EVALUATION OF THE UNSTEADY EFFECTS FOR A CLASS OF WIND TURBINES

Thesis by

Jean-Luc Cornet

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1984

(Submitted October 25, 1983)

ACKNOWLEDGMENT

I would like to express my sincere appreciation to Professor Theodore Yao-Tsu Wu, whose guidance, help and encouragement were indispensable to the successful completion of this work. For his patience and understanding, I am most sincerely grateful.

Special thanks are also due to Dr. George T. Yates for his friendship and for his interesting discussions on some of the theoretical issues. His help in the final correction of this manuscript was greatly appreciated.

I am deeply indebted to Miss Helen Burrus, who courageously undertook the laborious task of typing this thesis, for her excellent work and helpful suggestions.

I gratefully acknowledge financial support from the French "Ministère des affaires étrangères", the National Science Foundation, the Office of Naval Research and the California Institute of Technology.

Finally, the deepest appreciation of all goes to my wife, Arlette, and my daughter, Sandrine, who have silently borne the burden imposed by my graduate studies and who sacrificed the most to make this a reality. Without their continuing love and encouragement, this work might not have been undertaken.

ABSTRACT

An investigation of a class of vertical axis wind turbines is carried out with the unsteady effects due to the rotating blade motion fully taken into account. The work is composed of two parts.

In part one, a hydromechanical theory is developed which proceeds from the point of view of unsteady airfoil theory. A rotor comprised of a single blade is used and a two-dimensional analysis is applied to a cross section of the rotor in the limiting mode of operation wherein $U \ll \Omega R$. Use of linearized theory and of the acceleration potential allows the problem to be expressed in terms of a Riemann-Hilbert boundary value problem. The method of characteristics is used to solve for the remaining unknown function. A uniformly valid first order solution is obtained in closed form after some approximation based on neglecting the variations in the curvature of the path. Explicit expressions of the instantaneous forces and moments acting on the blade are given and the total energy lost by the fluid and the total power input to the turbine are determined.

In part two, the lift acting on a wing crossing a vortex sheet is evaluated by application of a reciprocity theorem in reverse flow. This theorem follows from Green's integral theorem and relates the circulation around a blade having impulsively crossed a vortex wake to the lift acting on a blade continuously crossing a vortex wake. A solution is obtained which indicates that the lift is composed of two parts having different rates of growth, each depending on the apparent flow velocity before and after the crossing.

TABLE OF CONTENTS

TABLE OF CONTENTS (cont'd)

TABLE OF CONTENTS (cont'd)

LIST OF FIGURES

LIST OF FIGURES (cont'd)

PART I

EVALUATION OF THE UNSTEADY EFFECTS FOR A CLASS OF VERTICAL AXIS WIND TURBINES

1. Introduction

Although Wind Energy was one of the first energy resources to be harnessed by man, and was extensively used by most of the known civilizations for propulsion or energy extraction up to the beginning of the industrial era, it seems to have been rediscovered only a few decades ago. Consequently the development of advanced theories on extraction of energy from the wind is only beginning. Horizontal axis wind turbines have benefited directly from the extensive investigations carried out for airplane propellers, as illustrated by the works of Glauert [1], Theodorsen [2] and Goldstein [3], and from the experiments on propeller type windmills performed in the past [4], [5]. Vertical axis wind turbines have only recently been studied and the theories developed for them are quite far from matching the sophisticated vortex theories currently used for horizontal axis machines.

Vertical axis wind turbines offer several advantages over more conventional wind turbines: they do not require an initial orientation into the wind, can conveniently have the power generator located at the ground level and may even eliminate the need for a supporting tower. These advantages should make possible the development of a lighter and cheaper structure, and thus make the vertical axis wind turbine a primary candidate in wind energy technology.

Among several designs of vertical axis wind turbines currently investigated the most promising is the Darrieus rotor (fig. 1), which was invented and patented by the independent french inventor Darrieus [6] in 1931. New interest in this design has been aroused by South and Rangi [7] at the National Research Council of Canada in Ottawa. The

Aerodynamics of the Darrieus rotor and models for performance prediction have since been formulated by Templin [8], Strickland [9], Shankar [10], Wilson and Lissaman [11], Holme [12] and others. Except for the work of Holme , all these formulations are based on variations of the actuator disk theory developed by Betz [13] for the horizontal axis windmill. In this theory the momentum flux through the stream tube enclosing the turbine is equated to the time averaged force on the blade elements. Templin [8] used a single stream tube and took the flow velocity at the rotor to be the arithmetic mean of the velocity far in front and the velocity far behind the turbine. Multiple stream tubes have been used to account for the variation of the velocity across the stream tube, [9], [10], [11], but only Holme [12] considered the variations of flow velocity in both the streamwise and transverse directions. Common to all these formulations is the assumption of quasi-steady motion in computing the forces acting on the blade elements, thereby neglecting the perturbation velocities induced by the wake, the added mass of the blades, and more significantly the variations of the leading edge suction induced by the unsteady flow. At the low values of the reduced frequency usually encountered in studies of a Darrieus rotor, variations of the lift due to the unsteady effects are known to be small; however the leading edge suction variations are much larger and since a vertical axis wind turbine derives its torque from the thrust acting on the blade elements, these variations may prove to be of significant importance. Other effects neglected in these previous investigations include the stream curvature effects, the interaction of the rear blades with the vortex wake shed from the forward blades as well as the three-dimensional effects.

The purpose of the present work is to develop a two-dimensional model for a general class of vertical axis wind turbines. The investigation proceeds from the point of view of unsteady airfoil theory and adopts many of the ideas developed by Wu [14], [15], [16] in his study of the motion of a heaving and pitching airfoil. The contribution of the unsteady effects on the power extraction capability of a turbine are fully evaluated, including the kinetic energy imparted to the fluid, hence lost, by the blades. This should make feasible the maximization of the mechanical efficiency of a turbine by a proper optimization of the blade motion. In addition this model could be used with some elements of momentum theory to provide a better approximation of the maximum power coefficient of a turbine than is now available. The basic theory is also applicable, with some change of parameters, to a large class of cycloidal propellers used in naval architecture (Schneider propellers). These propellers are lift oriented devices primarily intended for marine vehicles operating in restricted waters where their ability to produce thrust in any direction greatly enhances ship maneuverability.

2. General Formulation

In the present investigation we assume the fluid to be incompressible and inviscid. The effects of viscosity are only implicitly inferred in invoking the Kutta condition and in turn in the formation of vortex wakes behind the blades. No fluid singularities are allowed outside this wake and the flow is irrotational and has a scalar potential.

The rotor to be investigated consists of a number of high aspect-ratio blades, of half chord c, symmetrical in shape and of constant cross section along the span (fig. 2). The blades are placed at regular intervals along the generators of a cylinder of radius R and rotate about the cylindrical axis with angular velocity Ω. The rotor is placed in a flow having a constant velocity U at infinity perpendicular to the rotor axis. A cross section of the rotor is taken at mid span and a two-dimensional analysis is applied. Due to the high aspect-ratio of the blades this approximation is expected to provide good quantitative results. A rotor composed of a single blade is investigated, but the results can be extended to multi bladed systems if the mutual interference between the blades is small. This assumption may seem restrictive for the study of a wind energy extracting turbine, where the mutual interference between the blades is usually not small. However, since we restricted the analysis to lightly loaded turbines and are mainly interested in the influence of the unsteady effects on the overall performance of a turbine, it is a reasonable assumption.

The wake crossing effects are neglected in Part I of this study, since these effects are investigated as a new problem in Part II. The usual assumptions of linearized wing theory are used, i. e.:

i. Any movement of the wing relative to the apparent direction of the incident flow is small so that the perturbation velocity components are small compared to the apparent velocity of the undisturbed flow;

ii. The Kutta condition holds, i. e., both the pressure and the velocity are required to be finite at the trailing edge of the blade;

iii. The perturbation velocity is asymptotically zero at infinity, except in the region of the starting vortex and in the vortex wake.

To satisfy these assumptions, small perturbations superimposed on the "perfect" flow field of a flexible wing sliding on its trajectory are assumed. This simplification is applicable to a rigid blade if the curvature of the trajectory is small compared to the blade chord, and puts some restrictions on the relative value of ΩR and U:

i. When $\Omega R \simeq U$, the trajectory of the blade (described in the fluid frame of reference fixed with the fluid at infinity) is a common cycloid (fig. 3a). This case is excluded since there exist isolated regions of large trajectory curvature.

ii. When $\Omega R \ll U$, the trajectory is a curtate cycloid and resembles a sinusoidal path (fig. 3b). In this case the trajectory curvature is small, even when the radius of the rotor is of the same order as the chord of the blade. However, this case is not very interesting for energy extraction since a large number of blades would be necessary to cover the sweep area, and each blade allowed to rotate a full 360° around its axis if the angle of attack was to be kept small. Furthermore, this problem is equivalent to a simple heaving and pitching blade in a flow of approximately constant velocity and has been solved by Wu [14].

iii. We are therefore left with the case $\Omega R \gg U$. The trajectory in this case is a prolate cycloid (fig. 3c) and looks like a circular path being slightly displaced after each revolution. This case is of much greater practical interest since the tip speed of the turbine is considerably higher than the wind velocity, which is a necessary condition for a significant amount of energy to be extracted. It is also relevant to the experiments done with rotors of fixed geometry where tip speeds of the order of 3 times the wind velocity are necessary before any energy can be extracted [7]. The curvature of the path in this case is of order $\frac{1}{R}$ so that we also require $R \gg c$ for the curvature to be small. This is also relevant to the current design of Darrieus rotor, where the value of $\frac{c}{R}$ is usually around 0.1 or smaller.

We therefore restrict our investigation to the case

. $R \gg c$

. $\Omega R \gg U$.

3. Kinematics of the motion

3.1 The coordinate systems

We define three frames of reference. The first one (x_o, y_o) is the structure frame of reference. It is an inertial frame of reference, is fixed in space, its origin O_o is taken at the center of the rotor, and the x_o-axis points in the direction of the undisturbed flow velocity. The second frame of reference (x_1, y_1) rotates in the counterclockwise sense with an angular velocity Ω with respect to the structure frame and they share the same origin. The third one (x, y) is the body frame of reference, its origin O is located along the y_1-axis at a distance R from O_o and the x-axis is taken to be tangential to the relative velocity of the undisturbed flow at O, and points in the same direction. The three frames of reference are represented in figure 4.

We introduce θ, λ and η as the relative position angles of the three frames of reference, respectively (*fig.* 4), with all the angles assuming positive values in the counterclockwise directions.

θ is the angle between $\underset{\sim}{e}_{x_o}$ and $\underset{\sim}{e}_{x_1}$, such that

$$\sin \theta = (\underset{\sim}{e}_{x_o} \times \underset{\sim}{e}_{x_1}) \cdot \underset{\sim}{e}_z ,$$

where $\underset{\sim}{e}_z$ is a unit vector directed upward from the plane (x_o, y_o).

λ is the angle between $\underset{\sim}{e}_{x_1}$ and $\underset{\sim}{e}_x$, such that

$$\sin \lambda = (\underset{\sim}{e}_{x_1} \times \underset{\sim}{e}_x) \cdot \underset{\sim}{e}_z .$$

η is the angle between $\underset{\sim}{e}_{x_o}$ and $\underset{\sim}{e}_x$, such that

$$\sin \eta = (\underset{\sim}{e}_{x_o} \times \underset{\sim}{e}_x) \cdot \underset{\sim}{e}_z ,$$

and

$$\eta = \theta + \lambda .$$

The angular velocity of the body frame of reference with respect to the structure frame of reference is denoted by ω, so that

$$\omega = \eta_t = \lambda_t + \theta_t \quad ,$$

and (1.1)

$$\Omega = \theta_t \quad ,$$

where the subscript t denotes the time differentiation executed in the structure frame of reference. We also define the two vectors $\underset{\sim}{\omega}$ and $\underset{\sim}{\Omega}$ by

$$\underset{\sim}{\omega} = \omega \underset{\sim}{e}_z \quad ,$$

and

$$\underset{\sim}{\Omega} = \Omega \underset{\sim}{e}_z \quad .$$

We call $\underset{\sim}{U}$ the velocity of the undisturbed fluid in the structure frame of reference, and $\underset{\sim}{V}$ the relative velocity of the undisturbed fluid at the center of the body frame of reference. The following relationships can then be written

$$\begin{aligned}
\underset{\sim}{U} &= U \underset{\sim}{e}_{x_o} \quad , \\
\underset{\sim}{\Omega} \times \underset{\sim}{R} &= -\Omega R \underset{\sim}{e}_{x_1} \quad , \\
\underset{\sim}{V} &= V \underset{\sim}{e}_x \quad , \\
\underset{\sim}{V} &= \underset{\sim}{U} - \underset{\sim}{\Omega} \times \underset{\sim}{R} \quad .
\end{aligned} \tag{1.2}$$

The components of $\underset{\sim}{V}$ in (x_1, y_1) are

$$\underset{\sim}{V} \cdot \underset{\sim}{e}_{x_1} = V \cos\lambda = \Omega R + U \cos\theta \quad ,$$

$$\underset{\sim}{V} \cdot \underset{\sim}{e}_{y_1} = V \sin\lambda = -U \sin\theta \quad ,$$

so that the magnitude of $\underset{\sim}{V}$ is

$$V = (U^2 + \Omega^2 R^2 + 2UR\Omega \cos \theta)^{1/2} \quad , \tag{1.3}$$

and for the angle λ we have

$$\tan \lambda = \frac{-\sin \theta}{\cos \theta + \Omega R/U} \quad . \tag{1.4}$$

The body frame of reference was chosen for the convenience of describing small perturbations. In this frame of reference, with the variation of λ prescribed by (1.4), the undisturbed flow at the origin O is always parallel to the x-axis and thus small movement of a blade about this mean position can be considered in the framework of linearized wing theory. It must be noted that in the body frame of reference, a rotor of fixed geometry with respect to (x_1, y_1) would appear as having an unsteady motion. For instance a blade which would be centered at O with a fixed angle with respect to $\underset{\sim}{e}_{x_1}$ would appear as having a pure pitching motion.

3.2 Trajectory

The trajectory of the origin of the body frame of reference in the fluid can be more easily described with the help of a new frame of reference (x_2, y_2) which is translating with the undisturbed fluid at infinity, and will be referred to as the fluid frame of reference. Its axis are chosen with the same orientation as (x_o, y_o) and its origin O_2 is chosen to coincide with O_o at $t = 0$ and is translating in the positive x_o direction with speed U (see fig. 4).

In the fluid frame of reference the origin O appears to be moving with the velocity $-\underset{\sim}{V}(t)$. We call $\underset{\sim}{X}(t)$ the position vector of O in the fluid frame of reference and $\underset{\sim}{e}_s(t)$ the unit vector tangent to the trajectory in the direction of motion (see fig. 5). We can then write the following relations:

$$-\underset{\sim}{V}(t) = V(t)\underset{\sim}{e}_s(t) \quad , \tag{1.5}$$

$$\underset{\sim}{e}_s(t) = -\underset{\sim}{e}_x(t) \quad , \tag{1.6}$$

$$\frac{\partial \underset{\sim}{X}(t)}{\partial t} = -\underset{\sim}{V}(t) \quad . \tag{1.7}$$

In the body frame of reference, the trajectory of the origin O in space can be described by

$$\underset{\sim}{X}_c(t', t) = \underset{\sim}{X}(t') - \underset{\sim}{X}(t) \quad , \tag{1.8}$$

where t is the present time and t' an arbitrary instant of time serving as a parameter. Differentiating $\underset{\sim}{X}_c(t', t)$ with respect to t' and using (1.7) and (1.8) leads to

$$\frac{\partial \underset{\sim}{X}_c(t', t)}{\partial t'} = -\underset{\sim}{V}(t') \quad .$$

Integrating this relation with respect to t', we obtain the parametric representation of the trajectory in the body frame of reference as

$$\underset{\sim}{X}_c(t', t) = -\int_t^{t'} \underset{\sim}{V}(\tilde{t})d\tilde{t} \quad . \tag{1.9}$$

We can decompose $\underset{\sim}{e}_s(t')$ in the body frame of reference, using (1.6), we obtain

$$e_{s_1}(t') = -\underset{\sim}{e}_x(t) \cdot \underset{\sim}{e}_x(t') = -\cos\tilde{\eta}(t',t) \quad , \tag{1.10a}$$

$$e_{s_2}(t') = -\underset{\sim}{e}_y(t) \cdot \underset{\sim}{e}_x(t') = -\sin\tilde{\eta}(t',t) \quad , \tag{1.10b}$$

where $\tilde{\eta}(t',t)$ is defined by

$$\tilde{\eta}(t',t) = \eta(t') - \eta(t) = \int_t^{t'} \omega(\tilde{t})d\tilde{t} \quad . \tag{1.11}$$

Then, using (1.5), (1.9) and (1.10), we can decompose $\underset{\sim}{X}_c(t',t)$ into its components in the body frame of reference as follows

$$X_c(t',t) = -\int_t^{t'} V(\tilde{t})\cos\tilde{\eta}(\tilde{t},t)d\tilde{t} \quad , \tag{1.12a}$$

$$Y_c(t',t) = -\int_t^{t'} V(\tilde{t})\sin\tilde{\eta}(\tilde{t},t)d\tilde{t} \quad . \tag{1.12b}$$

Finally the curvature of the trajectory can be readily obtained from (1.10) and (1.11) as

$$\kappa(t') = \frac{1}{V(t')} \left| \frac{\partial \underset{\sim}{e}_s(t')}{\partial t'} \right| = \frac{\omega(t')}{V(t')} \quad . \tag{1.13}$$

The curvature of the path simply takes the form of the ratio of the two velocities characterizing the motion of the blade with respect to the fluid.

3.3 Boundary conditions

The velocity expressed in the structure frame of reference of a point fixed in the body frame of reference is

$$\underset{\sim}{C} = (\underset{\sim}{\Omega} \times \underset{\sim}{R}) + (\underset{\sim}{\omega} \times \underset{\sim}{x}) \quad , \tag{1.14}$$

where $\underset{\sim}{x}$ is the position vector of that point in the body frame of reference. The perturbation velocity $\underset{\sim}{u}$ and the total velocity $\underset{\sim}{Q}$, expressed in the structure frame of reference, are related by

$$\underset{\sim}{Q} = \underset{\sim}{U} + \underset{\sim}{u} \quad . \tag{1.15}$$

Further, in the body frame of reference the velocity of the fluid is

$$\underset{\sim}{q} = \underset{\sim}{Q} - \underset{\sim}{C} \quad . \tag{1.16}$$

Then, using (1.2), (1.14) and (1.15), we can express $\underset{\sim}{q}$ as

$$\underset{\sim}{q} = \underset{\sim}{u} + \underset{\sim}{V} - (\underset{\sim}{\omega} \times \underset{\sim}{x}) \quad . \tag{1.17}$$

The blade is represented in the body frame of reference by its transverse displacement from the x-axis given by

$$y = h(x, t) \quad .$$

A blade of chord 2 is chosen, extending from $x = -1$ to $x = +1$, and the thickness of the blade is neglected (fig. 6).

The normal velocity of the flow on the blade must always be equal to the normal component of the blade velocity

$$\underset{\sim}{v}_B \cdot \underset{\sim}{n} = \underset{\sim}{q} \cdot \underset{\sim}{n} \quad , \tag{1.18}$$

where $\underset{\sim}{n}$ is the unit vector normal to the blade. In the body frame of reference we can express $\underset{\sim}{n}$ as

$$\underset{\sim}{n} = \nabla(y-h)/|\nabla(y-h)| \quad ,$$

and $\underset{\sim}{v}_B$ as

$$\underset{\sim}{v}_B = \frac{\partial h}{\partial t} \underset{\sim}{e}_y \quad .$$

Therefore the kinematic boundary condition (1.18) can be expressed as

$$\frac{\partial h}{\partial t} + \underset{\sim}{q} \cdot \nabla(h-y) = 0 \quad , \qquad (1.19)$$
$$\text{on } y = h(x,t);\ -1 < x < +1 \quad .$$

We finally define u and v to be the x-component and y-component of the perturbation velocity $\underset{\sim}{u}$,

$$\underset{\sim}{u} = u\underset{\sim}{e}_x + v\underset{\sim}{e}_y \quad .$$

Then, using (1.17) and (1.2) in (1.19) we obtain

$$\frac{\partial h}{\partial t} + (u + V + \omega y)\frac{\partial h}{\partial x} + \omega x - v = 0 \qquad (1.20)$$
$$\text{on } y = h(x,t);\ -1 < x < +1 \quad .$$

3.4 <u>The osculating case</u>

If we consider the motion of a flexible blade sliding entirely on the trajectory left by its leading edge, then the blade will not disturb the inviscid fluid, because every point of the blade moves only tangentially to the boundary of the fluid with which the blade makes contact.

This conclusion follows from the absence of a mechanism by which the shear stress can be transmitted to the fluid in potential flow. Let's consider the trajectory of the leading edge 0 which is followed in osculation by the motion of a flexible blade. The blade displacement from the x–axis is then

$$h_c(x, t) = Y_c(t', t) \quad .$$

The boundary condition (1.20) can be written as

$$\frac{\partial}{\partial t}\left[Y_c(t'(X_c, t), t)\right]_{X_c} + (u + V + \omega Y_c)\left.\frac{\partial Y_c}{\partial X_c}\right|_t + \omega X_c - v = 0 \quad .$$

Using the relation

$$\left(\frac{\partial Y_c}{\partial t}\right)_{X_c} = \left(\frac{\partial Y_c}{\partial t}\right)_{t'} + \left(\frac{\partial Y_c}{\partial t'}\right)_t \left(\frac{\partial t'}{\partial t}\right)_{X_c} \quad ,$$

$$\left(\frac{\partial t'}{\partial t}\right)_{X_c} = -\left(\frac{\partial X_c}{\partial t}\right)_{t'} \Big/ \left(\frac{\partial X_c}{\partial t'}\right)_t \quad ,$$

$$\left(\frac{\partial Y_c}{\partial X_c}\right)_t = \left(\frac{\partial Y_c}{\partial t'}\right)_t \Big/ \left(\frac{\partial X_c}{\partial t'}\right)_t \quad ,$$

and the expression for $X_c(t', t)$ and $Y_c(t', t)$ given by (1.12), the kinematic boundary condition reduces to

$$\frac{v}{u} = \tan \tilde{\eta}\,(t', t) \quad .$$

This relation implies that the flow is everywhere tangential to the blade surface, a fact which can also be expressed as

$$\frac{\partial \varphi}{\partial n} = \underset{\sim}{u} \cdot \underset{\sim}{n} = 0 \quad ,$$

where φ is the velocity potential such that $\underset{\sim}{u} = \nabla \varphi$.

If we assume that the fluid is initially at rest and that any subsequent disturbance vanishes at infinity, then on all the flow boundaries, including the vortex sheet, the following relationship is satisfied.

$$\frac{\partial \varphi}{\partial n} = 0 \quad .$$

The unique solution of the Laplace equation, $\nabla^2 \varphi = 0$, satisfying this boundary condition is φ=const; consequently $\underset{\sim}{u} = \underset{\sim}{0}$ everywhere in the fluid and the proof is complete.

To ensure that only small disturbances are generated and imparted to the fluid, we must restrict the blade to small displacements from the trajectory of the origin O; namely we require that

$$h(x, t) - h_c(x, t) \ll c \quad .$$

The function $h_c(x, t)$ defined in section 3.4 depends directly on the value of the curvature of the path relative to the chord of the blade. For a blade of unit half chord, $h_c(x, t)$ can be kept small by taking

$$R \gg 1 \quad ,$$

$$\Omega R \gg U \quad .$$

As stated earlier, these two conditions are not really restrictive since they correlate well with the practical design and operating conditions of a Darrieus rotor. The only additional condition needed is

$$h(x, t) \ll 1 \quad ,$$

which puts a limitation on the relative movements of the blade in the body frame of reference.

3.5 Linearized boundary conditions

We now restrict our analysis to the case of small perturbations. In addition to the afore mentioned restrictions, we also assume the perturbation velocity $\underset{\sim}{u}$ to be small, as well as the derivatives of the blade displacement. The complete set of assumptions, with the half chord normalized to unity, is then

$$\begin{aligned} &|h(x, t)|, \ \left|\frac{\partial h}{\partial x}(x, t)\right| \ll 1 \quad , \\ &\left|\frac{\partial h}{\partial t}(x, t)\right|, \ u, v \ll V \quad , \\ &R, \ \frac{\Omega R}{U} \gg 1 \quad . \end{aligned} \tag{1.21}$$

Recalling equation (1.20) and neglecting the quadratic and higher order terms, we obtain the linearized boundary condition

$$\begin{aligned} &v(x, t) = \left(\frac{\partial}{\partial t} + V(t)\frac{\partial}{\partial x}\right) h(x, t) + \omega(t)x \quad , \\ &\text{on} \quad y = 0^{\pm}, \quad |x| < 1 \quad . \end{aligned} \tag{1.22}$$

We can see from this equation that the fluid disturbance arises from three sources:

i. $\frac{\partial h}{\partial t}$ which represents the disturbance induced by the rate of change in lateral motion of the blade in the body frame of reference;

ii. $V(t)\frac{\partial h}{\partial x}$ which is the disturbance induced by the displacement of the blade from its zero "effective incidence" position;

iii. $\omega(t)x$ which is the disturbance due to the rotation of the body coordinates with respect to the fixed system and the rigidity of the blade.

4. Dynamics

4.1 The Bernoulli equation

For the flow of an incompressible and inviscid fluid in the absence of external forces, the Euler equation of motion is

$$\frac{d}{dt_o}(\underset{\sim}{Q}) = -\frac{1}{\rho}\nabla_o p \quad , \tag{1.23}$$

where the subscript o denotes the derivatives taken in an inertial frame of reference. Assuming the flow to be irrotational in the inertial frame of reference, a first integral of this equation can be obtained, the result of which is the Bernoulli equation which can be expressed in the present case by

$$\frac{\partial\varphi}{\partial t_o} + \frac{1}{2}(\underset{\sim}{U} + \nabla_o\varphi)^2 + \frac{p}{\rho} = \frac{p_\infty}{\rho} + \frac{1}{2}U^2 \quad , \tag{1.24}$$

where φ represents the potential of the perturbation velocity $\underset{\sim}{u}$, i.e.,

$$\underset{\sim}{u} = \nabla\varphi \quad , \tag{1.25}$$

and p_∞ is the hydrodynamic pressure in the fluid at infinity. We can express equation (1.24) in the body frame of reference by

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + \underset{\sim}{V}(t) \cdot \nabla\varphi - (\underset{\sim}{\omega} \times \underset{\sim}{x}) \cdot \nabla\varphi = \frac{p_\infty - p}{\rho} \quad . \tag{1.26}$$

In deriving this expression, use has been made of (1.2), (1.4) and the following relationships

$$\nabla = \nabla_o \; ; \; \frac{\partial}{\partial t_o} = \frac{\partial}{\partial t} - \underset{\sim}{C} \cdot \nabla \quad . \tag{1.27}$$

Linearizing (1.26) by neglecting $(\nabla\varphi)^2$ and developing it yields

$$\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial x}(V + \omega y) - \frac{\partial\varphi}{\partial y}\omega x = \frac{p_\infty - p}{\rho} \quad . \tag{1.28}$$

This is our required linearized Bernoulli equation expressed in the body frame of reference. This equation shows that variations of pressure arise from three types of blade motion:

i. $\frac{\partial\varphi}{\partial t}$, which represents the class of motion of the blade that appears to be unsteady in the body frame of reference;

ii. $V\frac{\partial\varphi}{\partial x}$, which depends on the rectilinear motion of the body frame of reference;

iii. $\omega(y\frac{\partial\varphi}{\partial x} - x\frac{\partial\varphi}{\partial y})$, which arises from the angular motion of the body frame of reference.

The equation of continuity for an incompressible fluid is $\nabla_o \cdot \underset{\sim}{Q} = 0$. As usual in incompressible potential flow, this equation together with (1.25) leads to the Laplace equation

$$\nabla^2\varphi = 0 \quad . \tag{1.29}$$

Equations (1.28) and (1.29) therefore provide the equations of motion.

4.2 The acceleration potential

A different expression of the Euler equation of motion can be found by defining a new function

$$\Phi = \frac{p_\infty - p}{\rho} \quad , \tag{1.30}$$

which allow (1.23) to be expressed as

$$\underset{\sim}{a} = \nabla_o \Phi \quad , \tag{1.31}$$

where $\underset{\sim}{a}$ is the acceleration vector of the local field, $\underset{\sim}{a} = \frac{d\underset{\sim}{Q}}{dt_o}$.
Φ is accordingly called the acceleration potential, and has the main advantage of being continuous everywhere in the fluid, except across the blade itself. Especially noteworthy is its being continuous across the vortex sheet shed from the blade trailing edge, due to the continuity of the pressure across the wake.

