QUEUES OF QUEUES
IN
COMMUNICATION NETWORKS

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Abstract

The concept of a camp-on queueing system is related to the idea of having systems of multiple hierarchical queues. Customers requesting service at a service center are queued at one of different queueing stages based on the location of the customer's intended server within the service hierarchy. In many instances, customers in a camp-on model exhibit a dual function ***customer-server***, giving rise to a system with ***queues of queues***. For this model, we assume Poisson distributed arrivals with different classes of customers for each queueing level. The service completion process is regarded as exponentially distributed, a standard assumption for many communication systems.

Here we discuss a stationary model for such a Markovian camp-on system. Closed-form solutions are derived for various state occupancy distributions of interest (e.g., joint probability distribution of queue lengths, marginal distributions for subsystems, accumulated workload, etc.), in systems with finite and infinite storage capacity and two queueing levels. Most of these results are also extended to multilevel queueing systems. It is found that this camp-on model is stable whenever all the distinct queues, in isolation, behave as stable systems. The form of the joint probability distribution of queue lengths is not a product of the independent contributions from each subsystem, since it must also account for the relative position of the queues with respect to the initial service center, the root of the service hierarchy.

Two particular applications are discussed in detail: 1) PBX-like communication services, and 2) broadcast delivery services. Performance statistics such as waiting time distributions, blocking probabilities and mean response time are derived. These results show that we do not pay too large a penalty for introducing two or more levels of queueing, and under very extreme conditions (heavy traffic) the delay in response increases only linearly with the number of queueing stages. Broadcast service strategies provide even better performance than conventional point-to-point service, though a broadcast medium is required.

TABLE OF CONTENTS

LIST OF FIGURES AND TABLES

To my parents

INTRODUCTION

As the next generation of communication systems unveils, more customer-oriented services and computer applications are making use of hierarchical strategies to handle their decision-making algorithms and other resource management problems. These new tendencies within the communications market call for a better understanding of hierarchical approaches to queueing distribution problems in communication systems. The approach to be considered in this work is a tree-based queueing strategy. New job requests entering a service center can be scheduled and queued on top of previously queued jobs. Rather than doing so in a single queue, however, the customers are distributed in an array of queues based on their order of arrival and class and type of service requested. Such a queueing strategy, borrowed from the *camp-on* service in telephony, offers a variety of applications in areas such as system resource management, networking, scheduling and routing, etc.

So far, current services and applications have been taking advantage of well-established queueing concepts and sophisticated task management schemes developed for both single node and multinode queueing systems. The main core of this research has been aimed at systems with an arbitrary but fixed number of nodes or service centers. These have been within the general context of networks of queues. Key issues under consideration have been multiple classes of customers, queueing and service strategies, routing schemes, distribution of workload, etc. An extensive survey of the most important results in this area can be found in the specialized literature[9,25,28]. Some of these will be discussed in detail as we review the different techniques used for queueing system analysis in Chapter 1. Our goal here is to evaluate some of the queueing issues under a camp-on strategy, where users are queued at different levels within the queueing system, which are related to the user's end-point server. Besides being a natural extension of present queueing models, camp-on systems can also be seen as a model for *queues of queues.* Such representation permits modeling systems with a random number of service nodes or active queues, which are defined by the queueing schemes adopted. Some of the multilevel queueing issues can even be interpreted as problems of population size in a "genealogy tree."

Most of the specific problems in the area of networks of queues, such as those mentioned above, seem to stand in need of a general theoretical framework within which models and questions of system behavior can be appropriately formulated and addressed. However, a great amount of insight can still be gained through the use of techniques such as continuous time Markov chains, generating function methods and the use of notions such as balance equations and time reversibility of Markovian processes.

In this work we introduce the concept of a camp-on system. Specifically, we focus our attention on the equilibrium behavior of queues in a camp-on model. In Chapter I we describe the queueing strategy associated with a camp-on system and compare this model with other queueing systems. Chapters II and III are devoted to the study of two-level camp-on systems. The basic assumptions for the model and the underlying equilibrium balance equation for the joint probability distribution of queue lengths are derived in Chapter II. These are based on a Markovian model for the state occupancy problem in a two-level camp-on system. Defecting or reneging from the camp-on model is also contemplated. Chapter III addresses the problem of finding joint probability equilibrium distributions, some marginal distributions, and the total workload accumulated in the queueing stages. Closed-form solutions for different state occupancy distributions are provided for non-reneging camp-on systems. In Chapter IV we extend many of the concepts analyzed in the two-level case to multilevel camp-on systems. Finally, Chapter V presents analytic performance results for camp-on systems in two distinct communication applications: i) PBX-like communication services as in an enhanced office environment, and ii) broadcast delivery services as in Videotex. Chapter VI summarizes the scope of these results and suggests some open problems for further studies.

CHAPTER I:

CAMP-ON SYSTEMS

A camp-on system is a multilevel, multiqueue system where waiting lines are organized in a hierarchical manner. The hierarchy levels represent the number of queueing stages a customer must visit *before* his service process is initiated. Each level is seen as an ordered collection of waiting lines and each waiting line contributes to the next level with its own set of queues, spanning a tree-like queueing structure. Service takes place in the system on a level-by-level basis. This chapter introduces the basic concepts behind the camp-on model. Section 1 presents the queueing strategy associated with the camp-on model and provides a detailed description of the customer handling within this multilevel queueing system. Section 2 gives a historical account of some important results for systems with multiple queues as they relate to the camp-on model. The most interesting analogies are found within the context of networks of queues. Emphasis has been place on models that reduce the statistical behavior of the queueing system to the independent contributions of the network nodes.

I.1. Queueing in a Camp-on System

Consider a traffic stream S_0 that is initially offered to some service center, the originating service facility, from an infinite source of subscribers. If the service center is busy and buffering space is available, a queue will start to build up as in any conventional queueing system. For our purposes, this initial queue will be referred to as the *First-Level Queue* . Assume now that every single customer present at the first-level queue is also offered his own independent, infinite source of subscribers S_1, $S_2, \ldots,$ S_n. If buffering space is made available for these traffic streams, queues associated with these streams will build up, one for each of the n customers at the first-level queue. Let us refer to them as the *Second-Level Queues.* Once again, customers at this second level of queues can also be offered their own independent traffic streams from infinite pools of subscribers. New queues, when buffers are provided, will continue to build up and they will form what will be referred to as the *Third-Level Queues.* This process can go on indefinitely as customers join the system. We define the queueing system resulting from this queueing scheme to be a *camp-on system.* This is in analogy with the "call camping" service in telephony in which telephone calls to a busy station can be put on hold to be answered later rather that being immediately rejected by the local switching office.

After a customer commences his service period, we expect him to leave the service facility only when all of his requested task has been completed. However, service disciplines that preempt the customer in service in favor of newly arrived customers and even customers' defections before service completion will be considered. Upon his departure, the serviced customer will leave the functioning service facility, taking with him all of his associated second and upper-level queues, none of which have yet begun service. The next first-level customer in the order of service, as determined by the queueing discipline, is then serviced. The departing first-level customer, meanwhile, starts serving his own first-level queue, a second-level queue in the previous stage of the camp-on system. The associated camp-on depth, that is, the number of queueing levels in the branch of the service hierarchy for the departing customer, is reduced by one unless this customer were at the bottom of the hierarchy and were not also a server. In that case, the number of levels remains zero. The sequence in Figures 1 and 2 illustrate this queueing scheme in a camp-on system with two queueing stages.

The basic idea behind this camp-on queueing scheme is to allow customers in a multiservice environment to join one of different waiting lines that are organized in a hierarchical fashion. This hierarchy is given as a tree representation of the various stages of queues that servers and customers must go through during a communication session with a service node. Inherent is the assumption that users can not only demand service for their own jobs but can also provide service to other customers. New job requests entering the camp-on system will then have their service scheduled with respect to their end-point server, i.e., the actual service center for that job. But a customer's real position inside this queueing system will depend on the relative position of his intended server within the queueing hierarchy. Hence, customers appear as if they queue up or *camp* at different levels of a hierarchical multilevel queueing system.

Thus, the key point in the camp-on strategy is that customers need not be considered as simple customers in a traditional sense, e.g., ones that merely ask for their jobs to be done, but rather, they can also exercise control over the execution of other job requests being submitted. The way this control process is exercised determines to a large extent the complexity of the camp-on model. Still,

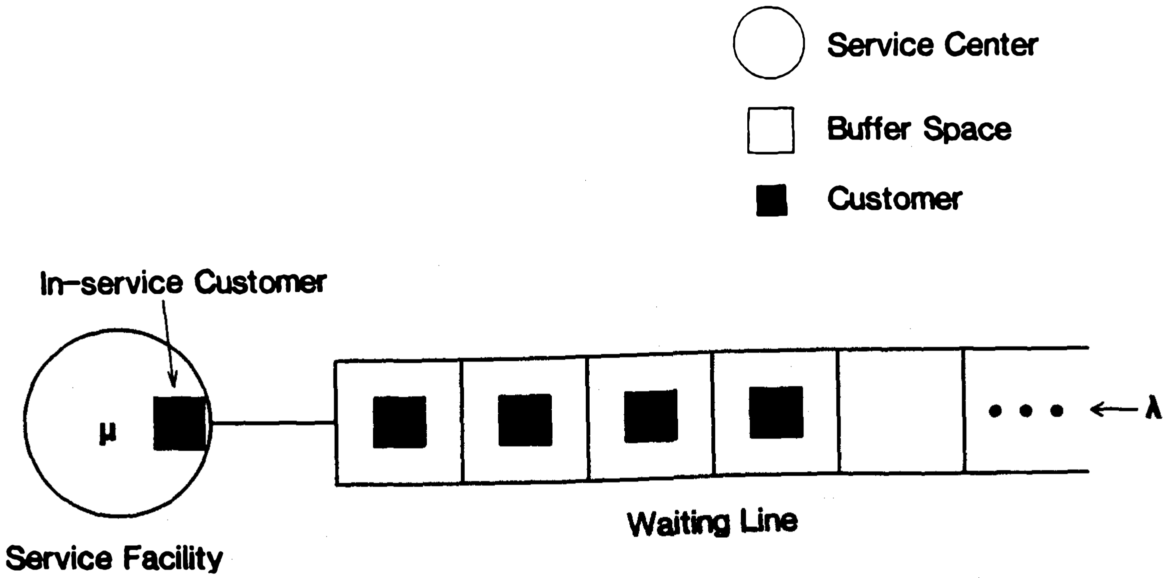

Figure 1.a: Typical $M/M/1/N$ queueing system (one queueing stage).

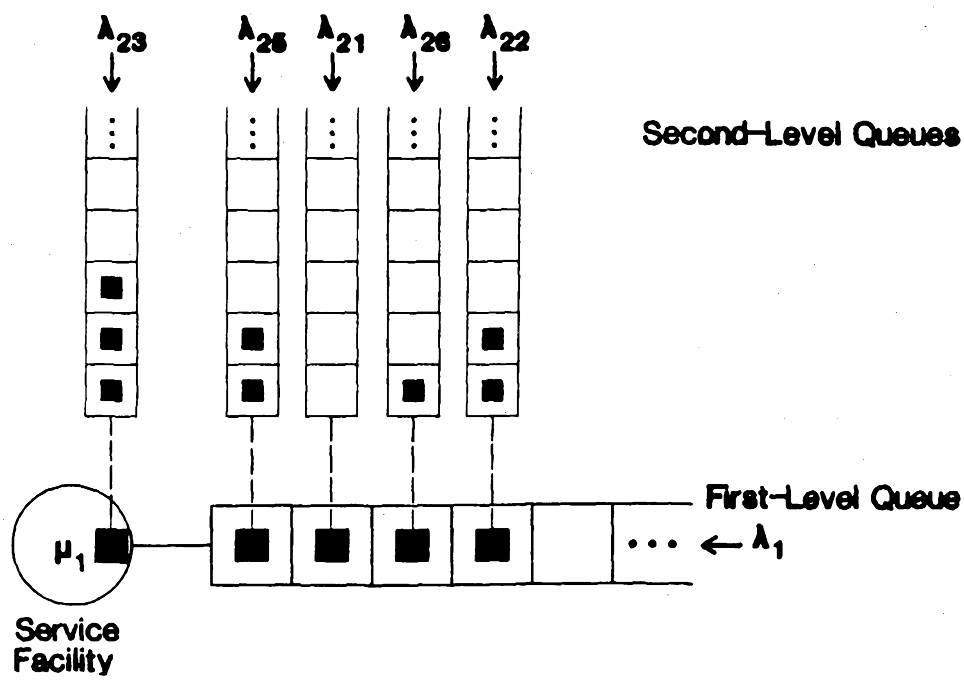

Figure 1.b: A camp-on system with two queueing stages.

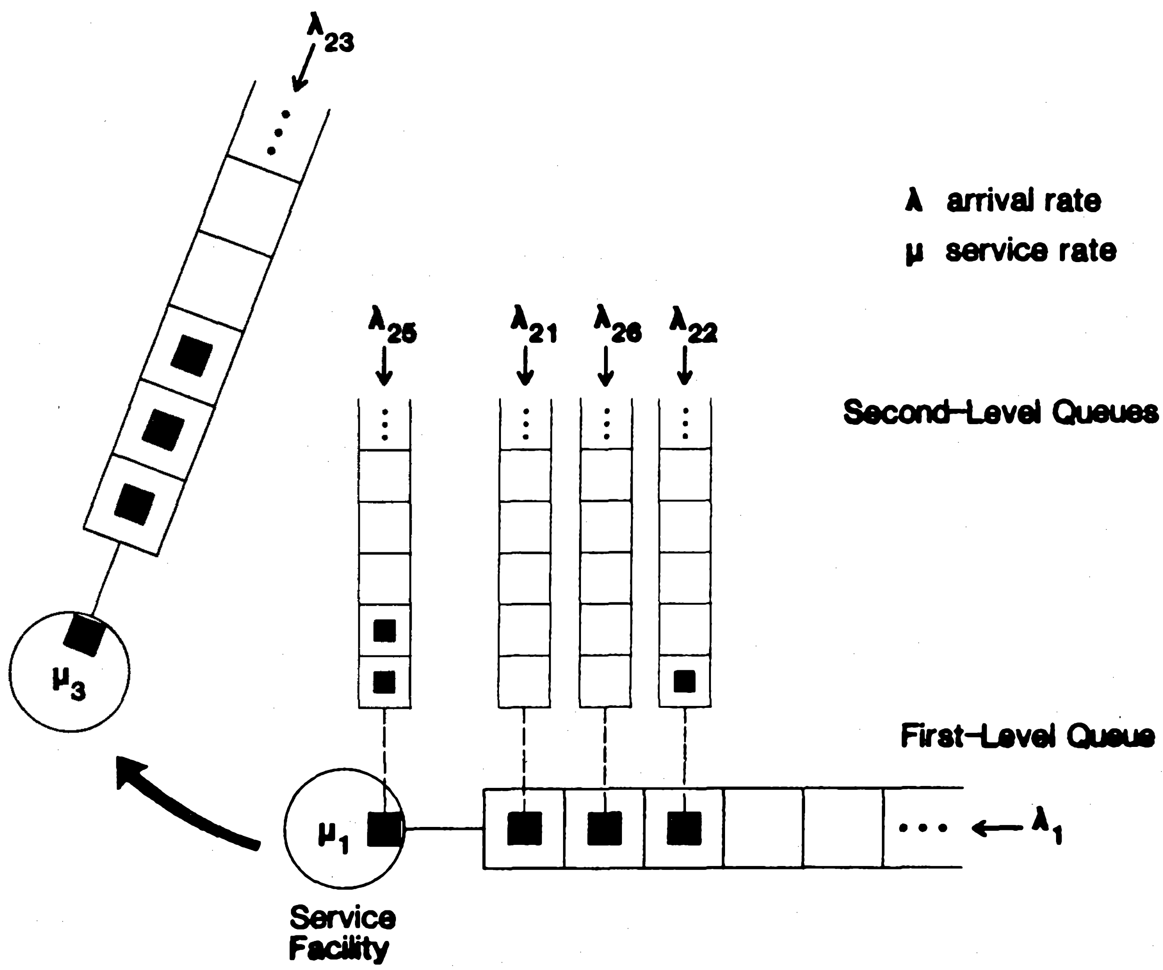

Figure 2: First-level customer departure, FCFS service discipline.

every customer-queue pair could be considered, in isolation, as a separate queueing system, or as a subsystem within the camp-on queueing model.

Ideally, one would like to include all sorts of service and schedule interrelations among queues, servers and even classes of customers, that can possibly take place in a communications environment such as in multiplexing or routing nodes, PBX's, inquiry-oriented systems, or any other networking application. Of course, this approach could quickly prove to be intractable and its usefulness therefore argued. Here, a simplified queueing scheme based on an independent-branch approach is proposed, which covers various queueing strategies of interest and paves the way to the study of more complex queueing situations.

We assume that each of the different sources of users is independent. We also require that job scheduling for upper-level customers does not interfere with the arrival/departure pattern for the lower-level customers, for they belong to subsystems outside their service path. Such a queueing strategy implies a decoupling between the different queue levels in the camp-on systems and will allow simpler representations for various state-occupancy problems of interest. More general schemes for schedule management await future analysis.

The camp-on system concept arises in enhanced telephone services. Incoming phone calls to busy telephone premises can be kept on hold by means of the standard PBX philosophy, while the waiting customers are attended to in the usual first-come-first-served basis. If third parties try to call any of the queued-up customers, one could also put them on hold instead of using the conventional telephone service procedure, which would block and clear all incoming calls from the system. By this means, a queueing system is created wherein any queued customer would eventually become a functioning service facility. These concepts may apply more realistically to computers as the calling and called parties rather than to people, given the relative "willingness" of computers to stay on hold as compared to people.

I.2. Camp-On Within the Queueing Systems Context

An extensive effort has been put into studying equilibrium distributions for queueing systems, especially those of the network-of-queues type. We will devote this section to a review of those queueing models that can be represented as a continuous time Markov chain with a countable state space. We deal with the issue of multiqueueing systems to provide further insight into what we should expect from the camp-on model.

A model for a network of queues was first introduced in a paper by Jackson[16], widely regarded as the departing point in the theory of multiqueueing systems. This model is based on a generalization of the classical $M/M/s$ queue: memoryless customer interarrival time, memoryless or exponential service time, and s servers, to an arbitrary interconnected *open* network with exponential servers, Poisson external input and a first-come-first-served service discipline. In his paper, Jackson proved that whenever an equilibrium condition exists, each node in the network behaves as if it were an independent $M/M/s$ queue with Poisson input; i.e., if $\pi(\mathbf{x}_N)$ is the equilibrium probability of the network of N nodes' being in state $\pi(\mathbf{x}_N) = (x_1, \cdots, x_N)$, then

$$\pi(\mathbf{x}_N) = \psi(x_1) \cdots \psi(x_N),$$

where $\psi(x_i)$ is the equilibrium probability of having x_i customers in an $M/M/s_i$ queue. This particular form of solution has come to be known as the *product form.* The appeal for this form of solution is obvious as it permits us to characterize the system behavior through the independent behavior of its nodes and allows a computationally efficient analysis of large networks.

In contrast to the open network of queues introduced by Jackson[16], Gordon and Newell[13] considered a *closed* Markovian network in which a fixed and finite number of customers, say M, circulate through the network and no external arrivals or departures are permitted. They proved that the equilibrium distribution has a product form, though the behavior at the various nodes can no longer be regarded as independent, since $x_1 + \cdots + x_N = M$.

Later, Jackson[17] and Posner and Bernholtz[29,30], respectively, introduced general open and closed Markovian network models that allow the total external arrival rate to depend upon the total number of customers in the system (open), the exponential service rate to be a function of the number of customers at the node, and the travel time between any two nodes in the system to have an arbitrary distribution (closed). Once again, it was demonstrated that the equilibrium distribution for the number of customers at the various nodes (and in transit) is of the product form. For closed networks, it was even permitted to have different classes of customers, with a different set of service rates and routing probabilities. These results are based on solutions to a steady-state *balance relation* equating the equilibrium rate of flow out of state $\mathbf{x}_n$ with the equilibrium rate of flow into state $\mathbf{x}_n$.

However, the above approaches had one limitation: granting that one could manage to guess the correct form of the equilibrium probabilities, verifying that the probabilities satisfy the balance relation was not still entirely trivial. Fortunately, the method of *partial balance equations* introduced by Whittle[40–41] provided in many cases simpler means to get around this inconvenience. In this approach, one attempts to decompose the balance equation into smaller sets of partial balance equations and then to show that the steady-state probability distribution $\pi(\mathbf{x}_N)$ satisfies the simpler equations. Using this technique, more general equilibrium results have been obtained as summarized by Baskett et al.[2] and Reiser and Kobayashi[34]. These authors allow a variety of customer classes and different kinds of service nodes in order to model central processors, data channels, terminals and routing delays in computer systems. It is important to note that for any given model, one has no assurance *a priori* that the set of partial balance equations is consistent. But, it is clear that any solution one finds for a set of partial balance equations will indeed satisfy the global balance equation, too.

In an effort to expand product form results for open systems, particularly with regard to routing behavior, Kelly[19–21] used a combination of the partial balance technique with the notion of *time reversal* or *reversibility*. Let $\{C(t),\ 0 \le t < \infty\}$ denote the Markov process describing the state of the system, and suppose the transition rates of $\{C(t),\ 0 \le t < \infty\}$ to be given by $q(\mathbf{x}_n;\mathbf{y}_m)$. Further, suppose we believe the probability $\pi(\mathbf{x}_N)$ to be the stationary distribution of $\{C(t),\ 0 \le t < \infty\}$. Then the reversed process $\{C(-t),\ 0 \le t < \infty\}$ also forms a Markov process. Moreover, the two processes, $C(t)$ and $C(-t)$, $0 \le t < \infty$, must have the same equilibrium distribution, with the transition rates $q'(\mathbf{x}_n;\mathbf{y}_m)$ for the reversed process given by

$$\pi(\mathbf{x}_n)q(\mathbf{x}_n;\mathbf{y}_m) \;=\; \pi(\mathbf{y}_m)q'(\mathbf{y}_m;\mathbf{x}_n),$$

and

$$\sum_{\mathbf{y}_m \neq \mathbf{x}_n} q(\mathbf{x}_n;\mathbf{y}_m) \;=\; \sum_{\mathbf{y}_m \neq \mathbf{x}_n} q'(\mathbf{x}_n;\mathbf{y}_m).$$

Hence, external *arrivals* for the reversed process correspond to external *departures* for the original process, and past *departures* from the system in the original process correspond to future external *arrivals* in the reversed process. This condition guarantees that the current state of the system in the original process is independent of past departures from the system, and this proves to be a powerful tool for analyzing many complex queueing systems.

Barbour[3] extended these results to nodes with arbitrary distributed service times through an argument invoking weak convergence methods. However, both of these works place a lot of constraints on the service discipline that can be implemented at each node. In fact, for the most part, a large share of the research undertaken in this direction has been aimed at enlarging the scope of product-form solutions to very special queueing networks. Chandy et al.[7] did work on the notion of ***station balance***, which provides a good summary of these trends. But in many cases, not even the popular FCFS discipline satisfies these constraints.

Lately, some authors have proposed more general approaches to analyze interconnected networks of nodes. Hence, Chandy, Herzog and Woo[8] studied the relationship between queueing networks and electrical networks and introduced the Norton's theorem approach to network analysis. Here, a closed network, a system that has a single node as input and a single node as output, is replaced by an "equivalent" network in which all queues are replaced by a single composite queue. Walrand[39] presented a probabilistic argument to explain the product form, the output theorems, and the Poisson character of the flows in order to provide a more intuitive justification of those properties. Finally, Lazar and Robertazzi[26,27] related the product-form solution for the probability distribution of Markovian queueing networks to the geometric and algebraic structure of the associated state-transition diagram. Using the consistency graph, necessary and sufficient conditions for the equivalence of the global balance equation have been given.

In perspective, a product-form type of solution for the camp-on queueing model would be ideal, since it would fit nicely within the mainstream of results for locally balance networks. It would also permit us to take advantage of the many computationally efficient algorithms already designed for these systems[6,31–34]. However, there are clear differences between the proposed camp-on model and the models proposed for networks of queues. First of all, the number of service centers in the camp-on model is a random variable, whereas it is regarded a fixed parameter in the study of networks of queues. Secondly, the networks-of-queues models do not contemplate multiple departures from the systems, whereas such departures are inherent to the camp-on model. An immediate consequence of the bulk nature of the effective departure process in a camp-on model is that the transition rate from state $\mathbf{x}_n$ to state $\mathbf{y}_m$ may be zero, $q(\mathbf{x}_n;\mathbf{y}_m) = 0$, while the reverse transition rate $q(\mathbf{y}_m;\mathbf{x}_n)$ is not zero. This result prevents us from taking advantage of the usefulness of the time reversal notion to tackle occupancy problems in the camp-on model, and demands rethinking the characteristics of the output flow.

Besides the above considerations, the fact that the upper-level service centers are not always considered active centers also precludes the existence of an independent product-form solution for this camp-on model in the same sense as the one for network of queues. In short, none of the previous results obtained from the theory of networks of queues applies directly to the camp-on model. Even though techniques such as partial balance equations are still convenient tools to get around some of the system complexity issues, their implications do not favor product-form solutions even in the most simple situations. We believe the study presented here will help us understand the behavior of these and other more general multiqueueing models.

CHAPTER II:

TWO-LEVEL CAMP-ON SYSTEMS

Although a general camp-on model would permit a queue to be formed at every single queued customer in the system, during the next two chapters we will focus only on *two-level* camp-on systems; i.e., no queueing will be allowed in the camp-on model beyond the second level of queues. If any such arrivals take place, they will be assumed to be *blocked* and *cleared* from the camp-on system. Figure 3 shows a typical distribution of queues in a two-level camp-on system. Two-level camp-on systems are very useful tools in themselves, for they can provide good models (as we will show in Chapter V) for many real-life applications such as inquiry-oriented networks, teleconferencing, Videotex, and other network database management systems.

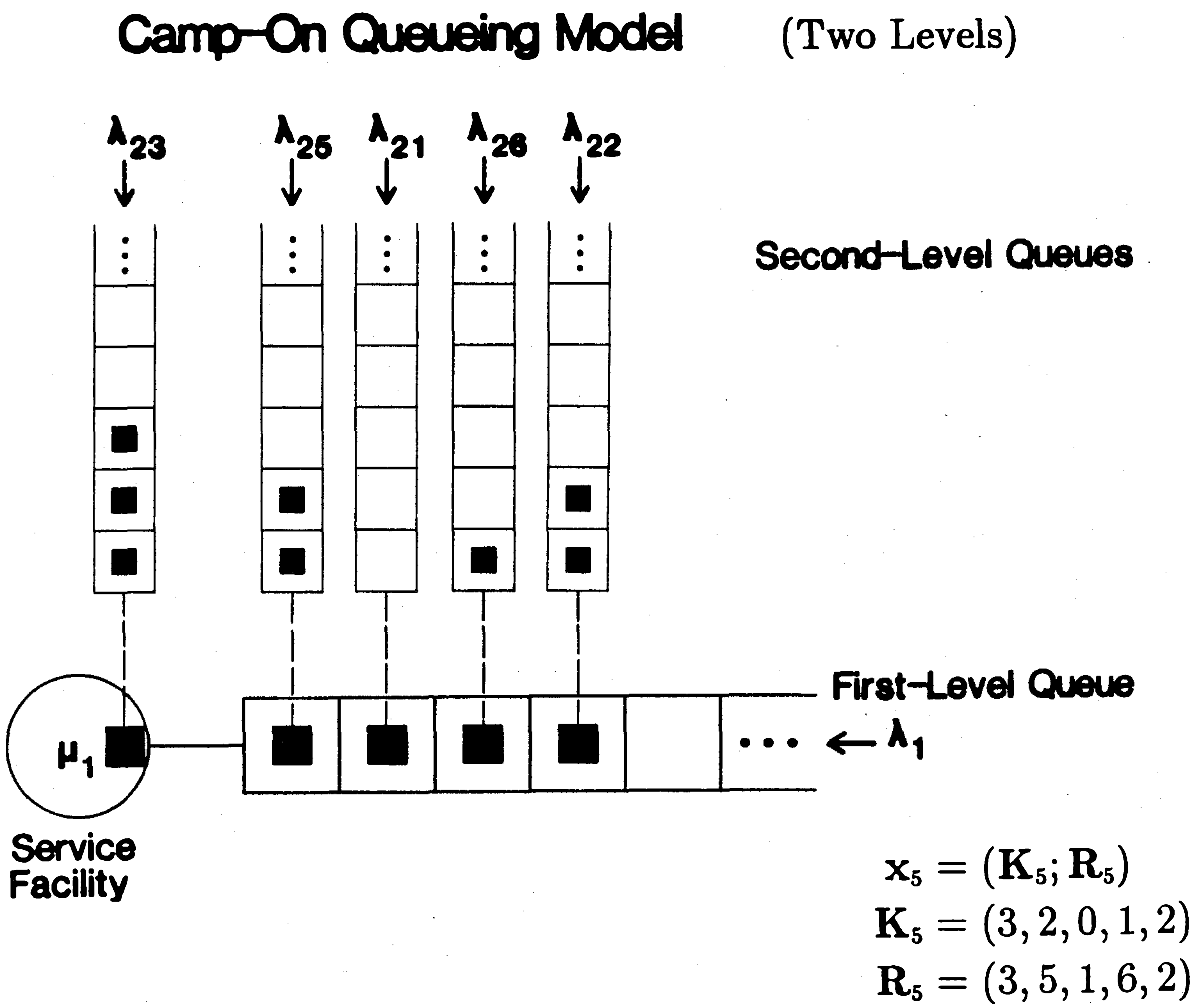

Figure 3: Typical distribution of queues in a two-level camp-on system.

II.1. Mathematical Model

The mathematical model to be used in this work will be based on a Markovian or quasi-Markovian approach to queueing systems. Both the customer arrival and the customer departure processes will be chosen such as to ensure a memoryless distribution for the amount of time spent in any of the states of the model. Various service disciplines will be considered, all of them a subset of the *work-conserving* disciplines[23,25]; i.e., no server is idle if there are still unserviced job requests within the same service center.

1.1. Customer Arrival Processes

Besides the two levels of queues, we will also consider R classes of customers. Every second-level system will be associated with a particular class-r group of customers ($1 \leq r \leq R$), which are drawn from an infinite source of subscribers. The classes can be used to represent the distinct conditions prevailing at each subsystems, e.g., the arrival rates to the individual nodes in a network, requests for particular pages of information from a database, traffic intensity at customer's premises, etc. The queue-class matching for the second-level systems implies a finite storage capacity at the first-level system because of the finite number of classes. Even though this matching does not imply any restriction on the storage capacity N for the first-level queue, including in-service and queued customers, we will consider only the case $R \geq N$.

The customer arrival process to the first-level system will be assumed conditionally Poisson, conditioned on the size of the first-level queue, with a mean arrival rate λ_{1n} ($0 \leq n < N$), when the system queue size is n customers (counting the customer in service). Many queueing schemes that encourage/discourage arrivals based on the queue size can be modeled through this variable arrival rate[23–25,37,38]. The customer arrival processes to each second-level queue will be assumed as independent Poisson processes with mean arrival rate λ_{2r} for a second-level queue associated with class-r customers.

The arrival process for first-level customers is regarded as independent from the arrival processes for the second-level systems. However, a simple type of interdependence between the first-level and the second-level arrival processes can also be introduced. The arrival process for the first-level system can be considered as state-dependent in the sense that it will be conditioned on the arrival processes to the second-level systems. The constraint is that the total arrival rate to the camp-on system is kept constant. That is, if $(r_1, \cdots, r_n)$ represents the class assignment for customers to the n second-level systems, then

$$\lambda_{1n} = \begin{cases} \lambda_{1n}, & \text{class-1 systems;} \\ \sum_{r=1}^{R} \lambda_{2r} - \sum_{i=1}^{n} \lambda_{2r_i}, & \text{class-2 systems.} \end{cases} \tag{2.1}$$

A class-1 camp-on system could be interpreted as a system that effectively has $R+1$ classes of customers, R for the second-level systems and one for the first-level system. This case corresponds to an increasing external-arrival rate and a random assignment of classes among the second-level systems. Class-2 camp-on systems, however, have only R classes of customers for both the first-level and the second-level systems. But a particular class of customers will be available to the second-level systems only if it is already present at the first-level queue.

