post-Newtonian limit. Existing and proposed solar-system experiments cannot hope to distinguish between such theories in the foreseeable future. There is, however, a strong element of hope: that new theories[10-13] and GRT differ markedly in the observable properties of their gravitational waves. With this motivation, we have embarked upon a program to develop a theoretical foundation for the analysis of gravitational waves in arbitrary metric theories of gravity — a foundation which is theory-independent and analogous to the PPN framework. (Gravitational-wave phenomena fall outside of the PPN framework.) We feel that experiments to detect gravitational waves from astronomical sources can prove to be a powerful experimental tool, in the foreseeable future, for ruling out gravitation theories.

The idea of building a theory-independent framework for analyzing gravitational-wave experiments was first conceived in mid-1972 by Robert V. Wagoner.[14] At about the same time, and independently, our group was analyzing the gravitational-wave properties of a particular metric theory — one that two of us had recently invented.[13] When our analysis was near completion (several months after we learned of Wagoner's ideas), we suddenly realized that our theory exhibits the most general type of gravitational wave admitted by any metric theory — and that, therefore, with a mere change of viewpoint, our analysis would become the general framework that Wagoner had proposed constructing. Upon contacting Wagoner we discovered that he and Clifford M. Will had already proceeded far toward the construction of this same framework. We therefore published a brief account of the framework jointly with them in Physical Review Letters.[15] This paper presents a more detailed account of our "Caltech" version of the framework.

In a future paper we hope to treat the generation of waves by particular sources in arbitrary theories and thereby "move in from the far field."

Our fundamental results are that the most general null or nearly null wave has six independent polarization modes, which can be classified according to their behavior under Lorentz transformations. Various theories admit some subset (perhaps all) of the six possible modes. If the wave direction is known, the modes can be resolved uniquely by feasible experiments; if the direction of the wave is not known, partial but not complete resolution can be obtained. In either case detection information limits the correct theory of gravity.

Section II summarizes the properties of the general waves while Sec. III gives the details of derivations. Section IV discusses application to particular theories and their classification within the formalism; Sec. V gives a complete prescription of how to analyze and classify waves that are observed by means of gravitational-wave detectors. (For a review of the prospects of gravitational-wave astronomy, we refer the reader to Ref. 16.)

II. PROPERTIES AND CLASSIFICATION OF WEAK, PLANE, NULL WAVES: A SUMMARY OF RESULTS

A. Definition of Gravitational Waves in Metric Theories

In any metric theory of gravity,[7] just as in GRT, the response of matter to gravity is determined solely by a universal, covariant coupling to the physical metric $\underset{\sim}{g}$ (Einstein's Equivalence Principle[7]). The equation of motion of matter is given by[17]

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{T} = 0,$$

where $\underset{\sim}{\nabla}$ is the covariant derivative associated with $\underset{\sim}{g}$, and $\underset{\sim}{T}$ is the matter stress-energy tensor. This equation ensures that test particles and photons travel along time-like and null geodesics of $\underset{\sim}{g}$, respectively. Metric theories differ only in the manner that matter acts back to generate $\underset{\sim}{g}$ — i.e., only in their gravitational field equations. Some theories postulate auxiliary gravitational fields, such as the scalar field ϕ in Dicke-Brans-

Jordan theory,[5] which enter into the field equations but do not act on matter directly.

It is the universality of the coupling to the metric that permits a theory-independent discussion of the propagation and detection of gravitational waves for metric theories. On the other hand, the emission of gravitational waves involves the detailed structure of field equations, and is therefore theory-dependent. Emission will not be treated in this paper.

Consider an experiment employing matter of negligible self-gravity in a local region to measure the static or wavelike gravitational fields from faraway sources. One cannot define the absolute acceleration due to gravity at a point in the region (Einstein's Equivalence Principle[7]); only the relative, tidal acceleration between two points has observable significance. The Riemann tensor $\underset{\sim}{\text{Riem}}$, formed from $\underset{\sim}{g}$, determines these relative accelerations, and is the sole locally observable imprint of gravity.

Consider a freely falling observer at any fiducial point P in the region. Let him set up an approximately Lorentz, normal coordinate system

$$\{x^{\mu}\} = \{t, x^{i}\} \ ,$$

with P as origin. For a particle with spatial coordinates x^{i} at rest or with nonrelativistic velocity in the region, the acceleration relative to P is (for sufficiently small $|x^{j}|$)

$$a_{i}^{\text{GRAV}} = - R_{i0j0} \, x^{j} \ , \tag{1}$$

where R_{i0j0} are the so-called "electric" components of the $\underset{\sim}{\text{Riem}}$ due to waves or other external gravitational influences.

A gravitational wave in a metric theory involves the metric field $\underset{\sim}{g}$ and any auxiliary gravitational fields that might exist. But the resultant $\underset{\sim}{\text{Riem}}$ is the only measurable field. So for this paper we define a "gravitational wave" in terms of its $\underset{\sim}{\text{Riem}}$: A "weak, plane, null wave" in a metric theory is a weak, propagating, vacuum gravitational field characterized, in some nearly Lorentz coordinate system, by a linearized $\underset{\sim}{\text{Riem}}$ with components that depend only upon a null "retarded time," $u \equiv t - z/c$:

$$R_{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau}(u).$$

$\underset{\sim}{\nabla} u$, which is proportional to the wave vector, is null with respect to the physical metric $\underset{\sim}{g}$: $\underset{\sim}{\nabla} u \cdot \underset{\sim}{\nabla} u = 0$. In "$u = t - z/c$," c is the speed of light, and the coordinates are oriented such that the wave travels in the +z direction.

Two restrictions appear in this definition: (i) Waves must travel at exactly the local speed of light, (ii) waves must be exactly plane. These restrictions turn out to be good approximations in feasible experiments for all viable metric theories of gravity; see Sec's. III and IV for a discussion of these points.

The fundamental properties of these waves follow immediately from the algebraic and differential identities that $\underset{\sim}{\text{Riem}}$ obeys. There are six algebraically independent components of $\underset{\sim}{\text{Riem}}$ in vacuum, (Sec. III proves this assertion and succeeding ones), which correspond to six modes of polarization. In a given, nearly Lorentz coordinate frame of the above type, group these six components into amplitudes of definite helicity s (where $s = 0, \pm 1, \pm 2$) under rotations about the z-axis. There arise two real amplitudes

$$\Psi_2(u),\ (s = 0)\ ;\ \Phi_{22}(u),\ (s = 0)\ ,$$

and two complex amplitudes

$$\Psi_3(u),\ (s = \pm 1)\ ;\ \Psi_4(u)\ ,\ (s = \pm 2).$$

Here and throughout this paper one complex amplitude is equivalent to two real amplitudes. We will always describe a gravitational wave by its six amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ in the six polarization modes of a given coordinate frame.

These amplitudes are related to the "electric" components of $\underset{\sim}{\text{Riem}}$, which govern relative accelerations through Eq. (1), by

$$\Psi_2(u) = -\frac{1}{6} R_{z0z0}(u)\ , \tag{2a}$$

$$\Psi_3(u) = -\frac{1}{2} R_{x0z0} + \frac{i}{2} R_{y0z0}\ , \tag{2b}$$

$$\Psi_4(u) = -R_{x0x0} + R_{y0y0} + 2i\, R_{x0y0}\ , \tag{2c}$$

$$\Phi_{22}(u) = -R_{x0x0} - R_{y0y0}\ . \tag{2d}$$

Figure 1 shows the displacement that each polarization mode induces on a sphere of test particles; Ψ_4 and Φ_{22} are purely transverse, Ψ_2 is purely longitudinal, and Ψ_3 is mixed. If an experimenter knows the wave direction, he can uniquely determine $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ by measuring the driving forces in his detector (see Sec. V for further details), and he can reconstruct $\underset{\sim}{\text{Riem}}$. Therefore, currently feasible detectors can obtain all the measurable information in the most general wave permitted by any metric theory.

B. Lorentz-Invariant E(2) Classification of Plane Waves

In any metric theory, the local nongravitational laws of physics are those of special relativity. So it is fruitful to sort waves into Lorentz-invariant classes, depending on the behavior of the amplitudes under Lorentz transformations. Observers in different Lorentz frames (e.g., in relative motion) can then agree on the classification of any wave.

Rather than use the entire Lorentz group relating observers in all frames, we choose a restricted set of "standard observers" such that (i) each observer sees the wave travelling in his +z direction, and (ii) each observer sees the same Doppler shift, e.g., each measures the same frequency for a monochromatic wave. These standard observers are related by the subgroup of Lorentz transformations that leaves the vector $\underset{\sim}{\nabla} u$ invariant ["little group, E(2)"]. The parts of the Lorentz group left out of the little group are (a) [due to requirement (i)] pure rotations of $\underset{\sim}{\nabla} u$ which merely change the direction of wave propagation, and (b) [due to requirement (ii)] pure boosts along $\underset{\sim}{\nabla} u$ which merely change the observed frequency and scale each amplitude up or down independently. Without requirement (ii), different observers would see the wave travelling along the +z direction, but generally at different Doppler shifts. The subgroup relating the standard observers would be bigger (4 dimensional), but the invariant classes would be the same.

The six amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ of a wave are generally observer-dependent; their transformation law is given in Sec. III. However, there are certain "invariant" statements about them that are true for all standard observers if they are true for any one. These statements characterize <u>invariant "E(2) classes" of waves</u>: (Notation is explained in Sec. III.)

Class II_6. $\Psi_2 \not\equiv 0$. All standard observers measure the same nonzero amplitude in the Ψ_2 mode. (But the presence or absence of all other modes is observer-dependent.)

Class III_5. $\Psi_2 \equiv 0 \not\equiv \Psi_3$. All standard observers measure the absence of Ψ_2 and the presence of Ψ_3. (But the presence or absence of Ψ_4 and Φ_{22} is observer-dependent.)

Class N_3. $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \not\equiv 0 \not\equiv \Phi_{22}$. Presence or absence of all modes is independent of observer.

Class N_2. $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \not\equiv 0 \equiv \Phi_{22}$. Independent of observer.

Class O_1. $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \equiv 0 \not\equiv \Phi_{22}$. Independent of observer.

Class O_0. $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \equiv 0 \equiv \Phi_{22}$. Independent of observer. All standard observers measure no wave.

Class II_6 is the most general. As one demands that successive amplitudes vanish identically, one descends to less and less general classes. Figure 2 exhibits these relations of generality among the classes. In this paper, "more (or less) general" for classes always refers to Fig. 2. (For example: O_1 is less general than N_3, III_5, and II_6, but neither more nor less general than N_2.) The E(2) class of a particular metric theory is defined as the class of its most general wave (see Sec. IV for illustrations).

The fundamental theoretical implication of our paper is that the class of the "correct" theory of gravity is at least as general as the class of any observed wave.

Once theorists are confident of a particular classical theory of gravity, they will wish to quantize it. Then it should be possible to associate the amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ with massless quanta of definite and Lorentz-invariant helicity. Section III demonstrates that the helicity content of class II_6 is not Lorentz-invariant, nor is that of III_5.

Furthermore, an associated pathology arises for these classes: The amplitudes form a nonunitary representation of the inhomogeneous Lorentz group, contradicting the tenets of relativistic quantum mechanics.[18] Attempts to quantize theories of Class II_6 or III_5 will therefore face grave difficulties.

These difficulties do not arise for theories of class N_3 or less general: There Ψ_4 and Φ_{22} act like massless quantum fields with $s = \pm 2$ and 0.

III. DERIVATIONS

This section may be skipped without essential loss of continuity.

A. Tetrad Components of Riem for Waves

A quasiorthonormal, null tetrad basis[19] is especially suitable for discussing null waves. At any point P, the null tetrad $(\underset{\sim}{k}, \underset{\sim}{\ell}, \underset{\sim}{m}, \overline{\underset{\sim}{m}})$ is related to the cartesian tetrad introduced in Sec. II by

$$\underset{\sim}{k} = (2)^{-\frac{1}{2}} (\underset{\sim}{e}_{\hat{t}} + \underset{\sim}{e}_{\hat{z}}) \quad , \tag{3a}$$

$$\underset{\sim}{\ell} = (2)^{-\frac{1}{2}} (\underset{\sim}{e}_{\hat{t}} - \underset{\sim}{e}_{\hat{z}}) \quad , \tag{3b}$$

$$\underset{\sim}{m} = (2)^{-\frac{1}{2}} (\underset{\sim}{e}_{\hat{x}} + i\underset{\sim}{e}_{\hat{y}}) \quad , \tag{3c}$$

$$\overline{\underset{\sim}{m}} = (2)^{-\frac{1}{2}} (\underset{\sim}{e}_{\hat{x}} - i\underset{\sim}{e}_{\hat{y}}) \quad . \tag{3d}$$

Throughout this section we follow Sec. II in orienting the axes such that the wave travels in the +z direction; $u \equiv t - z/c$. Equivalently, we choose $\underset{\sim}{k}$, one of the tetrad legs, proportional to the vector $\underset{\sim}{\nabla} u$. It is easily verified from Eqs. (3) that the tetrad vectors obey the relations:

$$-\underset{\sim}{k} \cdot \underset{\sim}{\ell} = \underset{\sim}{m} \cdot \overline{\underset{\sim}{m}} = 1 \quad , \tag{4}$$

while all other dot products vanish.

We adopt the following notation for null-tetrad components of tensors $\underset{\sim}{X}$:

$$X_{abc\ldots} = X_{\mu\nu\sigma\ldots} \, a^{\mu} b^{\nu} c^{\sigma} \ldots \quad , \tag{5}$$

where $(a,b,c\ldots)$ range over $(k,\ell,m,\overline{m})$.

Central to our later discussions will be the transformation properties of the components of $\underset{\sim}{\text{Riem}}$ under the action of some subgroup of the Poincaré group. In view of this, we first split $\underset{\sim}{\text{Riem}}$ into irreducible parts: the Weyl tensor, the traceless Ricci tensor and the Ricci scalar. We follow Newman and Penrose[19] in naming their tetrad components Ψ, Φ, and Λ respectively.

In general, the ten ψ's, nine ϕ's, and Λ are all algebraically independent. When we restrict ourselves to nearly plane waves, however, we find that the differential and algebraic properties of $\underset{\sim}{\text{Riem}}$ reduce the number of independent components to six by the following arguments:

Consider a weak, plane, null wave. It is characterized by the fact that the components of its $\underset{\sim}{\text{Riem}}$ are functions of the retarded time u only. Of their derivatives, only those with respect to the retarded time u will be nonvanishing:

$$R_{abcd,p} = 0 \quad , \tag{6}$$

where (a,b,c,d) range over $(k,\ell,m,\overline{m})$ while $(p,q,r,\ldots)$ range over $(k,m,\overline{m})$ only.

The covariant differential Bianchi identities and the symmetry properties of $R_{\mu\nu\sigma\tau}$ are necessary and sufficient to guarantee that the linearized $\underset{\sim}{\text{Riem}}$ is derivable from a metric perturbation,[20]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ . \tag{7}$$

Using Eq. (6) we see that these identities imply the relations

$$R_{ab[pq,\ell]} = 0 = \frac{1}{3} R_{abpq,\ell} \quad , \tag{8}$$

where ℓ is a fixed index. Equation (8) implies that

$$R_{abpq} = 0 = R_{pqab} \; , \tag{9}$$

except for a trivial, nonwavelike constant. Consequently, all nonvanishing components of Riem must have the form $R_{p\ell q\ell}$. Taking into account the symmetries of Riem, we thus see that there are only six independent, non-vanishing components. Corresponding simplifications are induced among the Newman-Penrose quantities. For a plane wave, they are[19]:

i) Weyl Tensor

$$\Psi_0 = \Psi_1 = 0 \; , \tag{10a}$$

$$\Psi_2 = -\frac{1}{6} R_{\ell k \ell k} \; , \tag{10b}$$

$$\Psi_3 = -\frac{1}{2} R_{\ell k \ell \bar{m}} \; , \tag{10c}$$

$$\Psi_4 = - R_{\ell \bar{m} \ell \bar{m}} \quad , \tag{10d}$$

ii) Traceless Ricci Tensor

$$\Phi_{00} = \Phi_{01} = \Phi_{10} = \Phi_{02} = \Phi_{20} = 0 \; , \tag{11a}$$

$$\Phi_{22} = - R_{\ell m \ell \bar{m}} \; , \tag{11b}$$

$$\Phi_{11} = \frac{3}{2} \Psi_2 \; , \tag{11c}$$

$$\Phi_{12} = \bar{\Phi}_{21} = \bar{\Psi}_3 \; , \tag{11d}$$

iii) Ricci Scalar

$$\Lambda = -\frac{1}{2} \Psi_2 \; . \tag{12}$$

As indicated in Sec. II, we shall choose the set $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ (Ψ_3 and Ψ_4 complex) to describe, in a given null frame, the six independent components of a wave in the generic metric theory. Equations (10) and (11) give the members of this set in terms of the null-tetrad components of the Riemann tensor. Equations (2) give the members of the set in terms of the directly observable "electric" components of the Riemann tensor.

In those cases where one calculates the Riemann tensor from a metric perturbation $h_{\mu\nu}$,[21] Eq. (7), the relation between $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ and derivatives of h_{ab} may be found in Appendix 1.

B. Behavior of Tetrad Components under Lorentz Transformation

Consider two standard observers O and O', with tetrads $(\underset{\sim}{k}, \underset{\sim}{\ell}, \underset{\sim}{m}, \underset{\sim}{\overline{m}},)$ and $(\underset{\sim}{k}', \underset{\sim}{\ell}', \underset{\sim}{m}', \underset{\sim}{\overline{m}}')$; then $\underset{\sim}{k} = \underset{\sim}{k}' \propto \underset{\sim}{\nabla} u$. Suppose O has measured the amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ of a wave; how do we predict the amplitudes $\{\Psi_2',$ $\Psi_3', \Psi_4', \Phi_{22}'\}$ measured by O'?

In group-theoretic language, we are asking the transformation properties of the amplitudes under the "little group" of Lorentz transformations that leaves the wave vector fixed. The various group representations formed by the amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ provide us with a means for classifying waves.

The most general proper Lorentz transformation relating the tetrads that keeps $\underset{\sim}{k}$ fixed is[22]

$$\underset{\sim}{k}' = \underset{\sim}{k} \, , \tag{13a}$$

$$\underset{\sim}{m}' = \exp(i\varphi)(\underset{\sim}{m} + \alpha \underset{\sim}{k}) \, , \tag{13b}$$

$$\underset{\sim}{\overline{m}}' = \exp(-i\varphi)(\underset{\sim}{m} + \overline{\alpha} \underset{\sim}{k}) \, , \tag{13c}$$

$$\underset{\sim}{\ell}' = \underset{\sim}{\ell} + \overline{\alpha}\underset{\sim}{m} + \alpha\underset{\sim}{\overline{m}} + \alpha\overline{\alpha}\underset{\sim}{k} \, , \tag{13d}$$

where α is an arbitrary complex number that produces "null rotations,"[23] (particular combinations of boosts and rotations), while φ, which ranges from 0 to 2π, is an arbitrary real phase that produces a rotation about $\underset{\sim}{e}_{\hat{z}}$. The transformations described in Eqs. (13) form a subgroup of the Lorentz group which is globally isomorphic to the abstract Lie group E(2), the group of proper rigid motions in the Euclidean 2-plane. In the latter group, φ represents the rotations in the plane and α, the translations. We denote a particular element of E(2) in Eqs. (13) by (φ,α). The law of composition is $(\varphi',\alpha')(\varphi,\alpha) = (\varphi' + \varphi,\ \alpha' + \exp(i\varphi')\alpha)$.

The transformation induced on the amplitudes of a wave by (φ,α) is

$$\Psi_2' = \Psi_2 , \tag{14a}$$

$$\Psi_3' = e^{-i\varphi} (\Psi_3 + 3\bar{\alpha}\Psi_2) , \tag{14b}$$

$$\Psi_4' = e^{-2i\varphi} (\Psi_4 + 4\bar{\alpha}\Psi_3 + 6\bar{\alpha}^2\Psi_2) , \tag{14c}$$

$$\Phi_{22}' = \Phi_{22} + 2\alpha\Psi_3 + 2\bar{\alpha}\bar{\Psi}_3 + 6\alpha\bar{\alpha}\Psi_2 . \tag{14d}$$

Now consider a set of observers related to one another by z-axis rotations $(\varphi,0)$. A quantity M that transforms under these rotations as $M' = \exp(is\varphi)\,M$ is said to have helicity s as seen by these observers. We see from Eqs. (14) that the amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ are helicity eigenstates. Furthermore, their helicity values can be read off easily from Eqs. (14), (setting $\alpha = 0 = \bar{\alpha}$):

$$\Psi_2 : s = 0 \tag{15a}$$

$$\Psi_3 : s = -1 , \quad \bar{\Psi}_3 : s = +1 , \tag{15b}$$

$$\Psi_4 : s = -2 , \quad \bar{\Psi}_4 : s = +2 , \tag{15c}$$

$$\Phi_{22} : s = 0 . \tag{15d}$$

C. E(2) Classification of Waves

It is evident from Eqs. (14) that the various amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ cannot be specified in an observer-independent manner. [Example: O may measure a wave to have as its only nonvanishing amplitude Ψ_2(helicity 0), while O', in relative motion with respect to O, may conclude that the wave has, in addition, Ψ_3 and Ψ_4 components (helicities 0, 1, and 2).] We classify waves in an E(2)-invariant manner by uncovering all representations of E(2) embodied in Eqs. (14). Each such representation, in which certain of the amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ vanish identically, is a distinct, invariant class. The name of each class is composed of the Petrov type of its nonvanishing Weyl tensor[24] (except that we do not distinguish between II and D) and the maximum number of nonvanishing amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ as seen by any observer (dimension of representation). Both the Petrov type and the dimension of representation are independent of observer.