Unlike the velocity potential φ, which is always a harmonic function for irrotational motion of an incompressible fluid, (i.e., $\nabla_o^2 \varphi = 0$), the acceleration potential is generally not harmonic. In fact, taking the divergence of (1.31), using (1.24) and (1.29), we easily find

$$\nabla_o^2 \Phi = \nabla_o^2 \left(\frac{\underset{\sim}{u}^2}{2}\right) \quad .$$

However, within the framework of our linearized theory, we can neglect the second order terms, and consider Φ to be a harmonic function, i.e.,

$$\nabla^2 \Phi = 0 \quad . \tag{1.32}$$

4.3 Formulation in the complex plane

Since both the velocity and the acceleration potential are harmonic functions, the complex variable theory will be used in the subsequent analysis. We therefore recast the problem in complex form. First, we define ψ and Ψ to be the functions conjugate to φ and Φ respectively, such that the Cauchy-Riemann equations are satisfied

$$\begin{cases} \varphi_x = \psi_y \\ \varphi_y = -\psi_x \end{cases} \qquad \begin{cases} \Phi_x = \Psi_y \\ \Phi_y = -\Psi_x \end{cases} . \tag{1.33}$$

We further define $\mathcal{F}$ to be the complex accleration potential and f to be the complex velocity potential so that

$$\mathcal{F}(z, t) = \Phi(x, y, t) + i\Psi(x, y, t) \quad , \tag{1.34a}$$

and

$$f(z, t) = \varphi(x, y, t) + i\psi(x, y, t) \quad , \tag{1.34b}$$

where $z = x + iy$, and $i = \sqrt{-1}$ is the imaginary unit. Finally we define $W(z, t)$ to be the complex velocity

$$W(z, t) = \frac{d}{dz} f(z, t) = u - iv \quad . \tag{1.35}$$

The Bernoulli equation (1.28) can be written as

$$\frac{\partial \varphi}{\partial t} + u(V + \omega y) - v(\omega x) = \Phi \quad ,$$

or, in complex form

$$\frac{\partial f}{\partial t} + VW - i\omega z W = \mathcal{F} \quad . \tag{1.36}$$

As usual in complex theory, only the real part of equation (1.36) represents the original Bernoulli equation. Differentiating (1.36) with respect to z yields

$$\frac{\partial \mathcal{F}}{\partial z} = \frac{\partial W}{\partial t} + (V - i\omega z)\frac{\partial W}{\partial z} - i\omega W \quad . \tag{1.37}$$

It is convenient at this point to change the time variable from the absolute time t in the body frame to a new one reflecting the arc length traversed along the trajectory in time t.

$$\tau(t) = \int_0^t V(\tilde{t})d\tilde{t} \quad . \tag{1.38}$$

As we may notice, $\tau(t)$ is directly a measure of the arc length along the blade trajectory. Performing this change of variables on (1.37) leads to

$$\frac{1}{V}\frac{\partial \mathcal{F}}{\partial z} = \left(\frac{\partial W}{\partial \tau} + \frac{\partial W}{\partial z}\right) - i\frac{\omega}{V}\frac{\partial}{\partial z}(zW) \quad . \tag{1.39}$$

We can further simplify this relation by defining two new functions

$$F(z, \tau(t)) = \frac{\mathcal{F}(z,t)}{V(t)} \quad , \tag{1.40a}$$

and

$$\kappa(\tau(t)) = \frac{\omega(t)}{V(t)} \quad , \tag{1.40b}$$

where $F(z, \tau)$ is analytic and $\kappa(\tau)$ represents the curvature of the trajectory. This allows us to express the complex equation of motion (1.39) as

$$\frac{\partial F}{\partial z}(z, \tau) = \left(\frac{\partial W}{\partial \tau} + \frac{\partial W}{\partial z}\right) - i\kappa(\tau)\frac{\partial}{\partial z}(zW) \quad . \tag{1.41}$$

In the sequel, we will use the new time variable $\tau(t)$ instead of t, so that, for instance, $W(z, t)$ can be written as

$$W^{*}(z, \tau) = W(z, t) \quad .$$

However, we will keep our old notation W for simplicity.

4.4 A Riemann-Hilbert problem

So far we have defined an analytic function $F(z, t)$ continuous everywhere in the fluid except across the blade, and subject to the following conditions.

i. $F(z, \tau) \rightarrow 0$ as $z \rightarrow \infty$. (1.42)

ii. $F(z, \tau)$ and $W(z, \tau)$ are related by

$$\frac{\partial F}{\partial z} = \left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial z}\right)W(z, \tau) - i\kappa(\tau)\frac{\partial}{\partial z}(zW) \quad .$$

iii. $W(z, \tau) \rightarrow 0$ as $z \rightarrow \infty$, except in the region of the vortex wake.

iv. $v(x, t) = \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)h(x, t) + \omega x$,

on $y = 0^{\pm}$; $|x| < 1$. (1.43)

These conditions allow us to define a Riemann-Hilbert problem for $\frac{\partial F}{\partial z}$. However, it is more convenient to work with $F(z, \tau)$ itself, which we can obtain by integrating (1.41) with respect to z. Taking the integration along the trajectory from upstream infinity $(\xi = -\infty)$ to the blade $(\xi = z)$ yields

$$[F(\xi, \tau)]_{\xi=-\infty}^{\xi=z} = \frac{\partial}{\partial \tau} \int_{-\infty}^{z} W(\xi, \tau) d\xi + [W(\xi, \tau)]_{\xi=-\infty}^{\xi=z} - i\kappa(\tau)[\xi W(\xi, \tau)]_{\xi=-\infty}^{\xi=z} . \tag{1.44}$$

Performing the integration from upstream infinity and neglecting the effects due to the crossing of the wake, we find that both $W(z, \tau)$ and $F(z, \tau)$ tend asymptotically to zero as z goes to $-\infty$. Furthermore, the Kelvin's circulation theorem states that the total circulation around the wing-wake system is always zero; this in turn implies that in the far-field at upstream infinity $W(z, \tau)$ decays at least as $0(|z|^{-2})$. Making use of these conditions in (1.44) we obtain

$$F(z, \tau) = \beta(z, \tau) W(z, \tau) + \frac{\partial}{\partial \tau} \int_{-\infty}^{z} W(\xi, \tau) d\xi \quad , \tag{1.45}$$

where $\beta(z, \tau)$ is a new function defined by

$$\beta(z, \tau) = 1 - i\kappa(\tau) z \quad . \tag{1.46}$$

The problem described above can now be formulated as a Riemann-Hilbert problem for $F(z, \tau)$. We seek a solution for $F(z, \tau)$, required to be holomorphic in the region D defined by C_o and C_∞, where C_o encloses the blade and its wake and C_∞ is a closed contour circumventing the point of infinity (fig. 7), and subject to the following boundary conditions on C_o and C :

i. on C_∞ $\quad F(z, \tau) \to 0$;

ii. on C_o, for $|x| > 1$,

$$F^+(z, \tau) = F^-(z, \tau) \quad , \tag{1.47}$$

where F^+ and F^- refer to the value of F as z approaches the boundary from above and from below, respectively;

iii. on C_o, for $|x| < 1$,

now, the symmetry of the flow and the Cauchy-Riemann equations (1.25) imply that $v^\pm$ is even in y and $u^\pm$ odd in y; therefore $W^+ + W^- = -2iv^+$ on the blade and the boundary condition for F can be written from (1.45) as

$$(F^+ + F^-) = 2i[f_1(x, \tau) + A_o(\tau)] \quad , \tag{1.48}$$

where $f_1(x, \tau)$ and $A_o(\tau)$ are defined by

$$f_1(x, \tau) = -\beta(x, \tau)v^+(x, \tau) - \frac{\partial}{\partial \tau}\int_{-1}^{x} v^+(\xi, \tau)d\xi \quad , \tag{1.49}$$

and

$$iA_o(\tau) = \frac{\partial}{\partial \tau}\int_{-\infty}^{-1} W^+(z, \tau)dz \quad . \tag{1.50}$$

It is apparent from these expressions that $f_1(x, \tau)$ is completely defined by the motion of the blade $h(x, t)$ and the kinematic boundary condition (1.22), and that $A_o(\tau)$ is an unknown function of time only. It now remains to solve the Riemann-Hilbert problem for $F(z, \tau)$.

4.5 Solution of the Riemann-Hilbert problem

In order to solve this problem, we introduce two analytic functions $H(z)$ and $Q(z, \tau)$ such that

$$\begin{aligned} H^+ + H^- &= 0 \quad \text{for} \quad |x| < 1 \text{ on } C_o \ , \\ H^+ - H^- &= 0 \quad \text{for} \quad |x| > 1 \text{ on } C_o \ , \end{aligned} \tag{1.51}$$

and

$$Q(z, \tau) = \frac{F(z, \tau)}{H(z)} \quad . \tag{1.52}$$

The boundary conditions on C_o for $Q(z, \tau)$ can then be written as

$$\begin{aligned} Q^+ - Q^- &= (F^+ + F^-)/H^+ \ , \quad |x| < 1 \ , \\ Q^+ - Q^- &= 0 \ , \quad |x| > 1 \ . \end{aligned}$$

The solution to this problem is readily available (see ref. [17], §5); it consists of a particular solution and a complementary solution appropriate to the corresponding homogeneous problem. The particular solution can be expressed as

$$Q(z, \tau) = \frac{1}{2\pi i} \int_{C_o} \frac{Q^+ - Q^-}{\xi - z} \, d\xi \quad ,$$

which provides us with a particular solution for $F(z, \tau)$

$$F(z,\tau) = \frac{H(z)}{2\pi i} \int_{-1}^{+1} \frac{F^+(\xi, \tau) + F^-(\xi, \tau)}{H^+(\xi, \tau)} \frac{d\xi}{\xi - z} ,$$

in which $H(z)$, a solution of the homogeneous boundary-value problem, (1.51), can assume the form

$$H(z) = R(z) \left(\frac{z-1}{z+1}\right)^{1/2} ,$$

where $R(z)$ is a rational function of z and $H(z)$ is defined to be one-valued in the cut z-plane with a branch cut from $z = -1$ to $z = +1$ along the real z-axis, so that $H(z) \to R(z)$ as $|z| \to \infty$ for all arg z. In addition, the function $H(z)$ can also serve as a complementary solution of $F(z, \tau)$ if the rational function $R(z)$ is purely imaginary on the real z-axis since then the boundary conditions (1.47) and (1.48) automatically remain fulfilled. The uniqueness of the solution then depends on the required properties of $F(z, \tau)$:

i. $F(z, \tau)$ must be regular everywhere in D and must vanish at infinity;

ii. $F(z, \tau)$ is expected to have a leading edge singularity of order $(z + 1)^{-1/2}$, as shown by Wu [15];

iii. $F(z, \tau)$ must be finite (actually vanishes) at the trailing edge to satisfy the Kutta condition.

For the particular solution of $F(z, \tau)$, these properties imply that $R(z)$ cannot have a pole or a zero at finite z and $R(z) \simeq 0(1)$ near $z = \pm 1$. The unique function satisfying these conditions is

$$R(z) = \text{constant} \quad .$$

For the complementary solution for $F(z, \tau)$, these same properties (i) - (iii) required of F clearly leave no other possibilities for $R(z)$ but $R(z) = 0$, and hence a trivial complementary solution for F.

Consequently, the unique solution for $F(z, \tau)$ is

$$F(z, \tau) = \frac{1}{\pi i} \left(\frac{z-1}{z+1}\right)^{1/2} \int_{-1}^{+1} \left(\frac{1+\xi}{1-\xi}\right)^{1/2} \left[\frac{f_1(\xi, \tau) + A_o(\tau)}{\xi - z}\right] d\xi \quad . \tag{1.53}$$

The only unknown remaining in this equation is the function $A_o(\tau)$. This function is affected by the unsteadiness of the motion and is related to the strength of the leading edge singularity, as will be shown later. It appears in (1.50) in the form of a definite integral of the complex velocity.

In the derivation of (1.53), the Euler equation of motion (1.41) was used for the sole purpose of providing the boundary conditions for $F(z, \tau)$. Making use of our solution for $F(z, \tau)$ in equation (1.41) will enable us to obtain an integral equation for $W(z, \tau)$, or equivalently for $A_o(\tau)$. To this end we need an expression of $W(z, \tau)$ in terms of $F(z, \tau)$, which we can obtain by integrating (1.41) using the method of characteristics.

4.6 Integration of the Euler equation along its characteristics

We recall the expression of the Euler equation of motion (1.41)

$$\frac{\partial}{\partial z} F(z, \tau) = \left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial z}\right) W(z, \tau) - i\kappa(\tau) \frac{\partial}{\partial z} [zW(z, \tau)] \quad .$$

Using (1.46) we can rewrite this expression as

$$\frac{\partial F}{\partial z} = \frac{\partial W}{\partial \tau} + \beta(z, \tau) \frac{\partial W}{\partial z} - i\kappa(\tau)W \quad . \tag{1.54}$$

The characteristics of this equation are given by

$$\frac{dz}{d\tau} = \beta(z, \tau) \quad , \tag{1.55}$$

along which the following relationship is satisfied

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + \beta \frac{\partial}{\partial z} \quad . \tag{1.56}$$

Using (1.55) in (1.54) then yields

$$\frac{dW}{d\tau} - i\kappa(\tau)W = \frac{\partial F}{\partial z} \quad , \tag{1.57}$$

$$\text{along} \quad \frac{dz}{d\tau} = \beta(z, \tau) \; .$$

It may be noted that we could have used $\frac{d\tau}{dz}$ for the characteristics instead of $\frac{dz}{d\tau}$. The two formulations are equivalent and we can exchange variables by using

$$\frac{d}{dz} = \frac{1}{\beta} \frac{d}{d\tau} \quad . \tag{1.58}$$

4.6.1 <u>The characteristic lines</u>

Recalling the definition of $\beta(z, \tau)$ given by (1.46), we can write (1.55) as

$$\frac{dz}{d\tau} + i\kappa(\tau)z = 1 \quad .$$

By inspection, we can rewrite this equation as

$$\frac{d}{d\tau}(bz) = b \quad , \tag{1.59}$$

provided that

$$\frac{db}{d\tau} = ib\kappa(\tau) \quad . \tag{1.60}$$

Equation (1.60) can of course be integrated directly, but a simpler result can be found by recalling from (1.40b) and (1.1) that

$$\kappa(\tau) = \frac{\omega(\tau)}{V(\tau)}$$

and

$$\omega(t) = \frac{d\eta(t)}{dt} \quad .$$

Using these two relations, we have

$$\kappa(\tau) = \frac{d\eta}{d\tau} \quad , \tag{1.61}$$

hence (1.60) can be readily integrated to obtain

$$b(\tau) = e^{i\eta(\tau)} \quad .$$

Replacing this result in (1.59) yields

$$\frac{d}{d\tau}[ze^{i\eta(\tau)}] = e^{i\eta(\tau)} \quad , \tag{1.62}$$

which can be easily integrated to obtain

$$z'(z, \tau, \tau') = ze^{i\eta(\tau, \tau')} - \int_{\tau'}^{\tau} e^{i\eta(\tilde{\tau}, \tau')} d\tilde{\tau} \quad ,$$

where $\eta(\tau, \tau')$ is defined similarly to $\tilde{\eta}(t', t)$(eq. 1.11) by

$$\eta(\tau, \tau') = \eta(\tau) - \eta(\tau') \quad . \tag{1.63}$$

This result can be expressed in a more elegant form by using the equation of the trajectory, as defined in (1.12) by

$$X_c(t', t) = -\int_t^{t'} V(\tilde{t}) \cos \tilde{\eta}(\tilde{t}, t)\, d\tilde{t} \quad ,$$

$$Y_c(t', t) = -\int_t^{t'} V(\tilde{t}) \sin \tilde{\eta}(\tilde{t}, t)\, d\tilde{t} \quad .$$

Expressing the equation of the trajectory in complex form yields

$$Z_c(\tau', \tau) = X_c(\tau', \tau) + iY_c(\tau', \tau) = -\int_\tau^{\tau'} e^{i\eta(\tilde{\tau}, \tau)} d\tilde{\tau} \quad , \tag{1.64}$$

and since

$$\int_{\tau'}^{\tau} e^{i\eta(\tilde{\tau}, \tau')} d\tilde{\tau} = Z_c(\tau', \tau) e^{i\eta(\tau, \tau')} \quad ,$$

we obtain the general expression for the characteristic lines as

$$z'(z, \tau, \tau') = e^{i\eta(\tau, \tau')}[z - Z_c(\tau', \tau)] \quad . \tag{1.65}$$

4.6.2 Integration along the characteristics

By analogy with (1.62), we can rewrite equation (1.57) as

$$\frac{d}{d\tau}[We^{-i\eta(\tau)}] = e^{-i\eta(\tau)} \frac{\partial F}{\partial z} \quad . \tag{1.66}$$

Using (1.56) and integrating (1.66) from 0 to τ, with $W(z,0) = 0$, yields

$$W(z,\tau)e^{-i\eta(\tau)} = \int_0^\tau \frac{e^{-i\eta(\tilde{\tau})}}{\beta(z',\tilde{\tau})}\left(\frac{dF}{d\tilde{\tau}}\right)d\tilde{\tau} - \int_0^\tau \frac{e^{-i\eta(\tilde{\tau})}}{\beta(z',\tilde{\tau})}\left(\frac{\partial F}{\partial \tilde{\tau}}\right)d\tilde{\tau} \quad .$$

Finally, integrating the first integral by parts, we obtain

$$\beta(z,\tau)W(z,\tau) = F(z,\tau) - i\beta(z,\tau)\int_0^\tau \frac{F(z',\tilde{\tau})z'\dot{\kappa}(\tilde{\tau})e^{-i\eta(\tilde{\tau},\tau)}}{\beta^2(z',\tilde{\tau})}d\tilde{\tau}$$

$$-\beta(z,\tau)\int_0^\tau \frac{\partial F}{\partial \tilde{\tau}}(z',\tilde{\tau})\,\frac{e^{-i\eta(\tilde{\tau},\tau)}}{\beta(z',\tilde{\tau})}d\tilde{\tau} \quad , \tag{1.67}$$

where $z'(z,\tau,\tilde{\tau})$ is given by (1.65).

This equation and the equation of the characteristic lines (1.65) provide the general relationship between $W(z,\tau)$ and $F(z,\tau)$. Substitution of the solution (1.53) found for $F(z,\tau)$ into (1.67) results in an integral equation for $W(z,\tau)$ and the problem is therefore reduced to solving a single integral equation involving only one variable. A closed form solution cannot be obtained at this point, but the integral equation (1.67) can be solved by numerical methods.

Some further simplifications can be made since we restricted ourself to the case $\Omega R \gg U$, the trajectory is a prolate cycloid (fig. 3c) and the path curvature is nearly constant and of order $(1/R)$. An explicit evaluation to leading order of the terms in the Euler equation (1.57) is given in Appendix A. This evaluation suggests that it should be a good approximation to neglect the variation of the path curvature.

4.6.3 Simplified expression of the characteristic equation

Taking the derivative of $\beta(z, \tau)W(z, \tau)$ along the characteristics, using (1.46) and (1.58), we obtain

$$\frac{d}{dz}(\beta W) = \beta \frac{dW}{dz} - i\kappa W\left[1 + \frac{z\dot{\kappa}}{\beta\kappa}\right] , \qquad (1.68)$$

where $\dot{\kappa}$ stands for $\frac{d\kappa(\tau)}{d\tau}$. It is shown in Appendix A that, within the framework of our linearized theory, the term $\frac{z\dot{\kappa}}{\beta\kappa}$ is at least of order $(1/R)$, and is of order $(1/R^2)$ for z of order unity. This suggests that we may neglect it in comparison with unity in (1.68), which is equivalent to neglecting the variation of the curvature of the path and should lead to a uniformly valid approximation of the solution at first order. Performing this simplification and using (1.58), we can express (1.68) as

$$\frac{d}{d\tau}(\beta W) = \beta\left[\frac{dW}{d\tau} - i\kappa W\right] . \qquad (1.69)$$

Replacing (1.57) in (1.69), we obtain the approximate expression for the Euler equation of motion in terms of the characteristics

$$\frac{d}{d\tau}(\beta W) = \beta \frac{\partial F}{\partial z} \qquad \text{on} \qquad \frac{dz}{d\tau} = \beta(z, \tau) . \qquad (1.70)$$

Integrating this expression from 0 to τ, using the initial condition $W(z, 0) = 0$, yields

$$\beta(z, \tau)W(z, \tau) = \int_0^\tau \beta(z', \tilde{\tau}) \frac{\partial F}{\partial z'}(z', \tilde{\tau})d\tilde{\tau} ,$$

where $z'(z, \tau, \tilde{\tau})$ is given by (1.65). Finally, integrating this expression by parts, using (1.56), (1.65) and the initial condition $F(z, 0) = 0$, leads to

$$\beta(z, \tau)W(z, \tau) = F(z, \tau) - \int_0^\tau \frac{\partial F}{\partial \tilde{\tau}}(z', \tilde{\tau})d\tilde{\tau} \quad , \tag{1.71}$$

where $z'(z, \tau, \tilde{\tau})$ is given by (1.65).

As can be seen easily, this expression is much simpler than (1.67) while still accurate to first order. It expresses the dependence of the local velocity of the fluid at any point on the instantaneous pressure at this point together with the retarded pressure which was present along the characteristic line emanating from this point. Substitution of the solution (1.53) for $F(z, \tau)$ into this simplified expression results again in an integral equation for $W(z, \tau)$ or equivalently an integral equation for $A_o(\tau)$.

4.7 <u>Development of the integral equation for $A_o(\tau)$</u>

By comparing the two integral forms of the Euler equation (1.45) and (1.71) we deduce the following relationship

$$\frac{\partial}{\partial \tau} \int_{-\infty}^{z} W(\xi, \tau)d\xi = \int_0^\tau \frac{\partial F}{\partial \tilde{\tau}}[z'(z, \tau, \tilde{\tau}), \tilde{\tau}]d\tilde{\tau} \quad , \tag{1.72}$$

where $z'(z, \tau, \tilde{\tau})$ is given by (1.65). Using (1.50) and (1.72), we can then express $A_o(\tau)$ as

$$iA_o(\tau) = \int_0^\tau \frac{\partial F}{\partial \tilde{\tau}}[z'(-1, \tau, \tilde{\tau}), \tilde{\tau}]\, d\tilde{\tau} \quad . \tag{1.73}$$

It is convenient at this point to rewrite the solution (1.53) for $F(z, \tau)$ in a form which isolates the contribution of the singularity at the leading edge. Using the identity

$$\frac{1}{\xi - z}\left(\frac{1+\xi}{1-\xi}\right)^{1/2} = (1 - \xi^2)^{-1/2}\left[\frac{1+z}{\xi - z} + 1\right] ,$$

we can rewrite (1.53) as

$$F(z, \tau) = -\frac{i}{2} a_o(\tau)\left(\frac{z-1}{z+1}\right)^{1/2} + \frac{1}{\pi i}\int_{-1}^{+1}\left(\frac{z^2-1}{1-\xi^2}\right)^{1/2}\left[\frac{f_1(\xi, \tau)+A_o(\tau)}{\xi - z}\right]d\xi ,$$

where $a_o(z)$ is defined by

$$\frac{1}{2} a_o(\tau) = \frac{1}{\pi}\int_{-1}^{+1}\left[\frac{f_1(\xi, \tau) + A_o(\tau)}{(1 - \xi^2)^{1/2}}\right]d\xi . \qquad (1.74)$$

We can simplify this further by using the identity

$$\int_{-1}^{+1}\left(\frac{z^2-1}{1-\xi^2}\right)^{1/2}\frac{d\xi}{\xi - z} = -\pi ,$$

which holds valid for arbitrary z, not lying on the branch cut extending from $z = -1$ to $z = +1$ along the real z axis,

$$F(z, \tau) = iA_o(\tau) - \frac{i}{2} a_o(\tau)\left(\frac{z-1}{z+1}\right)^{1/2} + \frac{1}{\pi i}\int_{-1}^{+1}\left(\frac{z^2-1}{1-\xi^2}\right)^{1/2}\frac{f_1(\xi, \tau)}{\xi - z}\, d\xi . \qquad (1.75)$$

It may be noted that the only term of this equation which is singular at the leading edge is $a_o(\tau)(z-1)^{1/2}(z+1)^{-1/2}$. We can therefore consider

$a_o(\tau)$ as representing the strength of the leading edge singularity. Making use of this expression for $F(z, \tau)$ in (1.73), we obtain

$$iA_o(\tau) = \int_o^\tau \frac{\partial}{\partial\tilde{\tau}}[iA_o(\tilde{\tau})]d\tilde{\tau} - \frac{i}{2}\int_o^\tau \frac{\partial}{\partial\tilde{\tau}}\left[\left(\frac{z'-1}{z'+1}\right)^{1/2} a_o(\tilde{\tau})\right]d\tilde{\tau}$$

$$+ \frac{1}{\pi i}\int_o^\tau d\tilde{\tau}\int_{-1}^{+1}\left(\frac{z'^2-1}{1-\xi^2}\right)^{1/2} \frac{f_1(\xi, \tilde{\tau})}{\xi - z'}d\xi \quad .$$

Finally, integrating the first integral term in this expression, using $A_o(0) = 0$, leads to

$$\frac{i}{2}\int_o^\tau \frac{\partial}{\partial\tilde{\tau}}\left[\left(\frac{z'-1}{z'+1}\right)^{1/2} a_o(\tilde{\tau})\right]d\tilde{\tau} = \frac{1}{\pi i}\int_o^\tau d\tilde{\tau}\ \frac{\partial}{\partial\tilde{\tau}}\int_{-1}^{+1}\left(\frac{z'^2-1}{1-\xi^2}\right)^{1/2} \frac{f_1(\xi, \tilde{\tau})}{\xi - z'}d\xi, \tag{1.76}$$

where the integrations are to be performed along the characteristic line emanating from $z = -1$.

This expression provides us with an integral equation for $a_o(\tau)$ which, if solved exactaly, would lead to a uniformly valid solution of $F(z, \tau)$. Unfortunately this equation, though considerably simpler than the original integral equation for $A_o(\tau)$, still needs to be solved numerically due to the complicated shape of the characteristic lines. A closed form solution for $a_o(\tau)$ can only be obtained by performing the integration along some simplified characteristic lines.

4.8 Approximated characteristic lines

The exact expression of the characteristic lines was given in (1.65) by

$$z'(z, \tau, \tau') = e^{i\eta(\tau, \tau')}[z - Z_c(\tau', \tau)] ,$$

in which $Z_c(\tau', \tau)$ represents the trajectory

$$Z_c(\tau', \tau) = X_c + iY_c = -\int_{\tau}^{\tau'} e^{i\eta(\tilde{\tau}, \tau)} d\tilde{\tau} .$$

Noting that

$$e^{i\eta(\tau, \tau')} Z_c(\tau' \tau) = \int_{\tau'}^{\tau} e^{i\eta(\tilde{\tau}, \tau')} d\tilde{\tau} = -Z_c(\tau, \tau') ,$$

we see that the characteristic lines look like "reversed" trajectories further distorted by the addition of a cyclic perturbation $z e^{i\eta(\tau, \tau')}$ (fig. 8).

An approximation of these lines can be made by the same simplification adopted in the development of equation (1.71) which amounts to neglecting the variation of the curvature of the path, which may be regarded as small in the present case. From Appendix A, $\kappa(\tau)$ can be expressed as

$$\kappa(\tau) = \frac{1}{R}[1 - 2\epsilon' \cos\theta + 0(\epsilon'^2)] ,$$

where by definition $\epsilon' = U/\Omega R$, here taken to be small, $\epsilon' \ll 1$.

From this expression, we see that $\kappa(\tau)$ can be approximated to the leading order by a constant representing the curvature of the path traversed in the inertial frame of reference:

$$\kappa(\tau) \simeq \frac{1}{R} . \tag{1.77}$$

Using this approximation in (1.61) and integrating from O to τ, using (1.63), we obtain

$$\eta(\tau, \tau') \approx \frac{1}{R}(\tau - \tau') \quad . \qquad (1.78)$$

The expression of the approximated characteristic lines can then be found by using (1.78) in (1.64), integrating (1.64), and replacing it in (1.65),

$$z'(z, \tau, \tau') = z\, e^{i\frac{(\tau - \tau')}{R}} - iR[1 - e^{i\frac{(\tau - \tau')}{R}}] \quad , \qquad (1.79)$$

which can also be expressed by simple inversion as

$$(\tau - \tau') = \frac{1}{i\epsilon} \log_e [\frac{1 - i\epsilon z'}{1 - i\epsilon z}] \quad , \qquad (1.80)$$

where $\epsilon \ll 1$ is defined by

$$\epsilon = \frac{1}{R} \quad . \qquad (1.81)$$

These curves approximate the characteristics at order $O(\frac{1}{R})$ for z' of O(1) and are still accurate at leading order for higher values of z'. They only diverge slowly from the true characteristics as the retarded time increases, since the true characteristics drift away from the airfoil after each revolution, while the approximated curves are periodic and close on themselves at $z = -1$ (as shown in fig. 8). This periodic behavior is indeed the major problem associated with this approximation, which should only be used with caution. In the case of a blade starting from rest, the curves can be used directly during most of the first revolution. If we are interested in the flow field a long time after the start of the motion, then we must restrict ourselves to some limited portion of these lines in order to avoid coming back in the area of strong disturbances around the leading edge. In doing so, we obviously neglect the portion of the characteristics associated with a large retarded time. The physical reasoning behind this simplification lies in the influence of the retarded pressure upon the instantaneous velocity field. Intuitively it is logical that the retarded pressures having the most influence are those occurring in the immediate past history. This assumption is further supported by the relative influence of the wake vorticity, which represents the link between the past history of the motion and the present state of the flow. Von Kármán and Sears have shown that the effect of the wake vorticity on the induced vorticity distribution on a blade is very much dependent on the distance from the wake vortex to the blade (see figure 2 and reference 2 of part II), and becomes practically negligible when the vortex is a few chords away from the blade. It is therefore quite clear that a small portion of the wake is really responsible for the local velocity field in the vicinity of the blade. Obviously, the more general flow field around the complete turbine is induced by the total wake vorticity, but this induced velocity appears as a constant in the immediate vicinity of the blade and therefore merely changes the apparent free flow velocity of the fluid around the blade. Similarly, the retarded pressures occurring at large values of the retarded time reflect the state of the general flow field around the turbine and have little influence on the instantaneous flow field in the vicinity of the blade. If we recall that the present theory can be considered as a correction of the quasi steady momentum theory, with the purpose of evaluating the effect of the unsteadiness on the blade loading, the above simplification can be seen as a way to provide a leading order approximation of this correction.