For each one of these first-level arrival cases, one can derive the effective rate at which new second-level systems incorporate into the camp-on model. This is denoted as γ_n, the first-level arrival rate for second-level systems as seen by the service center:

$$\gamma_{n+1} = \begin{cases} \dfrac{\lambda_{1n}}{R-n}, & \text{class-1 systems;} \\ \lambda_{2r_{n+1}}, & \text{class-2 systems,} \end{cases} \tag{2.2}$$

where r_n indicates the class of the newly arrived customer.

1.2. Customer Departure Processes

The customer service completion process will be regarded as exponential, or memoryless, with a mean service rate μ. It will be considered independent of the associated second-level stage (customer class, size, etc.). This has been a standard assumption in many communication systems. However, some computer applications would be better modeled by a service time distribution with constant holding times, such as for packet-switching applications. Nonetheless, memoryless service processes are always considered to be good reference models even in those cases where they may not exactly apply.

Customers will also be allowed to *renege* or defect from their queues any time prior to the start or completion of their service period. The customer reneging processes from the first-level and second-level queue will be regarded as independent and exponentially distributed, as is the service completion process. For the first-level customers, the reneging rate will be conditioned on the customer position inside the first-level queue for reasons that will soon be explained. The reneging rate will be called ν_i for the customer at the i^{th} $(1 \le i \le n)$ position in the first-level queue. The customer reneging processes from the second-level queues will be regarded as exponentially and identically distributed with a common mean reneging rate η. One expects that for many applications, customers will not know where they are in the queues, so this is not unreasonable.

The position-dependent reneging rate for first-level customers permits more general service situations to be handled, such as multiserver centers and variable-speed servers, or to discourage long queues in applications, such as public networking. If a first-level customer chooses to renege from his queue, he will take with him his associated second-level system and initiate its service procedure, as in the case of service completion. The customer arrival processes, service completion processes and customer-reneging processes will be regarded as statistically independent processes.

The multiserver center can be readily modeled through a generalization of the reneging parameters. Let s denote the total number of servers available at the first-level service center, and let each of these servers provide service for $1/\mu_i$ $(1 \leq i \leq s)$ units of times per customer, on the average. Then the multiserver center can be covered by considering the generalized departure rate

$$\mu_i' = \mu_i + \nu_i \qquad \text{for } 1 \leq i \leq s,$$

where μ_i' represents the effective departure rate from the i^{th} server and ν_i represents the reneging rate prior to his service completion.

We can always choose $\nu_i = 0$ if customers depart from the camp-on system only after all the requested task has been done (i.e., no reneging is permitted during the service period). This approach is based on the fact that for memoryless departures, to someone outside the queueing system, reneging and served customers look alike.

1.3. Service Strategies

The service strategy for first-level customers plays a decisive role in the evolution of the second-level systems, as it indirectly controls the relative sizes of the second-level queues. We will consider five distinct types of service strategies for the first-level service facility:

Type 1: The service discipline is first-come-first-served (FCFS); all customers are served in the exact order of arrival. Multiple servers are allowed at such a service center.

Type 2: There is a single server at the service center and the service discipline is last-come-first-served non-preemptive (LCFS-NP); the last customer to join is the next one in turn for service at such a center.

Type 3: There is a single server and the service discipline is last-come-first-served preemptive-resume (LCFS-PR); the last customer to join the system is immediately served while the in-service customer is queued as the next in turn. This is a "push down" stack.

Type 4: There are infinitely many servers available at such a center (IS); simultaneous service for second-level customers is allowed if their service center is of the same type.

Type 5: The service discipline is broadcast delivery (BD); first-level customers and associated second-level customers are served simultaneously, in a broadcast fashion. There is a single server at such a center.

All first-level customers are assumed to have the same service distribution. Type 4 and 5 centers are special examples of service strategies in the camp-on model, where second-level customers can receive service while their associate first-level customer is still in the queue.

The service-completion processes for the second-level customers can be completely arbitrary, as far as this camp-on system model is concerned. To see this, observe that once the first-level customer is served, his associated second-level queue leaves the camp-on model and becomes a single-level system. The camp-on model merely implies that the initial state for the new single-level system formed after a first-level customer departure will not necessarily be the empty state. However, for an ergodic system, the equilibrium probability distribution of the state of the model is independent of the initial state. The camp-on model does not change the equilibrium probability distribution that any particular second-level system would have shown if it had been considered as an independent and isolated queueing system on its own. Thus, well-established concepts from the classical analysis of queueing systems can be used to derive the pertinent information regarding this single stage. Nonetheless, one would have to account for the extra delays incurred while customers were waiting at the second-level stage. This issue will be further discussed, once we introduce the subject of stability in two-level camp-on systems (Chapter III) and analyze the performance of more concrete systems (Chapter V).

1.4. Other Considerations

Finally, once a given customer has joined a particular queue, he will be considered busy for all aspects related to customer handling inside this camp-on model. Because of this restriction, no customer will be allowed to be present in more than one queue at the same time as, for example, waiting for service both as a first-level and as a second-level customer. Also, no service will be provided by any first-level customer to any second-level customers until the first-level customer leaves the system. (This last restriction applies only to type 1-3 centers.)

We could have considered a more complex scenario for our camp-on system model, but it will be better to wait until we can fully appreciate the benefits and drawbacks of this simpler model. Further considerations on this topic will be discussed in Chapter VI.

II.2. State Occupancy

The camp-on model guarantees that the system stays in a particular state without notion of the time elapsed. Thus, once the queueing system reaches equilibrium, the states of the model are represented as vectors $(x_1, \cdots, x_n)$, where x_i denotes the condition prevailing at the second-level system in position i with respect to service center, when the first-level size is n. These states can be seen as the states of a continuous mutidimensional Markov chain for the size of the system queues. In this section, we discuss the state occupancy representation for the camp-on model.

2.1. States of the Model

Let n, a non-negative integer, denote the number of customers currently waiting for service at the first-level queue, including the ones in service at the service center. Let $\mathbf{R}_n = (r_1, \cdots, r_n)$ represent a particular ordering of n classes of customers, chosen out of possible R's, such that r_i denotes the class of the customers associated with at the i^{th} second-level system. Also, let $\mathbf{K}_n = (k_1, \cdots, k_n)$ represent the sizes of the second-level systems for that fixed ordering of customer classes $\mathbf{R}_n$ among the second-level systems, with k_i denoting the size of the system associated with the i^{th} first-level customer, i.e, the length of the i^{th} second-level queue. From the Markovian interpretation of the customer arrival/departure processes, the states of the camp-on model can then be completely specified by the $2n$-tuple $(\mathbf{K}_n; \mathbf{R}_n)$ $(0 \leq n \leq N)$. This $2n$-tuple contains information about the sizes of the first-level and the second-level systems as well as the particular class ordering among these queues.

More precisely, the state $\mathbf{x}_n$ of the camp-on system is a $2n$-dimensional vector with non-negative components $k_1, \ldots, k_n$ and $r_1, \cdots, r_n$, or in vectorial form, $\mathbf{x}_n = (\mathbf{K}_n; \mathbf{R}_n)$, where the pair (k_i, r_i) designates the size and customer class of the second-level system in position i with respect to a size-n first-level system. These states represent a finite-dimensional Markov chain for the queue sizes in the camp-on system model for every permutation $\mathbf{R}_n$ of n classes of customers.

Note that from the interpretation of the system states in the case of a single service center, the component k_1 stands for the length of the second-level queue associated with the first-level customer who is currently receiving service from the functioning service facility. When the system is empty, then $n = 0$; the null vector $\mathbf{K}_0$ represents this empty element in the discrete space of system states.

For instance, for the two-level camp-on system shown in Figure 3, we have are five customers in the first-level system, one in service and four queued. The distinct second-level queue sizes $k_1, \ldots, k_5$ consist of 3,2,0,1,2 customers with classes of types $3,5,1,6,2$, respectively. Therefore, the corresponding state vector representation for the camp-on system is $\mathbf{x}_n = (\mathbf{K}_5; \mathbf{R}_5)$, where $\mathbf{K}_5 = (3,2,0,1,2)$ and $\mathbf{R}_5 = (3,5,1,6,2)$.

The state vectors in the camp-on model do not have a fixed dimension, unlike the models for networks of queues mentioned in Chapter I. The dimension of each state is determined by the total number of customers in the first-level system. In the event of an infinite storage capacity at the first-level center, we will then have an infinite-dimensional Markov chain.

2.2. Neighboring States

Two states, $\mathbf{x}_n$ and $\mathbf{y}_m$, are considered *neighbors* if there is a one-step transition connecting state $\mathbf{y}_m$ to state $\mathbf{x}_n$. Clearly, the set of neighboring states is strongly dependent on the service strategy implemented at the service center. The ensuing derivation is compatible for types 1, 4 and 5 service centers. The system behavior under type-2 and type-3 centers will then be derived after we study the equilibrium behavior of the camp-on model for those service centers.

Let $\mathbf{x}_n = (\mathbf{K}_n; \mathbf{R}_n)$ represent the present state vector of the queueing system and let $\mathbf{y}_m = (\mathbf{K}'_m; \mathbf{R}'_m)$ denote any of the various state vectors achievable within the camp-on system. Under the above-stated conditions for the camp-on process, only one of the following one-step transitions could take the camp-on system from the state vector $\mathbf{y}_m$ into the state vector $\mathbf{x}_n$:

i) an arrival to the first-level system:

$$\mathbf{y}_m = \mathbf{x}_{n-1} = (\mathbf{K}_{n-1}; \mathbf{R}_{n-1}), \tag{2.3}$$

for the arrival will increase the first-level queue size by one. Such a transition would be possible only if $k_n = 0$ for state $\mathbf{x}_n$ and the associated customer class is r, since a second-level queue at the n^{th} first-level customer does not yet exist.

ii) an arrival to an i^{th} second-level system:

$$\mathbf{y}_m = \mathbf{x}_n^{-i} = (\mathbf{K}_n^{-i}; \mathbf{R}_n), \tag{2.4}$$

with

$$\mathbf{K}_n^{-i} = (k_1, \ldots, k_i - 1, \ldots, k_n);$$

that is, $\mathbf{y}_m$ is a state vector with one fewer customer at the i^{th} second-level queue than when in the state vector $\mathbf{x}_n$. Here, of course, $k_i \geq 1$.

iii) a departure from the service center due to service completion:

$$\mathbf{y}_m = \mathbf{x}_{n+1,1} = (\mathbf{K}_{n+1,1}; \mathbf{R}_{n+1,1}), \tag{2.5}$$

with

$$\mathbf{K}_{n+1,1} = (k_0, k_1, \ldots, k_n).$$

Here $\mathbf{y}_m$ is a state vector with $n+1$ first-level customers, and the customer whose service period was in progress had a second-level queue of length k_0* $(0 \leq k_0 < \infty)$ and class r_0 $(r_0 \neq r_i)$.

iv) a defection from the i^{th} at the first-level queue:

$$\mathbf{y}_m = \mathbf{x}_{n+1,i} = (\mathbf{K}_{n+1,i}; \mathbf{R}_{n+1,i}). \tag{2.6}$$

Here,

$$\mathbf{K}_{n+1,i} = (k_1, \ldots, k_{i-1}, k_0, k_i, \ldots, k_n),$$
$$\mathbf{R}_{n+1,i} = (k_1, \ldots, k_{i-1}, k_0, k_i, \ldots, k_n),$$

where $\mathbf{y}_m$ is a state vector with $n+1$ first-level customers, while the i^{th} first-level customer has an associated second-level queue of length k_0 and class r_0 distinct from the other r_i's.

v) a defection from the i^{th} second-level queue:

$$\mathbf{y}_m = \mathbf{x}_n^{+i} = (\mathbf{K}_n^{+i}; \mathbf{R}_n), \tag{2.7}$$

with

$$\mathbf{K}_n^{+i} = (k_1, \cdots, k_i + 1, \ldots, k_n).$$

Thus, $\mathbf{y}_m$ is a state vector with one customer more than the vector state $\mathbf{x}_n$ has at the i^{th} second-level queue.

Any other transition from a state $\mathbf{y}_m$ to a state $\mathbf{x}_n$ can be expressed as a series of one-step transitions linking the end states $\mathbf{y}_m$ and $\mathbf{x}_n$ through a chain of intermediate neighboring states, where the transitions are in the forms described from i) to v).

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* In this context, k_0 is used as a dummy variable indicating the size of the previous second-level system before its departure from the camp-on system.

2.3. State Transition Rates

The differential or dynamical interpretation of the customer arrival process is that if the camp-on model is in state $\mathbf{x}_n$ at time t, then the probability that a customer from the i^{th} stream source, S_i $(0 \le i \le n)$, will arrive between time t and $t+h$ is $h\lambda_{lr} + o(h)$, where l indicates the queue level and r the customer class. For the service completion process, the probability that the service of a customer's being served at time t will be completed by time $t+h$ is $h\mu + o(h)$. With respect to the customer reneging processes, the probability that the i^{th} first-level customer at time t will have defected from the first-level queue by time $t+h$ is $h\nu_i + o(h)$. Similarly, the probability that a customer at the i^{th} second-level queue at time t will have defected from his queue by time $t+h$ is $h\eta + o(h)$ as well. The probability of two or more events in the same time interval of small length h is also $o(h)$.

The state transition rate for this camp-on model $q(\mathbf{x}_n;\mathbf{y}_m)$, that is, the equilibrium rate of flow from the state $\mathbf{x}_n = (\mathbf{K}_n;\mathbf{R}_n)$ into the state $\mathbf{y}_m = (\mathbf{K}'_m;\mathbf{R}'_m)$, can be readily derived from the differential interpretation of the various processes involved as t goes to infinity. These infinitesimal generators for this Markov process are as follows*:

i) An arrival to the first-level system:

$$q(\mathbf{x}_n;\mathbf{y}_m) \;=\; \gamma_n\ \chi_{(k_n=0)}, \qquad \text{if } \mathbf{y}_m \;=\; \mathbf{x}_{n-1}. \tag{2.8}$$

ii) An arrival to the i^{th} second-level system:

$$q(\mathbf{x}_n;\mathbf{y}_m) \;=\; \lambda_{2r}\ \chi_{(k_i>0)}, \qquad \text{if } \mathbf{y}_m \;=\; \mathbf{x}_n^{-i}. \tag{2.9}$$

* χ_E stands for the characteristic or indicator function of the event or condition E. χ_E equals 1 if the condition holds and 0 otherwise.

iii) A departure from the service center (service completion):

$$q(\mathbf{x}_n;\mathbf{y}_m) \;=\; \mu\, \chi_{(n<N)}, \qquad \text{if } \mathbf{y}_m \;=\; \mathbf{x}_{n+1,1}. \tag{2.10}$$

iv) Reneging from the i^{th} position in the first-level queue:

$$q(\mathbf{x}_n;\mathbf{y}_m) \;=\; \nu_i\, \chi_{(n<N)}, \qquad \text{if } \mathbf{y}_m = \mathbf{x}_{n+1,i}. \tag{2.11}$$

v) Reneging from the i^{th} second-level queue:

$$q(\mathbf{x}_n;\mathbf{y}_m) \;=\; (k_i+1)\eta\, \chi_{(n>0)}, \qquad \text{if } \mathbf{y}_m \;=\; \mathbf{x}_n^{+i}. \tag{2.12}$$

Based on this information, we will be able to write down the equilibrium equations that govern the probabilistic behavior for the camp-on queueing model.

II.3. Mathematical Formulation

If this queueing system is to have a non-trivial behavior in steady state, the rate of flow into state $\mathbf{x}_n$ must be compensated by the rate of flow out of state $\mathbf{x}_n$; otherwise, there is an absorbing state and thus a single-point distribution. From this steady-state condition, we proceed to derive the equilibrium equations for the camp-on model as well as the generalized n-dimensional Z-transform for the joint probability distribution of queue lengths. However, these global balance equations prove to be insufficient to fully unravel the state occupancy distribution for the camp-on model. Here we propose a complementary set of partial balance equations that will help elucidate the distribution of customers among the queues.

3.1. Equilibrium Equations

It has been rightfully argued[21,23,41] that for an equilibrium probability distribution to exist, a probability flow conservation requirement must be imposed from every state into the others. Because of the disjoint decomposition of the state probabilities into mutually exclusive and exhaustive transitions through intermediate states, such flow conservation condition in queueing systems is usually expressed as

$$\sum_{\mathbf{y}_m \in \Omega} p(\mathbf{x}_n) q(\mathbf{x}_n;\mathbf{y}_m) = \sum_{\mathbf{y}_m \in \Omega} p(\mathbf{y}_m) q(\mathbf{y}_m;\mathbf{x}_n), \tag{2.13}$$

where

$$\Omega = \text{Set of all vector states } \mathbf{x}_n.$$

Let $p(\mathbf{x}_n) = p(\mathbf{K}_n;\mathbf{R}_n)$ denote the equilibrium joint probability distribution of queue lengths in the camp-on system. From the flow conservation requirement under stationary conditions in Equation (2.13) and the Markovian interpretation of transition rates among neighboring states, it follows that the equilibrium equation for the joint probability distribution of queue lengths is of the form

$$\begin{aligned}
\Big[\sum_{i=1}^{n} \big[\lambda_{2r_i} + \nu_i + k_i\eta\big]\, \chi_{(n>0)} + \lambda_{1n}\chi_{(n<N)} + \mu\chi_{(n>0)} \Big]\, p(\mathbf{x}_n) = {} & \\
& \gamma_n\, \chi_{(k_n=0)}\, p(\mathbf{x}_{n-1}) \\
& + \sum_{i=1}^{n} \lambda_{2r_i}\, \chi_{(k_i>0)}\, p(\mathbf{x}_n^{-i}) \\
& + \sum_{i=1}^{n} (k_i+1)\eta\chi_{(n>0)}\, p(\mathbf{x}_n^{+i}) \qquad (2.14) \\
& + \sum_{r_0=1}^{R} \chi_{(r_0\neq r_i)} \sum_{k_0=0}^{\infty} \mu\, \chi_{(n<N)}\, p(\mathbf{x}_{n+1,1}) \\
& + \sum_{r_0=1}^{R} \chi_{(r_0\neq r_i)} \sum_{i=1}^{n+1} \sum_{k_0=0}^{\infty} \nu_i\, \chi_{(n<N)}\, p(\mathbf{x}_{n+1,i}), \\
& (0 \le n \le N),
\end{aligned}$$

where

$$\gamma_{n+1} = \begin{cases} \dfrac{\lambda_{1n}}{R-n} & \text{for class-1 systems;} \\ \lambda_{2r_{n+1}} & \text{for class-2 systems.} \end{cases} \tag{2.15}$$

In order to ensure that $p(\mathbf{x}_n)$ is a properly defined probability distribution, we must add the normalizing probability relation

$$\sum_{\mathbf{x}_n \in \Omega} p(\mathbf{x}_n) = 1. \tag{2.16}$$

For class-1 systems, Equation (2.15) shows that a transition from state $\mathbf{x}_n$ to state $\mathbf{x}_{n+1}$ occurs with probability $1/(R-n)$, since there are $R-n$ choices for the customer class in the second-level system, and these are independent from the first-level processes. For class-2 systems, Equation (2.15) shows that the same transition is possible only if an arrival from the specific customer class has occurred as indicated by the class assignment $\mathbf{R}_n$.

For such a camp-on model with stationary transitions, which corresponds to an ergodic Markov chain, it can be shown[5,10,11,22] that a positive solution exists for all state vector probabilities $p(\mathbf{x}_n)$. The converse of this statement is equally true. We are going to establish later under what conditions this queueing system represents an ergodic Markov chain; basically, we must find the point when a stable solution exists for this set of equilibrium equations.

The set of difference equations in (2.14) could also be thought of as coming from a state-transition-rate diagram describing the flow rates into and out of any particular state vector $\mathbf{x}_n$. Under equilibrium conditions, it is intuitively clear that the total flow between neighboring states in the diagram must be conserved in such a form that the input flow to any given state vector must equal exactly the output flow; otherwise, the probability of finding the camp-on system in such a state will tend to zero or one as t goes to infinity. A pictorial representation of the total probability flow into a typical state $\mathbf{x}_n$, the left side of Equation (2.14), is shown in Figure 4.

3.2. Global and Partial Balance Equations

Equation (2.14) represents what is usually called the *global balance equation* for the queueing system. In some cases, it is possible to define special equilibrium relations, or *partial balance relations*, among groups of neighboring states. One instance of these special equilibrium relations is the *local balance equations*, which state the equilibrium conservation of flow among every pair of neighboring states:

$$p(\mathbf{x}_n)q(\mathbf{x}_n;\mathbf{y}_m) \;=\; p(\mathbf{y}_m)q(\mathbf{y}_m;\mathbf{x}_n).$$

Summed up together, these partial balance equations must yield the global balance equation for the queueing system. In general, we cannot foresee whether an arbitrary decomposition of the global balance equation into partial balance equations will provide a consistent description of the system behavior unless they happen to satisfy Kolgomorov's criterion[21,31] for reversibility and local balance.

Unfortunately, one cannot rely on the existence of local balance equations for this camp-on model. This is suggested by the asymmetry displayed in the state-transition-rate diagram for the system. For example, one may be able to go in a one-step transition from state $\mathbf{x}_n$ into state $\mathbf{y}_m$ but not from state $\mathbf{y}_m$ into state $\mathbf{x}_n$. This is a consequence of the effective bulk departure processes associated with this model. The total number of customers departing the camp-on model is also a random variable because of the random size of the second-level queues. Thus,

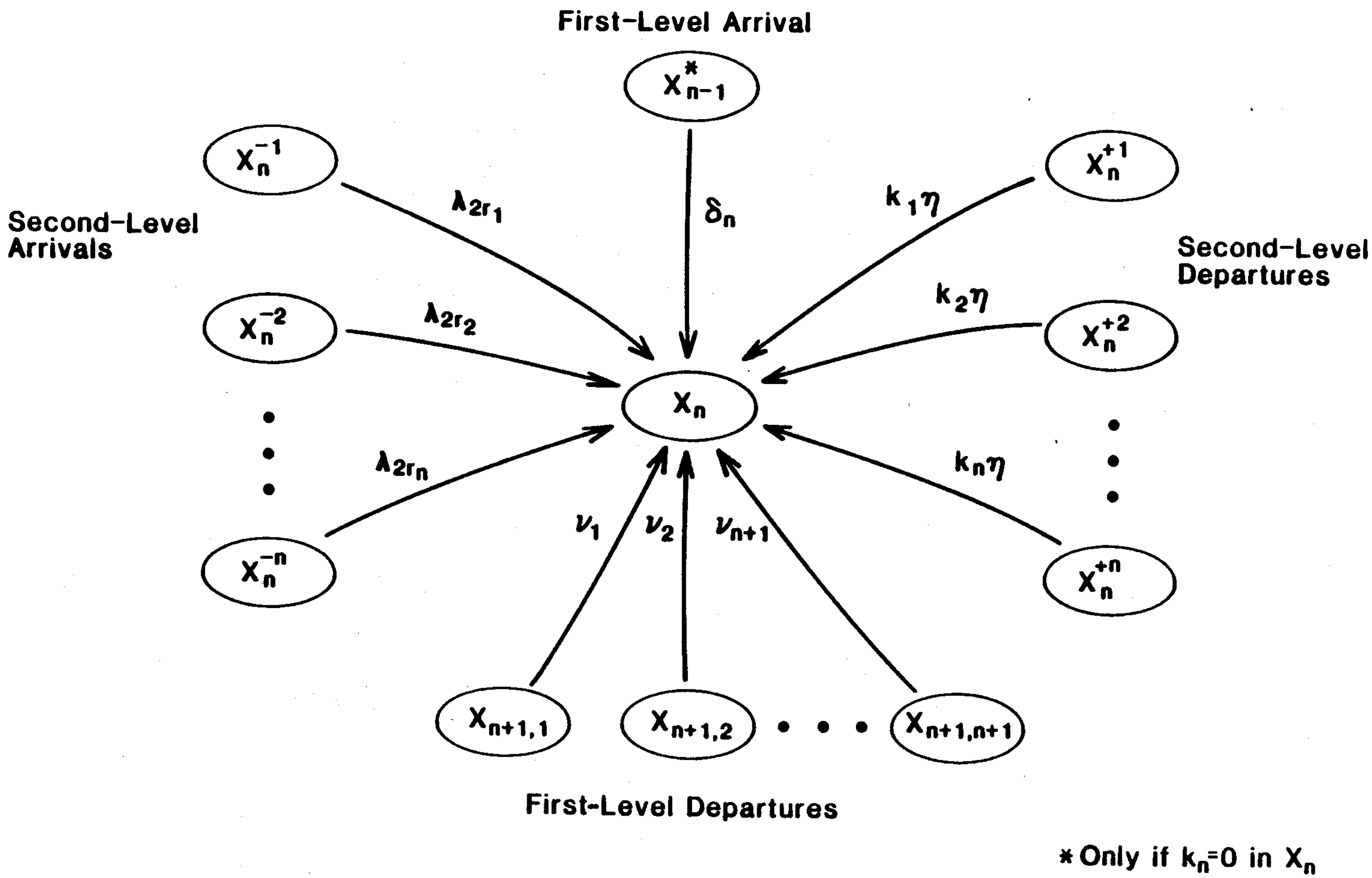


Figure 4: Flow into state x_n of the camp-on model.

Equation (2.14), even though it synthesizes the occupancy behavior of the camp-on model, still remains hopelessly inextricable.

However, because of the homogeneity of the processes for the customers at the second-level systems and the homogeneity of the departure processes from the first-level system, one should expect to find some other forms of independent balance equations in equilibrium by invoking complementary conditions for conservation of flow among some of the neighboring states. In fact, think of any arbitrary sequence of customer arrivals/departures from the second-level systems for any fixed first-level queue size. Because of the decoupling between first-level and second-level processes, the service center cannot tell whether the entire sequence of events takes place when the first-level queue size is n or when it is $n+1$. Hence, the service center has no knowledge of how potential new arrivers or renegers behave inside the second-level queues, regardless of the size of its own queue.

The above observation suggests that a partial balance relation must exist between the flow out of the first queueing stage and the flow into the first queueing stage. These general ideas can be summarized in a set of independent balance equations for the homogeneous camp-on model:

$$\begin{aligned}\Big[\sum_{i=1}^{n}\big[\lambda_{2r_i}+\nu_i+k_i\eta\big]\,\chi_{(n>0)}+\mu\chi_{(n>0)}\Big]\,p(\mathbf{x}_n) \;=\; & \\ & \gamma_n\,\chi_{(k_n=0)}\,p(\mathbf{x}_{n-1}) \qquad\qquad (2.17)\\ & +\sum_{i=1}^{n}\lambda_{2r_i}\,\chi_{(k_i>0)}\,p(\mathbf{x}_n^{-i})\\ & +\sum_{i=1}^{n}(k_i+1)\eta\,\chi_{(n>0)}\,p(\mathbf{x}_n^{+i}),\\ (0\le n\le N). &\end{aligned}$$

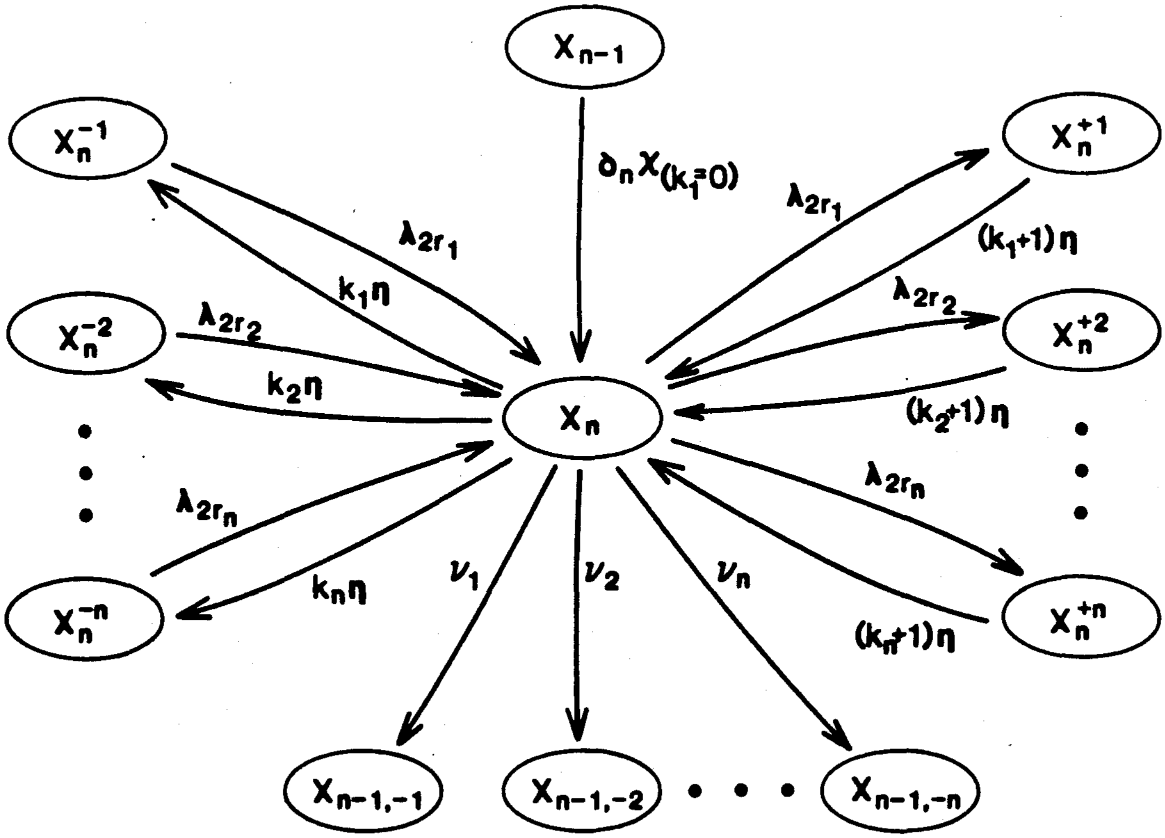
and

$$\lambda_{1n}\, p(\mathbf{x}_n) = \mu \sum_{r_0=1}^{R} \chi_{(r_0 \neq r_i)} \sum_{k_0=0}^{\infty} p(\mathbf{x}_{n+1,1}) + \sum_{r_0=1}^{R} \chi_{(r_0 \neq r_i)} \sum_{i=1}^{n+1} \sum_{k_0=0}^{\infty} \nu_i\, p(\mathbf{x}_{n+1,i}), \quad (0 \leq n < N). \tag{2.18}$$

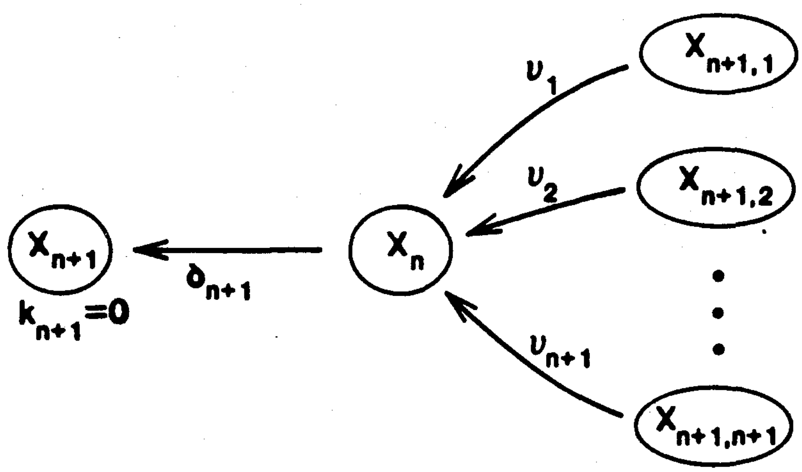
The interpretation of this independent balance equation is that under stationary conditions, the conservation of flow among neighboring states works in such a way that the rate of growth for the first-level queue must be compensated just by its rate of decrease, for a fixed distribution of the second-level systems.

Equations (2.17) and (2.18) are equivalent to balance equations found in the classical theory of queueing systems. There, as here, the customer arrival processes and the service completion processes are memoryless. Basically, the total flow between each pair of state vectors involving adjacent first-level customers must be preserved. Figures 5.a and 5.b show state-transition-rate diagrams depicting the independent balance equations in the camp-on model.