The various classes were delineated in Sec. II, they are:

Class II_6. $\Psi_2 \not\equiv 0$.

Class III_5. $\Psi_2 \equiv 0 \not\equiv \Psi_3$.

These two classes form reducible, indecomposable representations of E(2). (See Appendix 2 for a brief resumé of the relevant group — theoretic concepts.) The maximal invariant proper subspace is the 3-dimensional one spanned by Ψ_4 and Φ_{22}. The helicity content of classes II_6 and III_5 is observer-dependent.

Class N_3. $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \not\equiv 0 \not\equiv \Phi_{22}$.

Class N_2. $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \not\equiv 0 \equiv \Phi_{22}$.

Class O_1. $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \equiv 0 \not\equiv \Phi_{22}$.

Classes N_3, N_2, and O_1 form decomposable representations of E(2) which decompose into 1-dimensional invariant subspaces spanned by Ψ_4 and Φ_{22} respectively. Each of these invariant subspaces forms a unitary, massless-particle representation of definite, Lorentz invariant, helicity (spin). They are well studied as they occur in relativistic quantum field theory.[25]

Class O_0. $\Psi_2 \equiv 0 \equiv \Psi_3$; $\Psi_4 \equiv 0 \equiv \Phi_{22}$.

Class O_0 forms the trivial representation.

The foregoing classification scheme is patterned closely after Wigner's classic analysis[26] of wave functions of relativistic quantum particles as members of unitary, irreducible representations of the Poincaré group.[27] Wigner showed that each such wave function may be taken to have a definite 4-momentum $\underset{\sim}{q}$, and to transform as a member of some unitary, irreducible representation of the little group that leaves $\underset{\sim}{q}$ invariant. One determines the "spin" of the particle from the eigenvalues of the helicity operator and its square; the spin of the particle is completely determined once the representation formed by its associated wave functions under the little group is known.

For our gravitational waves, $\underset{\sim}{\nabla} u$ is null and nonvanishing, and the little group is E(2). Unfortunately, Wigner's analysis does not apply since we are not restricted to unitary representations of E(2). In fact, as we have seen, the representations generated by $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ are in general nonunitary and indecomposable. The amplitudes in classes II_6 and III_5 cannot be identified with massless particle fields. Consequently, it is impossible to give a spin decomposition for these waves.

A representation which is reducible and indecomposable can never be unitary. This applies to the little group E(2), and hence also to the

Poincaré group. In relativistic quantum theory, all invariance groups must be realized by unitary representations.[18] We therefore obtain the following result: If a theory is of class II_6 or III_5, it is impossible to quantize it in a way that is Poincaré invariant with respect to the local Lorentz metric.

D. Spherical Waves

Thus far, we have based our discussions on the properties of plane waves. The most physically satisfactory definition of a radiation field is one that carries energy off to infinity from a bounded source. For metric theories of gravity, this corresponds to that part of the Riemann tensor that falls off as 1/(distance) asymptotically. Far away from radiating sources, one may locally approximate these approximately spherical waves as plane waves. The following argument shows in a theory-independent manner that the plane wave approximation will not affect the classification scheme.

Adopt a (u,r,θ,φ) coordinate system in the wave zone, which is assumed to be almost Minkowskian. The line element is given by

$$ds^2 = -du^2 + 2dudr + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2). \qquad (16)$$

Place the origin of the coordinate system somewhere inside the source. Single out the 1/r part of the outgoing spherical waves:

$$R_{abcd} = \frac{1}{r} S_{abcd} (u,\theta,\varphi) + O\left(\frac{1}{r^2}\right). \qquad (17)$$

In the wave zone, observer O $[r = r_0, \theta = \varphi = 0]$ carries with himself a Cartesian tetrad $(\underset{\sim}{e}_{\hat{t}}, \underset{\sim}{e}_{\hat{x}}, \underset{\sim}{e}_{\hat{y}}, \underset{\sim}{e}_{\hat{z}})$ oriented such that $\underset{\sim}{e}_{\hat{z}}$ is along the incident direction of the wave. The two coordinate systems are related by

$$u = t - z \; , \tag{18a}$$

$$r = z + r_0 , \tag{18b}$$

$$\theta = \frac{x}{r_0} + O\left(\frac{1}{r_0^{\;2}}\right) \; , \tag{18c}$$

$$\varphi = \frac{y}{r_0} + O\left(\frac{1}{r_0^{\;2}}\right) \; . \tag{18d}$$

Thus O would measure

$$R_{abcd} = \frac{1}{r_0} S_{abcd}\left(u, \frac{x}{r_0}, \frac{y}{r_0}\right) + O\left(\frac{1}{r_0^{\;2}}\right) . \tag{19}$$

The differential Bianchi identities then imply

$$0 \equiv R_{ab[pq;c]} = O(1/r_0^{\;2}), \quad \text{if } c \neq \ell \; , \tag{20a}$$

$$0 \equiv R_{ab[pq;\ell]} = \frac{1}{3}\frac{1}{r_0} S_{abpq,\ell} + O(1/r_0^{\;2}) \; , \tag{20b}$$

where semicolon and comma denote covariant and partial differentiation respectively. It follows immediately from Eqs. (20) that the classification scheme based on the $1/r$ part of the Riemann tensor is identical to that based on the plane waves.

IV. APPLICATIONS TO PARTICULAR THEORIES

A. Two-Metric Theories

In all of the preceding discussion we have assumed that the components of the Riemann tensor are functions of the retarded time associated with the "physical metric" $g_{\alpha\beta}$, i.e.,

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}(u) \ , \tag{21a}$$

where

$$u_{,\alpha} u_{,\beta} g^{\alpha\beta} = 0 \quad . \tag{21b}$$

This is indeed the proper approach, since the physical metric is associated with the physical local Lorentz frames, which are in turn the basis for our classification scheme. In some theories of gravity,[10,13] however, gravitational waves travel along null geodesics of a "flat space," global, background metric $\tilde{\eta}$, while electromagnetic waves (and neutrinos) travel along null geodesics of the physical metric $\tilde{g}$. Equations (21) are then not rigorously satisfied. On the other hand, if $\tilde{g}$ differs from $\tilde{\eta}$ locally by only a small amount in the above-mentioned theories, Eqs. (21) are approximately correct and all of the formalism developed in Secs. II and III is applicable to a high degree of accuracy. In all such "two-metric" theories that we have studied, present experimental limits on "preferred-frame effects"[1,11] require, in the mean rest frame of the solar system,

$$\frac{|g_{\alpha\beta} - \eta_{\alpha\beta}|}{|\eta_{\alpha\beta}|} < 10^{-2} \ , \tag{22}$$

where $|\eta_{\alpha\beta}|$ refers to the magnitude of a typical element of $\eta_{\alpha\beta}$, etc. In fact, if the difference between $g_{\alpha\beta}$ and $\eta_{\alpha\beta}$ is due entirely to solar system or galactic matter, then the 10^{-2} in Eq. (22) becomes 10^{-7}. Equation (22) is equivalent to the relation, again as measured in the mean rest frame of the solar system,

$$\frac{|c_g - c_{em}|}{c} < 10^{-2} \ , \tag{23}$$

where c_g and c_{em} are the speeds of gravitational and electromagnetic waves respectively. Thus, for all Lorentz observers who move at low speeds ($v << c$) with respect to the mean rest frame of the solar system, two-metric theories that are viable [in the sense of no preferred frame effects and so compliance with Eq. (22)] may be included in the formalism of Secs. II and III.

A further important point is that Eq. (23), a distinctive feature of two-metric theories, suggests that a search for time delays between simultaneously emitted gravitational and electromagnetic bursts could prove a valuable experimental tool. An experimental limit of $\lesssim 10^{-8}$ for $|c_g - c_{em}|/c$ would disprove most "two-metric" theories and would stringently constrain future theory-building. If current experimental efforts continue unabated, by 1980 one may detect gravitational-wave bursts from supernovae in the Virgo cluster ($\sim$ 3 supernovae per year). Then a limit of

$$|c_g - c_{em}|/c \lesssim 10^{-9} \times \text{(time-lag precision)}/\text{(1 week)}$$

will be possible.

B. Degrees of Freedom Versus Polarization Modes

We have enumerated the various independent gravitational wave modes in the general metric theory. This does not mean, however, that for a given theory the maximum number of nonvanishing modes for any observer is equal to the number of dynamical degrees of freedom[28] in the gravitational field. For a given theory, there may be <u>fewer</u> or <u>more</u> degrees of freedom than the number of modes; if fewer, amplitudes in the various modes are linearly dependent in a manner dictated by the detailed structure of the theory (see discussion following Stratified Theories below).

C. Classification of Particular Theories

Table I gives the E(2) classification (see Secs. II and III) of some metric theories in the literature (some of which have already been ruled out, e.g., the conformally flat and stratified theories[29]). The classification procedure involves examining the far-field, linearized, vacuum field equations of a theory and is illustrated below by several examples. In the examples, the relevant approximated vacuum equations of a theory will be quoted whenever necessary.

1. General Relativity[4]

$$R_{\alpha\beta} = 0 \tag{24a}$$

From Eqs. (10), (11), and (A1.3) one can deduce that

$$R_{\ell k \ell k} = R_{\ell m \ell \bar{m}} = R_{\ell k \ell m} = R_{\ell k \ell \bar{m}} = 0, \tag{24b}$$

or

$$\Psi_2 = \Psi_3 = \Phi_{22} = 0. \tag{24c}$$

Since there are no further constraints, $\Psi_4 \neq 0$ and the E(2) classification is N_2.

2. Dicke-Brans-Jordan Theory[5]

$$\Box\varphi = 0, \tag{25a}$$

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta}R = \omega\varphi^{-2} (\varphi_{,\alpha}\varphi_{,\beta} - \tfrac{1}{2} g_{\alpha\beta}\, \varphi_{,\gamma}\varphi^{,\gamma}) + \varphi^{-1}\, \varphi_{,\alpha,\beta}\, , \tag{25b}$$

$$R = \omega\varphi^{-2}\, \varphi_{,\gamma}\varphi^{,\gamma}\, . \tag{25c}$$

The monochromatic plane wave solution to Eq. (25a) is[30]

$$\varphi = \varphi_0 + \varphi_1\, e^{i\underset{\sim}{q} \cdot \underset{\sim}{x}} \tag{25d}$$

where φ_0 and φ_1 are constants and the wave vector $\underset{\sim}{q}$ is null. The quantity φ_0 is the cosmological boundary value of the scalar field, and φ_1 is a small amplitude of a wave (work only to first order in φ_1). Then from Eq. (25c),

$$R = 0, \tag{25e}$$

and Eq. (25b) yields

$$R_{\alpha\beta} = - \varphi_0^{-1} \varphi_1 e^{i\underset{\sim}{q} \cdot \underset{\sim}{x}} q_\alpha q_\beta . \tag{25f}$$

Thus $R_{\ell\ell}$ is the only nonvanishing tetrad component of the Ricci tensor and one can conclude that

$$R_{\ell k \ell k} = R_{\ell k \ell m} = R_{\ell k \ell \bar{m}} = 0 \neq R_{\ell m \ell \bar{m}} , \tag{25g}$$

or

$$\Psi_2 = \Psi_3 = 0 , \quad \Phi_{22} \text{ and } \Psi_4 \neq 0 . \tag{25h}$$

Therefore for the Dick-Brans-Jordan theory, the E(2) classification is N_3.

3. Will-Nordtvedt Theory[11]

$$\Box K_\alpha = 0 \tag{26a}$$

$$R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} = K_{\alpha,\gamma} K_\beta{}^{,\gamma} + K_{\gamma,\alpha} K^\gamma{}_{,\beta} - \tfrac{1}{2} g_{\alpha\beta} K^\gamma{}_{,\delta} K_\gamma{}^{,\delta}$$
$$+ \tfrac{1}{2}\left[K^\gamma (K_{\alpha,\beta} + K_{\beta,\alpha}) - K_\alpha (K^\gamma{}_{,\beta} + K_\beta{}^{,\gamma}) - K_\beta (K^\gamma{}_{,\alpha} + K_\alpha{}^{,\gamma})\right]_{,\gamma} . \tag{26b}$$

The plane wave solution to Eq. (26a) is

$$K_\alpha = A_\alpha e^{i\underset{\sim}{q} \cdot \underset{\sim}{x}} + B_\alpha , \tag{26c}$$

where A_α and B_α are constant vectors and the wave vector q is null. Again, assume A_α is small and work only to linear order in that quantity. The vector B_α is of cosmological origin. Taking the trace of Eq. (26b) and using Eqs. (26c), (A1.2b), and (A1.4), we obtain

$$R = 0 = \Psi_2 \quad . \tag{26d}$$

Equation (26b) then reads

$$R_{\alpha\beta} = e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}}\left[(\underset{\sim}{q}\cdot\underset{\sim}{A})\, q_{(\alpha}B_{\beta)} - (\underset{\sim}{B}\cdot\underset{\sim}{q})\, A_{(\alpha}q_{\beta)}\right] \quad . \tag{26e}$$

Equation (26e) indicates the relations

$$R_{\ell m} \neq 0 \ , \quad R_{\ell\bar{m}} \neq 0 \ , \quad R_{\ell\ell} \neq 0 \quad , \tag{26f}$$

or, from Eqs. (A1.3),

$$\Psi_3 \neq 0 \ , \quad \Phi_{22} \neq 0 \quad . \tag{26g}$$

Using Eqs. (26g), Eq. (26d), and the fact that there are no other constraints on the Riemann tensor ($\Psi_4 \neq 0$), one concludes that for the Will-Nordtvedt theory, the E(2) classification is III_5.

4. Stratified Theories[29]

$$\Box\, \varphi = 0 \quad , \tag{27a}$$

$$g = e^{2h(\varphi)}\, \underset{\sim}{\eta} + \left(e^{2f(\varphi)} - e^{2h(\varphi)}\right) \underset{\sim\sim}{dt} \otimes \underset{\sim\sim}{dt} \quad , \tag{27b}$$

or

$$g_{\alpha\beta} = e^{2h}\, \eta_{\alpha\beta} + (e^{2f} - e^{2h})\, \delta^0{}_\alpha\, \delta^0{}_\beta \quad , \tag{27c}$$

in a particular coordinate system, where f and h are given, unequal functions of the scalar field φ and $\underset{\sim\sim}{dt}$ is a time-like one-form. The wave solution to Eq. (27a) is

$$\varphi = \varphi_0 + \varphi_1 \, e^{i\underset{\sim}{q} \cdot \underset{\sim}{x}} \quad , \tag{27d}$$

as in Eq. (25d) and one can compute the Riemann tensor from $g_{\alpha\beta}$ using Eqs. (A1.1), (27c), and (27d). Contraction with $g_{\alpha\beta}$ then gives the linearized Ricci tensor:

$$R_{\beta\delta} = \varphi_1 \, e^{i\underset{\sim}{q} \cdot \underset{\sim}{x}} \left[(f' + g') \, q_\beta q_\delta - 2(f' - g') \, q^0 \, \delta^0{}_{(\delta} q_{\beta)} \right] \quad , \tag{27e}$$

where $f' \equiv df/d\varphi$, etc. From Eq. (27e) one finds

$$R = - 2\varphi_1 (f' - g') \, e^{i\underset{\sim}{q} \cdot \underset{\sim}{x}} (q^0)^2 \neq 0 \quad . \tag{27f}$$

From Eq. (27f), one concludes $\Psi_2 \neq 0$ [cf. Eq. (A1.4)], and consequently. for stratified theories, the E(2) classification is II_6.

Here we have a perfect example of a discrepancy between the number of dynamical degrees of freedom and the number of nonzero modes in the E(2) classification. Stratified theories clearly have only one dynamical degree of freedom, arising from the scalar field φ — yet some Lorentz observers see all six gravitational wave modes. The reason for this apparent paradox is that the "prior geometric"[7] one-form $\underset{\sim}{\underset{\sim}{d}}t$ introduces another vector into the problem in addition to the wave vector $\underset{\sim}{q}$ — a vector which transforms in a complicated way under the Lorentz transformations which leave $\underset{\sim}{q}$ fixed. The Ricci tensor does not "point" only along the $\underset{\sim}{q}$ direction [cf., Eq. (27e)] and any pure mode feeds all the other modes under Lorentz transformations.

V. EXPERIMENTAL DETECTION AND CLASSIFICATION OF WAVES

A. The Ideal Detection Experiment

An experimenter attempting any foreseeable experiment to detect gravitational waves[16] faces two fundamental limitations which hinder the E(2) classification of detected waves: (i) He can measure only the six "electric" components R_{i0j0} of $\underset{\sim}{Riem}$, not all twenty.[31] (ii) He may not know that wave direction a priori; he may be hoping to infer it from his data, as does Weber.[32] We will find that the consequences of these limitations are that the experimenter can generally classify a wave unambiguously only if he knows the direction a priori, and that he can never determine the direction using a single detector. Other limitations (antenna pattern, noise, time-resolution, bandwidth, need for coincidence detection) complicate the task further, but to treat the heart of the classification problem, we will ignore them.

Consider an ideal detection experiment: The experimenter uses the coordinate system of Sec. II. He measures the relative accelerations of test masses and obtains via Eq. (1) the six components R_{i0j0} of $\underset{\sim}{Riem}$, with perfect accuracy and infinite time-resolution. He expresses his data as a 3×3, symmetric, "driving-force matrix" $\underset{\sim}{S}(t)$, with components

$$S_{ij}(t) \equiv R_{i0j0}(u) \quad ;$$

here t is his proper time, and he takes his spatial origin at his detector, so $t = u$.

The experimenter knows, by time-coherence of the signal or by some other means, that the wave originates in a single, localized source. He denotes the wave direction (which he may or may not know a priori) by a spatial unit vector $\vec{k}$. (In previous sections we have taken $\vec{k} = \vec{e}_{\hat{z}}$; here it is arbitrary.)

Let us rename, for this section only, the amplitudes of a wave with direction $\vec{k}$, measured at the detector:

$$p_1(\vec{k},t) \equiv \Psi_2(u) \quad , \tag{28a}$$

$$p_2(\vec{k},t) \equiv \mathrm{Re}\ \Psi_3(u) \quad , \tag{28b}$$

$$p_3(\vec{k},t) \equiv \mathrm{Im}\ \Psi_3(u) \quad , \tag{28c}$$

$$p_4(\vec{k},t) \equiv \mathrm{Re}\ \Psi_4(u) \quad , \tag{28d}$$

$$p_5(\vec{k},t) \equiv \mathrm{Im}\ \Psi_4(u) \quad , \tag{28e}$$

$$p_6(\vec{k},t) \equiv \Phi_{22}(u) \quad . \tag{28f}$$

Let the index A = 1, 2, ... 6 run over these six modes. The amplitudes $p_A(\vec{k},t)$ are real.

For the case $\vec{k} = \vec{e}_{\hat{z}}$, Eqs. (2) imply

$$\underset{\sim}{S} = \begin{pmatrix} -\frac{1}{2}(p_4 + p_6) & \frac{1}{2}p_5 & -2p_2 \\ \frac{1}{2}p_5 & \frac{1}{2}(p_4 - p_6) & 2p_3 \\ -2p_2 & 2p_3 & -6p_1 \end{pmatrix} ,$$

or

$$\underset{\sim}{S}(t) = \Sigma_A\ p_A(\vec{e}_{\hat{z}},t)\ \underset{\sim}{E}_A(\vec{e}_{\hat{z}}) \quad , \tag{29}$$

where "basis polarization matrices" $\underset{\sim}{E}_A(\vec{e}_{\hat{z}})$ belonging to wave direction $\vec{k} = \vec{e}_{\hat{z}}$ are defined by

$$\underset{\sim}{E}_1(\vec{e}_{\hat{z}}) = -6\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \underset{\sim}{E}_2(\vec{e}_{\hat{z}}) = -2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} ,$$

$$\underset{\sim}{E}_3(\vec{e}_{\hat{z}}) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \underset{\sim}{E}_4(\vec{e}_{\hat{z}}) = -\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$\underset{\sim}{E}_5(\vec{e}_{\hat{z}}) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underset{\sim}{E}_6(\vec{e}_{\hat{z}}) = -\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{30}$$

Equation (29) represents $\underset{\sim}{S}(t)$ as a superposition of modes with $\vec{k} = \vec{e}_{\hat{z}}$.

