A MODEL FOR THE STUDY OF VERY NOISY CHANNELS, AND APPLICATIONS

Thesis by

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ABSTRACT

Very Noisy channels (such as the wideband gaussian channel well-known in deep space communications) have the interesting property that although the maximum number of bits transmitted per symbol is close to zero, the maximum number of bits transmitted per second is not! Furthermore, recent results on the ultimate limits of information density indicate that some channels perform better when pushed to their very noisy limit.

We present a general mathematical model of Very Noisy channels which provides an insight in their behavior, and in some interesting cases, tells us about the limiting behavior of the larger class of noisy channels.

Two classes of Very Noisy Channels are identified and efficient algorithms that compute their capacity are presented. We show that for some Very Noisy broadcast channels, the time-shared coding strategy performs as well as the optimal strategy known as broadcast coding in the limit. Finally, with the help of our model, we derive a tight lower bound on the amount of information lost in a Channel Reduction or Data Compression.

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To my parents:

Bernard and Gail.

INTRODUCTION TO "VERY NOISY" CHANNELS

"Very noisy" channels are, by definition, channels with "very low" capacity. Such channels are of great interest to communications, although their low capacities seem to be a handicap. First, in deep space communications, channels such as the Wideband Gaussian channel, or more generally all channels with low signal-to noise ratios (SNR), are commonly dealt with, and are indeed "very noisy" channels. Fortunately, Shannon [9] showed that as long as the capacity of a channel is non-zero, arbitrarily reliable communication is possible! Furthermore, Abdel-Ghaffar and McEliece [2] give examples of channels (such as those modeling read/write operations in magnetic recording) for which, if noise scaling is possible, it is preferable to push them to their "very noisy" limit in order to achieve the largest information density, i.e., the maximum number of information bits per inch for a magnetic tape. Let us now give a brief overview on "very noisy" channels (which we will denote by VNCs).

In an early paper where he analyzed binary channels and their cascades, Silverman [19] mentions the "exceptional behavior" of "channels with very low capacity". The examples he cites are the binary symmetric channel (BSC) and the binary asymmetric channel (BAC) for which he finds for instance that, the BSC has a higher capacity in cascade than a BAC with the same capacity, "unless their capacities are very low", in which case that behavior is reversed: the BAC has the highest capacity in cascade. It is interesting that the two examples he gives of VNCs characterize two strikingly different classes of VNCs, which will be identified later in the thesis.

There are infinitely many other "very noisy" channels (VNCs) under our initial definition. Reiffen [18] was the first to attempt a broad definition of VNCs to model "many physical channels operating at low signal-to-noise ratios". His definition applies however only to channels with a continuous output space, and as a consequence, does not cover the "very noisy" BAC mentioned by Silverman [19]. Reiffen [18] then gave an approximation of the capacity of such channels, and also showed that the zero-rate exponent R_0 [21] was approximately one-half the capacity C. Later Gallager [15] used Reiffen's definition and computed the random and convolutional coding exponent-rate functions for Reiffen's VNCs, which had the unusual property of depending solely on their capacity C.

Recently, McEliece, Posner and Swanson [22],[23] showed that for the Wideband Gaussian degraded broadcast channel (a VNC covered by Reiffen's definition), the capacity region is the time-sharing region in the limit as the signal-to-noise ratios go to zero.

In this thesis, we propose a general mathematical model of "very noisy" channels covering all discrete channels with very low capacity. Furthermore, we introduce a *noise scaling parameter* which we denote by ϵ, which is always very small for VNCs and drives their capacity to zero as it goes to zero. This scaling parameter will allow us to define families of "very noisy" channels with the same limiting behavior, but different values of ϵ.

In chapter 1, after giving a formal definition of "very noisy" channels, we identify two separate classes of VNCs, give approximations of their average mutual information and present new algorithms that compute their capacity. Finally, we compute the random and convolutional coding exponents of VNCs.

In chapter 2, we focus on binary VNCs and compare the two classes of VNCs identified in chapter 1, after which we present a rigorous proof by Howard Rumsey of a result conjectured by Silverman [19], as well as new conjectures about the effect of signalling at a uniform input distribution instead of at capacity.

In chapter 3, we identify the binary-input "very noisy" degraded broadcast channels for which the capacity region is no larger than the well-known time-sharing region, in the limit as the noise scaling parameter goes to zero.

Finally, in chapter 4, we solve a problem by Abu-Mostafa published in [24], about the largest fraction of mutual information in a channel reduction or data compression, which is achieved only by a very particular family of VNCs.

CHAPTER 1:

DEFINITION AND PROPERTIES OF "VERY NOISY" CHANNELS

I - DEFINITION OF A VNC

A Discrete Memoryless Channel (DMC) is defined by a finite input alphabet $\mathcal{X}$, a finite output alphabet $\mathcal{Y}$, and a transition probability matrix $P_{Y|X} = \Big(p(y|x)\Big)$. The input X to the DMC is characterized by the probability distribution $q(x)$, and the output Y of the DMC is characterized by the probability distribution $p(y)$, which can be derived from $q(x)$ and $p(y|x)$:

$$p(y) = \sum_{u \in \mathcal{X}} q(u)p(y|u).$$

The entropy $H(Y)$ of a discrete random variable Y, which is a measure of uncertainty about the random variable Y, where $p(y) = \text{Prob}(Y = y)$, is defined by:

$$H(Y) = \sum_{y \in \mathcal{Y}} p(y) \log \frac{1}{p(y)}.$$

The conditional entropy $H(Y|X)$ of the random variable Y given X, also a measure of the uncertainty about the random variable Y after having observed X, is defined by:

$$H(Y|X) = \sum_{x \in \mathcal{X}} q(x) \sum_{y \in \mathcal{Y}} H(Y|X = x) = \sum_{x \in \mathcal{X}} q(x) \sum_{y \in \mathcal{Y}} p(y|x) \log \frac{1}{p(y|x)}.$$

The difference between $H(Y)$ and $H(Y|X)$, i.e. the decrease in the uncertainty about Y after the observance of X, is called the average mutual information between the random variables X and Y :

$$I(X;Y) = H(Y) - H(Y|X) = \sum_{x \in \mathcal{X}} q(x) \sum_{y \in \mathcal{Y}} p(y|x) \log \frac{p(y|x)}{p(y)}. \tag{1.1}$$

Clearly, $I(X;Y)$ is a function of the input probability distribution $q(x)$ and the channel transition probabilities $p(y|x)$. If we maximize $I(X;Y)$ over all possible input probability distributions, we obtain a quantity that is characteristic of the channel, that is, the capacity:

$$C = \max_{q(x)} I(X;Y).$$

Intuitively, "Very Noisy" Channels (VNCs) are channels with "very low" capacity. Since the capacity of a VNC is close to zero, it is first necessary to better understand the mathematical properties of channels with zero capacity.

I-1. Definition of a zero-capacity channel

Here are two definitions of a zero-capacity channel and we will show that they are equivalent.

Definition 1.1: A channel, defined by its transition probabilities $p(y|x)$, and with capacity C, is a zero-capacity channel if and only if

$$C = 0.$$

Definition 1.2: A channel, defined by its transition probabilities $p(y|x)$ is a zero-capacity channel if and only if, for all x and all y,

$$p(y|x) = p(y).$$

Theorem 1.1: *Definition 1.1 is equivalent to Definition 1.2.*

Proof of Theorem 1.1:

Since for all distributions $q(x)$, $C \geq I(X;Y) \geq 0$, then Definition 1.1 is equivalent to

$$I(X;Y) = 0, \quad \text{for all } q(x).$$

Let us now see what this is equivalent to in terms of transition probabilities.

$$\begin{aligned} I(X;Y) &= \sum_{x \in \mathcal{X}} q(x) \sum_{y \in \mathcal{Y}} p(y|x) \log \frac{p(y|x)}{\sum_{u \in \mathcal{X}} q(u) p(y|u)}, \\ &= \sum_{x \in \mathcal{X}} q(x) \sum_{y \in \mathcal{Y} \;:\; p(y|x) \neq 0} p(y|x) \log \frac{p(y|x)}{\sum_{u \in \mathcal{X}} q(u) p(y|u)}. \end{aligned}$$

Now, using the inequality $\log(x) \geq 1 - \frac{1}{x}$ for $x \neq 0$,

$$\begin{aligned} I(X;Y) &\geq \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}\ :\ p(y|x)\neq 0} p(y|x)\left(1 - \frac{\sum_{u\in\mathcal{X}} q(u)p(y|u)}{p(y|x)}\right) \\ &\geq \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}\ :\ p(y|x)\neq 0} \left(p(y|x) - \sum_{u\in\mathcal{X}} q(u)p(y|u)\right) \\ &\geq 1 - \sum_{x\in\mathcal{X}} q(x) \sum_{u\in\mathcal{X}} q(u) \sum_{y\in\mathcal{Y}\ :\ p(y|x)\neq 0} p(y|u). \end{aligned}$$

Therefore, our first inequality is:

$$I(X;Y) \geq 1 - \sum_{x\in\mathcal{X}} q(x) \sum_{u\in\mathcal{X}} q(u) \sum_{y\in\mathcal{Y}\ :\ p(y|x)\neq 0} p(y|u),$$

with equality if and only if, for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $p(y|x) \neq 0$,

$$p(y|x) = \sum_{u\in\mathcal{X}} q(u)p(y|u).$$

Now, using the fact that, given any $x \in \mathcal{X}$ and for all $u \in \mathcal{X}$,

$$\sum_{y\in\mathcal{Y}\ :\ p(y|x)\neq 0} p(y|u) \leq 1,$$

we have:

$$1 - \sum_{x\in\mathcal{X}} q(x) \sum_{u\in\mathcal{X}} q(u) \sum_{y\in\mathcal{Y}\ :\ p(y|x)\neq 0} p(y|u) \geq 1 - \sum_{x\in\mathcal{X}} q(x) \sum_{u\in\mathcal{X}} q(u) = 1 - 1 = 0,$$

which yields our second inequality:

$$1 - \sum_{x\in\mathcal{X}} q(x) \sum_{u\in\mathcal{X}} q(u) \sum_{y\in\mathcal{Y}\ :\ p(y|x)\neq 0} p(y|u) \geq 0,$$

with equality if and only if:

$$\sum_{y\in\mathcal{Y}\ :\ p(y|x)\neq 0} p(y|u) = 1, \quad \text{for all } x \in \mathcal{X} \text{ and } u \in \mathcal{X},$$

that is:

$$\text{if } \exists x \in \mathcal{X} \text{ and } \exists y \in \mathcal{Y}\ :\ p(y|x) = 0, \quad \Rightarrow p(y|u) = p(y) = 0, \quad \text{for all } u \in \mathcal{X}.$$

Both inequalities lead us to

$$I(X;Y) \geq 0,$$

with equality if and only if:

$$p(y|x) = p(y) = \sum_{u \in \mathcal{X}} q(u)p(y|u).$$

Definition 1.2 tells us that a zero-capacity channel is a channel for which the output is completely independent from the input, and therefore that the rows of the transition probability matrix of a zero-capacity channel are identical. Now that we have a definition for the transition probabilities of a zero-capacity channel, we would like to have such a definition for a VNC, since zero-capacity channels are limiting cases of VNCs. To develop a better intuition about VNCs, let us first examine two real life channels whose capacities can be made arbitrarily close to zero.

I-2 Examples of "Very Noisy" Channels.

I-2.1. A "Very Noisy" Binary Symmetric Channel.

Let us transmit two equally likely antipodal signals (s_0 and s_1) over a channel for which the noise is modeled by additive gaussian noise. At the output of the channel, we make a hard decision ($\hat{s}_0$ or $\hat{s}_1$) about which signal was sent through the channel. The resulting channel is a binary symmetric channel (figure 1.1), and we know that the error probability (the cross-over probability) is [1]:

$$p_e = p(\hat{s}_1|s_0) = p(\hat{s}_0|s_1) = Q\left(\sqrt{\frac{2E_S}{N_0}}\right),$$

$$\text{where } Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\gamma^2/2} d\gamma$$

and $\dfrac{E_S}{N_0}$ is the signal-to-noise ratio.

For convenience, let $\epsilon = \sqrt{\frac{E_S}{\pi N_0}}$.

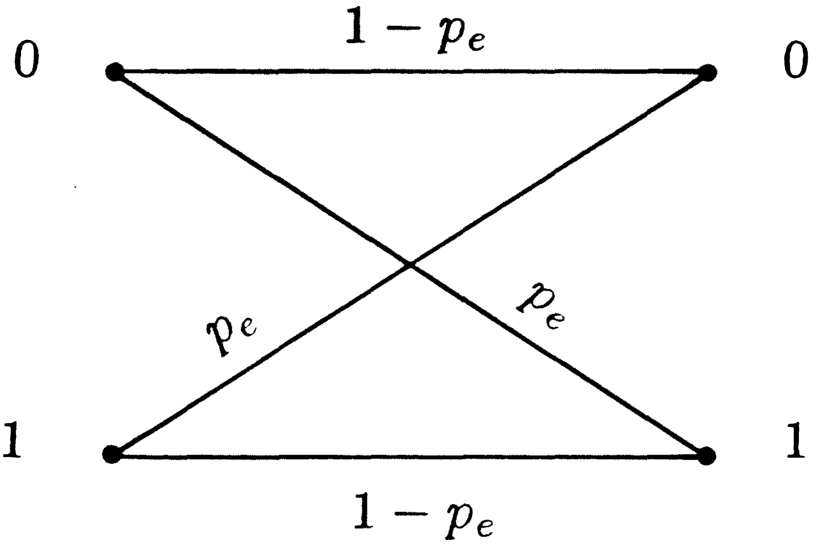


Figure 1.1: The Binary Symmetric Channel with cross-over probability p_e.

As the signal-to-noise ratio goes to 0, ϵ goes to 0, the error probability p_e goes to $\frac{1}{2}$, and the capacity C of the channel goes to 0.

For $x \approx 0$,

$$\begin{aligned}
Q(x) &= \frac{1}{2} - \int_0^x \frac{1}{\sqrt{2\pi}} e^{-\gamma^2/2} d\gamma, \\
&= \frac{1}{2} - \int_0^x \frac{1}{\sqrt{2\pi}} \Big(1 + O(\gamma^2)\Big) d\gamma, \\
&= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \Big(x + O(x^3)\Big), \\
&= \frac{1}{2} - \frac{x}{\sqrt{2\pi}} + O(x^3),
\end{aligned}$$

and for $x \approx 0$ and $\mathcal{H}(q)$ is the binary entropy function ($0 \leq q \leq 1$),

$$\begin{aligned}
\mathcal{H}\Big(\frac{1}{2} - x\Big) &= - \Big(\frac{1}{2} - x\Big) \log\Big(\frac{1}{2} - x\Big) - \Big(\frac{1}{2} + x\Big) \log\Big(\frac{1}{2} + x\Big), \\
\mathcal{H}'\Big(\frac{1}{2} - x\Big) &= \log\left(\frac{\frac{1}{2} - x}{\frac{1}{2} + x}\right), \\
\mathcal{H}''\Big(\frac{1}{2} - x\Big) &= \frac{-1}{(\frac{1}{2} - x)(\frac{1}{2} + x)},
\end{aligned}$$

and therefore,

$$\begin{aligned}
\mathcal{H}\Big(\frac{1}{2} - x\Big) &= \mathcal{H}\Big(\frac{1}{2}\Big) + x \cdot \mathcal{H}'\Big(\frac{1}{2}\Big) + \frac{x^2}{2} \cdot \mathcal{H}''\Big(\frac{1}{2}\Big) + O(x^3), \\
&= \log 2 - 2x^2 + O(x^3).
\end{aligned}$$

Therefore,

$$\begin{aligned}
p_e &\approx \frac{1}{2} - \epsilon, \\
C &= \log 2 - \mathcal{H}(p_e) \approx 2\epsilon^2.
\end{aligned} \tag{1.2}$$

Therefore, as the signal-to-noise ratio of this channel goes to 0, our channel becomes a "very noisy" channel ($C \to 0$). It is interesting to note that in this case, the capacity is proportional to ϵ^2.

I-2.2. A "Very Noisy" Z-Channel

Assume that we wish to transmit a stream of binary information using a light source and a window (see figure 1.2). The light source continuously emits photons.

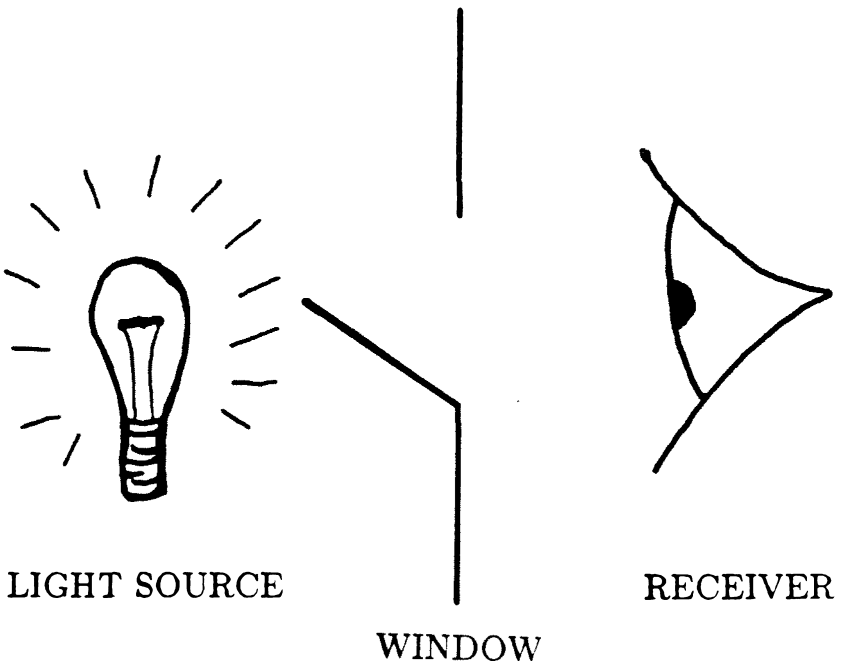


Figure 1.2: Example of a practical Z-channel.

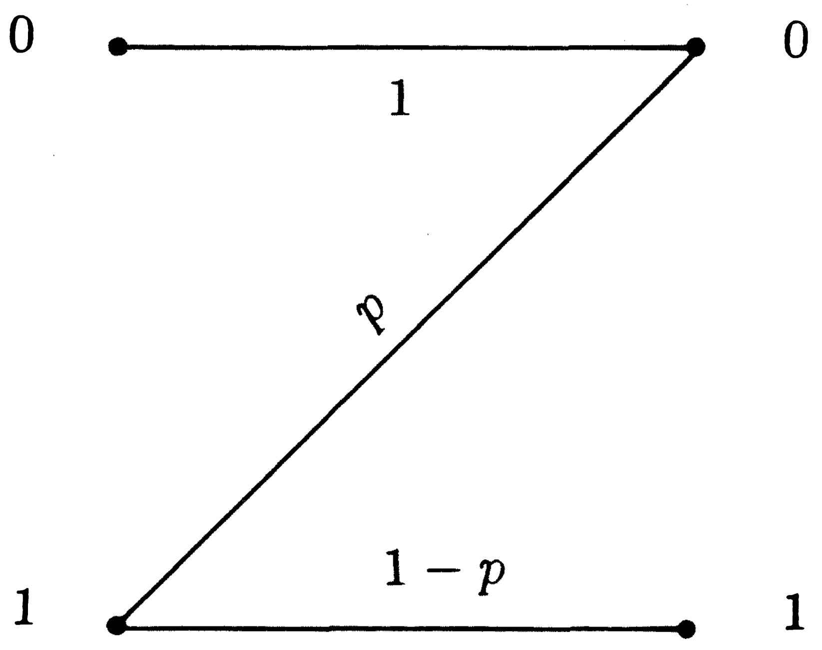


Figure 1.3: The Z-channel.

To transmit a "0", we keep the window closed, and no light will ever be seen by the receiver. To transmit a "1", we keep the window open for some fixed amount of time, and the photons emitted are seen by the receiver. This channel can be modeled by what is known as a *Z-channel,* i.e., a binary channel where one of the two inputs can only be received by one of the two outputs (see figure 1.3). Here, as we have seen, a "0" can only be received as a "0". A "1" will be received as a "0" only in the case the window is open and no photon is emitted while the window is open.

Let ϵ be the average number of photons transmitted per symbol. The probability distribution of the actual number of photons transmitted per symbol can be modeled by a Poisson distribution:

$$P[k] = \frac{\epsilon^k}{k!} e^{-\epsilon}.$$

Therefore, the probability that a "1" is received as a "0" is the probability that the source emits no photon although the window is open:

$$p(0|1) = P[0] = e^{-\epsilon}.$$

As we push the source to its limit and let ϵ go to 0, then:

$$p(0|1) \simeq 1 - \epsilon. \tag{1.3}$$

The capacity of the Z-Channel is [20]:

$$C = \max_q \left(H(q) - (1-q) H(\epsilon) \right),$$

where $q = \text{Prob}(X = 0)$.

As ϵ goes to 0,

$$C \approx \frac{\epsilon}{e}.$$

Again, as the average number of photons emitted per symbol by the source becomes very low ($\epsilon \ll 1$), the channel becomes a very noisy channel ($C \to 0$). It is interesting to note that, compared to the case described in I-2.1., in this case, the capacity is linear in ϵ in the limit.

I-3. Definition of a VNC

In the previous examples, we have been able to find a channel parameter ϵ such that:

- ϵ is a measure of the noisiness of the channel, with the capacity going to zero as ϵ goes to zero,
- the transition probabilities (and therefore the capacity) are expressed in the form of a power series of ϵ (see (1.2) and (1.3)),
- ϵ is a *noise scaling* parameter, i.e., a physical parameter that directly affects the noisiness of the channel (such as the average number of photons transmitted per symbol, or the signal-to-noise ratio);

In light of these remarks and of our study of zero-capacity channels, we want to define VNCs in terms of:

- the noise scaling parameter ϵ,
- the zero-capacity channel it approaches as ϵ goes to zero,
- how it approaches that zero-capacity channel.

Definition 1.3: A Discrete Memoryless Channel (DMC), defined by its transition probabilities $p(y|x)$ and with capacity $C > 0$, is a *"Very Noisy" Channel* if and only if there exists a noise scaling parameter $\epsilon > 0$ and real numbers $w(y)$ and $\lambda(x, y)$ such that:

$$p(y|x) = w(y) + \epsilon \cdot \lambda(x, y) + O(\epsilon^2), \quad \text{for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}, \tag{1.4}$$

and which satisfy the following properties:

- $w(y)$ is a probability distribution:

$$w(y) \geq 0 \text{ and } \sum_{y} w(y) = 1, \quad \text{for all } y \in \mathcal{Y}, \tag{1.5}$$

- the components of the perturbation matrix satisfy:

$$\sum_{y \in \mathcal{Y}} \lambda(x, y) = 0, \quad \text{for all } x \in \mathcal{X}, \tag{1.6}$$

- $\epsilon \approx 0$.

These properties can easily be derived from the fact that $p(y|x)$ are transition probabilities: $\sum_{y \in \mathcal{Y}} p(y|x) = 1$, for all $x \in \mathcal{X}$. Also, it is clear that the VNC approaches a zero-capacity channel as ϵ goes to zero:

$$\lim_{\epsilon \to 0} C = 0.$$

Note that if we fix the $w(y)$'s and the $\lambda(x, y)$'s, we are actually defining a *family* of VNCs the members of which can have any small value for ϵ. We do not consider however zero-capacity channels to be VNCs, although they are limiting channels of families of VNCs.

We can also define the transition probability matrix $P_{Y|X}$ of a VNC in the following way:

$$P_{Y|X} = \Omega_Y + \epsilon \cdot \Lambda_{X,Y} + O(\epsilon^2),$$

where:

- Ω_Y is the transition probability matrix of a zero-capacity channel; all rows of the matrix are equal.
- $\Lambda_{X,Y}$ is the perturbation matrix describing the limiting behavior of the VNC; the sum of the elements of any row of the matrix is equal to 0.
- ϵ is a noise scaling parameter which is measure of the noisiness of the channel

Note that in this case, the probability distribution $w(y)$ is not exactly the output probability distribution $p(y)$. However, $w(y)$ is the limiting output probability distribution as ϵ goes to zero:

$$w(y) = \lim_{\epsilon \to 0} p(y).$$

Finally, for technical reasons, we do not include in our study, channels which we call "extremely noisy" channels and whose perturbation matrices $\Lambda_{X,Y}$ have identical rows, although they satisfy Definition 1.3. We have included several examples of "extremely noisy" channels in Appendix A for the curious reader. Also, section II will clarify the need for such an exclusion.

Examples of VNCs:

For the "very noisy" B.S.C. (figure 1.4),

$$\Omega_Y = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ and } \Lambda_{X,Y} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

For the "very noisy" Z-Channel (figure 1.5),

$$\Omega_Y = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \Lambda_{X,Y} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

I-4. Two Classes of VNCs.

Now that we have a definition of VNCs, we wish to distinguish between two disjoint classes of VNCs. The justification for this distinction will become clear in section II, where we will find that the average mutual information will have totally different expressions, and each class will require different algorithms to compute capacity.

Definition 1.4: A VNC is a Class I VNC if and only if *there does not exist any* output $y \in \mathcal{Y}$ such that $w(y) = 0$.

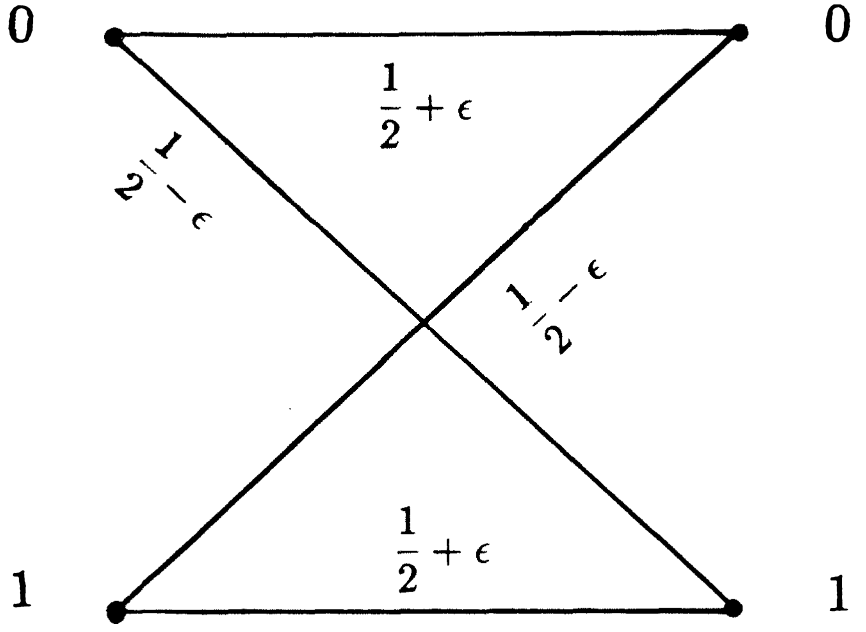


Figure 1.4: The "very noisy" BSC ($\epsilon \approx 0$).

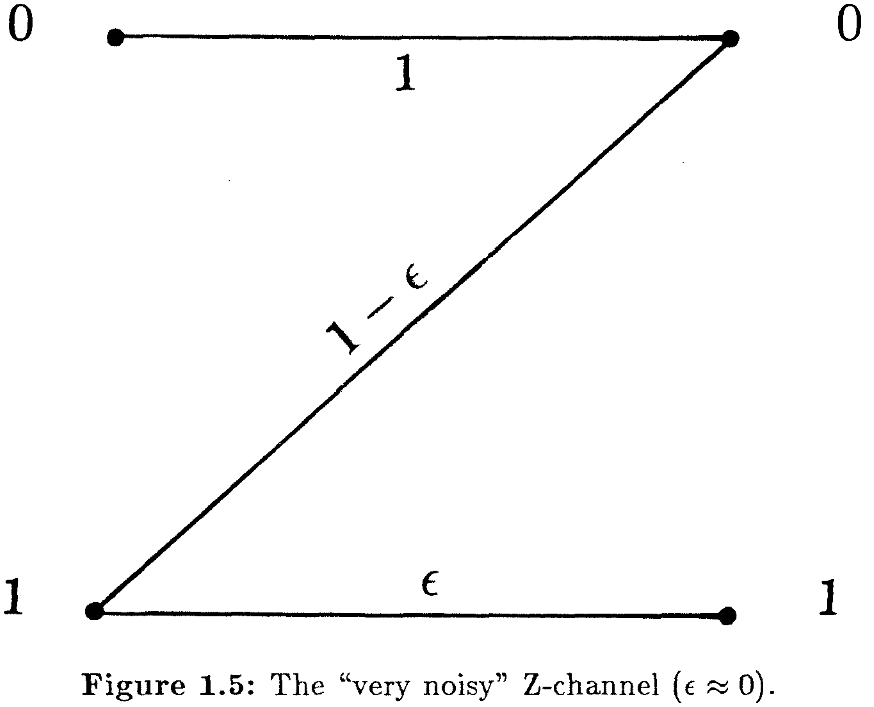


Figure 1.5: The "very noisy" Z-channel ($\epsilon \approx 0$).

Definition 1.5: A VNC is a Class II VNC if and only if *there exists at least one* output $y \in \mathcal{Y}$ such that $w(y) = 0$.

If we partition the set of outputs into two sets $\mathcal{Y}_1$ and $\mathcal{Y}_2$, where:

$$\mathcal{Y}_1 = \{y \in \mathcal{Y} \ : \ w(y) \neq 0\},$$

$$\mathcal{Y}_2 = \{y \in \mathcal{Y} \ : \ w(y) = 0\},$$

then we can say that our distinction is based on whether $\mathcal{Y}_2$ is empty or not.

I-5. Reiffen's Definition of VNCs.

In a note published in 1963, Reiffen [18] gives the following definition of a "very noisy" channel which he limits to discrete-input, continuous-output channels (which we do not study here), and which can be extended to many discrete memoryless channels.

Definition 1.6: Given a DMC with input probabilities $q(x)$, transition probabilities $p(y|x)$, and output probabilities $p(y) = \sum_{x \in \mathcal{X}} q(x)p(y|x)$, it is defined as "very noisy" if, for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$,

$$\epsilon(x,y) = \frac{p(y) - p(y|x)}{p(y)}, \quad \text{and } |\epsilon(x,y)| \ll 1. \tag{1.7}$$

He also notes that this definition corresponds to "many physical channels operating at low signal-to-noise ratio." Equation (1.7) can be rewritten as a definition of the transition probabilities for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$:

$$p(y|x) = p(y)\Big(1 - \epsilon(x,y)\Big), \quad \text{where } |\epsilon(x,y)| \ll 1.$$

Later, Gallager [15] relaxed this definition, replacing $p(y)$ in (1.7) by $v(y)$, where the $v(y)$'s need not exactly be the output probabilities, but only *an approximation* of the output probabilities:

$$\epsilon(x,y) = \frac{v(y) - p(y|x)}{v(y)}, \quad \text{and } |\epsilon(x,y)| \ll 1. \tag{1.8}$$

His definition involved "very noisy" channels, "in the sense that the probability of receiving a given output is almost independent of the input."

In fact, Reiffen and Gallager were defining a class of "very noisy" channels which we have called in this thesis Class I VNCs. There does exist, as we have shown, another class of VNCs that satisfy the requirement that "the probability of receiving a given output be almost independent of the input," which is equivalent to our requirement that "the capacity of the channel be almost zero." The "very noisy" Z-channel is a member of this other class of VNCs, which we defined as Class II VNCs. The question we now ask is: why aren't Class II VNCs encompassed by either Reiffen or Gallager's definition?

Consider any VNC for which there exists an input-output pair (x, y) such that:

$$p(y|x) = 0.$$

Therefore, according to (1.8), there exists a pair (x, y) such that:

$$\epsilon(x, y) = \frac{v(y) - 0}{v(y)} = 1,$$

but without having:

$$|\epsilon(x, y)| \ll 1.$$

Clearly, there is at least one class of channels (the one defined by channels for which there exists a pair (x, y) such that $p(y|x) = 0$), that is not covered by Reiffen and Gallager.

We will now show that all Class I VNCs are covered by (1.7), while no Class II VNC is covered.

A Class I VNC is defined by its transitions probabilitites:

$$p(y|x) = w(y) + \epsilon \cdot \lambda(x, y) + O(\epsilon^2)$$

where $w(y) \neq 0$ for all $y \in \mathcal{Y}$, and $|\frac{\epsilon \cdot \lambda(x,y)}{w(y)}| \ll 1$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

The expression for the output probabilities $p(y)$ is:

$$p(y) = w(y) + \epsilon \cdot \lambda(y) + O(\epsilon^2)$$

where $w(y) \neq 0$ for all y, $\lambda(y) = \sum_{x \in \mathcal{X}} \lambda(x,y)$ and $|\frac{\epsilon \cdot \lambda(y)}{w(y)}| \ll 1$ for all $y \in \mathcal{Y}$.

Therefore we obtain the expression for the ratio:

$$\frac{p(y) - p(y|x)}{p(y)} = \frac{\epsilon \cdot \Big(\lambda(y) - \lambda(x,y)\Big)}{w(y)} + O(\epsilon^2),$$

the absolute value of which, for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ is clearly negligeable with respect to 1. Therefore all Class I VNCs are covered by Reiffen's definition.

Consider now any Class II VNC:

$$p(y|x) = \epsilon \cdot \lambda(x,y) + O(\epsilon^2), \quad \text{for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}_2,$$

$$\text{and } p(y) = \epsilon \cdot \lambda(y) + O(\epsilon^2), \quad \text{for all } y \in \mathcal{Y}_2.$$

If we limit ourselves to $y \in \mathcal{Y}_2$, the expression of the ratio is now, for all $x \in \mathcal{X}$:

$$\frac{p(y) - p(y|x)}{p(y)} = 1 - \frac{\lambda(x,y)}{\lambda(y)} + O(\epsilon),$$

the absolute value of which is not negligible with respect to 1 (unless $\lambda(x,y) = \lambda(y)$ for all $y \in \mathcal{Y}_2$, which is characteristic of "extremely noisy" channels and which we have purposely excluded from our study). Therefore, Class II VNCs are not covered by Reiffen's definition.

The correspondence between the Reiffen-Gallager VNCs and our Class I VNCs now being established, all results in this thesis shown for Class I VNCs also hold for the Reiffen-Gallager VNCs.

Finally, the realization that the "very noisy" Z-channel is a rather interesting VNC must be credited to Silverman [19] who discovered, for instance, its large capacity in cascade compared to the "very noisy" BSC.

II - CAPACITY OF "VERY NOISY" CHANNELS

II-1. Average Mutual Information of VNCs

The expression of the average mutual information $I(X;Y)$, as we will see, depends critically on whether the set $\mathcal{Y}_2$ defined in I-4. is empty or not.

The general expression for the average mutual information, according to (1.1), is:

$$\begin{aligned} I(X;Y) &= \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}} p(y|x) \log\left(\frac{p(y|x)}{\sum_{u\in\mathcal{X}} q(u)p(y|u)}\right), \\ &= I_1(X;Y) + I_2(X;Y), \end{aligned}$$

where:

$$\begin{aligned} I_1(X;Y) &= \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}_1} p(y|x) \log\left(\frac{p(y|x)}{\sum_{u\in\mathcal{X}} q(u)p(y|u)}\right), \\ I_2(X;Y) &= \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}_2} p(y|x) \log\left(\frac{p(y|x)}{\sum_{u\in\mathcal{X}} q(u)p(y|u)}\right). \end{aligned}$$

Let us define for simplicity of notation the quantity:

$$\lambda(y) = \sum_{u\in\mathcal{X}} q(u)\lambda(u,y), \quad \text{for all } y \in \mathcal{Y}.$$

Using our definition of VNCs (1.4), we will evaluate $I(X;Y)$ in two cases: when $\mathcal{Y}_2$ is empty, and when it isn't. Let us first assume $\mathcal{Y}_2 = \emptyset$.

II-1.1. Average mutual information of Class I VNCs.

The transition probabilities of a Class I VNC satisfy:

$$p(y|x) = w(y) + \epsilon\cdot\lambda(x,y) + O(\epsilon^2), \quad \text{for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y},$$

where:

$$w(y) > 0, \quad \text{for all } y \in \mathcal{Y}.$$

However, the computation of $I(X;Y)$ in this section requires a longer expansion of $p(y|x)$ in terms of ϵ although, as we will see, the final expression of $I(X;Y)$ (its first order approximation) does not contain any of the new terms $\mu(x,y)$:

$$p(y|x) = w(y) + \epsilon \cdot \lambda(x,y) + \frac{\epsilon^2}{2} \cdot \mu(x,y) + O(\epsilon^3)$$

We have indicated in definition 1.3 that $\epsilon \approx 0$. We meant by that that we wanted $w(y) + \epsilon \cdot \lambda(x,y)$ to be a "good enough" approximation for $p(y|x)$. One way to express that mathematically is to require that:

$$|\epsilon \cdot \lambda(x,y)| \ll w(y), \quad \text{for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}. \tag{1.9}$$

Under that assumption we can write:

$$\begin{aligned} I(X;Y) = f(\epsilon) = \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} & q(x) \Big(w(y) + \epsilon \cdot \lambda(x,y) + O(\epsilon^2)\Big) \\ & \cdot \log \left(\frac{w(y) + \epsilon \cdot \lambda(x,y) + \frac{\epsilon^2}{2}\mu(x,y) + O(\epsilon^3)}{w(y) + \epsilon \cdot \lambda(y) + \frac{\epsilon^2}{2}\mu(y) + O(\epsilon^3)} \right), \end{aligned}$$

where:

$$\mu(y) = \sum_{u\in\mathcal{X}} q(u)\mu(x,y)$$

We can also express $f(\epsilon)$ as a power series in ϵ :

$$I(X;Y) = f(0) + \epsilon \cdot f'(0) + \frac{\epsilon^2}{2} \cdot f''(0) + O(\epsilon^3).$$

Clearly, $f(0) = 0$.