As a consequence of these homogeneous processes, a very simple coupling arises between the probabilistic behavior of the camp-on model when the first-level queue size is n and its behavior when the queue size is $n-1$. Given a particular state vector $\mathbf{x}_n$, the camp-on system has no memory as to whether it visited other state vectors $\mathbf{y}_m$ with $m > n$ before arriving at its current state. From a purely combinatorial viewpoint, given a system state vector $\mathbf{x}_n$, it suffices for this homogeneous case to count all the possible sequences of events by which such a particular set of second-level queue lengths k_i $(1 \leq i \leq n)$, can be achieved for a fixed first-level queue size.



a.



b.

Figure 5: State-transition-rate diagrams for the independent balance equations in the two-level camp-on model.

3.3. Generating Function for the Size of the Second-Level Systems

The computation of the equilibrium or steady-state response for the homogeneous camp-on model directly from Equations (2.17) and (2.18) can result in a cumbersome task even if we want to estimate only the first few terms of the joint probability distribution of queue lengths. Instead of solving that set of difference equations, it will be more convenient to resort to transform methods. Let $P(\mathbf{Z}_n;\mathbf{R}_n)$ denote the n-dimensional Z-transform or probabilistic generating function of the equilibrium state probability distribution of the second-level queue lengths:

$$\begin{aligned} P(\mathbf{Z}_n;\mathbf{R}_n) &= P(z_1,\ldots,z_n;\mathbf{R}_n), \\ &= \sum_{k_1,\ldots,kn=0}^{\infty} p(\mathbf{K}_n;\mathbf{R}_n) \prod_{i=1}^{n} z_i^{k_i}. \end{aligned} \tag{2.19}$$

This transform is certainly well defined within the n-dimensional hypersphere $0 \le |z_i| \le 1 \quad (1 \le i \le n)$. This is because each state probability $p(\mathbf{K}_n;\mathbf{R}_n)$ is strictly bounded by 1. Therefore, applying this transformation to Equations (2.17) and (2.18), i.e., multiplying these equations by $\prod_{i=1}^{n} z_i^{k_i}$ and summing over each of the indices k_i from zero to infinity, one can obtain equivalent conditions for this set of equations to represent an ergodic Markov chain (see Appendix I). The result is

$$\begin{aligned} [\mu + \sum_{i=1}^{n} \nu_i\,]\; P(\mathbf{Z}_n;\mathbf{R}_n) &= \gamma_n\; P(\mathbf{Z}_{n-1};\mathbf{R}_{n-1}) + \\ & \sum_{i=1}^{n}(z_i - 1)\left[\eta\frac{\partial}{\partial z_i}P(\mathbf{Z}_n;\mathbf{R}_n) \;-\; \lambda_{2r_i}P(\mathbf{Z}_n;\mathbf{R}_n)\right], \end{aligned} \tag{2.20a}$$

and

$$\lambda_{1n}\; P(\mathbf{Z}_n;\mathbf{R}_n) = \sum_{r_0=1}^{R} \chi_{(r_0 \ne r_i)} \left[\mu P(\mathbf{Z}_{n+1,1};\mathbf{R}_{n+1,1}) + \sum_{i=1}^{n+1} \nu_i\; P(\mathbf{Z}_{n+1,i};\mathbf{R}_{n+1,i})\right], \tag{2.20b}$$

for all $0 \le |z_i| \le 1$ and $1 \le n \le N$. Here, $P(\mathbf{Z}_{n+1,i};\mathbf{R}_{n+1,i})$ stands for

$$P(\mathbf{Z}_{n+1,i};\mathbf{R}_{n+1,i}) \;=\; P(z_1,\ldots,z_{i-1},1,z_i,\ldots,z_n\,;\,r_1,\ldots,r_{i-1},r_0,r_i,\ldots,r_n),$$

and $r_0 \ne r_i$ $(1 \le i \le n)$.

Observe that Equations (2.20) provide us with a set of linear, first-order inhomogeneous partial differential equations on the n independent variables z_i and the n dependent variables $P(\mathbf{Z}_n;\mathbf{R}_n)$ $(1 \le i \le n)$. This set of partial differential equations, combined with the normalizing probability relation in Equation (2.16),describes completely the equilibrium probabilistic behavior of the camp-on system from the state occupancy viewpoint, because of the uniqueness of the generalized n-dimensional Z-transform. Once we solve the transformed equilibrium equations (2.20a) and (2.20b), the problem of the equilibrium joint probability distribution of the queue lengths for this camp-on model is therefore essentially solved, too. If we wish to compute the probability of any particular state vector $\mathbf{x}_n$, one need only resort to the general inverse transform relation:

$$p(\mathbf{K}_n;\mathbf{R}_n) \;=\; \prod_{i=1}^{n} \frac{1}{k_i!}\,\frac{\partial^{k_i}}{\partial z_i^{k_i}} P(\mathbf{Z}_n;\mathbf{R}_n)\Big|_{\mathbf{Z}_n=\underline{0}}. \qquad (2.21)$$

The effect of the existence of a set of independent balance equations rather than only a set of global balance equations for the camp-on system is fully reflected in the transformed version of the independent balance equation. Instead of solving a set of first-order partial differential equations on n independent variables z_i and n dependent variables $P(\mathbf{Z}_i;\mathbf{R}_i)$ $(1 \le i \le n)$, one need only solve a set of first-order partial differential equations one at a time with only *one* dependent variable $P(\mathbf{Z}_n;\mathbf{R}_n)$.

CHAPTER III:

EQUILIBRIUM DISTRIBUTIONS IN TWO-LEVEL CAMP-ON SYSTEMS

In Chapter II we developed the basic mathematical framework necessary to analyze various state-occupancy problems of interest in two-level camp-on systems. In this chapter we exploit this mathematical representation to derive important closed-form expressions that describe the probabilistic behavior queues and the overall workload for this two-stage camp-on model. First, we present the general solution for the transformed equilibrium joint probability distribution of queue lengths. From these, other joint probability distributions for specific queueing environments as well as some marginal distributions for first-level and second-level systems are derived. The stability of the two-stage model is studied and is found to be closely controlled by the stability of the system at the first-level stage. Even though an infinite-state Markov chain is assumed for this analysis, many of these results are also extended to the finite-state camp-on model.

III.1. Joint Distributions for Queue Lengths

Section 1 deals with the problem of the equilibrium joint probability distribution of queue lengths in the camp-on queueing model. Two theorems are presented in this section. One of them relates to the transformed equilibrium state distribution $P(\mathbf{Z}_n;\mathbf{R}_n)$, while the other one relates to the actual steady-state probability distribution $p(\mathbf{x}_n)$ for the most important single case in Theorem 1: the non-reneging two-level camp-on system. Five different service disciplines are considered: first-come-first-served (FCFS), last-come-first-served non-preemptive (LCFS-NP), last-come-first-served preemptive-resume (LCFS-PR), infinite servers (IS) and broadcast delivery (BD). The ergodicity of the camp-on model is found to be tied to the stability of the first-level system. We view the queue levels as cyclical processes with respect to customer position in the service hierarchy. Here, first-level customers and associated subsystems migrate down the service path in the queueing hierarchy as in-service customers complete their jobs and second-level systems become first-level systems. Thus, the stability of the first-level system also implies the stability of each subsystem as an isolated queueing system.

1.1. Transformed Joint Probability Distribution for the Second-Level Queue Lengths

The mathematical model that we have considered allows for either infinite ($\lambda_{1n} \neq 0,\ \forall\, n \geq 0$) or finite ($\lambda_{1n} = 0,\ \forall\, n >$ some finite N) queue sizes at the first-level queue. Denote by $P_N(\mathbf{Z}_n;\mathbf{R}_n)$ the family of state probability distributions in a camp-on system with at most N customers at the first-level queue. Suppose we have another similar system but with a finite-storage capacity M and corresponding transformed state probability distribution $P_M(\mathbf{Z}_n;\mathbf{R}_n)$. From the interpretation of the independent balance equation in (2.20), if such decomposition is plausible, then both $P_N(\mathbf{Z}_n;\mathbf{R}_n)$ and $P_M(\mathbf{Z}_n;\mathbf{R}_n)$ must satisfy the *same* set of partial differential equations for all first-level queue sizes n in the range $1 \leq n \leq min(M,N)$. By uniqueness of the solution of partial differential equations, $P_N(\mathbf{Z}_n;\mathbf{R}_n)$ and $P_M(\mathbf{Z}_n;\mathbf{R}_n)$ must be identical within this queue length range except for a multiplicative factor. In this instance, the factor can be found to be a quotient of the corresponding p_0's, the probability of an empty queueing system, a function of the available storage capacities N and M at each of the respective first-level stages. This proportionality of the transformed state distributions $P_N(\mathbf{Z}_n;\mathbf{R}_n)$ and $P_M(\mathbf{Z}_n;\mathbf{R}_n)$ implies that of the state probability distributions $p_N(\mathbf{x}_n)$ and $p_M(\mathbf{x}_n)$, because of the one-to-one relationship between the transform pairs. Thus, as one might expect, the camp-on model has no recollection of whether the queued customers visited any of the system states tied to first-level queues larger than n customers before reaching its current state. All the current state knows is that the first-level queue capacity must be at least n. This motivates the following result.

Theorem 1 : For a two-level camp-on system with a type-i service center ($i = 1,4,5$), unconstrained queue sizes at the second queueing stage and finite-storage capacity N at the first queueing stage, the general solution for the transformed equilibrium joint probability distribution of queue lengths is of the form

$$P(\mathbf{Z}_n;\mathbf{R}_n) = p_0 \prod_{i=1}^{n} \Psi_i(\mathbf{Z}_i;\mathbf{R}_i), \qquad 0 \leq n \leq N, \tag{3.1}$$

where

$$\Psi_i(\mathbf{Z}_i;\mathbf{R}_i) = \frac{\gamma_i}{\eta} \sum_{l_i=0}^{\infty} \frac{Y_i^{l_i}}{(\alpha_i)_{l_i+1}}. \tag{3.2}$$

Here, the parameters Y_n and α_n are given by

$$Y_n = \sum_{i=1}^{n} \frac{\lambda_{2r_i}}{\eta}(z_i - 1), \tag{3.3}$$

$$\alpha_n = \frac{\mu_n}{\eta} + \sum_{i=1}^{n-1} l_i, \tag{3.4}$$

$$\mu_n = \mu + \sum_{i=1}^{n} \nu_i, \tag{3.5}$$

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \tag{3.6}$$

If we use the notation $P_i(\mathbf{Z}_n;\mathbf{R}_n)$ to indicate the transformed state probability distribution for a type-i service center, then

$$P_i(\mathbf{Z}_n;\mathbf{R}_n) = P(\mathbf{Z}_n;\mathbf{R}_n), \qquad i = 1,4,5.$$

To arrive at Equation (3.1), one could recursively compute $P(\mathbf{Z}_n;\mathbf{R}_n)$ from $P(\mathbf{Z}_{n-1};\mathbf{R}_{n-1})$ by solving the set of partial differential equations in (2.20a) through the use of analytical considerations for the solution of the transformed state probability distribution inside the hypersphere $|z_i| \leq 1$, and the use of known series expansions for gamma function[4,14]. Special attention must be paid in this procedure to the verification of the consistency of the state-distribution solution with the independent balance Equation (2.20b). One could also verify the validity of Equation (3.1) directly through the algebraic reduction of $P(\mathbf{Z}_n;\mathbf{R}_n)$ by means of the set of independent balance Equations (2.20), as shown in Appendix II.

1.2. Two-Level Camp-on and Stability

It can be argued that if the camp-on model represents a stable queueing system, then every single subsystem in the camp-on model in isolation, that is, the first-level system and each of the second-level systems, must behave as an ergodic queueing system. In such a situation, every second-level system, after becoming an independent queueing system functioning as a queueing system on its own, represents an ergodic Markov chain, where the first state in the chain has a random number of customers in its queue. In general, this queue size need not be zero, so the initial state in the chain is not always the empty state. However, for ergodic Markov chains, the equilibrium probability distribution is independent of the initial state of the system. Consequently, after departure from the camp-on model, each second-level subsystem is guaranteed, in the long run, to behave as if the camp-on stage was never there.

So far, there has not been any restriction imposed on $P(\mathbf{Z}_n;\mathbf{R}_n)$ that affects the stability of this queueing model. But we still need to prove that $p(\mathbf{x}_n)$ is a properly defined probability distribution and satisfies the normalizing condition for the state probability distribution in Equation (2.16). This normalizing condition can be expressed in terms of the transformed state probability distribution $P(\mathbf{Z}_n)$ as

$$\sum_{\mathbf{x}_n \in \Omega} p(\mathbf{K}_n;\mathbf{R}_n) = \sum_{n=0}^{N} \sum_{\Omega_n} P(\mathbf{Z}_n;\mathbf{R}_n)\Big|_{\mathbf{Z}_n=\underline{1}} = 1, \tag{3.7}$$

where

$$\Omega_n = \text{set of all states } \mathbf{x}_n \text{ with dimension } n.$$

Evaluating $P(\mathbf{Z}_n)$ at the vector $\mathbf{Z}_n = \underline{1}$ with all components $z_i = 1$ $(1 \le i \le n)$, it follows directly from the transformed state probability in Equation (3.1) that

$$P(\mathbf{Z}_n;\mathbf{R}_n)\Big|_{\mathbf{Z}_n=\underline{1}} = p_0 \prod_{i=1}^{n} \frac{\gamma_i}{\mu + \sum_{k=1}^{i} \nu_k}. \tag{3.8}$$

We define as *infinite storage capacity* those instances of the camp-on system wherein the model provides for infinite-size buffers in all the subsystems to allocate and handle the incoming traffic demand for the different queues. We define as *semi-infinite storage capacity* those instances wherein only the second-level queues have infinite-size buffers, but not the first-level queue. For a camp-on system with homogeneous second-level transitions and infinite storage capacity at the first-level queue, one can assert from Equations (3.7) and (3.8) that a necessary and sufficient condition for ergodicity of the queueing system and the existence of the probability distributions $P(\mathbf{Z}_n; \mathbf{R}_n)$ and $p(\mathbf{x}_n)$ are

$$\sum_{n=0}^{\infty} \sum_{\Omega_n} \prod_{k=1}^{n} \frac{\gamma_k}{\mu_k} < \infty. \tag{3.9}$$

For class-1 systems (refer to Section 1, Chapter II), this last condition turns out to be equivalent to the simpler relation

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\lambda_{1k-1}}{\mu_k} < \infty. \tag{3.10}$$

This result is not surprising. Each second-level system, in isolation, can be thought as an $M/M/\infty$ queueing system with arrival rate λ_{2r_i} and departure rate η, and hence is stable for all finite state transition-rates. Even more, since second-level systems leave the camp-on model at the same instant as their associated first-level customer departs from its queue, we expect that second-level queues have a negligible chance of growing unbounded and becoming a bottleneck issue in the camp-on queueing model. Therefore, the stability of the overall camp-on system rests on the stability of the first-level system. Equation (3.10) is merely the equivalent condition for ergodicity of the embedded birth-death process at the first-level service center.

For all the queueing disciplines mentioned in Chapter II, the camp-on model can be viewed as a cyclical process with respect to the customer's position in the service hierarchy. This is particularly true because every second-level system eventually becomes a first-level system as its associated first-level customer leaves the camp-on stage. Thus, each second-level system must also satisfy Equation (3.10); otherwise, it will not be a stable queueing system when operating alone. In this case, μ_k must correspond to the departure rate of the isolated second-level system. As time evolves, customers are promoted from the second-level stage into the first-level stage, and the conditions imposed on a first-level system are progressively transferred to the second-level systems, depending on the particular queueing scheme implemented.

If we have a camp-on system with semi-infinite storage capacity (i.e., finite queue sizes only at the first-level queue), the condition for ergodicity in Equation (3.9) boils down to the requirement of finite state-transition rates, as must be for any finite-state Markov chain.

For an ergodic camp-on system, it also follows from Equations (3.7) and (3.8) that the empty-system probability p_0 is given by

$$p_0 = \left[\sum_{n=0}^{\infty} \sum_{\Omega_n} \prod_{k=1}^{n} \frac{\gamma_k}{\mu_k} \right]^{-1} . \tag{3.11}$$

There will be two distinct formulas for the empty-state probability, depending on whether one refers to a class-1 or a class-2 camp-on system. For each of these cases, it is easy to verify, considering all possible class assignments $\mathbf{R}_n$ for second-level systems in Equation (3.8), that

$$p_0 = \begin{cases} \left[1 + \sum_{n=1}^{N} \prod_{k=1}^{n} (\frac{\lambda_{1k-1}}{\mu_k})\right]^{-1}, & \text{for class-1 systems;} \\ \left[\sum_{n=0}^{N} n! \sum_{\Omega_n} \prod_{r=1}^{R} \left(\frac{\lambda_{2r}}{\mu_n}\right)^{l_r}\right]^{-1}, & \text{for class-2 systems.} \end{cases} \tag{3.12}$$

Here, one has chosen the index l_i from Ω_n and the second-level class assignment $\mathbf{R}_n$ such that

$$l_i = \begin{cases} 1, & \text{if } r_k = i \text{ for some } i; \\ 0, & \text{otherwise.} \end{cases} \tag{3.13}$$

Thus, $l_1 + \cdots + l_R = n$. This result is consistent with the decoupling assumption between the first-level and the second-level processes, for the first-level facility is transparent in this model to the second-level subsystems.

As an extreme case of the class-1 camp-on system, if the arrival rate to the service center is Poisson and the storage capacity is unlimited, we have from Equation (3.12) is

$$p_0 = \begin{cases} 1 - \rho_1, & \text{for type-1,2,3,5 centers;} \\ e^{-\rho_1}, & \text{for type-4 centers.} \end{cases}$$

1.3. Transformed Distributions for Non-Reneging Camp-on Systems

Here we want to focus on a camp-on system in which customers do not defect from their queues but rather they wait in line until a free server is available and their service requests are completed. This situation arises in many communication systems involving computers or computer processes and other enhanced services where "customers" (people, machine jobs, processes, peripherals, etc.) can be programmed to stay on hold for any required period of time (or until told to disconnect to avoid deadlock). Without loss of generality, we can assume that the common reneging rate η for second-level customers is zero and that the reneging rate ν_i for first-level customers refers only to the individual service rates of the i^{th} server in a multiserver environment.

Accordingly, let the common reneging rate from the second-level stage η go to zero. Thus, we see that as $\eta \to 0$, the camp-on parameters in Equations (3.3) to (3.6) tend to

$$(\alpha_i)_n \quad \longrightarrow \quad \left(\frac{\mu_i}{\eta}\right)^n,$$

$$\frac{(Y_i)^n}{(\alpha_i)_n} \quad \longrightarrow \quad \left(\frac{\tilde{Y}_i}{\mu_i}\right)^n,$$

where we have defined $\tilde{Y}_n$ as

$$\tilde{Y}_n = \sum_{i=1}^{n} \lambda_{2r_i}(z_i - 1). \tag{3.14}$$

Thus, the function $\Psi_i(\mathbf{Z}_i;\mathbf{R}_i)$ in Equation (3.1) becomes independent of $\Psi_j(\mathbf{Z}_j;\mathbf{R}_j)$ for $i \neq j$, since α_i is no longer related to the summation index l_j for $1 \le j < i$. Therefore, in the limit as η tends to zero, the transformed state-probability distribution $P(\mathbf{Z}_n;\mathbf{R}_n)$ has an independent product-form solution given recursively by

$$\begin{aligned} P(\mathbf{Z}_n;\mathbf{R}_n) = &\quad p_0 \prod_{i=1}^{n} \frac{\gamma_i}{\mu_i} \sum_{l_i=0}^{\infty} \left(\frac{\tilde{Y}_i}{\mu_i}\right)^{l_i} \\ = &\ p_0 \prod_{i=1}^{n} \frac{\gamma_i}{\mu_i - \tilde{Y}_i} \quad = \quad \frac{\gamma_n}{\mu_n - \tilde{Y}_n} P(\mathbf{Z}_{n-1};\mathbf{R}_{n-1}). \end{aligned} \tag{3.15}$$

Here μ_n is regarded as defined in Equation (3.3). The region of convergence for the transformed state probability distribution in this non-reneging camp-on system turns out to be the intersection of the regions indexed by n for which

$$\sum_{i=1}^{n} \lambda_{2r_i}(z_i - 1) \leq \mu_n, \quad \forall\, \mathbf{x}_n \in \Omega.$$

The form of Equation (3.15) calls for an explanation. The interpretation of the above equation for the transformed joint distribution of queue lengths is as follows. Let $P(\mathbf{Z}_n;\mathbf{R}_n) = G(\mathbf{Z}_n;\mathbf{R}_n)P(\mathbf{Z}_{n-1};\mathbf{R}_{n-1})$ be defined as in Equation (3.15). Then the state probability distribution $p(\mathbf{K}_n;\mathbf{R}_n)$ can be found as the $(n-1)$-fold convolution of the joint probability distribution for the first $n-1$ second-level queues $p(\mathbf{K}_{n-1};\mathbf{R}_{n-1})$ with $g(\mathbf{K}_n;\mathbf{R}_n)$, the probability of the arrival vector $\mathbf{K}_n$ to the second-level stage in between the arrivals of the n^{th} and $(n+1)^{th}$ customers to the first-level queue. Here, $g(\mathbf{K}_n;\mathbf{R}_n)$ is the inverse Z-transform of $G(\mathbf{Z}_n;\mathbf{R}_n)$. Even more, $g(\mathbf{K}_n;\mathbf{R}_n)$ can also be thought of as a function representing the probability of moving from the state vector $\mathbf{y}_{n-1} = (\mathbf{J}_{n-1};\mathbf{R}_{n-1})$ to the state vector $\mathbf{x}_n = (\mathbf{K}_n;\mathbf{R}_n)$ with $j_i \leq k_i$ $(1 \leq i < n)$. In this case, it is not too difficult to prove from Equation (3.15) that

$$g(\mathbf{K}_n;\mathbf{R}_n) \;=\; \frac{\gamma_n}{\mu_n + \sum_{j=1}^{n} \lambda_{2r_j}} \prod_{i=1}^{n} \binom{k_i + \cdots + k_n}{k_i} \left(\frac{\lambda_{2r_i}}{\mu_n + \sum_{i=1}^{n} \lambda_{2r_i}}\right)^{k_i}.$$

We mentioned in Section 1, Chapter II, about the possibility of implementing service disciplines different from FCFS (type 1) and its extensions for infinite servers, IS (type 4) and broadcast delivery BD (type 5). In order to do so, we start with a more general version of the global balance equation for the non-reneging camp-on model:

$$\sum_{i=1}^{n} \mu_i\, P(\mathbf{Z}_n;\mathbf{R}_n) \;=\; \gamma_{nj}\, P(\mathbf{Z}^{j}_{n-1};\mathbf{R}^{j}_{n-1}) + \sum_{i=1}^{n}(1 - z_i)\lambda_{2r_i} P(\mathbf{Z}_n;\mathbf{R}_n), \tag{3.16a}$$

$$\lambda_{1n}\ P(\mathbf{Z}_n;\mathbf{R}_n) \;=\; \sum_{r_0=1}^{R} \chi_{(r_0\neq r_i)} \sum_{i=1}^{n+1} \mu_i\ P(\mathbf{Z}_{n+1,i};\mathbf{R}_{n+1,i}), \tag{3.16b}$$

with

$$\gamma_{nj} \;=\; \begin{cases} \dfrac{\lambda_{1n}}{N-n+1}, & \text{if a class-1 system;} \\ \lambda_{2r_j}, & \text{if a class-2 system.} \end{cases} \tag{3.16c}$$

This is a natural extension of independent balance Equations (2.20a) and (2.20b). However, we now have customer arrivals to any arbitrary position in the first-level queue $P(\mathbf{Z}^j_{n-1};\mathbf{R}^j_{n-1})$, and customer departures from any of the multiple servers $P(\mathbf{Z}_{n+1,i}\,;\,\mathbf{R}_{n+1,i})$. This condition allows for more general service disciplines than plain FCFS or IS. However, we do not intend to solve Equation (3.16) but only to study two special cases: i) single-server last-come-first-served non-preemptive resume (type 2), and ii) single-server last-come-first-served preemptive-resume (type 3). We will need to redefine the neighboring state associated with the transition from the first-level to the second-level stage $q_j(\mathbf{x}_{n-1};\mathbf{x}_n)$ and the transformed probability $P(\mathbf{Z}^j_{n-1};\mathbf{R}^j_{n-1})$ for each type of service center. These results are presented in the following two Corollaries.

<u>Corollary 1.1:</u> For a non-reneging camp-on system with one single server and a LCFS-NP service discipline (type 2):

$$P_2(\mathbf{Z}_n;\mathbf{R}_n) \;=\; P(z_1,z_n,\cdots,z_2\,;\,r_1,r_n,\cdots,r_2).$$

Proof : For a type 2 center, the newly arrived first-level customer is placed next in turn for service in the 2^{nd} position in the LCFS-NP queue, since the in-service customer is not pre-empted until his job is done. Therefore, we have $j=2$, $\gamma_{nj}=\gamma_{n2}$, and the server-dependent parameters in Equation (3.16) are given by

$$P_2(\mathbf{Z}^2_{n-1};\mathbf{R}^2_{n-1}) \;=\; P_2(z_1,z_3,\cdots,z_n\,;\,r_1,r_3,\cdots,r_n),$$

$$\mu_i(\mathbf{R}_n) = \begin{cases} \mu, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, by recursion on Equation (3.16a),

$$P_2(\mathbf{Z}_n; \mathbf{R}_n) = \frac{\gamma_{n2}}{\mu + \sum_{i=1}^{n} \lambda_{2r_i}(1 - z_i)} P_2(z_1, z_3, \cdots, z_n; r_1, r_3, \cdots, r_n)$$

$$= \prod_{i=1}^{n} \frac{\gamma_n}{\mu + \lambda_{2r_1}(1 - z_1) + \sum_{k=n-i+2}^{n} \lambda_{2r_k}(1 - z_k)}.$$

Comparing this last result with Equation (3.15), we notice that the above distribution is of the same form as the distribution for $P(\mathbf{Z}_n; \mathbf{R}_n)$ in Equation (3.14), except that the locations of the second-level systems associated with the classes $r_2, \cdots, r_n$ have been reversed to $r_n, \cdots, r_2$. Thus, the conclusion in Corollary 1.1 follows immediately.

Corollary 1.2: For a non-reneging camp-on system with one single server and a LCFS-PR service discipline (type 3):

$$P_3(\mathbf{Z}_n; \mathbf{R}_n) = P(z_n, \cdots, z_1; r_n, \cdots, r_1).$$

Proof : For a type 3 center, the newly arrived first-level customer is served immediately, for he is the last customer to join the service center. The previously in-service customer is placed next in turn for service, that is, the 2^{nd} position in the LCFS-PR queue, since he is pre-empted and placed in the first-level waiting line as the new customer arrives. Therefore, we are back to a similar situation as in Corollary 1.1, except that $j = 1$, $\gamma_{nj} = \gamma_{n1}$ and

$$P_3(\mathbf{Z}_{n-1}^1; \mathbf{R}_{n-1}^1) = P_3(z_2, z_3, \cdots, z_n; r_2, r_3, \cdots, r_n).$$

By recursion on Equation (3.16a),

$$P_3(\mathbf{Z}_n;\mathbf{R}_n) \;=\; \prod_{i=1}^{n} \frac{\gamma_n}{\mu + \sum_{k=n-i+1}^{n} \lambda_{2r_k}(1-z_k)}.$$

Again, this distribution is similar to the distribution $P(\mathbf{Z}_n;\mathbf{R}_n)$, except that the location of the second-level subsystems associated with the classes $r_1,\cdots,r_n$ has been reversed to $r_n,\cdots,r_1$. The conclusion in Corollary 1.2 follows immediately from this last result.

1.4. Stationary Distributions for Non-Reneging Camp-On Systems

Using the definition of $\tilde{Y}_i$ in Equation (3.14) and the inverse transform relation in Equation (2.21), one could try to derive the joint probability distribution for queue lengths $p(\mathbf{x}_n)$ for this non-reneging model. However, such an antitransform procedure will prove not to be an easy task. Using a less orthodox procedure, one can prove (see Appendix III) the following theorem:

Theorem 2: In a stationary camp-on system with no reneging allowed from the second-level stage ($\eta = 0$) and type-i service centers ($i = 1,4,5$), the equilibrium joint probability distribution of queue lengths is given by

$$p_i(\mathbf{x}_n) \;=\; p_0\ \psi_1(\mathbf{x}_1)\cdots\psi_n(\mathbf{x}_n), \qquad i = 1,4,5, \tag{3.17}$$

where

$$\psi_i(\mathbf{x}_i) \;=\; \frac{\gamma_i}{\xi_i} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \binom{k_i+l_i}{l_i} \left[\frac{\lambda_{2r_i}}{\xi_i}\right]^{k_i} \left[\frac{\xi_{i-1}}{\xi_i}\right]^{l_i}, \tag{3.18}$$

and

$$\xi_n \;=\; \xi_n(\mathbf{x}_n) \;=\; \mu_n + \sum_{i=1}^{n} \lambda_{2r_i}. \tag{3.19}$$

Notice that Equation (3.17) is not an independent product solution, since $\psi_i(\mathbf{x}_i)$ is still linked to $\psi_{i-1}(\mathbf{x}_{i-1})$ through the parameter l_{i-1}.

Equation (3.9) holds only if $\gamma_n < \mu_n$ for all $n > M$, finite; otherwise, the system is unstable (non-ergodic). The condition for convergence in Equation (3.15) suggests that $\sum_{i=1}^{n} |\lambda_{2r_i}| \leq \mu_n$ for all $\mathbf{x}_n \in \Omega$. However, it is shown in Appendix III that the requirement in Equation (3.9) is a necessary and sufficient condition for stability.

In a general sense, Equation (3.17) can be interpreted as the collection of all possible combinations of events that, starting from the empty state $\mathbf{x}_0$, would yield as a final result a state vector $\mathbf{x}_n$. Though this camp-on system model presents some similarities with the problem of networks of queues (the presence of multiple queues, different types of customers, etc.), this queueing model does not present the typical product form-solution found in this multinode model. This difference can be attributed to an effective bulk departure process for second-level customers, which results in a lack of reversibility between the arrival and the departure processes. It also results from the random character of the size and number of queues within the second-level system as opposed to the fixed character of these parameters in the case of a network of queues.

The following two corollaries are immediate extensions of Corollaries 1.1 and 1.2 to type 2-3 service centers.

<u>Corollary 2.1:</u> For a non-reneging camp-on system with one single server and a LCFS-NP service discipline (type 2):

$$p_2(\mathbf{K}_n; \mathbf{R}_n) \;=\; p_1(k_1, k_n, \cdots, k_2\,; r_1, r_n, \cdots, r_2).$$

Corollary 2.2: For a non-reneging camp-on system with one single server and a LCFS-PR service discipline (type 3):

$$p_3(\mathbf{K}_n;\mathbf{R}_n) \;=\; p_1(k_n,\cdots,k_1\,;\,r_n,\cdots,r_1).$$

Both of these results follow as a direct consequence of Theorem 2 after reordering the position of the second-level systems with respect to the first-level center.

1.5. Systems with Finite Storage Capacity

The systems previously discussed fall within the category of infinite or semi-infinite camp-on systems, for all the second-level queues can grow infinitely long. However, most of the queueing systems found in real life applications are finite-size queueing systems, typically because of buffer limitations in any practical implementation. The following results apply only to non-reneging camp-on systems.

Let N denote the storage capacity at the first-level queue and let N_{r_i} denote the storage capacity for the second-level system in position i with class-r_i customers. Let us proceed as in the infinite storage case, but now considering that when $k_i = N_{r_i}$, any incoming customers to the i^{th} second-level system will be blocked and cleared from this hierarchical queueing system. After making the necessary modifications to the state-transition rates in Equations (2.8) to (2.12) to accommodate the limited buffer sizes, it is clear that the equilibrium joint probability distribution of queue lengths for the finite-state camp-on model $p^*(\mathrm{x}_n)$ must satisfy the following set of difference equations. This set of equations closely resembles Equations (2.8) to (2.12) except for the finite number of system states and the boundary conditions at the boundary states:

$$\left[\sum_{i=1}^{n}(\lambda_{2r_i}\ \chi_{(k_i<N_{r_i})}+\nu_i)+\lambda_{1n}\ \chi_{(n<N)}+\mu\chi_{(n>0)}\right]p^*(\mathbf{x}_n)=$$

$$\begin{aligned}&\gamma_n\ \chi_{(k_n=0)}\ p^*(\mathbf{x}_{n-1})\\&+\sum_{i=1}^{n}\lambda_{2r_i}\ \chi_{(k_i>0)}\ p^*(\mathbf{x}_n^{-i})\\&+\sum_{r_0=1}^{R}\chi_{(r_0\neq r_i)}\sum_{k_0=0}^{N_0}\mu\chi_{(n<N)}\ p^*(\mathbf{x}_{n+1,1})\\&+\sum_{r_0=1}^{R}\chi_{(r_0\neq r_i)}\sum_{i=1}^{n+1}\sum_{k_0=0}^{N_0}\nu_i\ \chi_{(n<N)}\ p^*(\mathbf{x}_{n+1,i}),\end{aligned}\qquad(3.20)$$

$$0\le k_i\le N_{r_i},\qquad 0\le i\le n\le N.$$

The next theorem, which applies to camp-on systems with finite storage capacity and no defections from the second-level queues, shows how to compute $p^*(\mathbf{x}_n)$ from $p(\mathbf{x}_n)$, the equilibrium state probability distribution for a camp-on system with infinite storage capacity given by Equation (3.17).