For any other $\vec{k}$, just rotate these matrices: Let $\underset{\sim}{R}$ be a 3 × 3 rotation matrix[33] that takes $\vec{e}_{\hat{z}}$ into $\vec{k}$:

$$\vec{k} = \underset{\sim}{R}\vec{e}_{\hat{z}} \quad .$$

Define unit polarization matrices $\underset{\sim}{E}_A(\vec{k})$ for wave direction $\vec{k}$ by

$$\underset{\sim}{E}_A(\vec{k}) = \underset{\sim}{R}\ \underset{\sim}{E}_A(\vec{e}_{\hat{z}})\ \underset{\sim}{R}^T \quad .$$

Then for any $\underset{\sim}{S}(t)$ and any $\vec{k}$, there is the unique representation

$$\underset{\sim}{S}(t) = \Sigma_A\ p_A(\vec{k},t)\ \underset{\sim}{E}_A(\vec{k}) \quad ; \tag{31}$$

the amplitudes $p_A(\vec{k},t)$ may be extracted from $\underset{\sim}{S}(t)$ by

$$p_A(\vec{k},t) = C_A\ \mathrm{Trace}\Big(\underset{\sim}{E}_A(\vec{k})\ \underset{\sim}{S}(t)\Big) \quad , \tag{32}$$

where C_A are normalization constants:

$$C_A = (\tfrac{1}{36}\ ,\ \tfrac{1}{8}\ ,\ \tfrac{1}{8}\ ,\ 2,\ 2,\ 2) \quad .$$

Equation (32) follows from Eq. (31) and an orthogonality property of the $\underset{\sim}{E}_A(\vec{k})$:

$$C_A\ \mathrm{Trace}\Big(\underset{\sim}{E}_A(\vec{k})\ \underset{\sim}{E}_B(\vec{k})\Big) = \delta_{AB} \quad .$$

Equations (31) and (32) embody an important principle: Any measured $\underset{\sim}{S}(t)$ can be represented uniquely as a superposition of the six modes belonging to any arbitrary wave direction $\vec{k}$. Equation (32) specifies the amplitude in each mode of this wave. This wave is generally of class II_6, but it can be less general for certain $\underset{\sim}{S}(t)$ and certain $\vec{k}$.

The classification procedure now splits into two cases: $\vec{k}$ known and $\vec{k}$ unknown.

B. The Case of Known Direction

The experimenter knows $\vec{k}$ _a priori_ if the source of a gravitational wave that he detects can be identified with an object observed by means of electromagnetic radiation (light, radio, X-ray). There are also purely gravitational methods for determining $\vec{k}$. For example, if several detectors a distance $\gtrsim$ D apart, each with time-resolution $\ll$ D/c, detect a sharp wave burst with pulse-width $\ll$ D/c, then experimenters can determine $\vec{k}$ from the relative time-of-arrival at each detector. For D ~ radius of Earth, D/c ~ 13 msec.

Knowing $\vec{k}$, the experimenter extracts from $\underset{\sim}{S}(t)$ the amplitudes $p_A(\vec{k}, t)$ by Eq. (32). Knowing the amplitudes, he classifies the wave unambiguously, using the prescription given in Sec. II. The theoretical implications of his results are discussed in subsection E below.

C. The Case of Unknown Direction

If the experimenter does not know $\vec{k}$ _a priori_, he cannot hope to determine it from $\underset{\sim}{S}(t)$ without further assumptions; he can fit $\underset{\sim}{S}(t)$ equally well for any $\vec{k}$ in the sky by using Eqs. (31) and (32). Neither can he extract the p_A unambiguously. However, knowledge of $\underset{\sim}{S}(t)$ always provides information which limits the E(2) class of the wave and also the class of the correct theory of gravity (see E below).

He limits the possible class of the wave in the following way: For each arbitrary $\vec{k}$ in the sky, he computes the $p_A(\vec{k}, t)$ via Eq. (32) and determines the E(2) class associated with that $\vec{k}$. By letting $\vec{k}$ range all over the sky, he obtains the set of possible E(2) classes for that wave.

For a given $\underset{\sim}{S}(t)$, the following recipe yields a complete analysis of the possible E(2) classes of the wave. One distinguishes several cases according to the form of $\underset{\sim}{S}(t)$. Figure 3 diagrams this recipe as a flow-chart.

Case 1. Driving forces remain in a fixed line. There is a fixed coordinate system in which

$$\underset{\sim}{S}(t) = \begin{pmatrix} \lambda(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{33}$$

Pattern of forces is as in Fig. 1(d); but propagation direction need not be as in Fig. 1(d). Conclusion: Wave is II_6 or N_3.

Case 2. Driving forces remain in a fixed plane: There is a fixed coordinate system in which

$$\underset{\sim}{S}(t) = \begin{pmatrix} \lambda(t) & \mu(t) & 0 \\ \mu(t) & \nu(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{34}$$

but none in which Eq. (33) holds. Wave may always be II_6. In addition, two separate determinations must be made: (a) Can the wave be O_1, N_2, or N_3? (b) Can the wave be III_5?

Test 2.a; for O_1, N_2, or N_3.

Subcase 2.a.i. Driving forces are "pure monopole":

$$\lambda(t) \equiv \nu(t) , \quad \mu(t) \equiv 0 . \tag{35}$$

Pattern of forces is as in Fig. 1(c); but wave need not be pure Φ_{22}. Conclusion: Wave may be O_1. (Furthermore, wave cannot be III_5; test (b) is always failed.)

Subcase 2.a.ii. Driving forces are "pure quadrupole":

$$\lambda(t) \equiv - \nu(t) . \tag{36}$$

Pattern of forces is as in Fig. 1(a) (and the principal axes may rotate with time in the transverse plane); but propagation direction need not be as in Fig. 1(a). Conclusion: Wave may be N_2.

Subcase 2.a.iii. Driving forces are neither "pure monopole" nor "pure quadrupole": Neither Eq. (35) nor Eq. (36) holds. Conclusion: Wave may be N_3.

Test 2.b; for III_5. Wave may be III_5 if and only if there exists a fixed unit vector $\vec{k}$ not normal the plane of the forces [i.e.,

$$\vec{k} \neq \vec{e}_{\hat{z}} \quad ,$$

in the coordinates of Eq. (34)] such that

$$\vec{k} \cdot \underset{\sim}{S}(t) \cdot \vec{k} \equiv 0 \quad . \tag{37}$$

The complete set of possibilities for Case 2 is II_6 plus the outcomes of Test 2.a and Test 2.b.

Case 3. Driving forces do not remain in any fixed plane: Equation (34) does not hold in any fixed coordinate system. Wave may always be II_6. It may be III_5 if and only if there exists a fixed unit vector $\vec{k}$ such that

$$\vec{k} \cdot \underset{\sim}{S}(t) \cdot \vec{k} \equiv 0 \quad . \tag{38}$$

Note that when the driving forces do not occur in one plane and Eq. (38) is violated, the wave must be II_6.

D. Guessing $\vec{k}$

We have emphasized that $\vec{k}$ can never be extracted from $\underset{\sim}{S}(t)$. However, the fact that a certain $\underset{\sim}{S}(t)$ can be fitted by a wave of a certain class less general than II_6 must weigh as strong circumstantial evidence that the wave is actually of that class. If one is willing to assume that the simplest allowed classification is correct, then $\vec{k}$ is generally fixed uniquely (up to an inevitable antipodal ambiguity, $\vec{k} \to -\vec{k}$).

Referring to the recipe above, the information that one can guess in this way is as follows.

Case 1. If the wave is N_3, $\vec{k}$ lies anywhere in the plane spanned by $\vec{e}_{\hat{y}}$ and $\vec{e}_{\hat{z}}$ in the coordinates of Eq. (33).

Case 2. If the wave is O_1, N_2, or N_3, $\vec{k}$ is normal to the plane of the forces:

$$\vec{k} = \pm \vec{e}_{\hat{z}} \quad ,$$

in the coordinates of Eq. (34). If the wave is III_5, $\vec{k}$ is as in Eq. (37).

Case 3. If the wave is III_5, $\vec{k}$ is as in Eq. (38).

One can never limit the direction of a II_6 wave in this way.

E. Theoretical Implications of Experimental Results

The E(2) class of the correct theory of gravity is at least as general as that of any observed wave: This is always the fundamental implication of any observation. We must always qualify, "at least as general," because in any particular theory a particular source may couple poorly or not at all to some of the admissible modes, and therefore it may radiate only special classes of waves. But the observation of a wave of a certain class always rules out all theories of less general classes.

If the wave direction is unknown, an observed wave cannot be classified unambiguously (except for some waves of class II_6). However, there is always a least general possible class for each such wave, which limits the correct theory.

There are still sharper implications for particular theories. In the case of a well-understood source (e.g., binary star system), each particular theory should make a precise prediction about the mixture of modes radiated, leading to a crucial test. We shall discuss this point in a future paper. In the case of a theory for which the number of degrees of freedom is less than the dimension of the E(2) class (see Sec. IV.B), the various admissible modes should appear only in definite mixtures by any source, again leading to a crucial test. Finally, the difference in propagation speed for light and for gravitational waves leads to a crucial test for many theories (see Sec. IV.A).

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APPENDIX 1. USEFUL FORMULAE FOR PLANE WAVES

General linearized Riemann tensor in terms of flat space perturbation $h_{\mu\nu}$:

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}) \quad . \qquad \text{(A1.1)}$$

Tetrad components of Riemann tensor in terms of h_{ab}:

$$\Psi_2 \equiv -\frac{1}{6} R_{\ell k \ell k} = \frac{1}{12} \ddot{h}_{kk} \quad , \qquad \text{(A1.2a)}$$

$$\Psi_3 \equiv -\frac{1}{2} R_{\ell k \ell \bar{m}} = \frac{1}{4} \ddot{h}_{k\bar{m}} \quad , \qquad \text{(A1.2b)}$$

$$\Psi_4 \equiv - R_{\ell \bar{m} \ell \bar{m}} = \frac{1}{2} \ddot{h}_{\bar{m}\bar{m}} \quad , \qquad \text{(A1.2c)}$$

$$\Phi_{22} \equiv - R_{\ell m \ell \bar{m}} = \frac{1}{2} \ddot{h}_{m\bar{m}} \quad , \qquad \text{(A1.2d)}$$

(where $\ddot{h} \equiv d^2h/du^2$).

Tetrad components of Ricci tensor:

$$R_{\ell k} = R_{\ell k \ell k} \quad , \qquad \text{(A1.3a)}$$

$$R_{\ell \ell} = 2 R_{\ell m \ell \bar{m}} \quad , \qquad \text{(A1.3b)}$$

$$R_{\ell m} = R_{\ell k \ell m} \quad , \qquad \text{(A1.3c)}$$

$$R_{\ell \bar{m}} = R_{\ell k \ell \bar{m}} \quad . \qquad \text{(A1.3d)}$$

Ricci scalar:

$$R = -2 R_{\ell k} = -2 R_{\ell k \ell k} \quad . \qquad \text{(A1.4)}$$

APPENDIX 2. INDECOMPOSABLE GROUP REPRESENTATIONS

Let G be a group and $\mathcal{G}$ a linear representation of G on a linear space V. $\mathcal{G}$ is reducible, if it has an invariant proper subspace, $V_1 \subset V$. $\mathcal{G}$ is decomposable, if V is the direct sum of invariant proper subspaces. A decomposable representation is always reducible but not vice versa; $\mathcal{G}$ is indecomposable, if it is reducible but not decomposable. $\mathcal{G}$ is decomposable, if, and only if, there is a basis of V for which each $g \in G$ is represented by a block-triangular matrix

$$\begin{pmatrix} g_1 & 0 \\ g_3 & g_2 \end{pmatrix},$$

with not all g_3 vanishing.

Indecomposable representations never occur for a finite group G, for finite-dimensional representations of a semi-simple Lie group G, or for unitary representations of any Lie group G. Because of these facts, physicists are not well acquainted with indecomposable representations. For a physicist, indecomposable representations have two unpleasant attributes: (i) They are always nonunitary. (ii) There is no analog of Schur's lemma: An invariant operator is not generally constant on an indecomposable representation; e.g., "spin" is undefined.

See Ref. 27 or Ref. 34 for a discussion of these concepts.

For waves of E(2) class II_6 or III_5, we deal with 6- or 5-dimensional indecomposable representations of E(2). The only finite-dimensional decomposable representations of E(2) decompose to the familiar 1-dimensional unitary representations that describe a massless quantum particle of integral or half-integral helicity[25-27]; these representations arise for E(2) classes N_3, N_2, and O_1.

TABLE I. E(2) classification of various metric theories of gravity. See Sec. IV.

Theory	E(2) class	Degrees of freedom[28]	$c_g = c_{em}$?	Currently viable?	Equal to GRT in PPN limit[a]?
GRT[4]	N_2	2	yes	yes	yes
Dicke-Brans-Jordan[5,b]	N_3	3	yes	yes	no
Conformally flat theories[29]	O_1	1	yes	no	no
Stratified theories[29]	II_6	1	c	no	no
Will-Nordtvedt[11]	III_5	5	yes	yes	yes
Lightman-Lee[13]	II_6	6	no	yes	yes
Ni[10]	II_6	1	no	yes	yes
Hellings-Nordtvedt[12]	N_3	?	yes	yes	yes

[a] If a theory can be made to coincide with GRT in the PPN limit[6] by a particular choice of arbitrary constants and/or possible cosmological boundary values, we put a "yes" in this column.

[b] Typical of scalar-tensor theories.[29]

[c] Depends on the particular theory.

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[17] In this paper Latin indices range over 1, 2, 3, and Greek indices over 0, 1, 2, 3. We choose units such that $c = G = 1$ and we denote 4-vectors, 3-vectors, and tensors by $\underset{\sim}{v}$, $\vec{k}$, $\underset{\sim}{g}$ respectively. Other conventions are as in Ref. 5.

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[19] For details of the null-tetrad formalism, see, e.g., E. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962); see errata, J. Math. Phys. 4, 998 (1962). Note that we use a different convention in contracting the Riemann tensor to obtain the Ricci tensor, i.e., $R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}$, and the opposite signature; hence the sign differences in Φ's and Λ.

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[28] We define the "number of degrees of freedom" as being the number of independent components of dynamical variables obeying a wave equation, after constraints and coordinate arbitrariness have been subtracted.

[29] For a list of "stratified theories" and related discussion, and for a similar discussion of "conformally flat theories," see W.-T. Ni, Astrophys. J. **176**, 769 (1972).

[30] All of the coordinate components of Riemann and Ricci tensors are real, and one should always take the real part of expressions for these quantities. The reader should not confuse this with the complex tetrad components, obtained by projecting real coordinate components onto complex basis vectors.

31 Detectors have been proposed that measure the "magnetic" components R_{i0jk}, but none seem practical; see Ref. 16, also F. B. Estabrook and H. D. Wahlquist, J. Math. Phys. **5**, 1629 (1964). Using such a detector in conjunction with a conventional one, an experimenter could uniquely classify any wave and determine its direction.

32 J. Weber, Phys. Rev. Letters **22**, 1302 (1969); **24**, 276 (1969); **25**, 180 (1970).

33 There is actually a one-parameter family of such R; the members differ only in a final rotation about $\vec{k}$. This final rotation only changes the phase of Ψ_3 and Ψ_4 and hence cannot change the ultimate classification.

34 H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1931), Chap. III, Sec. 4. Weyl says "completely reducible" where we say "decomposable."

FIGURE CAPTIONS

Fig. 1. The six polarization modes of a weak, plane, null gravitational wave permitted in the generic metric theory of gravity. Shown is the displacement that each mode induces on a sphere of test particles. The wave is propagating in the +z direction (arrow at upper right) and has time dependence cos ωt. The solid line is a snapshot at $\omega t = 0$, the broken line one at $\omega t = \pi$. There is no displacement perpendicular to the plane of the figure.

Fig. 2. The E(2) classes of weak, plane, null waves, displayed in order of increasing generality toward the top. Descending along a line represents specializing the class by demanding that some amplitude vanish for all observers. One class is said to be more general than another if it is possible to descend from one to the other along lines.

Fig. 3. Prescription for finding possible E(2) classes for a wave of unknown direction $\vec{k}$, given the driving-force matrix $\underset{\sim}{S}(t)$. Boxes contain tests involving $\underset{\sim}{S}(t)$ and circles contain possible classes. See text of Sec. V.