The easiest way to practically avoid part of the approximated characteristic is to take an expansion in ϵ of (1.79), or (1.80), and only keep the lower order terms. An approximation at leading order leads to a straight line while a first order approximation includes the curvature and approximates the characteristic up to z' of order R. In any case the approximate curve would diverge from (1.79) before this line comes back in the region of higher disturbance, and therefore avoids the aforementioned problem.

4.9 Solution for $a_o(\tau)$

The integral equation for $a_o(\tau)$ was given in (1.76) as

$$\frac{i}{2}\int_o^{\tau} \frac{\partial}{\partial\tilde{\tau}}\left[\left(\frac{z'-1}{z'+1}\right)^{1/2} a_o(\tilde{\tau})\right] d\tilde{\tau} = \frac{1}{\pi i}\int_o^{\tau} d\tilde{\tau}\ \frac{\partial}{\partial\tilde{\tau}}\int_{-1}^{+1}\left(\frac{z'^2-1}{1-\xi^2}\right)^{1/2}\frac{f_1(\xi,\tilde{\tau})}{\xi-z'}d\xi,$$

where the integration is meant to be performed along $z'(-1,\tau,\tau')$. Using (1.58) and approximating $\beta(z,\tau)$ by

$$\beta(z,\tau) = 1 - i\kappa(\tau)z \simeq 1 - i\epsilon z = \beta(z) \quad ,$$

we can change the integration variable and express (1.76) as

$$\frac{\partial}{\partial\tau}\int_{-\infty}^{-1}\left(\frac{\tilde{z}-1}{\tilde{z}+1}\right)^{1/2} a_o(\tau')\ \frac{d\tilde{z}}{\beta(\tilde{z})} = -\frac{2}{\pi}\frac{\partial}{\partial\tau}\int_{-\infty}^{-1}\frac{d\tilde{z}}{\beta(\tilde{z})}\int_{-1}^{+1}\left(\frac{\tilde{z}^2-1}{1-\xi^2}\right)^{1/2}\frac{f_1(\xi,\tau')}{\xi-\tilde{z}}\ d\xi\ , \tag{1.82}$$

in which we have chosen the initial position to be $z_o \rightarrow -\infty$ and the integration path is along $\tau'(\tau, -1, \tilde{z})$. Using (1.80), the approximated characteristic $\tau'(\tau, -1, z')$ can be expressed as

$$\tau'(\tau, -1, z') \simeq \tau - \frac{1}{i\epsilon} \log_e \left[\frac{1 - i\epsilon z'}{1 + i\epsilon}\right] .$$

Expanding this expression for small ϵ yields

$$\tau'(\tau, -1, z') \simeq \tau + (1 + z') - \frac{i\epsilon}{2}(1 - z'^2) + 0(\epsilon^2) \quad ,$$

which can be expressed, after neglecting the higher order terms, as

$$\begin{aligned} \tau'(\tau, -1, z') &\simeq \tau + B(z') \quad , \\ B(z') &= (1+z') - \frac{i\epsilon}{2}(1-z'^2) . \end{aligned} \tag{1.83}$$

With this approximation equation (1.82) becomes

$$\frac{\partial}{\partial \tau} \int_{-\infty}^{-1} \left(\frac{z-1}{z+1}\right)^{1/2} \frac{a_o[\tau + B(z)]}{\beta(z)} dz =$$

$$- \frac{2}{\pi} \frac{\partial}{\partial \tau} \int_{-\infty}^{-1} \frac{dz}{\beta(z)} \int_{-1}^{+1} \left(\frac{z^2 - 1}{1 - \xi^2}\right)^{1/2} \frac{f_1(\xi, \tau + B(z))}{\xi - z} d\xi \quad . \tag{1.84}$$

This equation can now be solved by the method of Laplace transform.

Defining the Laplace transform by

$$\mathcal{L}[a_o(\tau)] = \tilde{a}_o(s) = \int_0^{\infty} e^{-s\tau} a_o(\tau) d\tau \quad , \qquad (\mathrm{Re}(s) > 0) \quad ,$$

and since $\mathrm{Re}[B(z)] < 0$, as the integration of (1.84) is performed from $z = -\infty$ to $z = +1$, we can use the shift property of this transformation to express the functions depending on $[\tau + B(z)]$:

$$\mathcal{L}\,[a_o(\tau + B(z))] = e^{sB(z)}\,\tilde{a}_o(s) \quad ,$$

and

$$\mathcal{L}\,[\, f_1(\xi, \tau + B(z))] = e^{sB(z)}\,\tilde{f}_1(\xi, s) \quad .$$

Taking the Laplace transform of (1.84) then leads to

$$s\tilde{a}_o(s)\int_{-\infty}^{-1}\left(\frac{z-1}{z+1}\right)^{1/2}\frac{e^{sB(z)}}{\beta(z)}\,dz =$$

$$-\frac{2s}{\pi}\int_{-\infty}^{-1}\frac{e^{sB(z)}}{\beta(z)}\,dz\int_{-1}^{+1}\left(\frac{z^2-1}{1-\xi^2}\right)^{1/2}\frac{f_1(\xi, s)}{\xi - z}\,d\xi \quad . \qquad (1.85)$$

As is readily apparent, the two integrals are completely defined and therefore $\tilde{a}_o(s)$ can be expressed in closed form from this equation. The remaining problem is in the evaluation of the two integrals and in the inverse transformation of $\tilde{a}_o(s)$. This evaluation is carried out in Appendix B up to order ϵ and the result is

$$a_o(s) = \frac{2}{\pi}\int_{-1}^{+1}\frac{\tilde{v}^{+}(\xi, s)}{(1-\xi^2)^{1/2}}\,[\xi + (1-\xi)\tilde{H}(s) + \frac{i\epsilon}{2}(\xi - 1)]d\xi \quad , \qquad (1.86)$$

where

$$\tilde{H}(s) = \frac{K_1(s)}{K_o(s) + K_1(s)} \quad , \qquad (1.87)$$

and $K_o(s)$ and $K_1(s)$ are modified Bessel's functions of the second kind.

Performing the inverse transformation of this relation leads to

$$a_o(\tau) = \frac{2}{\pi}\int_{-1}^{+1}\left[[\frac{v^+(\xi,\tau)}{(1-\xi^2)^{1/2}}][\xi + \frac{i\epsilon}{2}(\xi-1)] - \frac{(1+\xi)}{(1-\xi^2)^{1/2}}\int_o^{\tau} v^+(\xi,\tilde{\tau})H(\tau-\tilde{\tau})d\tau\right]d\xi, \tag{1.88}$$

where $H(\tau)$ is defined by the Mellin inversion integral as

$$H(\tau) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{s\tau}\tilde{H}(s)ds \quad , \qquad (\alpha > 0) \quad , \tag{1.89}$$

where the integration path is the Bromwich contour. Finally, we can express $A_o(\tau)$ from the definition of $a_o(\tau)$ given in (1.74); the result is

$$A_o(\tau) = \frac{1}{2}a_o(\tau) - \frac{1}{\pi}\int_{-1}^{+1}\frac{f_1(\xi,\tau)}{(1-\xi^2)^{1/2}}d\xi \quad , \tag{1.90}$$

where $f_1(\xi,\tau)$ is given by (1.49).

5. Forces

As is usual in potential flow, the only forces exerted on the blade by the fluid are due to the pressure. They consist of the pressure force, which arises from the difference in the pressure acting on both sides of the blade, and the leading edge suction, which arises from the singularity of the pressure at the nose of the airfoil.

5.1 The pressure force

This force is always normal to the blade for a flat plate blade and can be obtained directly from the integration of the pressure jump across the blade.

$$F_p = \int_{-1}^{+1}(\Delta p)dx \quad , \tag{1.91}$$

where $\Delta p = (p^- - p^+)$. The pressure jump across the blade can be expressed from (1.30) as

$$\Delta p(x,t) = \rho[\Phi^+(x,t) - \Phi^-(x,t)] \quad ,$$

and from equations (1.53), (1.40a) and (1.34a) we obtain

$$\Delta p(x,t) = \frac{2}{\pi}\rho V(t)\left(\frac{1-x}{1+x}\right)^{1/2}\int_{-1}^{+1}\left(\frac{1+\xi}{1-\xi}\right)^{1/2}\frac{\mathrm{Re}[f_1(\xi,t)+A_o(t)]}{\xi - x}\,d\xi \quad , \tag{1.92}$$

where $f_1(\xi,t)$ and $A_o(t)$ are the functions $f_1(\xi,\tau)$ and $A_o(\tau)$ expressed with the regular time t, and Re[ ] stands for the real part of a complex expression.

5.2 The leading edge suction

The leading edge suction arises from the singular pressure on the nose of the blade. It can be evaluated by applying the Blasius theorem on a contour enclosing a small neighborhood around the leading edge. The behavior of $F(z,\tau)$ as $z \to -1$ can be obtained from (1.53), or from (1.75) where the singularity is already singled out.

At this point it must be noted that in the linearized Bernoulli equation (1.28) the quadratic perturbation velocity term was neglected. This simplification is not valid in the neighborhood of the leading edge, where a singularity in $\underset{\sim}{u}$ exists, and therefore the solution obtained for $F(z,\tau)$ does not reflect the true pressure field in this area. However, the behavior of $W(z,\tau)$ can still be obtained from (1.75) as the particular solution of $F(z,\tau)$ was chosen, following Wu [15], so as to provide an integrable singularity in $W(z,\tau)$ at the leading edge.

Then, from equation (1. 45), we obtain

$$W(z, \tau) \simeq \frac{a_o(\tau)}{\beta(-1, \tau)} \quad \frac{1}{[2(z+1)]^{1/2}} + 0(1) \quad , \tag{1.93}$$

as $z \rightarrow -1$.

The general form of the Blasius Theorem is

$$X_s - iY_s = \frac{i\rho}{2} \oint_{L.\,E.} \widetilde{W}^2 dz + i\rho \oint_{L.\,E.} \frac{\partial\varphi}{\partial t} \, d\bar{z} \quad , \tag{1.94}$$

where X_s and Y_s represent the components of the singular force along $\underset{\sim}{e}_x$ and $\underset{\sim}{e}_y$ respectively and $\widetilde{W}$ is the total complex flow velocity composed of both the perturbation velocity W and the velocity induced by the nose of the blade in the immediate vicinity of the leading edge. The only term in the equation whose contribution does not vanish in the limit is the quadratic perturbation velocity term. Equation (1. 94) therefore reduces to

$$X_s - iY_s = \frac{i\rho}{2} \oint_{L.\,E.} W^2(z, \tau) dz \quad ,$$

which, when applied to (1. 93), yields

$$X_s = -\frac{\pi}{2} \rho \;\; \mathrm{Re} \left[\frac{a_o(\tau)}{\beta(-1, \tau)}\right]^2 \quad , \tag{1.95a}$$

$$Y_s = +\frac{\pi}{2} \rho \;\; \mathrm{IM} \left[\frac{a_o(\tau)}{\beta(-1, \tau)}\right]^2 \quad . \tag{1.95b}$$

As is readily apparent from these relations, the singular force is composed in the present case of both a tangential and a normal component. The presence of a normal component to the leading edge suction is a new phenomenon directly related to the curvature of the trajectory. Both the imaginary part of $\beta(-1, \tau)$, expressing the angular velocity ω of the body frame of reference, and the imaginary part of $a_o(\tau)$, reflecting the integration of the Euler equation of motion (1.71) over a curved characteristic line, are at the origin of Y_s. Since both terms are linear in $\kappa(\tau)$, and are therefore of order ϵ, the Y_s component itself is linear in $\kappa(\tau)$ and of order ϵ.

5.3 The total force

The total force is composed of F_p, X_s and Y_s. From (1.95a), we see that X_s is negative and therefore represents a local thrust. The direction of F_p depends on the position of the blade in the body frame of reference, and can induce either a thrust or a drag. The total force can be expressed by its components in the body frame of reference.

$$\underset{\sim}{F} = X \underset{\sim}{e}_x + Y \underset{\sim}{e}_y \quad , \tag{1.96}$$

where

$$X = X_s - F_p \sin \lambda' \quad , \tag{1.97a}$$

$$Y = Y_s + F_p \cos \lambda' \quad , \tag{1.97b}$$

and λ' represents the angle of the blade in the body frame of reference such that

$$\tan \lambda' = \frac{\partial h}{\partial x} \quad . \tag{1.98}$$

The force in the $-\underset{\sim}{e}_{x_1}$ direction, whose product with R (the moment arm) gives the final torque of the turbine, can be easily calculated

$$T_1 = -X \cos \lambda + Y \sin \lambda \quad . \tag{1.99}$$

Finally, the force in the $-\underset{\sim}{e}_{x_o}$ direction is given by

$$T_o = -X \cos \eta + Y \sin \eta \quad . \tag{1.100}$$

5.4 Moments

The moment of the total force about the origin of the body frame of reference is given by

$$M = \int_{-1}^{+1} x \Delta p \quad \cos \lambda' \, dx - Y_s \quad , \tag{1.101}$$

where $(\Delta p)\cos \lambda'$ represents the component of the pressure force in the $\underset{\sim}{e}_y$ direction.

The moment of the total force about the center of the turbine, representing the torque applied on the turbine axis by the blade, can be readily obtained from (1.99) as

$$M_o = T_1 R \quad . \tag{1.102}$$

6. Power and energy

6.1 Kinetic energy

The rate at which kinetic energy is imparted to the fluid by the blade can be easily obtained in the fluid frame of reference, in which the apparent velocity of the flow vanishes at infinity, by

$$\dot{E} = \frac{d}{dt} \int_D \frac{1}{2} \rho \underset{\sim}{u}^2 \, dV ,$$

where D represents the complete fluid region. Using (1.31) and the divergence free nature of $\underset{\sim}{u}$, it is possible to convert this volume integral into the following surface integral

$$\dot{E} = \int_{C_o + C_\infty} \rho \Phi \underset{\sim}{u} \cdot \underset{\sim}{n} \, ds ,$$

where C_o and C_∞ were defined in section 4.4 (fig. 7) and $\underset{\sim}{n}$ is the unit vector normal to the surface, pointing away from the fluid. Noting that $|\underset{\sim}{u} \cdot \underset{\sim}{n}|$ is continuous across the surfaces, that Φ is continuous everywhere except across the blade and that $\Phi(\underset{\sim}{u} \cdot \underset{\sim}{n})$ is of order $O(|z|^{-2})$ in the far field, we obtain for $\dot{E}$

$$\dot{E} = -\int_{-1}^{+1} v(x,t) \Delta p \cos \lambda' \, dx + \int_{LE} \rho \Phi (\underset{\sim}{\tilde{u}} \cdot \underset{\sim}{n}) ds . \qquad (1.103)$$

In this equation, the velocity $\underset{\sim}{\tilde{u}}$ represent the total fluid velocity with respect to the fluid frame of reference in the neighborhood of the leading edge. Using the kinematic boundary condition

$$\underset{\sim}{\tilde{u}} \cdot \underset{\sim}{n} = \underset{\sim}{V}_{LE} \cdot \underset{\sim}{n} ,$$

where $\underset{\sim}{V}_{LE}$ represents the velocity of the nose of the blade, we obtain the relation

$$\int_{LE} \rho \Phi (\underset{\sim}{\tilde{u}} \cdot \underset{\sim}{n}) ds = - \underset{\sim}{V}_{LE} [\int_{LE} \rho p \underset{\sim}{n} \, ds] .$$

Noting that the integral on the right hand side of the equation represents the leading edge suction, we can express (1.103) as

$$\dot{E} = -\int_{-1}^{+1} v(x,t)\, \Delta p \cos\lambda' \, dx - \underset{\sim}{V}_{LE} \cdot \underset{\sim}{F}_s \quad , \tag{1.104}$$

where $\underset{\sim}{V}_{LE}$ is given in the body frame of reference by

$$\underset{\sim}{V}_{LE} = -V(t)\, \underset{\sim}{e}_x + \left[\frac{\partial h}{\partial t} + \omega x\right]_{z=-1} \underset{\sim}{e}_y \quad . \tag{1.105}$$

Finally, using (1.105), (1.101), (1.98) and (1.91) in (1.104), we obtain

$$\dot{E} = +V(t)X_s - M\omega - V(t)F_p \sin\lambda' - \left[\frac{\partial h}{\partial t}\right]_{x=-1} Y_s - \int_{-1}^{+1} \left(\frac{\partial h}{\partial t}\right) \Delta p \cos\lambda' \, dx \quad . \tag{1.106}$$

6.2 Power output

The power input to the turbine, defined to be negative if energy is extracted from the fluid, consists of the energy necessary to sustain the rotational motion of the blade around the axis of the turbine, the energy necessary to maintain the rotational motion of the blade with respect to its mid chord and the energy necessary to overcome the hydrodynamic reaction to the blade motion in the body frame of reference. Hence the total power input is

$$P = -M_o \boldsymbol{\Omega} - M\omega - Y_s \left.\frac{\partial h}{\partial t}\right|_{x=-1} - \int_{-1}^{+1} \left(\frac{\partial h}{\partial t}\right) \Delta p \cos\lambda' \, dx \quad . \tag{1.107}$$

The power output of the turbine is then simply

$$P_{out} = -P \quad . \tag{1.108}$$

6.3 Energy balance

By comparison between (1.106) and (1.107), using (1.97a), we can immediately write

$$P - \dot{E} = -M_o \Omega - V(t)X \quad .$$

Combining then this expression with (1.102), (1.99) and (1.100) and using the following geometric relationships

$$\Omega R \sin\lambda = -U \sin\eta \quad ,$$

$$\Omega R \cos\lambda = V(t) - U \cos\eta \quad ,$$

we obtain

$$P = \dot{E} + T_o U \quad . \tag{1.109}$$

This relation expresses the principle of conservation of energy, by which the power input to the turbine must be equal to the rate of work done by the thrust, $T_o U$, plus the kinetic energy imparted to the fluid in unit time.

7. Conclusion

In this work, a hydromechanical theory was developed which proceeds from the point of view of unsteady airfoil theory. While primarily intended for energy extraction devices, such as the Darrieus rotor, this theory is applicable to a large class of vertical axis turbines, including cycloidal propellers of the Schneider type. The study was originally intended for the case of a fast rotating turbine in a slow stream, but since no limitation was applied to the range of the reduced frequency $\omega c/U$, the theory is inherently valid for other cases. Of particular interest, for instance, is the high speed propulsion mode of the Schneider propeller to which this theory can be applied with only minor changes. A uniformly valid first order solution has been obtained in closed form after making an approximation, which is based on neglecting the variation of the curvature of the path, thus approximating a prolate cycloid by a circle. Such an approximation should be physically sound since the retarded pressure having the strongest influence upon the velocity field is that occurring in the immediate past history. From the resulting value for $a_o(\tau)$, we find that the leading edge suction is composed in the present case of both tangential and a normal (to the blade chord) component. The normal component is linear in $\kappa(\tau)$ and reveals the first-order effects of the asymmetry of the flow field in the neighborhood of the leading edge due to the curvature of the flow. Unfortunately, no data are available to show the composition of this singular force acting on a blade in cycloidal motion and the existence of such a normal component has therefore never been observed. However, the existence of a normal force acting on a cylinder in cycloidal motion tends to support the possibility that such a component of the leading edge suction can occur. Finally, the contributions of the unsteady effects to the instantaneous force and moment acting on the blade, the total power output of the turbine and the total energy lost to the fluid have been fully evaluated. It is seen that the total power output is equal to the rate of work done by the thrust plus the kinetic energy imparted to the fluid. Contrary to the quasi-steady approach, where no energy is lost to the fluid, the unsteady approach allows a hydromechanical efficiency to be defined as the ratio of the useful power output to the total work done by the thrust (P/UT_o). This allows more flexibility in the design of a turbine by leaving the choice to the designer of maximizing either the hydromechanical efficiency, thus imparting the least disturbance to the fluid, or the mechanical efficiency (defined as the ratio of the power output to the power available in the stream), thus maximizing the total power output of the turbine. At the present time, no experimental data are available regarding the contribution of the curvature and of the unsteady effects on the total power output of a turbine and on the instantaneous blade loading. However, the inclusion of these effects is necessary for an accurate solution to be obtained for the entire range of the dynamic parameters involved in practical application; it is on this basis that the present theory is developed, which we hope will prove to be a useful tool in the engineering design of vertical axis wind turbines.

APPENDIX A

As the present study is limited to the case

$$\Omega R \gg U \ , \qquad R \gg c \ ,$$

we can define ϵ' and ϵ such that, with the half chord normalized to unity,

$$\epsilon' = U/\Omega R \ll 1 \ , \qquad \epsilon = 1/R \ll 1.$$

We furthermore assume Ω, R, and U to be constant. In order to carry out our evaluation, we need to recall the following relationships:

$$\beta(z, \tau) = 1 - i\kappa(\tau)z \quad ;$$

$$\kappa(\tau) = \frac{\omega}{V} \quad ;$$

$$V = |\underset{\sim}{U} + \underset{\sim}{\Omega} \times \underset{\sim}{R}| = (U^2 + \Omega^2 R^2 + 2U\Omega R \cos\theta)^{1/2} \quad ;$$

$$\omega = \lambda_t + \theta_t \quad ;$$

$$\Omega = \theta_t \quad ;$$

$$\tan\lambda = \frac{-\sin\theta}{\cos\theta + \Omega R/U} \quad .$$

We can then evaluate the following terms up to order ϵ (or ϵ'):

1) $$V = \frac{U}{\epsilon'}(1 + 2\epsilon' \cos\theta + \epsilon'^2)^{1/2} \simeq \Omega R(1 + \epsilon' \cos\theta) \quad .$$

2) $$\lambda = \mathrm{Arctan}\left[\frac{-\sin\theta}{\cos\theta + \frac{\Omega R}{U}}\right] \simeq \mathrm{Arctan}[-\epsilon' \sin\theta(1 - \epsilon' \cos\theta)] \ ,$$

$$\simeq \mathrm{Arctan}(-\epsilon' \sin\theta) \quad .$$

3) $\lambda_t \simeq \dfrac{1}{1+(-\epsilon\ \sin\theta)^2}\,[-\epsilon'\ \cos\theta\,\Omega] \simeq -\epsilon'\,\Omega\cos\theta$

4) $\omega = \lambda_t + \Omega \simeq \Omega(1 - \epsilon'\cos\theta)$.

5) $\kappa(\tau) = \dfrac{\omega}{V} \simeq \dfrac{\Omega(1-\epsilon'\cos\theta)}{\Omega R(1+\epsilon'\cos\theta)} \simeq \dfrac{1}{R}\,(1 - 2\epsilon'\cos\theta)$,

$$\simeq \epsilon\,(1 - 2\epsilon'\cos\theta) \quad .$$

6) $\dot{\kappa} = \dfrac{\kappa_t}{V} \simeq \dfrac{1}{R}\,\dfrac{2\Omega\,\epsilon'\sin\theta}{\Omega R(1+\epsilon'\cos\theta)} \simeq 2\epsilon'\epsilon^2(1-\epsilon'\cos\theta)\sin\theta$.

7) $\dfrac{z\dot{\kappa}}{\beta\kappa} \simeq (\dfrac{z}{\beta})\left[2\epsilon\epsilon'\sin\theta\ \dfrac{(1-\epsilon'\cos\theta)}{(1-2\epsilon'\cos\theta)}\right]$,

$$\simeq (\frac{z}{\beta})\,[2\epsilon\epsilon'\sin\theta(1+\epsilon'\cos\theta)] \quad .$$

For values of z of order unity, i. e., close to the blade, we have

$$(\frac{z}{\beta}) \simeq z \text{ and } (\frac{z\dot{\kappa}}{\beta\kappa}) \simeq 0(\epsilon\epsilon') \simeq 0(\frac{1}{R^2}) \quad .$$

For large values of z, we have

$$(\frac{z}{\beta}) \sim \frac{1}{\kappa} \text{ and } (\frac{z\dot{\kappa}}{\beta\kappa}) \sim 0(\epsilon') \simeq 0(\frac{1}{R}) \quad .$$

As is readily apparent from this equation, the value of $\frac{z\dot{\kappa}}{\beta\kappa}$ at any point on the characteristic line is always at least of order $(1/R)$. Furthermore for the region near the blade, which corresponds to large contributions to the unsteady effect, it is of order $(1/R^2)$.

It may be noted that the case $\beta = 0$, or $i\kappa z = 1$, correspond to $z = \frac{-i}{\kappa} \simeq -iR$ which is the center of the turbine, and that the characteristic lines do not go through this point, at least up to a very large retarded time. If this does happen for $\tau' \ll \tau$, the effect of the retarded pressure is small enough for the simplification not to be needed in the first place.

APPENDIX B

Laplace transform solution of $a_o(\tau)$

We recall (1.70)

$$s\tilde{a}_o(s)\int_{-\infty}^{-1}\left(\frac{z-1}{z+1}\right)^{1/2}\frac{e^{sB(z)}}{\beta(z)}\,dz = -\frac{2s}{\pi}\int_{-\infty}^{-1}\frac{e^{sB(z)}}{\beta(z)}\,dz\int_{-1}^{+1}\left(\frac{z^2-1}{1-\xi^2}\right)^{1/2}\frac{f_1(\xi,s)}{\xi-z}\,d\xi,$$

where $\quad B(z) = (1+z) - \frac{i\epsilon}{2}(1-z^2)$,

and $\quad \beta(z) = 1 - i\epsilon z$. We can rewrite this equation as

$$\tilde{a}_o(s)E_1(s) = -\frac{2}{\pi}E_2(s) \quad , \qquad \text{(B-1)}$$

where

$$E_1(s) = \int_{-\infty}^{-1}\left(\frac{z-1}{z+1}\right)^{1/2} e^{sz}\left[\frac{e^{\frac{i\epsilon s}{2}(z^2-1)}}{1-i\epsilon z}\right]dz \quad , \qquad \text{(B-2)}$$

and

$$E_2(s) = \int_{-\infty}^{-1} e^{sz}\left[\frac{e^{\frac{i\epsilon s}{2}(z^2-1)}}{1-i\epsilon z}\right]dz\int_{-1}^{+1}\left(\frac{z^2-1}{1-\xi^2}\right)^{1/2}\frac{f_1(\xi,s)}{\xi-z}\,d\xi. \qquad \text{(B-3)}$$

1. Evaluation of $E_1(s)$

Taking an expansion in small ϵ of the expression in the bracket of (B-2), and neglecting the second and higher order terms, we can express $E_1(s)$ as

$$E_1(s) = \int_{-\infty}^{-1}\left(\frac{z-1}{z+1}\right)^{1/2} e^{sz}\,dz + \frac{i\epsilon s}{2}\int_{-\infty}^{-1}(z^2-1)\left(\frac{z-1}{z+1}\right)^{1/2} e^{sz}\,dz + i\epsilon\int_{-\infty}^{-1} z\left(\frac{z-1}{z+1}\right)^{1/2} e^{sz}\,dz. \qquad \text{(B-4)}$$

The second integral in this expression can be integrated by part. Denoting it by $E_3(s)$, we obtain

$$E_3(s) = -\frac{i\epsilon}{2}\int_{-\infty}^{-1} e^{sz} \frac{d}{dz}\left[(z^2-1)\left(\frac{z+1}{z-1}\right)^{1/2}\right]dz \quad ,$$

which can be expressed as

$$E_3(s) = -i\epsilon\int_{-\infty}^{-1} z\left(\frac{z-1}{z+1}\right)^{1/2} e^{sz}\, dz - \frac{i\epsilon}{2}\int_{-\infty}^{-1}\left(\frac{z-1}{z+1}\right)^{1/2} e^{sz}\, dz \quad .$$

Replacing this expression in (B-4) leads to

$$E_1(s) = [1 - \frac{i\epsilon}{2}]\int_{-\infty}^{-1}\left(\frac{z-1}{z+1}\right)^{1/2} e^{sz}\, dz \quad .$$

Finally, using the identity (derived in Appendix C)

$$\int_{-\infty}^{-1}\left(\frac{z-1}{z+1}\right)^{1/2} e^{sz}\, dz = K_o(s) + K_1(s) \quad ,$$

we obtain

$$E_1(s) = [K_o(s) + K_1(s)](1 - \frac{i\epsilon}{2}) \quad , \qquad \text{(B-5)}$$

where $K_o(s)$ and $K_1(s)$ are the modified Bessel functions of the second kind.