Theorem 3 : Let $p^*(\mathbf{K}_n;\mathbf{R}_n)$ designate the equilibrium joint probability distribution of queue lengths in a non-reneging camp-on system with storage capacity of N customers at the first-level queue, N_{r_j} class-r_j customers at the j^{th} second-level queue and type-i ($i=1,4,5$) service center. Let $(l_i,\cdots,l_s)$ be indices denoting the set of second-level systems whose queues are already full, $k_l=N_{r_l}$. Then,

$$p^*(\mathbf{K}_n;\mathbf{R}_n)=\begin{cases}p(\mathbf{K}_n;\mathbf{R}_n) & \text{if } k_i<N_{r_i},\quad (0\le n\le N);\\[2ex] \displaystyle\sum_{j_{l_1}=N_{r_{l_1}}}^{\infty}\cdots\sum_{j_{l_s}=N_{r_{l_s}}}^{\infty}p(\mathbf{J}_n;\mathbf{R}_n) & \text{if } k_i=N_{r_i},\quad (1\le s\le n),\end{cases}\qquad(3.21)$$

while

$$p_0^* = p(\mathbf{x}_0).$$

Here $p(\mathbf{K}_n;\mathbf{R}_n)$ is the equilibrium joint probability distribution of queue lengths in a camp-on system with infinite or semi-infinite storage capacity with the same traffic parameters, and $(\mathbf{J}_n, \mathbf{R}_n)$ is a state vector with $j_i = k_i$ if $k_i \neq N_{r_i}$.

To prove this statement, start with the global balance equation for an infinite-storage camp-on system in Equation (2.14) and let $\eta = 0$. Define $\hat{p}(\mathbf{x}_n)$ as proposed in Equation (3.21), by summing the k_l's in the equation for $p(\mathbf{x}_n)$ for the unconstrained system, from $k_l = N_{r_l}$ to infinity. After some straightforward algebraic reduction (see Appendix V), it is easy to prove that the set of difference equations satisfied by $\hat{p}(\mathbf{x}_n)$ represents the same queueing system as the one described by Equation (3.20). It therefore follows that $\hat{p}(\mathbf{x}_n) = p^*(\mathbf{x}_n)$.

Accordingly, each time we increase the buffer capacity of any second-level system, we split the "old" state vector $\mathbf{x}_n$ with $k_i = N_{r_i}$ into two new state vectors $\mathbf{x}_n^1$ and $\mathbf{x}_n^2$, one with $k_i^1 = N_{r_i}$ and the other with $k_i^2 = N_{r_i} + 1$, in such a way that the sum of their probabilities equals the probability of the old state vector: $p(\mathbf{x}_n^1) + p(\mathbf{x}_n^2) = p(\mathbf{x}_n)$. Moreover, this redefinition of the state space has left unchanged the probability of the off-boundary states.

Tables 1 and 2 give closed-form expressions for the state probability distribution $p(\mathbf{x}_n)$ in some small-size systems, obtained from Theorem 3 with $N_r = 1$ and $N_r = 2$ for class-1 camp-on systems with single-class second-level customers and $\lambda_1 = \lambda_{2r}$. These results can be used to design enhanced customer services for many practical communication systems, as we will show in Chapter V.

TABLE 1

State Probabilities for Some Finite-Storage Camp-on Systems ($N_r = 1$)

N N_{r_i}	$p(\mathbf{K}_n)$
$N = 1$ $N_1 = 1$	$p_0 = \frac{\mu}{\lambda+\mu}$ $p(0) = \frac{\lambda}{\lambda+\mu} p_0$ $\quad p(1) = \frac{\lambda^2}{\mu(\lambda+\mu)} p_0$
$N = 2$ $N_1 = 1$ $N_2 = 1$	$p_0 = [1 + \lambda/\mu + \lambda^2/\mu^2]^{-1}$ $p(0) = \frac{\lambda}{\lambda+\mu} p_0$ $\quad p(1) = \frac{\lambda^2}{\mu(\lambda+\mu)} p_0$ $p(0,0) = \frac{\lambda^2}{(\lambda+\mu)(2\lambda+\mu)} p_0$ $\quad p(0,1) = \frac{\lambda^3}{(\lambda+\mu)^2(2\lambda+\mu)} p_0$ $p(1,0) = \frac{2\lambda^3}{\mu(\lambda+\mu)(2\lambda+\mu)} p_0$ $\quad p(1,1) = \frac{(2\lambda+3\mu)\lambda^4}{\mu^2(\lambda+\mu)^2(2\lambda+\mu)} p_0$
$N = 3$ $N_1 = 1$ $N_2 = 1$ $N_3 = 1$	$p_0 = [1 + \lambda/\mu + \lambda^2/\mu^2 + \lambda^3/\mu^3]^{-1}$ $p(0) = \frac{\lambda}{\lambda+\mu} p_0$ $\quad p(1) = \frac{\lambda^2}{\mu(\lambda+\mu)} p_0$ $p(0,0) = \frac{\lambda^2}{(\lambda+\mu)(2\lambda+\mu)} p_0$ $\quad p(0,1) = \frac{\lambda^3}{(\lambda+\mu)^2(2\lambda+\mu)} p_0$ $p(1,0) = \frac{2\lambda^3}{\mu(\lambda+\mu)(2\lambda+\mu)} p_0$ $\quad p(1,1) = \frac{(2\lambda+3\mu)\lambda^4}{\mu^2(\lambda+\mu)^2(2\lambda+\mu)} p_0$ $p(0,0,0) = \frac{\lambda^3}{(\lambda+\mu)(2\lambda+\mu)(3\lambda+\mu)} p_0$ $\quad p(0,0,1) = \frac{\lambda^4}{(\lambda+\mu)(2\lambda+\mu)^2(3\lambda+\mu)} p_0$ $p(0,1,0) = \frac{2\lambda^4}{(\lambda+\mu)^2(2\lambda+\mu)(3\lambda+\mu)} p_0$ $\quad p(1,0,0) = \frac{3\lambda^4}{\mu(\lambda+\mu)(2\lambda+\mu)(3\lambda+\mu)} p_0$ $p(0,1,1) = \frac{(5\lambda+3\mu)\lambda^5}{(\lambda+\mu)^3(2\lambda+\mu)^2(3\lambda+\mu)} p_0$ $\quad p(1,0,1) = \frac{2(3\lambda+2\mu)\lambda^5}{\mu(\lambda+\mu)^2(2\lambda+\mu)^2(3\lambda+\mu)} p_0$ $p(1,1,0) = \frac{2(3\lambda+4\mu)\lambda^5}{\mu^2(\lambda+\mu)^2(2\lambda+\mu)(3\lambda+\mu)} p_0$ $p(1,1,1) = \frac{12\lambda^3+40\lambda^2\mu+45\lambda\mu^2+15\mu^3}{\mu^3(\lambda+\mu)^3(2\lambda+\mu)^2(3\lambda+\mu)} p_0$

TABLE 2

State Probabilities for Some Finite-Storage Camp-on Systems ($N_r = 2$)

N N_{r_i}	$p(\mathbf{K}_n)$
$N=1$ $N_1 = 2$	$p_0 = \frac{\mu}{\lambda+\mu}$ $p(0) = \frac{\lambda}{\lambda+\mu} p_0$ $\quad p(1) = \frac{\lambda^2}{\mu(\lambda+\mu)} p_0$ $p(2) = \frac{\lambda^3}{\mu(\lambda+\mu)^2} p_0$
$N=2$ $N_1 = 2$ $N_2 = 2$	$p_0 = [1 + \lambda/\mu + \lambda^2/\mu^2]^{-1}$ $p(0) = \frac{\lambda}{\lambda+\mu} p_0$ $\quad p(1) = \frac{\lambda^2}{\mu(\lambda+\mu)} p_0$ $p(2) = \frac{\lambda^3}{\mu(\lambda+\mu)^2} p_0$ $p(0,0) = \frac{\lambda^2}{(\lambda+\mu)(2\lambda+\mu)} p_0$ $\quad p(0,1) = \frac{\lambda^3}{(\lambda+\mu)^2(2\lambda+\mu)} p_0$ $p(0,2) = \frac{\lambda^4}{(\lambda+\mu)^2(2\lambda+\mu)^2} p_0$ $\quad p(1,0) = \frac{2\lambda^3}{\mu(\lambda+\mu)(2\lambda+\mu)} p_0$ $p(1,1) = \frac{(2\lambda+3\mu)\lambda^4}{\mu^2(\lambda+\mu)^2(2\lambda+\mu)} p_0$ $\quad p(1,2) = \frac{2(3\lambda+\mu)\lambda^5}{(\lambda+\mu)^3(2\lambda+\mu)^3} p_0$ $p(2,0) = \frac{(4\lambda+3\mu)\lambda^4}{\mu(\lambda+\mu)^2(2\lambda+\mu)^2} p_0$ $\quad p(2,1) = \frac{2(4\lambda+3\mu)\lambda^5}{\mu(\lambda+\mu)^2(2\lambda+\mu)^3} p_0$ $p(2,2) = \frac{2(4\lambda^2+10\lambda\mu+5\mu^2)\lambda^5}{\mu(\lambda+\mu)^3(2\lambda+\mu)^3} p_0$

III.2. Other Special Distributions

In this section, we study some special cases of the probability distributions $P(\mathbf{Z}_n;\mathbf{R}_n)$ and $p(\mathbf{x}_n)$. First, we study two particular situations in the two-level camp-on model. One of them corresponds to single-class non-reneging systems. This is useful when the service center has no knowledge about the second-level system classes. The other corresponds to class-1 camp-on systems under heavy traffic conditions for the first-level queue, where $\lambda_1 \gg \mu$. The marginal distributions for the size of the second-level system in position i with respect to the first-level center and the total workload accumulated at each queueing stage are then derived also for non-reneging systems.

2.1. Single-Class Systems

The state vector representation for the camp-on model, $\mathbf{x}_n$, provides explicit information about the second-level class assignment, besides just the information about the queue sizes. Even if the number of classes R is small, computing $\mathbf{x}_n$ from Equationi (3.17) is tedious because of all possible permutations of customer classes among the second-level systems. In other instances, the central node of the network (first-level center) may not know beforehand the traffic statistics of all the nodes (second-level systems), and a routing decision has to be taken based only on the occupancy probabilities for the system. A great deal of simplicity and a corresponding reduction in computation can be achieved by looking at only single-class camp-on systems. That is, we assume $\lambda_{2r} = \lambda_2$ for all second-level systems. Then any information about the customer class is redundant, and it is just enough to specify the total number of customers in each of the second-level queues. Let $\mathbf{x}_n = \mathbf{K}_n = (k_1, \cdots, k_n)$ be the reduced state-vector representation for such a system and let $p(\mathbf{x}_n) = p(\mathbf{K}_n)$ be the corresponding state-probability distribution for queue lengths. It follows from Equation (3.17) that

$$p(\mathbf{K}_n) = \sum_{\mathbf{R}_n} p(\mathbf{K}_n; \mathbf{R}_n). \tag{3.22}$$

From this definition it can be seen that for the single-class camp-on system, the equilibrium joint probability distribution of queue lengths reduces to the simpler expression

$$p(\mathbf{K}_n) = \binom{R}{n} n!\, p(\mathbf{K}_n; \mathbf{R}_n), \tag{3.23}$$

with $p(\mathbf{x}_n; \mathbf{R}_n)$ as in Equation (3.17) but with a common arrival rate λ_2 for all the second-level systems. Here $p(\mathbf{x}_0)$ in (3.11) reduces to

$$p(\mathbf{x}_0) = \left[\frac{1-(\lambda_1/\mu)^{N+1}}{1-\lambda_1/\mu}\right]^{-1}. \tag{3.24}$$

There is quite a straightforward interpretation for $p(\mathbf{K}_n)$. Given that the size of the first-level system is n customers, we could have chosen only n classes of second-level customers out of the R available, but they could also have been chosen in any arbitrary order. This explain the form of Equation (3.23).

Figures 6 and 7 show the state probabilities for the first few terms of the state probability distribution $p(\mathbf{K}_n)$ in a single-class non-reneging camp-on system with infinite storage capacity as a function of the traffic intensity at the first-level service facility ($\rho_1 = \lambda_1/\mu$) for state vectors with $n = 1$ and $n = 2$. In this example, the system parameters were chosen such that $\lambda_{1n} = \lambda_1 = \lambda_{2r}$ and $\nu_i = 0$ ($1 \le i \le n < \infty$). Notice how $p(\mathbf{K}_n)$ increases and then decreases as the traffic intensity increases, as one should expect. This behavior exhibited by $p(\mathbf{K}_n)$ clearly indicates that for light traffic, those state vectors with small queue sizes mostly prevail, while

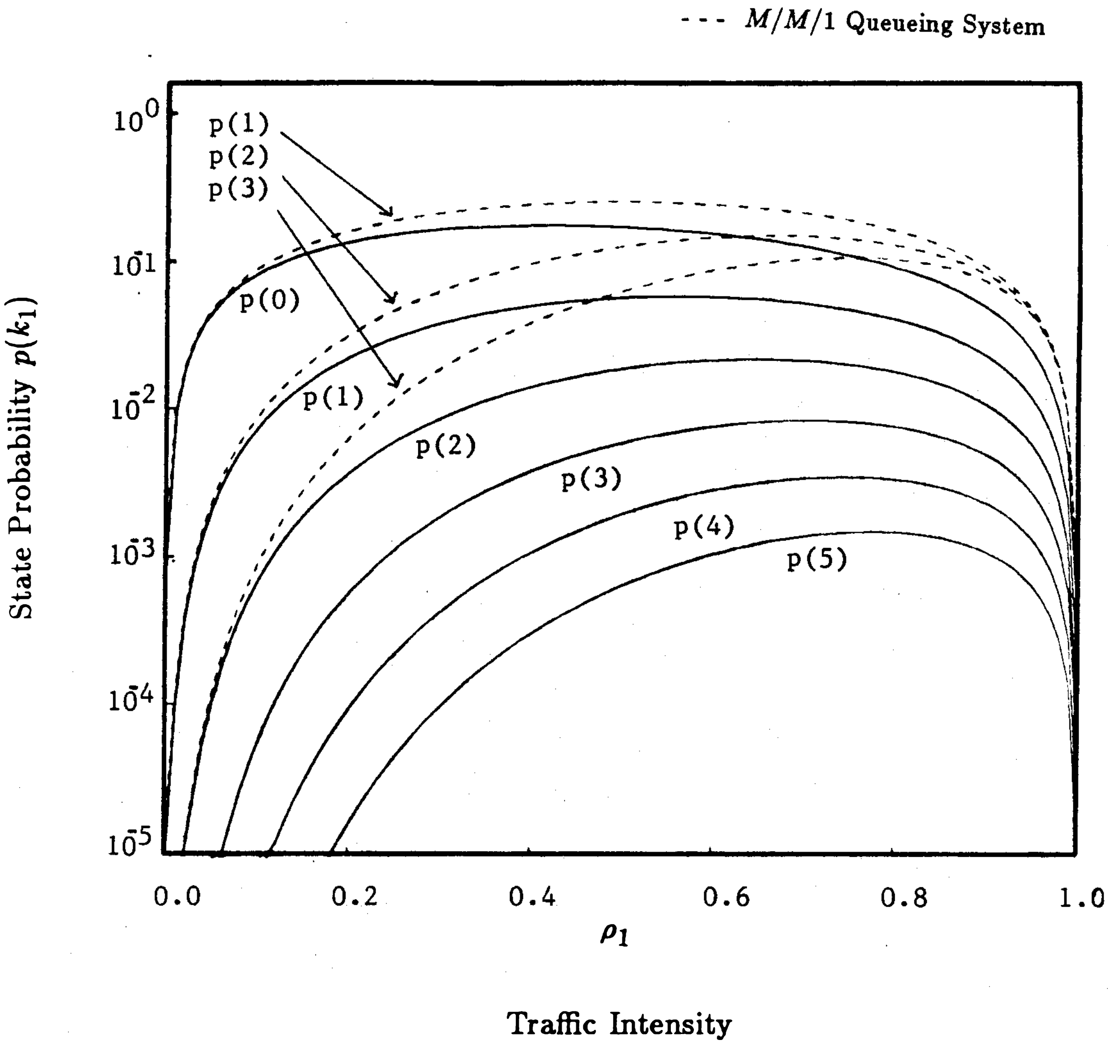


Figure 6: Equilibrium probabilities for some states $\mathbf{x}_n$ with $n = 1$ as a function of the traffic intensity at the first-level service center in a single-class non-reneging camp-on system with $\lambda_1 = \lambda_{2r}$.

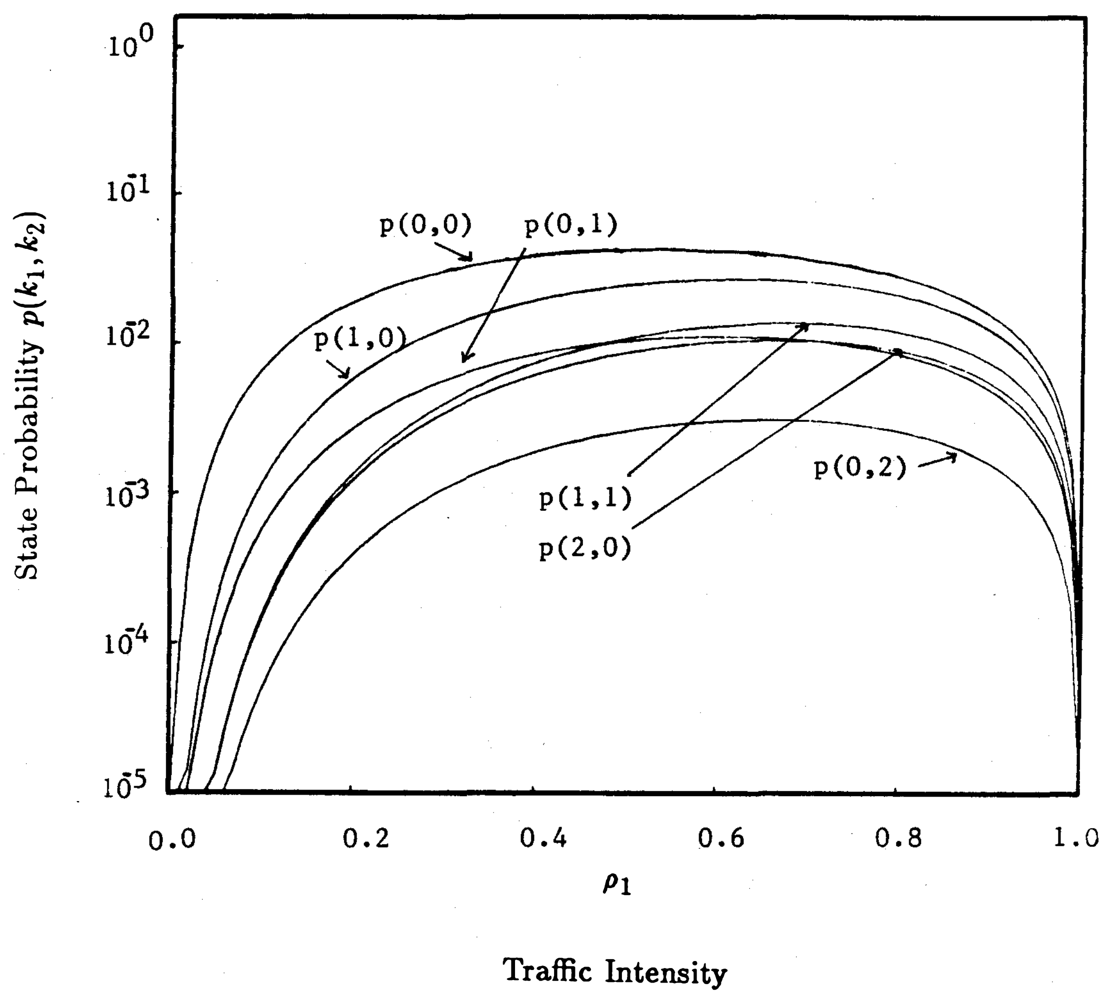


Figure 7: Equilibrium probabilities for some states $\mathbf{x}_n$ with $n = 2$ as a function of the traffic intensity at the first-level service center in a single-class non-reneging camp-on system with $\lambda_1 = \lambda_{2r}$.

for heavy traffic, those states with large queue sizes become much more significant. This traffic-dependent behavior is manifested by the progressive fan to the right of the curves for $p(\mathbf{K}_n)$ as a function of the total workload, $S(\mathbf{K}_n) = n + k_1 + \cdots + k_n$.

Figures 8 and 9 also show how the state probability $p(1,1)$ changes as a function of the traffic intensity at the first-level facility for a fixed traffic intensity at a second-level system (ρ_2 = 0.2, 0.4 0.6, 0.8, 1.0), and as a function of the traffic intensity at a second-level system for a fixed traffic-intensity at the first-level system (ρ_1 = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8). In Figure 8, we observe essentially the same type of behavior for the state probability $p(1,1)$ as in Figure 7. Basically, $p(1,1)$ is depicted as a shifted version of the state probability $\pi(2)$ in Figure 6. It also shows that the distribution of the size of the second-level queues changes very little for second-level traffic intensities in the range $\rho_2 \sim 0.3 - 1.0$. This is because of the strong influence of the first-level system on the initial structure of the second-level systems. Figure 9 shows that the state probability $p(\mathbf{K}_n)$ is relatively insensitive to changes in the traffic demand at the first-level center for medium traffic intensities ($\rho_1 \sim 0.4 - 0.7$).

2.2. Heavy Traffic at the First-Level Service Center

Another very interesting situation in a camp-on model is a class-1 system, where the arrival rate at the first-level systems is much larger than the service rate. Clearly, the queueing system can have either a finite storage capacity at the first-level queue or as many first-level servers as customers; otherwise, it will be unstable. Since $\lambda_1 \gg \mu_1$, it follows from Equation (3.1) that for any distribution of the second-level queue sizes, $P_{n-1}(\mathbf{Z}_{n-1};\mathbf{R}_{n-1}) \ll P_{n+1}(\mathbf{Z}_{n+1};\mathbf{R}_{n+1})$. Therefore, we can rewrite

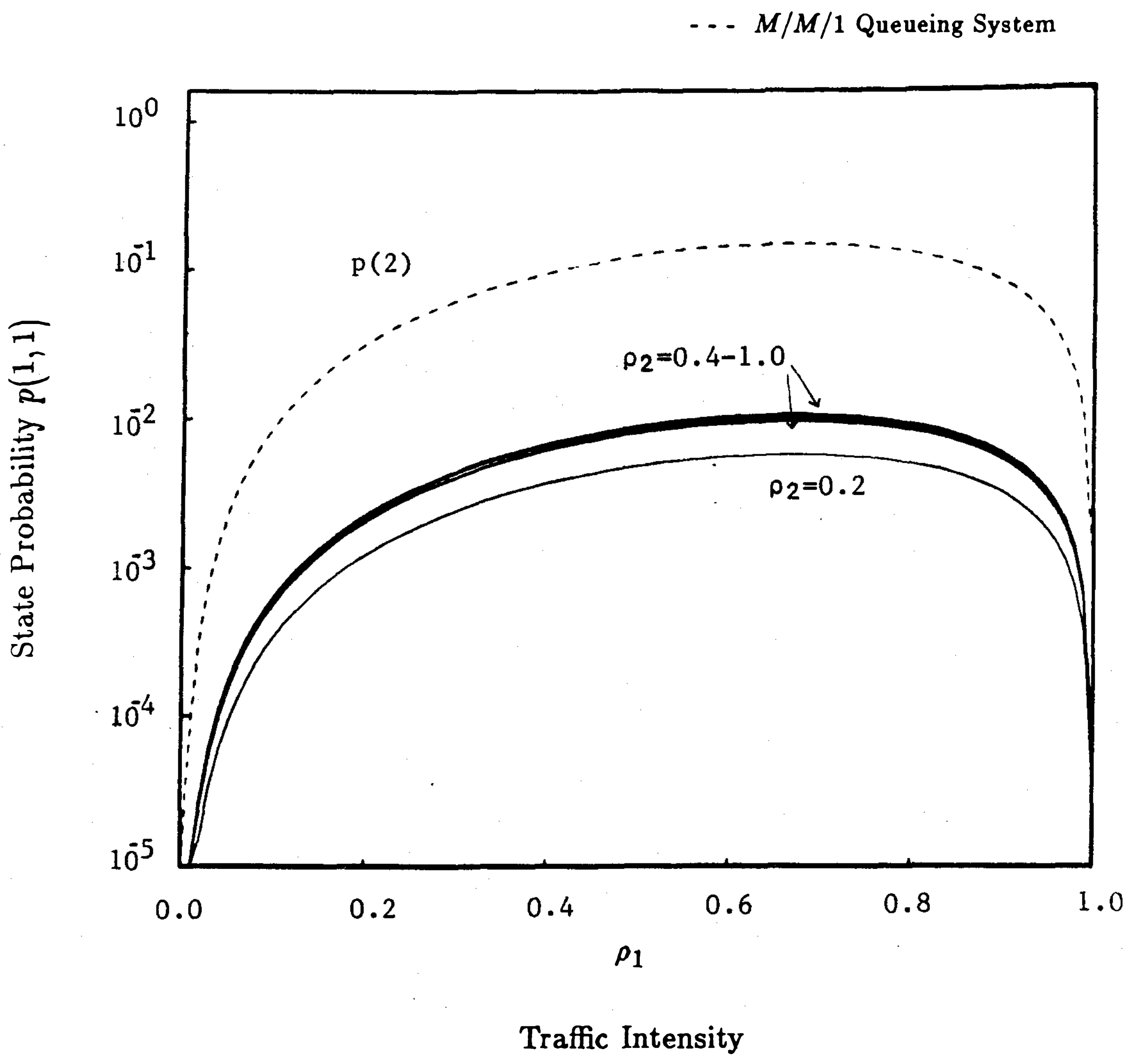


Figure 8: Equilibrium probability $p(1,1)$ as a function of the traffic intensity at the first-level service center in a single-class non-reneging camp-on system with λ_{2r} as a parameter.

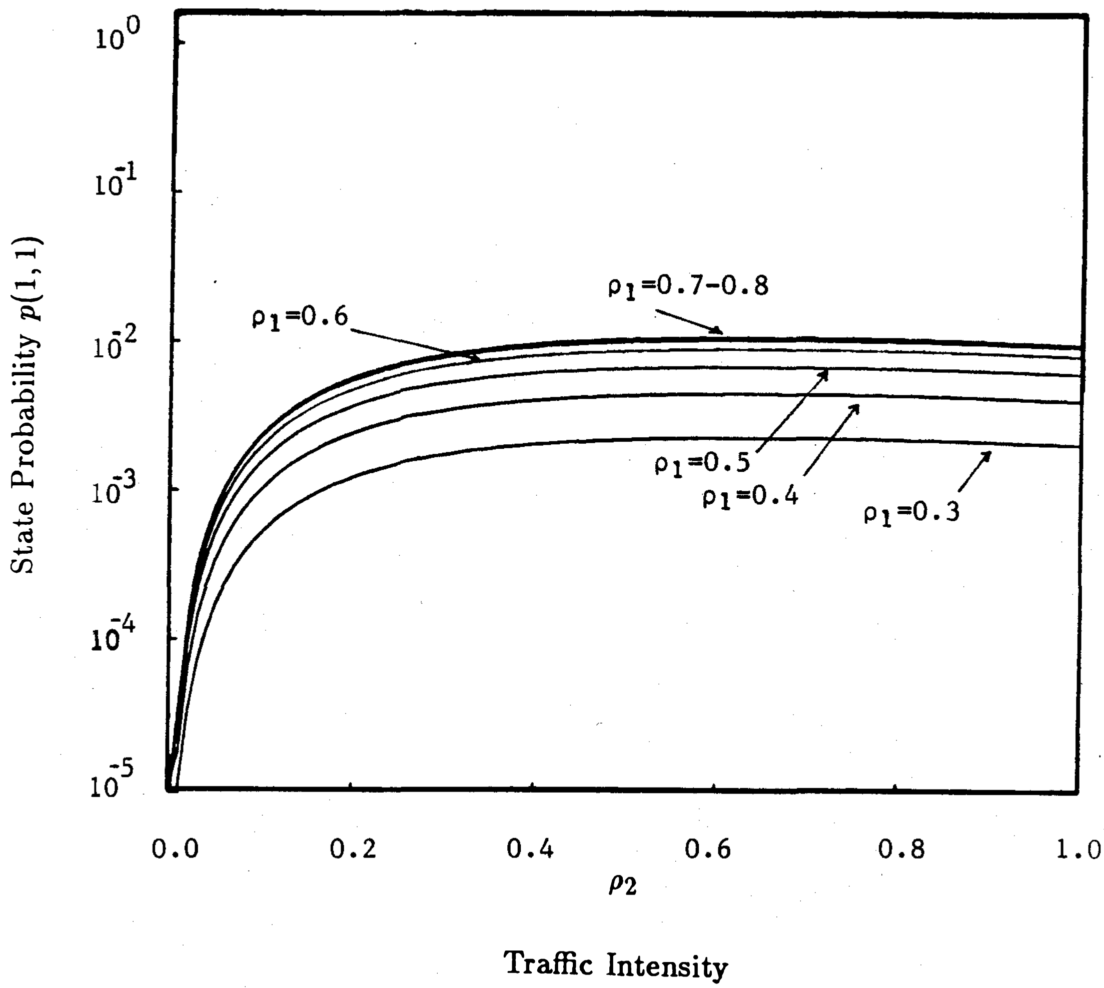


Figure 9: Equilibrium probability $p(1,1)$ as a function of the traffic intensity at the second-level service center in a single-class non-reneging camp-on system with λ_1 as a parameter.

the transformed balance equation for a camp-on system under heavy traffic conditions at the first-level service center as:

$$\sum_{i=1}^{n}(z_i-1)\left[\eta_{r_i}\frac{\partial}{\partial z_i}P(\mathbf{Z}_n;\mathbf{R}_n)-\lambda_{2r_i}\,P(\mathbf{Z}_n;\mathbf{R}_n)\right]+\lambda_{1n}\chi_{(n<N)}\,P(\mathbf{Z}_n;\mathbf{R}_n)=$$
$$\sum_{r_0=1}^{n+1}\chi_{(r_0\neq r_i)}\sum_{i=1}^{n+1}\nu_i\chi_{(n<N)}\,P(\mathbf{Z}_{n+1,i};\mathbf{R}_{n+1,i}).$$

Here we also chose the reneging rate from the second-level subsystems as dependent on the class assignment $\mathbf{R}_n$.

In this case, we could think of the reneging rates from the second-level systems μ_{r_i} as corresponding to an actual service being provided by the second-level subsystems rather than just to impatient second-level customers quitting the camp-on system. This represents an extension of the type-5 service center to the service centers for customers in the second-level stage of a non-reneging camp-on system. This is because of the similarity of the reneging and service in an infinite-server environment. In this case, a second-level customer could be serviced at the same instant as its associated first-level customer is being serviced or queued at the first-level waiting line. Assume that the second-level system behavior is that of a truly IS service center. Then the solution for the above partial balance equation would be

$$P(\mathbf{Z}_n;\mathbf{R}_n)=p_0\prod_{i=1}^{n}\frac{\gamma_i}{\mu_i}e^{\frac{\lambda_{2r_i}}{\eta_{r_i}}(z_i-1)},\qquad(3.25)$$

the product of the distributions for the size of the first-level system as a single-level system and the transformed distribution for the IS service centers at the second queueing stage.