Let us compute $f'(\epsilon)$:

$$\begin{aligned} f'(\epsilon) &= \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}} \Big(\lambda(x,y) + O(\epsilon)\Big) \log\left(\frac{w(y) + \epsilon \cdot \lambda(x,y) + O(\epsilon^2)}{w(y) + \epsilon \cdot \lambda(y) + O(\epsilon^2)}\right) \\ &+ \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}} \frac{\Big(\lambda(x,y) - \lambda(y)\Big)w(y) + \epsilon \cdot \Big(\mu(x,y) - \mu(y)\Big)w(y) + O(\epsilon^2)}{\Big(w(y) + \epsilon \cdot \lambda(y) + O(\epsilon^2)\Big)}, \end{aligned}$$

which yields:

$$\begin{aligned} f'(\epsilon) &= \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}} \Big(\lambda(x,y) + O(\epsilon)\Big) \log\left(\frac{w(y) + \epsilon \cdot \lambda(x,y) + O(\epsilon^2)}{w(y) + \epsilon \cdot \lambda(y) + O(\epsilon^2)}\right) \\ &+ \sum_{y\in\mathcal{Y}} \frac{w(y)\Big(\sum_{x\in\mathcal{X}} q(x)(\lambda(x,y) - \lambda(y)) + \epsilon \cdot \sum_{x\in\mathcal{X}} q(x)(\mu(x,y) - \mu(y))\Big) + O(\epsilon^2)}{w(y) + \epsilon \cdot \lambda(y) + O(\epsilon^2)}. \end{aligned}$$

But for all $y \in \mathcal{Y}$:

$$\begin{aligned}\sum_{x\in\mathcal{X}} q(x)\Big(\lambda(x,y) - \lambda(y)\Big) &= \sum_{x\in\mathcal{X}} q(x)\lambda(x,y) - \sum_{x\in\mathcal{X}} q(x)\lambda(y)\\ &= \sum_{x\in\mathcal{X}} q(x)\lambda(x,y) - \lambda(y) = 0,\end{aligned}$$

and, in the same way:

$$\sum_{x\in\mathcal{X}} q(x)\Big(\mu(x,y) - \mu(y)\Big) = 0.$$

Therefore,

$$f'(\epsilon) = \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}} \Big(\lambda(x,y) + O(\epsilon)\Big) \log\left(\frac{w(y) + \epsilon\cdot\lambda(x,y) + O(\epsilon^2)}{w(y) + \epsilon\cdot\lambda(y) + O(\epsilon^2)}\right) + O(\epsilon^2).$$

Clearly, $f'(0) = 0$, and we must therefore compute $f''(\epsilon)$:

$$f''(\epsilon) = \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}} \frac{\Big(\lambda(x,y) + O(\epsilon)\Big)\Big(\Big(\lambda(x,y) - \lambda(y)\Big)w(y) + O(\epsilon)\Big)}{\Big(w(y) + \epsilon\cdot\lambda(x,y) + O(\epsilon^2)\Big)\Big(w(y) + \epsilon\cdot\lambda(y) + O(\epsilon^2)\Big)} + O(\epsilon),$$

which yields:

$$\begin{aligned}f''(0) &= \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}} \lambda(x,y)\frac{\Big(\lambda(x,y) - \lambda(y)\Big)}{w(y)},\\ &= \sum_{y\in\mathcal{Y}} \frac{1}{w(y)}\left(\sum_{x\in\mathcal{X}} q(x)\Big(\lambda(x,y) - \lambda(y)\Big)^2\right).\end{aligned}$$

Therefore, when $\mathcal{Y}_2$ is empty, we can write:

$$I(X;Y) = I_1(X;Y) = \frac{\epsilon^2}{2}\cdot\left(\sum_{y\in\mathcal{Y}} \frac{1}{w(y)}\left(\sum_{x\in\mathcal{X}} q(x)\Big(\lambda(x,y) - \lambda(y)\Big)^2\right)\right) + O(\epsilon^3). \quad (1.10)$$

Although we originally had to use a larger expansion of $p(y|x)$ in terms of ϵ by introducing $\mu(x,y)$, the final result fortunately is not influenced by $\mu(x,y)$, which justifies our initial limited expansion of $p(y|x)$ as an appropriate definition for a VNC.

II-1.2. Average mutual information of Class II VNCs.

We know from the previous section, that $I_1(X;Y) = O(\epsilon^2)$. We will now show that when $\mathcal{Y}_2 \neq \emptyset$, $I_2(X;Y) = O(\epsilon)$, and therefore:

$$I(X;Y) = I_2(X;Y) + O(\epsilon^2).$$

For all $y \in \mathcal{Y}_2$:

$$p(y|x) = \epsilon \cdot \lambda(x,y) + O(\epsilon^2), \quad \text{for all } x \in \mathcal{X},$$

which yields:

$$\lambda(x,y) \geq 0, \quad \text{for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}_2.$$

Therefore,

$$I_2(X;Y) = \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}_2} \left(\epsilon \cdot \lambda(x,y) + O(\epsilon^2)\right) \log\left(\frac{\epsilon \cdot \lambda(x,y) + O(\epsilon^2)}{\sum_{u\in\mathcal{X}} q(u)\left(\epsilon \cdot \lambda(u,y) + O(\epsilon^2)\right)}\right),$$

which implies:

$$I_2(X;Y) = \epsilon \cdot \left(\sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}_2} \left(\lambda(x,y) + O(\epsilon)\right) \log\left(\frac{\lambda(x,y) + O(\epsilon)}{\lambda(y) + O(\epsilon)}\right)\right),$$

and finally:

$$I_2(X;Y) = \epsilon \cdot \left(\sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}_2} \lambda(x,y) \log\left(\frac{\lambda(x,y)}{\lambda(y)}\right)\right) + O(\epsilon^2).$$

Therefore, when $\mathcal{Y}_2$ is not empty, we can write:

$$I(X;Y) = \epsilon \cdot \left(\sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}_2} \lambda(x,y) \log\left(\frac{\lambda(x,y)}{\lambda(y)}\right)\right) + O(\epsilon^2). \tag{1.11}$$

II-1.3. Average mutual information of VNCs.

The expressions for $I(X;Y)$ obtained in (1.10) and (1.11) clearly identify two distinct classes of VNCs, depending on whether $\mathcal{Y}_2$ is empty or not:

• **Class I VNCs** ($\mathcal{Y}_2 = \emptyset$)

In this case, the expression for the average mutual information is:

$$I(X;Y) \approx \frac{\epsilon^2}{2} \cdot \left(\sum_{y \in \mathcal{Y}} \frac{1}{w(y)} \left(\sum_{x \in \mathcal{X}} q(x) \Big(\lambda(x,y) - \lambda(y) \Big)^2 \right) \right),$$

where:
- $q(x)$ is the input probability distribution
- $w(y)$ is a probability distribution where $w(y) > 0$ for all $y \in \mathcal{Y}$,
- $\sum_{y \in \mathcal{Y}} \lambda(x,y) = 0$ for all $x \in \mathcal{X}$.

Example: The "Very Noisy" B. S. C.

Then $\mathcal{Y}_2 = \emptyset$ and therefore the "Very Noisy" B.S.C. is a Class I VNC:

$$I(X;Y) \approx \frac{\epsilon^2}{2} \cdot 16\, q(x_1) \Big(1 - q(x_1) \Big).$$

• **Class II VNCs** ($\mathcal{Y}_2 \neq \emptyset$)

In this case, the expression for the average mutual information is:

$$I(X;Y) \approx \epsilon \cdot \left(\sum_{x \in \mathcal{X}} q(x) \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log \left(\frac{\lambda(x,y)}{\lambda(y)} \right) \right)$$

where:
- $q(x)$ is the input probability distribution
- $\lambda(x,y) \geq 0$ for all $y \in \mathcal{Y}_2$ and $x \in \mathcal{X}$,
- $\alpha(x) = \sum_{y \in \mathcal{Y}_2} \lambda(x,y)$ is not necessarily a constant.

It is important to note here that for a Class II VNC, the average mutual information does not depend on y's belonging to $\mathcal{Y}_1$: the information provided by the observance of y's belonging to $\mathcal{Y}_2$ is far greater!

Example: The "Very Noisy" Z-Channel

Here $\mathcal{Y}_2 \neq \emptyset$ and therefore the "Very Noisy" Z-Channel is a Class II VNC:

$$I(X;Y) \approx \epsilon \cdot \Big(1 - q(x_1) \Big) \log \left(\frac{1}{1 - q(x_1)} \right).$$

II-2. - Computing the Capacity of VNCs

The Arimoto-Blahut (AB) algorithm [3],[4] finds the input probability distribution vector $\hat{\mathbf{q}}$ that maximizes the average mutual information $I(X;Y)$, given the transition probabilities of the channel $p(y|x)$. It starts with an arbitrary input probability distribution vector $\mathbf{q_0}$ (usually the uniform probability density). Let $\mathbf{q_n}$ be the input probability density vector at the nth iteration of the AB algorithm. Let us now define the following intermediary quantities that are computed at each iteration n of the algorithm:

$$\begin{aligned}
p_n(y) &= \sum_{u\in\mathcal{X}} q_n(u)p(y|u), \quad \text{for all } y \in \mathcal{Y}, \\
I_n(x) &= \sum_{y\in\mathcal{Y}} p(y|x)\log\frac{p(y|x)}{p_n(y)}, \quad \text{for all } x \in \mathcal{X}, \\
C_n &= \sum_{x\in\mathcal{X}} q_n(x)I_n(x), \\
D_n &= \sum_{x\in\mathcal{X}} q_n(x)\exp\left(I_n(x)\right).
\end{aligned}$$

Then the AB algorithm tells how to compute $\mathbf{q_{n+1}}$ from $\mathbf{q_n}$:

$$q_{n+1}(x) = q_n(x)\frac{\exp\left(I_n(x)\right)}{D_n}, \quad \text{for all } x \in \mathcal{X}.$$

The capacity C of the channel is defined by:

$$C = \lim_{n\to\infty} C_n$$

and $\hat{\mathbf{q}}$ is defined by:

$$\hat{\mathbf{q}} = \lim_{n\to\infty} \mathbf{q_n}$$

The question we now want to answer is: how does the AB algorithm perform when computing the capacity of "Very Noisy" Channels?

Consider a VNC whose transition probabilities are given by

$$p(y|x) = w(y) + \epsilon\cdot\lambda(x,y) + O(\epsilon^2)$$

and we assume that $w(y)$ and $\lambda(x,y)$ are fixed while ϵ is allowed to vary.

Let us define, for a given ϵ, the optimizing input probability vector $\hat{\mathbf{q}}_\epsilon$, which we know we can obtain from the AB algorithm. Now, let us define $\hat{\mathbf{q}}_0$ by

$$\hat{\mathbf{q}}_0 = \lim_{\epsilon \to 0} \hat{\mathbf{q}}_\epsilon$$

How can we compute $\hat{\mathbf{q}}_0$? It is important that we obtain $\hat{\mathbf{q}}_0$ for then we would have an approximation of the capacity of the VNCs considered for *all* "small enough" values of ϵ.

We could compute $\hat{\mathbf{q}}_\epsilon$ for lower and lower values of ϵ until we estimate we have reached a satisfactory degree of convergence. This is not a very practical procedure in the first place, but there is another major problem: the number of iterations required by the AB algorithm for a given precision goes to infinity as ϵ goes to zero!

Another approach is needed to find $\hat{\mathbf{q}}_0$, and an efficient one is to directly try to maximize the quantities independent of ϵ in equations (1.10) and (1.11). Therefore we will directly maximize the following objective functions:

- for Class I VNCs:

$$\phi(\mathbf{q}) = \sum_{y \in Y} \frac{1}{w(y)} \left(\sum_x q(x)\lambda(x,y)^2 - \left(\sum_u q(u)\lambda(u,y) \right)^2 \right)$$

- for Class II VNCs:

$$\phi(\mathbf{q}) = \sum_x q(x) \sum_{y \in Y_2} \lambda(x,y) \log \frac{\lambda(x,y)}{\sum_u q(u)\lambda(u,y)}$$

II-3. Computing the capacity of Class I VNCs

II-3.1. General case

We want to find the input probability distribution $\hat{\mathbf{q}}_0$ that maximizes the objective function:

$$\phi(\mathbf{q}) = \sum_{y\in\mathcal{Y}} \frac{1}{w(y)}\Big(\sum_{x\in\mathcal{X}} q(x)\Big(\lambda(x,y)-\lambda(y)\Big)^2\Big),$$
$$= \sum_{y\in\mathcal{Y}} \frac{1}{w(y)}\Big(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)^2 - \Big(\sum_{u\in\mathcal{X}} q(u)\lambda(u,y)\Big)^2\Big).$$

Theorem 1.2: *The objective function $\phi(\mathbf{q})$ is convex $\cap$ in the input probability vector* $\mathbf{q}$.

Proof of Theorem 1.2:

Define:

$$\phi_y(\mathbf{q}) = \sum_{x\in\mathcal{X}} q(x)\lambda(x,y)^2 - \Big(\sum_{x} q(x)\lambda(x,y)\Big)^2.$$

Then we can express the objective function as:

$$\phi(\mathbf{q}) = \sum_{y\in\mathcal{Y}} \frac{1}{w(y)}\phi_y(\mathbf{q}).$$

Therefore,

$$\phi(r\mathbf{q}_1+\bar{r}\mathbf{q}_2) - r\phi(\mathbf{q}_1) - \bar{r}\phi(\mathbf{q}_2) = \sum_{y\in\mathcal{Y}} \frac{1}{w(y)}\Big(\phi_y(r\mathbf{q}_1+\bar{r}\mathbf{q}_2) - r\phi_y(\mathbf{q}_1) - \bar{r}\phi_y(\mathbf{q}_2)\Big).$$

It is therefore sufficient to show that, for all $y \in \mathcal{Y}$, $\phi_y(\mathbf{q})$ is convex $\cap$ in $\mathbf{q}$:

$$\phi_y(r\mathbf{q}_1+\bar{r}\mathbf{q}_2) - r\phi_y(\mathbf{q}_1) - \bar{r}\phi_y(\mathbf{q}_2) =$$

$$\sum_{x\in\mathcal{X}}\Big(rq_1(x)+\bar{r}q_2(x)\Big)\lambda(x,y)^2 - \Big(\sum_{x\in\mathcal{X}}\Big(rq_1(x)+\bar{r}q_2(x)\Big)\lambda(x,y)\Big)^2$$
$$-\,r\sum_{x\in\mathcal{X}} q_1(x)\lambda(x,y)^2 + r\Big(\sum_{x\in\mathcal{X}} q_1(x)\lambda(x,y)\Big)^2$$
$$-\,\bar{r}\sum_{x\in\mathcal{X}} q_2(x)\lambda(x,y)^2 + \bar{r}\Big(\sum_{x\in\mathcal{X}} q_2(x)\lambda(x,y)\Big)^2,$$

which yields:

$$\phi_y(r\mathbf{q}_1 + \bar{r}\mathbf{q}_2) - r\phi_y(\mathbf{q}_1) - \bar{r}\phi_y(\mathbf{q}_2) = r\bar{r}\Big(\sum_{x\in\mathcal{X}} q_1(x)\lambda(x,y) - \sum_{x\in\mathcal{X}} q_2(x)\lambda(x,y)\Big)^2,$$
$$\geq 0.$$

Since our objective function is convex $\cap$, any local maximum is a global maximum. We can also note that the objective function $\phi(\mathbf{q})$ is quadratic in $\mathbf{q}$:

$$\phi(\mathbf{q}) = \mathbf{d}^T\mathbf{q} - \frac{1}{2}\mathbf{q}^T C\mathbf{q}, \tag{1.12}$$

under the constraints: $A\mathbf{q} = b$ and $\mathbf{q} \geq 0$,

where: $A = (1,\ 1,\ldots,\ 1)$ and $b = (1)$,
$C = (c_{xx'})$ where $c_{xx'} = 2\sum_{y\in\mathcal{Y}} \frac{\lambda(x,y)\lambda(x',y)}{w(y)}$,
$\mathbf{d} = (d_x)$ where $d_x = \sum_{y\in\mathcal{Y}} \frac{\lambda(x,y)^2}{w(y)}$.

Also, since for all $\mathbf{q}$,

$$\mathbf{q}^T C\mathbf{q} = 2\sum_{y\in\mathcal{Y}} \frac{1}{w(y)}\Big(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)\Big)^2 \geq 0,$$

then C is positive semi-definite (as well as symmetric). This is no surprise, since C being a positive semi-definite matrix is equivalent to $\phi(\mathbf{q})$ being convex $\cap$ in $\mathbf{q}$.

Therefore, maximizing the average mutual information for Class I VNCs is done by solving a Quadratic Programming problem with a symmetric and positive definite matrix. Wolfe [5],[6] showed that such a problem can be solved by transforming it into an equivalent Linear Programming problem (using Barankin-Dorfman's procedure [7]), and then simply using the Simplex Method along with an exclusion rule. Here is the equivalent Linear Programming problem:

$$\begin{cases} A\mathbf{q} & = b \\ C\mathbf{q} - \mathbf{v} + A^T(u_1 - u_2) & = \mathbf{d} \end{cases}$$

where: $\mathbf{q} \geq 0,\ \mathbf{v} \geq 0,\ u_1 \geq 0,\ u_2 \geq 0$

under the "side condition": $\mathbf{v}^T\mathbf{q} = 0$

Note that, at each step of the simplex algorithm, the new basic solution has to satisfy a side condition, which is called the "exclusion rule". Wolfe showed that if the matrix C is positive definite, then (1.12) has a unique solution, and so the problem is solved for a positive definite matrix.

However, in the case there exists a probability vector $\mathbf{q}$ such that:

$$\Lambda_{X,Y}^T \mathbf{q} = \mathbf{0}, \quad (\text{which implies } \mathbf{q}^T C \mathbf{q} = 0)$$

then the matrix C is not positive definite. In this case, we used a method suggested by E.M.L. Beale [8], i.e., calculating the effect of a virtual perturbation of C by replacing c_{xx} by $c_{xx} + \gamma_x$ where $|\gamma_x| \ll |c_{xx}|$.

Many examples of capacity calculation are given in Appendix B. But there is a case, besides the well-known case of the symmetric channel, when we need not use an algorithm to find the optimizing input probability vector: for binary input Class I VNCs.

II-3.2. Capacity of Binary-Input Class I VNCs.

Theorem 1.3: *Given n real numbers $\lambda_1 < \lambda_2 < \ldots < \lambda_n$, the probability distribution $\hat{\mathbf{q}}$ that maximizes the variance $\sigma_{\mathbf{q}}$ of these numbers is:*

$$\begin{cases} \hat{q}_1 = \hat{q}_n = \frac{1}{2}, & \\ \hat{q}_i = 0 & \text{for } 1 < i < n. \end{cases}$$

Proof of Theorem 1.3:

Given a distribution $\mathbf{q}$, the variance is:

$$\sigma_{\mathbf{q}} = \sum_{i+1}^{n} q_i \lambda_i^2 - \left(\sum_i q_i \lambda_i\right)^2.$$

Using the Lagrange multiplier α to take into account the fact that we are dealing with a probability distribution, we have the following objective function to maximize:

$$\varphi(\mathbf{q}) = \sum_{i+1}^{n} q_i \lambda_i^2 - (\sum_{i=1}^{n} q_i \lambda_i)^2 + \alpha \sum_{i=1}^{n} q_i.$$

Using Kuhn-Tucker's theorem, we can differentiate the objective function with respect to all q_i's for which $\hat{q}_i \neq 0$:

$$\frac{\partial \varphi(\mathbf{q})}{\partial q_i} = \lambda_i^2 - 2\lambda_i (\sum_{j=1}^{n} q_j \lambda_j) + \alpha = 0.$$

Define the average number

$$\bar{\lambda}_{\mathbf{q}} = \sum_{j=1}^{n} q_j \lambda_j.$$

Then,

$$\lambda_i^2 - 2\lambda_i \bar{\lambda}_{\mathbf{q}} + \alpha = 0 \tag{1.13}$$

Let us first suppose there is only one $\hat{q}_i$ such that $\hat{q}_i \neq 0$. Then clearly, $\hat{q}_i = 1$, but then $\sigma_{\mathbf{q}} = 0$. Therefore there are at least two distinct indices i such that $\hat{q}_i \neq 0$. Take any two such indices, and name them i_1 and i_2 : They both satisfy (1.13), and therefore:

$$\begin{cases} \lambda_{i_1}^2 - 2\lambda_{i_1} \bar{\lambda}_{\mathbf{q}} + \alpha = 0 \\ \lambda_{i_2}^2 - 2\lambda_{i_2} \bar{\lambda}_{\mathbf{q}} + \alpha = 0 \end{cases}$$

Subtracting the second equation from the first, we obtain:

$$(\lambda_{i_1}^2 - \lambda_{i_2}^2) - 2\bar{\lambda}_{\mathbf{q}}(\lambda_{i_1} - \lambda_{i_2}) = 0$$

Since $\lambda_{i_1} \neq \lambda_{i_2}$, then this implies:

$$\frac{\lambda_{i_1} + \lambda_{i_2}}{2} = \bar{\lambda}_{\mathbf{q}}.$$

Is it possible to have a third distinct index i_3 such that $\hat{q}_{i_3} \neq 0$? If this were the case, we would have:

$$\frac{\lambda_{i_1} + \lambda_{i_2}}{2} = \frac{\lambda_{i_1} + \lambda_{i_3}}{2} = \frac{\lambda_{i_2} + \lambda_{i_3}}{2} = \bar{\lambda}_{\mathbf{q}},$$

which would yield the contradiction:

$$\lambda_{i_1} = \lambda_{i_2} = \lambda_{i_3}.$$

Therefore, there are exactly two numbers to which non-zero probabilities are assigned.

The expression of the variance now becomes:

$$\begin{aligned}\sigma_{\hat{\mathbf{q}}} &= q_{i_1}\lambda_{i_1}^2 + q_{i_2}\lambda_{i_2}^2 - (q_{i_1}\lambda_{i_1} + q_{i_2}\lambda_{i_2})^2 \\ &= q_{i_1}q_{i_2}\left(\frac{\lambda_{i_2} - \lambda_{i_1}}{2}\right)^2\end{aligned}$$

It is clear that the variance is maximized when a probability of $\frac{1}{2}$ is assigned to both λ_{i_1} and λ_{i_2}, and when λ_{i_1} and λ_{i_2} are as far apart from each other as possible, i.e., $\lambda_{i_1} = \lambda_1$ and $\lambda_{i_2} = \lambda_n$. The solution is globally optimal because $\sigma_{\mathbf{q}}$ is convex $\cap$ in $\mathbf{q}$.

Note: If we replace $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ by $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ and $\lambda_1 < \lambda_n$, the solution we gave in Theorem 1.3 is still a globally optimal solution although not the only one.

Theorem 1.4: *For all Binary Input Class I VNCs, the probability distribution* $\hat{\mathbf{q}}_0$ *that maximizes* $I(X;Y)$ *is:*

$$\hat{\mathbf{q}}_0 = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Proof of Theorem 1.4:

We want to maximize:

$$\phi(\mathbf{q}) = \sum_{y\in\mathcal{Y}} \frac{\phi_y(\mathbf{q})}{w(y)},$$

where:

$$\phi_y(\mathbf{q}) = \sum_{x\in\mathcal{X}} q(x)\lambda(x,y)^2 - \left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)\right)^2.$$

For a given y, $\phi_y(\mathbf{q})$ represents the variance of the numbers $\lambda(x,y)$, $x \in \mathcal{X}$. Using Theorem 3, $\phi_y(\mathbf{q})$ is therefore maximized by the uniform distribution $(\frac{1}{2}, \frac{1}{2})$, since there are only two possible inputs. Since this is true for all y, then it is clear that $\hat{\mathbf{q}}_0 = (\frac{1}{2}, \frac{1}{2})$ maximizes $\phi(\mathbf{q})$.

Theorem 1.3 gives an interesting insight in the solution for more than two inputs: each $\phi_y(\mathbf{q})$ is maximized by assigning $(\frac{1}{2}, \frac{1}{2})$ to the two inputs x_a and x_b such that:

$$\begin{cases} \lambda(x_a, y) \le \lambda(x, y), & \text{for all } x \\ \lambda(x_b, y) \ge \lambda(x, y), & \text{for all } x. \end{cases}$$

These two inputs x_a and x_b could be different for each output y, and this is what prevents us from generalizing the result from two inputs to more than two inputs.

Corollary 1.1: *The capacity C of a Binary Input Class I VNC is:*

$$C = \frac{\epsilon^2}{8} \cdot \left(\sum_{y \in Y} \frac{1}{w(y)} \Big(\lambda(x_1, y) - \lambda(x_2, y) \Big)^2 \right) + O(\epsilon^3)$$

This corollary follows directly from replacing the input distribution by the uniform distribution $(\frac{1}{2}, \frac{1}{2})$.

II-4. Computing the capacity of Class II VNCs.

We want to find the input probability distribution $\hat{\mathbf{q}}_0$ that maximizes the objective function:

$$\phi(\mathbf{q}) = \sum_{x \in \mathcal{X}} q(x) \sum_{y \in \mathcal{Y}_2} \lambda(x, y) \log \frac{\lambda(x, y)}{\sum_{u \in \mathcal{X}} q(u) \lambda(u, y)},$$

under the constraints:

$$\sum_{x \in \mathcal{X}} q(x) = 1,$$
$$q(x) \ge 0, \quad \text{for all } x \in \mathcal{X}.$$

Remember that for all $y \in \mathcal{Y}_2$,

$$\lambda(x, y) \ge 0.$$

This expression looks like that of average mutual information, except that the real numbers $\sum_{y \in \mathcal{Y}_2} \lambda(x, y)$ are not necessarily equal. Let us therefore define:

$$\alpha(x) = \sum_{y \in \mathcal{Y}_2} \lambda(x, y), \quad \text{for all } x \in \mathcal{X}.$$

We will see that the algorithm that finds the capacity of a Class II VNC depends critically on whether there exists an input $x \in \mathcal{X}$ such that $\alpha(x) = 0$, i.e., for all $y \in \mathcal{Y}_2$, $\lambda(x,y) = 0$. Let us define the two disjoint subsets of $\mathcal{X}$:

$$\begin{cases} \mathcal{X}_1 = \{x \in \mathcal{X} \ : \ \alpha(x) > 0\}, \\ \mathcal{X}_2 = \{x \in \mathcal{X} \ : \ \alpha(x) = 0\}. \end{cases}$$

The objective function now becomes:

$$\phi(\mathbf{q}) = \sum_{x \in \mathcal{X}_1} q(x) \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log \frac{\lambda(x,y)}{\sum_{u \in \mathcal{X}_1} q(u)\lambda(u,y)}.$$

At this point we will have to distinguish between two subclasses of Class II Very Noisy Channels:

- Class IIa VNCs, which verify:

$$\mathcal{X}_2 = \emptyset.$$

- Class IIb VNCs, such as the "Very Noisy" Z-Channel, which verify:

$$\mathcal{X}_2 \neq \emptyset.$$

The algorithms will be different in both cases, and therefore the two cases will be treated separately. However, they have one thing in common, in that they are derived from a generalization of the Arimoto-Blahut algorithm. Because of the similarity between the general expression of the average mutual information and the objective function that we have to maximize for Class II VNCs, we will use a *double-maximization* technique, similar to the one used by Arimoto and Blahut. This technique, as we will see, consists in broadening the definition of capacity to a larger maximization problem.

II-4.1. Computing the capacity of Class IIa VNCs

For Class IIa VNCs, $\mathcal{X} = \mathcal{X}_1$ and $\mathcal{X}_2 = \emptyset$. We will now consider a Class IIa VNC whose perturbation matrix is $\Lambda_{X,Y}$ and considered fixed. Let us define the function

[4]:

$$J(\mathbf{q},\mu) = \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}_2} \lambda(x,y) \log \frac{\mu(x,y)}{q(x)}$$

where: – $q(x)$ is the input probability distribution

– $\Lambda_{X,Y} = \Big(\lambda(x,y)\Big)$ is the perturbation matrix

– $\mu(x,y)$ are elements of a stochastic matrix, i.e.,

$\mu(x,y) \geq 0$ and for all y, $\sum_{x\in\mathcal{X}} \mu(x,y) = 1$.

Clearly, the function $J(\mathbf{q},\mu)$ is defined, when either $\mu(x,y) \neq 0$ or $\mu(x,y) = \lambda(x,y) = 0$. In the latter case, the corresponding term $\lambda(x,y)\log\frac{\mu(x,y)}{q(x)}$ may be dropped out of the sum.

We now have the following theorem:

Theorem 1.5: **a–** *The maximum value of $\phi(\mathbf{q})$ is a double maximum:*

$$\max_{q(x)} \phi(\mathbf{q}) = \max_{q(x)} \max_{\mu(x,y)} J(\mathbf{q},\mu)$$

b– *For fixed $q(x)$, $J(\mathbf{q},\mu)$ is maximized by:*

$$\mu(x,y) = \frac{q(x)\lambda(x,y)}{\sum_{u\in\mathcal{X}} q(u)\lambda(u,y)} \tag{1.14}$$

c– *For fixed $\mu(x,y)$, $J(\mathbf{q},\mu)$ is maximized by:*

$$q(x) = \exp\Big(\frac{1}{\alpha(x)} \sum_{y\in\mathcal{Y}_2} \lambda(x,y)\log\mu(x,y)\Big)\exp\Big(\frac{\alpha}{\alpha(x)} - 1\Big) \tag{1.15}$$

where α is the unique solution to:

$$f(\alpha) = \sum_{x\in\mathcal{X}} \bigg(\exp\Big(\frac{1}{\alpha(x)} \sum_{y\in\mathcal{Y}_2} \lambda(x,y)\log\mu(x,y)\Big)\exp\Big(\frac{\alpha}{\alpha(x)} - 1\Big)\bigg) = 1. \tag{1.16}$$

Proof of Theorem 1.5:

Proof of a– and b–:

We only need to show that:

$$\phi(\mathbf{q}) = \max_{\mu(x,y)} J(\mathbf{q},\mu).$$

and specify the optimizing stochastic matrix $\mu(x,y)$.

$$\phi(\mathbf{q}) - J(\mathbf{q},\mu) = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}_2} q(x)\lambda(x,y)\log\Big(\frac{\lambda(x,y)}{\sum_{u\in\mathcal{X}} q(u)\lambda(u,y)}\cdot\frac{q(x)}{\mu(x,y)}\Big)$$

If we now use $\log(x) \geq 1 - \frac{1}{x}$,

$$\begin{aligned}\phi(\mathbf{q}) - J(\mathbf{q},\mu) &\geq \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}_2} q(x)\lambda(x,y)\Big(1 - \frac{\mu(x,y)}{q(x)}\cdot\frac{\sum_{u\in\mathcal{X}} q(u)\lambda(u,y)}{\lambda(x,y)}\Big)\\ &\geq \Big(\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}_2} q(x)\lambda(x,y)\Big) - \sum_{y\in\mathcal{Y}_2}\Big(\sum_{u\in\mathcal{X}} q(u)\lambda(u,y)\Big)\Big(\sum_{x\in\mathcal{X}}\mu(x,y)\Big)\end{aligned}$$

But since $\sum_{x\in\mathcal{X}}\mu(x,y) = 1$, then

$$\phi(\mathbf{q}) - J(\mathbf{q},\mu) \geq \Big(\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}_2} q(x)\lambda(x,y)\Big) - \sum_{y\in\mathcal{Y}_2}\sum_{u\in\mathcal{X}} q(u)\lambda(u,y) = 0.$$

Therefore,

$$\phi(\mathbf{q}) \geq J(\mathbf{q},\mu)$$

with equality if and only if

$$\mu(x,y) = \frac{q(x)\lambda(x,y)}{\sum_{u\in\mathcal{X}} q(u)\lambda(u,y)}.$$

and so,

$$\phi(\mathbf{q}) = \max_{\mu(x,y)} J(\mathbf{q},\mu).$$

Proof of c–:

If for some y, $\mu(x,y) = 0$ and $\lambda(x,y) \neq 0$, then $q(x)$ should be set equal to 0 in order to maximize $J(\mathbf{q},\mu)$. Such an x can be deleted from the sum and dropped from further consideration. We are now left with the problem of finding the non-zero values of $q(x)$ that maximize $J(\mathrm{q},\mu)$. $J(\mathrm{q},\mu)$ can now be maximized over $q(x)$ by temporarily ignoring the constraint $q(x) \geq 0$, and using the Lagrange multiplier α to constrain $\sum_{x\in\mathcal{X}} q(x) = 1$.

For all $x \in \mathcal{X}$,

$$\frac{\partial}{\partial q(x)}\Big(\sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}_2} \lambda(x,y) \log \frac{\mu(x,y)}{q(x)} + \alpha\Big(\sum_{x\in\mathcal{X}} q(x) - 1\Big)\Big) = 0$$

$$\Leftrightarrow \sum_{y\in\mathcal{Y}_2} \lambda(x,y) \log \frac{\mu(x,y)}{q(x)} - \sum_{y\in\mathcal{Y}_2} \lambda(x,y) + \alpha = 0$$

$$\Leftrightarrow \Big(\sum_{y\in\mathcal{Y}_2} \lambda(x,y)\Big) \log q(x) = \sum_{y\in\mathcal{Y}_2} \lambda(x,y) \log \mu(x,y) - \Big(\sum_{y\in\mathcal{Y}_2} \lambda(x,y)\Big) + \alpha \qquad (1.17)$$

$$\Leftrightarrow q(x) = \exp\Big(\frac{1}{\alpha(x)} \sum_{y\in\mathcal{Y}_2} \lambda(x,y) \log \mu(x,y)\Big) \exp\Big(\frac{\alpha}{\alpha(x)} - 1\Big)$$

Clearly, since $q(x)$ is a probability distribution, then we obtain condition (1.16) on α by writing $\sum_{x\in\mathcal{X}} q(x) = 1$. Also, the value of α is unique since for all $x \in \mathcal{X}$, $\alpha(x) > 0$ and therefore $\exp(\frac{\alpha}{\alpha(x)} - 1)$ is strictly increasing in α, which implies that $f(\alpha)$ is strictly increasing in α. Also, it is clear that the values of $q(x)$ obtained are non-negative.

If we now combine the results obtained in b$-$ and c$-$, we obtain the following corollary:

Corollary 1.2: *If* $\hat{\mathrm{q}}$ *achieves capacity for a class IIa VNC, then all* $\hat{q}(x)$ *verify:*

$$\hat{q}(x) = \hat{q}(x) \exp\Big(\frac{1}{\alpha(x)} \sum_{y\in\mathcal{Y}_2} \lambda(x,y) \log \frac{\lambda(x,y)}{\sum_{u\in\mathcal{X}} \hat{q}(u)\lambda(u,y)}\Big) \exp\Big(\frac{\alpha}{\alpha(x)} - 1\Big)$$

where α *verifies:*

$$\sum_{x\in\mathcal{X}} \hat{q}(x)\Big(\exp\Big(\frac{1}{\alpha(x)} \sum_{y\in\mathcal{Y}_2} \lambda(x,y) \log \mu(x,y)\Big) \exp\Big(\frac{\alpha}{\alpha(x)} - 1\Big)\Big) = 1$$

and for all $x \in \mathcal{X}$,

$$\alpha(x) = \sum_{y\in\mathcal{Y}_2} \lambda(x,y) > 0.$$

Proof of Corollary 1.2:

If we replace in (1.15) $\mu(x,y)$ by its expression in (1.14), we obtain for $q(x) \neq 0$:

$$\begin{aligned} q(x) &= \exp\Big(\frac{1}{\alpha(x)} \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log \frac{q(x)\lambda(x,y)}{\sum_{u \in \mathcal{X}} q(u)\lambda(u,y)}\Big) \exp\Big(\frac{\alpha}{\alpha(x)} - 1\Big) \\ &= \exp\Big(\frac{1}{\alpha(x)} \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log q(x)\Big) \exp\Big(\frac{1}{\alpha(x)} \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log \frac{\lambda(x,y)}{\sum_{u \in \mathcal{X}} q(u)\lambda(u,y)}\Big) \\ &\qquad \cdot \exp\Big(\frac{\alpha}{\alpha(x)} - 1\Big) \\ &= q(x) \exp\Big(\frac{1}{\alpha(x)} \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log \frac{\lambda(x,y)}{\sum_{x \in \mathcal{X}} q(u)\lambda(u,y)}\Big) \exp\Big(\frac{\alpha}{\alpha(x)} - 1\Big) \end{aligned}$$

It is easy to see that the same equation is verified for $q(x) = 0$.

Also, it is clear from Theorem 1.5b that $\mu(x,y)$ can only be equal to zero when $\lambda(x,y)$ is also zero, which reassures us with respect to a proper definition of the $J(\mathbf{q},\mu)$.

We are able to use this corollary as in [4] to finally obtain the following algorithm that converges to $\hat{\mathbf{q}}_0$ (we can start with a uniform input distribution).

Algorithm 1:

At every step of the iteration,

1. We compute the following intermediary values:

$$\begin{aligned} \lambda_n(y) &= \sum_{u \in \mathcal{X}} q_n(u)\lambda(u,y) \\ I_n(x) &= \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log \frac{\lambda(x,y)}{\lambda_n(y)} \\ \phi_n &= \sum_{x \in \mathcal{X}} q_n(x) I_n(x) \end{aligned}$$

2. We compute α_n (using Newton's iteration method, which converges very fast in this case) such that:

$$f(\alpha_n) = \sum_{x \in \mathcal{X}} q_n(x) \exp\Big(\frac{I_n(x)}{\alpha(x)}\Big) \exp\Big(\frac{\alpha_n}{\alpha(x)} - 1\Big) = 1$$

3. Finally we obtain the next value for the input distribution:

$$q_{n+1}(x) = q_n(x)\exp\Big(\frac{I_n(x)}{\alpha(x)}\Big)\exp\Big(\frac{\alpha_n}{\alpha(x)} - 1\Big)$$

There is at least one particular case for which this algorithm simplifies, when $\alpha(x)$ is a positive constant independent of x (we can assume, without loss of generality, that this constant is 1); then the algorithm is exactly the AB algorithm where the normalizing factor

$$D_n = \frac{1}{\exp(\alpha - 1)},$$

independent of x in this case, need not be found using Newton's iteration method, and can be computed directly:

$$D_n = \sum_{x\in\mathcal{X}} q_n(x)\exp\Big(I_n(x)\Big)$$

In this case, the algorithm becomes:

$$q_{n+1}(x) = q_n(x)\frac{\exp\Big(I_n(x)\Big)}{D_n}$$

Examples of capacity computation for Class IIa VNCs:

We have run Algorithm 1 for various examples of Class IIa VNCs, which appear in Appendix C.

II-4.2. Computing the capacity of Class IIb VNCs

For Class IIb VNCs, $\mathcal{X}_2 \neq \emptyset$. Let us define

$$q_2 = \sum_{x\in\mathcal{X}_2} q(x).$$

We now have to maximize the objective function:

$$\phi(\mathrm{q}) = \sum_{x\in\mathcal{X}_1} q(x)\sum_{y\in\mathcal{Y}_2}\lambda(x,y)\log\frac{\lambda(x,y)}{\sum_{u\in\mathcal{X}_1} q(u)\lambda(u,y)}$$

under the constraint

$$\sum_{x\in\mathcal{X}_1} q(x) + q_2 = 1, \quad (\text{similarly, } \sum_{x\in\mathcal{X}_1} q(x) \leq 1) \tag{1.18}$$

This formulation of the problem makes it clear that we need only find $\hat{q}(x)$ and $\hat{q}_2$. Any distribution of the probability $\hat{q}_2$ among x's belonging to $\mathcal{X}_2$ will be optimal since the value of $\phi(\mathrm{q})$ and the constraint (1.18) are not affected by such an operation.