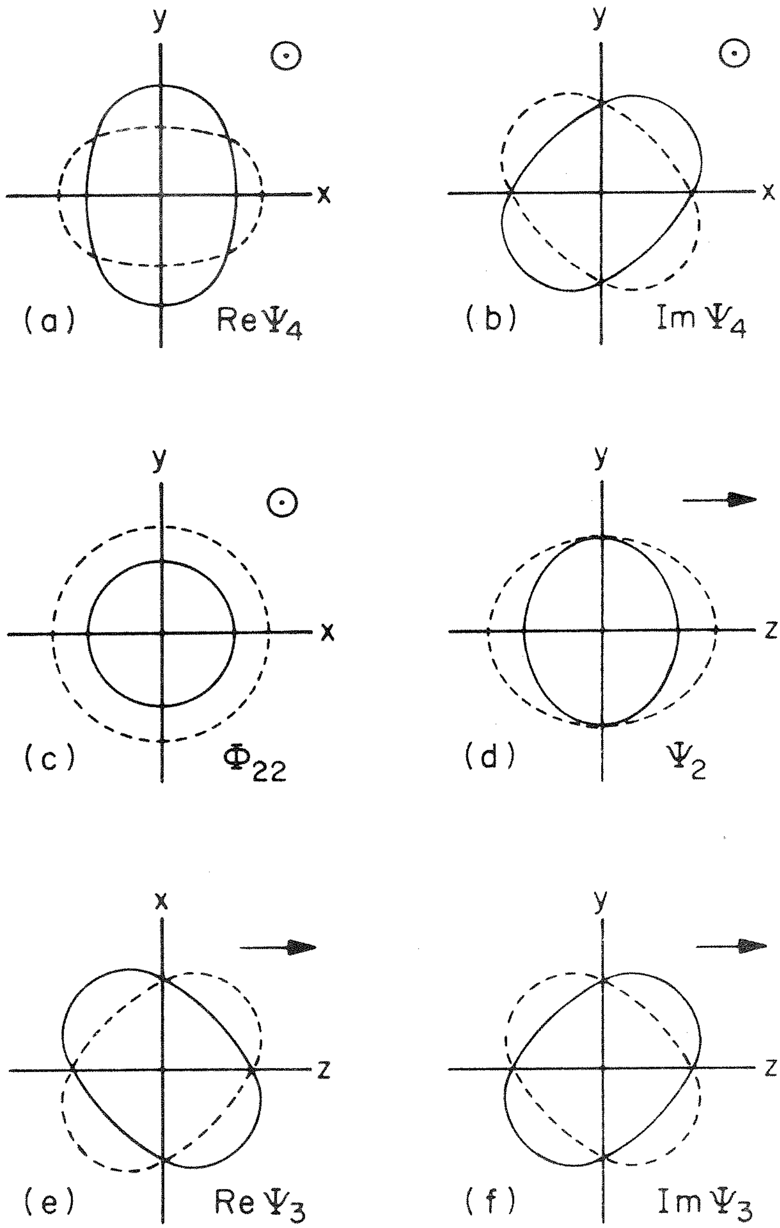

Fig. 1

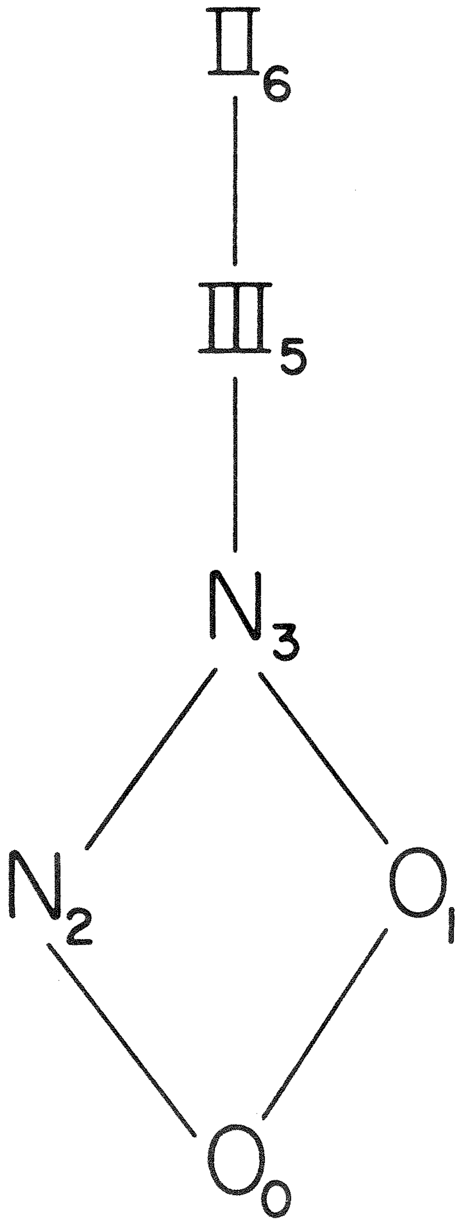

Fig. 2

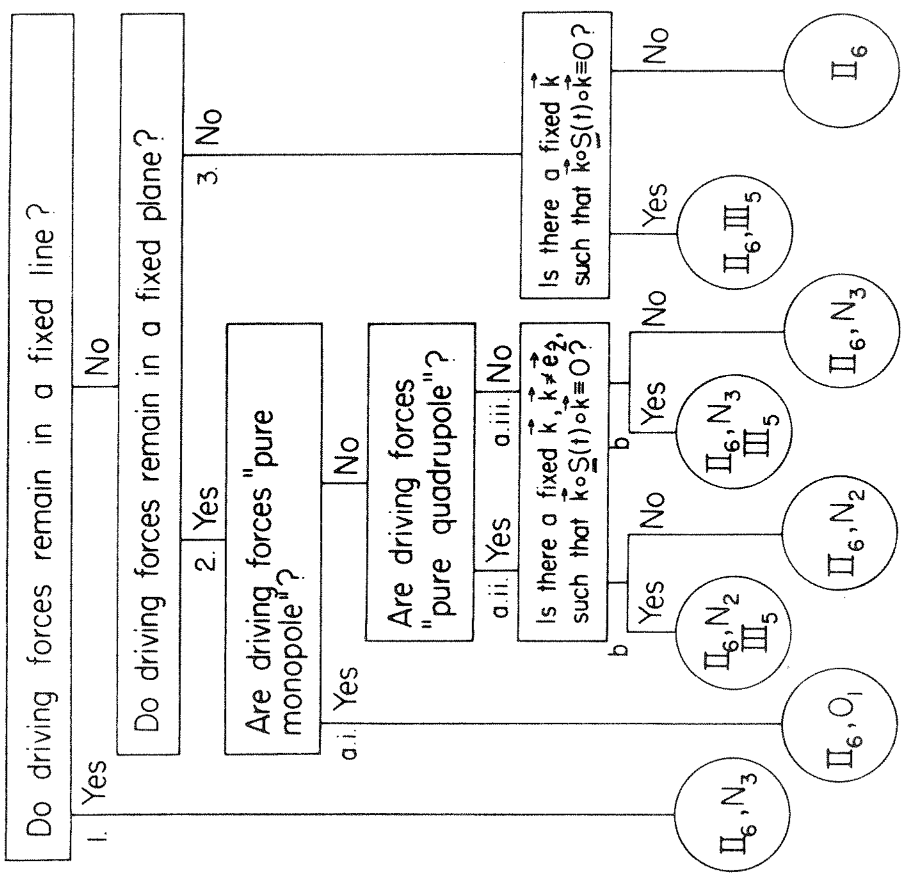
Do driving forces remain in a fixed line?
1. Yes
No
Do driving forces remain in a fixed plane?
2. Yes
3. No
Are driving forces "pure monopole"?
a.i. Yes
No
Are driving forces "pure quadrupole"?
a.ii. Yes
a.iii. No
Is there a fixed $\vec{k}$ such that $\vec{k}\circ \underline{S}(t)\circ\vec{k}\equiv 0$?
Is there a fixed $\vec{k}$, $\vec{k}\neq\vec{e}_2$, such that $\vec{k}\circ \underline{S}(t)\circ\vec{k}\equiv 0$?
b
Yes
No
II_6, N_3
II_6, O_1
II_6, N_2, III_5
II_6, N_2
II_6, N_3, III_5
II_6, N_3
II_6, III_5
II_6

Fig. 3

c) Variational Principles and Conservation Laws in Metric Theories of Gravity (Paper X; collaboration with D.L. Lee and W.T. Ni, submitted to Phys. Rev.D, 1974)

Variational Principles and Conservation Laws in Metric Theories of Gravity*

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ABSTRACT

Using the generalized Bianchi and Noether identities that apply to all Lagrangian-based theories, we specialize to Lagrangian-based, generally covariant metric theories of gravity ("LBGCM theories") and prove a number of theorems. Our most important results are the following: (i) The matter response equations $T^{\mu\nu}{}_{;\nu} = 0$ of any LBGCM theory are a consequence of the gravitational field equations iff the theory contains no absolute variables. (ii) Almost all LBGCM theories possess conservation laws of the form $\Theta_\mu{}^\nu{}_{,\nu} = 0$ (where $\Theta_\mu{}^\nu$ reduces to $T_\mu{}^\nu$ in the absence of gravity). (iii) For asymptotically flat systems the integral $P_\mu = \int \Theta_\mu{}^\nu d^3\Sigma_\nu$ is a conserved (hypersurface-independent) quantity which one naturally interprets as energy momentum. (iv) P_μ is expressible as a surface integral at spatial infinity, and thus can be measured by experiments confined to the asymptotically flat region outside the source, if $\Theta_\mu{}^\nu$ is expressible in terms of a superpotential,

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†Imperial Oil Predoctoral Fellow.

$\Theta_\mu{}^\nu = \Lambda_\mu{}^{[\nu\alpha]}{}_{,\alpha}$. In this case the existence of a conserved P_μ implies the existence of a conserved P^μ and vice versa. (v) While some LBGCM theories (e.g., general relativity and scalar-tensor theories) possess superpotentials, others may not. (vi) For a theory with a superpotential P_μ and P^μ (as measured at "infinity") transform as 4-vectors under Lorentz transformations, if the variables of the theory are all tensors, tensor densities, and affine connections. For other types of LBGCM theories, the P_μ constructed from a given $\Theta_\mu{}^\nu$ need not be a 4-vector. (vii) In Will's ten-parameter post-Newtonian ("PPN") formalism there exists a conserved P_μ if and only if the parameters obey 5 specific constraints; two additional constraints are needed for the existence of a conserved angular momentum $J_{\mu\nu}$. (This modifies and extends a previous result due to Will.) (viii) We conjecture that for metric theories of gravity, the conservation of energy-momentum is equivalent to the existence of a Lagrangian formulation; and using the PPN formalism, we prove the post-Newtonian limit of this conjecture. (ix) We present "stress-energy momentum complexes" $\Theta_\mu{}^\nu$ for a wide variety of specific theories of gravity.

I. INTRODUCTION AND SUMMARY

The variational principle is an elegant and compelling foundation upon which fundamental theories are formulated. In fact, most complete and self-consistent theories of gravity are derivable from variational principles — i.e., are "Lagrangian-based." In this paper, a member of a series[1,2,3,4] of papers which discuss general properties of gravitation theories, we specialize to Lagrangian-based, metric theories of gravity. It would be very helpful for the reader to have read Ref. 1 above (hereafter referred to as Paper I) for definitions of the terms and concepts used in this paper.[5]

Our discussion focuses on the identities and conservation laws that follow from a variational principle. We demonstrate that for the case when all fields present in the action are varied (when there are no absolute variables), the resulting Euler-Lagrange equations contain redundancies, i.e., identities. As a result of the specific form of these identities, we prove that the matter response equation $T_{\mu\ ;\ \nu}^{\ \nu} = 0$ is a consequence of the gravitational field equations if and only if no absolute variables are present. We also prove that all Lagrangian-based, generally covariant, metric theories in a certain broad class (denoted by "LBGCM*" — see Sec. III.E) have conservation laws, so that a conserved energy momentum P_μ can be defined. Furthermore, we show that if the conserved P_μ can be evaluated solely in terms of the asymptotic properties of the gravitational fields at asymptotic infinity, a conserved, contravariant, 4-energy momentum P^μ can be defined and vice versa. In such cases P_μ and P^μ transform as 4-vectors under Lorentz transformations in the asymptotically flat region, if the variables of the theory are all tensors, tensor densities, and affine connections.

In the weak field, post-Newtonian (PN) limit[6] we derive five constraints on the "PPN parameters" of LBGCM* theories. Our ability to explicitly construct a Lagrangian-based theory of gravity with five arbitrary parameters in the post-Newtonian limit particularly proves our conjecture that for metric theories of gravity, the existence of a conserved 4-energy momentum is equivalent to the existence of a Lagrangian formulation.

The fact that the action principle admits a covariance group can be expressed in the form of various differential identities. Excellent reviews on this subject abound.[7] We summarize the identities in Sec. II merely to set the framework for later discussions. We then specialize to metric theories of gravity in Sec. III, where because of a theorem proved in Paper I, the nongravitational part of the Lagrangian must have a simple, universal form. Section III.A sets up a model Lagrangian for metric theories, and Sec. III.B specializes the identities of Sec. II to such Lagrangians. In Sec. III.C the resulting field equations are derived symbolically and our results regarding absolute variables are proved. Section III.E makes use of the results of Sec. III.B to derive conservation laws. Section III.F discusses further the conservation laws derived in Sec. III.E, emphasizing in particular the role of the conserved energy momentum in asymptotically flat spacetime. Theories with "singular Lagrangians" — a topic somewhat unrelated to the rest of the paper — are discussed in Sec. III.D for completeness. Section IV specializes to the post-Newtonian limit.

Appendix A lists for various exemplary metric theories, the gravitational portion of the divergence-free stress-energy pseudo-tensor and whenever available, the corresponding superpotentials. Appendix B gives the "contravariant" and the "mixed-index" gravitational stress-energy pseudotensor that enters into conservation laws in the post-Newtonian limit.

Appendix C presents a new theory of gravity with conservation laws, and its post-Newtonian limit, which possesses the maximum allowed number of arbitrary parameters: 5.

II. CONSEQUENCES OF COVARIANT ACTION PRINCIPLES

In this section, we summarize some well-known identities resulting from the covariance of the mathematical representation of a given theory. There is a generalized Bianchi identity corresponding to each transformation of the covariance group. When specialized to the Manifold Mapping Group (MMG; that covariance group corresponding to arbitrary coordinate transformation), these identities can be written in different, but equivalent forms known as the Noether identities. For derivations of the cited identities, see any of Refs. 7.

Consider the action

$$I(\mathcal{Y}) = \int_R \mathcal{L}(\mathcal{Y}_A, \mathcal{Y}_{A,\mu}, \mathcal{Y}_{A,\mu\nu})\, d^4x \quad , \tag{1}$$

where for simplicity we assume the Lagrangian to be a functional of the geometric objects ("variables" of the representation) $\{\mathcal{Y}_A\}$ and their first and second order derivatives $\{\mathcal{Y}_{A,\mu}, \mathcal{Y}_{A,\mu\nu}\}$.[8] Let the action principle be invariant under some transformations characterized by the infinitesimal descriptors $\xi^i(x)$. The number of descriptors $\xi^i(x)$ is equal to the number of arbitrary functions characterizing the set of transformations (the covariance group). We assume henceforth that the functional change of $\{\mathcal{Y}_A\}$ [see, e.g., Eq. (6) of Paper I] has the form

$$\bar{\delta}\mathcal{Y}_A = d^{\mu}_{Ai}\xi^i{}_{,\mu} + C_{Ai}\xi^i \ , \qquad i = 1, \dots, n \quad , \tag{2}$$

where d^{μ}_{Ai} and C_{Ai} are functions of the $\mathcal{Y}_A$. This form is extremely general; it holds, for example, whenever the ξ^i are infinitesimal generators of MMG and the $\mathcal{Y}_A$ are tensors or tensor densities. Equation (2) and the subsequent discussions can be generalized to include a term containing the second derivatives of ξ^i for the case when $\mathcal{Y}_A$ is an affine connection field.

Bianchi identities

Corresponding to each transformation of the covariance group described by continuous functions ξ^i, there is a generalized Bianchi identity:

$$C_{Ai} \frac{\delta \mathcal{L}}{\delta \mathcal{Y}_A} - \left(d^{\mu}_{Ai} \frac{\delta \mathcal{L}}{\delta \mathcal{Y}_A}\right)_{,\mu} \equiv 0 , \qquad i = 1, \ldots, n , \tag{3}$$

where

$$\frac{\delta \mathcal{L}}{\delta \mathcal{Y}_A} \equiv \frac{\partial \mathcal{L}}{\partial \mathcal{Y}_A} - \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial \mathcal{L}}{\partial \mathcal{Y}_{A,\mu}}\right) + \frac{\partial^2}{\partial x^{\mu} \partial x^{\nu}} \left(\frac{\partial \mathcal{L}}{\partial \mathcal{Y}_{A,\mu\nu}}\right)$$

is the usual variational derivative of $\mathcal{L}$ with respect to $\mathcal{Y}_A$.

Noether identities

We now specialize to the case where the covariance group is MMG. Let ξ^{μ} be the descriptors of MMG:

$$x^{\mu'} = x^{\mu} + \xi^{\mu} . \tag{4}$$

In order that the action principle

$$\delta I(\mathcal{Y}) = 0$$

be invariant under MMG, the Lagrangian density $\mathcal{L}(\mathcal{Y}_A, \mathcal{Y}_{A,\mu}, \mathcal{Y}_{A,\mu\nu})$ must transform as a scalar density (modulo a total divergence $Q^{\rho}{}_{,\rho}$)

$$\bar{\delta}(\mathcal{L} + Q^{\rho}{}_{,\rho}) = - (\mathcal{L} + Q^{\rho}{}_{,\rho})\, \xi^{\mu}{}_{,\mu} - (\mathcal{L} + Q^{\rho}{}_{,\rho})_{,\mu} \xi^{\mu} . \tag{5}$$

On the other hand, the functional change of $\mathcal{L}$ is

$$\bar{\delta}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\mathcal{Y}_A}\bar{\delta}\mathcal{Y}_A + \frac{\partial\mathcal{L}}{\partial\mathcal{Y}_{A,\mu}}\bar{\delta}\mathcal{Y}_{A,\mu} + \frac{\partial\mathcal{L}}{\partial\mathcal{Y}_{A,\mu\nu}}\bar{\delta}\mathcal{Y}_{A,\mu\nu}$$

$$= \frac{\delta\mathcal{L}}{\delta\mathcal{Y}_A}\bar{\delta}\mathcal{Y}_A + \frac{\partial}{\partial x^\mu}\left\{\frac{\partial\mathcal{L}}{\partial\mathcal{Y}_{A,\mu}}\bar{\delta}\mathcal{Y}_A - 2\left(\frac{\partial\mathcal{L}}{\partial\mathcal{Y}_{A,\mu\nu}}\right)_{,\nu}\bar{\delta}\mathcal{Y}_A + \left(\frac{\partial\mathcal{L}}{\partial\mathcal{Y}_{A,\mu\nu}}\bar{\delta}\mathcal{Y}_A\right)_{,\nu}\right\} . \quad (6)$$

Combining Eqs. (5) and (6), we obtain the Noether identity:

$$\left(-\mathcal{L}\xi^\mu - \frac{\partial\mathcal{L}}{\partial\mathcal{Y}_{A,\mu}}\bar{\delta}\mathcal{Y}_A + 2\frac{\partial}{\partial x^\nu}\left(\frac{\partial\mathcal{L}}{\partial\mathcal{Y}_{A,\mu\nu}}\right)\bar{\delta}\mathcal{Y}_A - \frac{\partial}{\partial x^\nu}\left(\frac{\partial\mathcal{L}}{\partial\mathcal{Y}_{A,\mu\nu}}\bar{\delta}\mathcal{Y}_A\right)\right.$$

$$\left. - Q^\rho{}_{,\rho}\xi^\mu - \bar{\delta}Q^\mu\right)_{,\mu} \equiv \frac{\delta\mathcal{L}}{\delta\mathcal{Y}_A}\bar{\delta}\mathcal{Y}_A \quad , \quad (7)$$

for all arbitrary functions ξ^μ.

The above items require some discussion and clarification.

(i) The Bianchi identities [Eq. (3)] and the Noether identities [Eq. (7)] are satisfied by all "kinematically possible trajectories" (kpt; set of values for the components of all variables, unconstrained to satisfy the physical laws of the representation).

(ii) As an example of the Bianchi identities [Eq. (3)], consider the following Lagrangian density:

$$\mathcal{L}_{B-D}(g_{\mu\nu},\phi) = [R\phi^2 + 6\phi_{,\mu}\phi_{,\nu}g^{\mu\nu}]\sqrt{-g} \quad , \quad (8)$$

where

$R \equiv$ curvature scalar formed out of the metric $g_{\mu\nu}$.

$\phi \equiv$ scalar field.

Equation (8) is the gravitational Lagrangian of the Dicke-Brans-Jordan theory[9] for $\omega = -3/2$. The Lagrangian $\mathcal{L}_{B-D}$, in addition to being generally covariant, is also invariant under the "scale transformation of the second kind":

$$\begin{cases} g_{\mu\nu} = e^{-2\sigma} g'_{\mu\nu} \quad , & (9a) \\ \phi = e^{\sigma}\phi' \quad , & (9b) \end{cases}$$

where σ is some arbitrary spacetime function. The infinitesimal version of Eqs. (9) are

$$\begin{cases} \bar{\delta} g_{\mu\nu} = -2\sigma g_{\mu\nu} \quad , & (10a) \\ \bar{\delta}\phi = \sigma\phi \quad . & (10b) \end{cases}$$

Here σ plays the role of the descriptor of the transformation and comparison with Eq. (2) gives the C_{Ai} and $d_A{}^{\mu}{}_i$ (the latter being zero in this case). Further comparison with Eq. (3) yields the Bianchi identity corresponding to this "scale transformation of the second kind":

$$-2g_{\mu\nu} \frac{\delta \mathscr{L}_{B-D}}{\delta g_{\mu\nu}} + \phi \frac{\delta \mathscr{L}_{B-D}}{\delta \phi} \equiv 0 \quad . \tag{11}$$

(iii) The Bianchi identities [Eq. (3)] corresponding to ξ^i being the descriptors of the MMG [Eq. (4)] can be obtained from the Noether identities, Eqs. (7), by substituting in Eq. (2), performing an integration by parts, and utilizing the arbitrariness of the descriptors, ξ^i. Thus Eqs. (3) hold for any Lagrangian which is a scalar density (modulo a total divergence).

(iv) From Eqs. (7) and (3), we also obtain the identity

$$\left\{ -\mathscr{L}\xi^{\mu} - \frac{\partial \mathscr{L}}{\partial \mathscr{Y}_{A,\mu}} \bar{\delta}\mathscr{Y}_A + 2\left(\frac{\partial \mathscr{L}}{\partial \mathscr{Y}_{A,\mu\nu}}\right)_{,\nu} \bar{\delta}\mathscr{Y}_A - \left(\frac{\partial \mathscr{L}}{\partial \mathscr{Y}_{A,\mu\nu}} \bar{\delta}\mathscr{Y}_A\right)_{,\nu} - \frac{\delta \mathscr{L}}{\delta \mathscr{Y}_A} d_A{}^{\mu}{}_{\rho}\xi^{\rho} - Q^{\rho}{}_{\rho}\xi^{\mu} - \bar{\delta}Q^{\mu} \right\}_{,\mu} \equiv 0 \tag{12}$$

for all infinitesimal generators ξ^{μ} of MMG.

Equation (12) is known as a "strong conservation law" (see Ref. 7c) because it is an identity holding regardless of imposition of the field equations.