This claim can be immediately verified since in this case the first term on the right side of the transformed balance equation cancels out, and the resulting balance equation is that of the one-level queueing system. One can now prove from Equation (3.25) that

$$p(\mathbf{x}_n) \;=\; p_0 \prod_{i=1}^{n} \frac{\gamma_i}{\mu_i} \frac{\left(\lambda_{2r_i}/\eta_{r_i}\right)^{k_i}}{k_i!} e^{-\lambda_{2r_i}/\eta_{r_i}}. \tag{3.26}$$

Thus, each of the queues in the camp-on model behaves as an independent queueing system, and a product-form solution can be given to the state probability distribution. This is regardless of the bulk departure processes for the second-level customers, resulting from a service completion at a first-level service center. Notice that under these conditions, the second-level system reaches equilibrium before its associated first-level server can change his position with respect to the first-level queue. Hence, second-level customers perceive the first-level queue as a static collection of n servers rather than as a changing waiting line.

2.3. Marginal Distribution for the Size of the i^{th} Second-Level System

From the performance analysis viewpoint, the system description given by the state-vector probability $p(\mathbf{x}_n)$ may provide more information than one needs to know. In many cases, the marginal distribution for the size of the second-level systems more than suffices as a relevant description of the state of the camp-on system. For example, the probability that the second-level system in position i will have k_i class-r_i customers awaiting service, when the first-level system size is n, can be extracted from the transformed joint probability distribution of queue lengths by selecting only according to the size of that sole second-level system rather than on the whole joint distribution of the multiple subsystems in Equation (3.1).

Let $(l_1, \cdots, l_s)$ designate a subset of the second-level systems. For the fixed class assignment $\mathbf{R}_n$ of the customers queued at the second-level systems, the generating function $P(z_{l_1}, \cdots, z_{l_s}; \mathbf{R}_n)$ for the size of the i^{th} second-level system is found by evaluating the transformed state distribution $P(\mathbf{Z}_n; \mathbf{R}_n)$ in Equation (3.1) at the vector $\mathbf{Z}_n = \mathbf{z}$, with $z_i = 1$ if $i = l_k$ and $z_i = 1$, otherwise:

$$P(z_{l_1}, \cdots, z_{l_s}; \mathbf{R}_n) \;=\; P(1, \cdots, z_{l_1}, \cdots, z_{l_s}, \cdots, 1; \mathbf{R}_n).$$

An extreme case of this distribution results when we assume $z_k = 1$ for all k except for $k = i$. This case corresponds to the marginal distribution for the size of the second-level system which is in position i with respect to the first-level facility. Then,

$$\begin{aligned} P(z_i; \mathbf{R}_n) &= P(1, 1, \cdots, z_i, \cdots, 1; \mathbf{R}_n) \\ &= p_0 \prod_{k=1}^{i-1} \frac{\gamma_k}{\xi_k} \prod_{j=i}^{n} \frac{\gamma_j}{\xi_j} \sum_{l_j=0}^{\infty} \frac{\frac{\lambda_{2r_i}}{\eta}(z_i - 1)}{(\alpha_j)_{l_j+1}}. \end{aligned} \tag{3.27}$$

If we look a the particular example of a single-server non-reneging camp-on system ($\nu_i = \eta = 0$), it follows that the transformed marginal distribution for the size of the i^{th} second-level system, conditioned on the class assignment $\mathbf{R}_n$, reduces to

$$\begin{aligned} P(z; \mathbf{R}_n) &= p_0 \prod_{k=1}^{i-1} \frac{\gamma_{k-1}}{\mu} \prod_{j=i}^{n} \frac{\gamma_j}{\mu + \lambda_{2r_i}(1 - z)} \\ &= p_0 \left(\frac{\mu}{\mu + \lambda_{2r_i}} \right)^{n-i+1} \left(1 - \frac{\lambda_{2r_i}}{\mu + \lambda_{2r_i}} z \right)^{-(n-i+1)} \prod_{k=1}^{n} \frac{\gamma_k}{\mu}. \end{aligned} \tag{3.28}$$

This expression for $P(z; \mathbf{R}_n)$ can be easily antitransformed, using conventional Z-transform techniques. Through the straightforward computation of Equation (2.21) or by the use of an antitransform table[18], we see that the marginal distribution for the second-level systems in position i when the first-level queue has size n is of the form

$$p(k_i;\mathbf{R}_n) \;=\; \binom{n-i+k_i}{k_i}\beta_{r_i}^{n-i+1}\,\alpha_{r_i}^{k_i}\,\pi(\mathbf{R}_n). \tag{3.29}$$

Here

$$\pi(\mathbf{R}_n) \;=\; p_0\prod_{k=1}^{n}\frac{\gamma_k}{\mu} \tag{3.30}$$

is the probability that the first-level customers have class assignment $\mathbf{R}_n$ for its associated second-level waiting lines. The parameters α_{r_i} and β_{r_i} stand for

$$\alpha_{r_i} \;=\; \frac{\lambda_{2r_i}}{\mu+\lambda_{2r_i}}, \tag{3.31}$$

$$\beta_{r_i} \;=\; \frac{\mu}{\mu+\lambda_{2r_i}}. \tag{3.32}$$

The interpretation of the parameters α_{r_i} and β_{r_i} is that they represent the average arrival rate at the i^{th} second-level system and the average departure rate from position i to position $i-1$ in the second-level stage, respectively. A customer arrives at the i^{th} second-level system with probability α_{r_i} before a customer departure from the first-level occurs. Hence, $\alpha_{r_i}^{k_i}$ stems from the probability of exactly k_i class-r_i arrivals to the i^{th} second-level system between the i^{th} and the n^{th} arrivals to the first-level queue. Similarly, $\beta_{r_i}^{n-i}$ is the probability of exactly $n-i$ departures from the first-level system. Since the size of the first-level system is n, it is clear that an i^{th} second-level system will still exist at the second-level stage. Then, $p(k_i;\mathbf{R}_n)$ is the probability of exactly k_i arrivals at an i^{th} second-level system before this subsystem leaves the camp-on model. This explains the negative binomial distribution.

Based on the definition of $(l_1, \cdots, l_R)$ from Ω_n and $\mathbf{R}_n$ in Equation (3.13), it is convenient to rewrite $\pi(\mathbf{R}_n)$ as:

$$\pi(\mathbf{R}_n) = \begin{cases} p_0 \prod_{i=1}^{n} \dfrac{\lambda_{1i}}{(R-i+1)\mu}, & \text{for class-1 systems;} \\ \\ p_0 \prod_{r=1}^{R} \left(\dfrac{\lambda_{2r}}{\mu}\right)^{l_r}, & \text{for class-2 systems,} \end{cases} \tag{3.33}$$

which is equivalent to the probability distribution of an $M/M/1$ queueing system with multiple classes of customers. From this, $\pi(n)$, the unconstrained probability distribution for the size of the first-level system, is merely

$$\pi(n) = \sum_{\mathbf{R}_n} \pi(\mathbf{R}_n). \tag{3.34}$$

If we insist on the i^{th} second-level system's being of class r, then the probability that its size is k_{i_r} regardless of the composition of the first-level system is given by

$$p_n(k_{i_r}) = \sum_{\Omega_n} \chi_{(r_i=r)} p(k_i; \mathbf{R}_n). \tag{3.35}$$

We notice that for the special case of single-class class-1 camp-on systems, Equation (3.35) boils down to

$$p_n(k_i) = \binom{n-i+k_i}{k_i} \beta^{n-i+1} \alpha^{k_i} \pi(n), \tag{3.36}$$

with $\alpha = \lambda/(\lambda+\mu)$ and $\alpha+\beta = 1$.

The interpretation of Equation (3.31) is that $p_n(k_i)$ is the probability of k_i arrivals at the second-level system between arrivals i and $n+1$ at the first-level system. Figure 10 shows $p_n(k_i)$ as a function of the traffic intensity at the first-level system, with $\lambda_1 = \lambda_{2r}$ for the case $n = 3$ and $k_i = 0, 1$.

We will return to this expression for the marginal distribution of the i^{th} second-level system as we study multilevel camp-on systems in Chapter IV.

2.4. Workload Distribution Among the Queueing Stages

We already have a very complete description for the state occupancy problem in two-level camp-on systems through the transformed joint probability distribution of queue sizes $P(\mathbf{Z}_n; \mathbf{R}_n)$ in Equation (3.1) or the antitransformed form $p(\mathbf{x}_n)$ for non-reneging systems in Equation (3.17). We shall now exploit this description to derive the distribution of the total workload accumulated in the two queueing stages.

Let $P(z; \mathbf{R}_n)$ denote the generating function for the size of the second-level stage and let $p(k; \mathbf{R}_n)$ denote the joint probability distribution of the sizes of the system stages for a fixed class assignment $\mathbf{R}_n$. We obtain $P(z; \mathbf{R}_n)$ from Equation (3.1) by evaluating $P(\mathbf{Z}_n; \mathbf{R}_n)$ at the vector $\mathbf{Z}_n = \mathbf{z}$, which components $z_i = z$. To prove this statement, notice that

$$P(\mathbf{Z}_n; \mathbf{R}_n)\Big|_{\mathbf{Z}_n = \mathbf{z}} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} p(\mathbf{K}_n; \mathbf{R}_n) z^{k_1 + \cdots + k_n}$$

$$= \sum_{k_1 + \cdots + k_n \geq 0} p(\mathbf{K}_n; \mathbf{R}_n) z^{k_1 + \cdots + k_n}.$$

Here, $p(k; \mathbf{R}_n)$ stands for the joint probability distribution of the accumulated workload in both system stages. Hence,

$$p(k; \mathbf{R}_n) = \sum_{k_1 + \cdots + k_n = k} p(\mathbf{K}_n; \mathbf{R}_n).$$

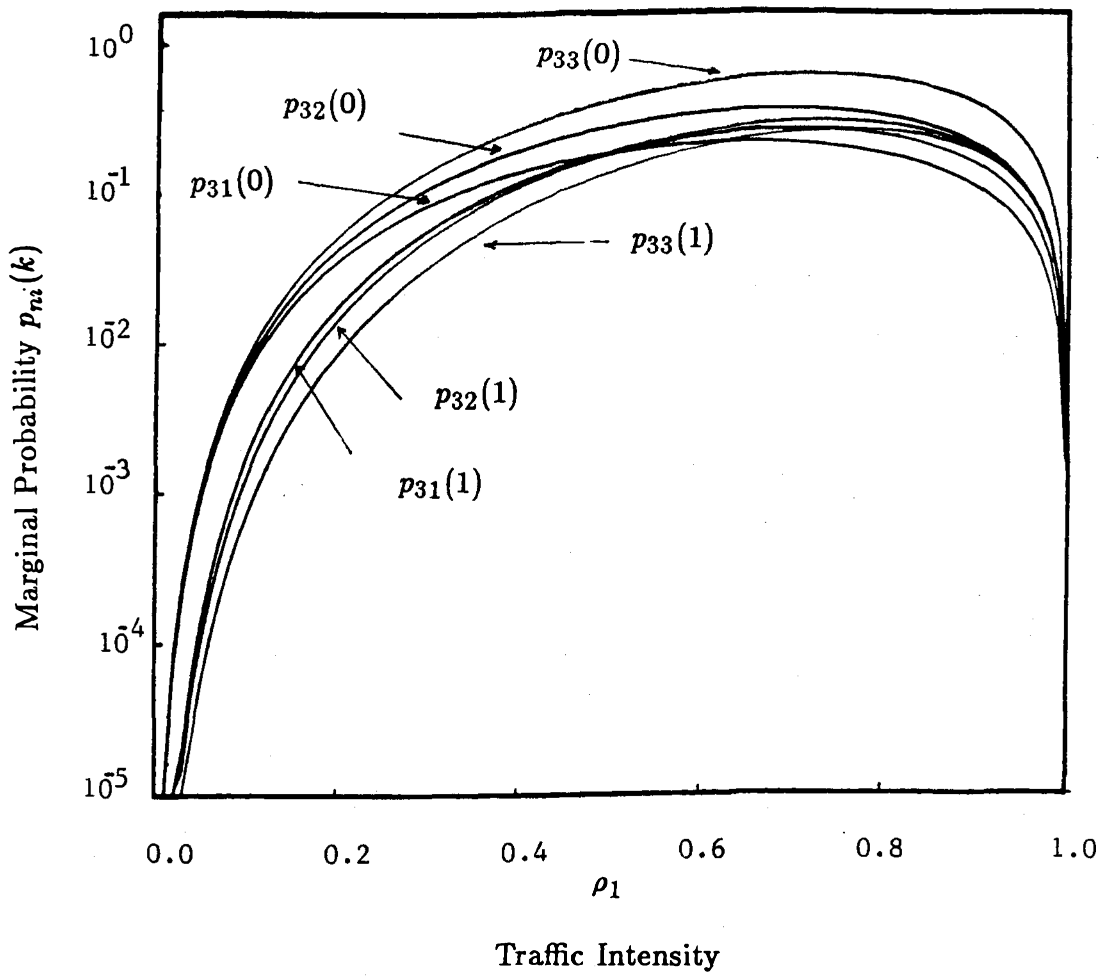


Figure 10: Marginal distribution for the size of the i^{th} second-level system $p_n(k_i)$ *vs*. the traffic intensity at the first-level system in a single-class camp-on system ($\lambda_{2r} = \lambda_1$).

It follows from the above equation that

$$\begin{aligned} P(z;\mathbf{R}_n) &= \sum_{k\geq 0} p(k;\mathbf{R}_n)z^k \\ &= P(\mathbf{Z}_n;\mathbf{R}_n)\Big|_{\mathbf{Z}_n=\mathbf{z}}. \end{aligned}$$

Accordingly, if we follow the above argument and evaluate $P(z;\mathbf{R}_n)$ from Equation (3.16), we conclude that the generating function for the distribution of the size of the second-level stage in a non-reneging camp-on system, conditioned on class assignment $\mathbf{R}_n$ for the second-level systems, is

$$P(z;\mathbf{R}_n) = p_0 \prod_{i=1}^{n} \frac{\gamma_i}{\mu + \sum_{j=1}^{i} \lambda_{2r_j}(1-z)}. \qquad (3.37)$$

For convenience, let us define the parameters $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ (they are not to be confused with α_{r_i} and β_{r_i} in the previous section) as:

$$\tilde{\alpha}_i = \frac{\hat{\lambda}_i}{\mu + \hat{\lambda}_i}, \qquad (3.38)$$

$$\tilde{\beta}_i = \frac{\mu}{\mu + \hat{\lambda}_i}. \qquad (3.39)$$

Here, $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ represent the average rate of arrival and departure into and out of the second-level stage, conditioned on a first-level system size of n customers.

After antitransforming Equation (3.37) (see Appendix VI), it can be shown that for a non-reneging camp-on system, the workload distribution among the two queueing stages is of the form

$$p(k;\mathbf{R}_n) = \sum_{i=1}^{n} \tilde{\beta}_i \tilde{\alpha}_i^k \pi(\mathbf{R}_n) \prod_{j\neq i}^{n} \frac{\tilde{\beta}_j \tilde{\alpha}_i}{(\tilde{\alpha}_i - \tilde{\alpha}_j)}, \qquad n \geq 0,\ k \geq 0. \qquad (3.40)$$

If we sum the above expression over all possible class assignments $\mathbf{R}_n$ for the second-level systems, we find that the distribution of total workload accumulated by the camp-on system at each stage $p_W(n,k)$, is

$$p_W(n,k) = \sum_{\mathbf{R}_n} p(k;\mathbf{R}_n).$$

In particular, for class-1 systems, when $\lambda_{2r} = \lambda_2$ for all r, we are back to the single-class camp-on system The above expression for $p_W(n,k)$ thus reduces to

$$p_W(n,k) = \sum_{i=1}^{n} \tilde{\beta}_i \tilde{\alpha}_i^k \binom{n}{i} \frac{(-1)^{n-i} i^n}{n!} \pi(n). \tag{3.41}$$

As a further check of Equation (3.41), let us compute the probability distribution for the size of the first-level stage, regardless of the size of the second-level stage. We have

$$\begin{aligned} p(n) &= \sum_{k=0}^{\infty} p(n,k) \\ &= \sum_{i=1}^{n} \binom{n}{i} \frac{(-1)^{n-i} i^n}{n!} \pi(n). \end{aligned}$$

But recall the combinatorial identity[35]

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^r = \begin{cases} 0 & \text{if } r < n; \\ n! & \text{if } r = n. \end{cases} \tag{3.42}$$

Consequently, the probability distribution for the size of the first-level system ends up being

$$p(n) = \frac{1}{n!}[n!\ \pi(n)] = \pi(n),$$

which is consistent with the interpretation of $\pi(n)$ in Equation (3.34).

From Equation (3.39), we could also derive an expression for the distribution of the total number of customers in the camp-on system. This is the overall system size, counting both first-level and second-level customers. Let $p_T(n)$ denote this probability distribution. It follows that

$$\begin{aligned} p_T(n) &= \sum_{k=1}^{n} p(k, n-k) \\ &= \sum_{k=1}^{n} \sum_{i=1}^{k} \tilde{\beta}_i \tilde{\alpha}_i^{n-k} \pi(\mathbf{R}_k) \prod_{j \neq i}^{k} \frac{\tilde{\alpha}_i \tilde{\beta}_j}{(\tilde{\alpha}_i - \tilde{\alpha}_j)}, \qquad n \geq 1. \end{aligned} \tag{3.43}$$

For a single-class non-reneging camp-on system, Equation (3.43) reduces to

$$p_T(n) = \sum_{k=1}^{n} \sum_{i=1}^{k} \tilde{\beta}_i \tilde{\alpha}_i^{n-k} \binom{k}{i} \frac{(-1)^{k-i} i^k}{k!} \pi(k). \tag{3.44}$$

Figure 11 shows how the workload distribution $p_W(n, k)$ behaves as a function of the incoming traffic to the first-level service center for a single-class non-reneging camp-on system with $\lambda_1 = \lambda_{2r}$ and $n, k = 1, 2, 3$. Figure 12 shows the distribution of the total number of customers as a function of the traffic intensity at the first-level service center in a single-class non-reneging camp-on system with $\lambda_1 = \lambda_{2r}$.

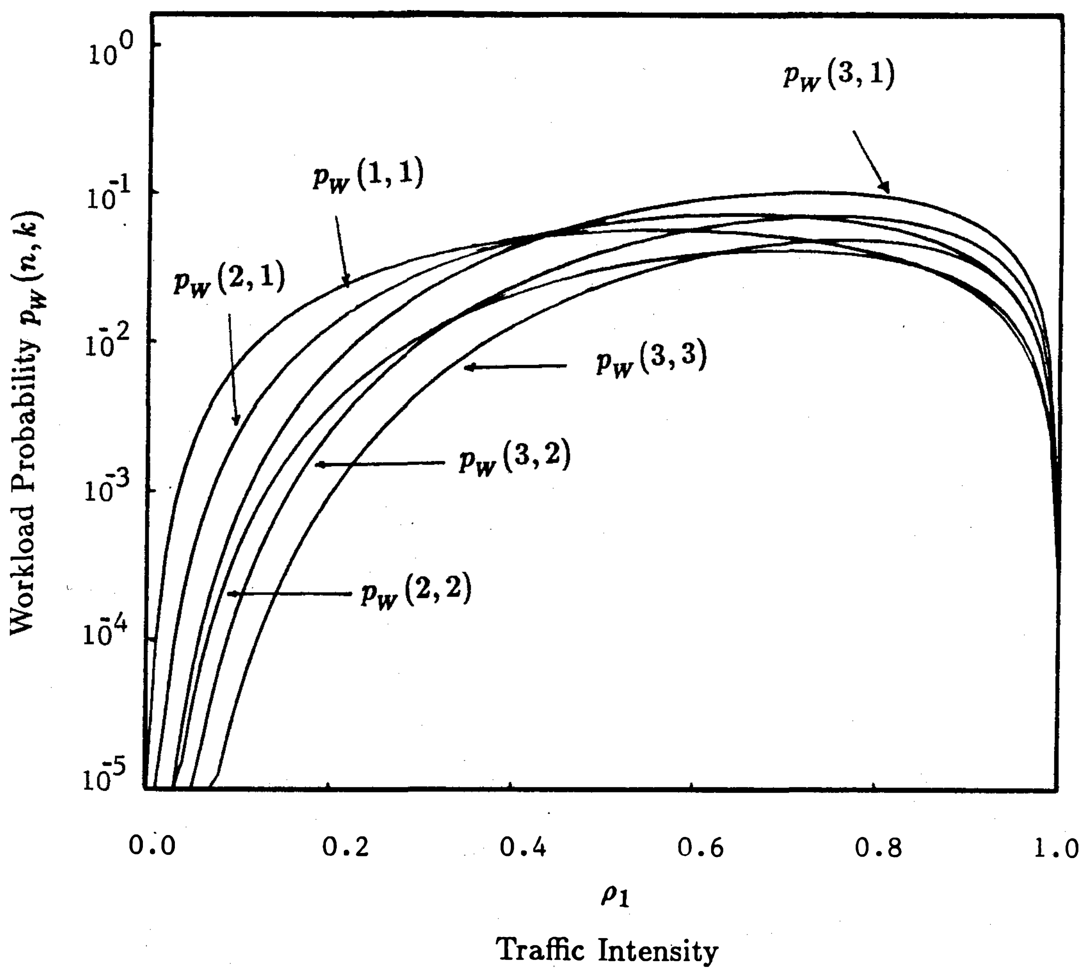


Figure 11: Workload probability distribution among the queueing stages $p_W(n,k)$ *vs.* the traffic intensity at the first-level service center in a single-class camp-on system with $\lambda_1 = \lambda_{2r}$ and $n, k = 1, 2, 3$.

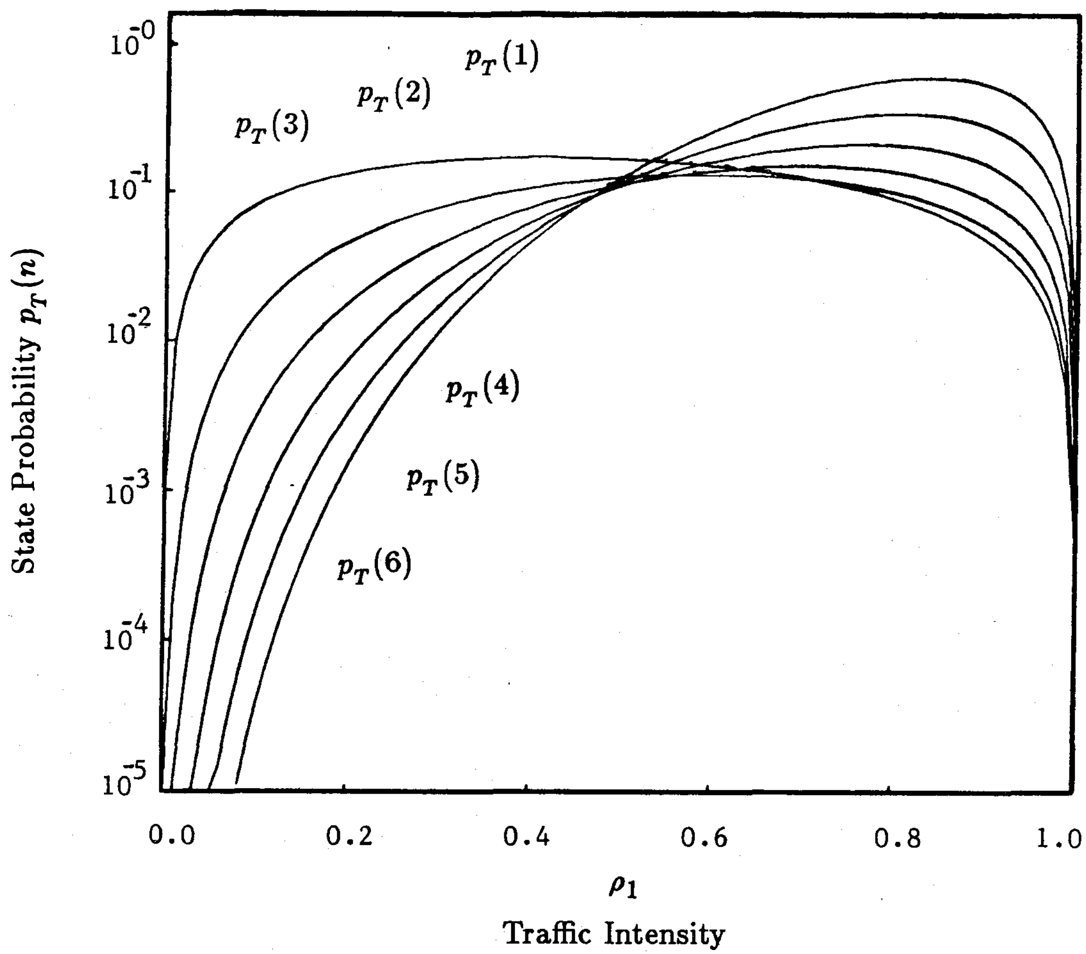


Figure 12: Probability distribution of the total number of customers $p_T(n)$ *vs.* the traffic intensity at the first-level service center in a single-class camp-on system with $\lambda_1 = \lambda_{2r}$ and $n = 1, \cdots, 6$.

CHAPTER IV:

MULTILEVEL CAMP-ON SYSTEMS

In Chapter IV, we extend the basic results found for two-level camp-on systems to systems with multiple levels of queues. We will concentrate mainly on the equilibrium distribution of queues in a non-reneging environment. Instead of trying to provide a description of the system that accounts for all queues in progress within the service hierarchy at a given point in time, we develop an alternative approach that looks only at some key subsystems of relevance. This approach provides a sizable reduction in computation. This is because it would require something in the neighborhood of an N^n-dimensional vector just to represent all the possible states in a camp-on system with n queueing stages and common storage capacity N for every subsystem. In Section 1, we first provide a detailed description of this queueing camp-on system. Then we propose an alternative reduced-state representation and derive an equivalent set of equilibrium balance equations for the reduced-state model. In Section 2, we find closed-form solutions for the reduced-state probability distribution. Both infinite and finite-state systems, that is, those with limited storage capacity, are considered.

IV.1. System Description

In the multilevel camp-on model, every queued customer is associated with two waiting lines, no matter the queueing stage. One is the waiting line where the customer is currently enrolled as a simple user. The other is the waiting line where the customer is being perceived as a service center for others. The main difference with respect to the two-level camp-on model studied in Chapters II and III is that these two queues are allowed to exist without any concern as to the total number of stages customers have to go through in order finally to be serviced by their intended service center. In the two-level camp-on model on the other hand, we allowed only two queueing stages, i.e., the first-level and the second-level queues. This multilevel queueing system is depicted in Figure 13.

Conventional computer services, such as job scheduling in batch processing, file printing, etc., or telephone services provided through PBX's are typical examples of one-level camp-on systems. However, current data communication services for job scheduling and management with a growing emphasis on networking and systems integration already display, at the very least, *two* levels of queueing. One example is shared peripherals in a network environment. Here, the first-level queue is located at the peripheral device itself; this queue consists of all the work requested by the different network nodes that can feed it. The second queueing stage includes the peripheral service requests made by individual users at the node level; the requests must first go to the local service node, which will later relay the request to the peripheral, when transmission facilities become available. Another simple example of a two-level camp-on system comprises enhanced telephone services such as call-waiting and teleconferencing, where three or more simultaneous, but not intermixed, calls can be placed at any given time. In the call camping example, the

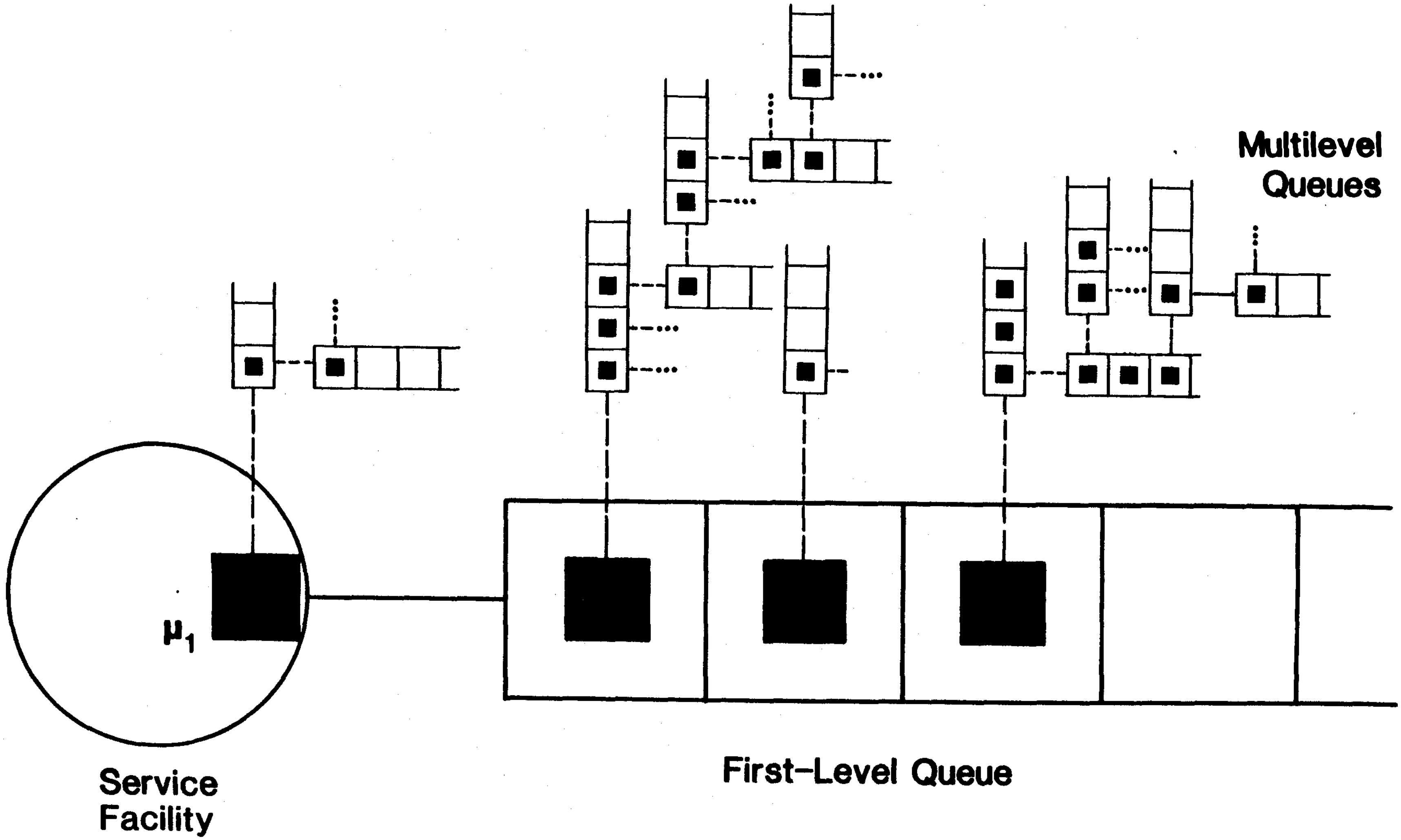


Figure 13: A multilevel camp-on system.

first queueing stage is composed of the calls being received by the called party. The second queueing stage is composed of the calls being received by the calling party. The teleconferencing example is just an extension of call camping with multiple calling and called parties.

Finally, with the advent of concurrent and parallel processing and of networks of LANs and other highly hierarchical systems, as well as with the increasing integration of voice, data and video, applications for multilevel queueing systems are certainly on the rise. For instance, we can envision LANs connected through another LAN in order to share very expensive peripherals or a common database. Concurrent processes will have to queue and wait if they are to exchange data with other processes already engaged in a data exchange. The motivation for this work is to better understand queueing in these hierarchical systems.

Many highly complex queueing systems can also be thought of as n-level "genealogical trees." For example, we can think of a data network built on a tree topology, where customers at the bottom of the network have to go through the different node levels to get to the central node. The number of levels, n, merely indicates the number of generations in the family tree. Hence, it is possible to establish a one-to-one correspondence between the size of a particular subsystem in the camp-on service hierarchy and the size of the branch in its tree representation in which the particular subsystem of interest is sitting. From this tree viewpoint, the size of subsystem x in the multilevel camp-on model is equivalent to the size of the progeny of descendant x in the family genealogy. Figure 13 shows a pictorial representation of a mutilevel camp-on system in the form of a tree.