2. Evaluation of $E_2(s)$

$f_1(x, \tau)$ was defined in (1.49) by

$$f_1(x,\tau) = -\beta(x,\tau)\,v^+(x,\tau) - \frac{\partial}{\partial\tau}\int_{-1}^{x} v^+(\xi,\tau)\,d\xi \quad .$$

Using the notation

$$G(x,\tau) = \int_{-1}^{x} v^+(\xi,\tau)\,d\xi \quad , \tag{B-6a}$$

and approximating $\beta(x,\tau)$ up to first order, we can express $\widetilde{f}_1(\xi, s)$ as

$$\widetilde{f}_1(\xi, s) = -\beta(\xi)\,\widetilde{v}^{\,+}(\xi, s) - s\,\widetilde{G}(\xi, s) \quad . \tag{B-6b}$$

Replacing this expression in (B-3), taking the expansion in ϵ of (B-3) and neglecting the second and higher order terms yields

$$E_2(s) = -\int_{-\infty}^{-1} e^{sz}\Big[1 + \frac{i\epsilon s}{2}(z^2-1) + i\epsilon z\Big]dz \int_{-1}^{+1} \Big(\frac{z^2-1}{1-\xi^2}\Big)^{1/2}\Big[\frac{\widetilde{v}^+(\xi, s) + s\widetilde{G}(\xi, s)}{\xi - z}\Big]d\xi$$

$$+\int_{-\infty}^{-1} e^{sz}dz \int_{-1}^{+1} \Big(\frac{z^2-1}{1-\xi^2}\Big)^{1/2}\Big[\frac{i\epsilon\,\xi\,\widetilde{v}^{\,+}(\xi, s)}{\xi - z}\Big]d\xi \quad .$$

This can be rewritten as

$$E_2(s) = -E_4(s) - i\epsilon\Big[\frac{s}{2}E_5(s) + E_6(s) - E_7(s)\Big] \quad , \tag{B-7}$$

where

$$E_4(s) = \int_{-\infty}^{-1} e^{sz}dz \int_{-1}^{+1} \Big(\frac{z^2-1}{1-\xi^2}\Big)^{1/2}\Big[\frac{\widetilde{v}^{\,+}(\xi, s) + s\widetilde{G}(\xi, s)}{\xi - z}\Big]d\xi \quad , \tag{B-8}$$

$$E_5(s) = \int_{-\infty}^{-1} e^{sz}(z^2-1)dz \int_{-1}^{+1} \left(\frac{z^2-1}{1-\xi^2}\right)^{1/2} \left[\frac{\widetilde{v}^+(\xi,s) + s\widetilde{G}(\xi,s)}{\xi-z}\right] d\xi \quad , \tag{B-9}$$

$$E_6(s) = \int_{-\infty}^{-1} e^{sz} z dz \int_{-1}^{+1} \left(\frac{z^2-1}{1-\xi^2}\right)^{1/2} \left[\frac{s\widetilde{G}(\xi,s)}{\xi-z}\right] d\xi \quad , \tag{B-10}$$

and

$$E_7(s) = \int_{-\infty}^{-1} e^{sz} dz \int_{-1}^{+1} \left(\frac{z^2-1}{1-\xi^2}\right)^{1/2} \left[\frac{\xi\widetilde{v}^+(\xi,s) - z\,\widetilde{v}^+(\xi,s)}{\xi-z}\right] d\xi \quad . \tag{B-11}$$

a. <u>Evaluation of $E_4(s)$</u>

We can rewrite (B-8) as

$$E_4(s) = \int_{-\infty}^{-1} e^{sz} dz \int_{-1}^{+1} \left(\frac{z^2-1}{1-\xi^2}\right)^{1/2} \frac{v^+(\xi,s)}{\xi-z} dz + \int_{-\infty}^{-1} s e^{sz} dz \int_{-1}^{+1} \left(\frac{z^2-1}{1-\xi^2}\right)^{1/2} \frac{\widetilde{G}(\xi,s)}{\xi-z} d\xi \ . \tag{B-12}$$

Denoting the last term of this expression by $E_8(s)$, we can integrate it by part in z. This leads to

$$E_8(s) = -\int_{-\infty}^{-1} e^{sz} dz \int_{-1}^{+1} \frac{\widetilde{G}(\xi,s)}{(1-\xi^2)^{1/2}} \frac{\partial}{\partial z}\left[\frac{(z^2-1)^{1/2}}{\xi-z}\right] d\xi \quad . \tag{B-13}$$

Using the identity

$$\frac{1}{(1-\xi^2)^{1/2}} \frac{\partial}{\partial z}\left[\frac{(z^2-1)^{1/2}}{\xi-z}\right] = \frac{1}{(z^2-1)^{1/2}} \frac{\partial}{\partial \xi}\left[\frac{(1-\xi^2)^{1/2}}{\xi-z}\right] ,$$

and the following relation which can be derived from (B-6a),

$$\frac{\partial}{\partial \xi} \; \widetilde{G}\,(\xi, s) \;=\; \widetilde{v}^{\,+}(\xi, s) \quad ,$$

we can integrate (B-13) by parts in ξ to obtain

$$E_8(s) \;=\; \int_{-\infty}^{-1} e^{sz}\, dz \int_{-1}^{+1} \left(\frac{1-\xi^2}{z^2-1}\right)^{1/2} \frac{\widetilde{v}^{\,+}(\xi, s)}{\xi - z}\, d\xi \quad .$$

Replacing this expression in (B-12) and using the identity

$$\frac{1}{\xi - z}\left[\left(\frac{z^2-1}{1-\xi^2}\right)^{1/2} + \left(\frac{1-\xi^2}{z^2-1}\right)^{1/2}\right] = \frac{-\,(z+\xi)}{(1-\xi^2)^{1/2}(z^2-1)^{1/2}} \quad ,$$

we obtain

$$E_4(s) \;=\; -\int_{-1}^{+1} \frac{\widetilde{v}^{\,+}(\xi, s)}{(1-\xi^2)^{1/2}}\, d\xi \int_{-\infty}^{-1} e^{sz}\; \frac{\xi - z}{(z^2-1)^{1/2}}\; dz \quad .$$

We can rewrite this expression as

$$E_4(s) = \int_{-1}^{+1} \frac{\widetilde{v}^{\,+}(\xi, s)}{(1-\xi^2)^{1/2}}\; d\xi \left[\xi \int_{1}^{\infty} e^{-sz}(z^2-1)^{-1/2}\, dz \;-\; \int_{1}^{\infty} z e^{-sz}(z^2-1)^{1/2}\, dz\right] ,$$

and, noting that (see Appendix C),

$$\int_{1}^{\infty} e^{-sz}(z^2-1)^{-1/2}\; dz \;=\; K_o(s) \quad ,$$

$$\int_{1}^{\infty} z e^{-sz}\,(z^2-1)^{-1/2}\; dz \;=\; K_1(s) \quad ,$$

we finally obtain

$$E_4(s) = \int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}} \left[\xi K_o(s) - K_1(s)\right] d\xi \quad . \tag{B-14}$$

b. Evaluation of $E_5(s)$ and $E_6(s)$

As we did for $E_4(s)$, we can rewrite (B-9) as

$$E_5(s) = \int_{-\infty}^{-1} e^{sz}(z^2-1)dz \int_{-1}^{+1} \left(\frac{z^2-1}{1-\xi^2}\right)^{1/2} \left[\frac{\tilde{v}^+(\xi, s)}{\xi - z}\right] d\xi + s\int_{-\infty}^{-1} e^{sz}(z^2-1)dz \int_{-1}^{+1} \left(\frac{z^2-1}{1-\xi^2}\right)^{1/2} \left[\frac{\tilde{G}(\xi, s)}{\xi - z}\right] d\xi . \tag{B-15}$$

Denoting the second part of this expression by $E_9(s)$, we can integrate it by parts in z and obtain

$$E_9(s) = \int_{-\infty}^{-1} e^{sz} dz \int_{-1}^{+1} \frac{\tilde{G}(\xi, s)}{(1-\xi^2)^{1/2}} \frac{\partial}{\partial z}\left[(z^2-1) \frac{(z^2-1)^{1/2}}{\xi - z}\right] d\xi \quad . \tag{B-16}$$

We can develop the partial derivative in z as

$$\frac{\partial}{\partial z}\left[(z^2-1) \frac{(z^2-1)^{1/2}}{\xi - z}\right] = 2z\left[\frac{(z^2-1)^{1/2}}{\xi - z}\right] + (z^2-1) \frac{\partial}{\partial z}\left[\frac{(z^2-1)^{1/2}}{\xi - z}\right] ,$$

so that

$$E_9(s) = -\int_{-\infty}^{-1} 2e^{sz} dz \int_{-1}^{+1} \left(\frac{z^2-1}{1-\xi^2}\right)^{1/2} \frac{\tilde{G}(\xi, s)}{\xi - z} d\xi - \int_{-\infty}^{-1} e^{sz}(z^2-1)dz \int_{-1}^{+1} \frac{\tilde{G}(\xi, s)}{(1-\xi^2)^{1/2}} \frac{\partial}{\partial z}\left[\frac{(z^2-1)^{1/2}}{\xi - z)}\right] d\xi .$$

By recalling (B-10), we note that the first part of this expression is equal to $-\frac{2}{s} E_6(s)$. The second part can be treated in a way similar to (B-13).

Integrating it by parts in ξ, using the same identities and combining $E_9(s)$ back in (B-15) leads to

$$E_5(s) = -\frac{2}{s}E_6(s) - \int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}}\, d\xi \int_{-\infty}^{-1} e^{s.z}(\xi+z)(z^2-1)^{1/2}\, dz \quad .$$

This expression can be rewritten as

$$E_5(s) = -\frac{2}{s}E_6(s) + \int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}} d\xi \left[\xi \int_1^{\infty} e^{-sz}(z^2-1)^{1/2}\, dz - \int_1^{\infty} z e^{-sz}(z^2-1)^{1/2}\, dz\right],$$

and, using the integral representations (see Appendix C)

$$\int_1^{\infty} e^{-sz}(z^2-1)^{1/2} dz = \frac{K_1(s)}{s} \quad \text{and} \quad \int_1^{\infty} z e^{-sz}(z^2-1)^{1/2} dz = \frac{K_2(s)}{s},$$

we finally obtain

$$E_5(s) = -\frac{2}{s}E_6(s) + \frac{1}{s}\int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}} [\xi K_1(s) - K_2(s)] d\xi \quad . \qquad \text{(B-17)}$$

c. Evaluation of $E_7(s)$

We can directly rewrite (B-11) as

$$E_7(s) = -\int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}}\, d\xi \int_{+1}^{+\infty} e^{-sz}(z^2-1)^{1/2}\, dz \quad ,$$

and noting that (see Appendix C)

$$\int_1^{\infty} e^{-sz}(z^2-1)^{1/2}\, dz = \frac{K_1(s)}{s} \quad ,$$

we obtain

$$E_7(s) = -\frac{1}{s}\int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}} K_1(s)d\xi \quad . \tag{B-18}$$

d. Evaluation of $E_2(s)$

Using (B-14), (B-17) and (B-18) in (B-7) leads to

$$E_2(s) = -\int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}}\left[\left(\xi K_o(s) - K_1(s)\right) + i\epsilon\left(\xi\frac{K_1(s)}{2} - \frac{K_2(s)}{2} + \frac{K_1(s)}{s}\right)\right]d\xi \quad ,$$

and using the recurrence formula

$$2K_1(s) = s[K_2(s) - K_o(s)] \quad ,$$

we finally obtain

$$E_2(s) = -\int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}} [(\xi K_o(s) - K_1(s)) + \frac{i\epsilon}{2}(\xi K_1(s) - K_o(s))]d\xi . \tag{B-19}$$

3. Evaluation of $\tilde{a}_o(s)$

Using (B-5) and (B-19) in (B-1) leads to

$$\tilde{a}_o(s)(1 - \frac{i\epsilon}{2})[K_o(s) + K_1(s)] = \frac{2}{\pi}\int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}}\Bigg[[\xi K_o(s) - K_1(s)] \ d\xi$$

$$+ \frac{i\epsilon}{2}[\xi K_1(s) - K_o(s)]\Bigg] d\xi \quad ,$$

from which we have

$$\tilde{a}_o(s) = \frac{2}{\pi}\int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}}\left[\frac{[\xi K_o(s) - K_1(s)] + \frac{i\epsilon}{2}[\xi K_1(s) - K_o(s)]}{(1-\frac{i\epsilon}{2})[K_o(s) + K_1(s)]}\right] d\xi \ .$$

Expanding the above integrand for small ϵ and neglecting the second and higher order terms, we obtain

$$\tilde{a}_o(s) = \frac{2}{\pi}\int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}}\left[\frac{[\xi K_o(s) - K_1(s)] + \frac{i\epsilon}{2}[\xi K_1(s) - K_o(s)] + \frac{i\epsilon}{2}[\xi K_o(s) - K_1(s)]}{[K_o(s) + K_1(s)]}\right] d\xi,$$

which is easily simplified to

$$\tilde{a}_o(s) = \frac{2}{\pi}\int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}}\left[\frac{[\xi K_o(s) - K_1(s)]}{[K_o(s) + K_1(s)]} + \frac{i\epsilon}{2}(\xi - 1)\right] d\xi \ .$$

As we can see, the first order correction turns out to be quite simple. Using the notation

$$\tilde{H}(s) = \frac{K_1(s)}{K_o(s) + K_1(s)} \ , \tag{B-20}$$

We can finally express $\tilde{a}_o(s)$ as

$$\tilde{a}_o(s) = \frac{2}{\pi}\int_{-1}^{+1} \frac{\tilde{v}^+(\xi, s)}{(1-\xi^2)^{1/2}}[\xi - (1+\xi)\tilde{H}(s) + \frac{i\epsilon}{2}(\xi - 1)] d\xi \ . \tag{B-21}$$

APPENDIX C

Evaluation of the integral relations used in Part I

1. General integral relation

If we call C a contour enclosing a branch cut extending from $z = -1$ to $z = +1$ along the real z-axis and we choose z not lying on the branch cut, we can write

$$\oint_C \left(\frac{z^2-1}{\xi^2-1}\right)^{1/2} \frac{d\xi}{\xi - z} = 2i \int_{-1}^{+1} \left(\frac{z^2-1}{1-\xi^2}\right)^{1/2} \frac{d\xi}{\xi - z} \quad .$$

Furthermore, the following relationship is also verified

$$\oint_C = \int_{\epsilon_\infty} - \int_{\epsilon} \quad ,$$

where ϵ_∞ is the contour at infinity and ϵ is a small contour enclosing the point z. Since the function possesses a double pole at infinity, the integral over ϵ_∞ does not contribute and, evaluating the integral over ϵ by the method of residue, we readily obtain

$$\int_{-1}^{+1} \left(\frac{z^2-1}{1-\xi^2}\right)^{1/2} \frac{d\xi}{\xi - z} = -\pi \quad .$$

2. Integral formulas involving Bessel functions

From reference [18], equation (3.387-3), we have

$$\int_1^\infty e^{-\mu x}(x^2-1)^{\nu-1}dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu) K_{\nu-1/2}(\mu) \quad ,$$

for $|\arg(\mu)| < \frac{\pi}{2}$ and $\mathrm{Re}(\nu) > 0$, where $K_{\nu - 1/2}(\mu)$ are the modified Bessel functions of the second kind. From this relation, we can readily find

a.
$$\int_1^\infty e^{-sz}(z^2-1)^{-1/2} dz = K_o(s) .$$

b.
$$\int_1^\infty e^{-sz}(z^2-1)^{1/2} dz = \frac{K_1(s)}{s} .$$

c.
$$\int_1^\infty z e^{-sz}(z^2-1)^{-1/2} dz = \int_1^\infty s e^{-sz}(z^2-1)^{1/2} dz = K_1(s) .$$

d.
$$\int_1^\infty z e^{-sz}(z^2-1)^{1/2} dz = \int_1^\infty \frac{s}{3} e^{-sz}(z^2-1)^{3/2} dz = \frac{K_2(s)}{s} .$$

e.
$$\int_{-\infty}^1 \left(\frac{z-1}{z+1}\right)^{1/2} e^{sz} dz = \int_1^\infty \left(\frac{z+1}{z-1}\right)^{1/2} e^{-sz} dz$$

$$= \int_1^\infty z(z^2-1)^{-1/2} e^{-sz} dz + \int_1^\infty (z^2-1)^{-1/2} e^{-sz} dz$$

$$= K_o(s) + K_1(s) .$$

APPENDIX D

1. Summary of the fundamental equations

This Appendix is a summary of the equations required for performing a numerical analysis in practical applications. The initial data consist of the flow velocity U, the rotor angular velocity Ω and the blade motion characterized by its position in the body frame of reference $y = h(x, t)$. The kinematic description of the motion can then be characterized from equations (1. 1), (1.3) and (1. 4) as

$$\theta = \Omega t \quad ,$$

$$V = (U^2 + \Omega^2 R^2 + 2U\Omega \cos \theta)^{1/2} \quad ,$$

$$\tan \lambda = - \sin \theta/(\cos \theta + \Omega R/U) \quad ,$$

$$\eta = \theta + \lambda \quad ,$$

$$\omega = \frac{\partial}{\partial t} \eta \quad .$$

The linearized boundary condition on the blade is given by equation (1. 22) as

$$v(x, t) = (\frac{\partial}{\partial t} + V(t) \frac{\partial}{\partial x}) h(x, t) + \omega(t)x \quad .$$

The linearized expression for the function $\beta(x, t)$ is obtained from (1. 46) and (1. 77) as

$$\beta(x) = 1 - i\frac{x}{R} \quad .$$

The function $f_1(x, t)$ is obtained from equation (1. 45) and expressed in terms of the real time t as

$$f_1(x,t) = -v^+(x,t)\beta(x) - \frac{1}{V(t)}\frac{\partial}{\partial t}\int_{-1}^{+1} v^+(\xi,t)d\xi \quad .$$

The function $A_o(t)$ is then obtained from equation (1.90) and expressed in terms of the real time t as

$$A_o(t) = \frac{1}{2}a_o(t) - \frac{1}{\pi}\int_{-1}^{+1} \frac{f_1(\xi,t)}{(1-\xi^2)^{1/2}}d\xi \quad .$$

The pressure jump across the blade is obtained from equation (1.92) as

$$\Delta p = \frac{2}{\pi}\rho V(t)\left(\frac{1-x}{1+x}\right)^{1/2} \int_{-1}^{+1} \left(\frac{1-\xi}{1+\xi}\right)^{1/2} \mathrm{Re}\left[\frac{f_1(\xi,t)+A_o(t)}{\xi - x}\right] d\xi \quad .$$

Finally the pressure force F_p and the singular force F_s are obtained from equation (1.91) and (1.95) as

$$F_p = \int_{-1}^{+1} (\Delta_p)dx \quad ,$$

$$X_s = -\frac{\pi}{2}\rho\,\mathrm{Re}\left[\frac{a_o(t)}{\beta(-1)}\right]^2 \quad ,$$

$$Y_s = \frac{\pi}{2}\rho\,\mathrm{Im}\left[\frac{a_o(t)}{\beta(-1)}\right]^2 \quad .$$

The only remaining unknown is the function $a_o(t)$. It can be determined from equation (1.88) in terms of τ

$$a_o(\tau) = \frac{2}{\pi}\int_{-1}^{+1}\left[\frac{v^+(\xi,\tau)}{(1-\xi^2)^{1/2}}\left(\xi + \frac{i}{2R}(\xi-1)\right) - \frac{1+\xi}{(1-\xi^2)^{1/2}}\int_{o}^{\tau} v^+(\xi,\tilde{\tau})H(\tau-\tilde{\tau})d\tilde{\tau}\right]d\xi ,$$

where

$$\tau(t) = \int_0^t v(\xi)d\xi \quad .$$

The problem of the evaluation of $a_o(\tau)$ is obtaining a uniformly valid simple expression for the Theodorsen function $H(\tau)$. However, $H(\tau)$ can be expressed in terms of the Wagner function $\Phi(\tau)$ for which numerical approximation are directly available (reference [6] of part II).

2. Note on the inverse Laplace transform of the Theodorsen function

It is well known that the steady-state response of any linear system to a periodic excitation of the form $e^{i\omega t}$ can be found from the Laplace transform of its response to a unit step function input by replacing in the latter response the Laplace variable s by $i\omega$, dividing it by $i\omega$ and multiplying the result by the input function. Applying this relation between the lift acting on an airfoil in oscillatory motion and the lift acting on an airfoil following an impulsive start, we obtain the relation

$$\widetilde{H}(s) = s\widetilde{\Phi}(s) \quad ,$$

where $\widetilde{H}(s)$ is the Theodorsen function in the Laplace variable s, and $\widetilde{\Phi}(s)$ is the Laplace transforms of the Wagner function $\Phi(\tau)$.

The inverse Laplace transform of $\widetilde{v}^+(\xi, s)\widetilde{H}(s)$ can therefore be obtained, using the differentiation rule, as

$$\mathcal{L}^{-1}[\widetilde{H}(s)\,\widetilde{v}^{+}(\xi, s)] = \mathcal{L}^{-1}[s\,\widetilde{v}^{+}(\xi, s)\,\widetilde{\Phi}(s)] \quad ,$$

$$= \mathcal{L}^{-1}[\widetilde{\dot{v}}^{+}(\xi, s)\,\widetilde{\Phi}(s) + v^{+}(\xi, o)\,\widetilde{\Phi}(s)] \quad ,$$

$$= v^{+}(\xi, o)\,\Phi(\tau) + \int_{o}^{\tau} \dot{v}^{+}(\xi, \widetilde{\tau})\,\Phi(\tau - \widetilde{\tau})d\widetilde{\tau} \quad ,$$

$$= v^{+}(\xi, o)\,\Phi(t) + \int_{o}^{t} \dot{v}^{+}(\xi, \widetilde{t})\,\Phi(t-\widetilde{t})d\widetilde{t} \quad ,$$

where $\dot{v}^{+}(\xi, \tau) = \frac{\partial}{\partial t}\, v^{+}(\xi, t)$. This allows us to express $a_o(t)$ as

$$a_o(t) = \frac{2}{\pi} \int_{-1}^{+1} \left[\frac{v^{+}(\xi, t)}{(1-\xi^2)^{1/2}} \left(\xi + \frac{i}{2R}\ (\xi - 1)\right) \right.$$

$$\left. - \frac{(1+\xi)}{(1-\xi^2)^{1/2}}\ v^{+}(\xi, o)\ \Phi(t) - \frac{(1+\xi)}{(1-\xi^2)^{1/2}} \int_{o}^{t} \dot{v}^{+}(\xi, \widetilde{t})\Phi(t-\widetilde{t})d\widetilde{t} \right] d\xi .$$

3. Example, a rigid blade

The case of a rotor with a rigid blade is treated here as an example. The blade is assumed to be fixed in (x_1, y_1), attached at right angle to $\underset{\sim}{R}$, to have its center coinciding with O. The blade therefore appears in the body frames of reference as having a pure pitching motion with an angle of attack α such that $\alpha = \lambda$. We then have

$$h(x, t) = -x \tan \lambda \quad ,$$

which yields the boundary condition that on the blade

$$v^+(x,t) = -V(t) \tan \lambda(t) + \Omega x \quad .$$

Defining $D(t)$ and $\dot{D}(t)$ by

$$D(t) = V(t) \tan \lambda(t) \quad ,$$

and

$$\dot{D}(t) = \frac{\partial}{\partial t} D(t) \quad ,$$

and assuming that the rotor started from rest, i. e. $V(o) = o$, we obtain, after some manipulation, the following expressions

$$\beta(-1) = 1 + \frac{i}{R} \quad ,$$

$$\mathrm{Re}[f_1(x,t)] = (D + \frac{\dot{D}}{V}) + x(\frac{\dot{D}}{V} - \Omega) \quad ,$$

$$\mathrm{Re}[A_o(t)] = \frac{1}{2}\mathrm{Re}[a_o(t)] - (D + \frac{\dot{D}}{V}) \quad ,$$

$$\mathrm{Im}[a_o(t)] = \frac{1}{R}(D + \frac{\Omega}{2}) \quad ,$$

$$\mathrm{Re}[a_o(t)] = \Omega(1 - \Phi(t)) + 2\int_o^t \dot{D}(\tilde{t})\Phi(t - \tilde{t})d\tilde{t} \quad .$$

The pressure force can then be expressed as

$$F_p = \pi\rho V \; \mathrm{Re}[a_o(t)] + \pi\rho[\dot{D} - \Omega V] \quad ,$$

and the singular force can be obtained from $\mathrm{Re}[a_o(t)]$, $\mathrm{Im}[a_o(t)]$ and $\beta(-1)$ by simple complex function operations.

REFERENCES

1. Glauert, H. 1947 The elements of aerofoil and airscrew theory. 2nd edition, Cambridge University Press, Cambridge, England.

2. Theodorsen, T. 1948 Theory of propellers. Mc Graw Hill Book Co., New York.

3. Goldstein, S. 1929 On the vortex theory of screw propellers. Proc. Byarl Soc., A 123, 440.

4. Putman, P. C. 1948 Power from the wind. Van Nostrand Co., New York.

5. Golding, E. W. 1950 Electricity from the wind. Discovery, 97-99.

6. Darrieus, G. J. M. 1931 U. S. Patent No. 1,835,018.

7. South, P. and Rangi, R. S. 1975 An experimental investigation of a 12 ft. diameter high-speed vertical-axis wind turbine. NRC, NAE report LTR-LA-166.

8. Templin, R. J. 1974 Aerodynamic performance theory for the NRC vertical-axis wind turbine. NRC, LTR-LA-166.

9. Strickland, J. H. 1975 The Darrieus turbine: a performance prediction model using multiple stream-tubes. Sandia Laboratories Energy Report, SAND 75-0431.

10. Shankar, P. N. 1975 On the aerodynamic performance of a class of vertical-axis windmills. Nat. Aeron. Lab., Bangalore Report No. TM AE-TM-13-75.

11. Wilson, R. E. and Lissaman, P. B. S. 1974 Applied aerodynamics of wind power machines. Oregon State Univ., Report NSF/RA/N-74113.

12. Holme, O. 1976 A contribution to the aerodynamic theory of vertical-axis wind turbine. Proc. Intern. Symp. on Wind Systems, St. John's College, Cambridge, England.

13. Betz, A. 1926 Windenergie und ihre ausnutzung durch windmuhlen. Vandenhoeck und Reprecht, Gottingen.

14. Wu, T. Y. 1971 Hydromechanics of swimming propulsion. Parts I, II, and III, Journal of Fluid Mechanics, 46, Part I, 337-355, Part II, 521-544 and Part III, 545-568.

15. Wu, T. Y. 1961 Swimming of a waving plate. Journal of Fluid Mechanics, 10, Part 3, 321-344. Also California Institute of Technology Engineering Division Report 97-1, August 1960.

16. Wu, T. Y. 1980 Extraction of energy from wind and ocean current. Proceedings of Thirteenth Symposium on Naval Hydrodynamics, Office of Naval Research, Tokyo, Japan, October 6-10.

17. Muskhelishvili, N. I. 1953 Singular integral equations. P. Noordhoff, Ltd., Groningen, Holland.

18. Gradshteyn, I. S. and Ryzhik, I. M. 1965 Table of integrals series and products. Academic Press, New York.

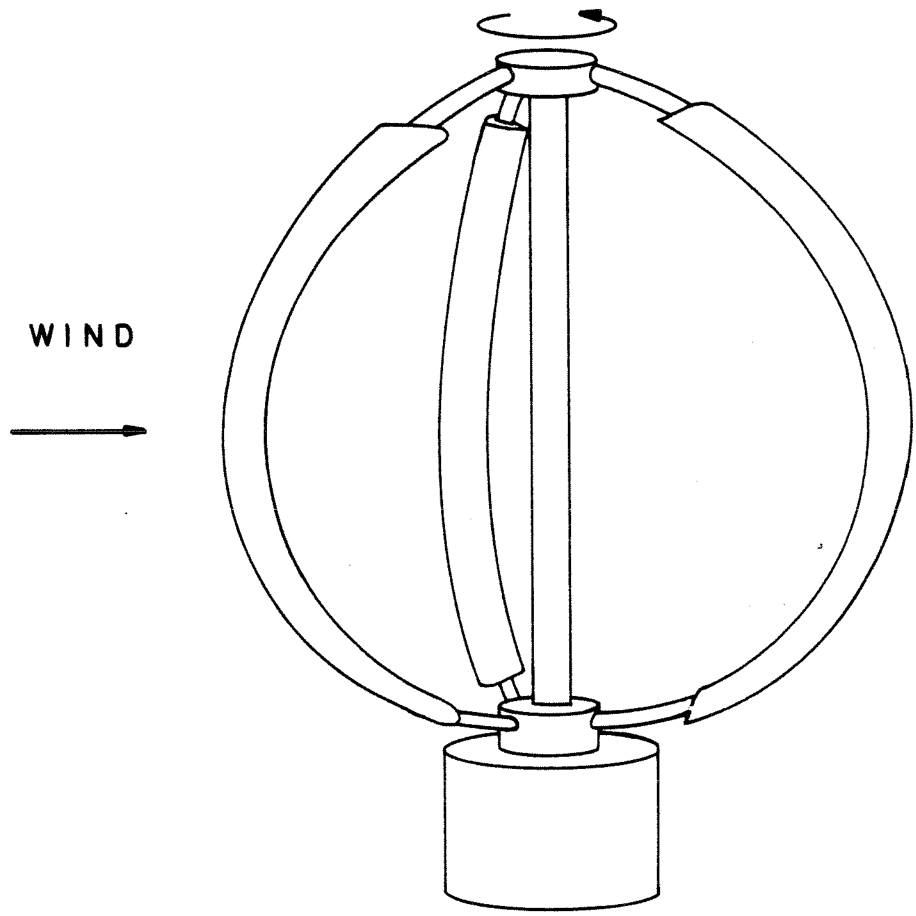


Figure 1. Initial design of the Darrieus rotor.