III. LAGRANGIAN-BASED METRIC THEORIES

A. The Lagrangian

We now apply the general discussions outlined in Sec. II. to generally covariant, Lagrangian-based metric theories of gravity. We first group the variables $\{\mathcal{Y}_A\}$ into three categories:

$$\{\mathcal{Y}_A\} = \{z_a\} + \{\phi_b\} + \{q_\lambda\} \quad , \tag{13}$$

where

$$\{z_a\} \equiv \text{dynamical gravitational fields, (identified in our notation by lower case a in symbolic sums),} \tag{14a}$$

$$\{\phi_b\} \equiv \text{nondynamical (absolute) gravitational fields, (identified by lower case b),} \tag{14b}$$

$$\{q_\lambda\} \equiv \text{nongravitational fields, (identified by } \lambda\text{).} \tag{14c}$$

The Lagrangian density can be separated into two parts, the gravitational part (containing no matter variables) and the nongravitational part:

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{NG} \quad . \tag{15}$$

In Paper I, we have shown that for relativistic metric theories, $\mathcal{L}_{NG}$ can contain only one gravitational field, the metric $g_{\mu\nu}$. To be more general, we assume that $g_{\mu\nu}$ may not actually appear in $\mathcal{L}_G$, but rather may be an algebraic function of $\{z_a\}$, $\{\phi_b\}$, and perhaps some $\{\psi_c\}$ that do not appear in $\mathcal{L}_G$ at all.[10] Symbolically,

$$\mathcal{L}_G = \mathcal{L}_G(z_a,\ \phi_b,\ z_{a,\rho},\ z_{a,\rho\mu},\ \phi_{b,\rho},\ \phi_{b,\rho\mu}) \tag{16a}$$

$$\mathcal{L}_{NG} = \mathcal{L}_{NG}(g_{\mu\nu},\ q_\lambda,\ g_{\mu\nu,\rho},\ g_{\mu\nu,\rho\tau},\ q_{\lambda,\rho},\ \ldots) \tag{16b}$$

$$g_{\mu\nu} = g_{\mu\nu}(z_a,\ \phi_b,\ \psi_c) \quad . \tag{17}$$

There will also be some "postulated" field equations of the form

$$F(\phi_b, \psi_c) = 0 \quad . \tag{18}$$

for the absolute objects ϕ_b and ψ_c. Our results do not preclude the possibility that $g_{\mu\nu}$ may in fact be identical to one of the z_a's.

B. The Identities and Their Consequences

In this subsection, we write down identities and relations for the Lagrangians in Eqs. (16), assuming they admit MMG as a covariance group. We take the functional changes of the variables to be

$$\bar{\delta} z_a = d_a{}^{\sigma}{}_{\rho} \xi^{\rho}{}_{,\sigma} - z_{a,\rho} \xi^{\rho} \quad , \tag{19a}$$

$$\bar{\delta} \phi_b = d_b{}^{\sigma}{}_{\rho} \xi^{\rho}{}_{,\sigma} - \phi_{b,\rho} \xi^{\rho} \quad , \tag{19b}$$

$$\bar{\delta} \psi_c = d_c{}^{\sigma}{}_{\rho} \xi^{\rho}{}_{,\sigma} - \psi_{c,\rho} \xi^{\rho} \quad , \tag{19c}$$

$$\bar{\delta} q_{\lambda} = d_{\lambda}{}^{\sigma}{}_{\rho} \xi^{\rho}{}_{,\sigma} - q_{\lambda,\rho} \xi^{\rho} \quad , \tag{19d}$$

and, in particular,

$$\bar{\delta} g_{\mu\nu} = - 2 g_{\rho(\mu} \delta^{\sigma}{}_{\nu)} \xi^{\rho}{}_{,\sigma} - g_{\mu\nu,\rho} \xi^{\rho} \quad , \tag{19e}$$

where the ξ^{ρ} are the descriptors of the coordinate transformation and are arbitrary functions. [Equations (19) must be generalized to include $\xi^{\rho}{}_{,\sigma\tau}$ if one of the variables is an affine connection field.] For simplicity, we further assume that $\mathcal{L}_G$ and $\mathcal{L}_{NG}$ are scalar densities (which is usually the case) so that the Q^{ρ} in Eqs. (7) and (12) may be set to zero. We now proceed to list some useful identities.

(i) Bianchi Identities for $\mathcal{L}_G$

Comparison of Eqs. (19) with Eq. (2) and use of the fact that $\mathcal{L}_G$ is a scalar density by itself reduces the Bianchi identities, Eqs. (3), to

$$- z_{a,\rho} \frac{\delta \mathcal{L}_G}{\delta z_a} - \phi_{b,\rho} \frac{\delta \mathcal{L}_G}{\delta \phi_b} - \left(d_{a}{}^{\sigma}{}_{\rho} \frac{\delta \mathcal{L}_G}{\delta z_a} + d_{b}{}^{\sigma}{}_{\rho} \frac{\delta \mathcal{L}_G}{\delta \phi_b}\right)_{,\sigma} \equiv 0 \quad . \tag{20}$$

(ii) Bianchi Identities for $\mathcal{L}_{NG}$

Similarly, Eqs. (3), when applied to $\mathcal{L}_{NG}$, yield

$$- \frac{1}{2} g_{\mu\nu,\rho}(-g)^{1/2} T^{\mu\nu} - q_{\lambda,\rho} \frac{\delta \mathcal{L}_{NG}}{\delta q_\lambda} - \left[- (-g)^{1/2} T_{\rho}{}^{\sigma} + d_{\lambda}{}^{\sigma}{}_{\rho} \frac{\delta \mathcal{L}_{NG}}{\delta q_\lambda}\right]_{,\sigma} \equiv 0 \quad , \tag{21}$$

where we have used the usual definition of the matter stress-energy tensor:

$$T^{\mu\nu} \equiv \frac{1}{2}(-g)^{1/2} \frac{\delta \mathcal{L}_{NG}}{\delta g_{\mu\nu}} \quad . \tag{22}$$

(iii) Noether Identity for $\mathcal{L}_G$

Equation (12), for $\mathcal{L}_G$, becomes

$$\Big\{- \mathcal{L}_G \xi^\sigma - \frac{\partial \mathcal{L}_G}{\partial z_{a,\sigma}} \bar{\delta} z_a - \frac{\partial \mathcal{L}_G}{\partial \phi_{b,\sigma}} \bar{\delta} \phi_b + 2 \left(\frac{\partial \mathcal{L}_G}{\partial z_{a,\rho\sigma}}\right)_{,\rho} \bar{\delta} z_a + 2 \left(\frac{\partial \mathcal{L}_G}{\partial \phi_{b,\rho\sigma}}\right)_{,\rho} \bar{\delta} \phi_b$$
$$- \left(\frac{\partial \mathcal{L}_G}{\partial z_{a,\rho\sigma}} \bar{\delta} z_a\right)_{,\rho} - \left(\frac{\partial \mathcal{L}_G}{\partial \phi_{b,\rho\sigma}} \bar{\delta} \phi_b\right)_{,\rho} - \frac{\delta \mathcal{L}_G}{\delta z_a} d_{a}{}^{\sigma}{}_{\rho} \xi^\rho - \frac{\delta \mathcal{L}_G}{\delta \phi_b} d_{b}{}^{\sigma}{}_{\rho} \xi^\rho \Big\}_{,\sigma} \equiv 0 \quad . \tag{23}$$

Since the ξ^σ are arbitrary functions, the coefficient of each derivative of ξ^σ in Eq. (23) must separately vanish. Using Eqs. (19) to write out $\bar{\delta} z_a$ and $\bar{\delta} \phi_b$ and equating to zero the coefficient of ξ^ρ yields the identity

$$\Big\{- \mathcal{L}_G \delta_\rho{}^\sigma + \frac{\partial \mathcal{L}_G}{\partial z_{a,\sigma}} z_{a,\rho} + \frac{\partial \mathcal{L}_G}{\partial \phi_{b,\sigma}} \phi_{b,\rho} - 2 \left(\frac{\partial \mathcal{L}_G}{\partial z_{a,\tau\sigma}}\right)_{,\tau} z_{a,\rho} - 2 \left(\frac{\partial \mathcal{L}_G}{\partial \phi_{b,\sigma\tau}}\right)_{,\tau} \phi_{b,\rho}$$
$$+ \left(\frac{\partial \mathcal{L}_G}{\partial z_{a,\tau\sigma}} z_{a,\rho}\right)_{,\tau} + \left(\frac{\partial \mathcal{L}_G}{\partial \phi_{b,\tau\sigma}} \phi_{b,\rho}\right)_{,\tau} - \frac{\delta \mathcal{L}_G}{\delta z_a} d_{a}{}^{\sigma}{}_{\rho} - \frac{\delta \mathcal{L}_G}{\delta \phi_b} d_{b}{}^{\sigma}{}_{\rho} \Big\}_{,\sigma} \equiv 0 \quad . \tag{24}$$

[Equation (24) can also be obtained from Eq. (23) by setting $\xi^\sigma = \delta_\rho{}^\sigma$.]

C. The Field Equations

Variation of the action yields the dynamical equations, which may be placed in two categories:

1. gravitational field equations

$$\frac{\delta \mathscr{L}_G}{\delta z_a} + \frac{\delta \mathscr{L}_{NG}}{\delta z_a} = 0 \quad , \tag{25a}$$

2. nongravitational field equations

$$\frac{\delta \mathscr{L}_{NG}}{\delta q_\lambda} = 0 \quad . \tag{25b}$$

If we impose the nongravitational field equations, Eq. (25b), on the Bianchi identities for $\mathscr{L}_{NG}$, Eq. (21), we obtain the "matter response equations"

$$T_\mu{}^\nu{}_{;\nu} = 0 \quad , \tag{26}$$

where the covariant divergence is with respect to the metric $g_{\alpha\beta}$. A further useful relation may be obtained if we impose the gravitational field equations, Eq. (25a), on Eqs. (20) to obtain

$$z_{a,\rho} \frac{\delta \mathscr{L}_{NG}}{\delta z_a} - \phi_{b,\rho} \frac{\delta \mathscr{L}_G}{\delta \phi_b} - \left(d_b{}^\sigma{}_\rho \frac{\delta \mathscr{L}_G}{\delta \phi_b} - d_a{}^\sigma{}_\rho \frac{\delta \mathscr{L}_{NG}}{\delta z_a} \right)_{,\sigma} = 0 \quad . \tag{27}$$

Equations (25a) and (25b), together with the prior-geometric constraints Eq. (18) and a possible decomposition Eq. (17) for $g_{\mu\nu}$ in terms of (z_a, ϕ_b, ψ_c), comprise the "physical laws" of the representation. These laws can determine the field variables z_a, ϕ_b, ψ_c, and q_λ only up to four arbitrary functions corresponding to coordinate freedom. In the case where no absolute variables are present, this means that the field equations, Eqs. (25), cannot be all independent of one another; the number of independent field equations must be fewer by four than the number of variables $\{z_a, q_\lambda\}$. This is the case, for example, in general relativity (GRT), where four of the gravitational field equations reduce to $T_{\mu\ ;\nu}^{\ \nu} = 0$. The same is true for all other theories that are devoid of absolute variables:

<u>Theorem</u>: The matter response equations $T_{\mu\ ;\nu}^{\ \nu} = 0$ of a Lagrangian-based, generally covariant, metric (LBGCM) theory of gravity follow from the gravitational field equations if and only if there exist no absolute variables in the theory (no ϕ_b and ψ_c).

<u>Proof</u>: The dynamical Eqs. (25a), plus Eqs. (22), plus the functional dependence of $\mathscr{L}_{NG}$ imply

$$\frac{\delta \mathscr{L}_G}{\delta z_a} = -\frac{\delta \mathscr{L}_{NG}}{\delta g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial z_a} = -\frac{1}{2}(-g)^{1/2} T^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial z_a} . \tag{28}$$

Also, one has identically

$$\bar{\delta} g_{\mu\nu} \equiv \frac{\partial g_{\mu\nu}}{\partial z_a} \bar{\delta} z_a + \frac{\partial g_{\mu\nu}}{\partial \phi_b} \bar{\delta} \phi_b + \frac{\partial g_{\mu\nu}}{\partial \psi_c} \bar{\delta} \psi_c , \tag{29}$$

which, when Eqs. (19) are used and the arbitrariness of the ξ^ρ is invoked, implies the relations

$$g_{\mu\nu,\rho} = \frac{\partial g_{\mu\nu}}{\partial z_a} z_{a,\rho} + \frac{\partial g_{\mu\nu}}{\partial \phi_b} \phi_{b,\rho} + \frac{\partial g_{\mu\nu}}{\partial \psi_c} \psi_{c,\rho} , \tag{30}$$

$$-2g_{\rho(\mu}\delta_{\nu)}^{\ \sigma} = \frac{\partial g_{\mu\nu}}{\partial z_a} d_{a\ \rho}^{\ \sigma} + \frac{\partial g_{\mu\nu}}{\partial \phi_b} d_{b\ \rho}^{\ \sigma} + \frac{\partial g_{\mu\nu}}{\partial \psi_c} d_{c\ \rho}^{\ \sigma} . \tag{31}$$

On the left-hand side of Eq. (31) we have used the explicit form for the $d_{a\ \rho}^{\ \mu}$ function belonging to the functional change in $g_{\mu\nu}$ [see Eq. (19e)]. If Eq. (28) is now multiplied by $z_{a,\rho}$ and then $d_{a\ \rho}^{\ \sigma}$, and Eqs. (30) and (31) are used, one obtains the two relations:

$$z_{a,\rho} \frac{\delta \mathscr{L}_G}{\delta z_a} = -\frac{1}{2}(-g)^{1/2} T^{\mu\nu} \left(g_{\mu\nu,\rho} - \frac{\partial g_{\mu\nu}}{\partial \phi_b} \phi_{b,\rho} - \frac{\partial g_{\mu\nu}}{\partial \psi_c} \psi_{c,\rho} \right) , \tag{32}$$

$$d_{a\ \rho}^{\ \sigma} \frac{\delta \mathscr{L}_G}{\partial z_a} = +\frac{1}{2}(-g)^{1/2} T^{\mu\nu} \left(2g_{\rho(\mu}\delta_{\nu)}^{\ \sigma} + \frac{\partial g_{\mu\nu}}{\partial \phi_b} d_{b\ \rho}^{\ \sigma} + \frac{\partial g_{\mu\nu}}{\partial \psi_c} d_{c\ \rho}^{\ \sigma} \right) . \tag{33}$$

Equations (32) and (33), when substituted into the identity, Eq. (20), yield

$$\frac{1}{2}(-g)^{1/2}T^{\mu\nu}\left(g_{\mu\nu,\rho} - \frac{\partial g_{\mu\nu}}{\partial\phi_b}\phi_{b,\rho} - \frac{\partial g_{\mu\nu}}{\partial\psi_c}\psi_{c,\rho}\right) - \phi_{b,\rho}\frac{\delta\mathscr{L}_G}{\delta\phi_b} - \left(d_b{}^\sigma{}_\rho\frac{\delta\mathscr{L}_G}{\delta\phi_b}\right)_{,\sigma}$$

$$- \frac{1}{2}\left[(-g)^{1/2}T^{\mu\nu}\left(2g_{\rho(\mu}\delta_{\nu)}{}^\sigma + \frac{\partial g_{\mu\nu}}{\partial\phi_b}d_b{}^\sigma{}_\rho + \frac{\partial g_{\mu\nu}}{\partial\psi_c}d_c{}^\sigma{}_\rho\right)\right]_{,\sigma} = 0 \quad . \tag{34}$$

Finally, using the identity

$$(-g)^{1/2}T_\alpha{}^\beta{}_{;\beta} \equiv \left(T_\alpha{}^\beta(-g)^{1/2}\right)_{,\beta} - \frac{1}{2}(-g)^{1/2}g_{\mu\nu,\alpha}T^{\mu\nu} \quad , \tag{35}$$

Eq. (34) becomes

$$(-g)^{1/2}T_\rho{}^\sigma{}_{;\sigma} = -\frac{1}{2}(-g)^{1/2}T^{\mu\nu}\left(\frac{\partial g_{\mu\nu}}{\partial\phi_b}\phi_{b,\rho} + \frac{\partial g_{\mu\nu}}{\partial\psi_c}\psi_{c,\rho}\right) - \phi_{b,\rho}\frac{\delta\mathscr{L}_G}{\delta\phi_b}$$

$$-\left(d_b{}^\sigma{}_\rho\frac{\delta\mathscr{L}_G}{\delta\phi_b}\right)_{,\sigma} - \frac{1}{2}\left[(-g)^{1/2}T^{\mu\nu}\left(\frac{\partial g_{\mu\nu}}{\partial\phi_b}d_b{}^\sigma{}_\rho + \frac{\partial g_{\mu\nu}}{\partial\psi_c}d_c{}^\sigma{}_\rho\right)\right]_{,\sigma} \quad . \tag{36}$$

Only the gravitational field Eq. (25a) and identities were used to obtain Eq. (36); hence it is equivalent to the gravitational field equations. Obviously, if there are no absolute variables ($\psi_c = \phi_b = 0$), the right-hand side of Eq. (36) vanishes and one obtains Eq. (26). On the other hand, if some of the ϕ_b and ψ_c do not vanish, the right-hand side of Eq. (36) does not in general vanish[11] and Eq. (27) is not implied. Thus the theorem is proved.

This theorem makes it clear that in theories with no absolute variables, one has four fewer independent field equations than variables, so the field equations leave the coordinate system unconstrained.

By contrast, generally convariant theories with absolute variables typically do not contain any redundancies among the field equations. In this case it is the responsibility of the prior geometric constraints (18) to avoid constraining the coordinate system. One must be able to satisfy them in any desired coordinate system — and after having picked a specific coordinate system, in which the absolute variables then take on specific forms, one can solve all of the field equations (which are now all independent) for the specific forms of all of the dynamical variables.

D. Singular Lagrangians

In the previous subsections, we have delineated the identities and field equations resulting from the particular form of the Lagrangian given in Eqs. (16). Throughout the discussions, and also in the proof of the theorem in Sec. III.C, we have tacitly assumed that the gravitational field equations are consistent with the nongravitational field equations. In general, Euler-Lagrange equations obtained from the variation of an action should be consistent among themselves. Anomalies may occur, however, when the action integral admits a partial gauge group[12] — i.e., when a portion, but not all, of the action integral is invariant under a group of transformations generated by arbitrary functions (called gauge group). We can find no general rule to detect such "singular Lagrangians" but shall illustrate with some examples. We will see that the "inconsistencies" can be expressed as some extraneous constraints on the field sources.

(i) Dicke-Brans-Jordan Theory with $\omega = -3/2$

$$I = \int [R\phi^2 + 6\,\phi_{,\mu}\phi_{,\nu}g^{\mu\nu}]\sqrt{-g}\,d^4x + \int \mathcal{L}_{NG}(g_{\mu\nu}, q_\lambda)\,d^4x\,. \tag{37}$$

The gravitational part of the Lagrangian has been considered in Sec. II. It is invariant under the "scale transformation of the second kind." This yields identity Eq. (11). The gravitational field equations are

$$\frac{\delta\mathcal{L}_G}{\delta g_{\mu\nu}} = -\frac{\sqrt{-g}}{2}\,T^{\mu\nu}\,, \quad \frac{\delta\mathcal{L}_G}{\delta\phi} = 0\,. \tag{38a}$$

Substituting these into Eq. (11) yields

$$g_{\mu\nu}T^{\mu\nu} = 0 \quad . \tag{38b}$$

This is definitely inconsistent with most of the sources one would want to put in $\mathcal{L}_{NG}$.

(ii) Lorentz Symmetric Spin-2 Theory

$$I = \int (\mathcal{L}_G + \mathcal{L}_{NG})\, d^4x \quad . \tag{39a}$$

where

$$\mathcal{L}_G = \eta^{\alpha\beta}\eta^{\gamma\delta}\eta^{\sigma\tau}(2\, g_{\alpha\sigma|\beta}g_{\gamma\tau|\delta} - g_{\alpha\sigma|\beta}g_{\delta\gamma|\tau} + g_{\alpha\beta|\sigma}g_{\gamma\delta|\tau}$$

$$- g_{\alpha\gamma|\sigma}g_{\beta\delta|\tau})\ (-\eta)^{1/2} \tag{39b}$$

$$\mathrm{Riem}(\underline{\eta}) = 0 \quad . \tag{39c}$$

We denote by a bar "|" covariant derivatives with respect to $\eta_{\mu\nu}$. The gravitational Lagrangian $\mathcal{L}_G$, admits a gauge group ($\mathcal{L}_G$ is unchanged under the transformation)

$$\overline{\delta} g_{\mu\nu} = -2\, \eta_{\rho(\mu}\delta_{\nu)}^{\sigma}\xi^{\rho}{}_{,\sigma} - \eta_{\mu\nu,\rho}\xi^{\rho} \quad . \tag{40}$$

This leads to a Bianchi identity corresponding to Eq. (3),

$$[\eta_{\mu(\alpha}\delta_{\beta)}{}^{\nu}\frac{\delta\mathcal{L}_G}{\delta g_{\alpha\beta}}]_{|\nu} \equiv 0 \quad . \tag{41}$$

Substituting in the gravitational field equations, we obtain a constraint on the matter stress-energy tensor:

$$[\eta_{\mu(\alpha}\delta_{\beta)}{}^{\nu}\,(-g)^{1/2}T^{\alpha\beta}]_{|\nu} = 0 \quad . \tag{42}$$

Equation (42) is inconsistent with the matter response Eqs. (26) for most sources $T^{\mu\nu}$.