The set of assumptions for the multilevel queueing model is basically an extension of the earlier assumptions for the two-level camp-on model. The customer arrival and departure processes are statistically independent. Customers come from an infinite pool of subscribers, and every queueing system in the camp-on model provides unlimited storage capacity. The arrival process at each subsystem is regarded as Poisson with mean arrival rate λ_i for any queueing subsystem at the i^{th} queueing stage of the camp-on model. The service completion process at the service center, the one serving first-level customers, is assumed memoryless, i.e., a negative exponential distribution with a mean service time μ_i. No renegings (defections) will be permitted from any of the system queues, and customers will be served with a FCFS policy. As before, these assumptions imply a decoupling among customers at different stages in the multilevel camp-on system, and even among customers located at the same stage, as long as they traverse distinct service paths in the service hierarchy. This is a direct consequence of the independence of the arrival and departure processes.

Finding the joint probability distribution of queue lengths for all possible queue configurations in the multilevel camp-on model, nonetheless, amounts to giving the family size distribution for every single descendant in the family tree. This is not a trivial task even if we have only a finite storage capacity of N customers per subsystem. If the number of queueing stages is n, it will require a vector of dimension on the order of N^n just to account for all the state vectors in such a sample space.

Because of the decoupling between the customer arrival processes in different paths of the service hierarchy along the queueing stages and the FCFS service strategy, we observe that if we pick any queued customer at random, he has only to be concerned with the amount of queueing he has to do before reaching his intended service facility. All other service paths outside its own do not affect his service expectations. From the family tree approach, this decoupling effect has a simple interpretation: to get to the family root, one need only look at the direct ancestors of a given generation, not to all the members of the family. Thus, we propose to choose as state representation a vector that includes the sizes of all subsystems in the path from the first-level service center to the last subsystem at the $(n+1)^{st}$ queueing stage. This reduced representation of the system's state is equivalent to a "depth-first" search along the family tree. In this, we search for the sizes of all subsystems along a root-to-leaf path of the service hierarchy. This interpretation of the multilevel camp-on system states is equivalent to giving the family sizes for all the ancestors of a family member up to the family root, starting from a member of the current generation.

Suppose the size of the first-level system, counting both queued and in-service customers, is i customers. Then the size of the first generation is i descendants. The distribution for the size of this first generation is well known to be $\pi(i) = (1-\rho_1)\rho_1^i$ with $\rho_1 = \lambda_1/\mu_1$, the probability distribution for the size of an M/M/1 queueing system, as in Equation (3.34). Similarly, let j_1, an index, denote one of the children of this first generation and let k_1 denote the size of his progeny. Then $p_i(j_1; k_1)$ is the joint distribution for the size of the family's first generation and the size of the first generation of descendant j_1. From our two-level camp-on system standpoint, $p_i(j_1; k_1)$ also represents the marginal distribution for the j_1^{st} second-level system in the two-level camp-on model. Therefore, using our earlier notation developed for two-level camp-on systems: $p_i(j_1; k_1) = p_i(k_{j_1} = k_1)$.

Let $j_1, \cdots, j_n$ and $k_1, \cdots, k_n$ be two sets of non-negative integers. Assume that we start with an i-member family, that is, a queueing system with i customers in the first-level queue. Extending the preceding reasoning to the next generation, that is, the next queueing stage in the multilevel model, let j_1 be a "child" from the family's first generation, that is, the customer at position j_1 in the first-level queue, and let k_1 denote the size of his, j_1's, progeny. In general, let j_i be a child from the first generation of j_{i-1}, the customer in position j_i at the i^{th} queueing stage, and let k_i denote the size of his progeny. Then the $(2n+1)$-tuple $(i, \mathbf{J}_n, \mathbf{K}_n)$ provides all the information required to know how much work is accumulated in any path of the service hierarchy, from the first-level service center to last subsystem at the $(n+1)^{st}$ queueing stage. But, because of the memoryless state transitions, this reduced-state representation suffices as well as a complete description of the current state, in equilibrium, of the multilevel camp-on system. In fact, these $(2n+1)$-tuples can be interpreted as the states of a homogeneous, irreducible and aperiodic Markov chain giving the "depth-first" search of the sizes of the subsystem's queues along a path of $n+1$ queueing stages in the multilevel camp-on model. Clearly, unless the last subsystem in a given service path is empty, $k_i \geq j_{i-1}$, or else we would be talking about $(m+1)$ queueing stages rather than $(n+1)$ queueing stages $(m < n)$. This representation as $(n+1)$-level search is illustrated in Figure 14.

In the next section, we derive an equivalent equilibrium balance equation for this reduced-state space. We find closed-form solutions for the equilibrium joint probability distribution of the queue sizes at each of the queueing stages.

IV.2. Stationary Multilevel Camp-on Model

Suppose that the current state in the multilevel camp-on system is given by the vector $\mathbf{x}_n = (i, \mathbf{J}_n; \mathbf{K}_n)$ and let $\mathbf{y}_m = (l, \mathbf{J}'_m; \mathbf{K}'_m)$ be any other permissible state in this queueing system. Also, let $p_M(\mathbf{x}_n)$ denote the joint probability distribution for

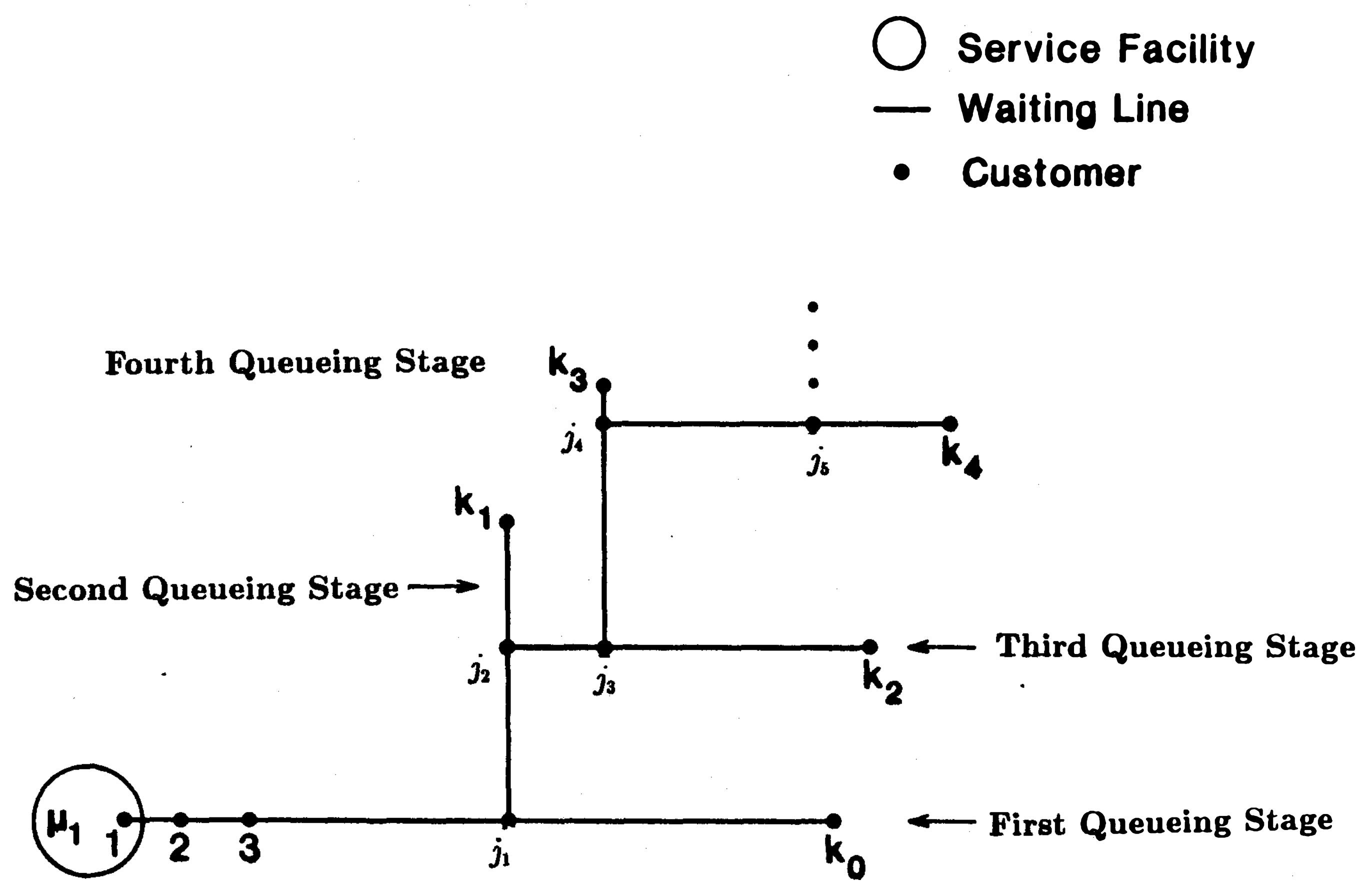


Figure 14: "Depth-first" search of queue sizes along a service path in a multilevel camp-on system.

the sizes of the subsystems along a service path with $(n+1)$ levels in the camp-on service hierarchy. If an equilibrium distribution exists for this queueing system, it must satisfy the global balance equation in (2.13):

$$p_M(\mathbf{x}_n) \sum_{\mathbf{y}_m \in \Omega} q(\mathbf{x}_n; \mathbf{y}_m) = \sum_{\mathbf{y}_m \in \Omega} p_M(\mathbf{y}_m) q(\mathbf{y}_m; \mathbf{x}_n), \tag{4.1}$$

where $q(\mathbf{y}_m; \mathbf{x}_n)$ gives the transition rate from state $\mathbf{y}_m$ into state $\mathbf{x}_n$ and Ω is the set of all permissible states.

In order to make the notation simpler, from now on we will consider $\mathbf{K}_n$ as an $(n+1)$-dimensional vector of the form $\mathbf{K}_n = (k_0, k_1, \cdots, k_n)$. Here, the k_i's $(0 \le i \le n)$ are non-negative integers with k_l designating the size of the first generation of the customer in position j_l at the level-l subsystem, and $k_0 = i$ designating the size of the first-level subsystem.

Looking at the state-transition rates in the global balance equation, one recognizes $\sum_{\mathbf{y}_m \in \Omega} q(\mathbf{x}_n; \mathbf{y}_m)$ as the total rate of departure from state $\mathbf{x}_n$. This rate is given by

$$\sum_{\mathbf{y}_m \in \Omega} q(\mathbf{x}_n; \mathbf{y}_m) = \mu_1 + \sum_{i=0}^{n} \lambda_i, \tag{4.2}$$

the sum of the arrival rates at each of the subsystems in state $\mathbf{x}_n$, plus the service rate at the first-level subsystem.

If the system enters state $\mathbf{x}_n$ because of a customer arrival, this state transition could have taken place only when the initial state $\mathbf{y}_m$ is a neighbor of state $\mathbf{x}_n$; i.e., state $\mathbf{y}_m$ has the same composition as state $\mathbf{x}_n$ except that one of its subsystem's queue is shorter by one customer. The customer arrival processes guarantee that no more than one single arrival can occur at any time. There are two distinct types of neighboring states with this property:

i) The subsystem at the $(i+1)^{st}$ queueing stage in state $\mathbf{y}_m$ is shorter by one customer than the same subsystem for the next state $\mathbf{x}_n$. However, the index j_{i+1} that identifies the customer in the level-$(i+1)$ queue and the service path towards the next queueing stage does not designate this last customer joining the camp-on system, for we would have $j_{i+1} = k_i$ and $k_{i+1} = 0$, so we would be back to $i+1$ queueing stages only. Here, $\mathbf{y}_m = \mathbf{x}_n^{-i}$ $(0 \leq i \leq n)$, where $\mathbf{x}_n^{-i}$ stands for the neighboring state with one fewer customer at the $(i+1)^{st}$ queueing stage:

$$\mathbf{x}_n^{-i} = (\mathbf{J}_n; \mathbf{K}_n^{-i}),$$
$$\mathbf{K}_n^{-i} = (k_0, \cdots, k_i - 1, \cdots, k_n).$$

This state transition occurs with rate λ_{i+1}, the arrival rate at the subsystem in the $(i+1)^{st}$ queueing stage. Hence,

$$q(\mathbf{x}_n^{-i}; \mathbf{x}_n) = \lambda_{i+1}\, \chi_{(j_{i+1} < k_i)}. \tag{4.3}$$

ii) The subsystem at the n^{th} queueing stage in state $\mathbf{y}_m$ is shorter by one customer than the same subsystem for the next state $\mathbf{x}_n$, but the index j_n identifying the customer at position j_n in the level-n subsystem does correspond to this last customer joining the camp-on system. Then $\mathbf{y}_m = \mathbf{x}_{n-1}^{-n+1}$, since the upper-level queue associated with this customer must be empty (he just joined the camp-on system). In this case, the state-transition rate is given by

$$q(\mathbf{x}_{n-1}^{-n+1}; \mathbf{x}_n) = \lambda_n\, \chi_{(j_n = k_{n-1},\ k_n = 0)}. \tag{4.4}$$

Notice that we have ruled out the possibility that $k_i = 0$ for $0 \leq i < n$. This condition will demand that the number of queueing stages along this service path is less than n, not n, as we originally demanded for the reduced-state representation $\mathbf{x}_n$.

There is only one single type of non-neighboring state transition allowed in multilevel camp-on systems, and this corresponds to the departure of a first-level customer by virtue of service completion. In this case, the previous system state must have been $\mathbf{y}_m = (\mathbf{J}_n^{+1}; \mathbf{K}_n^{+0})$, where

$$\begin{aligned}
\mathbf{J}_n^{+1} &= (j_1 + 1, j_2, \cdots, j_n), \\
\mathbf{K}_n^{+0} &= (k_0 + 1, k_1, \cdots, k_n);
\end{aligned}$$

i.e., the customer at position j_1 has been shifted one place closer to his service facility. This state transition occurs with rate μ_1, the service rate for customers in the first-level subsystem; hence,

$$q(\mathbf{y}_m; \mathbf{x}_n) = \mu_1. \tag{4.5}$$

Based on Equation (4.1) and the state-transition rates from Equations (4.2) to (4.5), we can write down the following global balance equation for the sizes of the subsystems in a given path of the service hierarchy of this reduced-state multilevel camp-on model:

$$\begin{aligned}
\left[\mu_1 + \sum_{i=0}^{n} \lambda_{i+1}\right] p_M(\mathbf{J}_n; \mathbf{K}_n) = & \sum_{i=0}^{n} \lambda_{i+1}\, \chi_{(j_{i+1} < k_i)}\, p_M(\mathbf{J}_n; \mathbf{K}_n^{-i}) \\
& + \lambda_n\, \chi_{(j_n = k_{n_1},\ k_n = 0)}\, p_M(\mathbf{J}_{n-1}; \mathbf{K}_{n-1}^{-n+1}) \qquad (4.6) \\
& + \mu_1\, p_M(\mathbf{J}_n^{+1}; \mathbf{K}_n^{+0}).
\end{aligned}$$

The next theorem is an extension of Theorem 2 to multilevel camp-on systems:

Theorem 4: Let $p_M(\mathbf{J}_n;\mathbf{K}_n)$ denote the equilibrium joint probability distribution of queue sizes for the subsystems along a service path with $(n+1)$ levels in a multilevel camp-on system. Then

$$p_M(\mathbf{J}_n;\mathbf{K}_n) = \prod_{i=0}^{n} \Psi_i(\mathbf{J}_i;\mathbf{K}_i), \tag{4.7}$$

where the function $\Psi_i(\mathbf{J}_i;\mathbf{K}_i)$ is of the form

$$\Psi_i(\mathbf{J}_i;\mathbf{K}_i) = \begin{cases} \pi(k_0), & \text{if } i=0; \\ 1, & \text{if } k_i = 0,\ 0<i<n; \\ \displaystyle\sum_{l_i=0}^{M_i} \binom{M_i + k_i - j_{i+1} - l_i}{k_i - j_{i+1}} \times \binom{j_{i+1} - 1 + l_i}{l_i} \beta_i^{l_i} \alpha_i^{j_{i+1}}, & \text{if } k_i > 0,\ 0<i<n; \\ \displaystyle\binom{M_n + k_n}{k_n} \beta_n^{1+k_0-j_1-\sum_{i=1}^{n-1} l_i} \times \prod_{i=1}^{n} \alpha_{in}^{k_i - j_{i+1}}, & \text{if } 0<i=n. \end{cases} \tag{4.8}$$

Here, the parameters M_i, β_i, α_i and α_{in} stand for:

$$M_i = \sum_{m=1}^{i} [k_{m-1} - j_m - l_m], \qquad k_i \geq j_{i+1}, \tag{4.9}$$

$$\beta_i = \frac{\mu}{\mu_1 + \sum_{k=2}^{i+1} \lambda_k}, \tag{4.10}$$

$$\alpha_i = \frac{\lambda_{i+1}}{\mu_1 + \sum_{k=2}^{i+1} \lambda_k}, \tag{4.11}$$

$$\alpha_{in} = \frac{\lambda_{i+1}}{\mu_1 + \sum_{k=2}^{n+1} \lambda_k}. \tag{4.12}$$

Of course, in this equation it is implicit that $k_i \neq 0$ $(0 \leq i < n)$; otherwise, it would then be a service path with $(i+1)$ levels instead of a service path with $(n+1)$ levels in the service hierarchy of this multilevel camp-on system.

Notice that β_i represents the average departure rate from the $(i+1)^{st}$ queueing stage into the i^{th} queueing stage, while α_i and α_{in} are the average arrival rates into the level-$(i+1)$ subsystem when the numbers of stages in the service path is i and n, respectively. Thus, $\alpha_i = \alpha_{ii}$.

Accordingly, from Equations (4.7) and (4.8), we can verify that for a single-level camp-on system,

$$\begin{aligned} p(\mathbf{J}_0; \mathbf{K}_0) &= \pi(k_0) \\ &= \pi(i), \end{aligned}$$

which is the marginal distribution for the first-level size, an M/M/1 queueing system, as it must be. Similarly, for $n = 1$ we observe that

$$\begin{aligned} p(\mathbf{J}_1; \mathbf{K}_1) &= \Psi_0(\mathbf{x}_0)\ \Psi_1(\mathbf{x}_1) \\ &= \binom{i - j_1 + k_1}{k_1} \beta_1^{i-j_1+1} \pi(i). \end{aligned}$$

This is again the marginal distribution for the j_1^{st} second-level system when the size of the first-level system is i customers, a two-level camp-on system as in (3.36). These provide a further check of the results obtained in Chapter III.

As for the case of a finite-state multilevel camp-on system, a result equivalent to the one presented in Theorem 3 for finite-state two-level camp-on systems can also be derived. Let N_i $(1 \le i \le n+1)$ designate the storage capacity of the subsystem located at the i^{th} queueing stage of the service hierarchy and let $l_1, \cdots, l_s$ designate those subsystems with $k_{i+1} = N_i$. We then have the following theorem relating the distribution with the storage constraint to the distribution obtained in Theorem 4 without storage constraint.

Theorem 5 : Let $p^*_M(\mathbf{K}_n;\mathbf{R}_n)$ denote the equilibrium joint probability distribution of queue sizes along a service path with $(n+1)$ levels in a multilevel camp-on system with storage capacity of N_i customers at the subsystem in the i^{th} queueing stage. Let $(l_i,\cdots,l_s)$ denote the set of subsystems with $k_l = N_{l-1}$. Then

$$p^*_M(\mathbf{J}_n;\mathbf{K}_n) \;=\; \begin{cases} p_M(\mathbf{x}_n), & \text{if } k_i < N_{i-1}, \\ \displaystyle\sum_{k_{l_1}=N_{l_1-1}}^{\infty} \cdots \sum_{k_{l_s}=N_{l_s-1}}^{\infty} p_M(\mathbf{J}_n;\mathbf{K}_n), & \text{if } k_i = N_{i-1}, \end{cases} \tag{4.15}$$

and

$$p_M(\mathbf{x}_0) \;=\; p(\mathbf{x}_0), \qquad (1 \le s \le n \le N),$$

where $p_M(\mathbf{J}_n;\mathbf{R}_n)$ is the state probability distribution for the same camp-on system without storage constraints.

The proof of Theorem 5 follows the same lines as the one for Theorem 3, except that we work with the equilibrium global balance equation for the finite-state multilevel camp-on system:

$$\begin{aligned}\left[\mu_1 + \sum_{i=0}^{n} \lambda_{i+1}\, \chi_{(k_i<N_{i+1})}\right] p_M(\mathbf{J}_n;\mathbf{K}_n) \;=\; & \sum_{i=0}^{n} \lambda_{i+1}\, \chi_{(j_{i+1}<k_i)}\, p_M(\mathbf{J}_n;\mathbf{K}_n^{-i}) \\ & + \lambda_n\, \chi_{(j_n=k_{n_1},\, k_n=0)}\, p_M(\mathbf{J}_{n-1};\mathbf{K}_{n-1}^{-n+1}) \\ & + \mu_1\, p_M(\mathbf{J}_n^{+1};\mathbf{K}_n^{+0}).\end{aligned} \tag{4.16}$$

Notice that the interpretation of Theorem 5 is compatible to Theorem 3 in the sense that the boundary states behave as absorbing states for the out-of-space states. They absorb the probabilities of all those states in the unconstrained model left out because of the finite buffering space.

IV.3. Multilevel Systems: Camp-On and Stability

As in the two-level camp-on system case, the ergodicity of the multilevel camp-on system is tied to the ergodicity of the first-level subsystem. However, every departing first-level customer leaves a camp-on system with $n + 1$ queueing stages and forms its own camp-on system with n queueing stages, which is a subset of the initial multilevel system. This last statement implies that customers are promoted from the $(i + 1)^{st}$ stage to the i^{th} stage, until they eventually reach their intended service facility. Since all the queueing subsystems in the service hierarchy are served with a FCFS discipline, the ergodicity of every subsystem as an isolated queueing system must be required. Otherwise, we cannot guarantee stability of every single subsystem after departing from the n-level camp-on model. Thus, most of the comments made in Section 1, Chapter III, about the stability behavior of two-level camp-on systems carry over to the multilevel model. The ergodicity condition then boils down to the condition in Equation (3.9):

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\lambda_k}{\mu_k} < \infty,$$

where we use λ_k instead of γ_k. In particular, we will be required to have $\lambda_n < \mu_n$ for a system with infinite storage capacity in its queue. Here, μ_n is the mean service rate for the subsystem at the n^{th} queueing stage, once it is ready to serve its own waiting line.

One interpretation of the stability condition in multilevel camp-on systems is that upper-level subsystems are promoted to the next queueing level before they can grow out of bounds. This behavior repeats as the subsystem progress towards its intended service center in the service hierarchy and reaches the first queueing stage. At this point, the customers in its waiting line start service. From the server standpoint, this is an $M/M/1$ queueing system with a non-empty-queue initial state. We conclude that its behavior in the long run must be that of an $M/M/1$ queueing system.

CHAPTER V:

COMMUNICATION APPLICATIONS

In Chapter 5, we study the performance of the two-level camp-on model in two different communication applications. The first example corresponds to a queueing system where both first-level and second-level customers are served in a FCFS basis. This can be interpreted as an extension of the PBX concept to hierarchical queueing systems. Subscribers calling premises attended by this PBX-like facility are eventually transferred to their end-point service centers. Here, the service requests are processed in an orderly manner, after they have been sorted with respect to some pre-established service criterion. This type of service is typical of inquiry-based communication systems.

The second example corresponds to a queueing system with broadcast delivery service. For this system, incoming customers are likewise presorted with respect to the class of job they request from the service center. Here the first-level queue stands for the different types of jobs submitted to the system's facility, while the second-level queues stand for the numbers of requests for every single type of job. The service philosophy here is to service all those jobs of the same class simultaneously, as in a broadcast system. In a sense, second-level customers can be regarded as outstanding job requests. These are job orders already taken, but not yet processed, by the first-level service center.

In these examples, performance statistics such as waiting time distribution, mean waiting time, and blocking probability, when defined, are derived. In both cases, it is found that the global system performance does not depart too much from that of the underlying $M/M/1$ queueing system, showing the expected dominance of the first-level system on the global behavior of the camp-on model.

V.1. PBX-like Communication Services

When we refer to a PBX-like communication service within the scope of the camp-on model, we mean a two-level, or perhaps multilevel, queueing system, where all service centers are type-1 centers, i.e., the service discipline implemented for all the subsystems is FCFS, no matter what their queueing stages. Essentially, this model can be used to represent any communication system where customers enrolled in a waiting line, the first-level queue, are not precluded form taking job orders from other users, the second-level customers, and they will eventually proceed to service those orders, once their engagement with the initial service center is through. The motivation for this type of services stems from the natural extension of the FCFS strategy to hierarchical multilevel queueing systems and its connection to the camp on concept in telephony as previously mentioned in Chapter II.

This service model can be extended to many inquiry-oriented communication services, in particular, those where some sort of clearence or preprocessing has to be completed before proceeding with a new batch of service orders. For example, a network for credit card inquiries from convenience stores is an extreme case of the finite-storage two-level camp-on model, where the number of second-level classes stands for the the number of stores in the network and the customer handling capacity is usually $N_r = 1$ for all r. Here, the credit card company's database plays the role of the first-level service center. Stores honoring the company's card constitute the source of first-level "customers," while shoppers buying merchandise and paying them with credit cards act as the system's second-level customers. We can more generally envision the inquiries made by a node to the central database as the first queueing stage of the camp-on model. These inquiries may deal with the authorization of certain customers to access application programs or other services

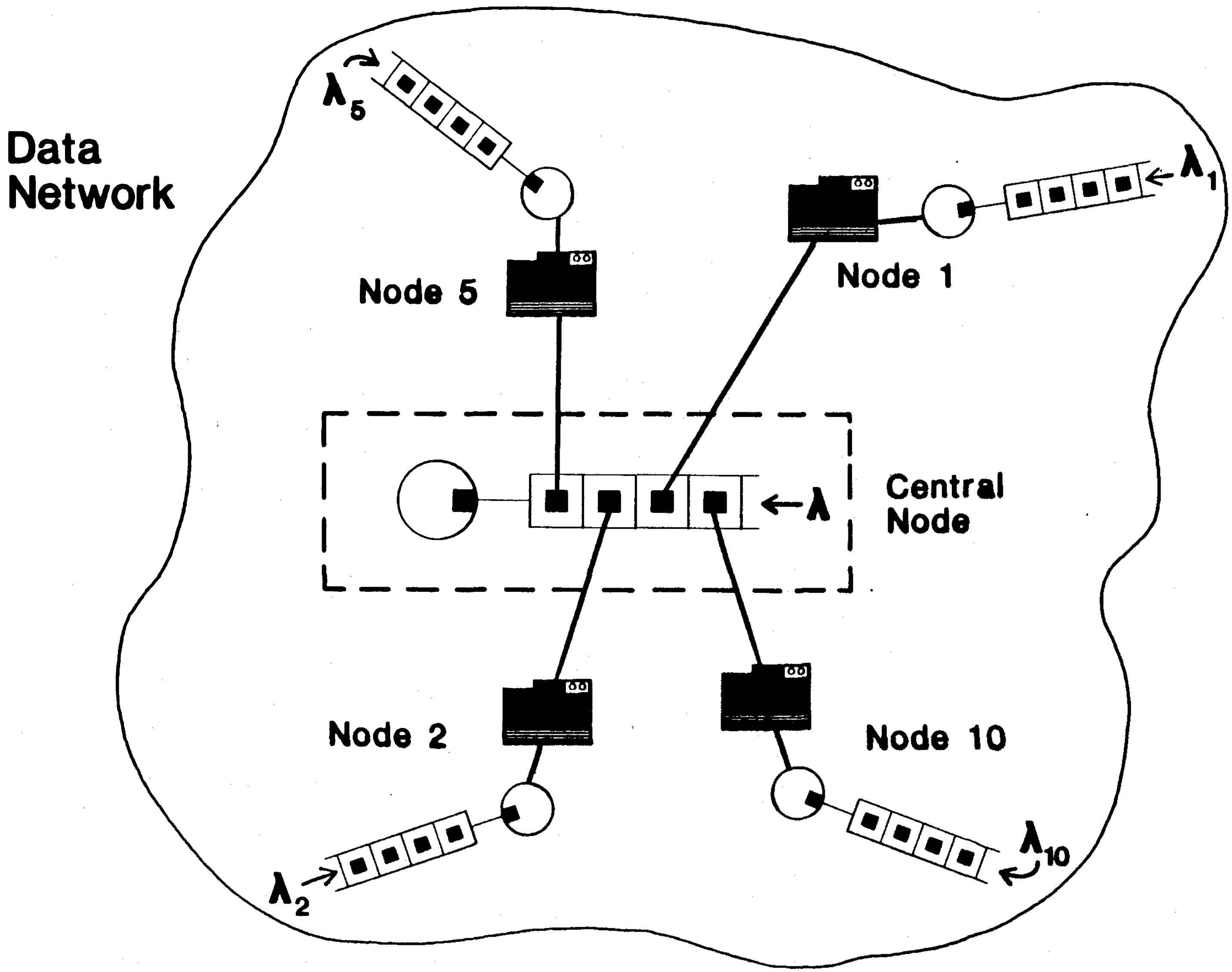


Figure 15: PBX-like communication services in a data network.

required by the node, e.g., maintenance information, special printing facilities, customer authentication, etc. Figure 15 depicts this PBX-like communication service in data network environment.

Typically, when we talk about the performance of a queueing system, we refer to statistics such as the amount of time a newly arrived customer must wait before starting his service period, either the waiting time density $W(t)$ or its cumulative distribution $W(>t)$. We may also be interested in the probability of being denied access to the queues because they are already full, i.e., the blocking probability. There are, of course, many other statistics of interest. For PBX-like communication services in infinite or finite storage-capacity systems, we are mainly concerned with the time customers spend waiting to be serviced, especially for those customers in the second-level stage. We will assume that the transmission time is neglegible compared to the queueing and processing time in the system.

1.1. Infinite Storage Systems

Let us first consider a non-reneging camp-on queueing system with infinite storage capacity. Blocking is not an issue, since the storage space is infinite. If we look at the two queueing stages of the camp-on model, we realize that the standard theory for $M/M/1$ queueing systems[15,23,24] tells what the waiting time density $W_1(t)$, the cumulative distribution $W_1(>t)$, or the mean waiting time probability $\overline{W}_1$ are for first-level customers when the newly arrived customer must wait:

$$W_1(t) = \mu_1\rho_1(1-\rho_1)e^{-\mu_1(1-\rho_1)t} \tag{5.1}$$

$$W_1(>t) = \rho_1 e^{-\mu_1(1-\rho_1)t} \tag{5.2}$$

$$\overline{W}_1 = \frac{\rho_1}{\mu_1(1-\rho_1)}. \tag{5.3}$$

Here μ_1 and λ_1 are the traffic parameters for the first-level queueing system; that is, μ_1 is the mean service rate at the first-level center and λ_1 the center's mean arrival rate.

We want to answer these same questions for those customers that must also wait at the second queueing stage. Let $W_2^n(j;t)$ denote the probability that an incoming class-r second-level customer arriving at the subsystem in position j of a size-n first-level system will have to wait for t units of time before its service period begins. If the j^{th} second-level system has k_j customers in his waiting line when the new customer arrives, the newly incoming customer will have a probability density function $b(j,k_j;t)$ for his waiting time. This accounts for the time period he will spend as a second-level customer, $b_2(k_j,t)$, plus the time he will spend as a first-level customer $b_1(j,t)$. The time spent in each of the queueing stages is given by an n-phase Erlang distribution[15,23], $E_n(t)$, because of the negative exponential distribution for the service time period in each of the queueing subsystems. That is,

$$E_n(t) \;=\; \frac{(\mu t)^{n-1}}{(n-1)!}\mu e^{-\mu t}, \tag{5.4}$$

where n is the effective queue length seen by the newly arrived customer at each of the queueing stages. With mean service rates for the first-level and second-level systems being μ_1 and μ_{2r}, we have

$$\begin{aligned} b(j,k_j;t) &= b_1(t) * b_2(t) \\ &= E_j^1(t) * E_{k_j}^r(t) \\ &= \frac{(\mu_1)^j(\mu_{2r_j})^{k_j}}{(j-1)!(k_j-1)!}\int_0^t \tau^{j-1}(t-\tau)^{k_j-1}d\tau. \end{aligned}$$

The service time distributions at each service center are, as usual, assumed statistically independent.