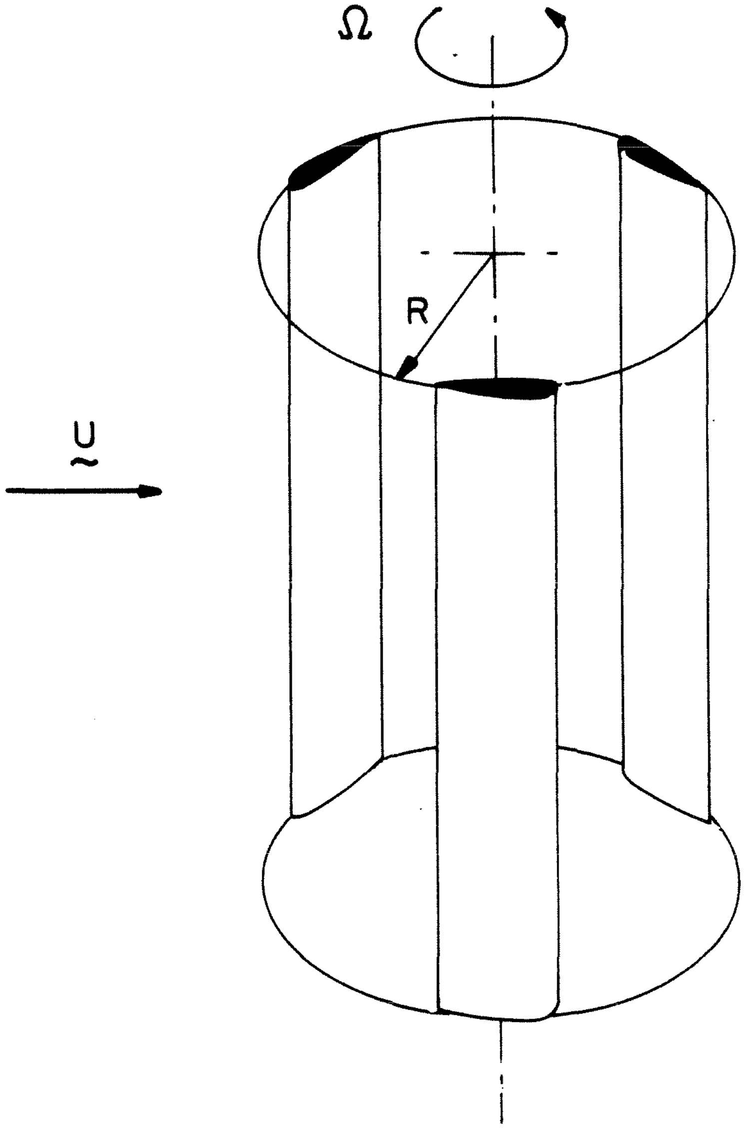
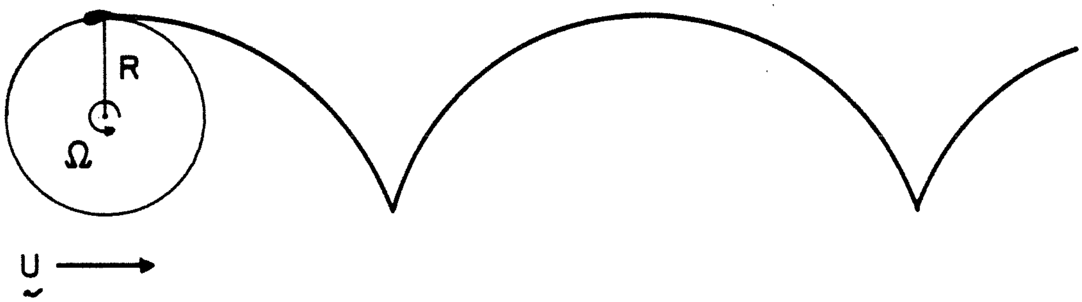
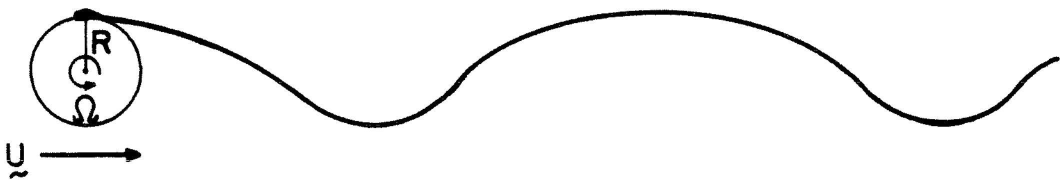


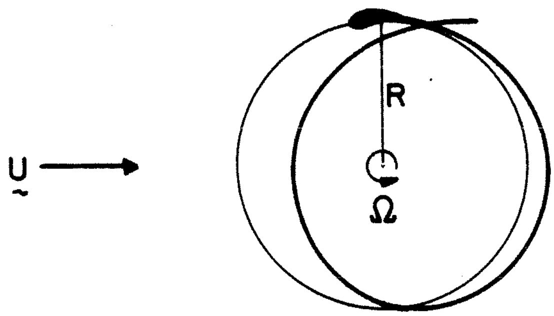
Figure 2. Simplified model of the Darrieus rotor used in the present investigation.



a) $\Omega R \simeq U$, common cycloid



b) $\Omega R \ll U$, curtate cycloid



c) $\Omega R \gg U$, prolate cycloid

Figure 3. Trajectories of the blade in the fluid frame of reference for different values of the tip speed to wind speed ratio. Only case (c) is of interest for energy extraction purpose and is therefore investigated.

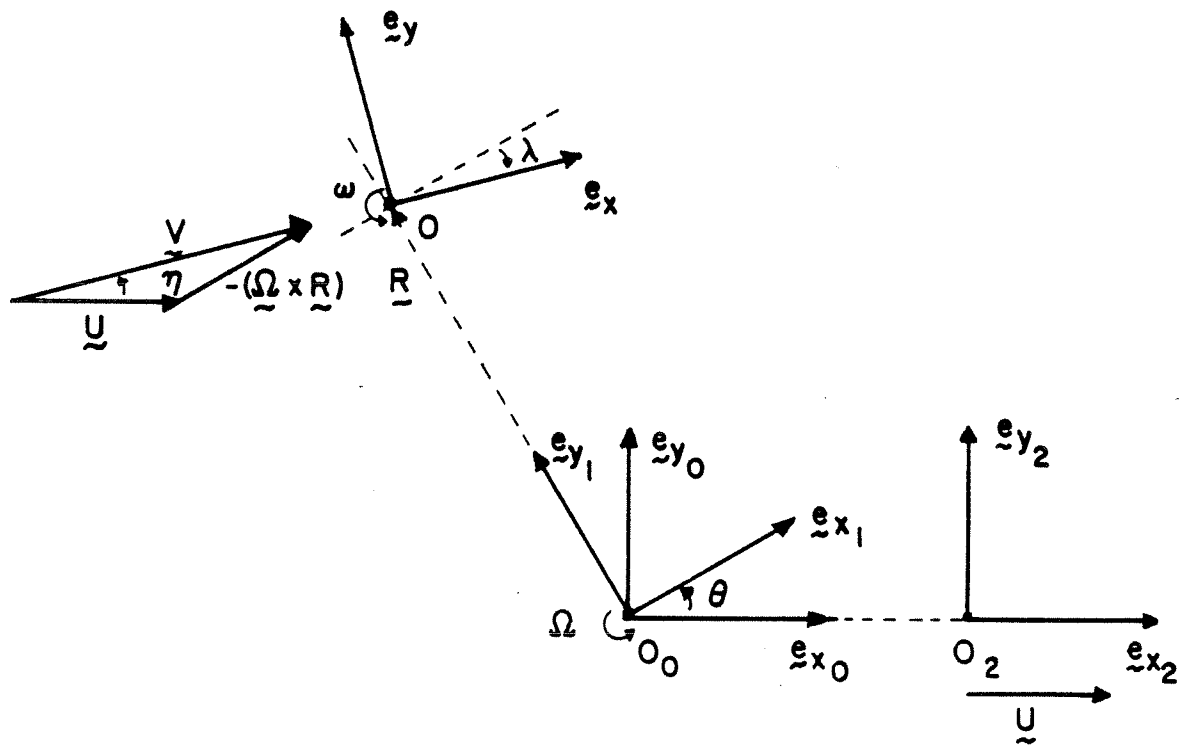


Figure 4. The coordinate systems comprising the structure frame of reference (x_o, y_o) fixed with the structure of the turbine, the body frame of reference (x, y) moving with the blade and defined with e_x always parallel to the apparent flow velocity V, and the fluid frame of reference (x_2, y_2) moving with the undisturbed fluid at infinity.

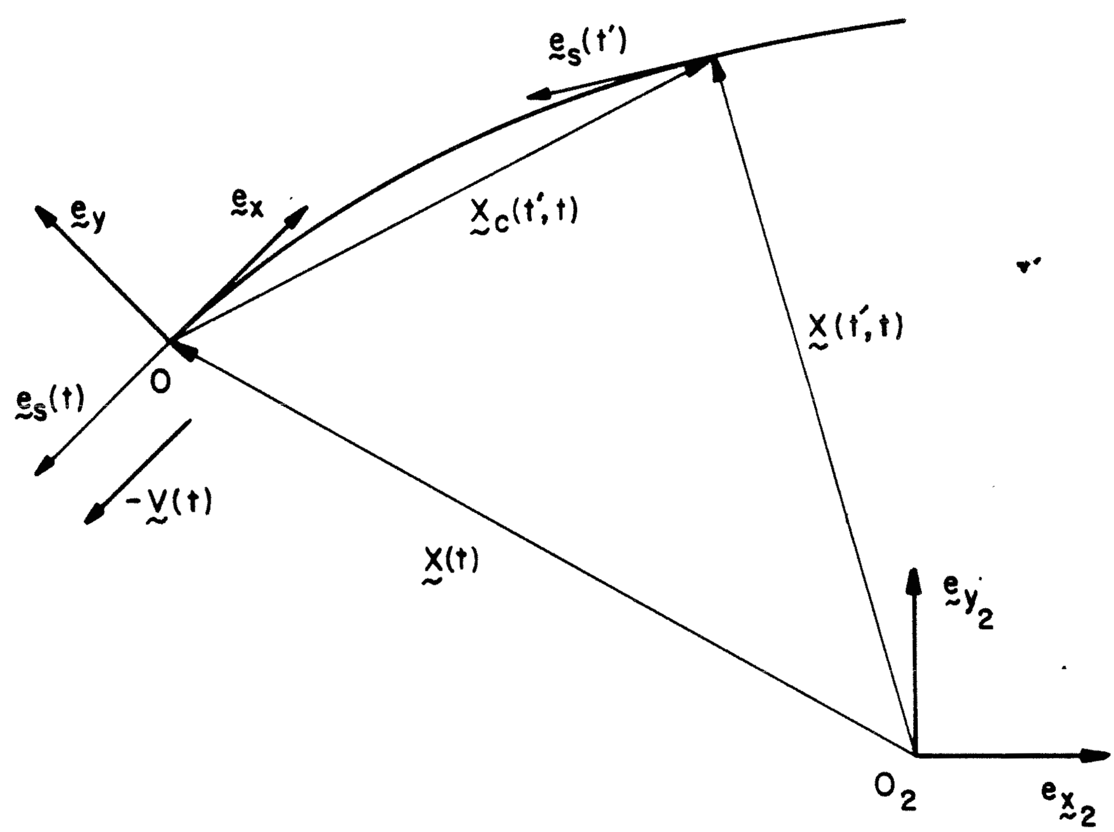


Figure 5. Description of the trajectory of the origin 0 in the fluid frame of reference (x_2, y_2).

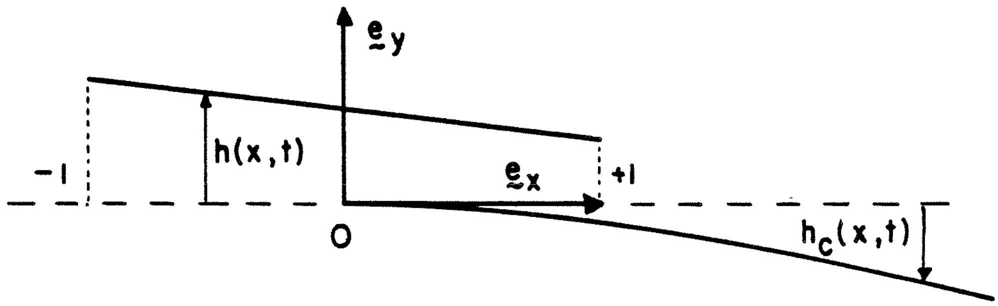


Figure 6. Description of the blade in the body frame of reference. The blade is represented by its transverse displacement from the x axis h(x, t) and the half chord is normalized to unity, with the blade extending from $x = -1$ to $x = +1$ along the x axis.

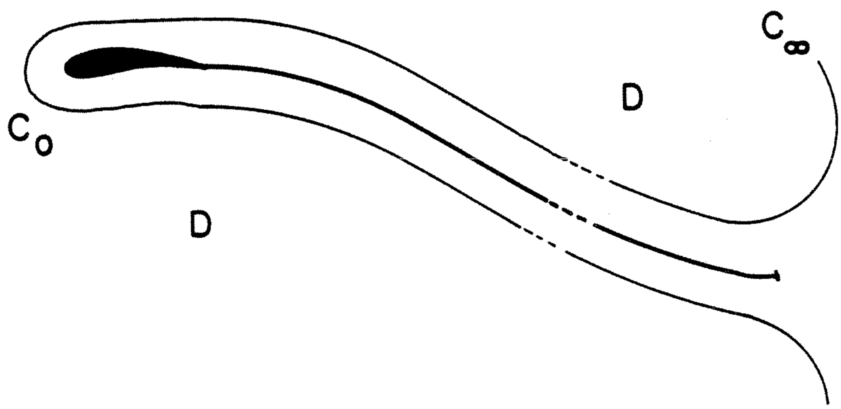


Figure 7. Representation of the contours C_o and C_∞ defining the region D. C_o encloses the blade and its wake and C_∞ is a closed contour circumventing the point at infinity.

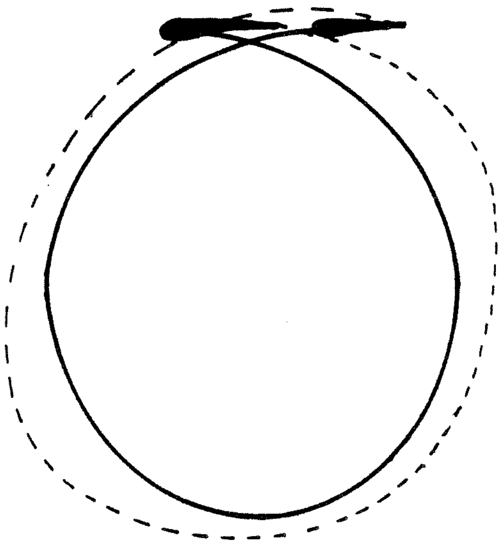


Figure 8. Representation of the exact characteristic lines (1. 65) for two values of z:

—— z = 0

- - - z = 0 .

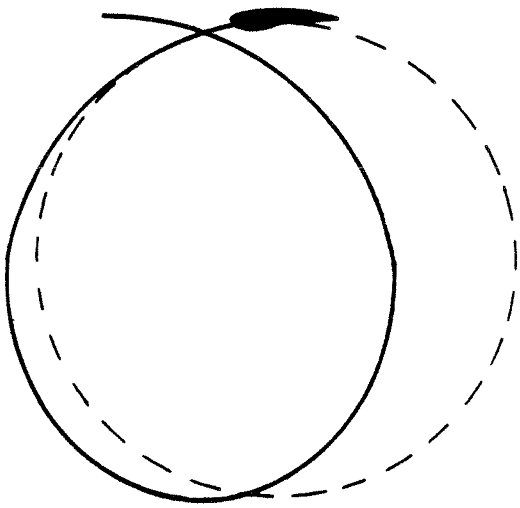


Figure 9. Representation of the approximate characteristic lines (1. 79). An expansion at first order in ϵ of (1. 79) is actually used to avoid coming back in the area of strong disturbance around the leading edge.

- - - Approximate characteristic line (1. 79)

—— Real characteristic lines (1. 65) .

PART II

AN APPROACH TO THE PROBLEM OF WAKE CROSSING

1. Introduction

This part of the thesis investigates variations of the lift experienced by a wing crossing a vortex wake. In the general theory developed in Part I, the effect of the wake crossing was neglected. This simplification is reasonable for a performance prediction model since this effect is expected to have a minimal influence on the average power output of a turbine. However, the periodic loading of the blades caused by successive crossings of vortex wakes can be important in the structural design of turbines. Furthermore, the problem of wake crossing is of basic importance, is not limited to wind turbines and a solution to this problem will be very useful in constructing a general solution for wings operating in shear flows.

The problem is directly related to that of a wing entering a sharp-edged vertical gust as investigated by Küssner [1], von Kármán [2], Sears [3], Miles [4], and Jones [5], among others. Its solution is also related to that for impulsive changes in the angle of attack of a wing, which was first investigated by Wagner [6]. Wagner used the method of conformal transformation to calculate the growth of circulation in the initial stage of motion. The results for oscillatory motion of a wing obtained by Theodorsen [7], along with the relationship between impulsive and oscillatory motion through the Laplace transformation, can also be used. This fundamental result and the principle of superposition facilitate the calculation of the lift for arbitrary motion of the wing, with the assumption of constant downwash on the blade.

The sharp-edged gust case cannot be solved directly using the Wagner function, since the downwash on the wing is not a constant, but rather a step function. The wing can be considered as being bent, with the bend moving along the chord. The solution for the lift of a bent airfoil is known [7], [8], [9] and the progress of the bend can be separated into two effects: the first appears as a change of the equivalent angle of attack, and can therefore be solved by superposition of the Wagner function, the second is the reaction caused by the acceleration of the non-circulatory part of the flow, and must be treated separately. The final solution was first worked out by Küssner [1] and corrected later by Sears [10] and von Kármán [2].

The major difference between the sharp-edged gust case and the one investigated here is the change of forward velocity as seen by the blade, in addition to the change of angle of attack, upon entering of the vortex sheet. This makes the direct application of the "gust" solution to the wake problem difficult. A breaking of the problem into one involving the accelerated mass of the fluid and another due to the change of angle of attack, as was done for the gust problem, is possible but would involve tedious calculations of the distribution of lift along the airfoil during the crossing, as well as complicated boundary conditions at the intersection of the wing and the wake.

A more elegant method has been developed here, based on the solution for the impulsive case (which can be reduced to a change of angle of attack or, as will be shown here, of quasi-steady circulation) and a reciprocity relation in reverse flow which links the circulation of the impulsive case with the lift of the continuous case. The solution to the impulsive crossing is found following von Kármán's approach [2]. This approach uses the change of momentum of the vortices representing the wing-wake system to find the forces acting on the wing. The present calculation along this approach is based on general momentum consideration for the system of moving vortices and the corresponding change of circulation around the wing. It does not differentiate whether this change is induced by a change of angle of attack or a change of velocity. Furthermore, the lift is obtained from a momentum balance, rather than by integration of the pressure on the blade, and therefore includes the leading edge suction.

It must be noted that the flow pattern in both cases is not completely irrotational due to the presence of the vortex sheet. However, we will assume that the methods of thin airfoil theory are still applicable. This in effect implies adoption of the assumption that the superposition principle is applicable, as the boundary conditions on the airfoil during the crossing of a given vortex are taken similar to that on a bent airfoil in a regular flow. Strictly speaking, the principle of superposition is only true for linear problems, but provided that all the additional velocities are small and that the deformation of the vortex sheet is neglected, the method yields results which are probably sufficiently accurate for the present problem.

2. General formulation

As we did in Part I, we assume the fluid to be incompressible and inviscid. The effect of viscosity is inferred only in the formation of a vortex sheet wake behind the blade, and no fluid singularities are allowed outside the wake. We assume the blade to have an infinite aspect ratio and consider the two-dimensional flow about a cross section of the blade.

The impulse of the flow field generated from rest by a system of vortices is derived in Appendix I and can be expressed as

$$\underset{\sim}{I} = \frac{1}{2}\rho \int_{V_B} (\underset{\sim}{x} \times \underset{\sim}{\omega})\, dV \quad , \tag{2.1}$$

where V_B is a volume containing all the vorticity so that the flow does not contain any singularities outside V_B, and $\underset{\sim}{\omega}$ is the local vorticity vector

$$\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u} \quad .$$

We consider the vortex system composed of a distribution of bound vortices representing the blade, and a trailing free vortex sheet forming its wake. The airfoil extends along the x-axis from -1 to +1 (fig. 1), and the wake is assumed to be represented by a plane vortex sheet extending from $x = +1$. The blade moves at a velocity $U(t)$ in the $-\underset{\sim}{e}_x$ direction and the wake is stationary with respect to the fluid at infinity. The usual assumptions of linearized two-dimensional airfoil theory are made, i.e.,:

i. The lateral movement of any part of the blade is small so that we may consider the airfoil and its wake to lie on the x-axis and neglect the second order terms;

ii. The Kutta condition holds, so that neither the velocity of the fluid, nor the pressure can be infinite at the trailing edge of the blade;

iii. The perturbation velocity is asymptotically zero at infinity except in the region of the starting vortex and in the vortex wake.

The motion is assumed to start at time $t = 0$ from a state of rest. We define two frames of reference:

i. (X, Y) is an inertial frame of reference and is stationary with respect to the fluid at infinity. Its origin is chosen to coincide with the blade center at $t = 0$.

ii. (x, y) is the body frame of reference. Its axes are chosen with the same orientation as (X, Y) and its origin is centered on the blade and moves with a velocity U in the $-\underset{\sim}{e}_x$ direction. Within this frame we denote the bound vortex associated with the blade by $\gamma(x, t)$ and the wake vortex by $\gamma(\xi, t)$, where γ is taken positive when its induced velocity has a clockwise sense of rotation. We can convert from one system of reference to the other by the formulas:

$$x = X + S(t) \quad ,$$

$$y = Y \quad ,$$

where

$$S(t) = \int_0^t U(\tau)d\tau \quad .$$

We denote by $\Gamma(t)$ the circulation around the blade,

$$\Gamma(t) = \int_{-1}^{+1} \gamma(x, t)dx \quad , \tag{2.2a}$$

$$= -\int_{+1}^{1+S(t)} \gamma(\xi, t)d\xi \quad , \tag{2.2b}$$

where the second equality follows from the Kelvin's circulation theorem, according to which the total circulation around the blade and its vortex wake vanishes at all time since it vanishes initially. The vorticity distribution on the blade can be decomposed into two parts:

$$\gamma(x,t) = \gamma_o(x,t) + \gamma_1(x,t) \quad , \tag{2.3a}$$

where $\gamma_o(x,t)$ is the quasi-steady vorticity distribution on the blade (i.e., the steady-state distribution for $U(t)$ without vortex shedding) and $\gamma_1(x,t)$ is the wake induced vorticity distribution. Their respective circulation $\Gamma_o(t)$ and $\Gamma_1(t)$ are defined as in (2.2a) with

$$\Gamma(t) = \Gamma_o(t) + \Gamma_1(t) \quad . \tag{2.3b}$$

From reference 9 the vorticity distribution on a wing due to a point vortex $\gamma(\xi)d\xi$ placed on the x-axis at a distance $\xi, \xi > 1$, from the center of the blade is

$$d\gamma_1(x) = \frac{1}{\pi}\frac{\gamma(\xi)d\xi}{\xi - x}\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{\xi+1}{\xi-1}} \quad .$$

By integrating of this expression over the wake, we obtain

$$\gamma_1(x,t) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\int_{+1}^{1+S(t)} \frac{\gamma(\xi,t)}{\xi - x}\sqrt{\frac{\xi+1}{\xi-1}}\, d\xi \quad . \tag{2.4}$$

The total impulse of the wing-wake system is given in (2.1) by

$$\underset{\sim}{I} = \rho\int_{-1}^{1+S(t)} x\gamma(x,t)dx \ \underset{\sim}{e}_y \quad ,$$

and the lift acting on the blade is

$$\underset{\sim}{L}(t) = -\rho \frac{d}{dt} \int_{-1}^{1+S(t)} x\gamma(x,t)dx \; \underset{\sim}{e}_y = L(t) \, \underset{\sim}{e}_y \quad .$$

By decomposition of $\gamma(x,t)$, we can express the lift as

$$L(t) = -\rho \frac{d}{dt} \left[\int_{-1}^{+1} x\gamma(x,t)dx + \int_{+1}^{1+S(t)} \xi\gamma(\xi,t)d\xi \right] .$$

In order to differentiate the part due to the wake with respect to time t, we recall that $\gamma(\xi,t)$ is not time dependent with respect to the fluid frame. This is due to the assumption that the wake vortex sheet remains stationary with the fluid. Performing this differentiation in the fluid frame of reference, using $\gamma(\xi,t) = \hat{\gamma}(X)$, leads to

$$\frac{d}{dt} \int_{+1}^{1+S(t)} \xi\gamma(\xi,t)d\xi = \frac{d}{dt} \int_{1-S(t)}^{1} [X + S(t)] \, \hat{\gamma}(X)dX$$

$$= \dot{S}(t)\hat{\gamma}(1-S(t)) + \int_{1-S(t)}^{1} \hat{\gamma}(x)\dot{S}(t)dX \quad ,$$

and since $\dot{S}(t) = U(t)$, we obtain in (x, y)

$$\frac{d}{dt} \int_{+1}^{1+S(t)} \xi\gamma(\xi,t)d\xi = U(t)\gamma(1,t) + U(t) \int_{+1}^{1+S(t)} \gamma(\xi,t)d\xi \quad .$$

Then, using (2.2b) and

$$\frac{d}{dt}\,\Gamma(t) = -U(t)\gamma(1,t) \quad , \tag{2.5}$$

we can express the lift as

$$L(t) = \rho U(t)\Gamma(t) + \rho\,\frac{d}{dt}\,\Gamma(t) - \rho\,\frac{d}{dt}\int_{-1}^{+1} x\gamma(x,t)dx \quad . \tag{2.6}$$

The integral term in this expression can be further developed by decomposing $\gamma(x,t)$ into $\gamma_o(x,t)$ and $\gamma_1(x,t)$. The contribution of $\gamma_1(x,t)$ can then be evaluated. Using (2.4), we have

$$\int_{-1}^{+1} x\gamma_1(x,t)dx = \int_{-1}^{+1} \frac{x}{\pi}\sqrt{\frac{1-x}{1+x}}\,dx \int_{1}^{1+S(t)} \frac{\gamma(\xi,t)}{\xi-x}\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi \;,$$

$$= \frac{1}{\pi}\int_{1}^{1+S(t)} \gamma(\xi,t)\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi \int_{-1}^{+1} \frac{x}{\xi-x}\sqrt{\frac{1-x}{1+x}}\,dx \quad ,$$

and using the identity

$$\int_{-1}^{+1} \frac{x}{\xi-x}\sqrt{\frac{1-x}{1+x}}\,dx = \pi\left[(\xi-1) - \xi\sqrt{\frac{\xi-1}{\xi+1}}\,\right] ,$$

we obtain

$$\int_{-1}^{+1} x\gamma_1(x,t)dx = \int_{1}^{1+S(t)} \gamma(\xi,t)\left[\sqrt{\xi^2-1} - \xi\right]d\xi \quad .$$

Differentiating this expression with respect to time, following the method outlined above, yields

$$\frac{d}{dt}\int_{-1}^{+1} x\gamma_1(x,t)dx = -U(t)\gamma(1,t) + U(t)\int_{1}^{1+S(t)} \frac{d}{d\xi}\left[\sqrt{\xi^2-1} - \xi\right]\gamma(\xi,t)d\xi \ ,$$

$$= -U(t)\gamma(1,t) - U(t)\int_{1}^{1+S(t)} \gamma(\xi,t)d\xi + U(t)\int_{1}^{1+S(t)} \frac{\xi\gamma(\xi,t)}{\sqrt{\xi^2-1}}d\xi \ ,$$

and combining this expression with (2.6), using (2.2b) and (2.5) we can express the lift as

$$L(t) = -\rho\,\frac{d}{dt}\int_{-1}^{+1} x\gamma_o(x,t)dx - \rho U(t)\int_{1}^{1+S(t)} \frac{\xi\gamma(\xi,t)}{\sqrt{\xi^2-1}}d\xi \ . \qquad (2.7)$$

We can further simplify the last integral in the above equation. From (2.4) we can express $\Gamma_1(t)$ as

$$\Gamma_1(t) = \int_{-1}^{+1} \gamma_1(x,t)dx = \frac{1}{\pi}\int_{1}^{1+S(t)} \gamma(\xi,t)\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi \int_{-1}^{+1} \frac{1}{\xi-x}\sqrt{\frac{1-x}{1+x}}\,dx \ .$$

Then, using the identity

$$\int_{-1}^{+1} \frac{1}{\xi-x}\sqrt{\frac{1-x}{1+x}}\,dx = \pi\left[1 - \sqrt{\frac{\xi-1}{\xi+1}}\right] \ ,$$

we obtain

$$\Gamma_1(t) = \int_{1}^{1+S(t)} \gamma(\xi,t)\left[\frac{\xi}{\sqrt{\xi^2-1}} + \frac{1}{\sqrt{\xi^2-1}} - 1\right] d\xi \ ,$$

which can be expressed as

$$\int_{1}^{1+S(t)} \frac{\xi\gamma(\xi,t)}{\sqrt{\xi^2-1}}\,d\xi = \Gamma_1(t) + \int_{1}^{1+S(t)} \gamma(\xi,t)d\xi - \int_{1}^{1+S(t)} \frac{\gamma(\xi,t)}{\sqrt{\xi^2-1}}\,d\xi \quad . \tag{2.8}$$

Combining this expression with (2. 7) and using (2. 3b) we obtain our final expression for the lift as

$$L(t) = -\rho \frac{d}{dt} \int_{-1}^{+1} x\gamma_o(x,t)dx + \rho U(t)\Gamma_o(t) + \rho U(t)\int_{1}^{1+S(t)} \frac{\gamma(\xi,t)}{\sqrt{\xi^2-1}}\,d\xi \;, \tag{2.9}$$

where $\gamma_o(x,t)$ is the quasi-steady vorticity on the airfoil, $\gamma(\xi,t)$ is the wake vorticity, and $\Gamma_o(t)$ is the quasi-steady circulation around the airfoil. As was to be expected, this expression reduces to the Kutta Joukowsky theorem ($L = \rho U_o \Gamma_o$) in the special case of steady state motion. The three parts of the contribution to the lift given in (2. 9) have the following significance.

i. The first part represents the reaction of the fluid to the acceleration of the virtual mass of the airfoil, as shown by von Kármán and Sears [2].

ii. The second part is the lift due to the circulation which would exist in the absence of the wake.

iii. The third part is the contribution of the wake.