Let us again consider the function:

$$J(\mathrm{q},\mu) = \sum_{x\in\mathcal{X}_1} q(x) \sum_{y\in\mathcal{Y}_2} \lambda(x,y) \log \frac{\mu(x,y)}{q(x)}.$$

Then we have the following theorem, identical in part to Theorem 1.5:

Theorem 1.6:

a– *The maximum value of $\phi(\mathrm{q})$ is a double maximum:*

$$\max_{q(x)} \phi(\mathrm{q}) = \max_{q(x)} \max_{\mu(x,y)} J(\mathrm{q},\mu)$$

b– *For fixed $q(x)$, $J(\mathrm{q},\mu)$ is maximized by:*

$$\mu(x,y) = \frac{q(x)\lambda(x,y)}{\sum_{u\in\mathcal{X}_1} q(u)\lambda(u,y)} \tag{1.19}$$

c– *For all $x \in \mathcal{X}_1$, and for fixed $\mu(x,y)$, compute the quantities*

$$q(x) = \exp\Big(\frac{1}{\alpha(x)} \sum_{y\in\mathcal{Y}_2} \lambda(x,y) \log \mu(x,y)\Big) \exp(-1) \tag{1.20}$$

→ *If they add up to a value less or equal than 1, then they maximize $J(\mathrm{q},\mu)$. Finally, set q_2 to $1 - \sum_{x\in\mathcal{X}_1} q(x)$.*

→ *If they add up to a value strictly greater than 1, then $J(\mathrm{q},\mu)$ is maximized by:*

for $x \in \mathcal{X}_2$, $\hat{q}(x) = 0$

for $x \in \mathcal{X}_1$, $\hat{q}(x) = \exp\Big(\frac{1}{\alpha(x)} \sum_{y\in\mathcal{Y}_2} \lambda(x,y) \log \mu(x,y)\Big) \exp\Big(\frac{\alpha}{\alpha(x)} - 1\Big)$

where α is such that:

$$f(\alpha) = \sum_{x\in\mathcal{X}_1}\left(\exp\left(\tfrac{1}{\alpha(x)}\sum_{y\in\mathcal{Y}_2}\lambda(x,y)\log\mu(x,y)\right)\exp\left(\tfrac{\alpha}{\alpha(x)}-1\right)\right) = 1$$

Proof of Theorem 1.6:

The proof of a– and b– is identical to that of parts a– and b– of Theorem 1.5. Before we go to the proof of part c–, we need to prove two other theorems.

Let us address the problem of finding, for fixed $\mu(x,y)$, the distribution $\hat{\mathbf{q}}$ that maximizes $J(\mathbf{q},\mu)$, i.e., the values of $\hat{q}(x)$ for $x \in \mathcal{X}_1$ and of $\hat{q}_2$. Ironically, the way in which the optimal $\hat{\mathbf{q}}$ is computed will depend on whether the optimal $\hat{q}_2$ is zero or not (cases corresponding to different Kuhn-Tucker conditions). Fortunately, there is a way to find this out, i.e., by first assuming that $\hat{q}_2 \neq 0$, and then confirming or disproving the validity of the assumption. Here are two theorems we will need to show this:

Theorem 1.7: *If we define $J(\mathbf{q},\mu)$ by*

$$J(\mathbf{q},\mu) = \sum_{x\in\mathcal{X}} q(x)\sum_{y\in\mathcal{Y}}\lambda(x,y)\log\frac{\mu(x,y)}{q(x)}$$

then, for any fixed $\mu(x,y)$, $J(\mathbf{q},\mu)$ is convex $\cap$ in $\mathbf{q}$.

Proof of Theorem 1.7:

Define the parameter θ :

$$0 \le \theta \le 1$$

We can also express $J(\mathbf{q},\mu)$ in the following way

$$\begin{aligned} J(\mathbf{q},\mu) &= \sum_{x\in\mathcal{X}} q(x)\sum_{y\in\mathcal{Y}}\lambda(x,y)\log\Big(\mu(x,y)\Big) + \sum_{x\in\mathcal{X}} q(x)\log\frac{1}{q(x)}\cdot\sum_{y\in\mathcal{Y}}\lambda(x,y) \\ &= \sum_{x\in\mathcal{X}} q(x)\sum_{x\in\mathcal{X}}\lambda(x,y)\log\Big(\mu(x,y)\Big) + \sum_{x\in\mathcal{X}}\alpha(x)\cdot q(x)\log\frac{1}{q(x)} \end{aligned}$$

where the first term is clearly linear in $\mathbf{q}$.

Therefore, dropping the linear term,

$$J(\theta\mathbf{q_1}+\bar{\theta}\mathbf{q_2},\mu)-\theta J(\mathbf{q_1},\mu)-\bar{\theta}J(\mathbf{q_2},\mu)$$

$$\begin{aligned}
&=\sum_{x\in\mathcal{X}}\alpha(x)\Bigg(\Big(\theta q_1(x)+\bar{\theta}q_2(x)\Big)\log\frac{1}{\theta q_1(x)+\bar{\theta}q_2(x)} \\
&\qquad\qquad -\theta q_1(x)\log\frac{1}{q_1(x)}-\bar{\theta}q_2(x)\log\frac{1}{q_2(x)}\Bigg) \\
&=\sum_{x\in\mathcal{X}}\alpha(x)\left(\theta q_1(x)\log\frac{q_1(x)}{\theta q_1(x)+\bar{\theta}q_2(x)}+\bar{\theta}q_2(x)\log\frac{q_2(x)}{\theta q_1(x)+\bar{\theta}q_2(x)}\right)
\end{aligned}$$

Now, using $\log x \geq 1-\frac{1}{x}$,

$$J(\theta\mathbf{q_1}+\bar{\theta}\mathbf{q_2},\mu)-\theta J(\mathbf{q_1},\mu)-\bar{\theta}J(\mathbf{q_2},\mu)$$

$$\begin{aligned}
&\geq\sum_{x\in\mathcal{X}}\alpha(x)\left(\theta q_1(x)\left(1-\frac{\theta q_1(x)+\bar{\theta}q_2(x)}{q_1(x)}\right)+\bar{\theta}q_2(x)\left(1-\frac{\theta q_1(x)+\bar{\theta}q_2(x)}{q_2(x)}\right)\right) \\
&\geq\sum_{x\in\mathcal{X}}\alpha(x)\Big(\theta\Big(q_1(x)-\theta q_1(x)-\bar{\theta}q_2(x)\Big)+\bar{\theta}\Big(q_2(x)-\theta q_1(x)-\bar{\theta}q_2(x)\Big)\Big) \\
&\geq\sum_{x\in\mathcal{X}}\alpha(x)\Big(\theta\bar{\theta}\Big(q_1(x)-q_2(x)\Big)+\bar{\theta}\theta\Big(q_2(x)-q_1(x)\Big)\Big) \\
&\geq\sum_{x\in\mathcal{X}}\alpha(x)\cdot 0 \\
&\geq 0.
\end{aligned}$$

Theorem 1.8: *If* $\mathbf{q}$ *is any real vector and if we define the function of* σ

$$K(\sigma)=\max_{\mathbf{q}\,:\,\sum_{x\in\mathcal{X}}q(x)=\sigma}J(\mathbf{q},\mu),$$

then, for any fixed $\mu(x,y)$, $K(\sigma)$ *is convex* $\cap$ *in* σ.

Proof of Theorem 1.8:

Given two real numbers σ_1 and σ_2, define the following vectors $\mathbf{q_1}$ and $\mathbf{q_2}$:

$$J(\mathbf{q_1},\mu)=K(\sigma_1)\text{ where }\sigma_1=\sum_{x\in\mathcal{X}}q_1(x)$$
$$J(\mathbf{q_2},\mu)=K(\sigma_2)\text{ where }\sigma_2=\sum_{x\in\mathcal{X}}q_2(x)$$

From Theorem 1.7, we know that

$$J(\theta\mathbf{q_1}+\bar{\theta}\mathbf{q_2},\mu)\geq\theta J(\mathbf{q_1},\mu)+\bar{\theta}J(\mathbf{q_2},\mu) \tag{1.21}$$

Clearly, the vector $\mathbf{q} = \theta\mathbf{q}_1 + \bar{\theta}\mathbf{q}_2$ qualifies as a vector for which

$$\sum_{x \in \mathcal{X}} q(x) = \theta\sigma_1 + \bar{\theta}\sigma_2$$

Therefore,

$$K(\theta\sigma_1 + \bar{\theta}\sigma_2) = \max_{\mathbf{q}\,:\, \sum_{x\in\mathcal{X}} q(x) = \theta\sigma_1 + \bar{\theta}\sigma_2} J(\mathbf{q}, \mu) \geq J(\theta\mathbf{q}_1 + \bar{\theta}\mathbf{q}_2, \mu) \tag{1.22}$$

If we combine (1.21) and (1.22), we obtain

$$K(\theta\sigma_1 + \bar{\theta}\sigma_2) \geq \theta J(\mathbf{q}_1, \mu) + \bar{\theta} J(\mathbf{q}_2, \mu) = \theta K(\sigma_1) + \bar{\theta} K(\sigma_2),$$

which yields Theorem 1.8.

Using Theorems 1.7 and 1.8, we are able to prove the last part of Theorem 1.6.

Proof of part c– of Theorem 1.6:

Let us now try to maximize $J(\mathbf{q}, \mu)$ for fixed $\mu(x, y)$. This has to be done under constraint (1.18).

Now, let us first assume that $\hat{q}_2 \neq 0$.

$\rightarrow \hat{q}_2 \neq 0$

In this case, the Kuhn-Tucker conditions are:

$$\frac{\partial}{\partial q(x)}\left(J(\mathbf{q}, \mu) + \alpha\Big(q_2 + \sum_{x \in \mathcal{X}_1} q(x) - 1\Big)\right) = 0 \text{ for all } x \in \mathcal{X}_1$$
$$\frac{\partial}{\partial q_2}\left(J(\mathbf{q}, \mu) + \alpha\Big(q_2 + \sum_{x \in \mathcal{X}_1} q(x) - 1\Big)\right) = 0$$

which is equivalent to

$$\frac{\partial J(\mathbf{q}, \mu)}{\partial q(x)} + \alpha = 0 \text{ for all } x \in \mathcal{X}_1$$
$$\alpha = 0$$

which, for all $x \in \mathcal{X}_1$ amounts to

$$\frac{\partial J(\mathbf{q},\mu)}{\partial q(x)} = -\alpha = 0.$$

Using the results of Theorem 1.5 but replacing the Lagrange multiplier α by zero, we obtain, for all $x \in \mathcal{X}_1$ and $q(x) \neq 0$:

$$q(x) = \exp\Big(\frac{1}{\alpha(x)} \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log \mu(x,y)\Big) \exp(-1)$$

Naturally, q_2 is chosen as follows:

$$q_2 = 1 - \sum_{x \in \mathcal{X}_1} q(x)$$

Clearly, for all $x \in \mathcal{X}_1$, the numbers $q(x)$ are all non-negative. However, nothing insures us that q_2 is non-negative. In fact, there are cases, as we will see later, where q_2, as defined here, *is* negative.

If $q_2 < 0$, then this implies

$$\sum_{x \in \mathcal{X}_1} q(x) > 1.$$

This means that the values of $q(x)$ that maximize $J(\mathbf{q},\mu)$ are such that they add up to $\sigma > 1$. But if we constrain them to add up to a number less or equal to 1, then because of the $\cap$-convexity of $K(\sigma)$, $J(\mathbf{q},\mu)$ is maximized by $\hat{q}(x)$'s that add up exactly to 1, which implies that $\hat{q}_2$ has to be equal to zero.

$\rightarrow \hat{q}_2 = 0$

In this case, the function to maximize is the same:

$$J(\mathbf{q},\mu) = \sum_{x \in \mathcal{X}_1} q(x) \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log \frac{\mu(x,y)}{q(x)}$$

but the constraint is different:

$$\sum_{x \in \mathcal{X}_1} q(x) = 1$$

But we already know how to solve this problem using part c– of Theorem 5, while restricting x to $\mathcal{X}_1$.

Corollary 1.3: *If* $\hat{\mathbf{q}}$ *achieves capacity for a Class IIb VNC, then, for all* $x \in \mathcal{X}_1$:

$$\textit{either } \hat{q}(x) = \hat{q}(x) \exp\left(\frac{1}{\alpha(x)} \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log \frac{\lambda(x,y)}{\sum_{u \in \mathcal{X}_1} \hat{q}(u)\lambda(u,y)}\right) \exp(-1) \quad (1.23)$$

$$\textit{or } \hat{q}(x) = \hat{q}(x) \exp\left(\frac{1}{\alpha(x)} \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log \frac{\lambda(x,y)}{\sum_{u \in \mathcal{X}_1} \hat{q}(u)\lambda(u,y)}\right) \exp\left(\frac{\alpha}{\alpha(x)} - 1\right) \quad (1.24)$$

where α *is such that:*

$$f(\alpha) = \sum_{x \in \mathcal{X}_1} \left(\exp\left(\tfrac{1}{\alpha(x)} \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log \mu(x,y)\right) \exp\left(\tfrac{\alpha}{\alpha(x)} - 1\right)\right) = 1$$

In both cases, the remaining probability $\hat{q}_2 = 1 - \sum_{x \in \mathcal{X}_1} q(x)$ *can be arbitrarily distributed among the* $\hat{q}(x)$*'s such that* $x \in \mathcal{X}_2$ *(in the second case,* $\hat{q}_2 = 0$*).*

Proof of Corollary 1.3:

The proof goes exactly among the same lines as Corollary 1.2 which was derived from Theorem 1.5. The first equation (1.23) is obtained by replacing every appearance of $\mu(x,y)$ in the first expression of $q(x)$ in part c– of Theorem 1.6, by its expression in part b– of Theorem 1.6. The second equation (1.24) is obtained by replacing every appearance of $\mu(x,y)$ in the second expression of $q(x)$ in part c– of Theorem 1.6, by its expression in part b– of Theorem 1.6. It is important to note that both equations of Corollary 1.3 are simultaneously satisfied only in the special case where $\alpha = 0$.

Algorithm 1.2:

At every step of the iteration,

1. We compute the following intermediary values:

$$\lambda_n(y) = \sum_{u\in\mathcal{X}_1} q_n(u)\lambda(u,y) \quad \text{for } y \in \mathcal{Y}_2$$

$$I_n(x) = \sum_{y\in\mathcal{Y}_2} \lambda(x,y)\log\frac{\lambda(x,y)}{\lambda_n(y)} \quad \text{for } x \in \mathcal{X}_1$$

$$\tilde{q}_n(x) = q_n(x)\exp\left(\frac{I_n(x)}{\alpha(x)}\right)\exp(-1) \quad \text{for } x \in \mathcal{X}_1$$

$$\tilde{\sigma}_n = \sum_{x\in\mathcal{X}_1} \tilde{q}_n(x)$$

2. We can now determine the next value for the input distribution:

a. If $\tilde{\sigma}_n \leq 1$, then

$$(q_2)_{n+1} = \sum_{x\in\mathcal{X}_2} q_{n+1}(x) = 1 - \tilde{\sigma}_n$$

$$q_{n+1}(x) = \tilde{q}_n(x) \text{ for } x \in \mathcal{X}_1$$

b. If $\tilde{\sigma}_n > 1$, then

$$(q_2)_{n+1} = \sum_{x\in\mathcal{X}_2} q_{n+1}(x) = 0$$

$$q_{n+1}(x) = \tilde{q}_n(x)\exp\left(\frac{\alpha_n}{\alpha(x)}\right) \text{ for } x \in \mathcal{X}_1$$

where α_n is computed so that:

$$f(\alpha_n) = \textstyle\sum_{x\in\mathcal{X}_1} \tilde{q}_n(x)\exp\left(\frac{\alpha_n}{\alpha(x)}\right) = 1$$

Example of capacity computation for Class IIb VNCs: Very Noisy Z-Channels of order N.

We will define them by defining their zero-capacity matrix and their perturbation matrix, which are $(N+1)\times(N+1)$ matrices, since these channels are $(N+1)-$input $(N+1)-$output channels:

- the zero-capacity matrix Ω_Y has identical rows defined by:

$$\begin{cases} w(y_1) = 1 \\ w(y_j) = 0 \quad \text{for } i = 2,\ldots,N+1 \end{cases}$$

- the perturbation matrix $\Lambda_{X,Y}$ is such that:

$$\begin{cases} \lambda(x_1,y_j) = 0 & \text{for } j = 1,\ldots,N+1 \\ \lambda(x_i,y_1) = -1 & \text{for } i = 2,\ldots,N+1 \\ \lambda(x_i,y_j) = \delta_{ij} & \text{for } i \geq 2 \text{ and } j \geq 2 \end{cases}$$

where δ_{ij} is defined by:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Clearly

$$\mathcal{X}_1 = \{x_2, x_3, \ldots, x_{N+1}\}$$

$$\mathcal{X}_2 = \{x_1\}$$

$$\mathcal{Y}_1 = \{x_1\}$$

$$\mathcal{Y}_2 = \{x_2, x_3, \ldots, x_{N+1}\}$$

Since $\mathcal{X}_2 \neq \emptyset$, then our channels are Class IIb VNCs. Let us compute the average mutual information $I(X;Y)$ for these channels:

$$\begin{aligned} I(X;Y) &= \sum_{x \in \mathcal{X}_1} q(x) \sum_{y \in \mathcal{Y}_2} \lambda(x,y) \log \frac{\lambda(x,y)}{\sum_{u \in \mathcal{X}_1} q(u)\lambda(u,y)} \\ &= \sum_{i=2}^{N+1} q(x_i) \sum_{j=2}^{N+1} \lambda(x_i, y_j) \log \frac{\lambda(x_i, y_j)}{\sum_{k=2}^{N+1} q(x_k)\lambda(x_k, y_j)} \\ &= \sum_{i=2}^{N+1} q(x_i) \sum_{j=2}^{N+1} \delta_{ij} \log \frac{\delta_{ij}}{q(x_j)} \\ &= \sum_{i=2}^{N+1} q(x_i) \log \frac{1}{q(x_i)} \end{aligned}$$

We can use Corollary 1.3 to find the optimizing distribution $\hat{\mathbf{q}}$; first, if we assume $\hat{q}_2 \neq 0$, the quantities $\hat{q}(x_i)$ have to verify, for $i = 2, \ldots, N+1$:

$$\hat{q}(x_i) = \hat{q}(x_i) \exp\left(\log \frac{1}{\hat{q}(x_i)}\right) \exp(-1) = \exp(-1) = \frac{1}{e}.$$

Therefore,

$$\hat{q}_2 = \hat{q}(x_1) = 1 - \sum_{i=2}^{N+1} \hat{q}(x_i) = 1 - \frac{N}{e}.$$

Clearly, if $N \leq 2$, then $\hat{q}_2 = \hat{q}(x_1) > 0$ and therefore the optimal solution is:

$$\begin{cases} \hat{q}(x_1) = 1 - \frac{N}{e} & \\ \hat{q}(x_i) = \frac{1}{e} & \text{for } i = 2, \ldots, N+1 \end{cases}$$

However, if $N \geq 3$, then $\hat{q}_2 = \hat{q}(x) < 0$, and therefore we have to use Corollary 1.2, knowing that $\hat{q}_2 = \hat{q}(x_1) = 0$. Fortunately in this case, the symmetry of the problem allows us to conclude that, for $N \geq 3$:

$$\begin{cases} \hat{q}(x_1) = 0 \\ \hat{q}(x_i) = \frac{1}{N} & \text{for } i = 2, \ldots, N+1. \end{cases}$$

We now have the following expression of the capacity C for Very Noisy Z-Channels of order N :

$$C \approx \begin{cases} \epsilon \cdot \frac{N}{e} & \text{for } N \leq 2 \\ \epsilon \cdot \log N & \text{for } N \geq 3 \end{cases} \tag{1.25}$$

We have run Algorithm 2 for some other cases of Class IIb VNCs which can be found in Appendix C.

III - CODING EXPONENTS FOR VNCs

We will compute two well-known coding exponents for "very noisy" channels: the random-coding exponent and the convolutional coding exponent.

III-1. The random coding exponent.

Consider all block codes in which N is the coding block length and M is the number of possible source outputs. Then the transmission rate R is defined by:

$$R = \frac{\log M}{N} \quad \text{(nats/channel symbol)}$$

Shannon [9] showed that, for any discrete-input memoryless channel, there exists at least one such code for which the average error probability P_E resulting from maximum likelihood decoding goes to zero as the coding block length N goes to infinity. Later [10]-[15], the following exponential bound on the error probability was found:

$$P_E < \mathrm{e}^{-NE_r(R)}$$

where $E_r(R)$, known as the *random coding exponent*, is defined by:

$$E_r(R) = \max_{0 \le \rho \le 1} \Big(E_0(\rho) - \rho R \Big), \tag{1.26}$$

where:

$$E_0(\rho) = \max_{q(x)} E_0(\rho, \mathbf{q}) \tag{1.27}$$

and:

$$E_0(\rho, \mathbf{q}) = -\log \sum_{y \in \mathcal{Y}} \Big(\sum_{x \in \mathcal{X}} q(x) p(y|x)^{\frac{1}{1+\rho}} \Big)^{1+\rho}. \tag{1.28}$$

Another important quantity known as the *computational cut-off rate* R_{comp}, or also the *zero-rate exponent* R_0, which is defined by:

$$R_0 = R_{\text{comp}} = E_0(1),$$

represents, among other things, the transmission rate above which sequential decoding of convolutional codes is unpractical due to buffer overflow.

III-2. The convolutional coding exponent.

Consider now all time-varying convolutional codes of rate

$$R = \log 2 \cdot \frac{b}{n} \quad \text{(nats/channel symbol)},$$

of constraint length K, and of arbitrary block length. Viterbi [16],[17] showed that, for any discrete-input memoryless channel, there exists such a time-varying convolutional code whose bit error probability P_b, resulting from maximum likelihood decoding, is bounded by:

$$P_b < (2^b - 1)\frac{2^{-Kb\frac{E_c(R)}{R}}}{\left(1 - 2^{-\delta b\frac{E_c(R)}{R}}\right)^2}$$

where δ is any positive real number and $E_c(R)$, known as the *convolutional coding exponent* is defined by:

$$E_c(R) = \begin{cases} E_0(1) & 0 \le R \le E_0(1)(1-\delta) \\ E_0(\rho) \;\; \text{where } R = (1-\delta)\max_{\mathbf{q}} \frac{E_0(\rho,\mathbf{q})}{\rho} & E_0(1)(1-\delta) \le R \le C(1-\delta) \end{cases}$$

We can also consider the function defined by taking the limit of $E_c(R)$ as δ goes to zero:

$$\lim_{\delta\to 0} E_c(R) = \begin{cases} E_0(1), & 0 \le R \le E_0(1), \\ E_0(\rho), \;\; \text{where } R = \max_{\mathbf{q}} \frac{E_0(\rho,\mathbf{q})}{\rho}, & E_0(1) \le R \le C. \end{cases} \tag{1.29}$$

The coding exponent-rate functions defined in (1.26) and (1.29) were computed by Gallager [15] in the case of Class I VNCs, who based his computations on Reiffen's definition of VNCs. We will now present Gallager's results, followed by the corresponding functions for Class II VNCs.

III-3. Coding exponents of Class I VNCs.

The capacity C of a Class I VNC is (see (1.10)):

$$C \approx \frac{\epsilon^2}{2} \cdot \max_{q(x)} \sum_{y \in \mathcal{Y}} \frac{1}{w(y)} \left(\sum_{x \in \mathcal{X}} q(x) \lambda(x,y)^2 - \left(\sum_{x \in \mathcal{X}} q(x) \lambda(x,y) \right)^2 \right).$$

III-3.1. General expression for $E_0(\rho)$.

Let us now evaluate $E_0(\rho, \mathbf{q})$ and $E_0(\rho)$ defined in (1.27) and (1.28). Using the fact that for Class I VNCs,

$$p(y|x) \approx w(y) + \epsilon \cdot \lambda(x,y), \quad \text{for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y},$$

where:

$$w(y) > 0 \quad \text{for all } y \in \mathcal{Y},$$

we obtain Gallager's result:

$$E_0(\rho, \mathbf{q}) \approx \frac{\rho}{1+\rho} \cdot \frac{\epsilon^2}{2} \cdot \sum_{y \in \mathcal{Y}} \frac{1}{w(y)} \left(\sum_{x \in \mathcal{X}} q(x) \lambda(x,y)^2 - \left(\sum_{x \in \mathcal{X}} q(x) \lambda(x,y) \right)^2 \right),$$

which, recognizing the expression of the average mutual information and using (1.27), yields:

$$E_0(\rho) \approx \frac{\rho}{1+\rho} \cdot C. \tag{1.30}$$

Therefore, for Class I VNCs, the low rate exponent is:

$$R_0 = E_0(1) \approx \frac{C}{2},$$

result which was originally found by Reiffen [18]. From (1.30), we can derive the expressions of both coding exponent-rate functions defined in (1.26) and (1.29).

III-3.2. The Random Coding exponent.

From (1.26), the random coding exponent is:

$$E_r(R) = \max_{0 \le \rho \le 1} \Big(E_0(\rho) - \rho R \Big)$$

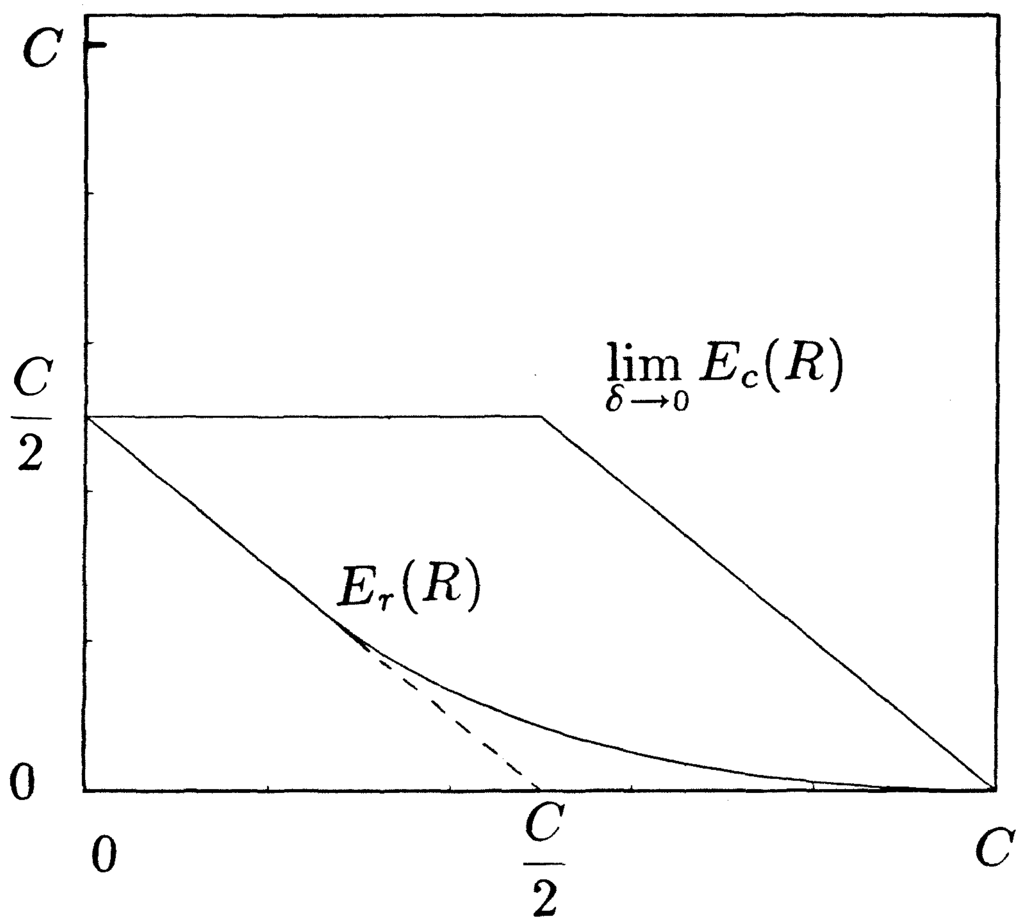


Figure 1.6: Coding exponent-rate functions for Class I VNCs.

We now obtain the random coding exponent for Class I VNCs (see figure 1.6):

$$E_r(R) \approx \begin{cases} C/2 - R & \text{for } 0 \leq \frac{R}{C} \leq \frac{1}{4} \\ \left(\sqrt{C} - \sqrt{R}\right)^2 & \text{for } \frac{1}{4} \leq \frac{R}{C} \leq 1 \end{cases} \tag{1.31}$$

III-3.3. The Convolutional Coding exponent.

From Gallager [15], we know that (1.29) becomes:

$$\lim_{\delta \to 0} E_c(R) = \begin{cases} E_0(1) & & \text{for } 0 \leq R < E_0(1) \\ E_0(\rho) & \text{where } R \approx E_0(\rho)/\rho & \text{for } E_0(1) \leq R < C \end{cases}$$

Since $E_0(1) \approx C/2$, then we have for Class I VNC's (see figure 1.6):

$$\lim_{\delta \to 0} E_c(R) \approx \begin{cases} C/2 & \text{for } 0 \leq \frac{R}{C} < \frac{1}{2} \\ C - R & \text{for } \frac{1}{2} \leq \frac{R}{C} < 1 \end{cases} \tag{1.32}$$

III-4. Coding exponents for Class II VNCs

For Class II VNCs, we obtained the following expression for the value of the capacity (see (1.11)):

$$C \approx \epsilon \cdot \max_{q(x)} \sum_{y \in \mathcal{Y}_2} \sum_{x \in \mathcal{X}} q(x)\lambda(x,y) \log \frac{\lambda(x,y)}{\sum_{u \in \mathcal{X}} q(u)\lambda(u,y)}$$

First we will try to find an expression for $E_0(\rho)$ in the general case of the Class II VNC, and then compute the coding exponents of a particular case of Class II VNCs which we have previously called the "Very Noisy" Z-channel of order N.

III-4.1. General expression of $E_0(\rho)$.

Let us first compute:

$$\exp\Big(-E_0(\rho, \mathbf{q})\Big) = \sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} q(x) p(y|x)^{\frac{1}{1+\rho}} \right)^{1+\rho}.$$

We have:

$$\begin{aligned}
\exp\Big(-E_0(\rho,\mathbf{q})\Big) &\approx \sum_{y\in\mathcal{Y}_1}\left(\sum_{x\in\mathcal{X}} q(x)\Big(w(y)+\epsilon\cdot\lambda(x,y)\Big)^{\frac{1}{1+\rho}}\right)^{1+\rho} \\
&\quad + \sum_{y\in\mathcal{Y}_2}\left(\sum_{x\in\mathcal{X}} q(x)\Big(\epsilon\cdot\lambda(x,y)\Big)^{\frac{1}{1+\rho}}\right)^{1+\rho} \\
&\approx \sum_{y\in\mathcal{Y}_1} w(y)\left(\sum_{x\in\mathcal{X}} q(x)\left(1+\frac{\epsilon\cdot\lambda(x,y)}{w(y)(1+\rho)}\right)\right)^{1+\rho} \\
&\quad + \epsilon\cdot\sum_{y\in\mathcal{Y}_2}\left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho} \\
&\approx \sum_{y\in\mathcal{Y}_1} w(y)\left(1+\frac{\epsilon\cdot\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)}{w(y)(1+\rho)}\right)^{1+\rho} \\
&\quad + \epsilon\cdot\sum_{y\in\mathcal{Y}_2}\left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho} \\
&\approx \sum_{y\in\mathcal{Y}_1} w(y)\left(1+\frac{\epsilon\cdot\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)}{w(y)}\right) \\
&\quad + \epsilon\cdot\sum_{y\in\mathcal{Y}_2}\left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho} \\
&\approx 1+\epsilon\cdot\sum_{x\in\mathcal{X}} q(x)\sum_{y\in\mathcal{Y}_1}\lambda(x,y)+\epsilon\cdot\sum_{y\in\mathcal{Y}_2}\left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho}.
\end{aligned}$$

But since for all $x\in\mathcal{X}$, $\sum_{y\in\mathcal{Y}_1}\lambda(x,y) = -\sum_{y\in\mathcal{Y}_2}\lambda(x,y)$, then:

$$\exp\Big(-E_0(\rho,\mathbf{q})\Big) \approx 1-\epsilon\cdot\sum_{y\in\mathcal{Y}_2}\left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)-\left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho}\right),$$

which yields:

$$E_0(\rho,\mathbf{q}) \approx \epsilon\cdot\sum_{y\in\mathcal{Y}_2}\left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)-\left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho}\right). \tag{1.33}$$

Therefore:

$$E_0(\rho) \approx \epsilon\cdot\max_{q(x)}\sum_{y\in\mathcal{Y}_2}\left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)-\left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho}\right). \tag{1.34}$$

If we now try to express $E_0(\rho)$ in terms of C and ρ, we find the rather complex expression:

$$E_0(\rho) = \frac{\max_{q(x)}\sum_{y\in\mathcal{Y}_2}\left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)-\left(\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho}\right)}{\max_{q(x)}\sum_{y\in\mathcal{Y}_2}\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)\log\frac{\lambda(x,y)}{\sum_{x\in\mathcal{X}} q(x)\lambda(x,y)}}\cdot C$$

There is no simple expression for $E_0(1)$ in terms of C and ρ either, and therefore we will now evaluate R_0 and the coding exponents for "Very Noisy" Z-channels of order N.

III-4.2. Coding Exponents for the "Very Noisy" Z-channel of order N.

For the Very Noisy Z-channel (VNZC) of order N (see (1.25)):

$$C \approx \begin{cases} \epsilon \cdot \frac{N}{e} & \text{for } N \leq 2, \\ \epsilon \cdot \log N & \text{for } N \geq 3. \end{cases}$$

Also, from (1.33) and the definition of VNZCs:

$$E_0(\rho, \mathbf{q}) \approx \epsilon \cdot \sum_{i=2}^{N+1} \left(q(x_i) - q(x_i)^{1+\rho} \right).$$

For reasons of symmetry, the input distribution that maximizes $E_0(\rho, \mathbf{q})$ satisfies:

$$\hat{q}(x_i) = \frac{1 - \hat{q}(x_1)}{N} \neq 0, \quad i = 2, \ldots, N+1.$$

Therefore,

$$\frac{\partial E_0(\rho, \mathbf{q})}{\partial q(x_i)} = 0, \quad i = 2, \ldots, N+1,$$

which yields:

$$\hat{q}(x_i) = \frac{1}{(1+\rho)^{\frac{1}{\rho}}}. \tag{1.35}$$

A little analysis shows that for all ρ such that $0 \leq \rho \leq 1$,

$$\frac{1}{3} < \frac{1}{e} < \frac{1}{(1+\rho)^{\frac{1}{\rho}}} \leq \frac{1}{2}.$$

Therefore equation (1.35) holds only when $N \leq 2$, and the distribution that maximizes $E_0(\rho, \mathbf{q})$ for $N \geq 3$ is:

$$\begin{cases} \hat{q}(x_1) = 0, \\ \hat{q}(x_i) = \frac{1}{N} & \text{for } i = 2, \ldots, N+1. \end{cases}$$

We now have the following expression for $E_0(\rho)$:

$$E_0(\rho) \approx \begin{cases} N\epsilon \cdot \frac{\rho}{(1+\rho)^{\frac{1+\rho}{\rho}}} & \text{for } N \leq 2, \\ \epsilon \cdot (1 - \frac{1}{N^\rho}) & \text{for } N \geq 3. \end{cases}$$

We can also express $E_0(\rho)$ as a function of C and ρ :

$$E_0(\rho) \approx \begin{cases} e \cdot C \cdot \frac{\rho}{(1+\rho)^{\frac{1+\rho}{\rho}}} & \text{for } N \leq 2, \\ \frac{C}{\log N} \cdot (1 - \frac{1}{N^\rho}) & \text{for } N \geq 3. \end{cases} \tag{1.36}$$

Note that the expressions are very different in both cases. This is due to the fact that the value of the distribution $\mathbf{q}$ that maximizes $E_0(\rho, \mathbf{q})$ depends on other channel parameters than the capacity C. If we now evaluate R_0 for the VNZC of order N we find:

$$R_0 = E_0(1) \approx \begin{cases} \frac{e}{4} \cdot C & \text{for } N \leq 2, \\ \frac{(1-\frac{1}{N})}{\log N} \cdot C & \text{for } N \geq 3. \end{cases} \tag{1.37}$$

Note that:

$$\lim_{N \to \infty} \frac{R_0}{C} = 0,$$

which indicates that VNZCs of high order are examples of Class II VNCs which perform particularly poorly. Such examples do not exist for Class I VNCs.

We will now directly present the expressions for both the random coding exponent and the convolutional coding exponent for the Very Noisy Z-channel of order N which are directly derived from $E_0(\rho)$. However, we do not present the details of the calculation.

- The Random Coding exponent

a. For $N \leq 2$, (see figure 1.7):

$$E_r(R) \approx \begin{cases} C \cdot \left(\frac{e}{4} - \frac{R}{C}\right) & \text{for } 0 \leq \frac{R}{C} \leq \frac{e \log 2}{4} \\ C \cdot \max_{0 \leq \rho \leq 1} \left(\frac{e\rho}{(1+\rho)^{\frac{1+\rho}{\rho}}} - \rho\frac{R}{C}\right) & \text{for } \frac{e \log 2}{4} \leq \frac{R}{C} \leq 1 \end{cases} \tag{1.37}$$

b. For $N \geq 3$, (see figure 1.8):

$$E_r(R) \approx \begin{cases} C \cdot \left(\frac{1-\frac{1}{N}}{\log N} - \frac{R}{C}\right) & \text{for } 0 \leq \frac{R}{C} \leq \frac{1}{N} \\ \frac{C}{\log N} \cdot \left(1 - \frac{R}{C} + \frac{R}{C}\log\frac{R}{C}\right) & \text{for } \frac{1}{N} \leq \frac{R}{C} \leq 1 \end{cases} \tag{1.38}$$

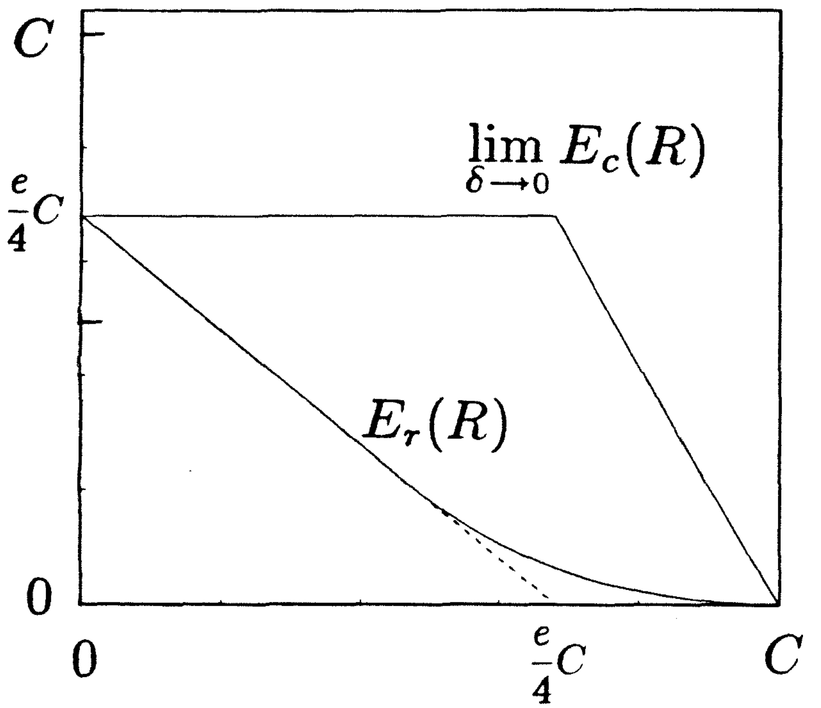


Figure 1.7: Coding exponent-rate functions for VNZCs of order $N \leq 2$.