E. Conservation Laws

We now derive conservation laws useful for defining a physical total energy momentum for matter and fields. We will be interested only in conservation laws of the forms

$$\Theta^{\mu\nu}{}_{,\nu} = 0 \quad , \tag{43a}$$

$$\Theta_{\mu}{}^{\nu}{}_{,\nu} = 0 \quad , \tag{43b}$$

where $\Theta^{\mu\nu}$ or $\Theta_{\mu}{}^{\nu}$ reduce to $T^{\mu\nu}$ or $T_{\mu}{}^{\nu}$ in flat spacetime ("in the absence of gravity"; see Paper I). In some cases, identities resulting from invariance under MMG can also be put in the form of a vanishing ordinary divergence [see e.g., Eq. (23)]. However, the quantity that has an identically vanishing divergence typically does not reduce to the matter stress-energy tensor in the absence of gravity. Hence Eqs. (23) and (24) do not directly yield the conservation laws we seek.

Once established, Eqs. (43) enable us to define conserved quantities,

$$P^{\mu} = \int_{\Sigma} \Theta^{\mu\nu} d^3\Sigma_{\nu} \quad , \tag{44a}$$

$$P^{\mu} = \int_{\Sigma} \Theta_{\mu}{}^{\nu} d^3\Sigma_{\nu} \quad . \tag{44b}$$

The integrals in Eqs. (44) vanish when taken over a closed three-dimensional hypersurface Σ_{ν}. If a coordinate system is chosen in which Σ_{ν} is a constant-time hypersurface and extends to asymptotically flat infinity in space, then P^{μ} and P_{μ} are time independent and are given by

$$P^{\mu} = \int \Theta^{\mu 0} d^3x \quad , \tag{45a}$$

$$P_{\mu} = \int \Theta_{\mu}{}^{0} d^3x \quad . \tag{45b}$$

If, in addition, $\Theta^{\mu\nu}$ is symmetric, we can likewise define the following set of conserved quantities

$$J^{\mu\nu} = 2\int_{\Sigma} x^{[\mu}\,\Theta^{\nu]\sigma} d^3\Sigma_\sigma = 2\int x^{[\mu}\,\Theta^{\nu]0} d^3x \quad . \tag{45c}$$

Since $\Theta^{\mu\nu}$ reduces to the matter stress-energy tensor in the absence of gravity, we can in fact interpret P^0 or P_0, as the total energy, P^i or P_i as the total momentum, and J^{ij} as the total angular momentum. J^{0i} determines the motion of the center of mass. (See, e.g., Box 5.6 of Ref. 13.) Note that for conserved angular momentum to exist, one must have a contravariant stress-energy "complex" $\Theta^{\mu\nu}$.

For general reference, and for purposes of clarifying the following theorem, we define the following:

LBGCM theory: Lagrangian-based, generally covariant, metric theory of gravity.

LBGCM* theory: LBGCM which has at least one symmetry group (group that produces $\bar{\delta}\phi_b = 0$ for all absolute variables ϕ_b) with these properties:

i) The group has at least 4 dimensions.

ii) If ξ^μ is a generator of the symmetry group, then

$$\xi^\mu \to \text{const.}$$

and

$$d_b{}^\sigma{}_\mu \frac{\partial g_{\alpha\beta}}{\partial \phi_b} \to g_{\mu(\beta}\delta^\sigma{}_{\alpha)}$$

where $\to$ denotes the limit to asymptotically flat infinity.

All LBGCM theories with no absolute variables are automatically LBGCM* theories. In all prior-geometric theories we have seen in the literature, constraints i) and ii) are obeyed; hence, the class of LBGCM* theories covers all LBGCM theories that we have seen.

It is well known that in general relativity, quantities $\Theta^{\mu\nu}$, P^{μ}, $J^{\mu\nu}$ can be found. The following theorem generalizes the result:

Theorem: Conservation laws of the form of Eq. (43b) exist for all LBGCM* theories.

The Lagrangians given in Eqs. (16) and (17) will be used for a model theory in the proof. They are general enough to include all specific metric theories known to us. The theorem will be proved in two steps: first for theories without absolute variables, then for theories with absolute variables.

Proof:

Case (i) No absolute variables are present.

In this case, Eq. (23) simplifies, with the help of the field equations, Eq. (25a), to become

$$\{-\mathcal{L}_G\xi^{\sigma} - \frac{\partial\mathcal{L}_G}{\partial z_{a,\sigma}}\bar{\delta}z_a + 2\left(\frac{\partial\mathcal{L}_G}{\partial z_{a,\sigma\tau}}\right)_{,\tau}\bar{\delta}z_a - \frac{\partial\mathcal{L}_G}{\partial z_{a,\sigma\tau}}(\bar{\delta}z_a)_{,\tau} + \frac{\delta\mathcal{L}_{NG}}{\delta z_a}d_a{}^{\sigma}{}_{\rho}\,\xi^{\rho}\}_{,\sigma} = 0\,. \quad \forall\xi^{\sigma}\,. \tag{46a}$$

This is already in the form [Eq. (43)] we seek because $\delta\mathcal{L}_{NG}/\delta z_a$ yields, among other things, the matter stress-energy tensor $T^{\mu\nu}$. We note that there are in fact an infinity of conservation laws embodied in Eq. (46a) since the ξ^{ρ}'s are completely arbitrary. This richness of conserved total energy-momentum complexes is to be associated with the absence of absolute variables, i.e., all gravitational fields are dynamical.

With some particular choice of ξ^{ρ}, we can rewrite Eq. (46a) in a more transparent form. Let $\xi^{\rho} = \delta^{\rho}{}_{\alpha}$. Then, with the help of Eq. (25a) and Eq. (33) [remembering that ϕ_b and ψ_c do not exist], we obtain

$$\{-\mathcal{L}_G\delta_\rho^{\ \sigma} + \frac{\partial\mathcal{L}_G}{\partial z_{a,\sigma}} z_{a,\rho} - 2(\frac{\partial\mathcal{L}_G}{\partial z_{a,\sigma\tau}})_{,\tau} z_{a,\rho} + \frac{\partial\mathcal{L}_G}{\partial z_{a,\sigma\tau}} z_{a,\rho\tau} - (-g)^{1/2}\, T_\rho^{\ \sigma}\}_{,\sigma} = 0 \ . \quad (46b)$$

This agrees with Einstein's prescription for obtaining the stress-energy pseudo-tensor in GRT.

Case (ii) Absolute variables present.

Letting $\mathcal{L}$ be the total Lagrangian in Eq. (3), use the dynamical field Eqs. (25a) to obtain

$$C_{b\mu}\frac{\delta(\mathcal{L}_G + \mathcal{L}_{NG})}{\delta\phi_b} - \left[d_b{}^\rho{}_\mu \frac{\delta(\mathcal{L}_G + \mathcal{L}_{NG})}{\delta\phi_b}\right]_{,\rho} = 0 \ . \qquad (47)$$

If Eq. (47) is now multiplied by arbitrary functions ξ^μ (and summed over μ) and Eq. (19c) is used, the result is

$$\left[\xi^\mu d_b{}^\sigma{}_\mu \frac{\delta(\mathcal{L}_G + \mathcal{L}_{NG})}{\delta\phi_b}\right]_{,\sigma} = \frac{\delta(\mathcal{L}_G + \mathcal{L}_{NG})}{\delta\phi_b}\,\bar{\delta}\phi_b \ . \qquad (48)$$

Equation (48) can be rewritten, with the help of Eq. (16b), as

$$\left\{\xi^\mu d_b{}^\sigma{}_\mu\left[\frac{\delta\mathcal{L}_G}{\delta\phi_b} + \frac{1}{2}(-g)^{1/2}\frac{\partial g_{\alpha\beta}}{\partial\phi_b}\, T^{\alpha\beta}\right]\right\}_{,\sigma} = \frac{\delta(\mathcal{L}_G + \mathcal{L}_{NG})}{\delta\phi_b}\,\bar{\delta}\phi_b \ . \qquad (49)$$

If one now chooses ξ^μ to be a generator of that symmetry group which appears in the definition of LBGCM*, i.e., a descriptor such that

$$\bar{\delta}\phi_b = 0 \ , \qquad (50)$$

and uses the defined properties of LBGCM* theories, then Eq. (49) takes on the form of Eq. (43b). In a coordinate system in which the absolute objects are constants, the total stress-energy tensor in brackets on the LHS of Eq. (49) reduces to a form identical to that in Eq. (46b).

As an example, consider conformally flat theories.[14] The absolute object is $\eta_{\mu\nu}$ and one has

$$\mathcal{L}_G = \mathcal{L}_G(\eta_{\alpha\beta}, \varphi) \quad , \tag{51a}$$

$$g_{\alpha\beta} = \eta_{\alpha\beta} f(\varphi) \quad , \tag{51b}$$

$$\mathrm{Riem}(\eta_{\alpha\beta}) = 0 \quad , \tag{51c}$$

where f is some function of the dynamical scalar field φ. For $\eta_{\mu\nu}$, the $d_b{}^\sigma{}_\alpha$ function is just

$$d_b{}^\sigma{}_\alpha = - 2\, \eta_{\mu(\alpha}\delta^\sigma{}_{\nu)} \quad . \tag{52}$$

If one now uses Eqs. (51) and (52), then Eq. (49), with its RHS zero, becomes

$$\left\{\xi^\mu \left[(-g)^{1/2} T_\mu{}^\sigma + 2\, \eta_{\mu\alpha} \frac{\delta \mathcal{L}_G}{\delta \eta_{\alpha\sigma}}\right]\right\}_{,\sigma} = 0 \quad . \tag{53}$$

Note that the conserved stress-energy tensor in Eq. (53) is a true tensor (density), as opposed to the corresponding quantity in theories without absolute objects (GRT, for example). Such "true" stress-energy tensors, which typically exist in prior geometric theories of gravity (theories with absolute objects), are associated with the symmetry group of the absolute objects.

To summarize: integral conservation laws are associated with the symmetries of the representation. When there are no absolute objects the symmetry group is MMG, the conservation laws are the result of covariance under coordinate transformations, and the "energy-momentum complexes" $\Theta_\mu{}^\nu$ are typically not tensor densities. On the other hand, when absolute objects are present, their symmetry group (smaller than MMG!) produces the conservation laws; and $\Theta_\mu{}^\nu$ typically are tensor densities.

F. Further Discussions

In Sec. III.E we obtained conserved energy and meomentum P_μ for LBGCM* theories in terms of a volume integral over $\Theta_\mu^{\ 0}$. We will limit our ensuing treatment of such quantities to their roles in asymptotically flat spacetime, because only there are they definable in a physically meaningful way.[15] To correspond as closely as possible to the experimental situation, we would like to know if we can evaluate these conserved quantities in the asymptotic region without any detailed knowledge of the near-field behavior.

It is clear from Eqs. (45) that if and only if $\Theta^{\mu\nu}$ and $\Theta_\mu^{\ \nu}$ are derivatives of a "superpotential":

$$\Theta^{\mu\nu} = \Lambda^{\mu\nu\alpha}{}_{,\alpha} \quad ; \quad \Lambda^{\mu\nu\alpha} = -\Lambda^{\mu\alpha\nu} \tag{54a}$$

$$\Theta_\mu^{\ \nu} = \Lambda_\mu^{\ \nu\alpha}{}_{,\alpha} \quad ; \quad \Lambda_\mu^{\ \nu\alpha} = -\Lambda_\mu^{\ \alpha\nu} \tag{54b}$$

can P_μ and P^μ be expressed as surface integrals:

$$P^\mu = \int \Lambda^{\mu[0\alpha]}{}_{,\alpha} d^3x = \oint \Lambda^{\mu 0 i} d^2\Sigma_i \quad , \tag{55a}$$

$$P_\mu = \int \Lambda_\mu^{\ [0\alpha]}{}_{,\alpha} d^3x = \oint \Lambda_\mu^{\ 0i} d^2\Sigma_i \quad . \tag{55b}$$

(Here square brackets [ ] denote antisymmetrized indices.) The general argument in Sec. III.E has no direct bearing on the existence of such superpotentials in LBGCM* theories. In fact, we do not at present know of any feature in the structure of the mathematical representations of a theory that is tied directly to the existence of superpotentials. While it is true that the existence of a divergenceless $\Theta^{\mu\nu}$ (or $\Theta_\mu^{\ \nu}$) in a certain region necessarily implies the existence of a superpotential from which the $\Theta^{\mu\nu}$ is derivable in that region (using the mathematics of differential forms), we have found[16] that such superpotentials either must be defined in the interior of the region, or are nonunique when defined on the boundary of the region. Consequently, no superpotential is guaranteed to exist which allows a unique P_μ to be defined in the asymptotically flat region around a gravitating source. Thus the existence of physically useful superpotentials associated with a divergenceless $\Theta^{\mu\nu}$ is theory-dependent (depends upon the detailed properties of $\Theta^{\mu\nu}$). Some conservative theories may have superpotentials and some may not.

One immediate consequence of superpotentials, when they exist, is that for every divergence-free $\Theta_\mu{}^\nu$ (and hence conserved P_μ), a corresponding divergence-free $\Theta^{\mu\nu}$ (and hence a conserved P^μ) can be constructed, and vice versa: Given a $\Theta_\mu{}^\nu$ (with a $\Lambda_\mu{}^{[\nu\alpha]}$), one simply defines a $\Lambda'^{\mu[\nu\alpha]}$ by, e.g.,

$$\Lambda'^{\mu[\nu\alpha]} \equiv g^{\mu\tau}\Lambda_\tau{}^{[\nu\alpha]} \quad , \tag{56a}$$

and a divergenceless $\Theta^{\mu\nu}$ is defined by

$$\Theta^{\mu\nu} = \Lambda'^{\mu[\nu\alpha]}{}_{,\alpha} \quad . \tag{56b}$$

Thus all LBGCM* theories that possess superpotentials have a divergence-free $\Theta^{\mu\nu}$ (and a conserved P^μ). The conservation of angular momentum hinges, however, on the symmetries of $\Theta^{\mu\nu}$, and thus far, our general arguments do not yield any useful information on this issue. In Sec. IV, we will take a different approach and derive empirical conditions in the post-Newtonian limit for the existence of a conserved angular momentum.

It was noted that Eq. (46a) gives an infinity of divergence-free $\Theta_\mu{}^\nu$. What about the corresponding P_μ; are there infinitely many of them? To seek insight into this question, one of us (DLL) has examined in detail the Dicke-Brans-Jordan theory and has found two conserved P_μ's that can be evaluated solely in terms of the asymptotic properties of the gravitational field.[17] This leads us to conclude that the P_μ's in general are not unique.

Once we know how to evaluate P^μ and P_μ in the asymptotic region, we would like to know their behavior under Lorentz transformations. From Eqs. (55) we see that, if in the general covariant mathematical representation of a theory the variables $\{y_A\}$ consist of nothing but scalars, vectors, tensors (and their respective densities) and affine connections, then the conserved P^μ and P_μ thus constructed will transform as 4-vectors under Lorentz transformations at asymptotic infinity. In Appendix A we give $\Theta^{\mu\nu}$ for various exemplary theories. In cases where superpotentials exist, we give them along with $\Theta^{\mu\nu}$. (As remarked earlier, there is no theory independent way of deriving superpotentials — those given in Appendix A are quoted from various references.) When $\Theta_\mu{}^\nu$ is given, we use the formulas derived in Sec. III.E.

IV. CONSERVATION LAWS IN THE POST-NEWTONIAN APPROXIMATION

In this section, we complement the analysis in Sec. III by discussing conservation laws in the larger domain of general metric theories, not necessarily Lagrangian-based, but restricted to the post-Newtonian approximation (gravity weak, stresses small compared to mass-energy density, and relative velocities small compared to that of light) In this domain the Paramatrized Post-Newtonian formalism[6] is applicable. Our analysis is patterned closely after the work by C. M. Will,[18] except that we consider a 10-parameter metric in the "PPN gauge" rather than the standard[6] 9-parameter metric. This 10-parameter metric was introduced recently in Ref. 19 by Will. It allows one to encompass in the PPN formalism the theories of Whitehead,[20] Deser and Laurent,[21] and Girotti and Wisnivesky,[22] theories requiring, in addition to the standard nine potential form of the metric, a "Whitehead term." To date, the 10-parameter metric encompasses all metric theories known to us.

Following C. M. Will, we will obtain the conditions at the post-Newtonian order for any metric theory to have a conserved P^{μ} (and P_{μ}). We will also obtain the appropriate conditions for there to exist a conserved $J^{\mu\nu}$.

We now proceed with the details.

A. The Metric

In the PPN coordinate system, we take the generic metric to have the form

$$g_{00} = 1 - 2U + 2\beta U^2 - 4\Phi + \zeta_1 \mathcal{A} + 2\zeta_w \Phi_w \quad , \tag{57a}$$

$$g_{0i} = \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1) V_i + \frac{1}{2}(1 + \alpha_2 - \zeta_1) W_i \quad , \tag{57b}$$

$$g_{ij} = -\delta_{ij}(1 + 2\gamma U) \quad , \tag{57c}$$

where

$$U(\vec{x}, t) \equiv \int \frac{\rho(\vec{x}', t)}{|x - x'|} d^3x'$$

$$\Phi(\vec{x}, t) = \Phi_1(\vec{x}, t) + \Phi_2(\vec{x}, t) + \Phi_3(\vec{x}, t) + \Phi_4(\vec{x}, t) \equiv \int \frac{\rho(\vec{x}', t)(\phi_1 + \phi_2 + \phi_3 + \phi_4) d^3x'}{|x - x'|}$$

$$\phi_1 \equiv \frac{1}{4}(\alpha_3 + 2\gamma + 2 + \zeta_1) v^2, \; \phi_2 \equiv \frac{1}{2}(\zeta_2 - 2\beta + 3\gamma + 1) U, \; \phi_3 \equiv \frac{1}{2}(\zeta_3 + 1) \Pi,$$

$$\phi_4 \equiv \frac{3}{2}(\zeta_4 + \gamma) p/\rho$$

$$\mathcal{A}(\vec{x}, t) \equiv \int \frac{\rho(\vec{x}', t)[(\vec{x} - \vec{x}') \cdot \vec{v}(\vec{x}')]}{|x - x'|^3} d^3x' \; , \; V_i(\vec{x}, t) \equiv \int \frac{\rho(\vec{x}', t)\, v_i(\vec{x}')}{|x - x'|} d^3x'$$

$$W_i(\vec{x}, t) \equiv \int \frac{\rho(\vec{x}', t)[\vec{v} \cdot (\vec{x} - \vec{x}')](x_i - x'_i)\, d^3x'}{|x - x'|^3}$$

$$\Phi_w(\vec{x}, t) = \int \frac{\rho(\vec{x}', t)\, \rho(\vec{x}'', t)}{|x - x'|^3} \left[\frac{(\vec{x}' - \vec{x}'')}{|x' - x''|} - \frac{(\vec{x} - \vec{x}'')}{|x' - x''|} \right] \cdot (\vec{x} - \vec{x}')\, d^3x' d^3x'' \quad .$$

Each metric theory is characterized in the PPN limit by its values for the ten PPN parameters β, γ, α_1, α_2, α_3, ζ_1, ζ_2, ζ_3, ζ_4, ζ_w. For simplicity we have chosen to work in a coordinate system which is at rest with respect to any "preferred frame" that may exist in the generic gravitation theory. Hence our metric of Eqs. (57) does not contain "w terms" (see Ref. 23 for further discussion).

B. Conservation of Energy Momentum

We attempt to construct quantities $\Theta^{\mu\nu}$ and $\Theta_\mu{}^\nu$ of the form

$$\Theta^{\mu\nu} = (1 - aU)(t^{\mu\nu} + T^{\mu\nu}) \quad , \tag{58a}$$

$$\Theta_\mu{}^\nu = (1 - aU)(t_\mu{}^\nu + T_\mu{}^\nu) \quad , \tag{58b}$$

satisfying

$$\Theta^{\mu\nu}{}_{,\nu} = \Theta_\mu{}^\nu{}_{,\nu} = 0 \quad . \tag{59}$$

In Eqs. (58) "a" is an undetermined constant and $t_\mu{}^\nu$ and $t^{\mu\nu}$ are the mixed-index and contravariant gravitational stress-energy pseudo-tensors. We will now sketch the calculations for a law of the form of Eq. (58a) and quote the results for a law of the form of Eq. (58b).

Using the matter response Eqs. (26) present in all metric theories and Eqs. (58), we find that $t^{\mu\nu}$ must satisfy the equation

$$t^{\mu\nu}{}_{,\nu} = aU_{,\nu}t^{\mu\nu} + \Gamma_\nu{}^\mu{}_\sigma T^{\nu\sigma} + \Gamma_\sigma{}^\nu{}_\nu T^{\mu\sigma} + aU_{,\nu}T^{\mu\nu} \quad , \tag{60}$$

where the $\Gamma^\alpha{}_{\beta\gamma}$ are the Christoffel symbols. The ability to construct a $t^{\mu\nu}$ and consequently a $\Theta^{\mu\nu}$ rests upon integrability conditions for Eq. (60).