It can be seen that for arbitrary motion of this class $\gamma_o(x,t)$ and $\Gamma_o(t)$ can be readily found from the solution of the corresponding stationary problem by the usual methods. Thus only the wake contribution is unknown. This contribution can be calculated by several methods. The first is to assume some convenient distribution of vorticity in the wake and integrate it over the wake. This is essentially the method used by Wagner [6] to solve the impulsive starting of a wing and by Theodorsen [7] for the periodic motion of a wing. Another method is to find the response to an impulsive change in the quasi-steady circulation and then use the principle of superposition to find the response for the general motion of a wing. To see the importance of the wake, the distribution of vorticity $\gamma_1(x)$ induced on a wing by a single vortex Γ' at various distances from its trailing edge is shown in figure 2 (from [2]). It can be seen that the influence of the wake is very strong for the parts near the airfoil, but diminishes rapidly as the distance increases.

We may also note, for later use, that the equation (2.8) yields, by making use of (2.2a), (2.2b) and (2.3b)

$$-\Gamma_o(t) = \int_1^{1+S(t)} \frac{\xi\gamma(\xi,t)}{\sqrt{\xi^2-1}}\, d\xi + \int_1^{1+S(t)} \frac{\gamma(\xi,t)}{\sqrt{\xi^2-1}}\, d\xi \quad ,$$

which we can rewrite as

$$\Gamma_o(t) + \int_1^{1+S(t)} \gamma(\xi,t)\sqrt{\frac{\xi+1}{\xi-1}}\, d\xi = 0 \ . \qquad (2.10)$$

3. <u>Impulsive crossing of a vortex sheet</u>

We consider a wing in steady translational motion at velocity $-U_2\, \underset{\sim}{e}_x$, with an angle of attack α_1, and assume that it impulsively crosses an inclined vortex sheet at time $t = 0$. The vortex sheet is assumed to have a constant strength $\tilde{\gamma}$ and is inclined with an angle θ

from the X-axis. We shall neglect in this section the transient unsteady motion during the crossing by approximating the crossing of the vortex sheet as being instantaneous. Unsteady effects prior to the crossing can be added later by superposition. The wake of the wing thus starts at the vortex sheet at time t (fig. 3). The quasi-steady circulation around the airfoil undergoes a sudden change at $t = 0$. Prior to the crossing we have

$$L_1 = \rho U_1 \Gamma_1 \quad ,$$

$$\Gamma_1 = 2\pi U_1 \alpha_1 \; .$$

Using the subscript 2 for the quantities after the crossing, the new velocity and angle of attack are

$$U_2 = \frac{U_1 + \tilde{\gamma} \cos \theta}{\cos \alpha'} \quad ,$$

$$\alpha_2 = \alpha_1 + \alpha' \quad ,$$

where

$$\alpha' = \operatorname{Arctan}\left[\frac{\tilde{\gamma} \sin \theta}{U_1 + \tilde{\gamma} \cos \theta} \right] .$$

Since the blade does not change its position or its velocity after the crossing, the new quasi-steady circulation is constant and equal to

$$\Gamma_2 = 2\pi U_2 \alpha_2 \quad .$$

The lift acting on the blade can be obtained from (2.9) as

$$L_2(t) = -\rho \frac{d}{dt} \int_{-1}^{+1} x\gamma_o(x_1 t)dx + \rho U_2 \Gamma_2 + \rho U_2 \int_{1}^{1+S(t)} \frac{\gamma(\xi, t)}{\sqrt{\xi^2 - 1}} d\xi ,$$

where $S(t) = U_2 t$. From reference [9], the vorticity distribution on a flat wing in steady state motion has the following expression

$$\gamma_o(x) = \frac{\Gamma_o}{\pi} \sqrt{\frac{1-x}{1+x}} .$$

Integrating this equation over the wing and using the identity

$$\int_{-1}^{+1} x \sqrt{\frac{1-x}{1+x}} dx = -\frac{\pi}{2} ,$$

we obtain

$$-\rho \frac{d}{dt} \int_{-1}^{+1} x\gamma_o(x)dx = \rho \frac{d}{dt} (\frac{\Gamma_o}{2}) .$$

Since Γ_o undergoes a step from Γ_1 to Γ_2 at $t = 0$, its derivative is an impulse function of value

$$\frac{d}{dt} (\frac{\Gamma_o}{2}) = \frac{1}{2} (\Gamma_2 - \Gamma_1)\delta(t) ,$$

where $\delta(t)$ is the Dirac function, i. e.,:

$$\begin{cases} \delta(t) = 0 & \text{for } t \neq 0 , \\ \int_{o}^{t} \delta(\tau)d\tau = 1 & \text{for } t > 0 . \end{cases}$$

The lift can therefore be expressed as

$$L_2(t) = \frac{\rho}{2}(\Gamma_2 - \Gamma_1)\delta(t) + \rho U_2 \Gamma_2 + \rho U_2 \int_1^{1+U_2 t} \frac{\gamma(\xi, t)}{\sqrt{\xi^2 - 1}}\, d\xi \quad . \tag{2.11}$$

In order to evaluate the wake contribution, we recall equation (2.10)

$$\Gamma_o + \int_1^{1+S(t)} \gamma(\xi, t)\sqrt{\frac{\xi+1}{\xi-1}}\, d\xi = 0 \quad .$$

Since this equation was obtained from the law of conservation of vorticity, we must in the present case replace Γ_o by $(\Gamma_2 - \Gamma_1)$. Expressing then (2.10) in the inertial frame of reference, and recalling that the wake is assumed stationary with the fluid at infinity, we can rewrite this equation as

$$\int_{1-U_2 t}^{1} \hat{\gamma}(X)\sqrt{\frac{X+U_2 t+1}{X+U_2 t-1}}\, dX = -(\Gamma_2 - \Gamma_1) \quad .$$

Using then the new variable $\chi = 1 - X = 1 + U_2 t - \xi$, we finally obtain

$$\int_o^{U_2 t} \gamma^*(\chi)\sqrt{\frac{2+U_2 t-\chi}{U_2 t-\chi}}\, d\chi = -(\Gamma_2 - \Gamma_1) \quad ,$$

where $\gamma^*(\chi) = \hat{\gamma}(X) = \gamma(\xi, t)$.

This integral equation for $\gamma^*(\chi)$ was solved by Wagner [6] for the unit step (i. e., $(\Gamma_2 - \Gamma_1) = 1$). The solution $\gamma^*(\chi)$, as well as the two following integrals

$$\Phi(Ut) = 1 - \phi(Ut) = 1 + \int_1^{1+Ut} \frac{\gamma(1+Ut-\xi)}{\sqrt{\xi^2-1}} d\xi \quad , \qquad (2.12)$$

and

$$\Psi(Ut) = -\int_1^{1+Ut} \gamma(1+Ut-\xi)d\xi \quad , \qquad (2.13)$$

are available in tabulated form.

The function $\Phi(Ut)$ is called the Wagner function and is given in Table I and shown in figure 4. The function $\phi(Ut)$ is needed for our expression of the lift and is called the lift deficiency function in the terminology of von Kármán [2]. The function $\Psi(Ut)$ represents the total circulation around the airfoil and is given in Table II and shown in figure 5. Both are expressed in terms of the distance from the start of motion, $\tau = Ut$, and both represent the response to a unit-step.

In the present case, scaling them with $(\Gamma_2 - \Gamma_1)$ and combining (2.12) and (2.11), we obtain the final expression of the lift

$$L_2(t) = \frac{\rho}{2}(\Gamma_2 - \Gamma_1)\delta(t) + \rho U_2\Gamma_2 - \rho U_2(\Gamma_2 - \Gamma_1)\phi(U_2t) \ ,$$

or, in terms of the Wagner function

$$L_2(t) = \frac{\rho}{2}(\Gamma_2 - \Gamma_1)\delta(t) + \rho U_2\Gamma_2\Phi(U_2t) + \rho U_2\Gamma_1[1 - \Phi(U_2t)] \ . \qquad (2.14)$$

The circulation around the airfoil can similarly be expressed, using (2.13), as

$$\Gamma_2(t) = \Gamma_2\Psi(U_2t) + \Gamma_1[1 - \Psi(U_2t)] \quad . \qquad (2.15)$$

4. Continuous crossing of a vortex sheet

The approach chosen to solve this problem is based on a reciprocity relation in reverse flow which can be proved by application of Green's theorem. This relation shows the equivalence between the lift of an airfoil entering a vortex sheet and the circulation of the same airfoil impulsively crossing the same vortex sheet. This equivalence was first noted by Sears [3] for the case of a sharp-edged gust, but no proof was given as this relation was established by comparison.

Reciprocity relations in reverse flows have been known for a long time [12], [13]. However, they have been mainly used for supersonic steady flows, for which they are particularly well suited. These relations are based on the same general concepts involved in many reciprocity theorems in physical sciences. They provide an indication of close connection between reciprocal theorems based on the principle of least action and the symmetric character of Green's theorem (see [12]). Von Kármán was the first to draw attention to such problems in aerodynamics when he proved the invariance of drag with forward or reverse direction of flight for a nonlifting symmetrical airfoil at supersonic speed [14]. Numerous publications have since appeared on the subject, a review of which can be found in [12]. Further examples of reciprocal theorems appear in the theories of electricity, magnetism, and optics.

4.1 Green's Integral Theorem

We consider the linear differential equation in a region of a m-dimensional Euclidian space $(x_1, x_2, \ldots, x_m)$

$$L(\Psi) = \sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij} (\Psi)_{ij} + B\Psi = 0 \quad , \tag{2.16}$$

where A_{ij} and B are constant and

$$(\Psi)_{ij} = \frac{\partial^2 \Psi}{\partial x_i \partial x_j} \quad .$$

Two arbitrary functions Ψ and ϕ can then be related by the integral formula (Green's theorem)

$$\iiint [\phi L(\Psi) - \Psi L(\phi)] \, dV = - \iint [\phi D_n \Psi - \Psi D_n \phi] dS \quad ,$$

where the integral on the left hand side is prescribed over a m-dimensional region, the integral on the right hand side is prescribed over the hyperspace S enclosing the given region and D_n stands for the directional derivative defined as

$$D_n \Psi = N \sum_{i=1}^{m} \nu_i \Psi_i = N \frac{\partial \Psi}{\partial \nu} \quad ,$$

with

$$\sum_{j=1}^{m} n_j A_{ij} = N\nu_i \quad .$$

In this expression $\underset{\sim}{n}$ is the normal to the hyperspace S, directed into the region and $\underset{\sim}{\nu}$ is called the conormal. Now, if both Ψ and ϕ are solutions of (2.16), i.e.,

$$L(\Psi) = 0 \quad ,$$

$$L(\phi) = 0 \quad ,$$

the foregoing integral reduces to

$$\iint_S \phi \, D_n \Psi \, dS = \iint_S \Psi \, D_n \phi \, dS \quad . \tag{2.17}$$

This relation expresses the functional dependence between any two solutions of the differential equation (2. 16).

4. 2 Reciprocity relation for unsteady incompressible potential flow

The continuity equation in this case reduces to the Laplace equation

$$\nabla^2 \varphi = 0 \quad ,$$

where φ is the velocity potential $\varphi(x, y, z, t)$. Since $\underset{\sim}{u} = \nabla \varphi$, the three velocity components also satisfy the Laplace equation, and so does the acceleration potential in the linearized case. In this case the directional derivative of equation (2. 17) is simply

$$D_n \varphi = \frac{\partial \varphi}{\partial n} \quad ,$$

and equation (2. 17) becomes

$$\iint_S \varphi \frac{\partial \Psi}{\partial n} \, dS = \iint_S \Psi \frac{\partial \varphi}{\partial n} \, dS \quad , \tag{2.18}$$

where Ψ can be any solution of the Laplace equation (i. e., the velocity components or the acceleration potential for the linearized case).

4. 3 Two-dimensional case

We now consider the application of (2. 18) to a two-dimensional lifting surface in unsteady flow for both direct and reverse motions.

We choose (x, y, t) as our frame of reference where $\underset{\sim}{e}_x$ is opposite to the direction of flight and the frame of reference is fixed with the fluid at infinity. The visualization of the time and geometrical relation is relatively easy in this case (fig. 6). The origin of our frame of reference is taken to coincide with the leading edge of the airfoil at time $t = 0$. An airfoil of chord 2 starts at time $t = 0$ and moves to the left at a constant velocity U_o, so that the traces of the leading and trailing edges in the (x, t) plane are respectively

$$\begin{cases} x = -U_o t \ , \\ x = 2 - U_o t \ . \end{cases}$$

In order to establish a reciprocity relation between direct and reverse flows, a reverse flow is generated by a second wing, starting from the final position of the first wing and moving to the right, at velocity U_o, to the starting point of the first wing. The plane form swept by the wings in the (x, t) plane will be called P. It must be noted that the term "reverse" applies to both the physical direction of flight and the <u>time</u> t, i. e.,: the second wing starts from the final position of the first wing in both space and time and proceeds back in "negative time" to the initial position of the first wing. The two wings are therefore superposed at all time instants, with the leading edge of the first wing on top of the trailing edge of the second wing and vice versa.

In the case of two-dimensional flows, the integral terms of equation (2. 18) reduce to line integrals. The path of integration that we consider consists of two semi-circles of large radius and two straight lines lying above and below the x-axis. The two lines will be later brought to coincide with the x-axis, but must first be considered displaced because the region of integration must be free of possible singularities. Among the different solutions of the Laplace equation that will be used in equation (2.18), the one with the lowest order at infinity is the velocity potential φ. Since we initially assumed φ to be regular at infinity, its far field representation is at least that of a doublet (i.e., of order $1/r$) and the products $\Psi \frac{\partial \varphi}{\partial n}$ and $\varphi \frac{\partial \Psi}{\partial n}$ are at least of order $1/r^2$. The contribution from the two semi-circles therefore vanishes in the limit and it only remains to evaluate the integral over the two straight lines $y = \pm \epsilon$. Consequently, integrating over the time t the two-dimensional form of equation (2.18), we obtain

$$\iint \varphi \frac{\partial \Psi'}{\partial y} \, dx \, dt = \iint \Psi' \frac{\partial \varphi}{\partial y} \, dx \, dt \quad , \tag{2.19}$$

where the prime denotes the reverse flow and the integral is valid for both the upper $(y = +\epsilon)$ and lower $(y = -\epsilon)$ planes.

4.4 Pressure and velocity relations

a) Since both φ and $\frac{\partial \varphi}{\partial t}$ are solutions of the Laplace equation, we can replace Ψ' by $\frac{\partial \varphi'}{\partial t}$ in (2.19), which yields

$$\iint \varphi \frac{\partial^2 \varphi'}{\partial y \partial t} \, dx \, dt = \iint \frac{\partial \varphi'}{\partial t} \frac{\partial \varphi}{\partial y} \, dx \, dt \quad .$$

Denoting the velocity components in x and y by u and v respectively, i.e., $\underset{\sim}{u} = (u, v)$, and using the relations $\underset{\sim}{u} = \nabla \varphi$ and $\frac{\partial \varphi}{\partial y} = v$, this expression becomes

$$\iint \varphi \frac{\partial v'}{\partial t} \, dx \, dt = \iint \frac{\partial \varphi'}{\partial t} \, v \, dx \, dt \quad .$$

Integrating the left hand side of this equation by parts yields

$$\iint \varphi \frac{\partial v'}{\partial t} \, dx \, dt = \int \Big[\varphi v' \Big]_{t=-\infty}^{t=+\infty} dx - \iint \frac{\partial \varphi}{\partial t} \, v' \, dx \, dt \quad ,$$

and since we have both direct and reverse flows

$$\Big[\varphi v' \Big]_{t=-\infty}^{t=+\infty} = 0 \quad ,$$

hence,

$$\iint v' \frac{\partial \varphi}{\partial t} \, dx \, dt = -\iint v \frac{\partial \varphi'}{\partial t} \, dx \, dt \quad . \tag{2.20}$$

b) Replacing now Ψ' by u' in (2.19), we obtain

$$\iint \varphi \frac{\partial u'}{\partial y} \, dx \, dt = \iint u' \frac{\partial \varphi}{\partial y} \, dx \, dt \quad .$$

Since the flow is irrotational, $\frac{\partial u'}{\partial y} = \frac{\partial v'}{\partial x}$, and we can express this relation as

$$\iint \varphi \frac{\partial v'}{\partial x} \, dx \, dt = \iint u'v \, dx \, dt \quad .$$

Integrating the left hand side of the equation by part yields

$$\iint \varphi \frac{\partial v'}{\partial x} dx \, dt = \int \Big[\varphi v' \Big]_{x=-\infty}^{x=+\infty} dt - \iint uv' \, dx \, dt \quad ,$$

and since

$$\left[\varphi v'\right]_{x = -\infty}^{x = +\infty} = 0 \ ,$$

for both the direct and reverse flows are considered, we obtain

$$\iint uv' \, dx \, dt = -\iint u'v \, dx \, dt \quad . \qquad (2.21)$$

c) The Bernoulli equation can be expressed with respect to our inertial frame of reference as

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\underset{\sim}{u})^2 = \frac{p_o - p}{\rho} \ ,$$

which yields after linearization

$$\frac{\partial\varphi}{\partial t} = \frac{p_o - p}{\rho}$$

Replacing this relation in (2.20), we obtain

$$\iint \left(\frac{p-p_o}{\rho}\right) v' \, dx \, dt = -\iint \left(\frac{p'-p_o}{\rho}\right) v \, dx \, dt \quad . \qquad (2.22)$$

The primary principal advantage of equation (2.22) lies in the fact that both p and v are continuous across the wake in linearized theory, while φ and u experience a jump across the wake. Since equation (2.22) is valid for both the upper and the lower planes $y = \pm\epsilon$, taking the difference between the integrals evaluated on these two planes eliminates the contributions of the wake. We are then only left with the plan form P in the (x, t) plane and the singularities at both trailing and leading edges. In the case of a lifting surface, square-root singularities in both p and v can occur at the leading edge. But since we assume the Kutta condition to hold for both the direct and reverse motions, and since at all time the leading edge of each wing is superposed on the trailing edge of the other wing, no combination of singularities occurs. Furthermore, since $\Delta p = 0$ everywhere outside the wing, we can restrict the integration surface to the plan form. We then obtain

$$\iint_P (\Delta p) v' \, dx \, dt = - \iint_P (\Delta p') v \, dx \, dt \quad , \tag{2.23}$$

where (Δp) denote the pressure jump on P, i. e.,:

$$\Delta p = p_{upper} - p_{lower} \quad .$$

So far the two sides of the equation are expressed in the same coordinates systems. For clarity, we now introduce two different systems corresponding to the two directions of motion. We will use the subscripts 1 and 2 with the following correspondence between the two systems:

$$\begin{cases} x_1 = -x_2 + \xi \quad , \\ y_1 = y_2 \quad , \\ t_1 = -t_2 + \tau \quad , \end{cases}$$

with ξ and τ fixing the relative origins. If we choose $t_1 = 0$ at the start of the direct motion and $t_2 = 0$ at the start of the reverse motion, then

$$\tau = T \quad \text{and} \quad \xi = 2 - U_o T \quad ,$$

where T is defined as the total time of the motion. Noticing that with this new notation $dx_1 = -dx_2$, $dt_1 = -dt_2$, we can rewrite (2.23) as

$$\iint_P \Delta p_1(x_1, t_1) v_2(x_1, t_1) dx_1 dt_1 = -\iint_P \Delta p_2(x_2, t_2) v_1(x_2, t_2) dx_2 dt_2 \quad . \tag{2.24}$$

This equation is valid for any unsteady motion satisfying the prescribed conditions. It can be used, for instance, to derive Munk's formula for the lift on an airfoil of arbitrary shape, using the known solution for a flat plate. In this case v_1 and Δp_1 are known, and the shape of the airfoil provides us with the value of v_2 which in turn allow L_2 to be evaluated.

4.5 Application to the crossing of a wake by a wing

Before proceeding with the analysis, some consideration regarding the validity of equation (2.24) is necessary. This equation was derived using the Green's integral theorem, which explicitely requires the flow field to be free of singularities. In the presence of a vortex sheet, this equation cannot be used if the velocity potential includes the velocity jump of the vortex sheet. We therefore have to consider φ as the potential of the perturbation velocity alone. This necessitates two additional assumptions. First we assume that the perturbation potential does not contain any singularities outside the airfoil and its wake and that this perturbation can be linearly superposed with the initial flow field, including the vortex sheet. Second, we assume that the initial flow field is not changed by the additional perturbation, which means that we assume no deformation or displacement of the vortex sheet.

It is well known from the work of Kelvin and Helmholtz that a vortex sheet is unstable, and that any perturbation applied to a vortex sheet grows exponentially. However, the time scale involved in the process is of order c/U_γ and therefore large compared to c/U_o. We can thus assume that when a blade crosses the newly formed wake of the preceding blade, it does so before any such instability would have had the time to develop. The assumption of fully potential flow for the perturbation velocity is more difficult to establish since it seems to violate the kinematic boundary conditions across the vortex sheet. Since we assume the pressure to be continuous across the vortex sheet, the kinematic boundary condition implies a motion of the vortex sheet in a direction normal to itself, which apparently contradicts our initial assumption that the vortex sheet is stationary in space. It must be pointed out however that the kinematic boundary condition has not been used in any of the publications known to the author regarding flows with localized vorticity. This includes the work of Helmholtz, which used a similar assumption to derive the induced motion of a vortex sheet, as well as publications by Küssner, von Kármán, Sears, Jones, Miles and others regarding airfoils flying in gusts. Similar limitations also apply in the study of airfoils in unsteady motion, since the kinematic boundary condition is similarly not satisfied. The validity of this assumption has been known through comparison with experiments and we thus expect the present analysis to provide a good estimation of the lift acting on an airfoil crossing a vortex sheet.

To apply equation (2.24) to the wake crossing case, we define two wings, one traveling in the direct and the other in the reverse direction. We choose the first wing traveling in the $-\underset{\sim}{e}_{x_1}$ direction and entering an inclined vortex sheet at time $t_1 = 0$. The vortex sheet is assumed to have a constant strength γ_1 and is inclined with an angle θ_1 from the x_1 axis. For simplicity we assume that the angle of attack of the wing prior to the crossing is zero. The velocity of the first wing is taken as $\frac{dx_1}{dt_1} = -U_o$. The second wing is traveling in the reverse direction and "Impulsively" crosses an inclined vortex sheet at time $t_2 = 0$. The vortex sheet is assumed to have a constant strength $\tilde{\gamma}_2$ and is inclined with an angle θ_2 from the x_2 axis. We also assume the angle of attack of the second wing to be zero prior to the crossing and its velocity to be $\frac{dx_2}{dt_2} = -U_o$ (fig. 7). In order to keep both angles of attack small, we assume both $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ to be small compared to U_o. We call α_1 the angle of attack and U_1 the new flow velocity as seen by the first wing after the crossing of the vortex sheet $\tilde{\gamma}_1$. Similarly, α_2 and U_2 are the new angle of attack and apparent flow velocity of the second wing after the impulsive crossing of the vortex sheet $\tilde{\gamma}_2$. All these parameters are dependent on the strength and direction of the vortex sheets and U_o. They are constant and their values were given in chapter 3. The boundary conditions on the two wings can then be expressed as follows:

a) <u>for the first wing</u>

The airfoil starts to cross the vortex sheet at time $t_1 = 0$ with a constant velocity $-U_o$. The vortex sheet is placed at $x_1 = 0$ so that only the portion of the wing with $x_1 < 0$ experiences its presence. The normal component of the flow velocity on the blade is therefore:

$$v_1 = -\alpha_1 U_1 \quad \text{for} \quad -U_o t_1 \leqslant x_1 < 0 \quad \text{and} \quad 0 < t_1 \leqslant 2/U_o \ ,$$

$$\text{and for} \quad -U_o t_1 \leqslant x_1 \leqslant 2-U_o t_1 \quad \text{and} \quad 2/U_o \leqslant t_1 \leqslant T \ ,$$

$$v_1 = 0 \quad \text{for} \quad 0 \leqslant x_1 \leqslant 2-U_o t_1 \quad \text{and} \quad 0 < t_1 \leqslant 2/U_o \ .$$

This provides the boundary conditions for v_1 everywhere on the plan form P. As already stated $\Delta p_2 = 0$ everywhere outside P so that v_1 is not needed outside P.

b) <u>for the second wing</u>

The airfoil crosses "impulsively" the vortex sheet $\tilde{\gamma}_2$ at time $t_2 = 0$ with a constant velocity $-U_o$. The normal component of the flow velocity on the blade is therefore

$$v_2 = -\alpha_2 U_2 \quad \text{for} \quad -U_o t_2 \leqslant x_2 \leqslant 2 -U_o t_2 \quad \text{and} \quad 0 < t_2 \leqslant T \ ,$$

which provides the boundary condition for v_2 on P. Since $\Delta p_1 = 0$ outside P, the value of v_2 is not needed in this region.

We can now apply these boundary conditions in equation (2.24) which we recall for clarity

$$\iint_P \Delta p_1(x_1, t_1) v_2(x_1, t_1) dx_1 \, dt_1 = -\iint_P \Delta p_2(x_2, t_2) v_1(x_2, t_2) dx_2 dt_2 \ . \tag{2.24}$$

We also recall that T represent the total time of the motion, and that $x_1 = 2 - U_o t_1$ is equivalent to $x_2 = -U_o t_2$. The first integral in equation (2. 24) can then be expressed as

$$\iint_P \Delta p_1 v_2 dx_1 dt_1 = -\alpha_2 U_2 \int_o^T dt_1 \int_{-U_o t_1}^{2-U_o t_1} \Delta p_1 dx_1 \quad ,$$

and since we defined $\Delta p = (p_{upper} - p_{lower})$, we can write

$$\int_{-U_o t_1}^{2-U_o t_1} \Delta p_1 \, dx_1 = -L_1(t_1) \quad ,$$

and we obtain

$$\iint_P \Delta p_1 v_2 dx_1 dt_1 = \alpha_2 U_2 \int_o^T L_1(t_1) dt_1 \quad . \tag{2.25}$$

The second integral term in equation (2. 24) can be more easily treated by expressing it in terms of the initial equation (2. 20)

$$\iint_P \Delta p_2 v_1 dx_2 dt_2 = -\iint_P \Delta\left(\frac{\partial \varphi_2}{\partial t_2}\right) v_1 dx_2 dt_2 \quad .$$

Applying the boundary condition to this expression then yields

$$\iint_P \Delta p_1 v_1 dx_2 dt_2 = \alpha_1 U_1 \iint_A \Delta\left(\frac{\partial \varphi_2}{\partial t_2}\right) dx_2 dt_2 \quad , \tag{2.26}$$

where A represents the part of region P that has been traversed by the first wing past the vortex sheet (fig. 7). Calling $\underset{\sim}{u}_{\gamma_2}$ the strength of the vortex sheet $\tilde{\gamma}_2$, and since the second blade has already crossed the vortex sheet, $\dfrac{\partial\varphi_2}{\partial t_2}$ can be expressed in linearized form as

$$\frac{\partial\varphi_2}{\partial t_2} = \frac{d\varphi_2}{dt_2} - \underset{\sim}{u}_2 \cdot \underset{\sim}{u}_{\gamma_2} \quad ,$$

which allows us to express equation (2.26) as

$$\iint_P \Delta p_2 v_1 dx_2 dt_2 = \rho\alpha_1 U_1\left[\iint_A \frac{d}{dt_2}(\Delta\varphi_2)dx_2 dt_2 - \iint_A (\underset{\sim}{u}_{\gamma_2} \cdot \Delta\underset{\sim}{u}_2)dx_2 dt_2\right]. \tag{2.27}$$

The second part of equation (2.27) can be integrated by noting that

$$\Delta\underset{\sim}{u}_2 = \gamma_2(x_2, t_2)\, \underset{\sim}{e}_{x_2} \quad ,$$

where $\gamma_2(x_2, t_2)$ represents the local vorticity on the blade. Then

$$\underset{\sim}{u}_{\gamma_2} \cdot \Delta\underset{\sim}{u}_2 = \gamma_2(\underset{\sim}{e}_{x_2} \cdot \underset{\sim}{u}_{\gamma_2}) \quad ,$$

and since U_2 is the velocity of the flow as seen by the airfoil, we have after linearization

$$(\underset{\sim}{e}_{x_2} \cdot \underset{\sim}{u}_{\gamma_2}) = (U_2 - U_o) \quad .$$

It follows that

$$\iint_A (\underset{\sim}{u}_{\gamma_2} \cdot \Delta\underset{\sim}{u}_2)dx_2 dt_2 = (U_2 - U_o)\iint_A \gamma_2(x_2, t_2)dx_2 dt_2 \quad ,$$

and equation (2.27) becomes

$$\iint_P \Delta p_2 v_1 dx_2 dt_2 = \rho \alpha_1 U_1 \left[\iint_A \frac{d}{dt_2} (\Delta\varphi_2) dx_2 dt_2 - (U_2 - U_o) \iint_A \gamma_2(x_2, t_2) dx_2 dt_2 \right] . \tag{2.28}$$

In order to integrate this equation we must distinguish between two cases, the first being when the airfoil is crossing the vortex sheet, and the second when the blade has completed passing across the vortex sheet (fig. 7a and 7b respectively). In the first case (fig. 7a) the first term on the right hand side of equation (2.28) yields

$$\iint_A \frac{d}{dt_2} (\Delta\varphi_2) dx_2 dt_2 = \int_{2-U_oT}^{2} dx_2 \int^{\frac{2-x_2}{U_o}} \frac{d}{dt_2} (\Delta\varphi_2) dt_2 ,$$

$$= \int_{2-U_oT}^{2} [\Delta\varphi_2(x_2, \frac{2-x_2}{U_o}) - \Delta\varphi_2(x_2, 0)] dx_2 .$$

In the second case (fig. 7b), it yields

$$\iint \frac{d}{dt_2} (\Delta\varphi_2) dx_2 dt_2 = \int_o^2 dx_2 \int_o^{\frac{2-x_2}{U_o}} \frac{d}{dt_2} (\Delta\varphi_2) dt_2$$

$$+ \int_{2-U_oT}^{o} dx_2 \int_{-\frac{x_2}{U_o}}^{\frac{2-x_2}{U_o}} \frac{d}{dt_2} (\Delta\varphi_2) dt_2 ,$$

$$= \int_o^2 \Delta\varphi_2(x_2, \frac{2-x_2}{U_o}) dx_2 - \int_o^2 \Delta\varphi_2(x_2, 0) dx_2$$

$$+ \int_{2-U_oT}^{o} \Delta\varphi_2(x_2, \frac{2-x_2}{U_o}) dx_2 - \int_{2-U_oT}^{o} \Delta\varphi_2(x_2, -\frac{x_2}{U_o}) dx_2 .$$

Since the jump in potential is zero at the leading edge and at $t_2 = 0$, both cases lead to the same result

$$\iint_A \frac{d}{dt_2}(\Delta\varphi_2)dx_2 dt_2 = \int_{2-U_oT}^{2} \Delta\varphi_2(x_2, \frac{2-x_2}{U_o})dx_2 \quad .$$

The second part of equation (2.28) can be integrated in the following way

$$\iint_A \gamma_2(x_2, t_2)dx_2 dt_2 = \iint_P \gamma_2(x_2, t_2)dx_2 dt_2 - \iint_{P-A} \gamma_2(x_2 t_2)dx_2 dt_2,$$

and since

$$\iint_P \gamma_2(x_2, t_2)dx_2 dt_2 = \int_o^T \Gamma_2(t_2)dt_2 \quad ,$$

it remains to evaluate the integration over (P-A).