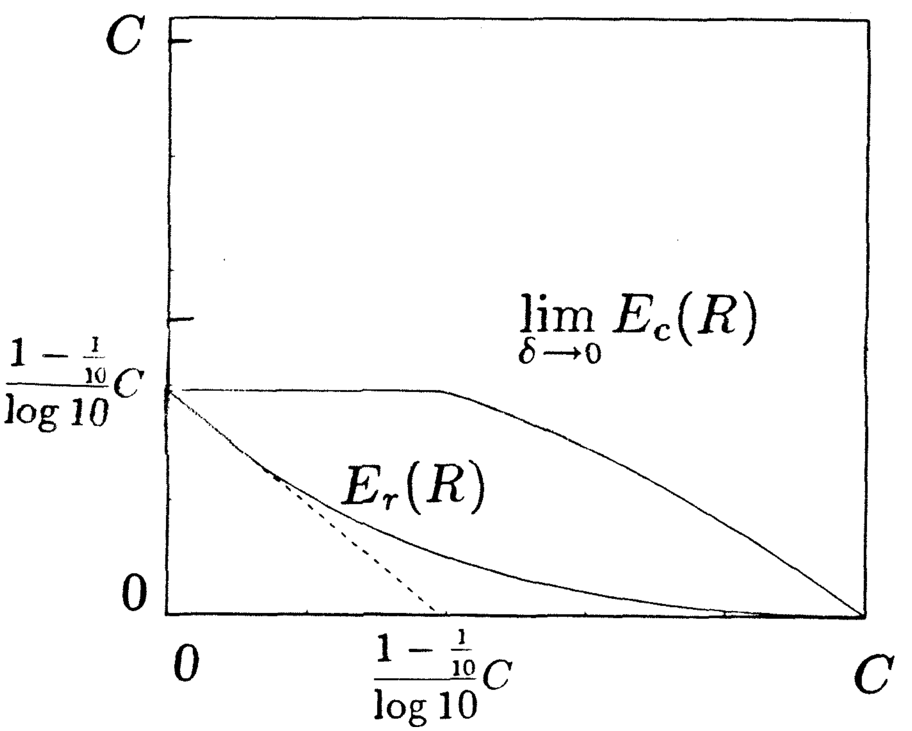


Figure 1.8: Coding exponent-rate functions for the VNZC of order $N = 10$.

• The Convolutional Coding exponent

a. for $N \leq 2$, (see figure 1.7):

$$\lim_{\delta \to 0} E_c(R) \approx \begin{cases} C \cdot \frac{e}{4} & \text{for } 0 \leq \frac{R}{C} \leq \frac{e}{4} \\ C \cdot \frac{e\rho}{(1+\rho)^{\frac{1+\rho}{\rho}}} \quad \text{where } \frac{R}{C} \approx \frac{e}{(1+\rho)^{\frac{1+\rho}{\rho}}} & \text{for } \frac{e}{4} \leq \frac{R}{C} \leq 1 \end{cases} \tag{1.39}$$

b. For $N \geq 3$, (see figure 1.8):

$$\lim_{\delta \to 0} E_c(R) \approx \begin{cases} C \cdot \frac{1-\frac{1}{N}}{\log N} & \text{for } 0 \leq \frac{R}{C} \leq \frac{1-\frac{1}{N}}{\log N} \\ C \cdot \frac{1-\frac{1}{N^\rho}}{\log N} \quad \text{where } \frac{R}{C} \approx \frac{1-\frac{1}{N^\rho}}{\rho \log N} & \text{for } \frac{1-\frac{1}{N}}{\log N} \leq \frac{R}{C} \leq 1 \end{cases} \tag{1.40}$$

III-5. Conclusion on Coding Exponents of VNCs.

Reiffen [18] and later Gallager [15] in more detail showed that, for Class I VNCs, R_0 and the coding exponent-rate functions depend directly on the capacity C and no other channel parameter, revealing a striking "universality" about them. What we showed, is that that "universal" behavior does not generalize to all VNCs, and Class II VNCs are such an example. Furthermore, we were able to give examples of Class II VNCs for which R_0 was better than for Class I VNCs (VNZC of order 1), as well as examples for which R_0 was far smaller (VNZC of large order).

CHAPTER 2:

BINARY VNCS AND SOME PROPERTIES OF BINARY CHANNELS

I - BINARY VNCS

It is our purpose in this section to compare Class I and Class II VNCs, but in the particular case of binary channels (two-input, two-output channels). First we define and compute the capacity of binary Class I and Class II VNCs. Then we consider any pair of binary Class I and Class II VNCs with the **same capacity**, and answer the two questions:

a. Which channel has the smallest average error probability in a received bit?

b. Which has the largest end-to-end capacity if its output terminals are connected to the input terminals of an identical channel?

Such questions have already been asked by Silverman [19], but only in a comparison between the "very noisy" binary symmetric channel and the "very noisy" Z-channel.

I-1. Binary Class I VNCs.

By definition, Class I VNCs have to verify (see page 24):

$$\begin{cases} w(y) > 0, \quad \text{for all } y \in \mathcal{Y}, \\ \sum_{y \in \mathcal{Y}} \lambda(x,y) = 0, \quad \text{for all } x \in \mathcal{X}. \end{cases}$$

Therefore, binary Class I VNCs are defined by:

$$\begin{cases} w = w(y_1) = 1 - w(y_2) \quad \text{where} \quad 0 < w < 1, \\ \lambda_1 = \lambda(x_1, y_2) = -\lambda(x_1, y_1), \\ \mu_1 = \lambda(x_2, y_2) = -\lambda(x_2, y_1), \end{cases}$$

and therefore,

$$\Omega_Y = \begin{pmatrix} w & \bar{w} \\ w & \bar{w} \end{pmatrix} \quad \text{and} \quad \Lambda_{X,Y} = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ -\mu_1 & \mu_1 \end{pmatrix},$$

and

$$P_{Y|X} \approx \begin{pmatrix} w - \lambda_1 \epsilon & \bar{w} + \lambda_1 \epsilon \\ w - \mu_1 \epsilon & \bar{w} + \mu_1 \epsilon \end{pmatrix}.$$

Let $(\hat{q}_\epsilon, 1 - \hat{q}_\epsilon)$ be the optimizing probability distribution that achieves capacity for a given small value of ϵ. Then, if we define

$$\hat{q}_0 = \lim_{\epsilon \to 0} \hat{q}_\epsilon,$$

we know from Theorem 1.4 that

$$\hat{q}_0 = \frac{1}{2},$$

and we know from Corollary 1 of chapter 1, that the capacity of a binary Class I VNC is:

$$\begin{aligned} C &= \frac{\epsilon^2}{8}(\lambda_1 - \mu_1)^2\left(\frac{1}{w} + \frac{1}{\bar{w}}\right) + O(\epsilon^3), \\ &= \epsilon^2\frac{(\lambda_1 - \mu_1)^2}{8w\bar{w}} + O(\epsilon^3). \end{aligned}$$

If, by convention and without loss of generality, we assume:

$$0 \leq \mu_1 < \lambda_1,$$

then we may define the ratio:

$$0 \leq r_1 = \frac{\mu_1}{\lambda_1} < 1,$$

and we can express the capacity C as:

$$C \approx (\lambda_1\epsilon)^2\frac{(1 - r_1)^2}{8w\bar{w}}. \tag{2.1}$$

I-2. Binary Class II VNCs.

By definition, Class II VNCs have to satisfy (see page 24):

$$\begin{cases} \exists y \in \mathcal{Y} \ : \ w(y) = 0, \\ \sum_{y \in \mathcal{Y}} \lambda(x, y) = 0, \quad \text{for all } x \in \mathcal{X}, \\ \lambda(x, y) \geq 0, \quad \text{for all } x \in \mathcal{X}, \text{ and } y \in \mathcal{Y}_2. \end{cases}$$

As a consequence, Binary Class II VNCs without loss of generality are defined by:

$$w(y_1) = 1 \quad \text{and} \quad w(y_2) = 0,$$

and,

$$\begin{cases} \lambda_2 = \lambda(x_1, y_2) = -\lambda(x_1, y_1) & \text{where } \lambda_2 \geq 0, \\ \mu_2 = \lambda(x_2, y_2) = -\lambda(x_2, y_1) & \text{where } \mu_2 \geq 0, \end{cases}$$

and therefore,

$$\Omega_Y = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and } \Lambda_{X,Y} = \begin{pmatrix} -\lambda_2 & \lambda_2 \\ -\mu_2 & \mu_2 \end{pmatrix},$$

and:

$$P_{Y|X} \approx \begin{pmatrix} 1-\lambda_2\epsilon & \lambda_2\epsilon \\ 1-\mu_2\epsilon & \mu_2\epsilon \end{pmatrix}.$$

Of course, any permutation of the rows or the columns of the transition probability matrix $P_{Y|X}$ is also a binary Class II VNC. We should always have $\lambda_2 \neq \mu_2$ (which implies that either λ_2 or μ_2 is non-zero), otherwise we would have a zero-capacity channel. By convention, we will assume that $\lambda_2 \neq 0$ and $\mu_2 < \lambda_2$ and define the ratio r_2 by:

$$0 \leq r_2 = \frac{\mu_2}{\lambda_2} < 1$$

If $q = q(x_1) = 1 - q(x_2)$, then the average mutual information is:

$$I(q) = (\lambda_2\epsilon) \cdot i(r_2, q) + O(\epsilon^2),$$

where:

$$i(r,q) = (q + \bar{q}r) \log \frac{1}{q+\bar{q}r} + \bar{q}r \log r.$$

Let us also define the function:

$$u(r) = -\frac{r}{1-r} \log r.$$

If we maximize $i(r_2, q)$ with respect to q, we obtain the following optimal value of q :

$$\hat{q}(r_2) = \frac{\frac{1}{e} r_2^{-\frac{r_2}{1-r_2}} - r_2}{1-r_2} = \frac{e^{u(r_2)-1} - r_2}{1-r_2},$$

and if we define the function:

$$c(r) = e^{u(r)-1} - u(r),$$

then the capacity is:

$$C \approx (\lambda_2\epsilon) \cdot c(r_2). \tag{2.2}$$

In figure 2.1, we have plotted $\hat{q}(r)$ and $c(r)$ as functions of r where $0 \leq r < 1$.

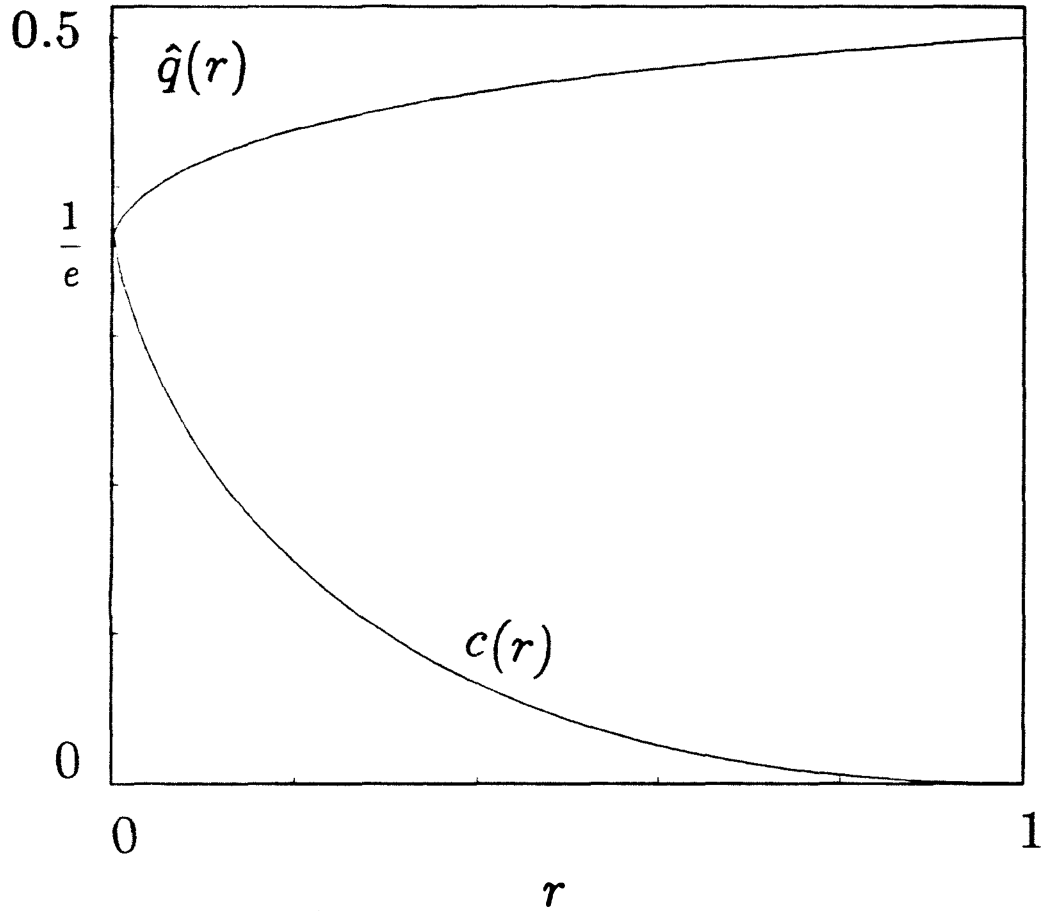


Figure 2.1: Plots of functions $\hat{q}(r)$ and $c(r)$.

I-3. Average error probability.

The true average error probability P_e for a binary channel is the minimum of the two following average error probabilities (since we must have: $P_e \leq \frac{1}{2}$):

$$P_e = \min\Big(qp(y_2|x_1) + \bar{q}p(y_1|x_2),\ qp(y_1|x_1) + \bar{q}p(y_2|x_2)\Big).$$

For a binary VNC, the limiting error probability is:

$$\lim_{\epsilon\to 0} P_e = \min\Big(qw(y_2) + \bar{q}w(y_1),\ qw(y_1) + \bar{q}w(y_2)\Big).$$

Therefore, for a binary Class I VNC, the limiting error probability at capacity, P_{e_I} is:

$$P_{e_I} = \min\left(\frac{1}{2}\bar{w} + \frac{1}{2}w,\ \frac{1}{2}w + \frac{1}{2}\bar{w}\right) = \frac{1}{2}.$$

We know that, for a binary Class I VNC,

$$\hat{q}_0 = \frac{1}{2} \Leftrightarrow r_2 = 1,$$

but since $0 \leq r_2 < 1$, then:

$$\min(\hat{q}_0,\ 1 - \hat{q}_0) < \frac{1}{2}.$$

For a binary Class II VNC, the limiting error probability $P_{e_{II}}$ is:

$$P_{e_{II}} = \min(q, \bar{q}).$$

If we are operating at capacity, then:

$$P_{e_{II}} = \min(\hat{q}_0, 1 - \hat{q}_0).$$

This result leads to the following theorem:

Theorem 2.1: *The limiting value of the average error probability of any binary Class II VNC ($P_{e_{II}}$) is strictly less than that of any binary Class I VNC with the* same *capacity ($P_{e_{II}}$):*

$$\frac{1}{e} \leq P_{e_{II}} < P_{e_I} = \frac{1}{2}.$$

The theorem still holds if the capacities are not equal, but then the comparison is of much less interest. We must point out that this theorem generalizes Silverman's results [19] which applied only to the "very noisy" BSC and the "very noisy" Z-Channel.

I-4. End-to-end capacity of the cascade of two identical VNCs.

Consider any binary Class I VNC. Its transition probability matrix is of the form:

$$P_{Y|X} \approx \begin{pmatrix} w - \lambda_1\epsilon_1 & \bar{w} + \lambda_1\epsilon_1 \\ w - \mu_1\epsilon_1 & \bar{w} + \mu_1\epsilon_1 \end{pmatrix},$$

and its capacity C_1, according to (1), is of the form:

$$C_1 \approx (\lambda_1\epsilon_1)^2 \frac{(1 - r_1)^2}{8w\bar{w}}.$$

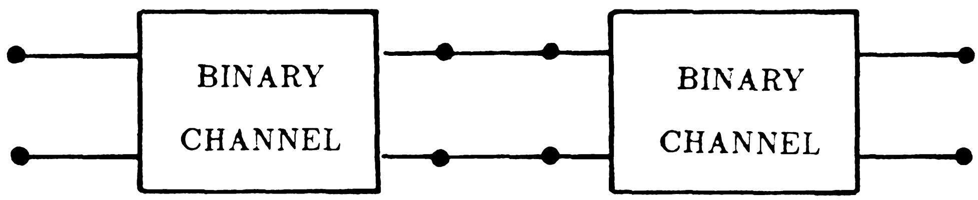
Consider any binary Class II VNC. Its transition probability matrix is of the form:

$$P_{Y|X} \approx \begin{pmatrix} 1 - \lambda_2\epsilon_2 & \lambda_2\epsilon_2 \\ 1 - \mu_2\epsilon_2 & \mu_2\epsilon_2 \end{pmatrix},$$

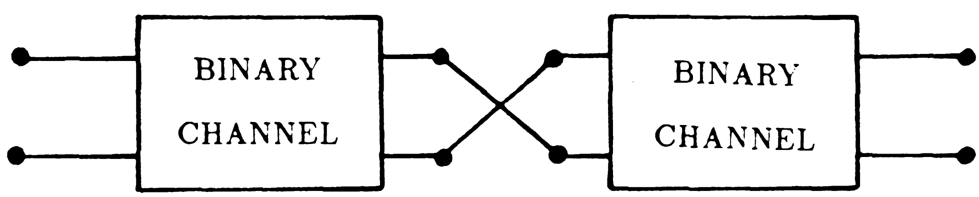
and its capacity C_2, according to (2), is of the form:

$$C_2 \approx (\lambda_2\epsilon_2) \cdot c(r_2).$$

In computing the end-to-end capacity of the cascade of two identical binary channels, we must first compute the average mutual information for two cascade configurations as depicted in figure 2.2: one defined by the connection of y_1 to x_1 and y_2 to x_2, and one defined by the connection of y_1 to x_2 and y_2 to x_1. We will then keep the configuration that yields the largest capacity. Also, we do not present the tedious details of these calculations, which are much less informative than the results themselves. We do present, however, the resulting expressions of the average mutual information in both cascade configurations.



Configuration (a).



Configuration (b).

Figure 2.2: Two cascade configurations.

I-4.1. Binary Class I VNC.

Here are the two possible configurations for the binary Class I VNC:

$$\text{a.}\quad \begin{pmatrix} w-\lambda_1\epsilon_1 & \bar{w}+\lambda_1\epsilon_1 \\ w-\mu_1\epsilon_1 & \bar{w}+\mu_1\epsilon_1 \end{pmatrix} \begin{pmatrix} w-\lambda_1\epsilon_1 & \bar{w}+\lambda_1\epsilon_1 \\ w-\mu_1\epsilon_1 & \bar{w}+\mu_1\epsilon_1 \end{pmatrix};$$

$$\text{b.}\quad \begin{pmatrix} w-\lambda_1\epsilon_1 & \bar{w}+\lambda_1\epsilon_1 \\ w-\mu_1\epsilon_1 & \bar{w}+\mu_1\epsilon_1 \end{pmatrix} \begin{pmatrix} w-\mu_1\epsilon_1 & \bar{w}+\mu_1\epsilon_1 \\ w-\lambda_1\epsilon_1 & \bar{w}+\lambda_1\epsilon_1 \end{pmatrix}.$$

The average mutual information obtained in both cases are exactly the same and equal to:

$$I(q) \approx q\bar{q}(\lambda_1\epsilon_1)^4 \frac{(1-r_1)^4}{2w\bar{w}}.$$

Therefore the end-to-end capacity of the cascade of two identical binary Class I VNCs which parameters are ϵ_1, λ_1 and $r_1 = \frac{\mu_1}{\lambda_1}$ is:

$$C_1' \approx (\lambda_1\epsilon_1)^4 \frac{(1-r_1)^4}{8w\bar{w}}.$$

Now, if we compute the ratio of C_1' by C_1^2, we find:

$$\frac{C_1'}{C_1^2} \approx 8w\bar{w},$$

and since $0 < w < 1$, the limit value of the ratio as ϵ_1 goes to zero satisfies the inequalities:

$$0 < \lim_{\epsilon_1 \to 0} \frac{C_1'}{C_1^2} = 8w\bar{w} \leq 2. \tag{2.3}$$

The largest value of the limit value of the ratio is clearly obtained when $w = \bar{w} = \frac{1}{2}$, i.e., for the "very noisy" BSC.

I-4.2. Binary Class II VNC.

We now turn to the two possible configurations for the binary Class II VNC:

$$\text{a.}\quad \begin{pmatrix} 1-\lambda_2\epsilon_2 & \lambda_2\epsilon_2 \\ 1-\mu_2\epsilon_2 & \mu_2\epsilon_2 \end{pmatrix} \begin{pmatrix} 1-\lambda_2\epsilon_2 & \lambda_2\epsilon_2 \\ 1-\mu_2\epsilon_2 & \mu_2\epsilon_2 \end{pmatrix}$$

$$\text{b.}\quad \begin{pmatrix} 1-\lambda_2\epsilon_2 & \lambda_2\epsilon_2 \\ 1-\mu_2\epsilon_2 & \mu_2\epsilon_2 \end{pmatrix} \begin{pmatrix} 1-\mu_2\epsilon_2 & \mu_2\epsilon_2 \\ 1-\lambda_2\epsilon_2 & \lambda_2\epsilon_2 \end{pmatrix}.$$

The average mutual information obtained in case a is:

$$I_a(q) \approx q\bar{q}\frac{(1-r_2)^4}{2}(\lambda_2\epsilon_2)^3.$$

In case b, the average mutual information is:

$$I_b(q) \approx \begin{cases} q\bar{q}\frac{(1-r_2)^4}{2r_2}(\lambda_2\epsilon_2)^3 & \text{for } r_2 \neq 0, \\ q\log\frac{1}{q}(\lambda_2\epsilon_2)^2 & \text{for } r_2 = 0. \end{cases}$$

Clearly, for both $r_2 = 0$ and $0 < r_2 < 1$, case b always yields the most average mutual information, and therefore the end-to-end capacity of the cascade of two identical binary Class II VNCs is:

$$C_2' \approx \begin{cases} \frac{(1-r_2)^4}{8r_2}(\lambda_2\epsilon_2)^3 & \text{for } r_2 \neq 0, \\ \frac{1}{e}(\lambda_2\epsilon_2)^2 & \text{for } r_2 = 0. \end{cases}$$

Now if we compute the ratio of C_2' and C_2^2, we find:

$$\frac{C_2'}{C_2^2} \approx \begin{cases} O(\epsilon_2) & \text{for } r_2 \neq 0, \\ e & \text{for } r_2 = 0. \end{cases}$$

Clearly, the limiting value of that ratio as $\epsilon \to 0$ satisfies:

$$\lim_{\epsilon_2 \to 0} \frac{C_2'}{C_2^2} = \begin{cases} 0 & \text{for } r_2 \neq 0, \\ e & \text{for } r_2 = 0. \end{cases} \tag{2.4}$$

Clearly, the largest value of the limit of the ratio is obtained when $r_2 = 0$, i.e., for the "very noisy" Z-channel.

I-4.3. Comparison of end-to-end capacities of Class I and Class II VNCs.

Consider now an arbitrary binary Class I VNC and an arbitrary binary Class II VNC, *but* with the same capacity, that is to say:

$$C_1 \approx (\lambda_1\epsilon_1)^2\frac{(1-r_1)^2}{8w\bar{w}} \approx (\lambda_2\epsilon_2)\cdot c(r_2) \approx C_2.$$

We now wish to compare their end-to-end capacities C_1' and C_2'. To do so, we can compute the limit value of the ratio of C_1' and C_2' as ϵ_1 and ϵ_2 go to zero but under the constraint:

$$\lim_{\epsilon_1,\epsilon_2 \to 0} \frac{C_1}{C_2} = 1.$$

The latter constraint is equivalent to:

$$\lim_{\epsilon_1,\epsilon_2\to 0}\left(\frac{C_1}{C_2}\right)^2 = 1.$$

Therefore,

$$\begin{aligned}
\lim_{\epsilon_1,\epsilon_2\to 0}\frac{C_1'}{C_2'} &= \lim_{\epsilon_1,\epsilon_2\to 0}\frac{C_1'}{C_2'}\cdot\left(\frac{C_2}{C_1}\right)^2,\\
&= \lim_{\epsilon_1,\epsilon_2\to 0}\frac{C_1'}{C_2'}\cdot\frac{C_2^2}{C_1^2},\\
&= \lim_{\epsilon_1,\epsilon_2\to 0}\frac{C_1'}{C_1^2}\cdot\frac{C_2^2}{C_2'},\\
&= \frac{\lim_{\epsilon_1,\epsilon_2\to 0}\frac{C_1'}{C_1^2}}{\lim_{\epsilon_1,\epsilon_2\to 0}\frac{C_2'}{C_2^2}}.
\end{aligned}$$

Using equations (2.3) and (2.4), we obtain the following results:

$$\lim_{\epsilon_1,\epsilon_2\to 0}\frac{C_1'}{C_2'} = \begin{cases} +\infty & \text{for } r_2 \neq 0,\\ \frac{2}{e} < 1 & \text{for } r_2 = 0.\end{cases}$$

This latter result directly yields the following theorem:

Theorem 2.2:

a: *Given any binary Class I VNC and any binary "very noisy" Z-channel ($r_2 = 0$) with the* same *capacity, the end-to-end capacity of the cascade of two identical "very noisy" Z-channels is, in the limit, strictly larger than that of the cascade of two identical Class I VNCs by a factor of $\frac{e}{8w\bar{w}}$, i.e., at least $\frac{e}{2} = 1.36$*

b: *Given any binary Class I VNC and any binary Class II VNC but a "very noisy" Z-channel (i.e. $r_2 \neq 0$) with the* same *capacity, the end-to-end capacity of the cascade of the two identical Class I VNCs is, in the limit, infinitely larger than that of the cascade of the two identical Class II VNCs (they are of totally different orders of magnitude).*

This theorem is interesting because it draws similarities between binary Class I VNCs and binary "very noisy" Z-channels (i.e. binary Class IIb VNCs) since if their capacities are the same, or of the same order of magnitude, their end-to-end capacities remain of the same order, and in fact, the "very noisy" Z-channel always performs better. However, the theorem draws clear dissimilarities between the channels we just mentioned and the binary Class IIa VNCs ($r_2 \neq 0$), which suffer a dramatic loss of capacity after cascade.

II - SOME PROPERTIES OF BINARY CHANNELS

In an early paper, Silverman [19] gave a detailed analysis of the general binary channel. One of his conjectures was that capacity cannot be achieved for a binary channel if either input symbol is transmitted at a probability lying outside the interval $(\frac{1}{e},\ 1-\frac{1}{e})$. In the first section, we give a rigorous analytical proof of this result, due to Howard Rumsey. In the second section, we present three conjectures suggested by the results in the first one.

II-1. Optimal input probabilities of binary channels

Consider an arbitrary binary channel. It is defined by its transition probability matrix:

$$P_{Y|X} = \begin{pmatrix} \alpha & \bar{\alpha} \\ \beta & \bar{\beta} \end{pmatrix}.$$

Then, if $q = q(x_1) = 1 - q(x_2)$, the value of the average mutual information is:

$$\begin{aligned} I(X;Y) = I(q) &= H(Y) - H(Y|X) \\ &= \mathcal{H}(q\alpha + \bar{q}\beta) - q\mathcal{H}(\alpha) - \bar{q}\mathcal{H}(\beta), \end{aligned}$$

where $\mathcal{H}(x)$ is the binary entropy function.

The probability q clearly satisfies the constraint:

$$0 \le q \le 1.$$

Define $\hat{q}$ to be the probability of the first input such that, if C is the capacity of the channel:

$$C = \max_q I(q) = I(\hat{q}).$$

The value of $\hat{q}$ can never be 0 or 1, since that would lead to $I(X;Y)$ being equal to zero. This allows us to differentiate $I(q)$ with respect to q and write:

$$\left(\frac{\partial I(q)}{\partial q}\right)_{q=\hat{q}} = 0.$$

Since

$$\frac{\partial I(q)}{\partial q} = (\alpha - \beta)\mathcal{H}'(q\alpha + \bar{q}\beta) - \mathcal{H}(\alpha) + \mathcal{H}(\beta),$$

then $\hat{q}$ satisfies:

$$\mathcal{H}'\Big(\hat{q}\alpha + (1-\hat{q})\beta\Big) = \frac{\mathcal{H}(\alpha) - \mathcal{H}(\beta)}{\alpha - \beta}. \tag{2.5}$$

If we define the quantity

$$\gamma = \hat{\alpha} + (1-\hat{q})\beta = \beta + \hat{q}(\alpha - \beta),$$

then equation (2.5) becomes:

$$\mathcal{H}'(\gamma) = \frac{\mathcal{H}(\alpha) - \mathcal{H}(\beta)}{\alpha - \beta}. \tag{2.6}$$

Equation (2.6) defines γ as a real number between α and β at which the value of the derivative of the binary entropy function is equal to the slope of the line that goes through the points $(\alpha,\ \mathcal{H}(\alpha))$ and $(\beta,\ \mathcal{H}(\beta))$. Because of the convexity of $\mathcal{H}(x)$, this number is unique (see figure 2.3).

Let us assume without loss of generality that:

$$0 \leq \beta < \alpha \leq 1,$$

and define the positive real number δ such that:

$$\delta = \alpha - \beta.$$

Then, from equation (2.6),

$$\mathcal{H}'(\gamma) = \int_0^1 \mathcal{H}'(\beta + \theta\delta)d\theta$$

$$\mathcal{H}'(\gamma) = \int_0^1 \log\frac{1}{\beta + \theta\delta}d\theta + \int_0^1 \log(1 - \beta - \theta\delta)d\theta. \tag{2.7}$$

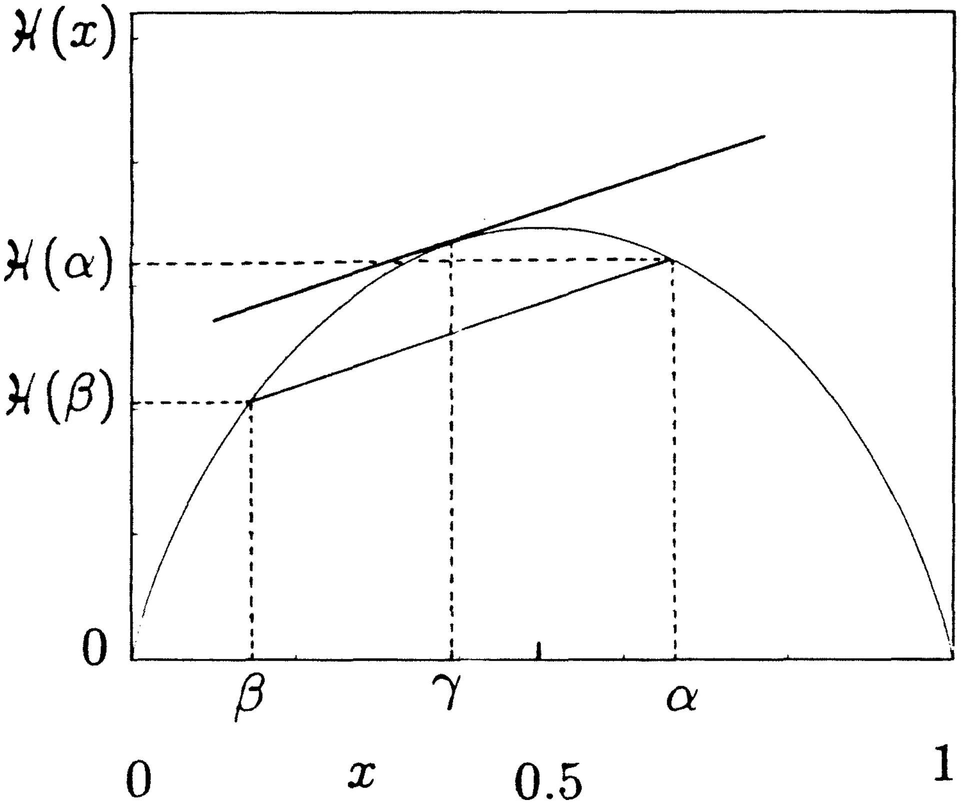


Figure 2.3: Convexity of the binary entropy function $\mathcal{H}(x)$.

Applying Jensen's inequality to the logarithm function:

$$\int_0^1 \log(1-\beta-\theta\delta)d\theta \le \log\left(\int_0^1 (1-\beta-\theta\delta)d\theta\right) \le \log\left(1-\beta-\frac{\delta}{2}\right). \tag{2.8}$$

Consider now the function:

$$f(\beta) = \int_0^1 \log\frac{1}{\beta+\theta\delta}d\theta - \log\frac{1}{\beta+\delta e^{-1}}.$$

In Appendix D, we show that:

$$f(\beta) \le 0, \quad \text{for all } \beta \ge 0,$$

and therefore:

$$\int_0^1 \log\frac{1}{\beta+\theta\delta}d\theta \le \log\frac{1}{\beta+\delta e^{-1}}. \tag{2.9}$$

Applying (2.8) and (2.9) to the two terms in (2.7), we obtain:

$$\mathcal{H}'(\gamma) \le \log\frac{1}{\beta+\delta e^{-1}} + \log(1-\beta-\frac{\delta}{2}).$$

But since $\frac{1}{2} \ge \frac{1}{e}$, then:

$$\log(1-\beta-\frac{\delta}{2}) \le \log(1-\beta-\frac{\delta}{e}),$$

which yields:

$$\mathcal{H}'(\gamma) \le \log\frac{1-\beta-\delta e^{-1}}{\beta+\delta e^{-1}} = \mathcal{H}'(\beta+\delta e^{-1}).$$

But since $\gamma = \beta + \delta\hat{q}$ and $\mathcal{H}'(x)$ is monotone decreasing, we have:

$$\beta+\delta\hat{q} \ge \beta+\delta e^{-1},$$

which yields:

$$\hat{q} \ge \frac{1}{e},$$

with equality if and only if $\delta = 0$. But since $\delta = 0$ corresponds to a trivial channel (a zero-capacity channel), we may discard that eventuality and conclude:

$$\hat{q} > \frac{1}{e}. \tag{2.10}$$

If we consider now the following channel defined by:

$$P_{Y|X} = \begin{pmatrix} \bar{\alpha} & \alpha \\ \bar{\beta} & \beta \end{pmatrix},$$

then clearly,

$$\begin{cases} \hat{q}(x_1) = \hat{q} \\ \hat{q}(x_2) = 1 - \hat{q}. \end{cases}$$

Since $\beta \leq \alpha$, then $\bar{\alpha} \leq \bar{\beta}$, and therefore (2.10) applies to $\hat{q}(x_2)$:

$$1 - \hat{q} > \frac{1}{e},$$

which yields:

$$\hat{q} < 1 - \frac{1}{e}. \tag{2.11}$$

Combining (2.10) and (2.11), we obtain:

$$\frac{1}{e} < \hat{q} < 1 - \frac{1}{e},$$

which yields the following theorem:

Theorem 2.3: *Given an arbitrary binary channel with positive capacity, any component $\hat{q}(x_i)$ of the optimal input probability distribution $\hat{\mathrm{q}}$ that achieves capacity satisfies the inequality:*

$$0.3679 = \frac{1}{e} < \hat{q}(x_i) < 1 - \frac{1}{e} = 0.6321 \tag{2.12}$$

II-2. Conjectures

A similar theorem on lower and upper bounds for optimal input probabilities would be welcome for other types of channels. We conjecture that Theorem 2.3 still holds for binary channels with more than two outputs:

Conjecture 1: *Given an arbitrary binary-input channel with positive capacity, any component $\hat{q}(x_i)$ of the optimal input probability distribution $\hat{\mathrm{q}}$ that achieves capacity satisfies the inequality:*

$$\frac{1}{e} < \hat{q}(x_i) < 1 - \frac{1}{e}.$$

However, for channels with more than two inputs, optimal input probabilities can be equal to zero: many such examples have already been presented in the literature [25]. The question that comes to mind is whether there exists a positive lower bound for non-zero optimal input probabilities. The answer is no. We give in Appendix E an example of a class of 3×3 channels for which one of the optimal input probabilities continuously varies from 0 to $\frac{1}{3}$ when one transition probability is made to continuously vary from 0 to 1 (see figure 2.4). All the results given here involve only channels with unique optimizing input probabilities. We conjecture however that the upper bound in (2.12) is still the same.

Conjecture 2: *Given an arbitrary channel with positive capacity, a unique optimizing input probability distribution and three or more inputs, any component $\hat{q}(x_i)$ of the optimal input probability distribution $\hat{\mathbf{q}}$ that achieves capacity satisfies the inequality:*

$$0 \leq \hat{q}(x_i) < 1 - \frac{1}{e}. \tag{2.13}$$

Finally, we present another conjecture suggested by Theorem 2.3 and computer simulations which reveals interesting potential properties of binary channels:

Conjecture 3: *Given an arbitrary binary channel with positive capacity, the average mutual information obtained by using a uniform input probability distribution is at least*

$$\frac{\frac{1}{2}\log 2}{\frac{1}{e}\log e} = 94.21\%$$

of the capacity.

That minimum is achieved (one can be arbitrarily close to it) in the well-known case of the "very noisy" Z-channel, as in Theorem 2.3. This result would be particularly useful for binary channels with time-varying transition probabilities (as in weather-dependent data acquisition), as well as a good justification for the use of linear codes in binary channels.

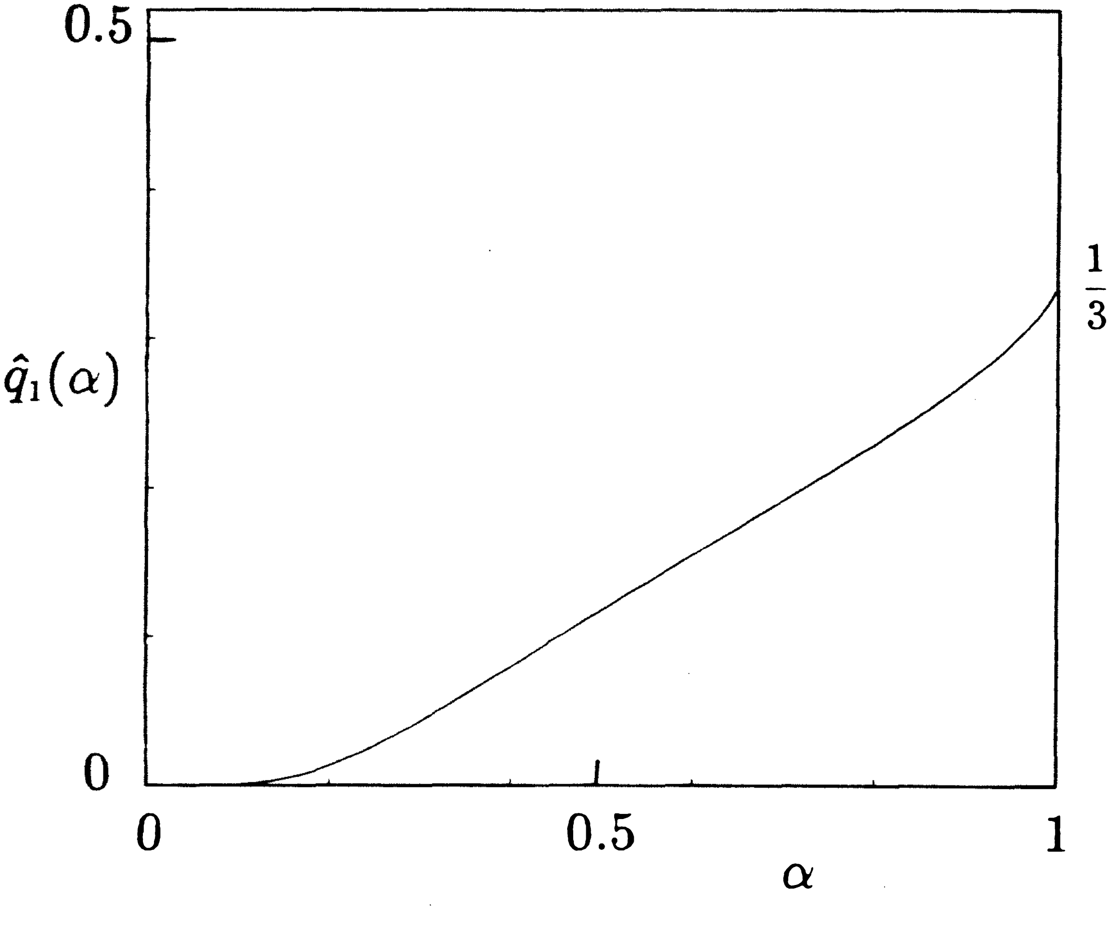


Figure 2.4: Optimal input probability $\hat{q}_1(\alpha)$.