We now calculate the Christoffel symbols to PPN order from the metric of Eqs. (57) and use the following identities

$$4\pi\rho f_{,i} \equiv -2(\partial/\partial x^i)[U_{,(i}f_{,j)} - \tfrac{1}{2}\delta_{ij}U_{,k}U_{,k}] + U_{,i}\nabla^2 f \quad \text{for any } f \quad , \tag{61a}$$

$$(4\pi)^{-1}U_{,i}\nabla^2\Phi_w \equiv -(8\pi)^{-1}U_{,i}|\nabla U|^2 - \frac{3}{4\pi}[U(U_{,i}U_{,j} - \frac{1}{2}\delta_{ij}|\nabla U|^2]_{,j}$$
$$+ U_{,i}[-2\rho U + \vec{\nabla}\rho \cdot \vec{\nabla}\chi + (4\pi)^{-1}\nabla^2(\vec{\nabla}U \cdot \vec{\nabla}\chi)] \quad , \qquad (61b)$$

$$\mathcal{A}_{,i} - \Phi_{1,i} - \chi_{,00i} + \int \frac{[\rho' U'_{,i} - p'_{,i}]}{|\vec{x} - \vec{x}'|} d^3x' - \int \frac{(\rho' U'_{,k} - p'_{,k})(x_k - x'_k)(x_i - x'_i)}{|\vec{x} - \vec{x}'|^3} d^3x$$
$$\equiv 0 \quad , \qquad (61c)$$

$$\chi(\vec{x}, t) = -\int \rho(\vec{x}', t)|\vec{x} - \vec{x}'| \, d^3x' \quad . \qquad (61d)$$

Then, Eq. (60) can be put into the form

$$4\pi t^{0\nu}{}_{,\nu} = 4\pi(t^{00}{}_{,0} + t^{0i}{}_{,i})$$
$$= \frac{\partial}{\partial t}[\frac{1}{2}|\nabla U|^2 (6\gamma + 2a - 5)]$$
$$- \frac{\partial}{\partial x^i}[(3\gamma + a - 2) U_{,0}U_{,i} + 2(3\gamma + a - 3) U_{,j}V_{[j,i]}] \quad , \qquad (62a)$$

$$4\pi t^{i\nu}{}_{,\nu} = 4\pi(t^{i0}{}_{,0} + t^{ik}{}_{,k})$$
$$= \frac{\partial}{\partial t}[\frac{1}{2}(\alpha_1 - 2\alpha_2 + 4\gamma + 2 + 2\zeta_1 + 4\zeta_w) U_{,0}U_{,i} - (5\gamma + a - 1) U\nabla^2 V_i$$
$$+ (\alpha_1 + 4\gamma + 4) U_{,j}V_{[j,i]}]$$
$$+ \frac{\partial}{\partial x^j}\Big\{[1 - \frac{1}{2}(2\zeta_2 - 2a - 3\zeta_w) U] \Gamma_{ij}(U)$$
$$+ (2\zeta_w - \zeta_1)\Gamma_{ij}(\mathcal{A}) + (\alpha_3 + 2\gamma + 2 + \zeta_1 + 2\zeta_w)\Gamma_{ij}(\Phi_1)$$
$$+ 2(\zeta_2 - 2\beta + 3\gamma + 1)\Gamma_{ij}(\Phi_2) + 2(\zeta_3 + 1)\Gamma_{ij}(\Phi_3)$$
$$+ [6(\zeta_4 + \gamma) + 4\zeta_w]\Gamma_{ij}(\Phi_4)$$
$$+ U_{,(i}[(\alpha_1 - \alpha_2 + 4\gamma + 3 + \zeta_1) V_{j)} + (\alpha_2 + 1 - \zeta_1) W_{j)}]_{,0} - 2\zeta_w\Gamma_{ij}(\Psi)$$
$$- \frac{1}{2}\delta_{ij}U_{,k}[(\alpha_1 - \alpha_2 + 4\gamma + 3 + \zeta_1) V_k + (\alpha_2 + 1 - \zeta_1) W_k]_{,0}$$
$$- \zeta_w\delta_{ij}(U_{,0})^2 \qquad (62b)$$

$$- 2(\alpha_1 + 4\gamma + 4)\ (V_{[i,k]}V_{[j,k]} - \frac{1}{4}\delta_{ij}V_{[k,\ell]}V_{[k,\ell]}) + 2\zeta_w\Gamma_{ij}(\chi_{,00})$$
$$- \frac{1}{4}\ (\alpha_1 + 2\alpha_2 + 4\gamma + 2 + 2\zeta_1)\ \delta_{ij}(U_{,0})^2 + 2\zeta_w\Gamma_{ij}(U_{,\ell}\chi_{,\ell})$$
$$+ (5\gamma + a - 1)\ U(\rho v^i v^j + \delta^{ij}p) + 8\pi\zeta_w\rho\chi_{,(i}U_{,j)} - \zeta_w\chi_{,ij}\phi_{k,k}$$
$$+ \zeta_w\delta_{ij}(\chi\phi_{k,k\ell})_{,\ell} - \zeta_w\chi\phi_{k,kij}\Big| + 4\pi Q^i \quad , \qquad \text{(62b con't.)}$$

where

$$\Gamma_{ij}(X) \equiv U_{,(i}X_{,j)} - \frac{1}{2}\delta_{ij}U_{,k}X_{,k} \quad , \qquad \text{(63a)}$$

$$\Psi(\vec{x}, t) \equiv \int \frac{\rho(\vec{x}', t)\ U'_{,j}(x_j - x'_j)}{|x - x'|}\ d^3x' \quad , \qquad \text{(63b)}$$

$$\phi_k\ (\vec{x}, t) = \int \frac{\rho' U'_{,k}}{|x - x'|}\ d^3x' \quad . \qquad \text{(63c)}$$

and

$$Q^i \equiv U_{,i}[\frac{1}{2}\ (\alpha_3 + \zeta_1 + 2\zeta_w)\ \rho v^2 + \frac{1}{8\pi}\ |\nabla U|^2(\zeta_2 - \zeta_w)$$
$$+ \zeta_3\rho\Pi + (3\zeta_4 + 2\zeta_w)\ p + \frac{1}{8\pi}\ \nabla^2 \mathcal{a}(\zeta_1 + 2\zeta_w)] \quad . \qquad \text{(63d)}$$

We have been utterly unable to write the terms in Q^i as a combination of gradients and time derivatives of matter variables and gravitational fields. Therefore the integrability conditions on $t^{i\nu}$ are that each term in Q^i must vanish separately, i.e.,

$$\frac{1}{2}\ (\alpha_3 + \zeta_1\ ) + \zeta_w = 0 \quad , \qquad \text{(64a)}$$

$$\zeta_2 - \zeta_w = 0 \quad , \qquad \text{(64b)}$$

$$\zeta_3 = 0 \quad , \qquad \text{(64c)}$$

$$3\zeta_4 + 2\zeta_w = 0 \quad , \qquad \text{(64d)}$$

$$\zeta_1 + 2\zeta_w = 0 \quad . \qquad \text{(64e)}$$

Equations (64) represent constraints that must be satisfied by the PPN parameters of a metric theory in order that there be conservation laws of the form of Eqs. (58a). A parallel calculation has been carried out for the integrability conditions on $t_\mu{}^\nu$ for conservation laws of the form of Eq. (58b); the result is that the same five constraints, Eqs. (64), must hold. The resulting $t^{\mu\nu}$ and $t_\mu{}^\nu$ are given in Appendix B. Will[18] obtained the results given in Eqs. (64), except without the ζ_w parameter appearing, since his generic metric did not contain the Whitehead term Φ_w.

Since we have proved in Sec. III.E that "mixed index" conservation laws of the form of Eqs. (58b), (59) exist for all LBGCM* theories, we can now state the following theorem:

Theorem: For all LBGCM* theories with metric given by Eq. (57), the PPN parameters satisfy the five constraints given in Eqs. (64).

A survey of the literature reveals that not only do all the Lagrangian-based metric theories satisfy the constraints in Eqs. (64), but there is no known theory satisfying these constraints that is not Lagrangian-based. We are thus persuaded to present the following conjecture:

Conjecture: For metric theories of gravity, the existence of a conserved energy momentum P^μ [defined by Eqs. (44a) and (43a)] is equivalent to the existence of a Lagrangian formulation.

From Eqs. (64) we see immediately that any metric theory admitting a conserved P^μ can have at most five arbitrary PPN parameters. To complement this result, we have generalized "Ni's New Theory"[24] to obtain a Lagrangian-based metric theory (see Appendix C) which has five arbitrary parameters in the post-Newtonian approximation. This, together with the theorem presented in Sec. III.E proves our conjecture at the post-Newtonian order.

C. Conservation of Angular Momentum

Equations (64) ensure that globally conserved energy-momentum vectors P^μ and P_μ exist. As mentioned previously, a conserved angular momentum tensor $J^{\mu\nu}$ can be defined if and only if $\Theta^{\mu\nu}$ (and hence $t^{\mu\nu}$) is symmetric. What constraints are required for a symmetric $t^{\mu\nu}$? An examination of Eqs. (62a), (62b) reveals that t^{ij} is manifestly symmetric, but t^{0i} is not equal to t^{i0}. However, Eqs. (62a) and (62b) determine $t^{\mu\nu}$ only up to a total divergence. We now seek quantities $S^{\mu\nu}$ such that

$$S^{\mu\nu}{}_{,\nu} = 0 \quad , \tag{65a}$$

and

$$t'^{\mu\nu} \equiv t^{\mu\nu} + S^{\mu\nu} = t'^{\nu\mu} \quad . \tag{65b}$$

Clearly, we can choose $S^{ij} = 0$. Setting $\mu = i$ in Eq. (65a) and using the fact that $t^{ij} = t^{ji}$, one concludes that $S^{i0} = 0$. An S^{0i} must then be found such that

$$S^{0i}{}_{,i} = (t^{i0} - t^{0i})_{,i} = - S^{00}{}_{,0} \quad . \tag{66}$$

With the help of Eqs. (62a) and (62b), Eq. (65) becomes

$$(4\pi)^{-1}[AU_{,i}U_{,0} + BU_{,j}(V_{j,i} - V_{i,j}) + CUV_{i,jj}]_{,i} = - S^{00}{}_{,0} \quad , \tag{67}$$

where

$$A \equiv \frac{1}{2}(\alpha_1 - 2\alpha_2 - 2 + 2\zeta_1) + 2\zeta_w + 5\gamma + a \quad , \tag{68a}$$

$$B \equiv \frac{1}{2}\alpha_1 + 5\gamma - 1 + a \quad , \tag{68b}$$

$$C \equiv - (5\gamma + a - 1) \quad . \tag{68c}$$

Now, using the identity

$$\frac{\partial}{\partial x^i}[U_{,i}U_{,0} - UV_{i,jj} + U_{,j}(V_{j,i} - V_{i,j})] + \frac{\partial}{\partial t}(4\pi\rho U - U_{,j}U_{,j}) = 0 \quad , \tag{69}$$

we see that an S^{00} exists satisfying Eq. (67) if and only if A = B = - C, or

$$\alpha_2 - \zeta_1 - 2\zeta_w = 0 \quad , \tag{70a}$$

$$\alpha_1 = 0 \quad . \tag{70b}$$

Equation (70a), when combined with Eq. (64e), demands that $\alpha_2 = 0$. Equations (70), in addition to Eqs. (64), represent 7 constraints which must be satisfied by the 10 PPN parameters in order that there be conserved energy momentum and conserved angular momentum in the PN approximation. Note that the constant "a" appearing in Eqs. (58) has been left unconstrained contrary to the results of previous calculations.[25]

D. Gauge Dependence of the Constraints

The metric given in Eqs. (57) is in the so-called "standard PPN gauge." This is the gauge in which all solar system gravity experiments have been analyzed. For prior-geometric theories, however, the "absolute frame"[26] is the most natural coordinate frame in which to solve gravitational field equations for the metric, to investigate the existence of globally conserved integrals, etc. We are thus prompted to redo the above calculations for a more general gauge (with two additional parameters σ and τ):

$$g_{00} = 1 - 2U + 2\beta U^2 - 4\Phi + \zeta_1 \mathcal{A} + 2\zeta_w \Phi_w + 2\sigma \chi_{,00} \quad , \tag{71a}$$

$$g_{0i} = \frac{1}{2} \left[(\alpha_1 - \alpha_2 + 4\gamma + 3 + \zeta_1) V_i + (\alpha_2 + 1 - \zeta_1) W_i \right] \quad , \tag{71b}$$

$$g_{ij} = - \delta_{ij}(1 + 2\gamma U) - 2\tau \chi_{,ij} \quad , \tag{71c}$$

where the χ potential has been defined in Eq. (61d). This metric form encompasses the post-Newtonian limit of all known metric theories in the absolute frame. The constraints [analogous to those in Eq. (64)] necessary for there to be a globally conserved P^μ are

$$\frac{1}{2}(\alpha_3 + \zeta_1) + \tau + \zeta_w = 0 \quad , \tag{72a}$$

$$\zeta_2 + 2\tau - \zeta_w = 0 \quad , \tag{72b}$$

$$\zeta_3 = 0 \quad , \tag{72c}$$

$$3\zeta_4 + 2\zeta_w + 2\tau = 0 \quad , \tag{72d}$$

$$\zeta_1 + 2\zeta_w + 2\tau = 0 \quad , \tag{72e}$$

while the additional constraints [analogous to Eqs. (70)] for there to be a conserved $J^{\mu\nu}$ are

$$\alpha_2 - \zeta_1 + 2\sigma - 4\tau - 2\zeta_w = 0 \quad , \tag{73a}$$

$$\alpha_1 - 8\tau = 0 \quad . \tag{73b}$$

The constraints [Eqs. (72) and (73a)] can be shown to be invariant under all gauge transformations that leave the form of the metric in Eqs. (71) unchanged. Many prior geometric theories (see Appendix A) have a symmetric $\Theta^{\mu\nu}$ in the absolute frame. One wonders if the existence of such symmetric quantities is independent of the coordinate system. The results in Eqs. (72) and (73a) have provided a partial answer to this question, i.e., <u>if the globally conserved P^μ and $J^{\mu\nu}$ (to the post-Newtonian order) exist in one coordinate frame, then they exist in all coordinate frames related by a gauge transformation that leaves the form of the metric in Eq. (71) unchanged.</u>

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APPENDIX A

In this appendix, we summarize the expressions for the gravitational portion of the divergence-free $\Theta^{\mu\nu}$ or $\Theta_{\mu}{}^{\nu}$ for some metric theories of gravity. We also tabulate the corresponding superpotentials whenever they exist.

(i) <u>GRT</u>:

$$(-g)(t^{\mu\nu}_{LL} + T^{\mu\nu}) = \Lambda^{\mu[\nu\alpha]}{}_{,\alpha} \quad . \tag{A1a}$$

The gravitational stress-energy pseudo-tensor is[27]

$$\begin{aligned} t^{\mu\nu}_{LL} = (16\pi)^{-1} \Big\{ & (2\Gamma_{\sigma}{}^{\alpha}{}_{\tau}\Gamma_{\alpha}{}^{\beta}{}_{\beta} - \Gamma_{\sigma}{}^{\alpha}{}_{\beta}\Gamma_{\tau}{}^{\beta}{}_{\alpha} - \Gamma_{\sigma}{}^{\alpha}{}_{\alpha}\Gamma_{\tau}{}^{\beta}{}_{\beta})(g^{\mu\sigma}g^{\nu\tau} - g^{\mu\nu}g^{\sigma\tau}) \\ & + g^{\mu\sigma}g^{\tau\alpha}(\Gamma_{\sigma}{}^{\nu}{}_{\beta}\Gamma_{\tau}{}^{\beta}{}_{\alpha} + \Gamma_{\tau}{}^{\nu}{}_{\alpha}\Gamma_{\sigma}{}^{\beta}{}_{\beta} - \Gamma_{\alpha}{}^{\nu}{}_{\beta}\Gamma_{\sigma}{}^{\beta}{}_{\tau} - \Gamma_{\sigma}{}^{\nu}{}_{\tau}\Gamma_{\alpha}{}^{\beta}{}_{\beta}) \\ & + g^{\nu\sigma}g^{\tau\alpha}(\Gamma_{\sigma}{}^{\mu}{}_{\beta}\Gamma_{\tau}{}^{\beta}{}_{\alpha} + \Gamma_{\tau}{}^{\mu}{}_{\alpha}\Gamma_{\sigma}{}^{\beta}{}_{\beta} - \Gamma_{\alpha}{}^{\mu}{}_{\beta}\Gamma_{\sigma}{}^{\beta}{}_{\tau} - \Gamma_{\sigma}{}^{\mu}{}_{\tau}\Gamma_{\alpha}{}^{\beta}{}_{\beta}) \\ & + g^{\sigma\tau}g^{\alpha\beta}(\Gamma_{\sigma}{}^{\mu}{}_{\alpha}\Gamma_{\tau}{}^{\nu}{}_{\beta} - \Gamma_{\sigma}{}^{\mu}{}_{\tau}\Gamma_{\alpha}{}^{\nu}{}_{\beta}) \Big\} \quad , \end{aligned} \tag{A1b}$$

while the superpotential is

$$\Lambda^{\mu[\nu\sigma]}{}_{,\sigma} = (16\pi)^{-1}[(-g)(g^{\mu\nu}g^{\sigma\tau} - g^{\mu\sigma}g^{\nu\tau})]_{,\sigma\tau} \quad . \tag{A1c}$$

(ii) <u>General Scalar-Tensor Theory by Bergman, Wagoner</u>:[28]

We know of two distinct, conserved, energy-momentum P^{μ} that arise from the following two conservation laws:

(a)

$$(-g)(\phi/\phi_0)(t^{\mu\nu} + T^{\mu\nu}) = \Lambda^{\mu[\nu\alpha]}{}_{,\alpha} \quad ,^{29} \tag{A2a}$$

where the gravitational stress-energy pseudo-tensor is

$$\begin{aligned} t^{\mu\nu} = \phi t^{\mu\nu}_{LL} + (8\pi\phi)^{-1} \Big\{ & [w(\phi) - 1]\, \phi^{,\mu}\phi^{,\nu} - \tfrac{1}{2}[w(\phi) - 2]\, g^{\mu\nu}\phi_{,\alpha}\phi^{,\alpha} \Big\} \\ & + (8\pi)^{-1}\phi_{,\alpha}[\Gamma_{\sigma}{}^{\mu}{}_{\tau}(g^{\sigma\tau}g^{\nu\alpha} - g^{\tau\alpha}g^{\sigma\nu}) + \Gamma_{\sigma}{}^{\nu}{}_{\tau}(g^{\sigma\tau}g^{\mu\alpha} - g^{\tau\alpha}g^{\sigma\mu}) \\ & + \Gamma_{\sigma}{}^{\tau}{}_{\tau}(2g^{\mu\nu}g^{\sigma\alpha} - g^{\mu\sigma}g^{\nu\alpha} - g^{\mu\alpha}g^{\nu\sigma}) \\ & + \Gamma_{\sigma}{}^{\alpha}{}_{\tau}(g^{\mu\sigma}g^{\nu\tau} - g^{\mu\nu}g^{\sigma\tau})] \quad . \end{aligned} \tag{A2b}$$

and the superpotential is

$$\Lambda^{\mu[\nu\alpha]}{}_{,\alpha} = (16\pi\phi_0)^{-1}[\phi^2(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})]_{,\alpha\beta} \quad . \tag{A2c}$$

[$t^{\mu\nu}_{LL}$ in Eq. (A2c) is defined by Eq. (A1b).]

(b)

$$(-g)(\phi_0/\phi)(U^{\mu\nu} + T^{\mu\nu}) = \Lambda^{\mu[\nu\alpha]}{}_{,\alpha} \quad ,^{17} \tag{A2d}$$

and

$$[(-g)(\phi_0/\phi)(U^{\mu\nu} + T^{\mu\nu})]_{,\nu} = 0 \quad , \tag{A2e}$$

where the gravitational stress-energy pseudo-tensor is

$$U^{\mu\nu} = \phi t^{\mu\nu}_{LL} + 8\pi w(\phi)\ \phi(\phi^{,\mu}\phi^{,\nu} - \tfrac{1}{2} g^{\mu\nu}\phi^{,\alpha}\phi_{,\alpha}) + (8\pi)^{-1}(\phi^{;\mu\nu} - g^{\mu\nu}g^{\alpha\beta}\phi_{;\alpha\beta}) \quad , \tag{A2f}$$

and

$$\Lambda^{\mu[\nu\alpha]}{}_{,\alpha} = \phi_0(16\pi)^{-1}[(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})]_{,\alpha\beta} \quad . \tag{A2g}$$

(iii) <u>Vector-Metric Theory of Will-Nordtvedt</u>:[30]

(a)

$$[t_\mu{}^\nu + (-g)^{1/2}\, T_\mu{}^\nu]_{,\nu} = 0 \quad , \tag{A3a}$$

where

$$t_\mu{}^\nu = -\mathcal{L}_G\delta_\mu{}^\nu + \frac{\partial\mathcal{L}_G}{\partial K_{\alpha,\nu}} K_{\alpha,\mu} + \frac{\partial\mathcal{L}_G}{\partial g_{\alpha\beta,\nu}} g_{\alpha\beta,\mu} - \frac{\partial\mathcal{L}_G}{\partial g_{\alpha\beta,\nu\sigma}} g_{\alpha\beta,\mu\sigma} - \left(\frac{\partial\mathcal{L}_G}{\partial g_{\alpha\beta,\nu\sigma}}\right)_{,\sigma} g_{\alpha\beta,\mu} \quad , \tag{A3b}$$

and

$$\mathcal{L}_G = (-g)^{1/2}[R + K_{\mu;\nu}K_{\sigma;\tau}g^{\mu\sigma}g^{\nu\tau}] \quad . \tag{A3c}$$

(R is the curvature scalar constructed out of $g_{\mu\nu}$ and the semi-colon denotes covariant derivative with respect to $g_{\mu\nu}$.)