Let $p_n(k_{jr})$ denote the probability that the j^{th} second-level system has k_j class-r customers in its waiting line when the size of the first-level system is n customers. This is the marginal distribution for the size of the j^{th} subsystem found in Chapter III. Then $W_{2r}^n(j;t)$ will be given by the sum of the probabilities over all possible sizes for the j^{th} subsystem $p_n(k_{jr})$ times their corresponding waiting time distribution for the waiting time period for the next incoming user:

$$W_2^n(j;t) \;=\; \sum_{k_j=0}^{\infty} p_n(k_{jr})\; b(j,k_j;t). \tag{5.5}$$

We consider here the special case of a balanced camp-on system, i.e., a queueing system where all service centers present the same traffic handling capacity ($\mu_1 = \mu_2 = \mu$). In this case, we can greatly simplify the computation of the communication system's performance, since

$$\begin{aligned} b(j,k_j;t) &= E_j(t) * E_{k_j}(t) \\ &= E_{j+k_j}(t), \end{aligned}$$

a $(j+k_j)$-phase Erlang distribution in Equation (5.4). In this special case, the arriving second-level customer does not perceive the queueing system as composed of two queueing stages but rather as a single-server center with a waiting line of $j+k_j$ customers.

We can now compute $W_2^n(t)$, using the expression for the marginal distribution for the size of the j^{th} subsystem $p_n(k_{jr})$ derived in Chapter III, Equation (3.36):

$$p_n(k_{jr}) \;=\; \binom{n-j+k}{k} \beta_r^{n-j+1} \alpha_r^k \; \rho_1^n(1-\rho_1). \tag{5.6}$$

Considering the above expression for $p_n(k_{jr})$ together with the waiting time function for the new second-level customers, one can write down the complete expression for the waiting time distribution of the class-r second-level customers in the j^{th} subsystem, when the first-level queue size is n customers:

$$W_{2r}^{n}(j;t) \;=\; \sum_{k=0}^{\infty} \binom{n-j+k_j}{k_j} \alpha^{n-j+1} \beta^{k} (1-\rho_1) \rho_1^{n} \frac{(\mu t)^{j+k-1}}{(j+k-1)!} \mu e^{-\mu t}, \qquad (5.7)$$

where

$$\begin{aligned} \alpha_r &= (1+\rho_{2r})^{-1}\rho_{2r}, \\ \beta_r &= (1+\rho_{2r})^{-1}, \qquad (5.8) \\ \rho_1 &= \frac{\lambda_1}{\mu}, \quad \rho_{2r} = \frac{\lambda_{2r}}{\mu}. \end{aligned}$$

These four parameters correspond respectively to the average arrival rate to a second-level subsystem, the average departure rate from the second-level stage into the first-level stage and the traffic intensities to the first-level system , and to the class-r second-level system.

The distribution for the waiting time period, $W_{2r}(t)$, for a class-r second-level customer is just the weighed contribution of his waiting time period at each of the subsystems where the newly arrived customer may have to camp on. We see that $W_{2r}(t)$ is obtained as a sum over all first-level queue sizes and all queue sizes and locations for the second-level subsystem. That is, we sum the waiting time probability for second-level customers at the j^{th} subsystem, $W_{2r}^{n}(j;t)$, times the probability that the new incoming customer arrives to his subsystem when it is located at the j position with respect to the size n first-level queue. We will now find these terms.

The probability that an incoming second-level customer will request service when his second-level subsystem is in the j^{th} position and the first-level system size has n customers is

$$\text{Prob}\,[j^{th} \text{ subsystem}/\ n \text{ second-level systems}] \;=\; 1/n,$$

for second-level customers have no *a priori* knowledge of the state of the first-level system. Therefore,

$$W_{2r}(t) \;=\; \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \frac{W_{2r}^{n}(j;t)}{n}.$$

After an algebraic reduction of this expression for $W_{2r}(t)$ (see Appendix VIII), we find that the probability distribution for the waiting time period of a class-r second-level customers in a camp-on queueing system with PBX-like communication service and infinite storage capacity is

$$W_{2r}(t) \;=\; \mu(1-\rho_1)e^{-\mu t} \int_0^{\rho_1} \frac{\beta_r}{(\beta_r x - \alpha_r)(1-x)} \left[x e^{\mu x t} - \gamma_r(x) e^{\mu \gamma_r(x) t} \right] dx, \quad (5.9)$$

with

$$\gamma_r(x) \;=\; \frac{\alpha_r}{1-\beta_r x}.$$

This expression for the waiting time distribution of the second-level customers is not as convenient to use as the one for the first-level customers in Equation (5.1). Nonetheless, other interesting statistics can be derived from this result. Among the most important is the probability of a second-level customer's having to wait for more than a certain period of time t before his service period begins. This cumulative distribution is given by (see also Appendix VIII)

$$W_{r2}(>t) \;=\; \int_t^{\infty} W_{2r}(\tau) d\tau$$

$$= \; (1-\rho_1) \left[\int_0^{\rho_1} f_r(x) \frac{x}{1-x} e^{-\mu(1-x)t} dx \; - \right.$$

$$\left. \int^{\rho_1} \cdots \frac{\gamma_r(x)}{1-\gamma_r(x)} e^{-\mu(1-\gamma_r(x))t} dx \right], \quad (5.10)$$

where

$$f_r(x) \;=\; \frac{\beta_r}{(\beta_r x - \alpha_r)(1-x)}.$$

Even though neither $W_{2r}(t)$ nor $W_{2r}(>t)$ is easy to compute, we still can obtain them through standard numerical techniques. Figure 16 shows $W_{2r}(t)$ as a function of time t in a balanced non-reneging camp-on system with $\lambda_{2r} = \lambda_1$.

Another important design parameter in the camp-on system performance is the mean waiting time for a class-r second-level customer. We compute from Equation (5.9) that in a PBX-like environment, the mean waiting time for an incoming second-level user is

$$\begin{aligned}\overline{W}_{2r} &= \int_0^\infty \tau\, W_{2r}(\tau)\, d\tau \\ &= \frac{2-\rho_1}{2\beta_r\mu(1-\rho_1)}\rho_1. \end{aligned} \qquad (5.11)$$

For the derivation of this expression, refer to Appendix VIII.

If we are more interested in the overall system performance because of all the distinct classes of second-level customers, we can compute this mean wait of an arriving customer to the second queueing stage as the weighted contribution from the different classes, e.g.,

$$\overline{W}_2 \;=\; \sum_{r=1}^{R} \frac{\lambda_{2r}}{\hat{\lambda}_2}\, W_{2r}, \qquad (5.12)$$

where $\hat{\lambda}_2 = \sum_{r=1}^{r} \lambda_{2r}$, R is the number of second-level classes and W_{2r} refers to any of the distributions and probabilities in equations from (5.9) to (5.11).

Figure 17 shows plots of $\overline{W}_1$ and $\overline{W}_{2r}$ versus the traffic intensity at the first-level system in a balanced non-reneging camp-on system for the cases when $\lambda_1 = \lambda_{2r}$ and when $\rho_{2r} = 0.2,\ 0.4,\ 0.6,\ 0.8$. Notice that $\overline{W}_{2r}$ follows very closely the curve for $\overline{W}_1$, showing that the second-level systems behave rather like an $M/M/1$ queueing

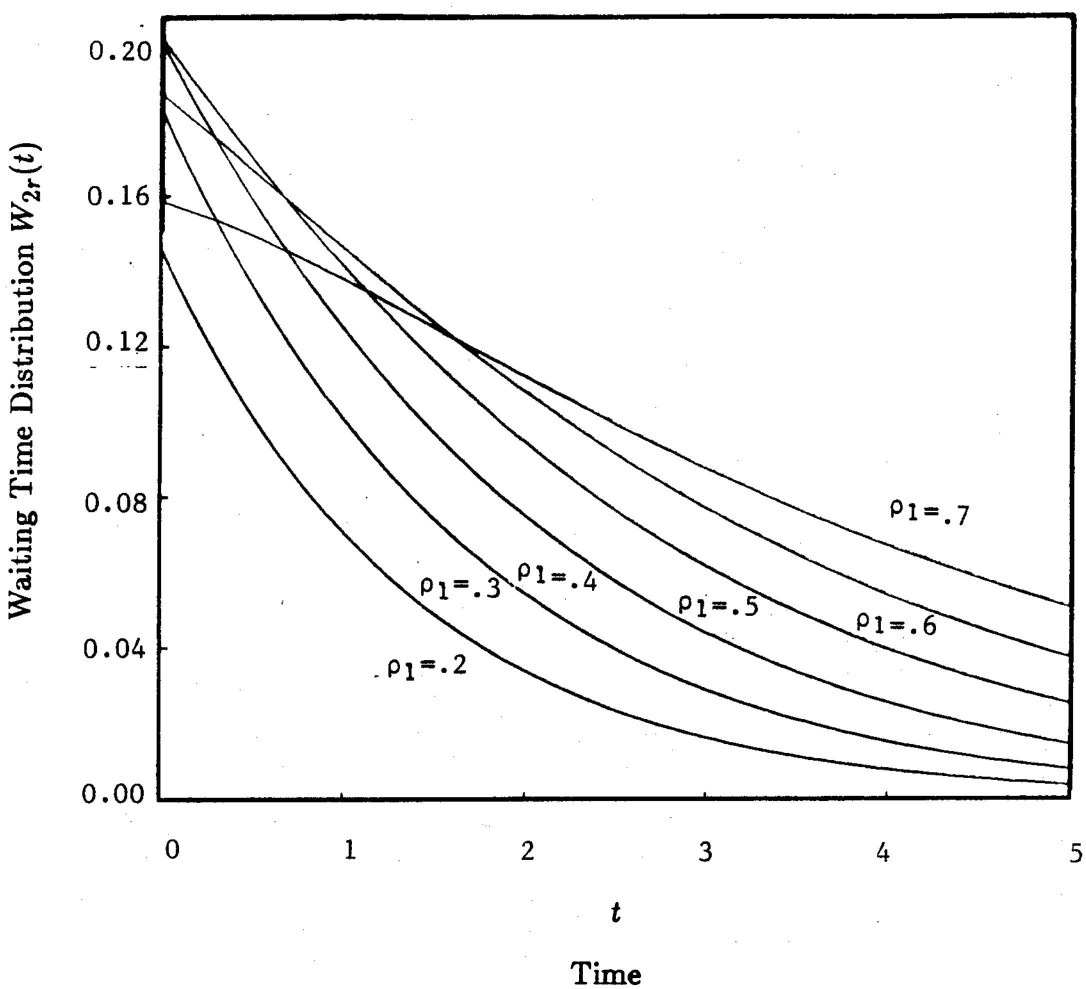


Figure 16: Waiting time distribution for a class-r second-level customer $W_{2r}(t)$ *vs.* time for different traffic intensities at the first-level service center in a balanced non-reneging camp-on system with $\lambda_{2r} = \lambda_1$.

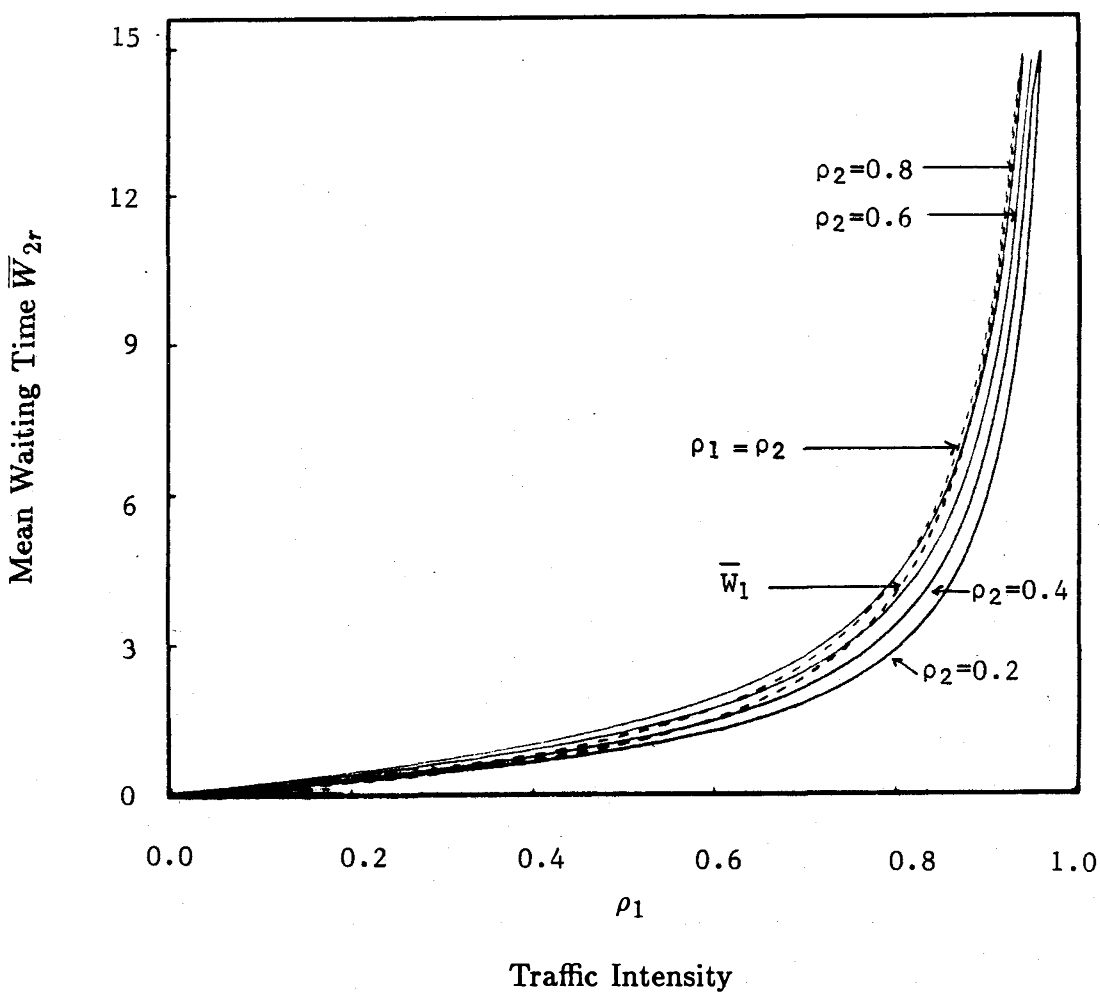


Figure 17: Mean waiting time for class-r second-level customers $\overline{W}_{2r}$ and for first-level customers $\overline{W}_1$ *vs.* the traffic intensity at the first-level service center in a balanced non-reneging camp-on system for the cases when $\lambda_{2r} = \lambda_1$ and when $\rho_{2r} = 0.2,\ 0.4,\ 0.6,\ 0.8$.

system despite the initial delay at the first-level queue. This is a reflection of the fact that the second-level systems move into the first-level stage before their queues can grow very much. Hence, the second-level systems never become a bottleneck in the camp-on model. The result also shows that we need not pay too large a penalty for introducing a second-level of queueing, contrary to what may have been our first impression.

1.2. Finite Storage Systems

So far we have been dealing with non-blocking queueing models. However, most queueing systems encountered in real life applications will have only a finite amount of memory space to handle the incoming traffic. Thus, we must extend the previous results to limited-storage systems. Recalling our notation from Section 3.1, Chapter III, for camp-on systems with finite storage capacity, let N be the maximum storage capacity for the first-level system, including both in-service and queued customers, and let N_{r_j} $(1 \leq j \leq N)$ be the storage capacity for the subsystem located in position j of the second-level stage associated with class-r_j customers. Following the same reasoning as in the above derivation for infinite-capacity systems, we see the waiting time distribution for a newly arrived class-r second-level customer to be of the form

$$W_{2r}(t) \;=\; \sum_{n=1}^{N}\sum_{j=1}^{n}\sum_{k_j=0}^{N_r} \frac{E_{j+k_j}(t)}{n} p_n^*(k_{jr}). \tag{5.13}$$

Here, $p_n^*(k_{jr})$ stands for the marginal distribution for the size of the j^{th} subsystem at the second-level stage in the finite-storage case. Similarly, the mean waiting time for this size-constrained system will be given by

$$\overline{W}_{2r} \;=\; \sum_{n=0}^{N}\sum_{j=1}^{n} \frac{(j+k_j)}{\mu n} \sum_{k_j=0}^{N_r} p_n^*(k_{jr}), \tag{5.14}$$

where we have assumed again the conditions of a balanced non-reneging camp-on system ($\mu_1 = \mu_{2r} = \mu$.)

From the analysis of the occupancy problem in camp-on systems with finite storage capacity, we can also recall that the probability that an incoming class-r customer will find his second-level system in position j of a size n first-level system when its waiting line is already full is

$$p_n^*(N_{r_j})\Big|_{r_j=r} = \sum_{k_j=N_r}^{\infty} p_n(k_{jr}) = \pi(n) - \sum_{k_j=0}^{N_r-1} p_n(k_{jr}). \quad (5.15)$$

Here as before, $\pi(n)$ is the equilibrium probability distribution for the size of the first-level system and $p_n(k_{jr})$ is the marginal distribution for the size of j^{th} subsystem with respect to a size-n first-level service center in the infinite storage case, as in Equation (5.6).

Also, notice that the total arrival rate to the system increases as the size of the first-level queue increases because of the new service centers' being incorporated into the queue. Thus, this PBX-like system has the structure of a class-1 camp-on system as described in Section 3, Chapter II. Therefore, from Equation (3.12), it follows that

$$\pi(n) = \rho_1^n \, p_0$$

with

$$p_0 = \left[\frac{1-\rho_1^{N+1}}{1-\rho_1}\right]^{-1}.$$

From Equations (5.15) and (5.16), we now find that the waiting time distribution and mean waiting time for a class-r second-level customer in the finite-storage camp-on model given in Equations (5.13) and (5.14) reduce to

$$\begin{aligned} W_{2r}(t) &= p_0\mu \sum_{n=1}^{N} \sum_{j=1}^{n} \sum_{k=0}^{N_r} \frac{\rho_1^n}{n} \binom{n-j+k}{k} \beta_r^{n-j+1} \alpha_r^k \frac{(\mu t)^{j+k-1}}{(j+k-1)!} + \\ & p_0\mu \sum_{n=1}^{N} \sum_{j=1}^{n} \frac{\rho_1^n}{n} \left[1 - \sum_{k=0}^{N_r} \binom{n-j+k}{k} \beta_r^{n-j+1} \alpha_r^k \right] \frac{(\mu t)^{j+N_r-1}}{(j+N_r-1)!}, \quad (5.16) \end{aligned}$$

$$\begin{aligned} \overline{W}_{2r} &= p_0 \sum_{n=0}^{N} \sum_{j=1}^{n} \sum_{k=0}^{N_r} \frac{\rho_1^n}{n\mu} (j+k) \binom{n-j+k}{k} \beta_r^{n-j+1} \alpha_r^k + \\ & p_0 \sum_{n=0}^{N} \sum_{j=1}^{n} \frac{\rho_1^n}{n\mu} (j+N_r) \left[1 - \sum_{k=0}^{N_r} \binom{n-j+k}{k} \beta_r^{n-j+1} \alpha_r^k \right]. \quad (5.17) \end{aligned}$$

From this equation we can obtain similar plots as before of $W_{2r}(t)$ and $\overline{W}_{2r}$ for a finite storage class-r camp-on system. The top and bottom lines in the expressions for $W_{2r}(t)$ and $\overline{W}_{2r}$ represent respectively the contribution to the waiting time due to a non-full and a full class-r second-level system. $\overline{W}_{2r}$ and $\overline{W}_1$ are plotted in Figure 18.

Another very important parameter for blocking systems is, of course, its blocking probability, since a customer may face a reasonably small waiting time once he has been admitted into the system but there is also a small probability of his being accepted into his second-level system.

An incoming second-level customer will be blocked, and so cleared from the camp-on system, whenever the second-level subsystem from which the incoming customer is demanding service has no further storage space to allocate new service requests. Let B_{2r} denote the probability that an incoming class-r second-level customer will be blocked and cleared from the second-level stage. This is given by the sum of the probabilities that the newly arrived class-r customer finds his waiting line in the second-level stage already full, averaged over all possible positions of this subsystem with respect to the first-level service center:

$$B_{2r} = \sum_{j=1}^{N} \sum_{n=j}^{N} \frac{p_n^*(N_{r_j})}{n}\bigg|_{r_j=r}. \tag{5.18}$$

Using the relation in Equation (5.13) for the marginal distribution for j^{th} second-level system in a size n first-level system, we get

$$B_{2r} = p_0 \sum_{j=1}^{N} \sum_{n=j}^{N} \frac{\rho_1^n}{n} \left[1 - \sum_{k=0}^{N_r-1} \binom{n-j+k}{k} \beta_r^{n-j+1} \alpha_r^k \right], \tag{5.19}$$

which is compatible with the interpretation of the bottom line of Equations (5.16) and (5.17).

Again, if we are concerned about the overall system performance, we can go back to Equation (5.12) and replace W_{2r} by any of the above expressions for the finite-storage system in Equations (5.16), (5.17) and (5.19) in order to to obtain the weighted contributions from the different classes of customers.

Figures 18 and 19 show curves for the mean waiting time and blocking probability for first-level and second-level customers as a function of the traffic intensity at the first-level service center in the cases when $\lambda_1 = \lambda_{2r}$ and when $\rho_{2r} = 2,\ 4,\ 6,\ 8$ in a balanced non-reneging camp-on system with $N = N_r = R = 10$ waiting spaces. These parameters have been chosen to reflect the behavior of a PBX-like communication system rather than to represent any real life system, where the number of different classes may be quite large. For example, a medium-sized department store might have around $\sim 25 - 40$ outlets (N), while the number of departments within the store N_r might vary from a couple to a few tens. From Figures 18 and

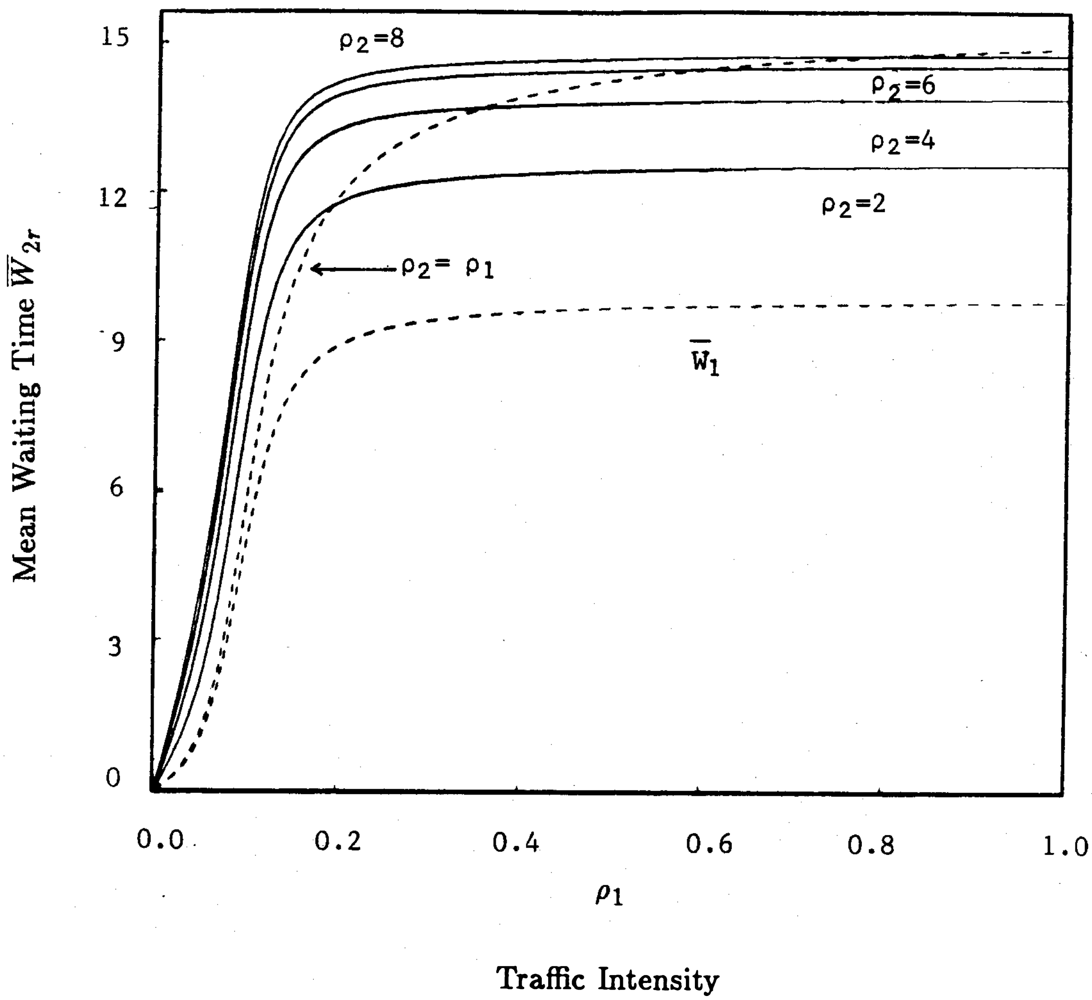


Figure 18: Mean waiting time $\overline{W}_{2r}$ *vs.* the traffic intensity at the first-level service center in the cases when $\lambda_1 = \lambda_{2r}$ and when $\rho_{2r} = 2, 4, 6, 8$ in a finite-storage non-reneging camp-on system with $N = N_r = R = 10$ waiting spaces.

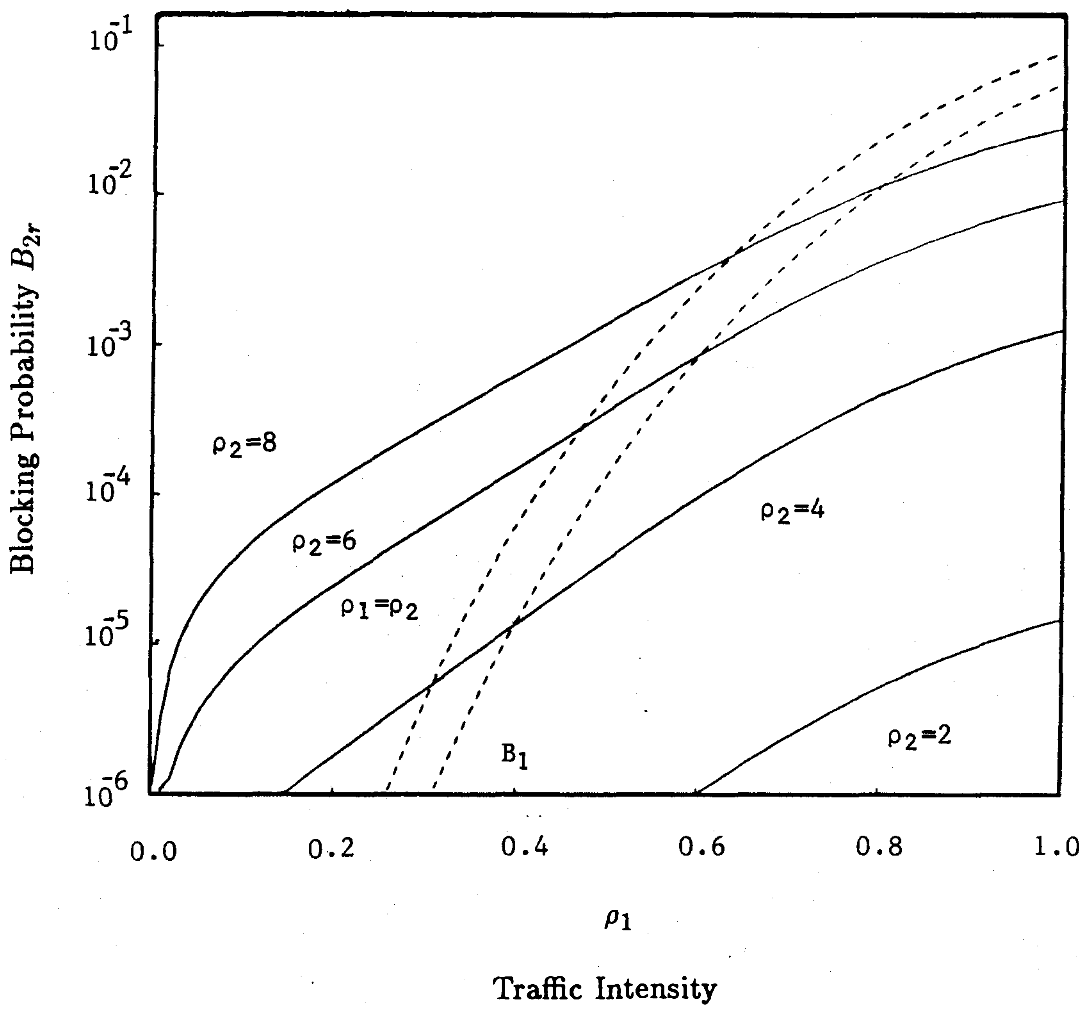


Figure 19: Blocking probability B_{2r} *vs.* the traffic intensity at the first-level service center in the cases when $\lambda_1 = \lambda_{2r}$ an when $\rho_{2r} = 2,\ 4,\ 6,\ 8$ in a finite-storage non-reneging camp-on system with $N = N_r = R = 10$ waiting spaces.

19 we observed that even under heavy traffic conditions the mean waiting time for second-level customers is no more than twice that of the first-level customers. This factor of two is an extreme situation of the heavy traffic condition, where the subsystems at the first-level and second-level stages have the same amount of customers in their waiting lines, i.e., $N = N_r$. Being a balanced system ($\mu_1 = \mu_{2r}$), customers experience about the same amount of queueing at each level, but second-level customers must visit two queueing stages. In fact, under heavy traffic conditions for both first-level and second-level systems, we have from Equation (5.17) that

$$\lim_{\rho_1=\rho_{2r}\to\infty} = \sum_{j=1}^{N} \frac{(j+N_r)}{N\mu} = \frac{N+1}{2\mu} + \frac{N_r}{\mu}.$$

This corresponds to $\overline{W}_{2r} = 15.4$ for the camp-on system in Figure 18. The factor $(N+1)/2$ stems from the observation that the subsystem associated with the arriving second-level customer could be located in any one of the N queueing positions at the first-level queue.

V.2. Broadcast Delivery Services

In many communication systems, different customers may request the same type of service from a common service center. Serving these requests unwisely may induce overload in the communication system, providing poor utilization of the system resources. Through the use of a broadcast delivery scheme, we can satisfy the communication needs of several users simultaneously and thus improve the performance of the queueing system for everyone.

The relationship between broadcast delivery services such as multiple-addressee electronic mail facilities or Videotex[12] and the two-level camp-on model is fairly straightforward. In a Videotex or multiple-addressee electronic mail system, requests submitted by terminal users are processed by a central computer, resulting in the retrieval of the desired piece of information from the system database, e.g., pages of a menu program, pictures, catalog information, manuals, etc. If the system is fairly large, it is likely that users will request the same piece of information within a short time interval, resulting in multiple message requests for the same work simultaneously.

Let us take a closer look at a typical electronic mail system. A message request from a terminal user arrives at the central computer, which is the first-level service center with respect to the camp-on model. The central host checks to see whether other requests for the same message are already in progress, e.g., being retrieved from the database to be transmitted over the network, or are about to be processed, which can mean, for example, being sent to queue in a waiting line. First, the service facility checks its first-level waiting line, where all pending messages are stored. If no other similar message is scheduled to be broadcasted in this queue, then the system is dealing with a new work order and the message request is placed on the system's first-level queue. If the message is already scheduled for broadcast, then the system is dealing with an outstanding message request and the new work order is placed in a second-level queue.