CHAPTER 3:

BINARY-INPUT "VERY NOISY" DEGRADED BROADCAST CHANNELS

In this chapter, we will study the capacity regions of "very noisy" degraded broadcast channels. The motivation for this study is a result by McEliece, Posner and Swanson [22],[23]. They showed that the capacity region of the Wideband Gaussian Broadcast Channel (a continuous-output VNC) was the time-sharing region in the limit as the signal-to-noise ratios went to zero. The question we will answer is whether this result generalizes to other VNCs. The answer is: yes for some classes of VNCs, and no for some others.

Since the computation of capacity regions involves the computation of channel capacities, and since there does not exist a general expression of a channel's capacity, except in some specific cases, we limit our study to binary-input VNCs for which, in most cases, such expressions exist. In the first section we review some important results about broadcast channels and degraded broadcast channels. In the second, we compute the capacity regions of various classes of binary-input "very noisy" degraded broadcast channels identified in chapter 1.

I - BROADCAST CHANNELS

I-1. Definitions

Consider a satellite broadcasting simultaneously to several stations (figure 3.1). This situation, where one single transmitter is sending data to two or more (say N) receivers, is known as the *broadcast channel.* Furthermore, we impose a "no-collaboration" restriction between the receivers connected to the different terminals of the broadcast channel.

Definition 3.1: A broadcast channel (BC) is therefore defined by:

- the input X and the corresponding alphabet $\mathcal{X}$;
- the N outputs Y_j and the corresponding alphabets $\mathcal{Y}_j$ (for $j = 1, \ldots, N$;)
- the marginal transition probabilities $p(y_j|x)$ of each of the N component channels: $A_1, \ldots, A_N$.

Let us denote by C_j the capacity of component channel A_j and assume by convention and without loss of generality that:

$$C_1 \geq C_2 \geq \ldots \geq C_N.$$

The capacity C_j of component channel A_j represents the maximum rate at which one can transmit through the channel reliably: Shannon showed that at any rate below capacity, communication can be as reliable as we want by using random codes of sufficient length.

For broadcast channels, we are rather interested in their *capacity region,* i.e., the set of all simultaneously achievable rates at which communication is reliable to all terminals of the broadcast channel. We will limit ourselves to the broadcast situation in which, for a given receiver Y_j, all information available to Y_{j+1} is also available to Y_j (for $j = 1, \ldots, N-1$).

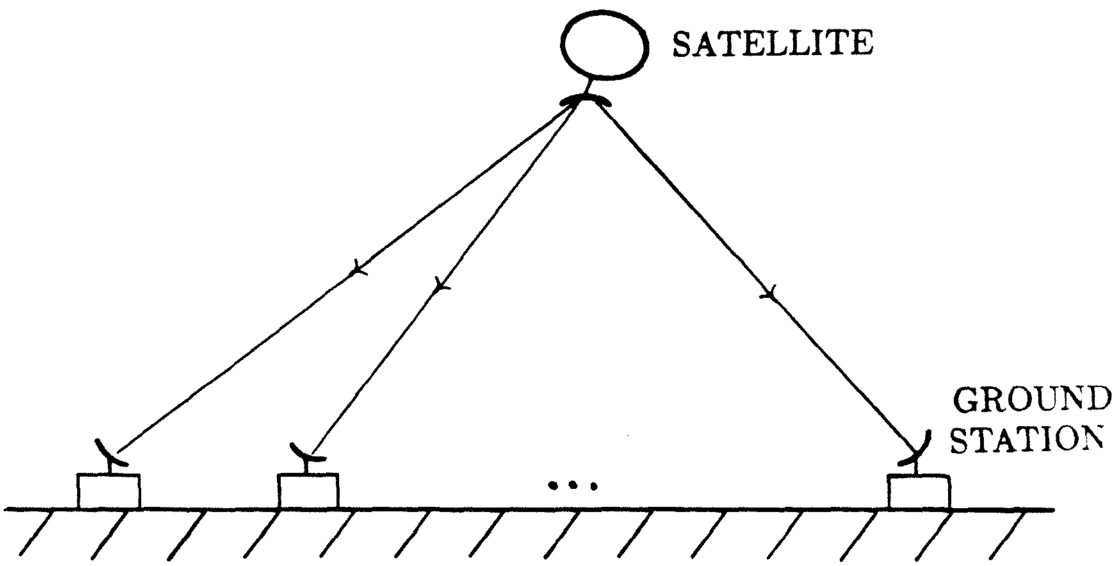


Figure 3.1: Example of a Broadcast Network.

In the case of two receivers, this situation was termed as (K, II) by van der Meulen [26].

Although an achievable rate region has been defined for the general broadcast channel, the problem of finding its capacity region has not yet been solved. However, Bergmans [27] and Gallager [28] solved the special case of the degraded broadcast channel, using a coding scheme that Cover [29] applied to the Binary Symmetric Broadcast Channel (BSBC) and the Gaussian Broadcast Channel (GBC). Here is the definition of the degraded broadcast channel, as well as the definition of a degraded version of a channel:

Definition 3.2: A broadcast channel is called *degraded* if every component channel A_j is a degraded version of A_{j-1}, for $j = 2, \ldots, N$.

Definition 3.3: A channel A_2 is a *degraded version* of a channel A_1, if there exists a third channel D, called the *degrading channel*, such that A_2 can be represented as the cascade of A_1 and D (see figure 3.2).

For convenience, we shall only consider in this thesis, degraded broadcast channels (DBCs) with two receivers ($N = 2$). They are defined by:

- the input X and the corresponding alphabet $\mathcal{X}$;
- the two outputs Y and Z, and the corresponding alphabets $\mathcal{Y}$ and $\mathcal{Z}$;
- the transition probabilities $p_1(y|x)$ and $p_2(z|x)$ satisfying the "degradability requirement":

  $$p_2(z|x) = \sum_{y \in \mathcal{Y}} p_d(z|y) p_1(y|x), \quad \text{for all } x \in \mathcal{X} \text{ and } z \in \mathcal{Z},$$

  where $p_d(z|y)$ is the transition probability of the degrading channel D.

We therefore have a "strong" channel A_1 and a "weak" channel A_2, where the capacity C_1 of A_1 and the capacity C_2 of A_2 satisfy:

$$C_1 \geq C_2.$$

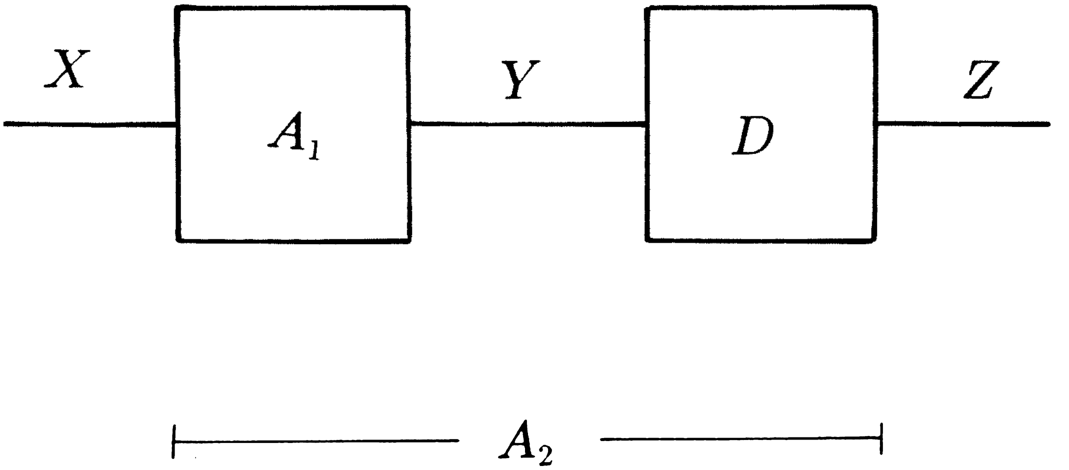


Figure 3.2: Degraded Broadcast Channels:

A_2 is a degraded version of A_1.

Given a coding technique, if R_1 is the transmission rate of channel A_1, and R_2 is the transmission rate of channel A_2, then the set of all rates (R_1, R_2) simultaneously achievable with that coding technique will define what we call an achievable rate region which we can represent here in a two-dimensional space. As we will see, different coding techniques (such as time-shared coding and broadcast coding) will yield different rate regions. We are primarily interested in the largest achievable rate region, also known as the *capacity region.* Naturally, all achievable rate regions will be enclosed in the region defined by:

$$0 \leq R_1 \leq C_1, \quad \text{and } 0 \leq R_2 \leq C_2.$$

In the following sections, we will first present the rate region achievable by time-shared coding (also known as the *time-sharing region,*) and finally we will present a theorem on the capacity region of the two-receiver DBC.

I-2. The Time-Sharing region

The time-sharing approach consists in allocating a proportion of time λ to sending at rate $(R_1, R_2) = (C_2, C_2)$, and a proportion of time $\bar{\lambda} = 1 - \lambda$ to sending at rate $(R_1, R_2) = (C_1, 0)$. The first point (C_2, C_2) corresponds to sending data at the maximum rate tolerated by the weak receiver Z : if A_1 is in some sense compatible with A_2, Y can reliably decode data transmitted at rate $C_2 \leq C_1$. The second point $(C_1, 0)$ corresponds to sending data at the maximum rate tolerated by the strong receiver Y, and therefore preventing the weak receiver Z from decoding any data reliably since $C_1 \geq C_2$. If we let the parameter λ continuously vary from 0 to 1, the set of rates obtained lies on a line segment known as the time-sharing line defined by:

$$(R_1, R_2) = (\lambda C_2 + \bar{\lambda} C_1, \lambda C_2), \quad \text{for } 0 \leq \lambda \leq 1. \tag{3.1}$$

The information transmitted to the two receivers can be interpreted as the combination of *common* information (here $R_2 = \lambda C_2$ is called the *common rate*) accessible to both receivers, and *bonus* information (here $R_1 - R_2 = \bar{\lambda} C_1$ is called the *bonus rate*) accessible only to the strong receiver.

We can rewrite (3.1) in terms of two new variables, the *normalized bonus rate* ρ_1, and the *normalized common rate* ρ_2 :

$$\rho_1 = \frac{R_1 - R_2}{C_1},$$

$$\text{and } \rho_2 = \frac{R_2}{C_2},$$

The equation of the time-sharing line (plotted in figure 3.3) now becomes:

$$\rho_1 + \rho_2 = 1. \tag{3.2}$$

I-3. The Broadcast Coding region

In a classic paper, Cover [29] showed that higher rates than those obtained by time-sharing could be achieved for the binary symmetric broadcast channel (BSBC) and the Gaussian Broadcast Channel (GBC), using a superposition random coding scheme. Here is a brief description of this coding scheme, known as *broadcast coding*:

Encoding:

- find a low rate random code for the weak receiver Z, where the rate R_2 is below capacity C_2;
- since the rate R_2 is below capacity, take advantage of the allowed extra tolerance to noise, by further randomizing the codewords previously chosen for Z into a small cloud of satellite codewords intended for the strong receiver Y;

Decoding:

- the weak receiver Z reliably decodes the cloud center;
- the strong receiver Y also decodes the cloud center reliably;
- once the cloud center has been decoded, the strong receiver Y decodes the satellite codeword.

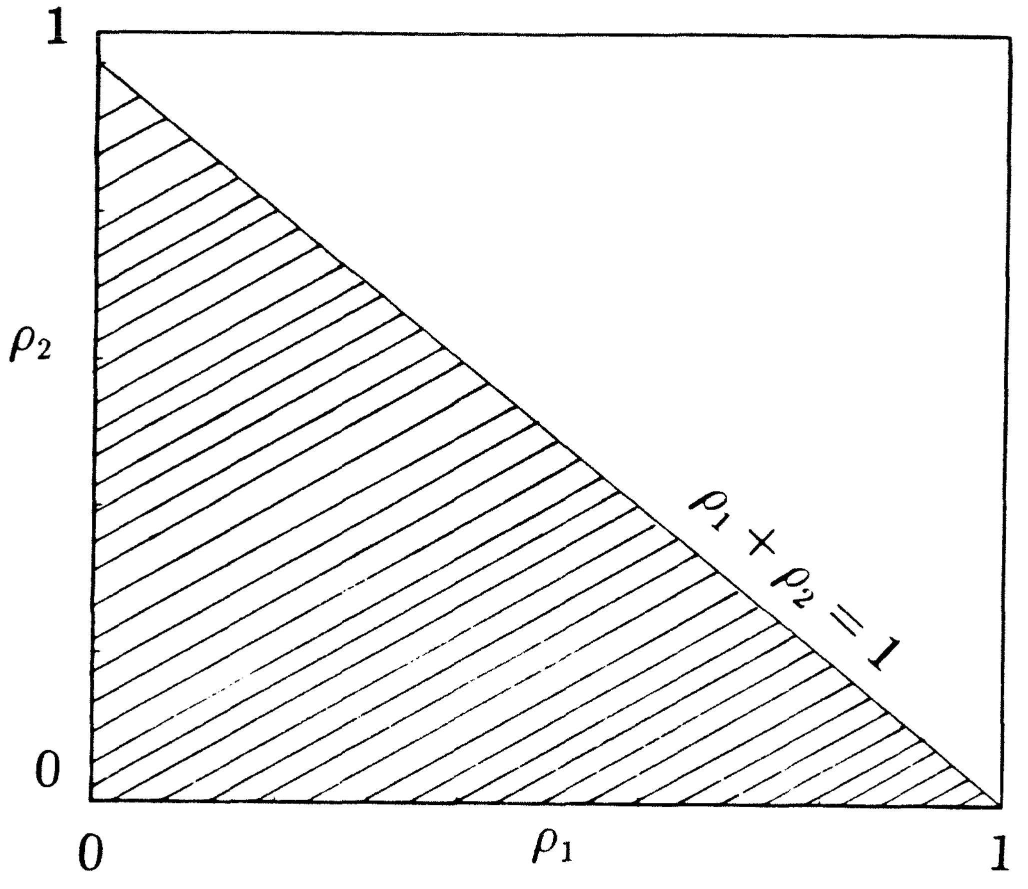


Figure 3.3: The Time-Sharing Region: $\rho_1 + \rho_2 \leq 1$.

Broadcast Coding has therefore the effect of superimposing information intended for the strong receiver Y (and which looks like noise to the weak receiver Z) on information intended for both channels.

The way this is modeled is by introducing an auxiliary sender U and an associated channel from U to X (figure 3.4):

- the auxiliary source U represents the common information intended for both receivers Y and Z;
- the associated noisy channel (also called the artificial "satellizing channel") models the superposition of the bonus information intended for the strong receiver Y on its input U; this superimposed information can only be seen as noise by the weak receiver Z.

The auxiliary source U is characterized by a probability distribution $\mathbf{q_2}$ with probabilities:

$$q_2(u), \quad u \in \mathcal{U},$$

and the associated channel by the transition probability distribution $\mathbf{q_1}$ with transition probabilities:

$$q_1(x|u), \quad u \in \mathcal{U}, \; x \in \mathcal{X}.$$

To each pair of distributions $(\mathbf{q_1}, \mathbf{q_2})$ corresponds a point in the two-dimensional space (R_1, R_2). The region delimited by the convex hull of all the points associated with all possible pairs of distributions $(\mathbf{q_1}, \mathbf{q_2})$, will be referred to as the rate region achievable by broadcast coding, or again the *broadcast coding region.*

The superposition scheme we described was then reformulated by Bergmans [27] for the general DBC (of which the BSBC and the GBC are particular examples) to obtain an achievable rate region. Later, the Bergmans region was shown to be the capacity region of the DBC by Gallager [38] and Ahlswede and Körner [30].

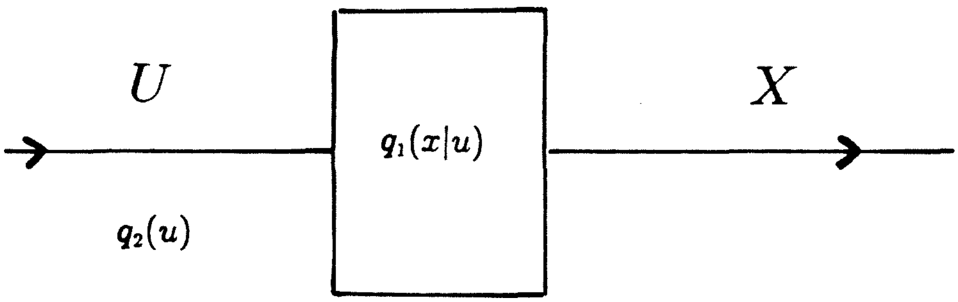


Figure 3.4: Modeling Broadcast Coding:

The noise present in the "satellizing" channel from U to X is the bonus information intended for receiver Y.

These results lead to the following theorem:

Theorem 3.1: *The capacity region of the degraded broadcast channel is the broadcast coding region. It is defined by the set of all points* (R_1, R_2) *defined by:*

$$0 \leq R_1 < R_2 + I(X;Y|U),$$
$$0 \leq R_2 < I(U;Z), \tag{3.3}$$

over all possible pairs $(\mathbf{q}_1, \mathbf{q}_2)$ *as long as:*

$$|\mathcal{U}| \leq \min\Big(|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{Z}|\Big). \tag{3.4}$$

Again, we can rewrite (3.3) in terms of ρ_1 and ρ_2 :

$$0 \leq \rho_1 < \frac{I(X;Y|U)}{C_1},$$
$$0 \leq \rho_2 < \frac{I(U;Z)}{C_2} \tag{3.5}$$

Since the time-sharing region (3.2) is included in the capacity region, the determination of the capacity region consists in determining the set of all points (ρ_1, ρ_2) satisfying:

$$\rho_1 + \rho_2 \geq 1.$$

If there is no point satisfying (3.5) and $\rho_1 + \rho_2 > 1$, then the time-sharing region *is* the capacity region.

II - BINARY-INPUT VERY NOISY DEGRADED BROADCAST CHANNELS (BIVNDBCs)

Having briefly reviewed the known results of broadcast coding theory in section I, we now wish to use them for a sub-class of VNCs: the binary-input VNC. McEliece, Posner and Swanson [22],[23] showed that, in the case of the Wideband Gaussian Broadcast Channel, in the limit as the signal-to-noise ratios of both receivers Y and Z go to zero, broadcast coding offers no advantage over time-shared coding and therefore time-shared coding is optimal. The Wideband Gaussian Channel belongs to the class of discrete-input continuous-output VNCs, which we have not covered here.

The question we now wish to answer in this section is whether, for VNCs, broadcast coding offers any advantage over time-shared coding, and perhaps identify sub-classes with that property. It is important to identify VNCs which share the property of the Wideband Gaussian Channel, because the classical time-sharing approach is simple to implement.

Computing capacity regions is a rather complex task, especially when there is no analytical expression for the capacity. We will therefore restrict our study to binary-input VNCs, for which in most cases, the capacity is known. Our study is divided in three parts: the Class I BIVNDBC, the Class IIa BIVNDBC, the Class IIb BIVNDBC. But first, we describe the methodology we use in the computation of the capacity regions.

II-1. Methodology - Notations.

Consider a class of channels $\mathcal{C}$. We first must define what we mean by classes of degraded broadcast channels.

Definition 3.4: We say that the channels A_1 and A_2 form a *binary-input two-receiver degraded broadcast channel of class* $\mathcal{C}$ if and only if the following conditions are met:

- A_1 is a $2 \times N_1$ channel belonging to $\mathcal{C}$, which implies that $|\mathcal{X}| = 2$ and $|\mathcal{Y}| = N_1$;
- A_2 is a $2 \times N_2$ channel belonging to $\mathcal{C}$, which implies that $|\mathcal{X}| = 2$ and $|\mathcal{Z}| = N_2$;
- there exists an $N_1 \times N_2$ channel D (not necessarily from $\mathcal{C}$) such that A_2 is the cascade of A_1 and D;

Since we intend to study only classes of "very noisy" two-receiver degraded broadcast channels, let us be more specific in defining the channels involved in the computation of their capacity region.

- Defining channel A_1.

A_1 is a $2 \times N_1$ VNC of class $\mathcal{C}$ with transition probabilities:

$$p_1(y|x) = w_1(y) + \epsilon \cdot \lambda_1(x,y) + O(\epsilon^2), \quad \text{for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}. \tag{3.6}$$

- Defining channel D.

The degrading channel D can be any $N_1 \times N_2$ channel, the cascade of A_1 and D has to be a member of class $\mathcal{C}$. Its transition probabilities are:

$$p_d(z|y), \quad \text{for all } y \in \mathcal{Y}, \text{ and } z \in \mathcal{Z}.$$

Note that the transition probabilities $p_d(z|y)$ can be functions of the parameter ϵ. However, we will consider them to be constants with respect to ϵ, since only their limiting value as $\epsilon \to 0$ determines which class A_2 belongs to. Naturally, we assume that the transition probability matrix does not have any column of zeros, which is a degenerate case.

• Defining channel A_2.

A_2 is a $2 \times N_2$ VNC of class $\mathcal{C}$ and the result of the cascade of A_1 and D, with transition probabilities:

$$p_2(z|x) = w_2(z) + \epsilon \cdot \lambda_2(x,z) + O(\epsilon^2), \quad \text{for all } x \in \mathcal{X} \text{ and } z \in \mathcal{Z},$$

where:

$$w_2(z) = \sum_{y \in \mathcal{Y}} p_d(z|y) w_1(y), \quad \text{for all } z \in \mathcal{Z}, \tag{3.7}$$

and:

$$\lambda_2(x,z) = \sum_{y} p_d(z|y) \lambda_1(x,y), \quad \text{for all } x \in \mathcal{X} \text{ and } z \in \mathcal{Z}. \tag{3.8}$$

• Modeling broadcast coding of X.

Theorem 3.1 tells us that we will obtain all achievable rates R_1 and R_2 (and therefore ρ_1 and ρ_2) if we introduce at the input of A_1 and A_2, an auxiliary source U and a "satellizing" channel $U \to X$: the alphabet size of U is constrained by inequality (3.3), which Gallager [28] showed to be a sufficient condition to obtain all achievable rates R_1 and R_2. Furthermore, since $|\mathcal{X}| = 2$, then inequality (3.3) becomes:

$$|\mathcal{U}| \leq \min(2, N_1, N_2) = 2.$$

But since $|\mathcal{U}| \geq 2$, then $|\mathcal{U}| = 2$ and $U \to X$ is a binary channel. Let us define the probabilities:

$$q_2 = \text{Prob}(U = u_1) = 1 - \text{Prob}(U = u_2),$$

$$q = \text{Prob}(X = x_1) = 1 - \text{Prob}(X = x_2),$$

and the transition probability matrix of the "satellizing" channel:

$$\begin{pmatrix} \alpha & \bar{\alpha} \\ \beta & \bar{\beta} \end{pmatrix}.$$

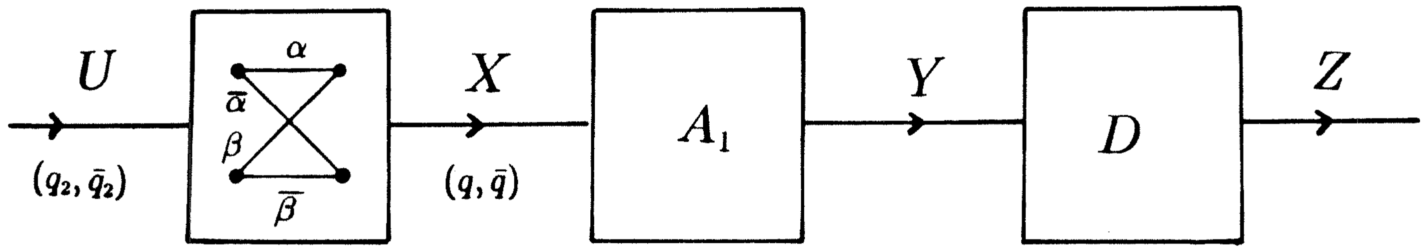


Figure 3.5: The binary-input "very noisy" degraded broadcast channel.

We will always assume that $\alpha \neq \beta$, the reason being that the particular case $\alpha = \beta$ corresponds to transmitting no common information ($\rho_2 = 0$) and only bonus information ($\rho_1 \leq 1$), which is only a point on or below the time-sharing line.

We can therefore write:

$$\begin{pmatrix} q \\ \bar{q} \end{pmatrix} = \begin{pmatrix} \alpha & \bar{\alpha} \\ \beta & \bar{\beta} \end{pmatrix}^T \begin{pmatrix} q_2 \\ \bar{q}_2 \end{pmatrix},$$

which yields:

$$q = q_2\alpha + \bar{q}_2\beta. \tag{3.9}$$

• The capacity region.

To cover all possible binary-input degraded broadcast channels of class C, we must span in our study all possible channels of type A_1, all possible channels of type D leading to channels of type A_2, and all possible values for q_2, α and β. The values of N_1 and N_2 are arbitrary but fixed. Figure 3.5 contains the diagram of the general binary-input degraded broadcast channel.

Since from now on we will be only dealing with VNCs, for given q_2, α and β, we consider ρ_1 and ρ_2 to be the *limiting* normalized bonus and common rates instead of their *exact* values:

$$\rho_1 = \lim_{\epsilon \to 0} \frac{I(X;Y|U)}{C_1},$$
$$\rho_2 = \lim_{\epsilon \to 0} \frac{I(U;Z)}{C_2}.$$

II-2. The capacity region of Class I BIVNDBCs.

• Defining channel A_1.

The transition probabilities of channel A_1 are defined by (3.6) where:

$$w_1(y) > 0, \quad \text{for all } y \in \mathcal{Y}. \tag{3.10}$$

• Defining channels D and A_2.

The transition probabilities $p_d(z|y)$ of channel D can be arbitrary as long as A_2 is a Class I BIVNDBC which is equivalent to:

$$w_2(z) > 0, \quad \text{for all } z \in \mathcal{Z}.$$

But because of (3.7) and (3.10), this is also equivalent to:

$$\text{for all } z \in \mathcal{Z}, \quad \exists y \in \mathcal{Y} \ : \ p_d(z|y) \neq 0. \tag{3.11}$$

• Computing the capacity of channels A_1 and A_2.

We know from our study of Class I VNCs, that:

$$C_1 \approx \frac{\epsilon^2}{8} \sum_y \frac{\Big(\lambda_1(x_1, y) - \lambda_1(x_2, y)\Big)^2}{w_1(y)}, \tag{3.12}$$

and that:

$$C_2 \approx \frac{\epsilon^2}{8} \sum_z \frac{\Big(\lambda_2(x_1, z) - \lambda_2(x_2, z)\Big)^2}{w_2(z)}. \tag{3.13}$$

• Computing $I(X;Y|U)$.

By definition,

$$I(X;Y|U) = q_2 I(X;Y|U = u_1) + \bar{q}_2 I(X;Y|U = u_2).$$

We also know from our study of Class I VNCs that the average mutual information between the input and the output of a Class I VNC, given that the probability of the first of the two inputs is q, is:

$$I(X;Y) \approx \frac{\epsilon^2}{2} q\bar{q} \sum_y \frac{\Big(\lambda_1(x_1, y) - \lambda_1(x_2, y)\Big)^2}{w(y)}. \tag{3.14}$$

Therefore,

$$\begin{aligned} I(X;Y|U) \approx & q_2 \frac{\epsilon^2}{2} \alpha\bar{\alpha} \sum_y \frac{\Big(\lambda_1(x_1, y) - \lambda_1(x_2, y)\Big)^2}{w(y)} \\ & + \bar{q}_2 \frac{\epsilon^2}{2} \beta\bar{\beta} \sum_y \frac{\Big(\lambda_1(x_1, y) - \lambda_1(x_2, y)\Big)^2}{w(y)}, \end{aligned}$$

which yields:

$$I(X;Y|U) \approx (q_2\alpha\bar{\alpha} + \bar{q}_2\beta\bar{\beta}) \cdot \frac{\epsilon^2}{2} \sum_y \frac{\Big(\lambda_1(x_1,y) - \lambda_1(x_2,y)\Big)^2}{w(y)}. \tag{3.15}$$

• Computing ρ_1.

Since

$$\rho_1 = \lim_{\epsilon \to 0} \frac{I(X;Y|U)}{C_1},$$

then, using (3.12) and (3.15) we obtain:

$$\rho_1 = 4 \cdot (q_2\alpha\bar{\alpha} + \bar{q}_2\beta\bar{\beta}). \tag{3.16}$$

• Computing $I(U;Z)$.

The transition probabilities of the channel with input U and output Z ($U \to Z$) are defined by:

$$\begin{aligned} p(z|u) &= \sum_x p_2(z|x) q_1(x|u) \\ &\approx \sum_x \Big(w_2(z) + \epsilon \cdot \lambda_2(x,z)\Big) q_1(x|u) \\ &\approx w_2(z) + \epsilon \cdot \sum_x \lambda_2(x,z) q_1(x|u). \end{aligned}$$

Therefore channel $U \to Z$ is also a Class I BIVNDBC since $\alpha \neq \beta$. Using (14), we obtain:

$$\begin{aligned} I(U;Z) &\approx \frac{\epsilon^2}{2} q_2\bar{q}_2 \sum_z \frac{\Big(\sum_x \lambda_2(x,z) q_1(x|u_1) - \sum_x \lambda_2(x,z) q_1(x|u_2)\Big)^2}{w_2(z)}, \\ &\approx \frac{\epsilon^2}{2} q_2\bar{q}_2 \sum_z \frac{\Big(\sum_x \lambda_2(x,z)\Big(q_1(x|u_1) - q_1(x|u_2)\Big)\Big)^2}{w_2(z)}. \end{aligned}$$

But since $U \to X$ is a binary channel,

$$I(U;Z) \approx \frac{\epsilon^2}{2} q_2\bar{q}_2 \sum_z \frac{\Big(\lambda_2(x_1,z) - \lambda_2(x_2,z)\Big)^2 (\alpha - \beta)^2}{w_2(z)},$$

which yields:

$$I(U;Z) \approx \frac{\epsilon^2}{2} q_2 \bar{q}_2 (\alpha - \beta)^2 \sum_z \frac{\Big(\lambda_2(x_1,z) - \lambda_2(x_2,z)\Big)^2}{w_2(z)}. \tag{3.17}$$

• Computing ρ_2.

Using (3.13) and (3.17), we obtain:

$$\rho_2 = 4 q_2 \bar{q}_2 (\alpha - \beta)^2. \tag{3.18}$$

• Computing the capacity region of Class I BIVNDBCs.

Adding (3.16) and (3.18), we obtain:

$$\rho_1 + \rho_2 = 4\Big(q_2 \alpha \bar{\alpha} + \bar{q}_2 \beta \bar{\beta} + q_2 \bar{q}_2 (\alpha - \beta)^2\Big).$$

By remembering the definition of q in (3.9), the reader can easily verify that:

$$\rho_1 + \rho_2 = 4 q \bar{q}, \tag{3.19}$$

and therefore,

$$\rho_1 + \rho_2 \leq 1,$$

with equality if and only if:

$$q = q_2 \alpha + \bar{q}_2 \beta = \frac{1}{2}.$$

This final result gives us the proof that the capacity region of Class I BIVND-BCs, in the limit as $\epsilon \to 0$, is completely enclosed in the time-sharing region, and therefore, the time-sharing region is the capacity region: nothing is gained by using broadcast coding. This yields the following theorem:

Theorem 3.2: *The capacity region of the Class I binary-input "very noisy" degraded broadcast channel, in the limit as $\epsilon \to 0$, is the time-sharing region.*

We will now turn to Class II BIVNDBCs.

II-3. The capacity region of Class II BIVNDBCs.

One major property of Class II VNCs is that some components of their transition probability matrix have no influence in the expression of the average mutual information, and therefore may be dropped out. Therefore we will define the following partitioning of the output alphabets $\mathcal{Y}$ and $\mathcal{Z}$:

$$\mathcal{Y} \ : \ \begin{cases} \mathcal{Y}_1 = \{y \in \mathcal{Y} \ : \ w_1(y) \neq 0\}, \\ \mathcal{Y}_2 = \{y \in \mathcal{Y} \ : \ w_1(y) = 0\}. \end{cases}$$
$$\mathcal{Z} \ : \ \begin{cases} \mathcal{Z}_1 = \{z \in \mathcal{Z} \ : \ w_2(z) \neq 0\}, \\ \mathcal{Z}_2 = \{z \in \mathcal{Z} \ : \ w_2(z) = 0\}. \end{cases}$$

We have previously identified two subclasses of Class II VNCs with different properties: Class IIa and Class IIb. We will therefore study separately Class IIa and Class IIb BIVNDBCs.

II-3.1. The capacity region of Class IIb BIVNDBCs.

• Defining channel A_1.

Since channel A_1 is a Class IIb VNC, its transition probabilities, without loss of generality, are defined by (3.6) and:

$$\begin{aligned} w_1(y) &= 0, \quad \text{for all } y \in \mathcal{Y}_2, \\ \lambda_1(x_1, y) &= 0, \quad \text{for all } y \in \mathcal{Y}_2, \end{aligned} \tag{3.20}$$

where $\mathcal{Y}_2 \neq \emptyset$.

• Defining channels D and A_2.

First, given that A_1 is a Class II VNC, let us see what is required of channel D for A_2 to be a Class II VNC or, equivalently, for $\mathcal{Z}_2$ to be a non-empty set. Using (3.7), we can rewrite the definition of $\mathcal{Z}_2$:

$$\mathcal{Z}_2 = \{z \in \mathcal{Z} \ : \ \sum_{y \in \mathcal{Y}} p_d(z|y) w_1(y) = 0\}.$$

But since for a Class II VNC, $w_1(y) = 0$ for all $y \in \mathcal{Y}_2$, then:

$$\mathcal{Z}_2 = \{z \in \mathcal{Z} \ : \ \sum_{y \in \mathcal{Y}_1} p_d(z|y) w_1(y) = 0\},$$

and finally, since $w_1(y) > 0$ for all $y \in \mathcal{Y}_1$:

$$\mathcal{Z}_2 = \{z \in \mathcal{Z} \ : \ p_d(z|y) = 0 \text{ for all } y \in \mathcal{Y}_1\}. \tag{3.21}$$

Therefore we have the theorem:

Theorem 3.3: *Given that A_1 is a Class II VNC, A_2 is a Class II VNC if and only if the following condition on channel D is satisfied:*

$$\exists z \in \mathcal{Z} \ : \ p_d(z|y) = 0 \ \text{ for all } y \in \mathcal{Y}_1. \tag{3.22}$$

Furthermore, if A_1 is a Class IIb VNC as described in (3.20), then using (3.8) we obtain:

$$\lambda_2(x_1, z) = \sum_{y \in \mathcal{Y}_1} p_d(z|y) \lambda_1(x_1, y), \ \text{ for all } z \in \mathcal{Z},$$

which yields, using (3.21):

$$\lambda_2(x_1, z) = 0, \ \text{ for all } z \in \mathcal{Z}_2.$$

Therefore we have the theorem:

Theorem 3.4: *Given that A_1 is a Class IIb VNC, A_2 is a Class IIb VNC if and only if the following condition on channel D is satisfied:*

$$\exists z \in \mathcal{Z} \ : \ p_d(z|y) = 0 \ \text{ for all } y \in \mathcal{Y}_1.$$

In other words, we find identical necessary and sufficient conditions for theorems 3.3 and 3.4.

- Computing the capacity of channels A_1 and A_2.

If q is the probability of the first input x_1, the average mutual information for binary-input Class IIb VNCs is:

$$I(X;Y) \approx \epsilon \cdot \bar{q} \log \frac{1}{\bar{q}} \cdot \alpha_i(x_2), \tag{3.23}$$

where

$$\alpha_i(x) = \sum_{y \in Y_2} \lambda_i(x,y).$$

Therefore, for binary-input Class IIb VNCs, the optimizing input probability distribution is:

$$\hat{\mathrm{q}} = (1 - \frac{1}{e}, \frac{1}{e}),$$

which yields the capacities of channels A_1 and A_2 :

$$C_1 \approx \frac{\epsilon}{e} \cdot \alpha_1(x_2), \tag{3.24}$$

and

$$C_2 \approx \frac{\epsilon}{e} \cdot \alpha_2(x_2). \tag{3.25}$$

Note that:

$$\begin{aligned}
\alpha_2(x_2) &= \sum_{z \in Z_2} \lambda_2(x_2, z), \\
&= \sum_{z \in Z_2} \sum_{y \in Y_2} p_d(z|y)\lambda_1(x_2, y), \\
&= \sum_{y \in Y_2} \lambda_1(x_2, y)\Big(\sum_{z \in Z_2} p_d(z|y)\Big), \\
&\leq \sum_{y \in Y_2} \lambda_1(x_2, y) = \alpha_1(x_2).
\end{aligned}$$

- Computing $I(X;Y|U)$.

By definition,

$$I(X;Y|U) = q_2 I(X;Y|U = u_1) + \bar{q}_2 I(X;Y|U = u_2).$$

Therefore, using (3.23):

$$I(X;Y|U) \approx q_2\Big(\epsilon \cdot \bar{\alpha} \log \frac{1}{\bar{\alpha}} \cdot \alpha_1(x_2)\Big) + \bar{q}_2\Big(\epsilon \cdot \bar{\beta} \log \frac{1}{\bar{\beta}} \cdot \alpha_1(x_2)\Big),$$

which yields:

$$I(X;Y|U) \approx \epsilon \cdot \alpha_1(x_2) \cdot \Big(q_2\bar{\alpha}\log\frac{1}{\bar{\alpha}} + \bar{q}_2\bar{\beta}\log\frac{1}{\bar{\beta}}\Big). \tag{3.26}$$

• Computing ρ_1.

Using (3.24) and (3.26), we obtain:

$$\rho_1 = e \cdot \Big(q_2\bar{\alpha}\log\frac{1}{\bar{\alpha}} + \bar{q}_2\bar{\beta}\log\frac{1}{\bar{\beta}}\Big). \tag{3.27}$$

• Computing $I(U;Z)$.