(b) This conservation law does not satisfy the requirements set out in Sec. III.E, but its superpotential allows a physical interpretation:

$$(-g)(T^{\mu\nu} + t^{\mu\nu}) = \Lambda^{\mu[\nu\alpha]}{}_{,\alpha} \quad , \tag{A3d}$$

and

$$[(-g)(T^{\mu\nu} + t^{\mu\nu})]_{,\nu} = 0 \quad . \tag{A3e}$$

The gravitational stress-energy pseudo-tensor is

$$t^{\mu\nu} = (1 + \frac{K^2}{2})^{-1} [(8\pi)^{-1} \Theta^{\mu\nu} + t^{\mu\nu}_{LL}] \quad , \tag{A3f}$$

where $\Theta^{\mu\nu}$ is defined in Eq. (A4) of Ref. 30, and $t^{\mu\nu}_{LL}$ is defined in Eq. (A1b). The superpotential is

$$\Lambda^{\mu[\nu\alpha]}{}_{,\alpha} = [16\pi(1 + \frac{1}{2} K^2)]^{-1} [(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})]_{,\alpha\beta} \quad . \tag{A3g}$$

(iv) <u>Hellings-Nordtvedt Theory:</u>[31] $[\omega \neq 0, \ \eta = 0]$

(a)

$$[t_\mu{}^\nu + (-g)^{1/2} T_\mu{}^\nu]_{,\nu} = 0 \quad , \tag{A4a}$$

where

$$t_\mu{}^\nu = -\mathcal{L}_G \delta_\mu{}^\nu + \frac{\partial \mathcal{L}_G}{\partial K_{\alpha,\nu}} K_{\alpha,\mu} + \frac{\partial \mathcal{L}_G}{\partial g_{\alpha\beta,\nu}} g_{\alpha\beta,\mu} + \frac{\partial \mathcal{L}_G}{\partial g_{\alpha\beta,\nu\sigma}} g_{\alpha\beta,\mu\sigma}$$
$$- \left(\frac{\partial \mathcal{L}_G}{\partial g_{\alpha\beta,\nu\sigma}}\right)_{,\sigma} g_{\alpha\beta,\mu} \quad , \tag{A4b}$$

and

$$\mathcal{L}_G = (-g)^{1/2}[R - (K_{\nu,\mu} - K_{\mu,\nu})(K_{\alpha,\beta} - K_{\beta,\alpha}) g^{\alpha\mu}g^{\beta\nu} - \omega K_\mu K_\nu R g^{\mu\nu} + \eta K_\mu K_\nu R^{\mu\nu}] \quad . \tag{A4c}$$

(b)

$$(1 + \omega\phi)^{-1} (-g)(T^{\mu\nu} + t^{\mu\nu}) = \Lambda^{\mu[\nu\alpha]}{}_{,\alpha} \quad . \tag{A4d}$$

The gravitational stress-energy pseudo-tensor is

$$t^{\mu\nu} = -(8\pi)^{-1}\omega(K^{\mu}K^{\nu}R + \phi^{;\mu\nu} - g^{\mu\nu}g^{\alpha\beta}\phi_{;\alpha\beta}) - (4\pi)^{-1}(F^{\mu\alpha}F_{\alpha}{}^{\nu} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})$$
$$+ (1 + \omega\phi)\, t^{\mu\nu}_{LL} \quad , \qquad \text{(A4e)}$$

and the superpotential is

$$\Lambda^{\mu[\nu\alpha]}{}_{,\alpha} = (16\pi)^{-1}[(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})]_{,\alpha\beta} \quad . \qquad \text{(A4f)}$$

(v) Ni's Lagrangian-Based Conformally Flat Theory:[14]

$$[t_{\mu}{}^{\nu} + (-g)^{1/2}\, T_{\mu}{}^{\nu}]_{,\nu} = 0 \quad . \qquad \text{(A5a)}$$

The gravitational stress-energy pseudo-tensor in the preferred frame [in which $\eta_{\mu\nu}$ = diag. (-1, 1, 1, 1)] is

$$t_{\mu}{}^{\nu} = 2f_1(\phi)\phi_{,\alpha}\phi_{,\beta}\eta^{\alpha\beta}\delta_{\mu}{}^{\nu} - 4f_1(\phi)\phi_{,\alpha}\phi_{,\mu}\eta^{\alpha\nu} \quad . \qquad \text{(A5b)}$$

(vi) Lightman-Lee Theory:[32]

$$[t_{\mu}{}^{\nu} + (-g)^{1/2}\, T_{\mu}{}^{\nu}]_{,\nu} = 0 \quad . \qquad \text{(A6a)}$$

The gravitational stress-energy pseudo-tensor in the absolute frame [in which $\eta_{\mu\nu}$ = diag. (-1, 1, 1, 1)] is

$$t_{\mu}{}^{\nu} = (16\pi)^{-1}[\delta_{\mu}{}^{\nu}(ah^{\gamma\sigma,\beta}h_{\gamma\sigma,\beta} + fh'^{,\alpha}h_{,\alpha}) - 2(ah^{\alpha\beta,\nu}h_{\alpha\beta,\mu} + fh_{,\mu}h'^{,\nu})] \quad . \qquad \text{(A6b)}$$

APPENDIX B1

Contravariant Gravitational Stress-Energy Complex for Metric Theories in the PN Limit [satisfying the constraints of Eqs. (64)]

If the Q^i is zero in Eq. (62b), then from Eqs. (62) one may read off t^{00} and t^{0i} [Eq. (62a)] and t^{i0} and t^{ik} [Eq. (62b)]. We therefore do not write down $t^{\mu\nu}$ here.

APPENDIX B2

Mixed-Index Gravitational Stress-Energy Complex for Metric Theories in the PN Limit [satisfying constraints of Eqs. (64)]

$$t_0{}^0 = (8\pi)^{-1}\,(6\gamma + 2a - 1)\,|\nabla U|^2 \quad . \tag{B2-1}$$

$$t_0{}^\ell = -(4\pi)^{-1}\,[2(3\gamma + a - 1)\,U_{,k}V_{[k,\ell]} + (a + 3\gamma)\,U_{,0}U_{,\ell}] \; , \tag{B2-2}$$

$$t_k{}^0 = (4\pi)^{-1}\,[(a + 3\gamma - 1)\,U\nabla^2{}_{,k} - 2\zeta_w U_{,0}U_{,k}] \quad . \tag{B2-3}$$

$$\begin{aligned}
t_k{}^\ell = {} & (1 - a - 3\gamma)\,U(\rho v_\ell v_k + p\delta_{\ell k}) - (4\pi)^{-1}\,[1 - (\zeta_2 + \zeta_w - a)]\,\Gamma_{k\ell}(U) \\
& - (\pi)^{-1}\,\Gamma_{k\ell}(\Phi) + (4\pi)^{-1}\,\zeta_1\Gamma_{k\ell}(\mathcal{A}) + (2\pi)^{-1}\,\zeta_w\Gamma_{k\ell}(\Phi_w) \\
& + (8\pi)^{-1}\,(\alpha_1 - \alpha_2 + \zeta_1 + 4\gamma + 3)(V_{m,k}V_{m,\ell} - \tfrac{1}{2}\delta_{\ell k}V_{m,s}V_{m,s}) \\
& + (8\pi)^{-1}\,(\alpha_2 + 1 - \zeta_1)(V_{m,\ell}V_{m,s} - V_m W_{m,\ell k} + 2V_m W_{[m,\ell]k} - \delta_{\ell k}V_{[r,s]}V_{[r,s]} \\
& \quad + V_\ell U_{,0k} + \tfrac{1}{2}\delta_{\ell k}U_{,0}{}^2) \\
& + (2\pi)^{-1}\,\zeta_w[\tfrac{1}{2}U_{,0}{}^2\delta_{\ell k} + \tfrac{1}{4}\chi\Omega_{,k\ell} - \tfrac{1}{4}\chi_{,\ell}\Omega_{,k} + \pi\chi\delta_{k\ell}(U_{,r}U_{,s})_{,rs} \\
& - \chi_{,\ell m}U_{,m}U_{,k} + \tfrac{1}{2}\chi_{,k}(U_{,m}U_{,\ell})_{,m} - \tfrac{1}{2}\chi_{,mk}U_{,m}U_{,\ell} + \tfrac{1}{2}\chi_{,rs}U_{,r}U_{,s}\delta_{\ell k}] \quad .
\end{aligned} \tag{B2-4}$$

The potential χ, and $\Gamma_{ij}(X)$ have been defined previously and

$$\Omega \equiv \frac{\partial^2}{\partial x^r \partial x^s} \int \frac{U(x')_{,r'}U(x')_{,s'}\,d^3x'}{|x - x'|} \quad .$$

APPENDIX C

A Lagrangian-Based Theory of Gravity

In this appendix we present a Lagrangian-based theory of gravity. It is a generalized version of Ni's New Theory;[24] and it is designed to have the maximum number (5) of unconstrained PPN parameters allowed for any theory with conserved P^{μ}.

a. Gravitational fields present: A flat background metric η, scalar fields ϕ and t, a vector field ψ, a symmetric tensor field h, and the physical metric g.

b. Arbitrary parameters and functions: Three arbitrary functions $f_1(\phi)$, $f_2(\phi)$, $f_3(\phi)$ and three arbitrary parameters e, k_1, k_2; in the post-Newtonian limit, with appropriate choice of the cosmological model, there are five arbitrary parameters: a, b, d, e, and (k_2/k_1).

c. Prior geometry: The following constraints are imposed, *a priori*, on the geometrical relationships among the gravitational fields:

(i) flatness of the metric η

$$\text{(Riemann tensor constructed from } \eta) = 0 \quad ; \qquad \text{(C1a)}$$

(ii) "meshing constraints" on t, η and ψ

$$t_{|\mu\nu} = 0 \quad , \qquad \text{(C1b)}$$

$$t_{,\mu} t_{,\nu} \eta^{\mu\nu} = -1 \quad . \qquad \text{(C1c)}$$

(Here and below a slash denotes a covariant derivative with respect to η, and $\eta^{\mu\nu}$ is the inverse of $\eta_{\mu\nu}$.)

$$t_{,\mu} \psi_{\nu} \eta^{\mu\nu} = 0 \quad ; \qquad \text{(C1d)}$$

$$t_{,\mu} h_{\nu} \eta^{\mu\nu} = 0 \quad . \qquad \text{(C1e)}$$

(iii) algebraic equation for the physical metric in terms of the "auxiliary gravitational fields" $\boldsymbol{\eta}$, ϕ, t, $\underline{\psi}$, $\underline{h}$

$$\mathbf{g} = f_2(\phi)\,\boldsymbol{\eta} + [f_1(\phi) - f_2(\phi)]\,\underline{dt} \otimes \underline{dt} + \underline{\psi} \otimes \underline{dt} + \underline{dt} \otimes \underline{\psi} + \underline{h} \quad . \qquad \text{(C1f)}$$

d. Preferred coordinate system: The prior-geometric constraints (C1) guarantee the existence of a preferred coordinate system in which (i) the time coordinate is equal to the scalar field t; (ii) the components of $\boldsymbol{\eta}$ are Minkowskiian

$$\eta_{\mu\nu} = \text{diagonal } (1,-1,-1,-1) \quad ; \qquad \text{(C2a)}$$

(iii) $\underline{\psi}$ is purely spatial

$$\psi_0 = 0 \quad ; \qquad \text{(C2b)}$$

(iv) $\underline{h}$ has only space-space parts non-vanishing

$$h_{0\mu} = h_{\mu 0} = 0 \quad ; \qquad \text{(C2c)}$$

(v) the physical line element $g_{\alpha\beta}dx^{\alpha}dx^{\beta}$ is

$$ds^2 = f_1(\phi)\,dt^2 - f_2(\phi)(dx^2 + dy^2 + dz^2)$$

$$+ 2\psi_1 dxdt + 2\psi_2 dydt + 2\psi_3 dzdt + h_{ij}dx^i dx^j \quad . \qquad \text{(C2d)}$$

e. Lagrangian: The field equations are determined by an action principle

$$\delta \int \mathcal{L}\, d^4x = 0 \qquad \text{(C3a)}$$

where the Lagrangian density $\mathcal{L}$ is

$$\mathcal{L} = L_{NG}\sqrt{-g} + 2\Big\{ (1/e)\,\psi_{\mu|\sigma}\psi_{\nu|\tau}\eta^{\mu\nu}\eta^{\sigma\tau} - \phi_{,\mu}\phi_{,\nu}\eta^{\mu\nu}$$

$$+ [f_3(\phi) + 1](\phi_{,\mu}t_{,\nu}\eta^{\mu\nu})^2$$

$$+ k_1 h_{\mu\nu|\sigma}h_{\gamma\delta|\tau}\eta^{\mu\gamma}\eta^{\nu\delta}\eta^{\sigma\tau} + k_2 h_{\mu\nu|\sigma}\phi_{,\tau}\eta^{\mu\tau}\eta^{\nu\sigma}\Big\}\sqrt{-\eta} \quad . \qquad \text{(C3b)}$$

Here L_{NG} is the standard interaction Lagrangian metric theories of gravity. The quantities g and η are the determinants of $||g_{\mu\nu}||$ and $||\eta_{\mu\nu}||$. In the action principle (C3) one is to vary the standard matter and non-gravitational fields that appear in L_{NG} and the gravitational fields ϕ and $\underline{\psi}$, while maintaining the prior-geometric constraints (C1). In the preferred coordinate system (C2) the Lagrangian density reduces to

$$\mathcal{L} = L_{NG}\sqrt{-g} + (\tfrac{2}{e})(\psi_{i,j}\psi_{i,j} - \psi_{i,t}\psi_{i,t}) + 2\phi_{,i}\phi_{,i} + 2f_3(\phi)\phi_{,t}\phi_{,t}$$
$$- 2k_1 h_{ij,k}h_{ij,k} + 2k_1 h_{ij,t}h_{ij,t} + 2k_2 h_{ij,j}\phi_{,i} \quad . \qquad \text{(C4)}$$

f. Field equations: The nongravitational field equations derived from this action principle take on their standard metric form. The gravitational field equations derived from the action principle are

$$\psi_{\mu|\nu}{}^{|\nu} = 2\pi e(\sqrt{-g}/\sqrt{-\eta})\; T^{\sigma\tau}(\partial g_{\sigma\tau}/\partial\psi_\nu)(\eta_{\mu\nu} - t_{|\mu}t_{|\nu}) \quad ,$$
$$\phi_{|\nu}{}^{|\nu} - [f_3(\phi) + 1]\,\phi_{,\mu}{}^{|\nu}t^{|\mu}t_{|\nu} + \frac{1}{2}f'_3(\phi)(\phi_{|\nu}t^{|\nu})^2 - \frac{1}{2}k_2 h_{\mu\nu}{}^{|\mu\nu}$$
$$- 2\pi\sqrt{-g}\; T^{\mu\nu}(\partial g_{\mu\nu}/\partial\phi) = 0$$
$$k_1 h_{\mu\nu|\sigma}{}^{|\sigma} = [\frac{1}{2}k_2\phi^{|\sigma|\tau} + 2\pi(\sqrt{-g}/\sqrt{-\eta})\; T^{\gamma\eta}(\partial g_{\gamma\eta}/\partial h_{\sigma\tau})]$$
$$\times(\eta_{\mu\sigma} - t_{|\mu}t_{|\sigma})(\eta_{\nu\tau} - t_{|\nu}t_{|\tau}) \quad . \qquad \text{(C5a)}$$

In the absolute coordinate system, these equations reduce to

$$\psi_{i,jj} - \psi_{i,tt} = 4\pi e\sqrt{-g}\; T^{0i}$$
$$\phi_{,ii} - f_3(\phi)\phi_{,tt} + \frac{1}{2}f'_3(\phi)\phi_{,t}\phi_{,t} = \frac{1}{2}k_2 h_{ij|ij} + 2\pi\sqrt{-g}\; T^{\mu\nu}(\partial g_{\mu\nu}/\partial\phi)$$
$$h_{ij,kk} - h_{ij,tt} = -\frac{1}{2}(k_2/k_1)\;\phi_{,ij} - 2\pi(k_1)^{-1}\sqrt{-g}\; T_{ij} \quad . \qquad \text{(C5b)}$$

Here

$$T_{\mu\nu} \equiv -\frac{1}{8\pi}\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\,L_{NG})}{\partial g^{\mu\nu}} ,$$

$$T^{\mu\nu} \equiv g^{\mu\alpha}g^{\nu\beta}T_{\alpha\beta} .$$

g. Post-Newtonian limit: The solution of Eqs. (C5a) or (C5b) proceed along the same lines as in Ref. 21. We present only the results here. In the preferred frame, the physical metric is

$$g_{00} = -\Big\{1 - 2U + 2bU^2 - 2[(1 + a - k_2/8k_1c)\,\Phi_1 + (3a - 2b + 1 - k_2/4ck_1)\Phi_2 + \Phi_3 + (3a - k_2/4ck_1)\,\Phi_4] - (k_2/4ck_1)\,\mathcal{A} - [d - 1 + c^2)/c^2]\chi_{,tt}\Big\} + O(6) \quad \text{(C6a)}$$

$$g_{0i} = + eV_i . \quad \text{(C6b)}$$

$$g_{ij} = \delta_{ij}(1 + 2aU) - (k_2/4ck_1)\,\chi_{,ij} . \quad \text{(C6c)}$$

We now perform a gauge transformation

$$x^{0\dagger} = x^0 - \frac{1}{2}\chi_{,t}(d - 1 + c^2)/c^2$$

$$x^{i\dagger} = x^i + (k_2/8ck_1)\,\chi_{,i} ,$$

and bring the metric into the "standard" form:

$$g^{\dagger}_{00} = 1 - 2U + 2(b + k_2/8ck_1)\,U^2 - 2(1 + a - k_2/8ck_1)\,\Phi_1 - 2(3a + 2b + 1 - k_2/8ck_1)\,\Phi_2 - 4\,\Phi_3 - 4(a - k_2)12ck_1)\,\Phi_4 - (k_2/4ck_1)\,\mathcal{A} + (k_2/4ck_1)\,\Phi_w , \quad \text{(C7a)}$$

$$g^{\dagger}_{0i} = [e - (d - 1 + c^2)/2c^2]\,V_i + (d - 1 + c^2)/2c^2\,W_i , \quad \text{(C7b)}$$

$$g^{\dagger}_{ij} = -\delta_{ij}(1 + 2aU) . \quad \text{(C7c)}$$

The PPN parameters are thus [cf., Eq. (C7)]

$$\beta = b + k_2/8ck_1, \quad \gamma = a , \quad \alpha_1 = - 2e - 4a - 4 ,$$

$$\alpha_2 = - 1 - k_2/(4ck_1) + (d - 1 + c^2)/c^2 , \quad \alpha_3 = 0 .$$

$$\zeta_1 = - k_2/4ck_1 , \quad \zeta_2 = k_2/8ck_2 , \quad \zeta_3 = 0 .$$

$$\zeta_4 = - k_2/12ck_1 , \quad \zeta_w = k_2/8ck_1 . \tag{C8}$$

Where a, b, c, d are defined by the power series expansions of the functions $f_1(\phi)$, $f_2(\phi)$ and $f_3(\phi)$:

$$f_1(\phi) = 1 - 2c\phi + 2bc^2\phi^2 + \dots , \tag{C9a}$$

$$f_1(\phi) = 1 + 2ac\phi + \dots , \tag{C9b}$$

$$f_3(\phi) = d + \dots , \tag{C9c}$$

and c is set to have the value

$$c = 1 + 4k_2^2/k_1^2 , \tag{C9d}$$

to obtain the correct Newtonian limit.

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