Again we have to distinguish the two cases of figure (7a) and (7b). In the first case (fig. 7a)

$$\iint_{P-A} \gamma_2(x_2, t_2)dx_2 dt_2 = \int_o^T dt_2 \int_{-U_o t_2}^{2-U_o t_2} \gamma_2(x_2, t_2)dx_2 \quad . \qquad (2.29)$$

In the second case (fig. 7b)

$$\iint_{P-A} \gamma_2(x_2, t_2)dx_2 dt_2 = \int_{T-\frac{2}{U_o}}^{T} dt_2 \int_{-U_o t_2}^{2-U_o t_2} \gamma_2(x_2, t_2)dx_2 \quad . \qquad (2.30)$$

In both cases the integrals are functions of T only and can be evaluated from the known distribution of vorticity $\gamma_2(x_2, t_2)$ on a blade following an impulsive crossing of a vortex sheet. For the time being we will call $\Theta(T)$ the value of this integral, i.e.,:

$$\iint_{P-A} \gamma_2(x_2, t_2) dx_2 dt_2 = \Theta(T) \quad ,$$

so that equation (2.28) can be written as

$$\iint_P \Delta p_2 v_1 dx_2 dt_2 = \rho \alpha_1 U_1 \left[\int_{2-U_o T}^{2} \Delta\varphi_2\left(x_2, \frac{2-x_2}{U_o}\right) dx_2 - (U_2 - U_o) \left[\int_o^T \Gamma_2(t_2) dt_2 - \Theta(T) \right] \right]. \tag{2.31}$$

Substituting (2.31) and (2.25) in (2.24) and differentiating with respect to T leads to

$$\alpha_2 U_2 L_1(T) = -\rho \alpha_1 U_1 \left[U_o \Delta\varphi_2(2 - U_o T, T) - (U_2 - U_o)[\Gamma_2(T) - \Theta'(T)] \right].$$

Since $\Delta\varphi_2$ is evaluated at the trailing edge of the airfoil at time T, it is equal to the circulation $\Gamma_2(T)$ around the airfoil at time T, thus we obtain

$$\alpha_2 U_2 L_1(T) = -\rho \alpha_1 U_1 U_o \Gamma_2(T) + \rho \alpha_1 U_1 (U_2 - U_o)[\Gamma_2(T) - \Theta'(T)] \; .$$

We now need to specify the two vortex sheets $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. The first one, $\tilde{\gamma}_1$, is arbitrarily chosen since it corresponds to the case under the present investigation. The second one, $\tilde{\gamma}_2$, is taken identical in strength to $\tilde{\gamma}_1$ but is displaced to the starting position of the second wing and oriented in the opposite direction. Therefore $\theta_1 = \theta_2$ and

$\underset{\sim}{u}_{\gamma_1} = -\underset{\sim}{u}_{\gamma_2}$ (fig. 8). With this definition of $\tilde{\gamma}_2$, the following linearized relationships are satisfied:

$$\alpha_1 U_1 = -\alpha_2 U_2 ,$$

$$(U_2 - U_o) = (U_o - U_1) .$$

The lift can therefore be expressed as

$$L_1(T) = \rho U_1 \Gamma_2(T) + \rho(U_o - U_1)\Theta'(T) , \tag{2.32}$$

where it only remains to evaluate $\Theta'(T)$.

4.6 <u>Evaluation of $\Theta'(T)$</u>

$\Theta(T)$ is defined for the two cases of figure 7a and figure 7b by equations (2.29) and (2.30) respectively. Differentiating these equations with respect to T yields:

i. In the first case (fig. 7a)

$$\frac{d}{dT}\int_o^T dt_2 \int_{-U_o t_2}^{2-U_o T} \gamma_2(x_2, t_2)dx_2 =$$

$$\int_{-U_o T}^{2-U_o T} \gamma_2(x_2, T)dx_2 + \int_o^T dt_2 \frac{d}{dT}\int_{-U_o t_2}^{2-U_o T} \gamma_2(x_2, t_2)dx_2 ,$$

$$= \int_{-U_o T}^{2-U_o T} \gamma_2(x_2, T)dx_2 - U_o \int_o^T \gamma_2(2-U_o T, t_2)dt_2 ,$$

and since

$$\int_{-U_oT}^{2-U_oT} \gamma_2(x_2, T)dx_2 = \Gamma_2(T) \quad ,$$

we obtain

$$\text{Ⓗ}'(T) = \Gamma_2(T) - U_o\int_o^T \gamma_2(2-U_oT, t_2)dt_2 \quad . \tag{2.33}$$

ii. In the second case (fig. 7b)

$$\frac{d}{dT}\int_{T-2/U_o}^{T} dt_2 \int_{-U_ot_2}^{2-U_oT} \gamma_2(x_2t_2)dx_2 =$$

$$\int_{-U_oT}^{2-U_oT} \gamma_2(x_2, T)dx_2 - \int_{2-U_oT}^{2-U_oT} \gamma_2(x_2, T-2U_o)dx_2$$

$$+\int_{T-2/U_o}^{T} dt_2 \frac{d}{dT}\int_{-U_ot_2}^{2-U_oT} \gamma_2(x_2, t_2)dx_2 \quad ,$$

which reduces to

$$\text{Ⓗ}'(T) = \Gamma_2(T) - U_o\int_{T-2/U_o}^{T} \gamma_2(2-U_oT, t_2)dt_2 \quad .$$

Defining a new function $\theta(U_oT)$ by changing the variable t_2 to $\tau = U_ot_2$ and using the notation

$$\gamma_2^*(x_2, U_ot_2) = \gamma_2(x_2, t_2) \quad ,$$

we can express $\Theta'(T)$ as

$$\Theta'(T) = \Gamma_2(T) - \theta(U_o T) \quad , \tag{2.34}$$

where $\theta(U_o T)$ is defined by

$$\theta(U_o T) = \int_o^{U_o T} \gamma_2^* (2-U_o T, \tau) d\tau \quad \text{for} \quad 0 \leqslant U_o T \leqslant 2 \ ,$$

$$\theta(U_o T) = \int_{U_o T-2}^{U_o T} \gamma_2^* (2-U_o T, \tau) d\tau \quad \text{for} \quad U_o T > 2. \tag{2.35}$$

As we can see, $\theta(o) = 0$ and $\theta(U_o T)$ tend to Γ_2 when T tends to infinity, with Γ_2 being the final value of the circulation around the airfoil in the impulsive case. The lift can now be expressed, using equation (2.23), as

$$L_1(T) = \rho U_1 \Gamma_2(T) + \rho(U_o - U_1)[\Gamma_2(T) - \theta(U_o T)] \quad ,$$

$$= \rho U_o \Gamma_2(T) + \rho(U_1 - U_o)\theta(U_o T) \quad .$$

As a further generalization of this expression, it remains to consider the general case for $\theta(U_o T)$. We already know that $\theta(U_o T)$ is a function of the distance between the blade and its instantaneous position held in the steady part of the fluid at the start of the crossing (i. e., just before the vortex sheet is reached). This indicates that $\theta(U_o T)$ can be expressed in indicial form by an appropriate scaling, thus allowing the lift formula to be used for any strength of the vortex sheet. Performing this transformation is easily done since we have already found that the limit of $\theta(U_o T)$ as T tends to infinity is the steady-state circulation around the airfoil. We therefore define a new function $\Lambda(U_o T)$ by

$$\Lambda(U_o T) = \int_o^{U_o T} \gamma^*(2-U_o T, \tau) d\tau \qquad \text{for } 0 \leqslant U_o T \leqslant 2 ,$$

$$\Lambda(U_o T) = \int_{U_o T-2}^{U_o T} \gamma^*(2-U_o T, \tau) d\tau \qquad \text{for } U_o T > 2 , \tag{2.36}$$

where $\gamma^*(x, U_o T)$ is the vorticity distribution on the blade following a unit step change in quasi-steady circulation. $\Lambda(U_o T)$ can be computed using the results of Wagner [6], and (2.35) can now be expressed as

$$\theta(U_o T) = \dot{\Gamma}_2 \Lambda(U_o T) \quad . \tag{2.37}$$

Substituting then (2.37) in (2.34), we obtain our final expression for $\Theta'(T)$ as

$$\Theta'(T) = \Gamma_2(T) - \Gamma_2 \Lambda(U_o T) , \tag{2.38}$$

where $\Lambda(o) = 0$ and $\Lambda(U_o T)$ tends to 1 as T tends to infinity .

4.7 <u>Solution for the lift</u>

Substituting (2.38) in (2.32), we obtain

$$L_1(T) = \rho U_o \Gamma_2(T) + \rho(U_1 - U_o)\Gamma_2 \Lambda(U_o T) \quad . \tag{2.39}$$

Recalling (2.15), we can express $\Gamma_2(T)$ in the present case as

$$\Gamma_2(T) = \Gamma_2 \Psi(U_1 T) \quad ,$$

which gives us the final expression for the lift as

$$L_1(T) = \rho\Gamma_2\Big[U_o\Psi(U_1 T) + (U_1 - U_o)\Lambda(U_o T)\Big]. \qquad (2.40)$$

This expression shows that the rate of growth of the lift acting during the crossing of a vortex sheet is composed of two terms. The first and most important contribution to the lift has a rate of growth determined by the apparent flow velocity experienced by the airfoil past the vortex sheet. The second term of equation (2.40) is a new result. It has a rate of growth determined by the velocity of flight of the airfoil and depends linearly on the difference between the two velocities U_1 and U_o. In the special case of a vertical vortex sheet, or "sharp-edged gust," $U_o = U_1$ and (2.40) reduces to the equivalence relationship found by Sears [3], namely $L_g(T) = \rho U_o \Gamma_i(T)$, where the subscripts g and i stand for "gust" and "impulsive start" respectively. Furthermore, the lift $L_1(T)$ tends to a steady state value $L_1 = \rho U_1 \Gamma_2$, which was to be expected since both the impulsive crossing and the continuous crossing tend to the same steady state.

It must be noted that equation (2.40) is only applicable to a wing devoid of initial circulation. Furthermore the superposition principle cannot be used in this instance to find the variation of lift for a wing having an initial circulation because of the quadratic nature of the lift in terms of the velocity. Unfortunately, the method used in this analysis breaks down if any initial circulation around the wing exists since the region of integration in this case consists of an infinite planeform, with both integrals diverging. A general solution for the lift cannot therefore be obtained at this point, nor can a superposable indicial solution for the lift be obtained in general in the case of variable velocity.

4.8 Evaluation of $\Lambda(U_o T)$

The evaluation of $\Lambda(U_o T)$ has been carried out numerically, using the approximation of Wagner [6] for the distribution of vorticity in the wake. The vorticity distribution on a wing following an impulsive start was first computed using von Kármán's approach, i. e.,

$$\gamma(x, t) = \gamma_o(x) + \gamma_1(x, t) \quad ,$$

where $\gamma_o(x)$ is the quasi-steady vorticity distribution on the blade given by

$$\gamma_o(x) = \frac{\Gamma_o}{\pi b} \left(\frac{b-x}{b+x}\right)^{1/2} \quad ,$$

and $\gamma_1(x, t)$ is the vorticity distribution on the blade induced by the vortex wake

$$\gamma_1(x, t) = \frac{1}{\pi} \left(\frac{b-x}{b+x}\right)^{1/2} \int_b^{b+S} \frac{\gamma(\xi)}{\xi - x} \left(\frac{\xi - b}{\xi + b}\right)^{1/2} d\xi \quad .$$

The calculation was carried out for a unit jump in quasi-steady circulation ($\Gamma_o = 1$) and for a blade of unit half chord. The resulting values of $\gamma_1(x, t)$ were then used to calculate $\Lambda(U_o T)$. The function was evaluated in terms of the distance from the impulsive start $U_o T$, expressed in half chord. The results are given in Table III and shown in figure 9. The negative part of the curve corresponds to the high negative values of the vorticity present near the trailing edge at the beginning of the motion.

5. Conclusion

In this part of the thesis, the variation of lift acting on a wing during and following its entrance into a vortex sheet has been evaluated. The initial angle of attack of the wing prior to the crossing was assumed to be zero and the solution was obtained by application of a reciprocity theorem in reverse flow relating the desired lift to the circulation around a wing which has impulsively crossed a vortex wake.

From the resulting solution we find that the lift is composed of two terms with different rates of growth. The first component can be expressed in terms of Sear's function $\Psi(U_1 T)$ and has a rate of growth determined by the apparent flow velocity after the crossing. The second component can be expressed in terms of a new function $\Lambda(U_o T)$; it depends linearly on the difference between the velocities U_o and U_1, and has a rate of growth determined by the velocity of flight of the airfoil. The physical significance of these two terms can be explained as follows. From the original integral forms of these two terms, we see that the first, also the more important term is related to the part of the blade's own wake that has already passed the vortex sheet while the second term is related to the part of the blade's wake left behind the vortex sheet. The different rates of growth of these two terms then follow logically from the two different transport rates of the two parts of the wake. Furthermore, the numerical values obtained for $\Lambda(UT)$ show that this function is at all times smaller than Sear's function $\Psi(UT)$. If we assume $U_1 > U_o$, this indicates that the total lift $L_1(T)$ is at all times smaller than the lift $L_3(T)$ which would act on the wing if the complete wake had passed the vortex sheet. This can easily be seen since the lift L_3 can be expressed as $\rho U_1 \Gamma_2 \Psi(U_1 T)$ and thus the difference $(L_3 - L_1)$ is given by

$$(L_3 - L_1) = \rho\Gamma_2(U_1 - U_o)[\Psi(U_o T) - \Lambda(U_1 T)] \quad .$$

This fact is physically meaningful since the part of the wake left behind the vortex sheet stays closer to the airfoil, and therefore has a stronger influence on the lift, than the equivalent part of the wake moving with the vortex velocity.

It has been noted earlier that the lift acting on a wing experiencing variations in its apparent flow velocity cannot be obtained by superposition of the indicial solution. While the superposition principle can be applied to obtain the velocity field for the modes of motion involved, the quadratic nature of the transfer function for the lift corresponding to the combined motion rule out this possibility. The existence of various rates of growth for the lift, as exemplified by the solution obtained here, only adds to the complexity of the problem. It is hoped that the solution obtained in this work for a wing without initial circulation will provide a good starting point toward a general solution of the very difficult problem of wings operating unsteadily in sheared flows of finite thickness.

APPENDIX A

Evaluation of the impulse of a vortical system

We consider an unbounded fluid with vorticity distributed within a bounded region V_B, and the impulsive force field $\tilde{\underset{\sim}{f}}(\underset{\sim}{x}, t)$ necessary to generate this given motion from rest. We define $\underset{\sim}{\mathcal{F}}(x)$ by

$$\mathcal{F}(\underset{\sim}{x}) = \lim_{\tau \to o} \int_o^{\tau} \tilde{\underset{\sim}{f}}(\underset{\sim}{x}, t)dt \quad , \tag{A-1}$$

and $P(\underset{\sim}{x})$ by

$$P(\underset{\sim}{x}) = \lim_{\tau \to o} \int_o^{\tau} p(\underset{\sim}{x}, t)dt \quad , \tag{A-2}$$

where $p(\underset{\sim}{x}, t)$ is the impulsive pressure field generated by the impulsive force field $\tilde{\underset{\sim}{f}}(\underset{\sim}{x}, t)$.

The Euler equation of motion for an inviscid, incompressible fluid can be expressed as

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \frac{1}{2} \nabla (\underset{\sim}{u}^2) + (\underset{\sim}{\omega} \times \underset{\sim}{u}) = -\frac{1}{\rho} \nabla p + \underset{\sim}{f} \quad ,$$

where $\underset{\sim}{u}(\underset{\sim}{x}, t)$ is the velocity, $\omega(\underset{\sim}{x}, t)$ the vorticity vector, $p(\underset{\sim}{x}, t)$ the pressure and $\underset{\sim}{f}(\underset{\sim}{x}, t)$ any external force field acting on the fluid. Taking the integral of this expression over a time period τ, and letting τ tends to zero, both the contributions of $(\nabla \underset{\sim}{u}^2)$ and $(\underset{\sim}{\omega} \times \underset{\sim}{u})$ disappear since these terms are bounded, and we obtain

$$\underset{\sim}{u}_{t=o^+} - \underset{\sim}{u}_{t=o} = -\frac{1}{\rho} \lim_{\tau \to o} \int_o^{\tau} \nabla p(\underset{\sim}{x}, t)dt + \lim_{\tau \to o} \int_o^{\tau} \underset{\sim}{f}(\underset{\sim}{x}, t)dt \quad .$$

If we assume the fluid to be initially at rest and the only external force field acting on the fluid to be the impulsive force field $\tilde{f}(\underset{\sim}{x}, t)$, this equation becomes

$$\underset{\sim}{u}_{t=o^+} = - \frac{\nabla P}{\rho} + \underset{\sim}{\mathfrak{F}} ,$$

which we can express as

$$\underset{\sim}{u}_{t=o^+} = \nabla \phi + \underset{\sim}{\mathfrak{F}} ,$$

where ϕ is defined by

$$\phi = - \frac{P}{\rho} . \tag{A-3}$$

This equation expresses the fact that any potential motion can be generated by an impulsive pressure acting on the fluid, but that vortical motions can only be generated by a system of non-conservative force field. In the presence of such a force field, an impulsive pressure field is generated whose characteristics are dependent on $\underset{\sim}{\mathfrak{F}}$. However, any potential motion can subsequently be superposed by addition of their own impulsive pressure field to P.

In the present case, we assume the vorticity to be restricted to the region V_B. We can therefore limit the force field to the same region. The velocity field outside V_B can be expressed as

$$\underset{\sim}{u}_o = \nabla \phi_o , \tag{A-4}$$

and inside V_B as

$$\underset{\sim}{u}_i = \nabla \phi_i + \underset{\sim}{\mathfrak{F}} . \tag{A-5}$$

The condition of continuous pressure and velocity across the boundary S_B of V_B gives us, on S_B

$$\begin{aligned} \underset{\sim}{u}_o &= \underset{\sim}{u}_i \quad , \\ \phi_o &= \phi_i \quad . \end{aligned} \tag{A-6}$$

Combining these relations with (A-4) and (A-5), we obtain

$$\nabla(\phi_o - \phi_i) = \underset{\sim}{\mathfrak{F}} \; .$$

Then, using the continuity of ϕ across S_B, we can express the boundary condition for $\underset{\sim}{\mathfrak{F}}$ as

$$\begin{aligned} &\underset{\sim}{n} \times \underset{\sim}{\mathfrak{F}} = 0 \quad , \\ &\text{on } S_B \quad , \end{aligned} \tag{A-7}$$

where $\underset{\sim}{n}$ is the vector normal to S_B.

The rate at which momentum is communicated to the fluid outside V_B by the region V_B is

$$\underset{\sim}{F}(t) = -\int_{S_B} \underset{\sim}{n}_o \, p(\underset{\sim}{x}, t) dS \quad .$$

We can therefore express the fluid impulse outside V_B, using (A-2) and (A-3) as

$$\underset{\sim}{I}_o(t) = \int_o^t \underset{\sim}{F}(t)dt = \int_{S_B} \rho\phi_o \underset{\sim}{n}_o dS \quad . \tag{A-8}$$

The momentum of the fluid within V_B is

$$\underset{\sim}{I}_i(t) = \int_{V_B} \rho \underset{\sim}{u}_i(\underset{\sim}{x}, t) dV \quad ,$$

which can be expressed, using (A-5) and the divergence theorem, as

$$\underset{\sim}{I}_i(t) = \int_{S_B} \rho\phi_i \underset{\sim}{n}_i dS + \int_{V_B} \rho \underset{\sim}{\mathcal{F}} dV \quad . \tag{A-9}$$

The total impulse of the fluid is therefore, noticing that $\underset{\sim}{n}_o = -\underset{\sim}{n}_i$ and using (A-8) and (A-9)

$$\underset{\sim}{I}(t) = \underset{\sim}{I}_o(t) + \underset{\sim}{I}_i(t) = \int_{V_B} \rho \underset{\sim}{\mathcal{F}} dV \quad . \tag{A-10}$$

Using the identity

$$\int_V \underset{\sim}{x} \times (\nabla \times \underset{\sim}{A}) dV = \int_S \underset{\sim}{x} \times (\underset{\sim}{n} \times \underset{\sim}{A}) dS + 2\int_V \underset{\sim}{A} dV \quad ,$$

we can express (A-10) as

$$\underset{\sim}{I}(t) = -\frac{1}{2}\rho \int_{S_B} \underset{\sim}{x} \times (\underset{\sim}{n} \times \underset{\sim}{\mathcal{F}}) dS + \frac{1}{2}\rho \int_{V_B} \underset{\sim}{x} \times (\nabla \times \underset{\sim}{\mathcal{F}}) dV \quad ,$$

and since

$$\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u} = \nabla \times \underset{\sim}{\mathcal{F}} \quad ,$$

we finally obtain, using (A-7)

$$\underset{\sim}{I}(t) = \frac{1}{2}\rho \int_{V_B} [\underset{\sim}{x} \times \underset{\sim}{\omega}(\underset{\sim}{x}, t)] dV \quad .$$

APPENDIX B

Evaluation of the integrals used in Part II

1) $$\int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}}\, dx = \int_{-1}^{+1} \frac{1-x}{\sqrt{1-x^2}}\, dx = \int_{-1}^{+1} \frac{dx}{\sqrt{1-x^2}} - \int_{-1}^{+1} \frac{xdx}{\sqrt{1-x^2}} ,$$

$$= \Big[\text{Arcsin } x \Big]_{-1}^{+1} + \Big[\sqrt{1-x^2} \Big]_{-1}^{+1} = \pi .$$

2) $$\int_{-1}^{+1} \frac{1}{\xi - x} \sqrt{\frac{1-x}{1+x}}\, dx = \int_{-1}^{+1} \frac{1-x}{\xi - x} \frac{1}{\sqrt{1-x^2}}\, dx ,$$

$$= \int_{-1}^{+1} \frac{dx}{\sqrt{1-x^2}} + (1-\xi) \int_{-1}^{+1} \frac{dx}{(\xi - x)\sqrt{1-x^2}} ,$$

$$= \pi + (1-\xi) \int_{-1}^{+1} \frac{dx}{(\xi - x)\sqrt{1-x^2}} .$$

From Appendix C of Part I, equation (2), we have

$$\sqrt{z^2-1} \int_{-1}^{+1} \frac{d\xi}{(\xi - z)\sqrt{1-\xi^2}} = -\pi ,$$

for $z \neq x, \quad |x| < 1.$

Using this identity we obtain, for $|\xi| > 1$,

$$\int_{-1}^{+1} \frac{1}{(\xi - x)} \sqrt{\frac{1-x}{1+x}}\, dx = \pi \left[1 - \sqrt{\frac{\xi - 1}{\xi + 1}} \right].$$

$$3) \int_{-1}^{+1} \frac{x}{(\xi - x)} \sqrt{\frac{1-x}{1+x}}\, dx = -\int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}}\, dx + \xi \int_{-1}^{+1} \frac{1}{\xi - x} \sqrt{\frac{1-x}{1+x}}\, dx \quad ,$$

$$= \pi \left[(\xi - 1) - \xi \sqrt{\frac{\xi - 1}{\xi + 1}} \right] \quad ,$$

which is valid when $|\xi| > 1$.

REFERENCES

1. Küssner, H. G. 1936 Zusammenfassender bericht uber den instationaren auftrieb von flugeln. Luftfahrtforschung, Bd. 13, Nr. 12, 20, 410.

2. Kármán, Th. von and Sears, W. R. 1938 Airfoil theory for non-uniform motion. Journal of the Aeronautical Sciences, 5, No. 10, 370.

3. Sears, W. R. 1940 Operational methods in the theory of airfoil in non-uniform motion. Journal of the Franklin Institute, 230, 95.

4. Miles, J. W. 1956 The aerodynamic force on an airfoil in a moving gust. Journal of the Aeronautical Sciences, 23, 1044.

5. Jones, R. T. 1940 The unsteady lift of a wing of finite aspect ratio. NACA Report 681.

6. Wagner, H. 1925 Uber die entstehung des dynamischen auftriebes von tragflugen. Z. F. A. M. M., Bd. 5, Heft 1, 17-35.

7. Theodorsen, T. 1935 General theory of aerodynamic instability and the mechanism of flutter. NACA Report 496.

8. Munk, M. M. 1929 Fundamental of fluid dynamics for aircraft designers. The Ronald Press Co., New York.

9. Durand, W. F. 1935 Aerodynamic theory. Julius Springer, Berlin, 2.

10. Sears, W. R. 1941 Some aspect of non-stationary airfoil theory and its practical application. Journal of Aeronautical Sciences, 8, 104.

11. Batchelor, G. K. 1970 An introduction to fluid dynamics. Cambridge University Press, Cambridge, England.

12. Heaslet, M. A. and Spreiter, J. R. 1953 Reciprocity relation in aerodynamics. NACA Report 1119.

13. Garrick, I. E. 1938 On some reciprocal relations in the theory of non stationary flows. NACA Report 629.

14. von Kármán, Th. 1947 Supersonic aerodynamics - principles and application. Journal of the Aeronautical Sciences, 14, No. 7, 373-409.

TABLE I

Wagner's function (Φ (Ut))

The function is given in terms of the distance from the impulsive start Ut, expressed in half chord.

Ut	Φ(Ut)	Ut	Φ(Ut)
0.0	0.5000	7	0.8412
0.5	0.5557	10	0.8750
1	0.6006	15	0.9148
1.5	0.6378	25	0.9501
2	0.6693	50	0.9767
2.5	0.6962	100	0.9891
3	0.7195	500	0.9980
4	0.7579	1000	0.9990
5	0.7882	∞	1.0000

TABLE II

Sear's function ($\Psi(Ut)$)

The function is given in terms of the distance from the impulsive start UT, expressed in half chord.

Ut	$\Psi(Ut)$	Ut	$\Psi(Ut)$
0.0	0.0000	5	0.7388
0.2	0.1980	6	0.7731
0.4	0.2757	7	0.8004
0.6	0.3324	8	0.8225
0.8	0.3782	9	0.8408
1	0.4167	10	0.8561
1.2	0.4500	16	0.9117
1.4	0.4794	26	0.9490
1.6	0.5057	51	0.9765
1.8	0.5293	100	0.9890
2	0.5508	500	0.9979
3	0.6351	1000	0.9990
4	0.6945	∞	1.0000

TABLE III

Lambda function $\Lambda(Ut)$

The function is given in terms of the distance from the start of the motion Ut, expressed in half chord.