For $z \in Z_2$, the transition probabilities of channel $U \rightarrow Z$ satisfy:

$$\begin{aligned} p(z|u_1) &= p_2(z|x_1)\alpha + p_2(z|x_2)\bar{\alpha}, \\ &\approx \epsilon \cdot \lambda_2(x_2,z)\bar{\alpha}. \end{aligned}$$

and:

$$\begin{aligned} p(z|u_2) &= p_2(z|x_1)\beta + p_2(z|x_2)\bar{\beta}, \\ &\approx \epsilon \cdot \lambda_2(x_2,z)\bar{\beta}. \end{aligned}$$

Note that $U \rightarrow Z$ is a Class IIb VNC. Therefore,

$$\begin{aligned} I(U;Z) \approx&\epsilon \cdot \left(q_2 \sum_{z\in Z_2} \lambda_2(x_2,z)\bar{\alpha}\log\frac{\lambda_2(x_2,z)\bar{\alpha}}{\lambda_2(x_2,z)(q_2\bar{\alpha}+\bar{q}_2\bar{\beta})}\right) \\ &+ \epsilon \cdot \left(\bar{q}_2 \sum_{z\in Z_2} \lambda_2(x_2,z)\bar{\beta}\log\frac{\lambda_2(x_2,z)\bar{\beta}}{\lambda_2(x_2,z)(q_2\bar{\alpha}+\bar{q}_2\bar{\beta})}\right) \\ \approx&\epsilon \cdot q_2\Big(\sum_{z\in Z_2}\lambda_2(x_2,z)\Big)\bar{\alpha}\log\frac{\bar{\alpha}}{q_2\bar{\alpha}+\bar{q}_2\bar{\beta}} \\ &+ \epsilon \cdot \bar{q}_2\Big(\sum_{z\in Z_2}\lambda_2(x_2,z)\Big)\bar{\beta}\log\frac{\bar{\beta}}{q_2\bar{\alpha}+\bar{q}_2\bar{\beta}} \\ \approx&\epsilon \cdot \alpha_2(x_2) \cdot \Big(q_2\bar{\alpha}\log\frac{\bar{\alpha}}{q_2\bar{\alpha}+\bar{q}_2\bar{\beta}} + \bar{q}_2\bar{\beta}\log\frac{\bar{\beta}}{q_2\bar{\alpha}+\bar{q}_2\bar{\beta}}\Big). \end{aligned}$$

Remembering the definition of q in (3.9), we can write:

$$I(U;Z) \approx \epsilon \cdot \alpha_2(x_2) \cdot \Big(\bar{q}\log\frac{1}{\bar{q}} + q_2\bar{\alpha}\log\bar{\alpha} + \bar{q}_2\bar{\beta}\log\bar{\beta}\Big). \tag{3.28}$$

• Computing ρ_2.

Using (3.25) and (3.28), we obtain:

$$\rho_2 = e\Big(\bar{q}\log\frac{1}{\bar{q}} - q_2\bar{\alpha}\log\frac{1}{\bar{\alpha}} - \bar{q}_2\bar{\beta}\log\frac{1}{\bar{\beta}}\Big). \tag{3.29}$$

• Computing the capacity region of Class IIb BIVNDBCs.

Adding (3.27) and (3.29), we obtain:

$$\rho_1 + \rho_2 = e\cdot\bar{q}\log\frac{1}{\bar{q}}, \tag{3.30}$$

and therefore,

$$\rho_1 + \rho_2 \le 1,$$

with equality if and only if:

$$q = 1 - \frac{1}{e}.$$

Again, as in the case of Class I BIVNDBCs, the capacity region of Class IIb BIVNDBCs , in the limit as $\epsilon \to 0$, is completely enclosed (see (3.30)) in the time-sharing region. We now have the theorem:

Theorem 3.5: *The capacity region of the Class IIb binary-input "very noisy" degraded broadcast channel, in the limit as $\epsilon \to 0$, is the time-sharing region.*

Note: If we had not put restrictions on D for A_2 to be a Class IIb VNC (as required by definition 3.4), then A_2 would have been a Class I VNC. Using equation (3.18) for ρ_2, we would have obtained:

$$\rho_1 + \rho_2 = e\cdot\left(q_2\bar{\alpha}\log\frac{1}{\bar{\alpha}} + \bar{q}_2\bar{\beta}\log\frac{1}{\bar{\beta}}\right) + 4q_2\bar{q}_2(\alpha-\beta)^2.$$

Here is an example where $\rho_1 + \rho_2 > 1$:

$$q_2 = 0.3698, \quad \alpha = 0., \quad \beta = 0.8549,$$

which yields:

$$\rho_1 = 0.4798, \quad \rho_2 = 0.6813, \quad \rho_1 + \rho_2 = 1.1611.$$

Therefore our definition of classes of broadcast channels requiring that A_2 be of the same class as A_1 is appropriate. Let us now finally examine the case of Class IIa BIVNDBCs.

II-3.2. The capacity region of Class IIa BIVNDBCs.

Unlike Class I and Class IIb BIVNDBCs, the time-sharing region is not always the capacity region in the limit as $\epsilon \to 0$. First we will show that the capacity region of all Class IIa BIVNDBCs such that A_1 is a *binary channel* ($N_1 = 2$), is the time-sharing region. Then we will give an example of Class IIa BIVNDBCs (where $N_1 \geq 3$) for which an achievable rate-region is strictly above the time-sharing line.

II-3.2.1. The capacity region of Class IIa BIVNDBCs when A_1 is a binary channel.

Let us first define, as in section I of Chapter 2, the following functions of r and q :

$$\begin{aligned} u(r) &= \frac{r}{r-1} \log r, \\ c(r) &= e^{u(r)-1} - u(r), \\ i(r,q) &= (q + \bar{q}r) \log \frac{1}{q + \bar{q}r} + \bar{q}r \log r, \end{aligned} \tag{3.31}$$

where:

$$\begin{cases} 0 \leq r < 1, \\ 0 \leq q \leq 1. \end{cases}$$

• Defining channel A_1.

Consider the following binary Class IIa VNC defined by its transition probability matrix:

$$\begin{pmatrix} 1 - \lambda_1\epsilon & \lambda_1\epsilon \\ 1 - \mu_1\epsilon & \mu_1\epsilon \end{pmatrix},$$

where we assume, without loss of generality, that:

$$0 < \mu_1 < \lambda_1, \tag{3.32}$$

and therefore we may define the ratio:

$$0 < r_1 = \frac{\mu_1}{\lambda_1} < 1.$$

• Defining channels D and A_2.

As shown in Theorem 3.3, for A_2 to be a Class II VNC given that A_1 is a Class IIa VNC (and a fortiori a Class II VNC), the transition probabilities of the binary-input channel D have to satisfy:

$$p_d(z|y) = 0, \quad \text{for all } y \in \mathcal{Y}_1 \text{ and } z \in \mathcal{Z}_2.$$

Since $Y_1 = \{y_1\}$, the condition becomes:

$$p_d(z|y_1) = 0, \quad \text{for all } z \in \mathcal{Z}_2. \tag{3.33}$$

Therefore, because of (3.33) and since D may not have any column of zeros:

$$p_d(z|y_2) \neq 0, \quad \text{for all } z \in \mathcal{Z}_2. \tag{3.34}$$

The result of the cascade of A_1 and D is channel A_2, with transition probabilities satisfying:

$$p_2(z|x) \approx \epsilon\lambda_2(x,z), \quad \text{for all } x \in \mathcal{X} \text{ and } z \in \mathcal{Z}_2,$$

where, using (3.8) and (3.33):

$$\lambda_2(x,z) = p_d(z|y_2) \cdot \lambda_1(x,y_2), \quad \text{for all } x \in \mathcal{X} \text{ and } z \in \mathcal{Z}_2. \tag{3.35}$$

Note that:

$$\text{for all } z \in \mathcal{Z}_2 : \begin{cases} \lambda_2(x_1,z) = p_d(z|y_2) \cdot \lambda_1, \\ \lambda_2(x_2,z) = p_d(z|y_2) \cdot \mu_1, \end{cases} \tag{3.36}$$

and using (3.35):

$$\lambda_2(x_2,z) = p_d(z|y_2) \cdot \lambda_1 r_1, \quad \text{for all } z \in \mathcal{Z}_2,$$

which yields:

$$\lambda_2(x_2, z) = r_1 \cdot \lambda_2(x_1, z), \quad \text{for all } z \in Z_2. \tag{3.37}$$

Now, because of (3.32), (3.34) and (3.36), we conclude:

$$\lambda_2(x, z) \neq 0, \quad \text{for all } x \in \mathcal{X} \text{ and } z \in Z_2,$$

which yields the following theorem:

Theorem 3.6: *Given that A_1 is a binary Class IIa VNC, A_2 is a Class IIa VNC if and only if condition* (3.26) *is satisfied.*

- Computing the capacity of channels A_1 and A_2.

If q is the probability of the first input x_1, the average mutual information of the binary Class IIa VNC A_1 is:

$$I(X;Y) \approx (\lambda_1 \epsilon) \cdot i(r_1, q), \tag{3.38}$$

where $i(r, q)$ has been defined in (3.31).

Since:

$$c(r) = \max_q i(r, q),$$

then the capacity of channel A_1 is:

$$C_1 \approx \lambda_1 \epsilon \cdot c(r_1). \tag{3.39}$$

Now, if q is the probability of the first input x_1, the average mutual information of the Class IIa BIVNDBC is:

$$\begin{aligned} I(X;Z) \approx \Bigg(& q\epsilon \sum_{z \in Z_2} \lambda_2(x_1, z) \log \frac{\lambda_2(x_1, z)}{q\lambda_2(x_1, z) + \bar{q}\lambda_2(x_2, z)} \\ & + \bar{q}\epsilon \sum_{z \in Z_2} \lambda_2(x_2, z) \log \frac{\lambda_2(x_2, z)}{q\lambda_2(x_1, z) + \bar{q}\lambda_2(x_2, z)} \Bigg). \end{aligned}$$

Using (3.37),

$$I(X;Z) \approx \epsilon \left(q \sum_{z \in Z_2} \lambda_2(x_1, z) \log \frac{1}{q + \bar{q}r_1} + \bar{q} \sum_{z \in Z_2} r_1 \lambda_2(x_1, z) \log \frac{r_1}{q + \bar{q}r_1} \right).$$

Let $a = \sum_{z \in Z_2} p_d(z|y_2) \leq 1$, then, using (3.32):

$$I(X;Z) \approx \epsilon\Big(qa\lambda_1 \log \frac{1}{q+\bar{q}r_1} + \bar{q}ar_1 \log \frac{r_1}{q+\bar{q}r_1}\Big),$$

which yields:

$$I(X;Z) \approx a\lambda_1\epsilon \cdot i(r_1, q). \tag{3.40}$$

Therefore, the capacity of C_2 is:

$$C_2 \approx a\lambda_1\epsilon \cdot c(r_1). \tag{3.41}$$

• Computing $I(X;Y|U)$.

By definition,

$$I(X;Y|U) = q_2 I(X;Y|U = u_1) + \bar{q}_2 I(X;Y|U = u_2).$$

Therefore, using (3.38):

$$I(X;Y|U) \approx q_2(\lambda_1\epsilon) \cdot i(r_1, \alpha) + \bar{q}_2(\lambda_1\epsilon) \cdot i(r_1, \beta),$$

which yields:

$$I(X;Y|U) \approx (\lambda_1\epsilon)\Big(q_2 i(r_1, \alpha) + \bar{q}_2 i(r_1, \beta)\Big). \tag{3.42}$$

• Computing ρ_1.

Using (3.39) and (3.42), we obtain:

$$\rho_1 = \frac{\lambda_1\epsilon \cdot \Big(q_2 i(r_1, \alpha) + \bar{q}_2 i(r_1, \beta)\Big)}{\lambda_1\epsilon \cdot c(r_1)},$$

which yields:

$$\rho_1 = \frac{q_2 i(r_1, \alpha) + \bar{q}_2 i(r_1, \beta)}{c(r_1)}. \tag{3.43}$$

• Computing $I(U;Z)$.

For $z \in Z_2$, the transition probabilities of channel $U \to Z$ are defined by:

$$\begin{aligned} p(z|u_1) &= p_2(z|x_1)\alpha + p_2(z|x_2)\bar{\alpha}, \\ &\approx \epsilon\lambda_2(x_1,z)\alpha + \epsilon\lambda_2(x_2,z)\bar{\alpha}, \\ &\approx \lambda_1\epsilon(\alpha + \bar{\alpha}r_1)p_d(z|y_2), \end{aligned}$$

and:

$$\begin{aligned} p(z|u_2) &= p_2(z|x_1)\beta + p_2(z|x_2)\bar{\beta}, \\ &\approx \lambda_1\epsilon(\beta + \bar{\beta}r_1)p_d(z|y_2). \end{aligned}$$

Clearly, $U \to Z$ is a binary-input Class IIa VNC satisfying:

$$\frac{p(z|u_2)}{p(z|u_1)} = \frac{\beta + \bar{\beta}r_1}{\alpha + \bar{\alpha}r_1}, \quad \text{for all } z \in Z_2.$$

Therefore, applying (3.40):

$$I(U;Z) \approx a\lambda_1\epsilon(\alpha + \bar{\alpha}r_1) \cdot i\Big(\frac{\beta + \bar{\beta}r_1}{\alpha + \bar{\alpha}r_1}, q_2\Big). \tag{3.44}$$

- Computing ρ_2.

Using (3.41) and (3.44), we obtain:

$$\rho_2 = \frac{(\alpha + \bar{\alpha}r_1) \cdot i(\frac{\beta+\bar{\beta}r_1}{\alpha+\bar{\alpha}r_1}, q_2)}{c(r_1)}.$$

In Appendix F, we show that:

$$(\alpha + \bar{\alpha}r_1) \cdot i\Big(\frac{\beta + \bar{\beta}r_1}{\alpha + \bar{\alpha}r_1}, q_2\Big) = i(r_1, q_2\alpha + \bar{q}_2\beta) - q_2 i(r_1\alpha) - \bar{q}_2 i(r_1, \beta), \tag{3.45}$$

and therefore:

$$\rho_2 = \frac{i(r_1, q_2\alpha + \bar{q}_2\beta) - q_2 i(r_1\alpha) - \bar{q}_2 i(r_1, \beta)}{c(r_1)}. \tag{3.46}$$

- Computing $\rho_1 + \rho_2$.

Adding (3.43) and (3.46), we obtain:

$$\rho_1 + \rho_2 = \frac{i(r_1, q_2\alpha + \bar{q}_2\beta)}{c(r_1)}, \tag{3.46}$$

and therefore,

$$\rho_1 + \rho_2 \leq 1,$$

with equality if and only if: $q_2\alpha + \bar{q}_2\beta$ (which is q according to (3.9)) is the optimizing input probability for channel A_1.

Again, the capacity region of Class IIa BIVNDBCs when A_1 is a binary channel, in the limit as $\epsilon \to 0$, is completely enclosed in the time-sharing region (see (3.47)). We now have the theorem:

Theorem 3.7: *The capacity region of Class IIa binary-input "very noisy" degraded broadcast channel when A_1 is a binary channel ($N_1 = 2$), in the limit as $\epsilon \to 0$, is the time-sharing region.*

II-3.2.2. An achievable rate region for some Class IIa BIVNDBCs.

We have just shown that if channel A_1 is a binary channel ($N_1 = 2$), the capacity region is the time-sharing region. However, when $N_1 \geq 3$, it is possible to find Class IIa BIVNDBCs for which there exist achievable rates strictly above the time-sharing line, and therefore, we cannot state as a general property of "very noisy" degraded broadcast channels that the capacity region always coincides with the time-sharing region. We give such an example in the case where $N_1 = N_2 = 3$.

- Defining channel A_1.

Consider the following 2×3 Class IIb VNC, defined by its transition probability matrix:

$$\begin{pmatrix} 1-\epsilon & \epsilon & 0 \\ 1-\epsilon & 0 & \epsilon \end{pmatrix}.$$

This channel is known as the "very noisy" binary symmetric erasure channel.

- Defining channel D.

Consider the following 3×3 channel, with transition probability matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & d & \bar{d} \\ 0 & \bar{d} & d \end{pmatrix},$$

where d is a real number satisfying:

$$0 < d < 1, \quad \text{and } d \neq \frac{1}{2}.$$

- Defining Channel A_2.

The result of the cascade of A_1 and D is the 2×3 channel A_2, with transition probability matrix:

$$\begin{pmatrix} 1-\epsilon & d\epsilon & \bar{d}\epsilon \\ 1-\epsilon & \bar{d}\epsilon & d\epsilon \end{pmatrix}.$$

- Defining channel $U \to X$.

We will consider the case of a symmetric $U \to X$ channel:

$$\begin{pmatrix} \alpha & \bar{\alpha} \\ \bar{\alpha} & \alpha \end{pmatrix}.$$

- Computing the average mutual information of a symmetric 2×3 Class IIb VNC.

Consider a symmetric 2×3 Class IIb VNC. It has the following transition probability matrix:

$$\begin{pmatrix} 1-(\lambda+\mu)\epsilon & \lambda\epsilon & \mu\epsilon \\ 1-(\lambda+\mu)\epsilon & \mu\epsilon & \lambda\epsilon \end{pmatrix},$$

where either λ and μ are two non-negative real numbers with at least one of them being positive. Using the general expression for the average mutual information of Class II VNCs, and if q is the probability of the first input, we obtain:

$$\begin{aligned} I(q,\lambda,\mu) \approx & q\left(\epsilon\lambda \log \frac{\lambda}{q\lambda+\bar{q}\mu} + \epsilon\mu \log \frac{\mu}{q\mu+\bar{q}\lambda}\right) \\ & + \bar{q}\left(\epsilon\mu \log \frac{\mu}{q\lambda+\bar{q}\mu} + \epsilon\lambda \log \frac{\lambda}{q\mu+\bar{q}\lambda}\right), \end{aligned}$$

$$I(q,\lambda,\mu) \approx \epsilon\left((q\lambda+\bar{q}\mu)\log\frac{1}{q\lambda+\bar{q}\mu}+(q\mu+\bar{q}\lambda)\log\frac{1}{q\mu+\bar{q}\lambda}+\lambda\log\lambda+\mu\log\mu\right). \tag{3.48}$$

Due to symmetry of (3.48) in q and $\bar{q}$, $I(q,\lambda,\mu)$ is maximized by a uniform input probability. Therefore, the capacity of a symmetric 2×3 Class IIb VNC is:

$$C \approx \epsilon\cdot\left((\lambda+\mu)\log\frac{2}{\lambda+\mu}+\lambda\log\lambda+\mu\log\mu\right).$$

Note that, if $\lambda+\mu=1$, then:

$$C \approx \epsilon\cdot c(\lambda), \tag{3.49}$$

where:

$$c(\lambda) = \log 2 - \mathcal{H}(\lambda),$$

where $\mathcal{H}(\lambda)$ is the binary entropy function.

Also, if $\lambda = 1$ and $\mu = 0$, then:

$$I(q,1,0) \approx \epsilon\left(q\log\frac{1}{q}+\bar{q}\log\frac{1}{\bar{q}}\right) = \epsilon\cdot\mathcal{H}(q),$$

which yields:

$$I(q,1,0) \approx \epsilon\cdot\left(\log 2 - c(q)\right). \tag{3.50}$$

- Computing the capacity of channels A_1 and A_2.

For channel A_1, using (3.49), we obtain:

$$C_1 \approx \epsilon\cdot c(1). \tag{3.51}$$

For channel A_2, using (3.49), we obtain:

$$C_2 \approx \epsilon\cdot c(d). \tag{3.52}$$

- Computing $I(X;Y|U)$.

By definition,

$$I(X;Y|U) = q_2 I(X;Y|U=u_1) + \bar{q}_2 I(X;Y|U=u_2).$$

Since

$$I(X;Y|U=u_1) = I(\alpha,1,0) = I(\bar{\alpha},1,0) = I(X;Y|U=u_2),$$

then, using (3.50):

$$I(X;Y|U) \approx \epsilon \cdot \Big(\log 2 - c(\alpha)\Big). \tag{3.53}$$

- Computing ρ_1.

Using (3.51) and (3.53),

$$\rho_1 = \frac{\log 2 - c(\alpha)}{c(1)}. \tag{3.54}$$

- Computing $I(U;Z)$.

If we define the operation:

$$a \star b = ab + \bar{a}\bar{b},$$

then the transition probability matrix of the channel from U to Z is:

$$\begin{pmatrix} 1-\epsilon & (\alpha \star d)\epsilon & \overline{(\alpha \star d)}\epsilon \\ 1-\epsilon & \overline{(\alpha \star d)}\epsilon & (\alpha \star d)\epsilon \end{pmatrix}.$$

Therefore, if we assume $q_2 = 1/2$ (the optimizing input distribution since we are only dealing with symmetric channels), and if we use (3.49), we obtain:

$$I(U;Z) \approx \epsilon \cdot c(\alpha \star d). \tag{3.55}$$

- Computing ρ_2.

Using (3.52) and (3.55), we obtain:

$$\rho_2 = \frac{c(\alpha \star d)}{c(d)}. \tag{3.56}$$

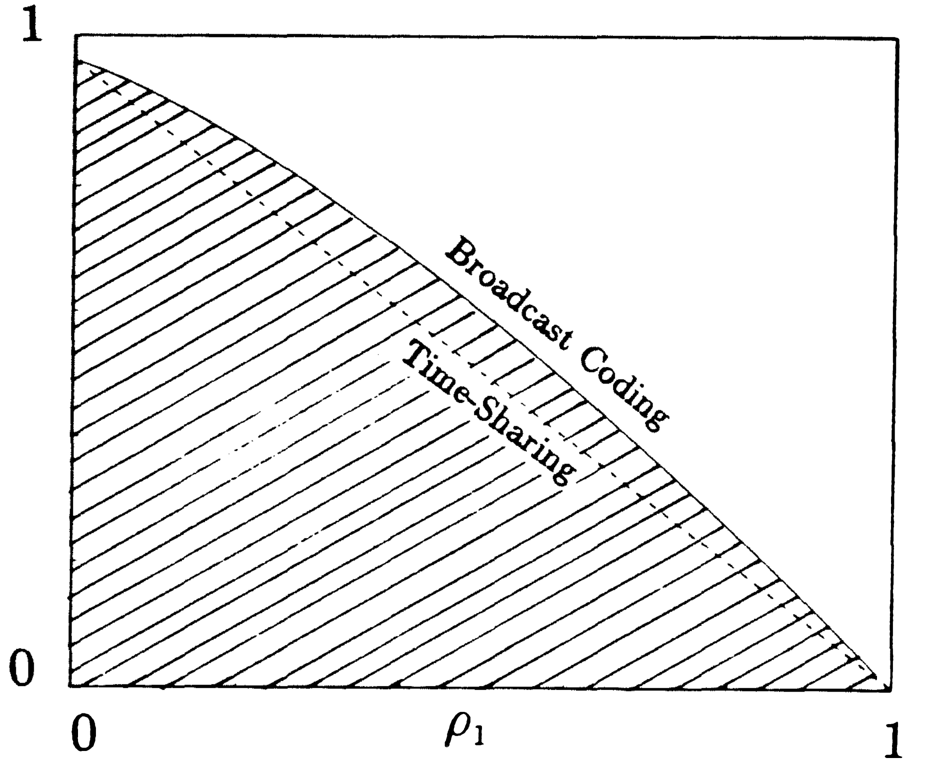


Figure 3.6: An achievable rate region for
the "very noisy" binary symmetric erasure channel.

• Computing $\rho_1 + \rho_2$.

Using (3.54) and (3.56), we obtain:

$$\rho_1 + \rho_2 = 1 - \frac{c(\alpha)}{c(1)} + \frac{c(\alpha \star d)}{c(d)}.$$

and therefore,

$$\rho_1 + \rho_2 - 1 = \frac{c(\alpha \star d)}{c(d)} - \frac{c(\alpha)}{c(1)}.$$

In figure 3.6, we plotted ρ_1 as a function of ρ_2 for $d = \frac{1}{4}$ as α varies from 0 to $\frac{1}{2}$. It is clear, in that example, that:

$$\rho_1 + \rho_2 \geq 1,$$

with equality if and only if:

$$\alpha = 0, \quad \text{or } \alpha = \frac{1}{2}, \quad \text{or } \alpha = 1.$$

And so we have found an example of Class IIb BIVNDBCs for which the set of rates achievable with broadcast coding is larger than with time-sharing.

II-4. Conclusion on the capacity region of BIVNDBCs.

We have shown the following results:

- The time-sharing region *is* the capacity region in the limit as $\epsilon \to 0$ for:
  - Class I BIVNDBCs,
  - Class IIb BIVNDBCs.
  - some Class IIa BIVNDBCs ($N_1 = 2$).
- The time-sharing region *is not necessarily* the capacity region in the limit as $\epsilon \to 0$ for:
  - some Class IIa BIVNDBCs (such as the "very noisy" binary symmetric channel).

What is remarkable here, is that in general, Class I and Class IIb VNCs share common properties (such as capacities of the same order in cascade, same capacity regions) whereas Class IIa VNCs usually stands out by behaving differently from Class I and Class IIb VNCs.

CHAPTER 4:

INFORMATION LOSS IN OPTIMAL CHANNEL REDUCTION AND DATA COMPRESSION

I - INTRODUCTION.

In many types of information channels encountered in real life, the set of channel outputs is far larger than the user would like and therefore has to be reduced. We may model this situation as the cascading of two channels : the *original channel* (with M inputs and N outputs, where N is large) followed by a *deterministic channel* which describes the decisions made by the user in order to reduce the number of outputs to K. The resulting channel is called the *reduced channel* as is described in figure 4.1. The input X to the original channel has a fixed probability distribution $q(x)$, and Y is the output. The original channel is fully defined by the transition probabilities $p(y|x)$.

In the reduction stage, the user has to find the optimal reduction with respect to some criterion. In the particular case where $K = M$, the reduction corresponds in fact to *estimating* the original value of X, and two classical criteria are *minimizing the average error probability* and *maximum likelihood* (these criteria are discussed in section V of this chapter).

In the more general case where K and M are not necessarily equal, there is another criterion known as *maximum mutual information* where we choose the optimal reduction $\hat{Y}$ that will maximize the average mutual information after reduction:

$$I(X;\hat{Y}) = \max_{\tilde{Y}} I(X;\tilde{Y}), \tag{4.1}$$

where $\tilde{Y}$ covers the range of all possible reductions from N to K outputs. Section V attempts a comparison of this and previous reduction criteria.

Clearly, due to the Data Processing Theorem [33], the mutual information after reduction is less than before reduction:

$$\frac{I(X;\hat{Y})}{I(X;Y)} \leq 1, \tag{4.2}$$

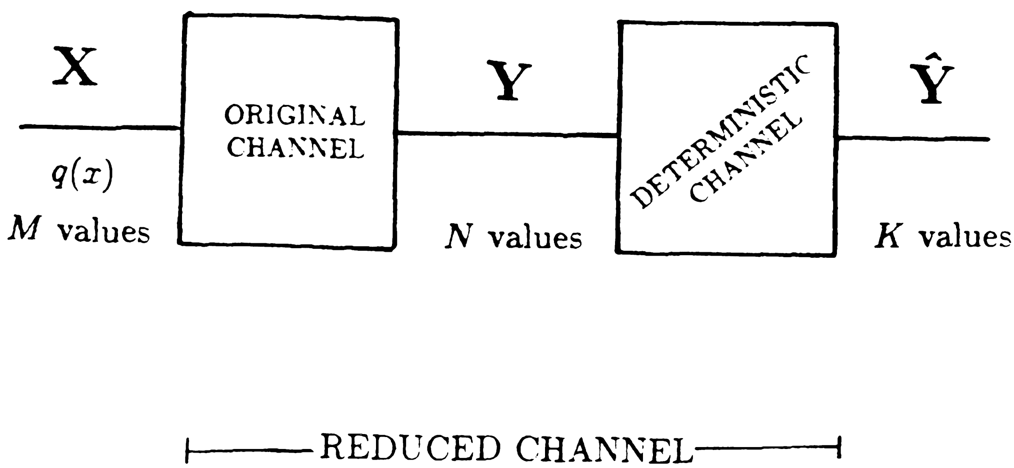


Figure 4.1: Channel Reduction or Data Compression:

Modeling the optimal reduction of the number N of outputs of a channel to K : the reduced channel is the cascade of the original channel and the deterministic channel corresponding to the optimal reduction.

where the ratio $\frac{I(X;\hat{Y})}{I(X;Y)}$ represents the fraction of mutual information that is left after reduction.

We wish, in this chapter, to answer two questions:

a. How do we find the optimal reduction $\hat{Y}$ and its corresponding deterministic channel?

b. What is, if there is one, the lower bound on the ratio

$$\frac{I(X;\hat{Y})}{I(X;Y)}$$

over all possible original channels (with positive capacity), for a fixed input distribution $q(x)$, and does that lower bound depend on the input distribution?

We will answer these questions in the case of binary-input channels ($M = 2$), and give conjectures for other channels ($M > 3$).

For binary-input channels, we will show that:

$$\frac{K-1}{N-1} < \frac{I(X;\hat{Y})}{I(X;Y)} \leq 1 \tag{4.3}$$

where the upper bound is achieved by *sufficiently reduceable* [31] channels and the lower bound can be approached arbitrarily close by a some Class IIa VNCs.

For that reason, the quantity

$$\frac{K-1}{N-1} \cdot I(X;Y)$$

is called the *essential mutual information*, as the amount of information that is never lost after an optimal reduction.

• Notations.

Let $\mathcal{X} = \{x_1, x_2\}$ be the set of input alphabet, $\mathcal{Y} = \{y_1, y_2, ..., y_N\}$ be the output alphabet and J be the set of output indices.

The mutual information may be expressed, as:

$$I(X;Y) = I(Y;X) = \sum_{i=1}^{2}\sum_{j=1}^{N} p(x_i, y_j) \log \frac{p(x_i|y_j)}{q(x_i)}.$$

We may also express the average mutual information as an average of the information provided by individual outputs y_j about the random variable X :

$$I(X;Y) = \sum_{j=1}^{N} p(y_j) I(X;y_j), \tag{4.4}$$

where,

$$I(X;y_j) = \sum_{i=1}^{2} p(x_i|y_j) \log \frac{p(x_i|y_j)}{q(x_i)}, \quad \text{for } j = 1, \ldots, N.$$

Now, let us define the probabilities:

$$q = q(x_1) = 1 - q(x_2),$$

$$r_j = p(y_j), \quad \text{for all } j \in J,$$

$$\alpha_j = p(x_1|y_j) = 1 - p(x_2|y_j), \quad \text{for all } j \in J.$$

Then, we have

$$I(X;y_j) = \alpha_j \cdot \log \frac{\alpha_j}{q} + \bar{\alpha}_j \cdot \log \frac{\bar{\alpha}_j}{\bar{q}}.$$

Clearly, for fixed q, $I(X;y_j)$ is a function of α_j alone and if we define the function of α :

$$F_q(\alpha) = \alpha \log \frac{\alpha}{q} + \bar{\alpha} \log \frac{\bar{\alpha}}{\bar{q}},$$

where $0 \leq \alpha \leq 1$, we have:

$$I(X;y_j) = F_q(\alpha_j),$$

and therefore:

$$I(X;Y) = \sum_{j \in J} r_j F_q(\alpha_j). \tag{4.5}$$

A little analysis shows that:

$$F_q'(\alpha) = \log \frac{\alpha}{\bar{\alpha}} \cdot \frac{\bar{q}}{q},$$

and,

$$F_q''(\alpha) = \frac{1}{\alpha\bar{\alpha}} > 0,$$

and therefore, $I(X; y_j)$ is a convex $\cup$ function of α_j independently of q. In figure 4.2, we plotted $F_q(\alpha)$ for $q = \frac{1}{2}$ and $q = 0.3$. We have therefore expressed the mutual information $I(X;Y)$ in (4.5) as a function of r_j and α_j. Note however, that because:

$$\sum_{j \in J} p(y_j) = 1,$$
$$q(x) = \sum_{j \in J} p(y_j) p(x|y_j),$$

the r_j's and the α_j's have to verify the following constraints:

$$\sum_{j \in J} r_j = 1, \tag{4.6}$$

$$\sum_{j \in J} r_j \alpha_j = q. \tag{4.7}$$

Furthermore, $r_j \geq 0$, $0 \leq \alpha_j \leq 1$, and $0 < q < 1$.

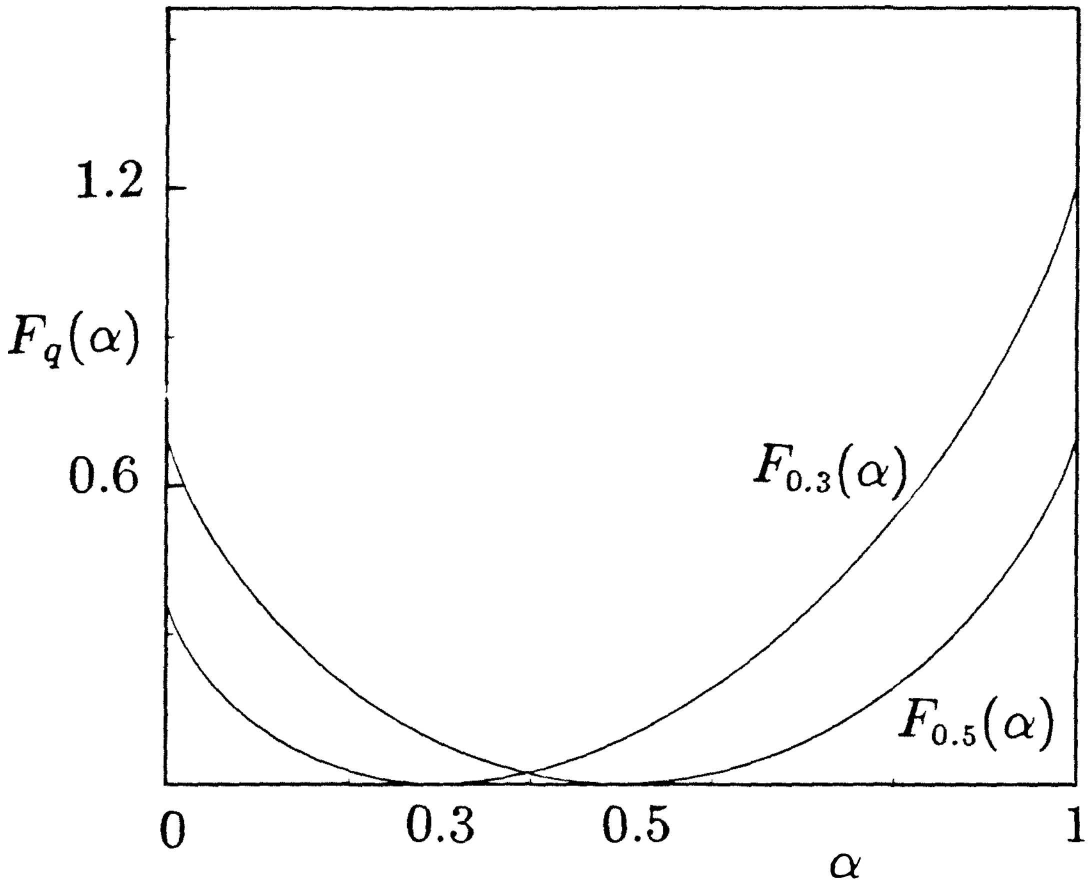


Figure 4.2: The function $F_q(\alpha)$ for $q = 0.5$ and $q = 0.3$.

II - ELEMENTARY REDUCTION

II-1. Definition of an Elementary Reduction.

By an elementary reduction, we mean that the reduced channel is obtained from the original channel by assigning two original outputs to one reduced output, while all other original outputs remain untouched, i.e., $K = N - 1$. Assume without loss of generality that we combine the first two outputs y_1 and y_2 into a reduced output which we denote y_{12}. Then if we define the probabilities:

$$r_{12} = p(y_{12}),$$

$$\alpha_{12} = p(x_1|y_{12}),$$

we want to find their expressions in terms of r_1, r_2, α_1, and α_2.

Theorem 4.1: If outputs y_1 and y_2, characterized respectively by (r_1, α_1) and (r_2, α_2) are merged into a single output y_{12}, characterized by (r_{12}, α_{12}), then we have:

$$\begin{aligned} r_{12} &= r_1 + r_2, \\ \alpha_{12} &= \frac{r_1\alpha_1 + r_2\alpha_2}{r_1 + r_2}. \end{aligned} \tag{4.6}$$

Proof of Theorem 4.1:

The elementary reduction amounts to adding the first two columns of the forward transition probability matrix:

$$\begin{cases} p(y_{12}|x_1) = p(y_1|x_1) + p(y_2|x_1), \\ p(y_{12}|x_2) = p(y_1|x_2) + p(y_2|x_2). \end{cases}$$

But we have, from Bayes' Law:

$$p(y|x) = \frac{p(y)p(x|y)}{q(x)},$$

which yields:

$$\begin{aligned} \frac{r_{12}\alpha_{12}}{q} &= \frac{r_1\alpha_1}{q} + \frac{r_2\alpha_2}{q}, \\ \frac{r_{12}\bar{\alpha}_{12}}{\bar{q}} &= \frac{r_1\bar{\alpha}_1}{\bar{q}} + \frac{r_2\bar{\alpha}_2}{\bar{q}}. \end{aligned}$$

Finally, after some calculation, we obtain:

$$r_{12} = r_1 + r_2,$$
$$\alpha_{12} = \frac{r_1\alpha_1 + r_2\alpha_2}{r_1 + r_2}.$$

So, an elementary reduction corresponds to *adding* the probabilities of the outputs that are combined and to *averaging* the backward probabilities with respect to the output probabilities.

Let us now present a theorem that can be found in [31], applied here to binary-input channels, on sufficient reductions:

Theorem 4.2: If $\tilde{Y}$ is any reduction of Y into $N-1$ outputs, and y_1 and y_2 are the outputs combined in the course of that reduction, then:

$$I(X;\tilde{Y}) \leq I(X;Y), \tag{4.7}$$

with equality if and only if:

$$\alpha_1 = \alpha_2.$$

If we compare the mutual information before and after an elementary reduction:

$$I(X;Y) = r_1 F_q(\alpha_1) + r_2 F_q(\alpha_2) + r_3 F_q(\alpha_3) + \cdots + r_N F_q(\alpha_N),$$
$$I(X;\tilde{Y}) = r_{12} F_q(\alpha_{12}) + r_3 F_q(\alpha_3) + \cdots + r_N F_q(\alpha_N),$$

which yields:

$$I(X;Y) - I(X;\tilde{Y}) = r_1 F_q(\alpha_1) + r_2 F_q(\alpha_2) - r_{12} F_q(\alpha_{12}). \tag{4.8}$$

Due to the $\cup$-convexity of the function $F_q(\alpha)$, the left hand of (4.8) is positive unless $\alpha_1 = \alpha_2$, in which case it is zero.

Theorem 4.2 is a direct proof of (4.2) since (4.7) also applies to the optimal reduction $\hat{Y}$. More generally, there is no loss of information in an optimal elementary reduction if and only if there exist a pair of outputs (y_1, y_2) such that $\alpha_1 = \alpha_2$ (the backward probabilities are equal).

II-2. Derivation of the lower bound

Finding the optimal elementary reduction turns out to be already a difficult task. Collins [32] showed that the optimal reduction can be found among $N-1$ particular elementary reductions, which we will call the *good elementary reductions* (Collins' result and its generalization will be discussed in detail in section IV). However, it is possible to derive a lower bound on the ratio $\frac{I(X;\hat{Y})}{I(X;Y)}$ which we give in the following theorem:

Theorem 4.3: If $\hat{Y}$ is the optimal reduction of Y into $N-1$ outputs, then:

$$\frac{N-2}{N-1} < \frac{I(X;\hat{Y})}{I(X;Y)}. \tag{4.9}$$

Proof of Theorem 4.3:

First, let us reorder the outputs of the original channel by increasing α_j:

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_N. \tag{4.10}$$

Now, let us define what we mean by good reductions.

Definition 4.1: A reduction $\tilde{Y}$ is called a *good reduction*, if under the convention specified in (4.10), it consists in the combining of outputs with two adjacent indices.

Naturally, if we limit ourselves to reductions to $N-1$ outputs, then there are exactly $N-1$ of them: the one that combines y_1 and y_2, the one that combines y_2 and y_3, ..., and the one that combines y_{N-1} and y_N. More generally, we will denote by $\tilde{Y}_k$ $(k = 1, 2, \cdots, N-1)$ to be the reduction obtained from the combination of y_k and y_{k+1} into $y_{k,k+1}$.

Then

$$\begin{aligned}
I(X;\tilde{Y}_1) &= r_{12}F_q(\alpha_{12}) + r_3F_q(\alpha_3) + \cdots + r_NF_q(\alpha_N),\\
I(X;\tilde{Y}_2) &= r_1F_q(\alpha_1) + r_{23}F_q(\alpha_{23}) + \cdots + r_NF_q(\alpha_N),\\
&\vdots\\
I(X;\tilde{Y}_{N-1}) &= r_1F_q(\alpha_1) + \cdots + r_{N-2}F_q(\alpha_{N-2}) + r_{N-1,N}F_q(\alpha_{N-1,N}).
\end{aligned}$$

If we add all these mutual informations, we obtain:

$$\begin{aligned}
\sum_{k=1}^{N-1} I(X;\tilde{Y}_k) =&(N-3)\Big(r_1F_q(\alpha_1) + \cdots + r_NF_q(\alpha_N)\Big)\\
&+\Big(r_1F_q(\alpha_1) + r_{12}F_q(\alpha_{12}) + \cdots + r_{N-1,N}F_q(\alpha_{N-1,N})\Big).
\end{aligned}$$

We recognize $I(X;Y)$ in the first term of the sum, and therefore:

$$\begin{aligned}
\sum_{k=1}^{N-1} I(X;\tilde{Y}_k) = (N-3)I(X;Y) + \Big(r_1F_q(\alpha_1)+r_{12}F_q(\alpha_{12}) + \cdots +\\
r_{N-1,N}F_q(\alpha_{N-1,N}) + r_NF_q(\alpha_N)\Big).
\end{aligned}$$

Now since we have from Theorem 4.1:

$$r_{k,k+1} = r_k + r_{k+1}, \quad \text{for } k = 1,2,\cdots,N-1 \tag{4.11}$$

then:

$$\begin{aligned}
r_1F_q(\alpha_1) + r_{12}F_q(\alpha_{12}) + \cdots + r_{N-1,N}F_q(\alpha_{N-1,N}) + r_NF_q(\alpha_N) =\\
r_1\Big(F_q(\alpha_1) + F_q(\alpha_{12})\Big) + r_2\Big(F_q(\alpha_{12}) + F_q(\alpha_{23})\Big) + \cdots +\\
r_{N-1}\Big(F_q(\alpha_{N-2,N-1}) + F_q(\alpha_{N-1,N})\Big) + r_N\Big(F_q(\alpha_{N-1,N}) + F_q(\alpha_N)\Big).
\end{aligned}$$

Now, let us compare:

$$F_q(\alpha_{i-1,i}) + F_q(\alpha_{i,i+1}) \text{ and } F_q(\alpha_i), \quad \text{for } i = 2,3,\ldots,N-1.$$

From (4.10) and (4.11) we can write:

$$\alpha_{i-1} \leq \alpha_{i-1,i} \leq \alpha_i \leq \alpha_{i,i+1} \leq \alpha_{i+1}.$$

Define λ_i so that:

$$\alpha_i = \lambda_i \alpha_{i-1,i} + (1-\lambda_i)\alpha_{i,i+1}.$$

Using the convexity of $F_q(\alpha)$:

$$F_q(\alpha_i) \leq \lambda_i F_q(\alpha_{i-1,i}) + (1-\lambda_i) F_q(\alpha_{i,i+1}) \leq F_q(\alpha_{i-1,i}) + F_q(\alpha_{i,i+1}).$$

Since also, $F_q(\alpha_{12}) \geq 0$ and $F_q(\alpha_{N-1,N}) \geq 0$, then:

$$r_1 F_q(\alpha_1) + r_{12} F_q(\alpha_{12}) + \ldots + r_{N-1,N} F_q(\alpha_{N-1,N}) + r_N F_q(\alpha_N) \geq$$

$$r_1 F_q(\alpha_1) + r_2 F_q(\alpha_2) + \ldots + r_N F_q(\alpha_N) = I(X;Y).$$

Therefore:

$$\sum_{k=1}^{N-1} I(X;\tilde{Y}_k) \geq (N-3) I(X;Y) + I(X;Y) = (N-2) \cdot I(X;Y).$$

But since for all $\tilde{Y}_k$, $I(X;\tilde{Y}_k) \leq I(X;\hat{Y})$, then:

$$(N-2) I(X;Y) \leq \sum_{k=1}^{N-1} I(X;\tilde{Y}_k) \leq (N-1) \cdot I(X;\hat{Y}),$$

which yields:

$$\frac{N-2}{N-1} \leq \frac{I(X;\hat{Y})}{I(X;Y)}.$$

When does equality hold ? If equality holds, then necessarily

$$F_q(\alpha_{12}) = F_q(\alpha_{N-1,N}) = 0,$$

since those two terms were among the ones dropped to derive the previous inequality and if we assume $r_1 \neq 0$ and $r_N \neq 0$. But $F_q(\alpha) = 0 \Leftrightarrow \alpha = q$. Therefore, $\alpha_{12} = \alpha_{N-1,N} = q$. But because of (4.10) and (4.11), this is possible if and only if:

$$\alpha_j = q, \quad \text{for all } j \in J,$$

which is equivalent to saying $I(X;Y) = 0$, which we have discarded. Equality therefore may never hold.

II-3. Approaching the lower bound

We have shown that the lower bound in (4.9) may not be achieved exactly since we excluded from our study channels for which $I(X;Y) = 0$. However, it may be approached arbitrarily close by some Class IIa VNCs. A necessary condition for this to happen is that $F_q(\alpha_{12}) \approx 0$ and $F_q(\alpha_{N-1,N}) \approx 0$, which yields, using the same argument as in II.2.:

$$I(X;Y) \approx 0.$$

Therefore we will express equations (4.5), (4.6) and (4.7) for the particular case of a VNC, and then give an example of Class IIa that approaches the bound arbitrarily close.

Consider a channel for which:

$$\alpha_j = q + \epsilon_j, \quad \text{for all } j \in J, \tag{4.12}$$

where:

$$|\epsilon_j| \ll q \text{ and } |\epsilon_j| \ll \bar{q}.$$

Then constraint (4.7) becomes:

$$\sum_{j=1}^{N} r_j \epsilon_j = 0. \tag{4.13}$$

Since $F_q(q) = F_q'(q) = 0$, then:

$$F_q(\alpha_j) = \frac{\epsilon_j^2}{2} \cdot \frac{1}{q\bar{q}} + O(\epsilon_j^4). \tag{4.14}$$

If we let $S = \frac{1}{2q\bar{q}}$, our very noisy channel has to satisfy (4.5), (4.13) and:

$$I(X;Y) = S \cdot \sum_{j \in J} r_j \epsilon_j^2 + O(\sum_{j \in J} r_j \epsilon_j^4). \tag{4.15}$$

Now let us define the positive number ϵ and the integer L such that:

$$|\epsilon| \ll \min(q, \bar{q}),$$

$$L = \lfloor \frac{N}{2} \rfloor,$$

and consider the VNCs for which:

$$r_1 = 1 - O(\epsilon),$$

$$\epsilon_1 = O(\epsilon^{L+1}),$$

and for $l = 1, 2, \ldots, L$:

$$r_{2l} = r_{2l+1} = \epsilon^{2L+1-2l} + O(\epsilon^{2L+2-2l}),$$

$$\epsilon_{2l} = -\epsilon_{2l+1} = \epsilon^{l} + O(\epsilon^{l+1})$$

(in the case where N is even, we simply drop y_{2L+1}). Examples of proper channels satisfying all the necessary conditions are given in Appendix G.

Let us compute the nutual information:

$$I(X;Y) = S \cdot \sum_{j=1}^{N} r_j \epsilon_j^2 + O(\sum_{j=1}^{N} r_j \epsilon_j^4).$$

But since $r_j \epsilon_j^2 = \epsilon^{2L+1} + O(\epsilon^{2L+2})$, except for $j = 1$ where $r_1 \epsilon_1^2 = O(\epsilon^{2L+2})$, then:

$$I(X;Y) = S \cdot (N-1)\epsilon^{2L+1} + O(\epsilon^{2L+2}). \tag{4.16}$$

Now, let us find an approximation for $I(X;\hat{Y})$ by looking at the possible values for $I(X;\tilde{Y})$ for all $\tilde{Y}$. Our outputs consist of one isolated output y_1 and L pairs of outputs $\{y_{2l}, y_{2l+1}\}$ for $l = 1, 2, \ldots, L$ (if N is even, then one element of the last pair will be missing). Since an elementary reduction consists in the combination of 2 outputs y_{j_1} and y_{j_2}, then:

$$I(X;Y) - I(X;\tilde{Y}) \approx S \cdot (r_{j_1}\epsilon_{j_1}^2 + r_{j_2}\epsilon_{j_2}^2 - r_{j_1,j_2}\epsilon_{j_1,j_2}^2). \tag{4.17}$$

Three cases of reduction may occur:

- a. Combining two outputs from the *same* pair: $j_1 = 2l$ and $j_2 = 2l + 1$.

Then $r_{j_1} = r_{j_2} = \epsilon^{2L+1-2l} + O(\epsilon^{2L+2-2l})$, $\epsilon_{j_1} = -\epsilon_{j_2} = \epsilon^{l} + O(\epsilon^{l+1})$, $r_{j_1,j_2} = 2\epsilon^{2L+1-2l} + O(\epsilon^{2L+2-2l})$ and $\epsilon_{j_1,j_2} = O(\epsilon^{l+1})$.

As a consequence, using (4.17):

$$I(X;Y) - I(X;\tilde{Y}) = S\Big(\epsilon^{2L+1} + O(\epsilon^{2L+2}) + \epsilon^{2L+1} + O(\epsilon^{2L+2}) - O(\epsilon^{2L+3})\Big),$$

which yields:

$$I(X;\tilde{Y}) = S \cdot (N-3)\epsilon^{2L+1} + O(\epsilon^{2L+2}).$$

• b. Combining two outputs from two *different* pairs: $j_1 \in \{2l, 2l+1\}$ and $j_2 \in \{2l', 2l'+1\}$ where $l > l'$.

Then $r_{j_1} = \epsilon^{2L+1-2l} + O(\epsilon^{2L+2-2l})$, $\epsilon_{j_1} = \pm\epsilon^{l} + O(\epsilon^{l+1})$, $r_{j_2} = \epsilon^{2L+1-2l'} + O(\epsilon^{2L+2-2l'})$, $\epsilon_{j_2} = \pm\epsilon^{l'} + O(\epsilon^{l'+1})$, $r_{j_1,j_2} = \epsilon^{2L+1-2l} + O(\epsilon^{2L+2-2l})$ and $\epsilon_{j_1,j_2} = \pm\epsilon^{l} + O(\epsilon^{l+1})$.

Therefore:

$$\begin{aligned} I(X;Y) - I(X;\tilde{Y}) &= S\left(\epsilon^{2L+1} + O(\epsilon^{2L+2}) + \epsilon^{2L+1} + O(\epsilon^{2L+2}) - \epsilon^{2L+1} + O(\epsilon^{2L+2})\right) \\ &= S \cdot \epsilon^{2L+1} + O(\epsilon^{2L+2}), \end{aligned}$$

which yields:

$$I(X;\tilde{Y}) = S \cdot (N-2)\epsilon^{2L+1} + O(\epsilon^{2L+2}).$$

• c. Combining output y_1 and any other output : $j_1 = 1$ and $j_2 \in \{2l, 2l+1\}$.

Then $r_{j_1} = 1 - O(\epsilon)$, $\epsilon_{j_1} = O(\epsilon^{L+1})$, $r_{j_2} = \epsilon^{2L+1-2l} + O(\epsilon^{2L+2-2l})$, $\epsilon_{j_2} = \pm\epsilon^{l} + O(\epsilon^{l+1})$, $r_{j_1,j_2} = 1 - O(\epsilon)$ and $\epsilon_{j_1,j_2} = \left(O(\epsilon^{L+1}) \pm \epsilon^{2L+1-l} + O(\epsilon^{2L+2-l})\right) / \left(1 - O(\epsilon)\right) = O(\epsilon^{L+1})$.

Therefore,

$$\begin{aligned} I(X;Y) - I(X;\tilde{Y}) &= S\left(O(\epsilon^{2L+2}) + \epsilon^{2L+1} + O(\epsilon^{2L+2}) - O(\epsilon^{2L+2})\right) \\ &= S \cdot \epsilon^{2L+1} + O(\epsilon^{2L+2}), \end{aligned}$$

which yields:

$$I(X;\tilde{Y}) = S \cdot (N-2)\epsilon^{2L+1} + O(\epsilon^{2L+2}).$$

Since the three cases investigated cover all possible reductions, then it is clear that the optimal reduction for this class of $2 \times N$ channels can only be found in reductions of type (b) or (c). Also, these reductions of type (b) and (c) are the same in the limit as ϵ goes to zero. Since the optimal reduction is among them, we call all reductions of type (b) and (c) *asymptotically optimal.*

Therefore we finally have:

$$I(X;\hat{Y}) = S \cdot (N-2)\epsilon^{2L+1} + O(\epsilon^{2L+2}).$$

It is crucial to note that in cases (b) and (c) :

$$\begin{aligned} r_{j1,j2} &= r_{j1}\Big(1 + O(\epsilon)\Big), \\ \epsilon_{j1,j2} &= \epsilon_{j1}\Big(1 + O(\epsilon)\Big). \end{aligned}$$

This means that the optimal reduction of a $2 \times N$ channel of the type we defined is a $2 \times (N-1)$ channel of the same type, since it corresponds to simply eliminating one of the two outputs combined. This important remark will be used later in the general case while using induction arguments. We have therefore found a class of channels for all N such that :

$$\frac{I(X;\hat{Y})}{I(X;Y)} = \frac{N-2}{N-1} + O(\epsilon). \tag{4.18}$$

III - GENERAL REDUCTION

We will now look at the case of a general reduction from N to K outputs. We will show the following theorem in this section:

Theorem 4.4: If $\hat{Y}$ is the optimal reduction of Y into K outputs, then:

$$\frac{K-1}{N-1} < \frac{I(X;\hat{Y})}{I(X;Y)}.$$

III-1. Derivation of the lower bound.

The reduction of the number of outputs from N to K can be viewed as the cascade of $N-K$ elementary reductions, from N to $N-1$ outputs, from $N-1$ to $N-2$ outputs, $\ldots$, and from $K+1$ to K outputs. This will allow us to induce properties of a general reduction from those obtained for elementary reductions.

Given a binary input X and an N-valued output Y, consider the following $N-K$ optimal elementary reductions, where $\hat{Y}_k$ denotes the optimal elementary reduction from $k+1$ to k outputs:

$$\begin{aligned} Y &\text{ is optimally reduced into } \hat{Y}_{N-1}, \\ \hat{Y}_{N-1} &\text{ is optimally reduced into } \hat{Y}_{N-2}, \\ &\vdots \\ \hat{Y}_{K+1} &\text{ is optimally reduced into } \hat{Y}_K. \end{aligned}$$

It is important to note that $\hat{Y}_K$ is not necessarily the optimal reduction of Y into K values: a greedy approach does not necessarily work and we give such an example in Appendix H. Let us therefore define by $\hat{Y}$ the truly optimal reduction of Y into K values.

Clearly, $I(X;\hat{Y}_K) \leq I(X;\hat{Y})$ by definition of $\hat{Y}$. And therefore,

$$\frac{I(X;\hat{Y}_K)}{I(X;Y)} \leq \frac{I(X;\hat{Y})}{I(X;Y)} \tag{4.19}$$

Also, we may rewrite $I(X;\hat{Y}_K)/I(X;Y)$ as

$$\frac{I(X;\hat{Y}_K)}{I(X;Y)} = \frac{I(X;\hat{Y}_K)}{I(X;\hat{Y}_{K+1})} \times \frac{I(X;\hat{Y}_{K+1})}{I(X;\hat{Y}_{K+2})} \times \cdots \times \frac{I(X;\hat{Y}_{N-1})}{I(X;Y)} \tag{4.20}$$

Applying Theorem 4.3 to the $N-K$ optimal elementary reductions defined above yields:

$$\frac{k-1}{k} < \frac{I(X;\hat{Y}_k)}{I(X;\hat{Y}_{k+1})}, \quad \text{where } k = K, K+1, \ldots, N-1. \tag{4.21}$$

The result follows from applying (4.21) to (4.20):

$$\frac{K-1}{K} \times \frac{K}{K+1} \times \cdots \times \frac{N-2}{N-1} < \frac{I(X;\hat{Y}_K)}{I(X;Y)} \leq \frac{I(X;\hat{Y})}{I(X;Y)}$$

which yields:

$$\frac{K-1}{N-1} < \frac{I(X;\hat{Y})}{I(X;Y)} \tag{4.22}$$

III-2. Approaching the lower bound

We now want to know if there is a channel that can approach the lower bound arbitrarily close. We have previously found a class of channels that does so in an elementary reduction. We also showed that the optimal elementary reduction of any channel from that class was still a channel of that class. Also, any general reduction can be decomposed in a sequence of elementary reductions. Since the lower bound in (4.22) can only be approached asymptotically if each of the elementary reductions achieve the lower bound in (4.21) asymptotically, then they each have to be asymptotically optimal. Is this possible? Yes. To achieve this, we need only choose a channel of the class previously defined and use the fact that the optimal general reduction of such a channel can be decomposed in a sequence of asymptotically optimal elementary reductions.

IV - FINDING THE OPTIMAL REDUCTION

Instead of having to search for the optimal reduction among all possible partitions of N elements into K non-empty sets (there is a number of them that is exponential in N), Collins [32] showed that the search may be limited to $\binom{N-1}{K-1}$ of them (a number that is polynomial in N), i.e., the partitions of N elements into K non-empty sets which only contain consecutive elements (according to the ordering defined in (4.10)). The following proofs are based on his original proof for $K = 2$.

However, the optimal reduction from N to K outputs is not necessarily the composition of the $N - K$ optimal elementary reductions from N to $N - 1$, $N - 1$ to $N - 2$, ... , and $K + 1$ to K which would make the search linear in N.

IV-1. Reduction from N to 2 outputs.

Consider the ordering of the outputs defined in (4.10) and the case where $K = 2$. Then we are looking for the optimal partition of the set of indices $J = \{1, 2, \ldots, N\}$ into $K = 2$ sets J_1 and J_2. Define the *good partitions* as the ones for which :

$$j_1 \in J_1 \text{ and } j_2 \in J_2 \Rightarrow j_1 < j_2 \tag{4.23}$$

We want to show the following theorem:

Theorem 4.5: The optimal partition of J into two sets corresponding to the optimal reduction of Y into two values is a good partition (defined in (4.23)).

We will show in two steps that, given any arbitrary partition, there is always a good partition that achieves a value for $I(X; \tilde{Y})$ at least as high, and therefore the optimal partition lies among the good ones.

- Step 1

We start with an arbitrary partition into 2 sets J_1 and J_2. Define the probabilities:

$$r_{J_1} = \sum_{j \in J_1} r_j,$$
$$r_{J_2} = \sum_{j \in J_2} r_j = 1 - r_{J_1},$$

and the backward probabilities:

$$\alpha_{J_1} = \frac{\sum_{j \in J_1} r_j \alpha_j}{r_{J_1}},$$
$$\alpha_{J_2} = \frac{\sum_{j \in J_2} r_j \alpha_j}{r_{J_2}}.$$

Then,

$$I(X;\tilde{Y}) = r_{J_1} F_q(\alpha_{J_1}) + r_{J_2} F_q(\alpha_{J_2}).$$

We assume without loss of generality that $\alpha_{J_1} \leq \alpha_{J_2}$. We now wish to create an artificial partition into 2 sets J_1' and J_2' where one of the outputs is "split" in two, and such that:

$$j_1 \in J_1' \text{ and } j_2 \in J_2' \Rightarrow \alpha_{j_1} \leq \alpha_{j_2} \tag{4.24}$$

Define the integer j_0 such that :

$$\sum_{j=1}^{j_0} r_j \leq r_{J_1} < \sum_{j=1}^{j_0+1} r_j$$

and the real number $\lambda_0 \in [0,1]$ such that :

$$\sum_{j=1}^{j_0} r_j + \lambda_0 r_{j_0+1} = r_{J_1}$$

Then $J_1' = \{1, 2, \ldots, j_0 + 1\}$, $J_2' = \{j_0 + 1, j_0 + 2, \ldots, K\}$, and y_{j_0+1} has been split into two: in J_1', α_{j_0+1} is assigned a probability of $\lambda_0 r_{j_0+1}$, and in J_2', α_{j_0+1} is assigned a probability of $(1 - \lambda_0) r_{j_0+1}$.

Therefore:

$$r'_{J_1} = \sum_{j=1}^{j_0} r_j + \lambda_0 r_{j_0+1}$$

$$r'_{J_2} = (1-\lambda_0) r_{j_0+1} + \sum_{j=j_0+2}^{N} r_j$$

$$\alpha'_{J_1} = \frac{\sum_{j=1}^{j_0} r_j \alpha_j + \lambda_0 r_{j_0+1} \alpha_{j_0+1}}{r'_{J_1}}$$

$$\alpha'_{J_2} = \frac{(1-\lambda_0) r_{j_0+1} \alpha_{j_0+1} + \sum_{j=j_0+2}^{N} r_j \alpha_j}{r'_{J_2}}$$

Since $r'_{J_1} = r_{J_1}$ and $r'_{J_2} = r_{J_2}$, because of (4.10),

$$\alpha'_{J_1} \leq \alpha_{J_1} \text{ and } \alpha'_{J_2} \geq \alpha_{J_2}$$

under the constraint: $r_{J_1}\alpha'_{J_1} + r_{J_2}\alpha'_{J_2} = r_{J_1}\alpha_{J_1} + r_{J_2}\alpha_{J_2}$.

This is equivalent to saying that:

$$\exists \mu_0 \geq 0 \ : \ \alpha'_{J_1} = \alpha_{J_1} - \mu_0 r_{J_2} \text{ and } \alpha'_{J_2} = \alpha_{J_2} + \mu_0 r_{J_1}.$$

Define the function of μ :

$$h(\mu) = r_{J_1} F_q(\alpha_{J_1} - \mu r_{J_2}) + r_{J_2} F_q(\alpha_{J_2} + \mu r_{J_1}).$$

Note that $h(\mu_0) = r_{J'_1} F_q(\alpha_{J'_1}) + r_{J'_2} F_q(\alpha_{J'_2})$. Then we have:

$$h'(\mu) = r_{J_1} r_{J_2} \Big(F'_q(\alpha'_{J_2}) - F'_q(\alpha'_{J_1}) \Big).$$

Since $F'_q(\alpha)$ is a monotonously increasing function of α, and $\alpha'_{J_1} \leq \alpha'_{J_2}$,, then $h'(\mu) \geq 0$, and therefore $h(\mu_0) \geq h(0) = I(X;\tilde{Y})$ which yields:

$$I(X;\tilde{Y}) \leq r'_{J_1} F_q(\alpha'_{J_1}) + r'_{J_2} F_q(\alpha'_{J_2}) \tag{4.25}$$

To summarize this first step, we have taken two sets J_1 and J_2, and performed the following operation : we replaced them by two artificial sets of the same probability ($r'_{J_1} = r_{J_1}$ and $r_{J'_2} = r_{J_2}$), increased the mutual information while satisfying

(4.24). However we had to split an element y_{j_0+1} into two using a parameter λ_0. We would have obtained a good partition if only λ_0 were 0 or 1: we call this an artificial good partition.

- Step 2.

However we can still increase the mutual information by replacing λ_0 by 0 or 1. Define the function of λ (where $\lambda \in [0,1]$):

$$\begin{aligned} f(\lambda) =& \Big(\sum_{j=1}^{j_0} r_j + \lambda r_{j_0+1}\Big) F_q\Big(\frac{\sum_{j=1}^{j_0} r_j\alpha_j + \lambda r_{j_0+1}\alpha_{j_0+1}}{\sum_{j=1}^{j_0} r_j + \lambda r_{j_0+1}}\Big) \\ &+ \Big((1-\lambda) r_{j_0+1} + \sum_{j=j_0+2}^{N} r_j\Big) F_q\Big(\frac{(1-\lambda) r_{j_0+1}\alpha_{j_0+1} + \sum_{j=j_0+2}^{N} r_j\alpha_j}{(1-\lambda) r_{j_0+1} + \sum_{j=j_0+2}^{N} r_j}\Big). \end{aligned}$$

Note that:

$$f(\lambda_0) = r'_{J_1} F_q(\alpha'_{J_1}) + r'_{J_2} F_q(\alpha'_{J_2}).$$

We show in Appendix I that $f(\lambda)$ is a convex $\cup$ function of λ, and therefore, $f(\lambda_0) \le \max(f(0), f(1))$, which yields, using (4.25):

$$\begin{aligned} I(X;\tilde{Y}) \le \max\Big(&(\sum_{j=1}^{j_0} r_j) F_q\Big(\frac{\sum_{j=1}^{j_0} r_j\alpha_j}{\sum_{j=1}^{j_0} r_j}\Big) + (\sum_{j=j_0+1}^{N} r_j) F_q\Big(\frac{\sum_{j=j_0+1}^{N} r_j\alpha_j}{\sum_{j=j_0+1}^{N} r_j}\Big), \\ &(\sum_{j=1}^{j_0+1} r_j) F_q\Big(\frac{\sum_{j=1}^{j_0+1} r_j\alpha_j}{\sum_{j=1}^{j_0+1} r_j}\Big) + (\sum_{j=j_0+2}^{N} r_j) F_q\Big(\frac{\sum_{j=j_0+2}^{N} r_j\alpha_j}{\sum_{j=j_0+2}^{N} r_j}\Big)\Big). \end{aligned} \tag{4.26}$$

This yields our final result, since the right hand of (4.26) is the value of $I(X;\tilde{Y})$ for one of two good partitions.

IV-2. Reduction from N to K outputs (Generalization)

Consider again the ordering of the outputs defined in (4.10). We are looking for the optimal partition of the set of indices $J = \{1, 2, \ldots, N\}$ into K sets $J_1, J_2, \ldots, J_K$. Define the good partitions as the ones for which:

$$k < k', j_k \in J_k \text{ and } j_{k'} \in J_{k'} \Rightarrow j_k < j_{k'}. \tag{4.27}$$

We will now show a generalization of Theorem 4.5 to the case of more than two outputs:

Theorem 4.6: The optimal partition of J into K sets corresponding to the optimal reduction of Y into K values is a good partition (defined in (4.27)).

We will again show in 2 steps that, given any arbitrary partition, its mutual information $I(X;\tilde{Y})$ can always be upper bounded by the mutual information corresponding to a good partition, and therefore the optimal partition lies among the good ones.

- Step 1.

For any 2 sets J_{k_1} and J_{k_2} where $k_1 < k_2$, consider the following operation. Replace J_{k_1} and J_{k_2} by 2 artificial sets J'_{k_1} and J'_{k_2} such that $r_{J'_{k_1}} = r_{J_{k_1}}, r_{J'_{k_2}} = r_{J_{k_2}}$, and for all $j_1 \in J'_{k_1}$ and $j_2 \in J'_{k_2}$, $\alpha_{j_1} \leq \alpha_{j_2}$. As a result of this operation, J'_{k_1} and J'_{k_2} may have at most one element in common, its probability being split among the 2 sets so as to satisfy the constraints above. This leads to the introduction of a splitting parameter that specifies the percentage of the probability assigned to the element common to the 2 sets. This operation has been described in detail in the case $K = 2$ and has been shown to yield an increase in mutual information.

Take any partition $\tilde{Y}$ into K sets. Then:

$$I(X;\tilde{Y}) = \sum_{k=1}^{K} r_{J_k} F_q(\alpha_{J_k}).$$

Consider the artificial partition $\tilde{Y}'$ into K sets $J'_1, J'_2, \ldots, J'_K$ such that :

$$r_{J'_1} = r_{J_1}, r_{J'_2} = r_{J_2}, \cdots, r_{J'_K} = r_{J_K} \tag{4.28}$$

$$k_1 < k_2, j_1 \in J'_{k_1} \text{ and } j_2 \in J'_{k_2} \Rightarrow \alpha_{j_1} \leq \alpha_{j_2} \tag{4.29}$$

To achieve (4.28), we need to split some elements in 2, and in this case, there are at most $K-1$ of them along with the same number of splitting parameters. This good partition is clearly unique, and we want to show that $I(X;\tilde{Y}') \geq I(X;\tilde{Y})$.

If we apply the elementary operation described previously to J_1 and J_2, we obtain $(J_1)'$ and $(J_2)'$. We then apply it again to $(J_1)'$ and $J_3, \ldots$, and so on until we reach J_K. The resulting first set is indeed the same J_1' defined in (4.28) and (4.29), and we define λ_1 to be the splitting parameter associated with J_1' and its complementary. We have therefore replaced $J_1, J_2, \ldots, J_K$ by $J_1', (J_2)', \ldots, (J_K)'$.

We are now left with $K-1$ sets $(J_2)', (J_3)', \ldots, (J_K)'$. We apply the same procedure to $(J_2)'$ to obtain the following sets: $J_2', (J_3)'', (J_4)'', \ldots, (J_K)''$. The first set is naturally the same J_2' defined in (4.28) and (4.29), and we define λ_2 to be the splitting parameter associated with J_2' and its complementary.

Continuing in this way, we obtain K sets $J_1', J_2', \ldots, J_K'$ as defined in (4.28) and (4.29) and $K-1$ splitting parameters $\lambda_1, \lambda_2, \ldots, \lambda_{K-1}$. We have therefore described a procedure to transform any partition $\tilde{Y}$ (defined by $J_1, J_2, \ldots, J_K$) into an artificial good partition $\tilde{Y}'$ (defined by $J_1', J_2', \ldots, J_K'$), which only involves elementary operations on pairs of sets. Since each of those elementary operations yields an increase in mutual information, $I(X;\tilde{Y}') \geq I(X;\tilde{Y})$.

- Step 2.

We have upperbounded $I(X;\tilde{Y})$ by a function of $K-1$ parameters $\lambda_1, \lambda_2, \ldots, \lambda_{K-1}$. Let one of the parameters λ_k vary. The upper bound is now a function of the type $f(\lambda_k)$ as described for $K=2$. Therefore, the mutual information is increased by replacing λ_k by either 0 or 1. This is true for all values of k and we end up with $I(X;\tilde{Y})$ for an arbitrary partition being upper bounded by the value of $I(X;\tilde{Y})$ for a good partition.

V - A COMPARISON OF MINIMUM ERROR, MAXIMUM LIKELIHOOD AND MAXIMUM MUTUAL INFORMATION DECODING

We now restrict ourselves to the case $K = M = 2$ and wish to estimate which of the 2 inputs (x_1 and x_2) was sent when we observe any of the N outputs $(y_1, y_2, \ldots, y_N)$. Let us briefly describe three decoding techniques before we compare them.

V-1. Minimum Error Decoding.

One strategy, known as Minimum Error Decoding (MED), is to minimize the average error probability which is the same as choosing for each output y the input x that maximizes $p(x|y)$.

Since for all $j \in \{1, 2, \ldots, N\}$, $p(x_1|y_j) + p(x_2|y_j) = 1$, and if $\alpha_j = p(x_1|y_j)$, then MED is the same as setting a threshold at $\frac{1}{2}$, choosing x_1 for all y_j such that $\alpha_j \geq \frac{1}{2}$ and choosing x_2 for all y_j such that $\alpha_j < \frac{1}{2}$.

V-2. Maximum Likelihood Decoding.

Another strategy is Maximum Likelihood Decoding (MLD), where for all y we choose the input x that maximizes $p(y|x)$, and it is also based on a threshold-type decision :

Let $r_j = p(y_j), q = q(x_1)$, and $\alpha_j = p(x_1|y_j)$. Then,

$$p(y_j|x_1) = r_j \cdot \frac{\alpha_j}{q},$$
$$p(y_j|x_2) = r_j \cdot \frac{\bar{\alpha}_j}{\bar{q}}.$$

Therefore:

$$p(y_j|x_1) \leq r_j \Leftrightarrow \alpha_j \leq q \Leftrightarrow 1 - \alpha_j \geq 1 - q \Leftrightarrow p(y_j|x_2) \geq r_j.$$

This yields:

$$\alpha_j \leq q \Leftrightarrow p(y_j|x_2) \geq p(y_j|x_1).$$

Therefore, MLD is the same as setting a threshold at q and choosing x_1 for all y_j such that $\alpha_j \geq q$ and x_2 for all y_j such that $\alpha_j < q$.

Now, take any channel defined by its transition probabilities $p(y|x)$. Then by definition, the MLD estimates are independent of the input probabilities $q(x)$ (here q), while the MED estimates are dependent.

V-3. Maximum Mutual Information Decoding.

Let us now introduce a third decoding strategy : Maximum Mutual Information Decoding (MMID). This consists in finding the reduction $\tilde{Y}$ of the channel to 2 outputs that maximizes $I(X;\tilde{Y})$. We already know that MMID is also a threshold-type decision on the backward probabilities $\alpha_j = p(x_1|y_j)$ which is dependent on all parameters of the channel.

V-4. Comparison.

First, we need to make sure that, in general, these decoding techniques are not equivalent: we give in Appendix E the example of a channel for which these 3 strategies give different optimal solutions.

We believe that although they have been shown to be different, MLD and MMID give the same solution in many cases and when not, they yield nearly the same average error probabilities.

We will prove this only for output-symmetric channels with an even number of outputs and with a uniform input probability (and in this case, we have MED=MLD). This example will allow us to study the extent of the difference between MMID and more classical and less complex practically such as MLD or MED.

Let $N = 2n$ be the number of outputs. Because of the symmetry of the class of channels we are studying,

$$r_j = r_{N+1-j}, \quad \text{and } \alpha_j = 1 - \alpha_{N+1-j}.$$

Also by convention, we will assume that:

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_N.$$

The optimal MLD partition is clearly

$$(1, 2, \ldots, n)|(n+1, \ldots, 2n).$$

Let $1 \leq m \leq n$ be such that the optimal MMID partition is :

$$(1, 2, \ldots, m)|(m+1, \ldots, n, n+1, \ldots, 2n).$$

Define the probabilities:

$$\hat{r}_1 = \sum_{j=1}^{m} r_j, \quad \text{and } \hat{r}_2 = \sum_{j=m+1}^{n} r_j.$$

Note that:

$$\hat{r}_1 + \hat{r}_2 = \frac{1}{2}.$$

Also define the average a posteriori probabilities:

$$\hat{\alpha}_1 = (\sum_{j=1}^{m} r_j \alpha_j)/\hat{r}_1,$$

$$\hat{\alpha}_2 = (\sum_{j=m+1}^{n} r_j \alpha_j)/\hat{r}_2.$$

Clearly, if $m = n$ then MMID is equivalent to MLD. We are interested in what happens when MMID is not equivalent to MLD, i.e., $m < n$.

Also, by definition of MMID,

$$I(X; \tilde{Y}_{MMID}) \geq I(X; \tilde{Y}_{MLD}).$$

Let us compute $I(X; \tilde{Y}_{MLD})$ and $I(X; \tilde{Y}_{MMID})$.

$$\begin{aligned} I(X; \tilde{Y}_{MLD}) &= \frac{1}{2} F_{\frac{1}{2}}\Big(\hat{r}_1 \hat{\alpha}_1 + \hat{r}_2 \hat{\alpha}_2\Big) + \frac{1}{2} F_{\frac{1}{2}}\Big(\hat{r}_1(1-\hat{\alpha}_1) + \hat{r}_2(1-\hat{\alpha}_2)\Big) \\ &= F_{\frac{1}{2}}(\hat{r}_1 \hat{\alpha}_1 + \hat{r}_2 \hat{\alpha}_2) \\ &= F_{\frac{1}{2}}\Big((\frac{1}{2} - \hat{r}_2)\hat{\alpha}_1 + \hat{r}_2 \hat{\alpha}_2\Big). \end{aligned}$$

Now, let us compute $I(X;\tilde{Y}_{MMID})$:

$$\begin{aligned}
I(X;\tilde{Y}_{MMID}) &= \hat{r}_1 F_{\frac{1}{2}}(\hat{\alpha}_1) + (\hat{r}_2+\hat{r}_2+\hat{r}_1) F_{\frac{1}{2}}\Big(\frac{\hat{r}_2\hat{\alpha}_2 + \hat{r}_2(1-\hat{\alpha}_2) + \hat{r}_1(1-\hat{\alpha}_1)}{\hat{r}_2+\hat{r}_2+\hat{r}_1}\Big) \\
&= \hat{r}_1 F_{\frac{1}{2}}(\hat{\alpha}_1) + (2\hat{r}_2+\hat{r}_1) F_{\frac{1}{2}}\Big(\frac{\hat{r}_2 + \hat{r}_1(1-\hat{\alpha}_1)}{2\hat{r}_2+\hat{r}_1}\Big) \\
&= (\frac{1}{2}-\hat{r}_2) F_{\frac{1}{2}}(\hat{\alpha}_1) + (\frac{1}{2}+\hat{r}_2) F_{\frac{1}{2}}\Big(\frac{\frac{1}{2}-(\frac{1}{2}-\hat{r}_2)\hat{\alpha}_1}{\frac{1}{2}+\hat{r}_2}\Big)
\end{aligned}$$

If we fix $\hat{r}_2$ and $\hat{\alpha}_1$, and let $\hat{\alpha}_2$ vary, then:

$$\begin{aligned}
\frac{\partial I(X;\tilde{Y}_{MLD})}{\partial \hat{\alpha}_2} &= \hat{r}_2 F'_{\frac{1}{2}}\Big((\frac{1}{2}-\hat{r}_2)\hat{\alpha}_1 + \hat{r}_2\hat{\alpha}_2\Big) < 0 \\
\frac{\partial I(X;\tilde{Y}_{MMID})}{\partial \hat{\alpha}_2} &= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
& I(X;\tilde{Y}_{MMID}) \geq I(X;\tilde{Y}_{MLD}) \\
\Leftrightarrow\ & (\frac{1}{2}-\hat{r}_2)\hat{\alpha}_1 + \hat{r}_2\hat{\alpha}_2 \geq F^{-1}_{\frac{1}{2}}\Big((\frac{1}{2}-\hat{r}_2) F_{\frac{1}{2}}(\hat{\alpha}_1) + (\frac{1}{2}+\hat{r}_2) F_{\frac{1}{2}}\Big(\frac{\frac{1}{2}-(\frac{1}{2}-\hat{r}_2)\hat{\alpha}_1}{\frac{1}{2}+\hat{r}_2}\Big)\Big) \\
\Leftrightarrow\ & \hat{\alpha}_2 \geq \hat{\alpha}_{2\min}(\hat{r}_2,\hat{\alpha}_1)
\end{aligned}$$

where:

$$\hat{\alpha}_{2\min}(\hat{r}_2,\hat{\alpha}_1) = \frac{1}{\hat{r}_2}\Big(F^{-1}_{\frac{1}{2}}\Big((\frac{1}{2}-\hat{r}_2) F_{\frac{1}{2}}(\hat{\alpha}_1) + (\frac{1}{2}+\hat{r}_2) F_{\frac{1}{2}}\Big(\frac{\frac{1}{2}-(\frac{1}{2}-\hat{r}_2)\hat{\alpha}_1}{\frac{1}{2}+\hat{r}_2}\Big)\Big) - (\frac{1}{2}-\hat{r}_2)\hat{\alpha}_1\Big)$$

Computer simulations show that for all $0 \leq \hat{r}_2 \leq \frac{1}{2}$ and $0 \leq \hat{\alpha}_1 \leq \frac{1}{2}$,

$$\hat{\alpha}_{2\min}(\hat{r}_2,\hat{\alpha}_1) \geq \hat{\alpha}_{2\min}(\hat{r}_2,0) \geq \hat{\alpha}_{2\min}(0.25253156,0) = 0.36796998$$

Therefore,

$$I(X;\tilde{Y}_{MMID}) \geq I(X;\tilde{Y}_{MLD}) \Rightarrow \hat{\alpha}_2 \geq 0.36796998 \tag{4.30}$$

This means that for MMID and MLD to be different, some outputs should be hard to distinguish statistically:

$$p(x_1|y_j) \simeq p(x_2|y_j).$$

Consequently, this leads us to believe that the difference between MMID and MLD is not very significant. Let us quantify this further by comparing the average error probabilities:

$$\begin{aligned} P_{eMLD} &= 2\hat{r}_1\hat{\alpha}_1 + 2\hat{r}_2\hat{\alpha}_2 = (1-2\hat{r}_2)\hat{\alpha}_1 + 2\hat{r}_2\hat{\alpha}_2 \\ P_{eMMID} &= \frac{1}{2}(2\hat{r}_1\hat{\alpha}_1) + \frac{1}{2}\Big(2\hat{r}_1\hat{\alpha}_1 + 2\hat{r}_2\hat{\alpha}_2 + 2\hat{r}_2(1-\hat{\alpha}_2)\Big) \\ &= (1-2\hat{r}_2)\hat{\alpha}_1 + \hat{r}_2 \end{aligned}$$

Clearly,

$$P_{eMLD} \geq 2\hat{r}_2\hat{\alpha}_2. \tag{4.31}$$

It is also easy to verify that:

$$2\hat{\alpha}_2 \leq \frac{P_{eMLD}}{P_{eMMID}} \leq 1 \tag{4.32}$$

The lower bound is achieved for $\hat{\alpha}_1 = 0$ or $\hat{r}_2 = \frac{1}{2}$.