Ut	$\Lambda(Ut)$	Ut	$\Lambda(Ut)$
0.0	0	5	0.6783
0.2	-0.0839	6	0.7269
0.4	-0.1041	7	0.7640
0.6	-0.1059	8	0.7929
0.8	-0.0946	9	0.8157
1.0	-0.0718	10	0.8345
1.2	-0.0373	12	0.8625
1.4	0.0102	14	0.8828
1.6	0.0747	16	0.8981
1.8	0.1663	18	0.9100
2	0.3866	20	0.9194
3	0.5246	∞	1
4	0.6142		

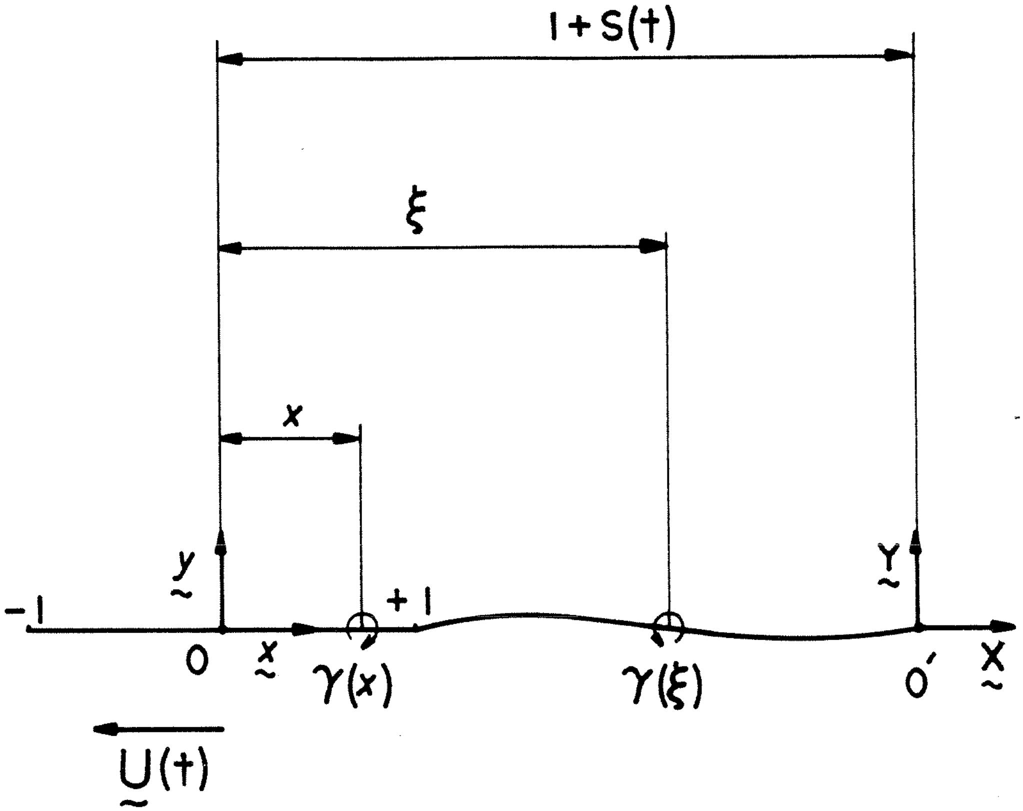


Figure 1. Description of the blade and the coordinate systems comprising the inertial frame of reference (X, Y), stationary with the fluid, and the body frame of reference (x, y), translating along the X-axis with velocity $-U_o$. The blade extends from $x = -1$ to $x = +1$ along the x-axis and the wake extends from $x = +1$ to $X = 0$.

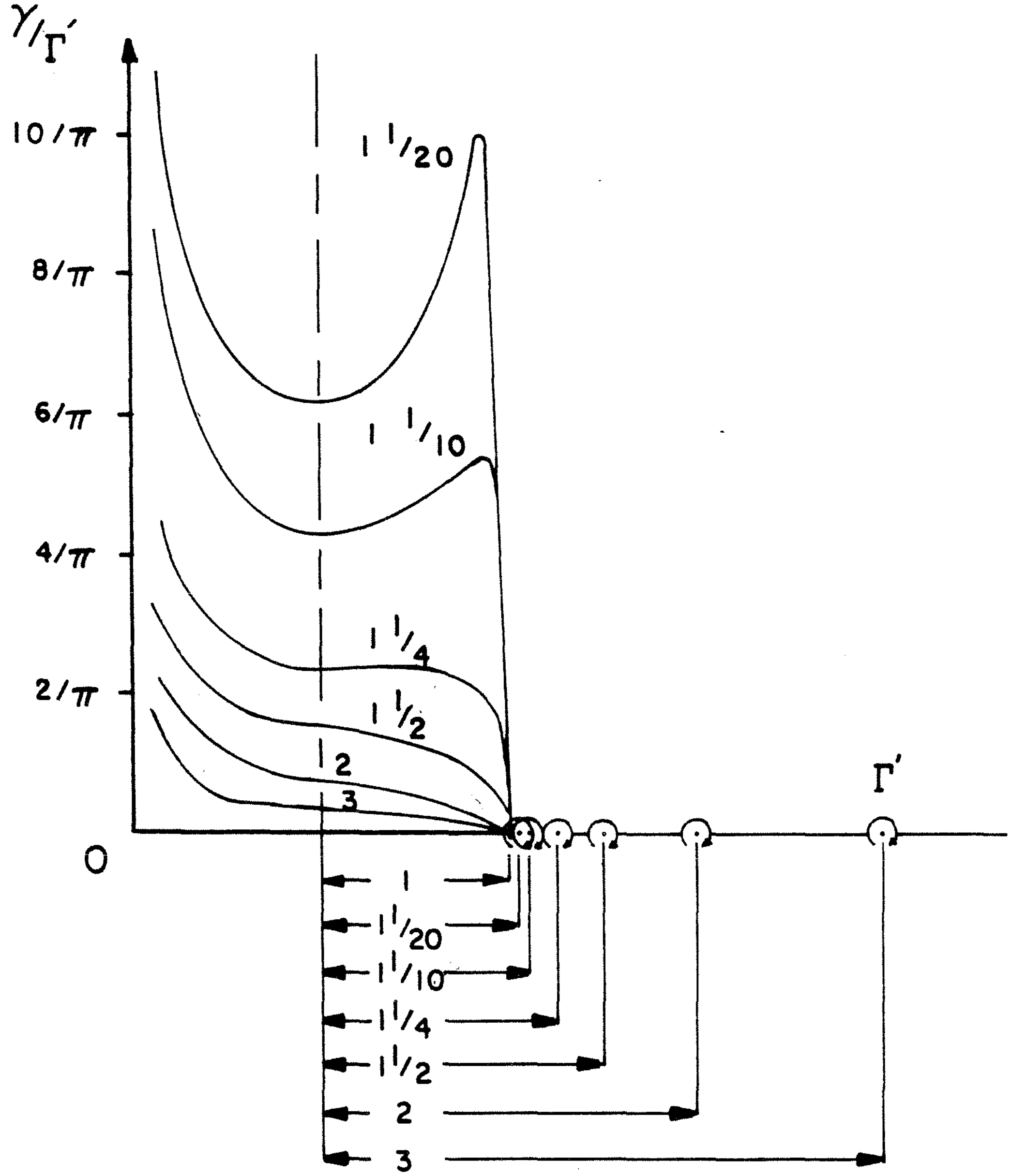


Figure 2. Vorticity distribution on a wing induced by a point vortex placed at various distances from the trailing edge on the wake.

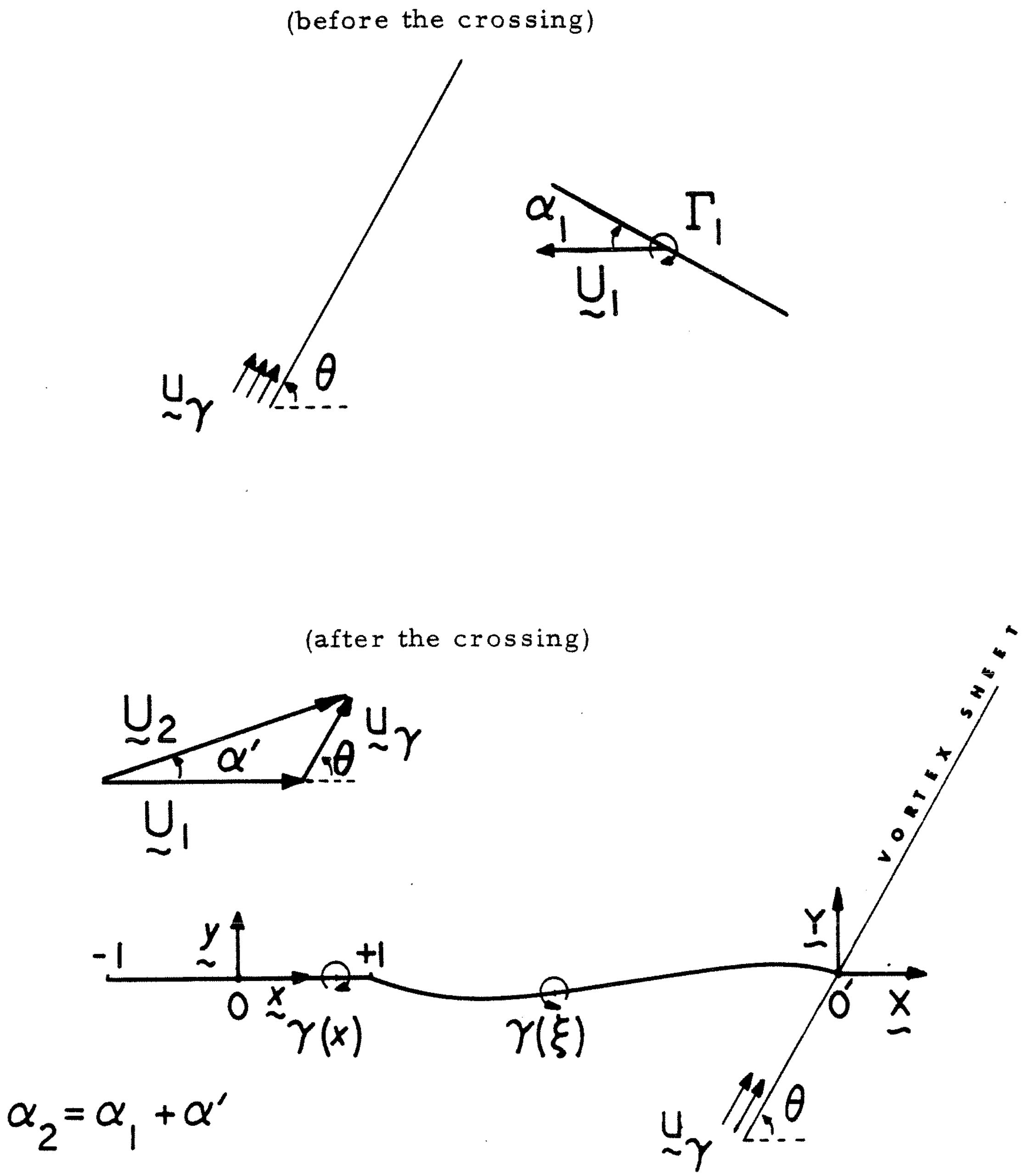


Figure 3. Description of the vortex sheet in the case of impulsive crossing of a vortex sheet by a wing. The vortex sheet crosses the X-axis at X = 0 and is inclined with an angle θ from the X-axis. The wing is translating along the X-axis with velocity -U and has an angle of attack α_1 prior to the crossing and α_2 after the crossing. U_2 represents the apparent flow velocity after the crossing.

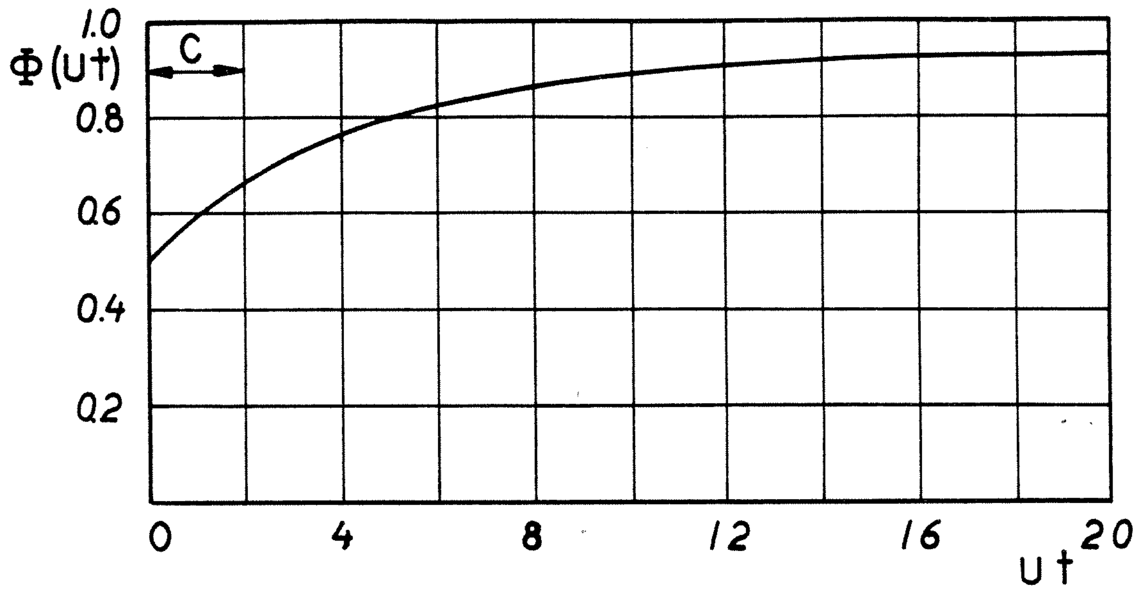


Figure 4. Wagner's function ($\Phi(Ut)$).

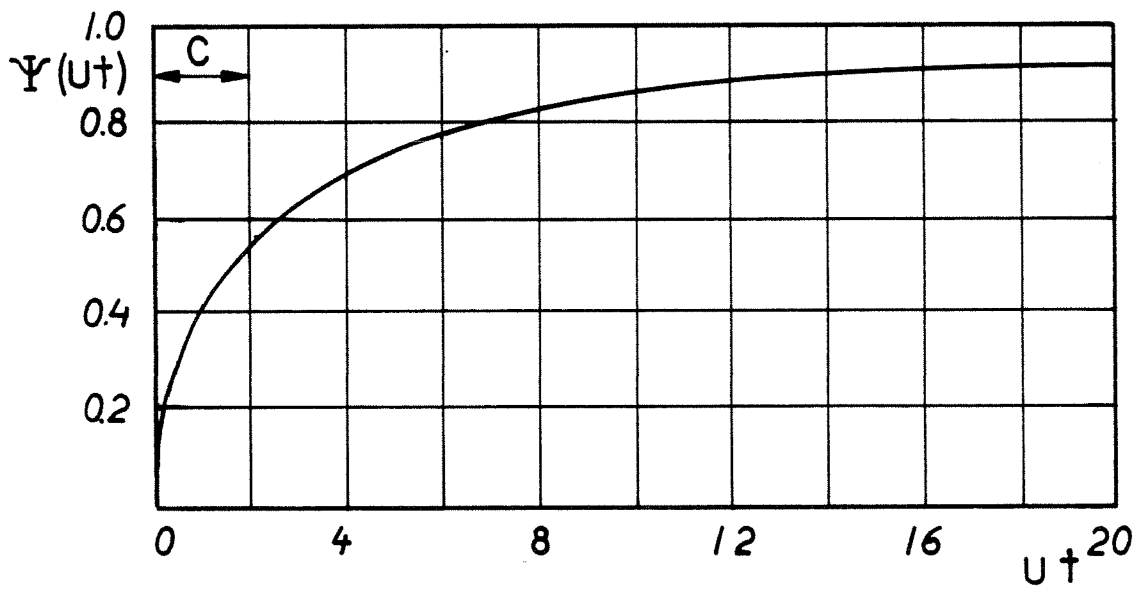


Figure 5. Sear's function ($\Psi(Ut)$).

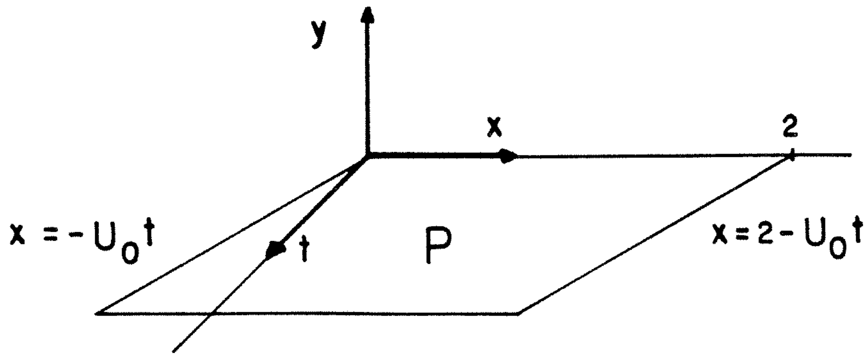
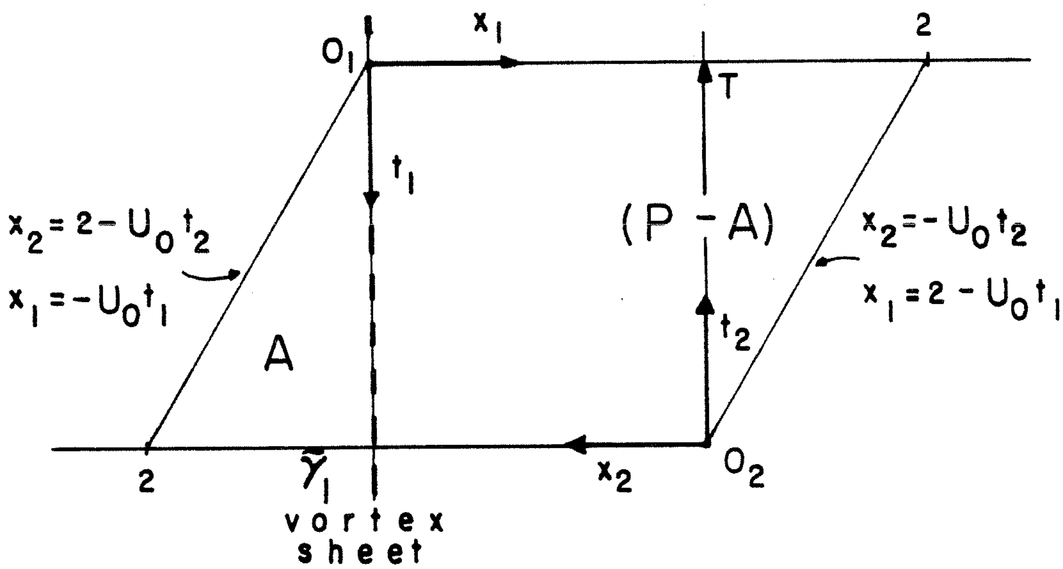


Figure 6. Description of the reference system used in the derivation of the reverse flow relations. The plan form P represents the space occupied by the blade in the (x, t) plane.



a) during the crossing of $\tilde{\gamma}_1$ by the first wing

Figure 7. Description of the coordinate systems in the (x, t) plane for both the direct and reverse motion. (x_1, t_1) is associated with the direct motion and (x_2, t_2) with the reverse motion. The vortex sheet $\tilde{\gamma}_1$ is placed at $x_1 = 0$. Two cases are represented.

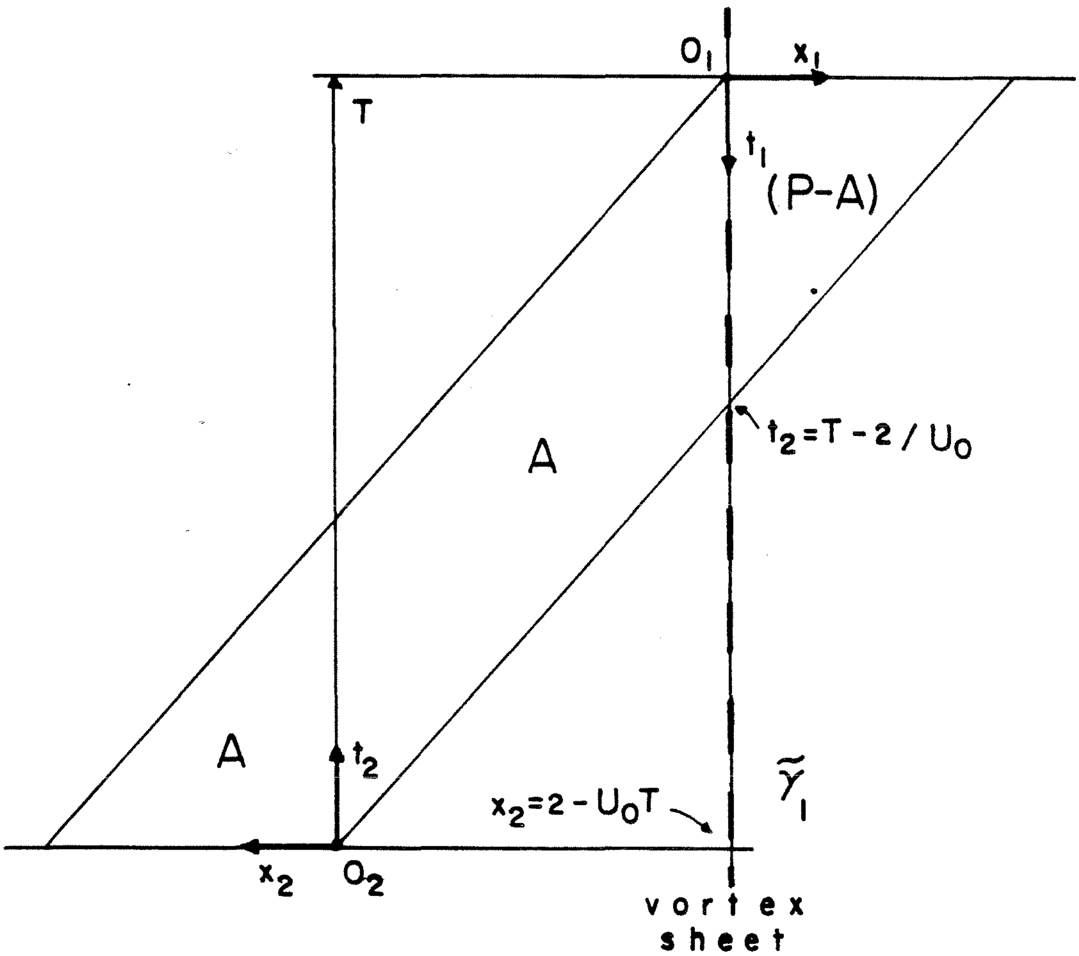


Figure 7. b) after completion of the crossing of $\tilde{\gamma}_1$ by the first wing.

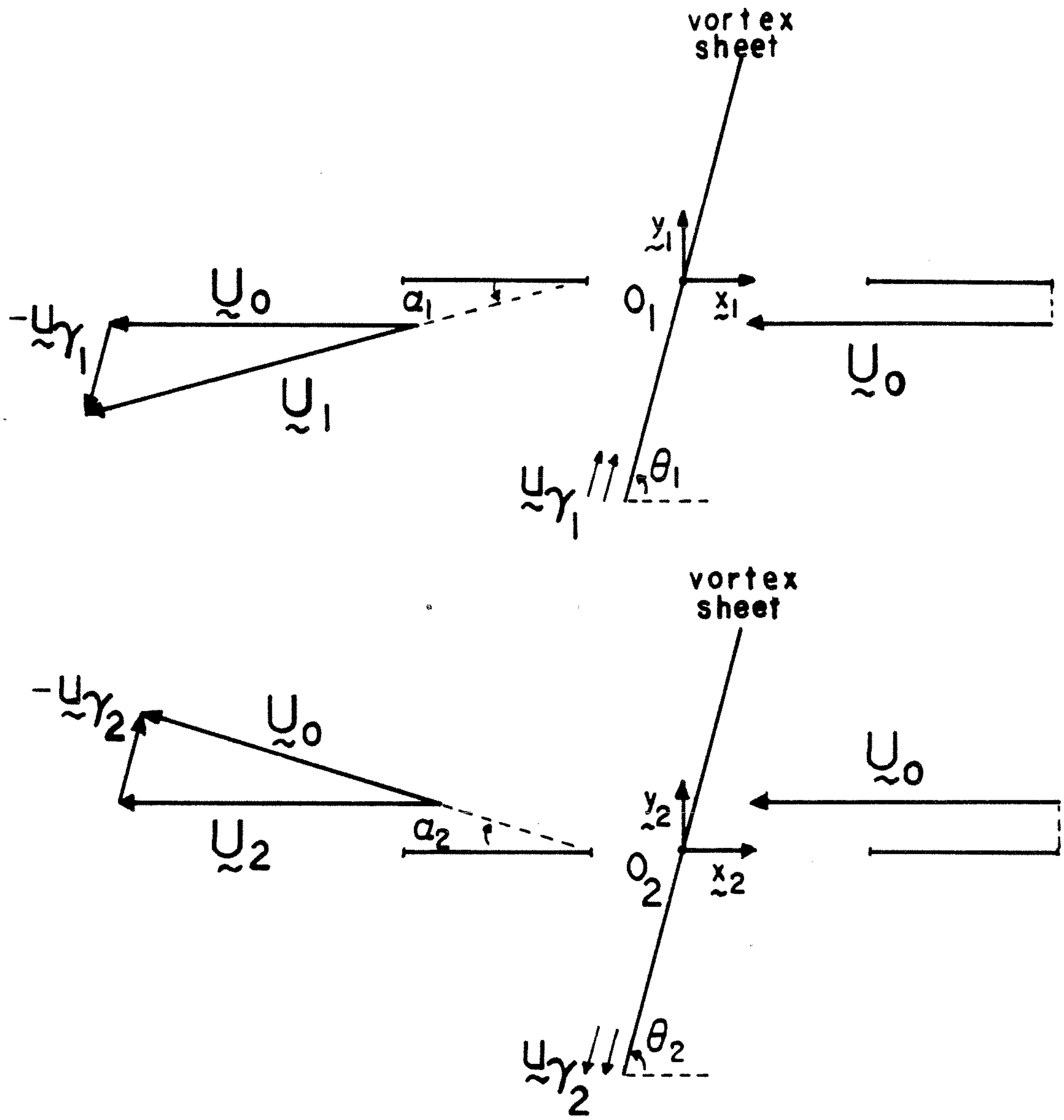


Figure 8. Description of the two vortex sheets $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. $\tilde{\gamma}_2$ is chosen with $\underset{\sim}{u}_{\gamma_2} = -\underset{\sim}{u}_{\gamma_1}$ and $\theta_1 = \theta_2$.

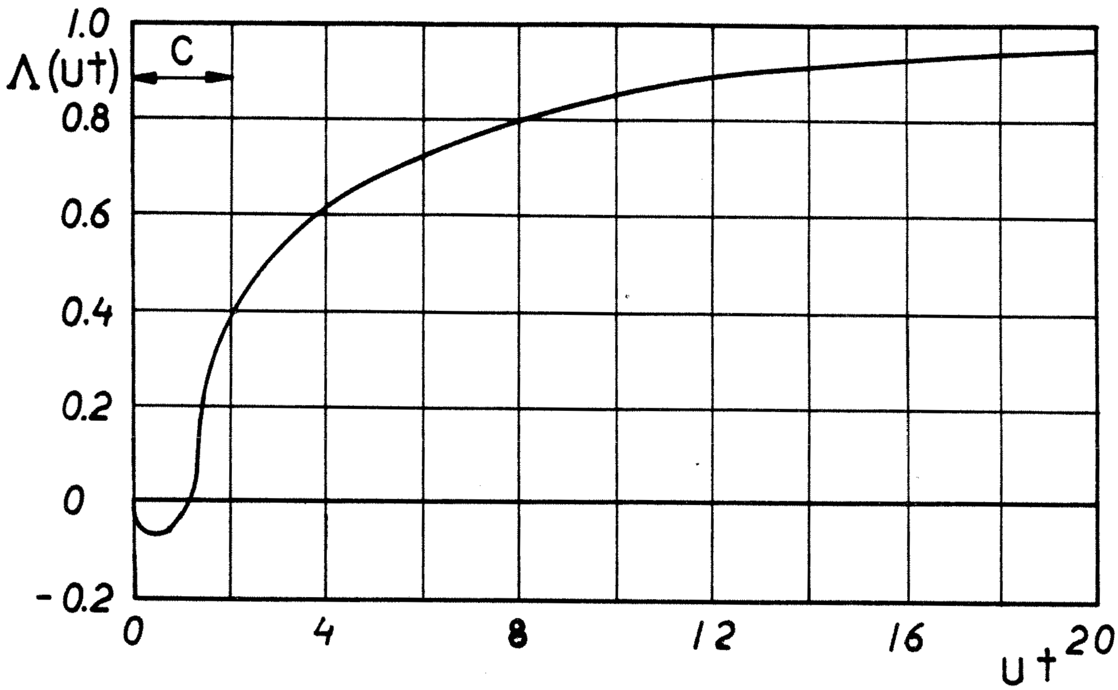


Figure 9. Lambda function $\Lambda(Ut)$.