Therefore, combining (4.30), (4.31) and (4.32), $I(X;\tilde{Y}_{MMID}) \geq I(X;\tilde{Y}_{MLD})$ yields :

$$0.7359 \cdot \hat{r}_2 \leq P_{eMLD}$$

$$0.7359 \leq \frac{P_{eMLD}}{P_{eMMID}} \leq 1 \tag{4.33}$$

The lower bound of (4.33) is clearly achieved for

$$\hat{\alpha}_1 = 0 \text{ and } \hat{r}_2 = 0.25253156$$

V-5. Conclusion

In summary this means that, at least for the symmetric channels of the type we just studied :

1. In some cases, MMID=MLD (especially for low error probabilities as (4.33)).
2. If MMID≠MLD, their respective error probabilities are approximately the same (as shown in (4.34)).

It is natural therefore, to conjecture that the same is true for other channels. Due to the enormous complexity of proving such results for all channels, our example can be considered as an indication that MED and MLD are reasonably good approximations of MMID.

VI - CONCLUSION

VI-1. Summary of results.

We have shown the following results for $M = 2$ and any value of the input probability distribution $q(x)$:

$$\frac{K-1}{N-1} < \frac{I(X;\hat{Y})}{I(X;Y)} \leq 1.$$

More precisely, we have found Class IIa VNCs which approach the lower bound arbitrarily close, and therefore we may write:

$$\inf_{p(y|x)} \frac{I(X;\hat{Y})}{I(X;Y)} = \frac{K-1}{N-1},$$

as well as:

$$\sup_{p(y|x)} \frac{I(X;\hat{Y})}{I(X;Y)} = 1.$$

VI-2. Generalization.

We do not know if, for $M > 2$, these above results are still valid. The difficulty is now (for $M > 2$) that good partitions no longer exist in the same way we defined them for $M = 2$. However, in the following theorem, we present a bound which we do not yet know to be tight:

Theorem 4.7: If $\hat{Y}$ is the optimal reduction of Y into K outputs for $M > 2$, then we have:

$$\frac{K(K-1)}{N(N-1)} < \frac{I(X;\hat{Y})}{I(X;Y)}.$$

Proof of Theorem 4.7:

In the proof of theorem 4.3 (generalized in Theorem 4.4) that takes place in section II-2., we add the mutual information after reduction of $N-1$ good reductions to find that they add up to at least $N-2$ times the mutual information before reduction by using the $\cup$-convexity of the function $F_q(\alpha)$.

Instead, let us add the mutual information after reduction of the N reductions $\tilde{Y}_k$ where $\tilde{Y}_k$ is defined as the elementary reduction corresponding to the combination of y_k and y_{k+1} into $y_{k,k+1}$ (k and $k+1$ are modulo N).

We have for $\tilde{Y}_k$:

$$I(X;Y) - I(X;\tilde{Y}_k) = r_k I(X;y_k) + r_{k+1} I(X;y_{k+1}) - r_{k,k+1} I(X;y_{k,k+1}),$$

where $r_{k,k+1} = r_k + r_{k+1}$.

Therefore, if we add these quantities for all k, we obtain:

$$N \cdot I(X;Y) - \sum_{k=1}^{N} I(X;\tilde{Y}_k) = 2 \cdot I(X;Y) - \sum_{k=1}^{N} r_{k,k+1} I(X;y_{k,k+1}),$$

which yields:

$$\sum_{k=1}^{N} I(X;\tilde{Y}_k) = (N-2) \cdot I(X;Y) + \sum_{k=1}^{N} r_{k,k+1} I(X;y_{k,k+1}).$$

Since $I(X;\hat{Y}) \geq I(X;\tilde{Y}_k)$ and $I(X;y_{k,k+1}) \geq 0$ for all k, then:

$$I(X;\hat{Y}) \geq \frac{N-2}{N} I(X;Y).$$

Using the same generalization argument than in III-1., we obtain:

$$I(X;\hat{Y}) \geq \frac{K(K-1)}{N(N-1)} I(X;Y).$$

Furthermore, if we let $M-2$ of the input probabilities go to zero, then the previous results we derived for $M=2$ in the limit, still hold for this particular case. We have no evidence that the same ratio of $(K-1)/(N-1)$ can be attained for $M>2$ for any input distribution, but we believe that, contrary to the case $M=2$, an attainable lower bound will depend on the input distribution.

CONCLUSION

1. Summary of results.

We have shown the following results:

- There exist two major classes of VNCs associated with different properties and behavior (which we denoted Class I and Class II VNCs), where Class II VNCs can be divided into two subclasses (Class IIa and Class IIb VNCs).
- The limiting value of capacity of VNCs can be computed, sometimes directly or through the use of an algorithm:
  - The capacity of binary-input Class I VNCs can be computed directly (using a uniform input distribution), while in general the capacity is computed using the Simplex Method.
  - The capacity of Class IIa VNCs is computed using a generalized version of the AB algorithm, while the capacity of Class IIb VNCs is computed using a similar but slightly more sophisticated algorithm, still based on a generalization of the AB algorithm.
- Coding exponent-rate functions have fundamental differences between Class I and Class II VNCs:
  - The coding exponent-rate functions of Class I VNCs depend only on the value of the capacity.
  - The coding exponent-rate functions of Class II VNCs depend on many other channel parameters than the capacity.
- When comparing binary Class I and Class II VNCs, we find that binary Class IIb VNCs perform slightly better than binary Class I VNCs, which in turn perform far better than Class IIa VNCs (the performance criteria used were capacity in

cascade and largest average error probability of channels with initially the same capacity).

- Binary Class IIb VNCs achieve the maximally asymmetric input probability distribution $(1/e,\ 1 - 1/e)$ at capacity of all binary channels.
- For binary-input "very noisy" two-receiver degraded broadcast channels (BIVNDBCs), the capacity region is:
  - the time-sharing region for Class I BIVNDBCs, Class IIb BIVNDBCs and some Class IIa BIVNDBCs (when the strong channel is a binary channel),
  - strictly larger than the time-sharing region for some Class IIb BIVNDBCs.
- The smallest fraction of mutual information left after reduction relative to the mutual information before reduction is achieved by some Class IIa VNCs.

2. Possible extensions.

We have seen throughout this thesis how VNCs have interesting limiting properties, and how bounds for the more general case of noisy channels are often achieved by VNCs, particularly Class II VNCs, whose properties and behavior had not been much noticed previous to our work. There are, we believe, more bounds to discover and achieve using Class II VNCs. Furthermore, some of the results we have presented here, we hope, can be generalized:

1. What is the capacity region of the general "very noisy" degraded broadcast channel (not limited to binary-input VNCs). Is it still the time- sharing region?
2. How much do we really lose, by signalling at a uniform input probability distribution instead of at capacity (see conjecture 3)?
3. What is the lower bound on the fraction of mutual information left after reduction for channels with more than two inputs?

APPENDICES

Appendix A: "Extremely Noisy" Channels (ENCs)

Our definition of VNCs is based on equation (1.4):

$$p(y|x) = w(y) + \epsilon \cdot \lambda(x, y) + O(\epsilon^2), \quad \text{for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}.$$

But let us now assume that the dependency of the output of the channel on its input is far less than in (1.4). This new assumption can be expressed by saying that $\lambda(x, y)$ is only a function of y and not of x, and therefore that the dependency on x is present in higher order terms of equation (1.4).

We shall call such channels "extremely noisy" channels (ENCs) because they result in capacities of higher orders than the VNCs we have studied so far. Let us now consider for simplicity only ENCs for which the dependency on the input is present in the ϵ^2 term. Then the transition probabilities of such ENCs will be characterized by the equation:

$$p(y|x) = w(y) + \epsilon \cdot \lambda(y) + \epsilon^2 \cdot \mu(x, y) + O(\epsilon^3).$$

The partitioning of ENCs into two classes is adequate here, and we need only look at the zero-capacity matrix Ω_Y to determine which class an ENC belongs to. If $Y_2 = \{y \in \mathcal{Y} \ : \ w(y) = 0\}$, then:

- Class I ENCs are characterized by $\mathcal{Y}_2 = \emptyset$
- Class II ENCs are characterized by $\mathcal{Y}_2 \neq \emptyset$

A further partitioning of Class II ENCs into two new classes is also appropriate, as we will see. If $\mathcal{Y}_2' = \{y \in \mathcal{Y}_2 \mid \exists x \ : \ \lambda(x, y) \neq 0\}$ and if $\mathcal{Y}_2'' = \{y \in \mathcal{Y}_2 / \lambda(x, y) = 0, \ \text{for all } x\}$, then:

- Class II-1 ENCs are characterized by $\mathcal{Y}_2' = \emptyset$
- Class II-2 ENCs are characterized by $\mathcal{Y}_2' \neq \emptyset$

We will now directly present the expressions for the average mutual information of these Class I, II-1 and II-2 ENCs, the derivations of which are left to the rigourous reader:

- for Class I ENCs:

$$I(X;Y) = \frac{\epsilon^4}{2} \cdot \sum_{y\in\mathcal{Y}} \frac{1}{w(y)} \left(\sum_{x\in\mathcal{X}} q(x) \Big(\mu(x,y) - \mu(y)\Big)^2 \right) + O(\epsilon^5),$$
$$\text{where } \mu(y) = \textstyle\sum_{\S} q(x)\mu(x,y).$$

example:

$$P_{Y|X} \approx \begin{pmatrix} \frac{1}{2} - \epsilon - \epsilon^2 & \frac{1}{2} + \epsilon + \epsilon^2 \\ \frac{1}{2} - \epsilon & \frac{1}{2} + \epsilon \end{pmatrix} \text{ and } I(X;Y) \approx 2q\bar{q}\epsilon^4.$$

- for Class II-1 ENCs:

$$I(X;Y) = \frac{\epsilon^3}{2} \cdot \sum_{y\in\mathcal{Y}_2} \frac{1}{\lambda(y)} \left(\sum_{x\in\mathcal{X}} q(x) \Big(\mu(x,y) - \mu(y)\Big)^2 \right) + O(\epsilon^4).$$

example:

$$P_{Y|X} \approx \begin{pmatrix} 1 - \epsilon - \epsilon^2 & \epsilon + \epsilon^2 \\ 1 - \epsilon & \epsilon \end{pmatrix} \text{ and } I(X;Y) \approx \frac{q\bar{q}}{2}\epsilon^3.$$

- for Class II-2 ENCs:

$$I(X;Y) = \epsilon^2 \cdot \sum_{x\in\mathcal{X}} q(x) \sum_{y\in\mathcal{Y}_2''} \mu(x,y) \log \frac{\mu(x,y)}{\mu(y)} + O(\epsilon^3).$$

example:

$$P_{Y|X} \approx \begin{pmatrix} 1 - \epsilon & \epsilon - \epsilon^2 & \epsilon^2 \\ 1 - \epsilon & \epsilon & 0 \end{pmatrix} \text{ and } I(X;Y) \approx (q \log \frac{1}{q})\epsilon^2.$$

Interestingly enough, the expressions of the average mutual information of ENCs are similar to the ones we obtained for VNCs, quadratic for Class I and Class II-1

ENCs (as for Class I VNCs), "average-mutual-information"-like for Class II-2 ENCs (as for Class II VNCs). The algorithms developed to compute the capacity of VNCs therefore can be used for ENCs. Also, we can conjecture that properties of Class I VNCs will apply to Class I and Class II-1 ENCs, as properties of Class II VNCs will apply to Class II-2 ENCs.

As for ENCs for which the dependency of the outputs on the inputs is present only in terms higher than ϵ^2, we can conjecture a generalization of the previous results. However, it seems that such a generalization would serve essentially mathematical purposes rather than practical ones stemming from real life examples of such channels.

Appendix B: Capacity of some Class I VNC's

Example 1

$$P_{Y|X} \approx \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} + \epsilon \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0.27777778 \\ 0.44444444 \\ 0.27777778 \end{pmatrix} \text{ and } C \approx 4.0074861\epsilon^2 \text{ bits}$$

Example 2

$$P_{Y|X} \approx \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} + \epsilon \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0.41975308 \\ 0.17901235 \\ 0.40123457 \end{pmatrix} \text{ and } C \approx 5.4880296\epsilon^2 \text{ bits}$$

Example 3

$$P_{Y|X} \approx \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} + \epsilon \begin{pmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ 3 & -4 & 1 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0. \\ 0.5 \\ 0.5 \end{pmatrix} \text{ and } C \approx 25.968511\epsilon^2 \text{ bits}$$

Example 4

$$P_{Y|X} \approx \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} + \epsilon \begin{pmatrix} -1 & 0 & 1 \\ -1 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0.453125 \\ 0.078125 \\ 0.468750 \end{pmatrix} \text{ and } C \approx 8.1715147\epsilon^2 \text{ bits}$$

Example 5

$$P_{Y|X} \approx \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} + \epsilon \begin{pmatrix} 2 & -2 \\ -1 & 1 \\ 3 & -3 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0. \\ 0.5 \\ 0.5 \end{pmatrix} \text{ and } C \approx 11.541560\epsilon^2 \text{ bits}$$

Example 6

$$P_{Y|X} \approx \begin{pmatrix} 0.33 & 0.33 & 0.34 \\ 0.33 & 0.33 & 0.34 \\ 0.33 & 0.33 & 0.34 \\ 0.33 & 0.33 & 0.34 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0. \\ 0.25944444 \\ 0.25944444 \\ 0.48111111 \end{pmatrix}$$

and $C \approx 7.7895601\epsilon^2$ bits

Example 7

$$P_{Y|X} \approx \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.1 & 0.3 \\ 0.1 & 0.2 & 0.3 & 0.1 & 0.3 \\ 0.1 & 0.2 & 0.3 & 0.1 & 0.3 \\ 0.1 & 0.2 & 0.3 & 0.1 & 0.3 \end{pmatrix} + \epsilon \begin{pmatrix} 2 & 1 & 0 & -1 & -2 \\ -3 & 0 & 3 & 0 & 0 \\ -4 & 1 & 1 & 1 & 1 \\ -1 & 2 & -3 & 4 & -2 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0.40429688 \\ 0. \\ 0.34179687 \\ 0.25390625 \end{pmatrix}$$

and $C \approx 87.761599\epsilon^2$ bits

Appendix C: Capacity of some Class II VNCs

Example 1

$$P_{Y|X} \approx \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -2 \\ 1 & 2 & -3 \\ 2 & 1 & -3 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0.61068519 \\ 0. \\ 0.19465740 \\ 0.19465740 \end{pmatrix} \text{ and } C \approx 1.6849876\epsilon \text{ bits}$$

Example 2

$$P_{Y|X} \approx \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 1 & 5 & -6 \\ 2 & 4 & -6 \\ 3 & 3 & -6 \\ 4 & 2 & -6 \\ 5 & 1 & -6 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0.53112264 \\ 0.23443868 \\ 0. \\ 0. \\ 0. \\ 0.23443868 \end{pmatrix} \text{ and } C \approx 4.0586823\epsilon \text{ bits}$$

Example 3

$$P_{Y|X} \approx \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 1 & 5 & -6 \\ 3.82 & 3.82 & -7.64 \\ 5 & 1 & -6 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0.53112264 \\ 0.23443868 \\ 0. \\ 0.23443868 \end{pmatrix} \text{ and } C \approx 4.0586823\epsilon \text{ bits}$$

Example 4

$$P_{Y|X} \approx \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 1 & 5 & -6 \\ 3.83 & 3.83 & -7.66 \\ 5 & 1 & -6 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0.63212056 \\ 0. \\ 0.36787944 \\ 0. \end{pmatrix} \text{ and } C \approx 4.0654518\epsilon \text{ bits}$$

Example 5

$$P_{Y|X} \approx \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 1 & 5 & -6 \\ 3 & 4 & -7 \\ 5 & 3 & -8 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0.60427503 \\ 0.09038845 \\ 0. \\ 0.30533652 \end{pmatrix} \text{ and } C \approx 4.3064777\epsilon \text{ bits}$$

Example 6

$$P_{Y|X} \approx \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & -2 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0.47151777 \\ 0.52848223 \end{pmatrix} \text{ and } C \approx 0.12295138\epsilon \text{ bits}$$

Example 7

$$P_{Y|X} \approx \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & -1 \\ 0.5 & -0.5 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0.47151777 \\ 0.52848223 \end{pmatrix} \text{ and } C \approx 0.06147569\epsilon \text{ bits}$$

Example 8

$$P_{Y|X} \approx \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & 2 & 3 & 4 & -10 \\ 9 & 1 & 8 & 4 & -22 \\ 1 & 4 & 7 & 4 & -16 \end{pmatrix} \Rightarrow \hat{\mathbf{q}} = \begin{pmatrix} 0.38285302 \\ 0.40628260 \\ 0.21086438 \end{pmatrix}$$

and $C \approx 3.84044717\epsilon$ bits

Appendix D: Proof of inequality (2.9)

Consider the function:

$$f(\beta) = \int_0^1 \log \frac{1}{\beta + \theta\delta} d\theta - \log \frac{1}{\beta + \delta e^{-1}}.$$

We intend to show that:

$$f(\beta) \leq 0, \quad \text{for all } \beta \geq 0.$$

Define $x = \frac{\beta}{\delta}$. Then,

$$\begin{aligned}
f(\beta) &= \int_0^1 \log \frac{\delta^{-1}}{x+\theta} d\theta - \log \frac{\delta^{-1}}{x + e^{-1}} \\
&= \int_0^1 \log \delta^{-1} d\theta + \int_0^1 \log \frac{1}{x+\theta} d\theta - \log \delta^{-1} - \log \frac{1}{x+e^{-1}} \\
&= \int_0^1 \log \frac{1}{x+\theta} d\theta - \log \frac{1}{x+e^{-1}}.
\end{aligned}$$

Clearly, $f(\beta)$ does not depend on the value of δ. We may therefore, without loss of generality, let δ equal to any positive number, such as 1 for example.

We will therefore study the function:

$$f(x) = \int_0^1 \log \frac{1}{x+\theta} d\theta - \log \frac{1}{x+e^{-1}}.$$

Let us compute the first and second derivatives of $f(x)$:

$$\begin{aligned}
f'(x) &= \int_0^1 \frac{-1}{x+\theta} d\theta + \frac{1}{x+e^{-1}} \\
&= \log \frac{x}{x+1} + \frac{1}{x+e^{-1}}.
\end{aligned}$$

$$\begin{aligned}
f''(x) &= \frac{1}{x} - \frac{1}{x+1} + \frac{-1}{(x+e^{-1})^2} \\
&= \frac{e^{-2} - x(1-2e^{-1})}{x(x+1)(x+e^{-1})^2}.
\end{aligned}$$

Clearly,

$$f''(x) \geq 0 \Leftrightarrow x \leq \frac{e^{-2}}{1-2e^{-1}} = \frac{1}{e(e-2)} = b = 0.5122,$$

and therefore:

$$\begin{cases} f''(x) > 0 & \text{for } 0 < x < b \\ f''(x) \leq 0 & \text{for } b \leq x. \end{cases}$$

Since

$$\lim_{x \to +\infty} f'(x) = 0,$$

and

$$f''(x) \leq 0 \quad \text{for } x \geq b,$$

therefore:

$$f'(x) \geq 0 \quad \text{for } x \geq b.$$

Also, since

$$f''(x) > 0 \quad \text{for } 0 < x < b,$$

then $f'(x)$ is monotone increasing for all x between 0 and b. Therefore $f'(x) = 0$ has a unique solution $a = 0.2358$ which is the solution to:

$$\log \frac{a}{a+1} + \frac{1}{a + e^{-1}} = 0.$$

Consequently,

$$\begin{cases} f'(x) < 0 & \text{for } 0 < x < a \\ f'(x) \geq 0 & \text{for } a \leq x. \end{cases}$$

This yields

$$f(x) \leq \min\Big(f(0),\ \lim_{x \to +\infty} f(x)\Big).$$

But since

$$f(0) = \int_0^1 \log \frac{1}{\theta} d\theta - \log e = 1 - 1 = 0,$$

and

$$\lim_{x \to +\infty} f(x) = 0,$$

we can conclude:

$$f(x) \leq 0 \quad \text{for all } x \geq 0.$$

Appendix E: Proof of lower bound in (2.13)

Consider the following 3×3 channel which depends on a parameter α such that $0 \leq \alpha \leq 1$:

$$P_{Y|X} = \begin{pmatrix} \alpha & \frac{\bar{\alpha}}{2} & \frac{\bar{\alpha}}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

If we compute the optimizing probability vector $\hat{\mathbf{q}}_0 = (\hat{q}_1,\ \hat{q}_2,\ \hat{q}_3)$, we find that the expression for the average mutual information is:

$$I(X;Y) = q_1\Big(\alpha \log \frac{1}{q_1} + \bar{\alpha} \log \frac{\bar{\alpha}}{q_1\bar{\alpha} + \bar{q}_1}\Big) + \bar{q}_1\Big(\log \frac{2}{q_1\bar{\alpha} + \bar{q}_1}\Big). \qquad (A.1)$$

where we assumed for reasons of symmetry that $q_2 = q_3$. By differentiating (A.1) with respect to q_1, we find:

$$\hat{q}_1 = \begin{cases} \frac{1}{\alpha + \left(\frac{2}{\bar{\alpha}^{\bar{\alpha}}}\right)^{\frac{1}{\alpha}}} & \text{for } \alpha \neq 0, \\ 0 & \text{for } \alpha = 0. \end{cases}$$

We plotted $\hat{q}_1$ as a function of α in figure 2.4, where clearly, as α increases monotonically from 0 to 1, $\hat{q}_1$ increases monotonically from 0 to $\frac{1}{3}$. This example generalizes to any number of inputs greater or equal to 3 and therefore the lower bound on $\hat{q}_i$ is indeed 0.

Appendix F: Proof of equation (3.45).

Given that:

$$i(r,q) = (q+\bar{q}r)\log\frac{1}{q+\bar{q}r} + \bar{q}r\log r$$
$$q = q_2\alpha + \bar{q}_2\beta,$$

we want to show:

$$(\alpha+\bar{\alpha}r)\,i\Big(\frac{\beta+\bar{\beta}r}{\alpha+\bar{\alpha}r}, q_2\Big) = i(r,q) - q_2 i(r,\alpha) - \bar{q}_2 i(r,\beta).$$

First,

$$\begin{aligned} q_2 + \bar{q}_2\frac{\beta+\bar{\beta}r}{\alpha+\bar{\alpha}r} &= \frac{q_2\alpha + q_2\bar{\alpha}r + \bar{q}_2\beta + \bar{q}_2\bar{\beta}r}{\alpha+\bar{\alpha}r}, \\ &= \frac{(q_2\alpha+\bar{q}_2\beta) + (q_2\bar{\alpha}+\bar{q}_2\bar{\beta})r}{\alpha+\bar{\alpha}r}, \\ &= \frac{q+\bar{q}r}{\alpha+\bar{\alpha}r}. \end{aligned}$$

Therefore:

$$\begin{aligned} i\Big(\frac{\beta+\bar{\beta}r}{\alpha+\bar{\alpha}r}, q_2\Big) &= \Big(q_2 + \bar{q}_2\frac{\beta+\bar{\beta}r}{\alpha+\bar{\alpha}r}\Big)\log\frac{1}{q_2+\bar{q}_2\frac{\beta+\bar{\beta}r}{\alpha+\bar{\alpha}r}} \\ &\quad + \bar{q}_2\frac{\beta+\bar{\beta}r}{\alpha+\bar{\alpha}r}\log\frac{\beta+\bar{\beta}r}{\alpha+\bar{\alpha}r}, \\ &= \frac{q+\bar{q}r}{\alpha+\bar{\alpha}r}\log\frac{\alpha+\bar{\alpha}r}{q+\bar{q}r} + \bar{q}_2\frac{\beta+\bar{\beta}r}{\alpha+\bar{\alpha}r}\log\frac{\beta+\bar{\beta}r}{\alpha+\bar{\alpha}r}, \end{aligned}$$

and:

$$\begin{aligned} (\alpha+\bar{\alpha}r)\,i\Big(\frac{\beta+\bar{\beta}r}{\alpha+\bar{\alpha}r}, q_2\Big) &= (q+\bar{q}r)\log\frac{\alpha+\bar{\alpha}r}{q+\bar{q}r} + \bar{q}_2(\beta+\bar{\beta}r)\log\frac{\beta+\bar{\beta}r}{\alpha+\bar{\alpha}r}, \\ &= (q+\bar{q}r)\log\frac{1}{q+\bar{q}r} - (q+\bar{q}r)\log\frac{1}{\alpha+\bar{\alpha}r} \\ &\quad + \bar{q}_2(\beta+\bar{\beta}r)\log\frac{1}{\alpha+\bar{\alpha}r} - \bar{q}_2(\beta+\bar{\beta}r)\log\frac{1}{\beta+\bar{\beta}r}, \\ &= (q+\bar{q}r)\log\frac{1}{q+\bar{q}r} - q_2(\alpha+\bar{\alpha}r)\log\frac{1}{\alpha+\bar{\alpha}r} \\ &\quad - \bar{q}_2(\beta+\bar{\beta}r)\log\frac{1}{\beta+\bar{\beta}r}, \\ &= i(r,q) - \bar{q}r\log r - q_2 i(r,\alpha) + q_2\bar{\alpha}r\log r \\ &\quad - \bar{q}_2 i(r,\beta) + \bar{q}_2\bar{\beta}r\log r, \\ &= i(r,q) - q_2 i(r,\alpha) - \bar{q}_2 i(r,\beta). \end{aligned}$$

Appendix G: VNCs achieving lower bound of (4.9).

We are giving here an example of a class of very noisy channels which approaches the lower bounds defined in (4.9) and (4.22).

First, for any odd value of $N = 2L + 1$:

$$r_1 = 1 - 2\epsilon - 2\epsilon^3 - \cdots - 2\epsilon^{2L-1}$$

$$\epsilon_1 = 0$$

and for $l = 1, 2, \ldots, L$

$$r_{2l} = r_{2l+1} = \epsilon^{2L+1-2l}$$

$$\epsilon_{2l} = -\epsilon_{2l+1} = \epsilon^l$$

It is clear that this channel belongs to the class of channels we defined previously and that $\sum_{j=1}^{N} r_j = 1$ and $\sum_{j=1}^{N} r_j \epsilon_j = 0$ are satisfied.

As an example of a $2 \times N$ channel where $N = 2L$ is even, then we simply choose the optimal elementary reduction of the $2 \times (2L + 1)$ channel defined above.

Appendix H: Example where the Greedy Algorithm fails

We intend to show in a particular case ($N = 4$) that the optimal reduction from 4 to 2 outputs is not the composition of the optimal elementary reduction from 4 to 3 outputs and the subsequent optimal elementary reduction from 3 to to 2 outputs.

Define the following channel : $r_1 = r_2 = r_3 = r_4 = \frac{1}{4}$, $q = \frac{1}{2}$ and $\alpha_1 = \frac{1}{2} - \epsilon_2$, $\alpha_2 = \frac{1}{2} - \epsilon_1$, $\alpha_3 = \frac{1}{2} + \epsilon_1$, $\alpha_4 = \frac{1}{2} + \epsilon_2$ where $0 < \epsilon_1 < \epsilon_2 \ll \frac{1}{2}$

It is easy to verify that (4.6) and (4.7) are satisfied.

A simple computation yields the following result :

$$I(X;Y) \simeq \frac{C}{2}\left(\epsilon_1^2 + \epsilon_2^2\right)$$

Let $\tilde{Y}_1$ and $\tilde{Y}_2$ be the following reductions from 4 to 3 outputs:

$\tilde{Y}_1 : 1234 \rightarrow 1(23)4$

$$\Rightarrow I(X;\tilde{Y}_1) \simeq \frac{C}{2}\epsilon_2^2.$$

$\tilde{Y}_2 : 1234 \rightarrow (12)34$ or $12(34)$

$$\Rightarrow I(X;\tilde{Y}_2) \simeq \frac{3C}{8}(\epsilon_1^2 + \epsilon_2^2) + \frac{C}{4}\epsilon_1\epsilon_2$$

Let $\tilde{Y}_3$ and $\tilde{Y}_3$ be the following reductions from 4 to 2 outputs:

$\tilde{Y}_3 : 1234 \rightarrow (123)4$ or $1(234)$

$$\Rightarrow I(X;\tilde{Y}_3) \simeq \frac{C}{3}\epsilon_2^2$$

$\tilde{Y}_4 : 1234 \rightarrow (12)(34)$

$$\Rightarrow I(X;\tilde{Y}_4) \simeq \frac{C}{4}(\epsilon_1 + \epsilon_2)^2$$

It is easy to show that $I(X;\tilde{Y}_1) > I(X;\tilde{Y}_2)$ if and only if $\epsilon_2 > 3\epsilon_1$. In this case, the optimal reduction from 4 to 3 outputs is $\tilde{Y}_1$, and therefore, the optimal reduction from 3 to 2 outputs is : $1(23)4 \rightarrow (123)4$ or $1(234)$. Therefore, the composition of those 2 optimal elementary reductions is $\tilde{Y}_3$. Now, in the limit when $\epsilon_1 \rightarrow \epsilon_2/3$, we have:

$$I(X;\tilde{Y}_3) \simeq \frac{3C}{9}\epsilon_2^2$$

$$I(X;\tilde{Y}_4) \simeq \frac{4C}{9}\epsilon_2^2$$

The greedy algorithm would give us $\tilde{Y}_3$ as an optimal solution, although the truly optimal solution is $\tilde{Y}_4$. In this particular case, the solution given by the greedy algorithm represents only 75% of the mutual information we would have with the truly optimal solution.

Appendix I: Convexity of $f(\lambda)$

Define the following function of λ where $\lambda \in [0,1]$, and $\alpha_1 \leq \alpha_2 \leq \alpha_3$:

$$f(\lambda) = (r_1 + \lambda r_2) F_q\Big(\frac{r_1\alpha_1 + \lambda r_2 \alpha_2}{r_1 + \lambda r_2}\Big) + \Big((1-\lambda) r_2\alpha_2 + r_3\alpha_3\Big) F_q\Big(\frac{(1-\lambda) r_2\alpha_2 + r_3\alpha_3}{(1-\lambda) r_2 + r_3}\Big)$$

Let us also define

$$g_{r_1,r_2}(\lambda) = (r_1 + \lambda r_2) F_q\Big(\frac{r_1\alpha_1 + \lambda r_2\alpha_2}{r_1 + \lambda r_2}\Big)$$

Then, $f(\lambda) = g_{r_1,r_2}(\lambda) + g_{r_3,r_2}(1-\lambda)$.

Let us show that $f(\lambda)$ is convex $\cup$ in λ.

First derivative of $g_{r_1,r_2}(\lambda)$

$$\begin{aligned} g'_{r_1,r_2}(\lambda) =& (r_1 + \lambda r_2)\Big(\frac{r_2\alpha_2(r_1+\lambda r_2) - (r_1\alpha_1 + \lambda r_2\alpha_2) r_2}{(r_1+\lambda r_2)^2}\Big) F'_q\Big(\frac{r_1\alpha_1 + \lambda r_2\alpha_2}{r_1 + \lambda r_2}\Big) \\ &+ r_2 F_q\Big(\frac{r_1\alpha_1 + \lambda r_2\alpha_2}{r_1 + \lambda r_2}\Big) \end{aligned}$$

$$g'_{r_1,r_2}(\lambda) = r_2 F_q\Big(\frac{r_1\alpha_1 + \lambda r_2\alpha_2}{r_1 + \lambda r_2}\Big) + \frac{r_1 r_2(\alpha_2 - \alpha_1)}{r_1 + \lambda r_2} F'_q\Big(\frac{r_1\alpha_1 + \lambda r_2\alpha_2}{r_1 + \lambda r_2}\Big)$$

Second derivative of $g_{r_1,r_2}(\lambda)$

$$\begin{aligned} g''_{r_1,r_2}(\lambda) =& r_2 \frac{r_1 r_2(\alpha_2 - \alpha_1)}{(r_1 + \lambda r_2)^2} F'_q\Big(\frac{r_1\alpha_1 + \lambda r_2\alpha_2}{r_1 + \lambda r_2}\Big) \\ &+ \frac{-r_1 r_2^2(\alpha_2 - \alpha_1)}{(r_1 + \lambda r_2)^2} F'_q\Big(\frac{r_1\alpha_1 + \lambda r_2\alpha_2}{r_1 + \lambda r_2}\Big) \\ &+ \frac{r_1 r_2(\alpha_2 - \alpha_1)}{r_1 + \lambda r_2} \frac{r_1 r_2(\alpha_2 - \alpha_1)}{(r_1 + \lambda r_2)^2} F''_q\Big(\frac{r_1\alpha_1 + \lambda r_2\alpha_2}{r_1 + \lambda r_2}\Big) \end{aligned}$$

$$g''_{r_1,r_2}(\lambda) = \frac{r_1^2 r_2^2(\alpha_2 - \alpha_1)^2}{(r_1 + \lambda r_2)^3} F''_q\Big(\frac{r_1\alpha_1 + \lambda r_2\alpha_2}{r_1 + \lambda r_2}\Big)$$

Second derivative of $f(\lambda)$

$$f''(\lambda) = g''_{r_1,r_2}(\lambda) + g''_{r_3,r_2}(1-\lambda)$$

$$f''(\lambda) = \frac{r_1^2 r_2^2(\alpha_2-\alpha_1)^2}{(r_1+\lambda r_2)^3}F_q''\Big(\frac{r_1\alpha_1+\lambda r_2\alpha_2}{r_1+\lambda r_2}\Big)+\frac{r_3^2 r_2^2(\alpha_2-\alpha_3)^2}{(r_3+(1-\lambda)r_2)^3}F_q''\Big(\frac{r_3\alpha_3+(1-\lambda)r_2\alpha_2}{r_3+(1-\lambda)r_2}\Big)$$

Since $F_q(\alpha)$ is a convex $\cup$ function of α, then $f''(\lambda) \geq 0$, and therefore, $f(\lambda)$ is a convex $\cup$ function of λ.

Appendix J: Classification analogy

Finding the optimal reduction from N to K outputs can also be viewed as finding the optimal classification of N objects into K classes with the following definitions.

Let us have N objects $Y_1, Y_2, \ldots, Y_N$ characterized by their probability r_j and their (statistical) feature α_j.

Any classification is a partition of the set of objects into K disjoint classes. The probability of a class is the *sum* of the probabilities of the elements of the class, and the (statistical) feature of the class is the *average* of the (statistical) features of the elements of the class w.r.t. the individual probabilities of the elements.

Therefore, the inter-object (or inter-class) distance between 2 objects (or between 2 classes of objects) is :

$$d(Y_i, Y_j) = r_i F_q(\alpha_i) + r_j F_q(\alpha_j) - r_{i,j} F_q(\alpha_{i,j}).$$

This definition satisfies the first two properties of distances :

(1) $d(Y_i, Y_j) \geq 0$

(2) $d(Y_i, Y_j) = 0 \Rightarrow \alpha_i = \alpha_j$ which means $Y_i = Y_j$

The 3^{rd} property is not necessarily satisfied. In fact we have a class of examples where the inequality is reversed.

Take $q = \frac{1}{2}$, $N = 3$, $r_1 = r_3 = r$, $\alpha_1 = 1 - \alpha_3 = \alpha$ and $\alpha_2 = \frac{1}{2}$.

Then $d(Y_1, Y_3) = 2rF_q(\alpha)$, $d(Y_1, Y_2) = d(Y_2, Y_3) = rF_q(\alpha) - (1-r)F_q(\frac{\frac{1-2r}{2} + r\alpha}{1-r}) \leq rF_q(\alpha)$.

Clearly, $d(Y_1, Y_3) \geq d(Y_1, Y_2) + d(Y_2, Y_3)$.

Note however that $d(Y_2, Y_3) \leq d(Y_2, Y_1) + d(Y_1, Y_3)$.

Appendix K: Examples where MED, MLD and MMID are different

Consider the following channel ($N = 4$ and $q = 0.3$) :

for all j, $r_j = \frac{1}{4}$ and $\alpha_1 = 0, \alpha_2 = 0.28, \alpha_3 = 0.32, \alpha_4 = 0.6$.

1. The optimal MED partition is (123)4

   This is because $\alpha_1 < \alpha_2 < \alpha_3 < \frac{1}{2} \leq \alpha_4$.

2. The optimal MLD partition is (12)(34)

   This is because $\alpha_1 < \alpha_2 < q = 0.3 < \alpha_3 < \alpha_4$.

3. The optimal MMID partition is 1(234)

   partition 1(234) : $I(X; \tilde{Y}) = 0.106$

   partition (12)(34) : $I(X; \tilde{Y}) = 0.063$

   partition (123)4 : $I(X; \tilde{Y}) = 0.067$

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