Thus, a second-level queue contains all the outstanding service requests for the same type of work. This explains the camp-on feature of the queueing model. Since the original message request in the first-level queue and the outstanding message requests in the associated second-level queue are all for the same work, i.e., retrieve and mail file X from the central host's database, we can service all of them simultaneously by broadcasting file X to all interested second-level customers when

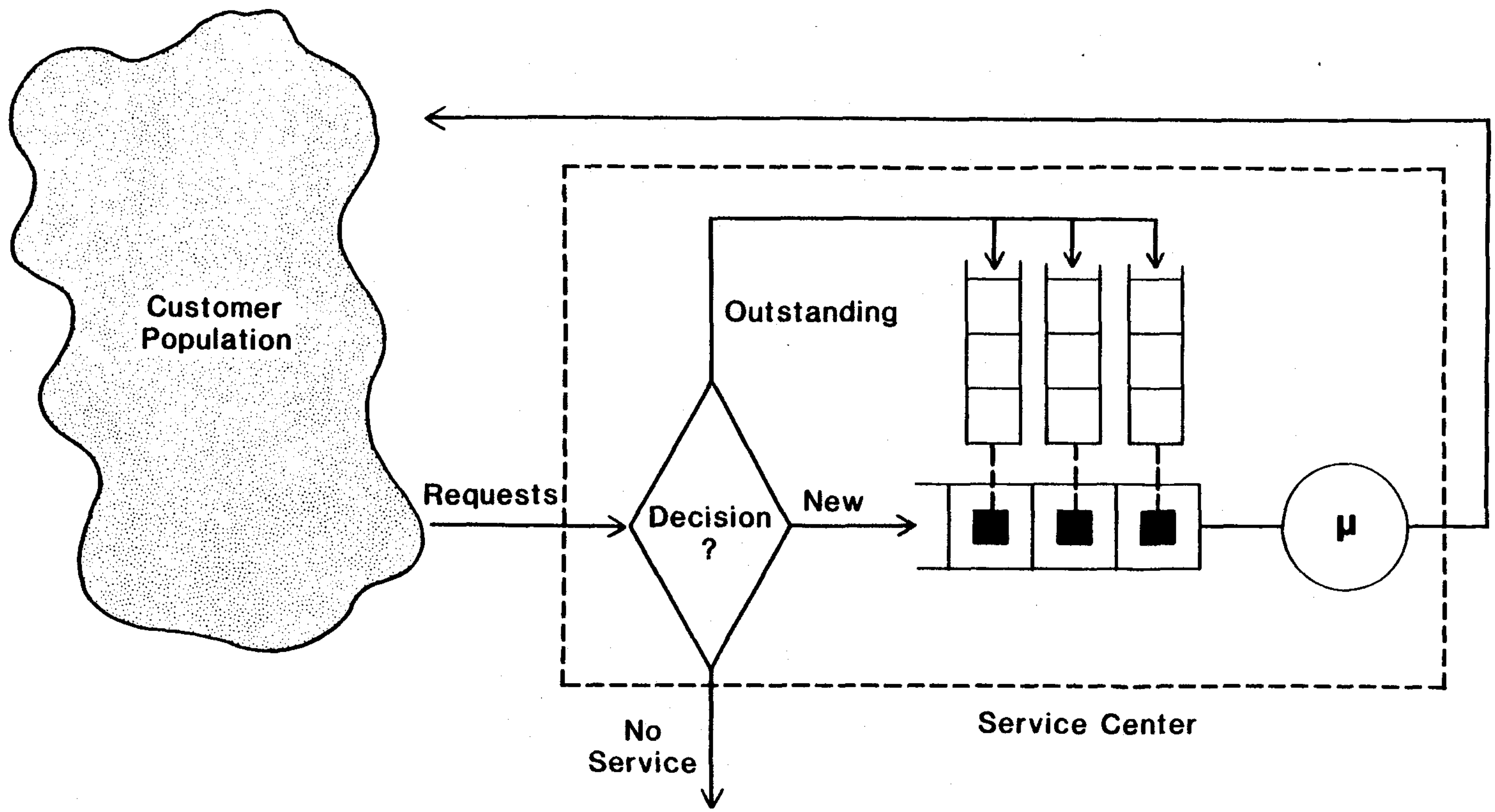


Figure Service strategy in a broadcast delivery system.

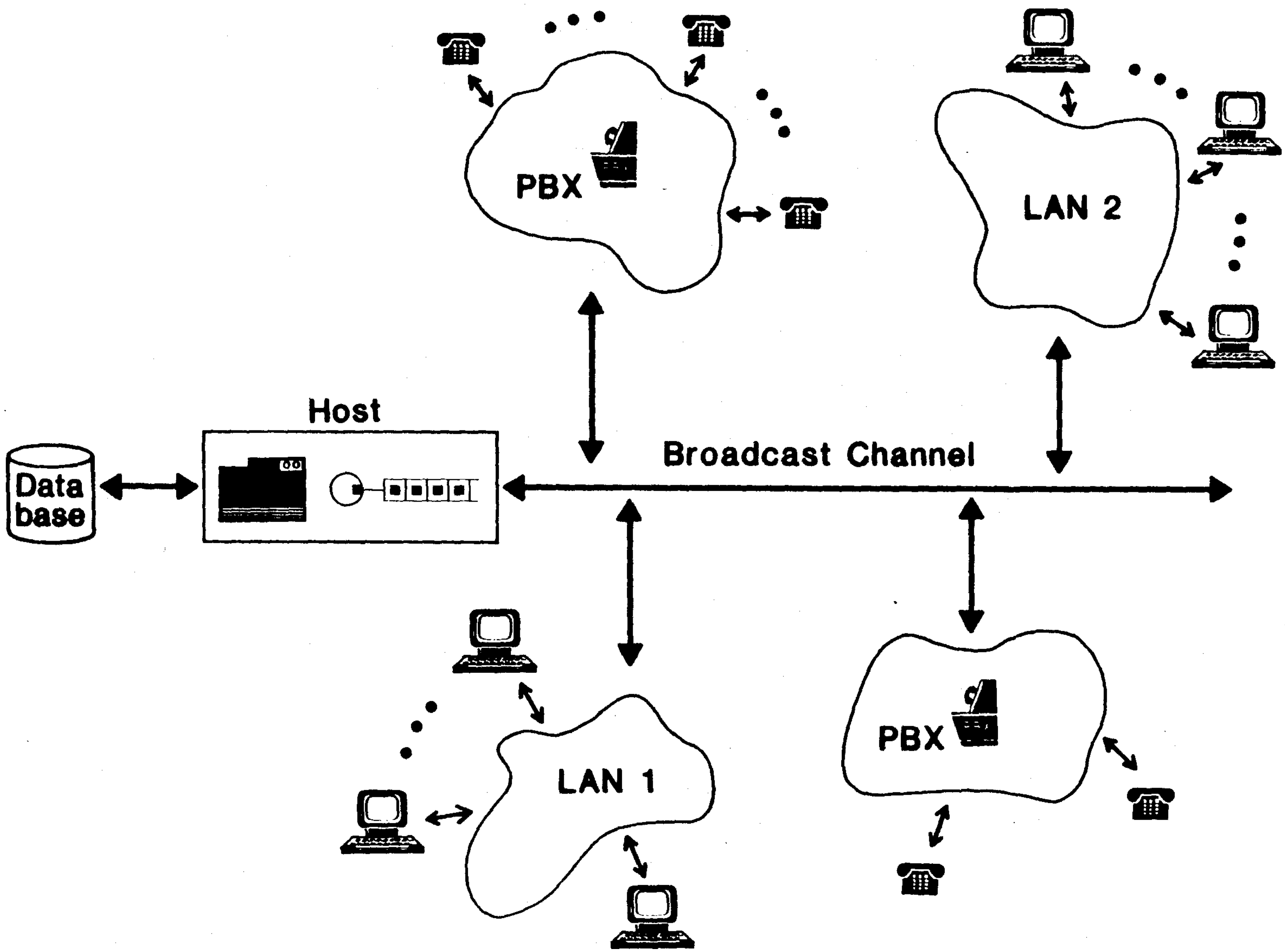


Figure A Videotex or electronic mail system with broadcast delivery service.

the service facility services the first-level message request. This is similar to the broadcast philosophy in radio transmission, which explains the broadcast aspect of the model. Figure 20 illustrates the basic concepts behind this broadcast delivery service. A Videotex or electronic mail system with a broadcast delivery strategy is provided in Figure 21. Here, a central host's database is made available through a broadcast channel to customers requesting pieces of information from nearby LANs or PBXs.

The main statistic of interest for broadcast delivery services is the response time of the broadcast delivery system: the time required to provide service simultaneously to all camp-on users. We do not consider the possibility here of deliberately delaying a broadcast in the hope that more requests for the same message will arrive.

In a Videotex system, message requests for pages of information are submitted to the host computer that fetch the file from the system's database. Hence, the system's response time refers to the time required by the host computer to fetch the information from the system database and broadcast it to the terminal users. Here we assume that first-level customers in a Videotex systems are served with a FCFS service discipline.

2.1. Response Time Distribution

Because of the broadcast nature of the system, only new message requests should matter, from the service center viewpoint, when computing the response time distribution. All outstanding message requests will be taken care of through the response to the initial demand, since service is parallel for all customers in the same second-level queue. Also, since the traffic demand to the central host is not expected to change because of the presence of outstanding requests, we will think of this broadcast system as a class-2 camp-on system as described in Section 1, Chapter II, where

$$\lambda_1 = \sum_{r=1}^{R} \lambda_{2r},$$

and λ_{2r} is the arrival rate for a class-r request.

Let us assume that the delays in the transmission part of the network are negligible. Let $S_r(t)$ denote the response time distribution for a class-r message in such a broadcast delivery system. From the two-level camp-on model for systems with finite storage capacity, we notice that there are two possible queueing situations: 1) there are n message requests submitted but none of them is a class-r message, and 2) there are n message requests submitted, and the j^{th} request is a class-r message.

If there are no class-r requests yet submitted, then the contribution to the response time distribution due to an incoming class-r request amounts to

$$S_r^1(t) = \sum_{n=0}^{N-1} \sum_{\Omega_{nr}} \pi(\mathbf{R}_n)\ E_{n+1}(t). \tag{5.20}$$

Here, as in Equation (5.4), $E_{n+1}(t)$ is the waiting time probability for an incoming request when the first-level queue size is n. Also, Ω_{nr}, a subset of Ω_n, is the set of all class assignments for the n first-level customers that do not include a message of class r. Finally, $\pi(\mathbf{R}_n)$ is the probability that messages of classes $r_1, \cdots, r_n$ are queued in that precise order in the first-level queue. For a class-2 camp-on system, we must choose the l_i's for the definition of $\pi(\mathbf{R}_n)$ in Equation (3.33) such that

$$l_i = \begin{cases} 0, & \text{if } i = r\text{; that is,} \\ & \quad \text{no class-}r\text{ request has yet been submitted;} \\ 1, & \text{if } r_j = i \text{ for some } j\text{; that is,} \\ & \quad \text{a class-}i\text{ request has been submitted to the system;} \\ 0, & \text{otherwise, that is,} \\ & \quad \text{no class-}i\text{ request has been submitted to the system.} \end{cases}$$

On the other hand, if the incoming class-r request is an outstanding message request, then

$$S_r^2(t) = \sum_{n=0}^{N} \sum_{j=1}^{n} \sum_{k_j=0}^{N_r} \frac{p_n^*(k_{jr}) E_j(t)}{n}. \tag{5.21}$$

This is because a previous request may have been already scheduled in position j of the first-level queue. Here $E_j(t)$ is its corresponding waiting time function for that request. The marginal distribution for the size of the j^{th} second-level system, $p_n^*(k_{jr})$, is taken from Equation (3.35).

Notice that in $S_r^2(t)$, only $p_n^*(k_{jr})$ is dependent on the number of outstanding requests already received by the broadcast delivery system. However, if we use Theorem 3 for this class-2 camp-on system, we find that

$$\begin{aligned}\sum_{k_j=0}^{N_r} p_n^*(k_{jr}) &= \sum_{k_j=0}^{\infty} p_n(k_{jr}) \\ &= \sum_{\Omega_n} \chi_{(r_j=r)} \pi(\mathbf{R}_n).\end{aligned}$$

This shows that only the total number of non-outstanding requests, or first-level customers, must be accounted for when computing the response time for the broadcast delivery system.

Define the functions $g(\mathbf{R}_n)$ and $g(\mathbf{R}_n)^{-i}$ as

$$g(\mathbf{R}_n) = \sum_{l_1+\cdots+l_R=n} \pi(\mathbf{R}_n) = \sum_{l_1+\cdots+l_R=n} \prod_{r=1}^{R} q_r^{l_r}, \tag{5.22}$$

$$g^{-i}(\mathbf{R}_n) = \sum_{\substack{l_1+\cdots+l_R=n \\ l_i=0}} \pi(\mathbf{R}_n) = \sum_{\substack{l_1+\cdots+l_R=n \\ l_i=0}} \prod_{r=1}^{R} q_r^{l_r}, \tag{5.23}$$

where $q_r = \lambda_{2r}/\lambda_1$ is the probability of a class-r request. The two functions represent, respectively, the probability of a size-n first-level queue regardless of the request classes, and the probability of a size-n first-level queue without a class-r request. It follows directly from Equations (5.20) and (5.21) that the response time for a class-r request in the broadcast delivery system is given by

$$\begin{aligned} S_r(t) &= S_r^1(t) + S_r^2(t) \\ &= p_0 \mu e^{-\mu t} \left[\sum_{n=0}^{N-1} (\rho \mu t)^n g^{-r}(\mathbf{R}_n) + \right. \\ &\qquad \left. \sum_{n=1}^{N} (n-1)! \rho^n q_r g^{-r}(\mathbf{R}_{n-1}) \sum_{j=1}^{n} \frac{(\mu t)^{j-1}}{(j-1)!} \right] . \end{aligned} \tag{5.24}$$

Using the fact that $\int_0^\infty x^n e^{-x}/(n-1)! dx = n$, we also obtain from Equation (5.24) that the mean response time for a class-r request in a broadcast delivery system is

$$\begin{aligned} \overline{S}_r &= \int_0^\infty t \, S_r(t) dt \\ &= \frac{p_0}{\mu} \left[\sum_{n=0}^{N-1} (n+1)! \rho^n g^{-r}(\mathbf{R}_n) + \right. \\ &\qquad \left. \sum_{n=1}^{N} 1/2(n+1)! \rho^n q_r g^{-}(\mathbf{R}_{n-1}) \right] . \end{aligned} \tag{5.25}$$

If we are interested in the overall broadcast system performance for all classes of requests, we must average the above distribution over all r. The mean response time $\overline{S}$ averaged over request class is given by

$$\overline{S} = \sum_{r=1}^{R} q_r \, \overline{S}_r .$$

Figures 22, 23 and 24 show the response time distribution $S(t)$ and the mean response time $\overline{S}$ as a function of the traffic intensity in a broadcast delivery system.

These examples are based in a camp-on system with $R = N = 30$, $\mu = 1.0$ and a Zipf's law[43] distribution for the class-r requests ($q_r = c/r$).

In Figures 22 and 23 we show the overall response time $S(t)$ as a function of time for traffic intensities $\rho \leq 1$ and $\rho \geq 1$, respectively. We notice that $S(t)$ is a weighted sum of the Erlang distributions $E_i(t)$ of requests being broadcast in position i in the Videotex system. For light traffic, the higher weights correspond to lower phase $E_i(t)$. This accounts for the almost exponential shape of $\overline{S}(t)$ in Figure 22. As the traffic increases, higher phases $E_i(t)$ dominate, accounting for the peak seen in Figure 23.

Figure 24 shows the mean response time behavior as a function of traffic intensity as well as the behavior of the class-1 and class-30 message requests, the two most extreme cases of delay for this broadcast delivery model. Here, we see that for heavy traffic the mean waiting time reaches a plateau. To understand this behavior we notice that as ρ increases, the probability that we find N requests in the system goes to one. A new request must, of course, wait for at most N/μ units of time on the average to be serviced. It is not hard to see from Equation (5.25) that, in fact,

$$\lim_{\rho \to \infty} \overline{S}_r = \sum_{j=1}^{N} \frac{1}{N}\left(\frac{j}{\mu}\right) = \frac{N+1}{2\mu}.$$

This is because the new message request can be broadcast in any of N broadcasting time slots, not necessarily the last one, since a previous request for the same message may already be in the system. For the above values we have $\overline{S} = 15.5$, as in Figure 24.

The response time distribution of a Videotex system using broadcast delivery systems has also been studied by Ammar and Wong[1,42]. There, a product-form approach was used to deal with the problem of the size of the demand for every class of

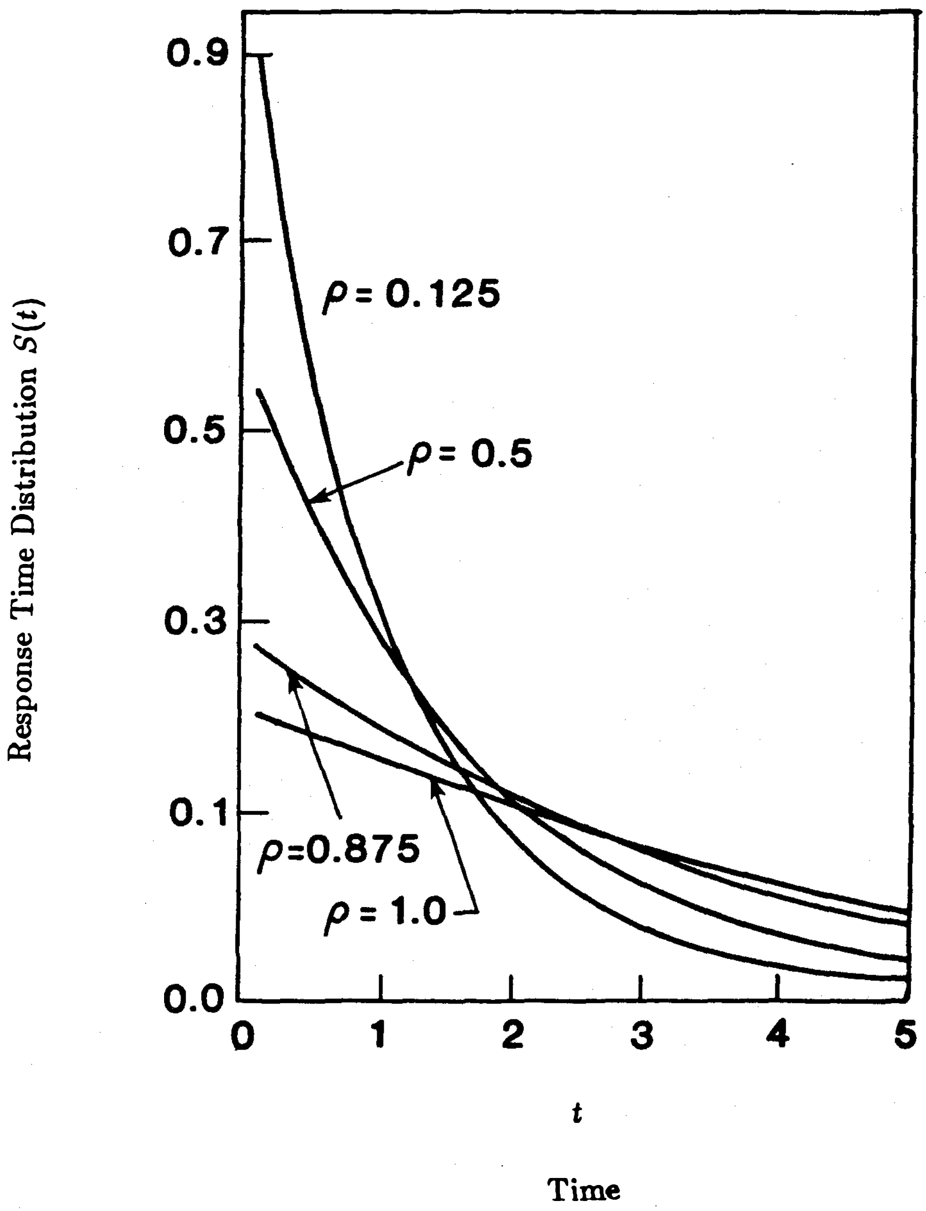


Figure 22: Response time distribution $S(t)$ *vs.* time in a camp-on system with $R = N = 30$, $\mu = 1.0$ and a Zipf's law distribution for the class-r request ($q_r = c/r$) for $\rho \leq 1$.

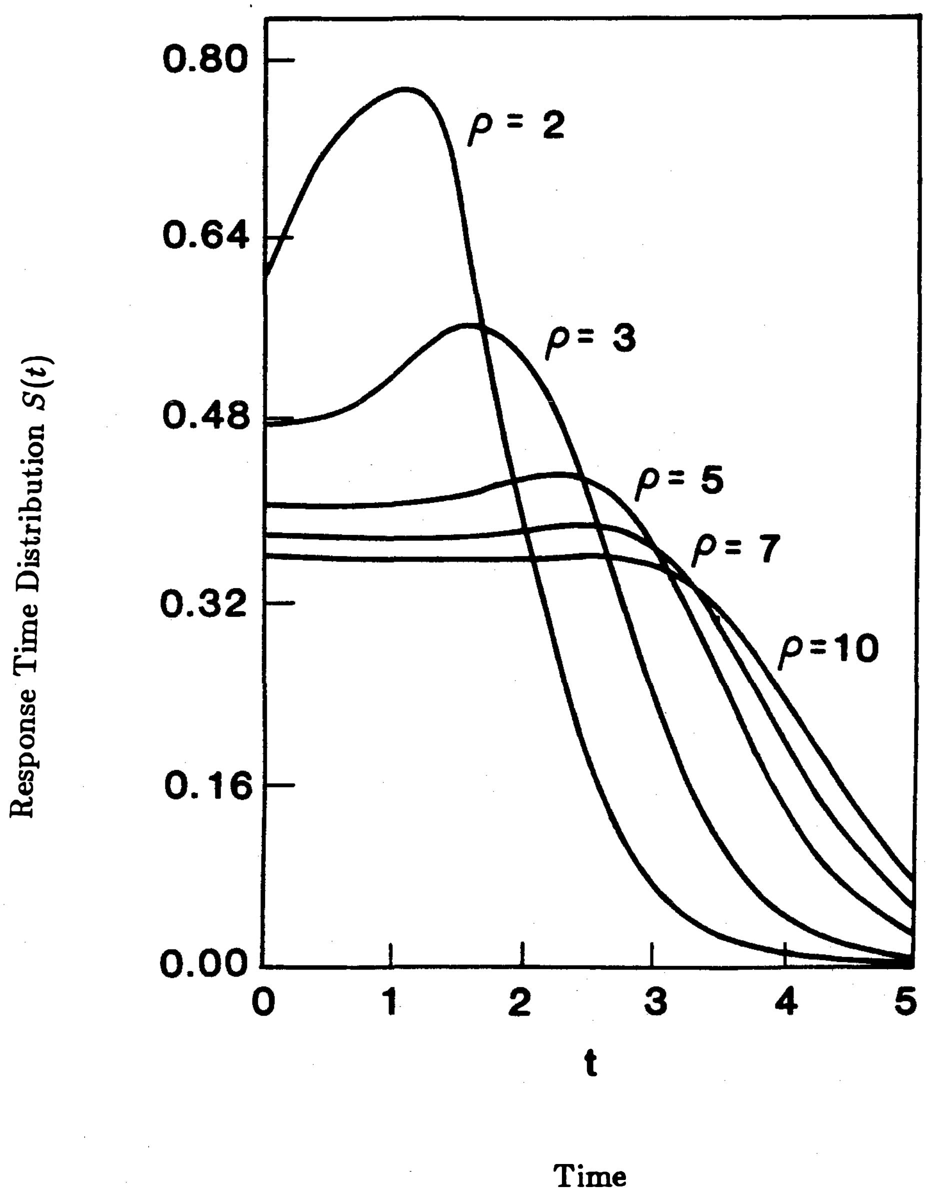


Figure 23: Response time distribution $S(t)$ *vs.* time in a camp-on system with $R = N = 30$, $\mu = 1.0$ and a Zipf's law distribution for the class-r request ($q_r = c/r$) for $\rho > 1$.

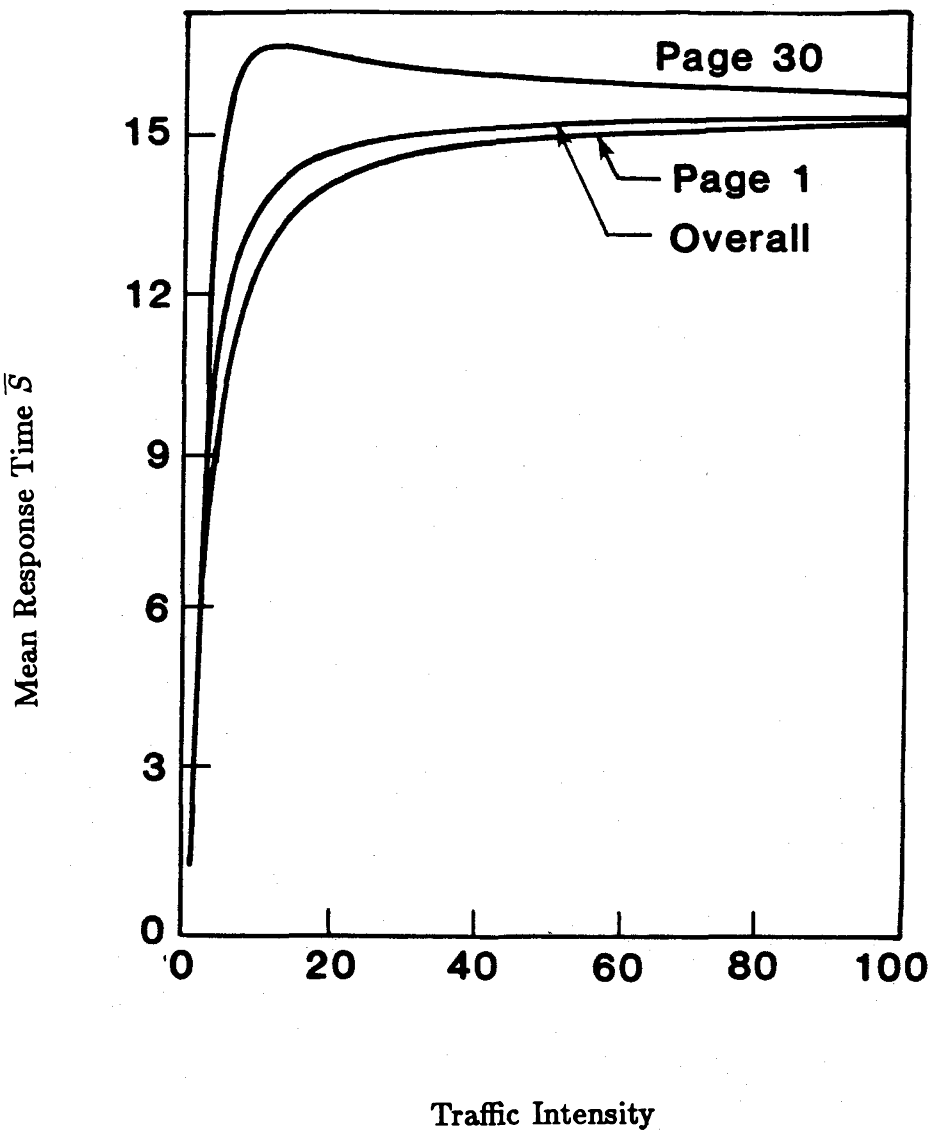


Figure 24: Mean response time $\overline{S}$ *vs.* the traffic intensity at the first-level service center in a camp-on system with $R = N = 30$, $\mu = 1.0$ and a Zipf's law distribution for the class-r request ($q_r = c/r$).

work. In order to model the queueing system from the product-form standpoint, the idea of *superfluous broadcasting*, that is, broadcasting the requested message more than once, was proposed. The idea here is to anticipate future service demand by broadcasting some of the outstanding message requests. When non-superfluous broadcast is used, Ammar and Wong's approach reduces to the response time distribution we found in Equation (5.24). We were able to derive this distribution from the actual state occupancy distribution of the classes of messages received by the service center with a broadcast delivery service. We expect a more general camp-on approach to the queueing and service strategy that takes advantage of the class and current size of the scheduled service to yield still better performance than the one obtained through superfluous broadcast delivery.

CHAPTER VI:

SUMMARY AND CONCLUSIONS

In the present work, a model for a stationary, multilevel camp-on system with reneging allowed from any of the system queues has been proposed. For the two-level camp-on model, customers were assumed to be drawn from an infinite pool of subscribers. The associated arrival processes were considered to be independent processes with mean arrival rate λ_{1n} for the first-level service facility when the queue size is n, and mean arrival rate λ_{2r_i} for the subsystem located at the i^{th} position in the second-level stage associated with class-r_i customers. For the multilevel camp-on model, the same assumptions hold, but now λ_i represents the arrival rate to the systems located at the i^{th} queueing stage. The service completion processes were considered to be independent and exponentially distributed with mean service rates μ_i. Similarly, the customer reneging processes from the main system queue and the second-level queues were also assumed to be independent and exponentially distributed with mean reneging rates ν_i $(1 \leq i \leq n)$ and η, respectively. Five service discipline were studied for the two-level camp-on model: first-come-first-served (FCFS), last-come-first-served non-preemptive (LCFS-NP), last-come-first-served preemptive-resume (LCFS-PR), infinite servers (IS) and broadcast delivery (BD). Only a first-come-first-served service discipline was considered for the multilevel case.

Closed-form solutions for the probabilistic generating function of the equilibrium joint probability distribution of the queue lengths, i.e., the generalized n-dimensional Z-transform of the second-level queue sizes conditioned on the size of the first-level system, were found for two-level camp-on systems with finite and semi-infinite storage capacity. In the special case of non-reneging queueing systems, closed-form solutions for the equilibrium joint probability distribution of the queue lengths were also given for both finite-storage and infinite-storage camp-on systems. For a multilevel camp-on system, closed-form solutions were found for the joint distributions of queue sizes along any service path with n queueing levels in the service hierarchy. This reduced representation of the system states is equivalent to a "depth-first" search of the queue sizes in the service tree.

The stability of the camp-on model was shown to depend primarily on the stability of the first queueing stage. This ergodicity condition for the first-level stage translates into a requirement for stability for all the subsystems as isolated queueing systems in the camp-on model. Such a requirement for stability is a consequence of the hierarchical structure of the camp-on model, since customers are continuously promoted to the next queueing stage. Other special distributions, such as the marginal distribution for the size of a subsystem in the second-level stage and the total workload accumulated in the queueing stages, were also derived from the more general distribution for the queue sizes.

The joint distribution of queue sizes in the camp-on model was found not to be of the product-form type, as would be typical of some other queueing systems with multiple queues, classes of customers and similar traffic parameters, except for very special traffic conditions such as heavy traffic at the first-level service center. However, the effect of the first-level stage on the second-level system is simple enough to provide ease of computation. This ease of computation is one of the best assets of this model. This result also suggests that more general interrelations between the first and second queueing stages may be proposed without losing too much in the simplicity of the camp-on system representation or in its computational tractability.

Two communication applications were discussed in details. One was for PBX-like communication services, such as inquiry-oriented networks, and the other for broadcast delivery services such as in Videotex or electronic mail systems. Performance statistics such as waiting-time distribution, response time distribution, mean waiting times and blocking probabilities were given for such services. From this work we can conclude that it is possible to implement two-level queueing systems in an infinite state space, without necessarily running into a deadlock problem, if the traffic parameters are adequately chosen. Moreover, if the reneging rates from the first-level and second-level systems are all positive, the camp-on systems are inherently stable. This is, however, obvious for finite-state camp-on systems with non-negative transition rates because they represent finite Markov chains.

In the case of non-reneging camp-on systems, those results provide a simple way to compute the equilibrium joint probability distribution of queue lengths that can be readily used for systems designs. The examples in Chapter V show that it is possible to achieve system performance close to the performance of a conventional queueing system for moderate traffic intensities, though this requires extra hardware and memory capacities. Applications of these results range from typical telephone networks and PBX systems and computer networking to task distribution and management in general multiqueueing systems.

The results presented in this work constitute only the first steps towards the understanding the statistical behavior of hierarchical queueing systems. There are two main areas wherein new alternatives can be explored for the two-level camp-on model. One corresponds to more general policies for queueing and service at the queueing stages, especially those that require a deeper coupling between the second-level composition and the type of service provided by the first-level service center. The other alternative corresponds to the study of the sensitivity of the present model to the service strategy implemented at the first-level stage, as well as to the other parameters of the system (traffic demands, storage capacity, etc.).

Priority service schemes based on the size of the second-level systems and/or customer classes are of great interest, since they better represent how service and routing are provided in many commercial systems. Simultaneous service at the multiple queueing stages is also attractive, since some of the jobs queued at the second-level stage could be completed without the participation of the first-level service center. This is particularly true in computer-related applications.

For PBX-like services, we found in Section 1, Chapter V, that even under heavy traffic conditions we increase the delay in service experienced by the second-level customers by a factor of only 2. This is definitely the worst-case scenario. We should expect that improved service strategies perform better than FCFS, producing even smaller service delays. For an electronic mail system using broadcast delivery, service strategies based on the actual probability distribution of queue lengths rather than on approximations from the product-form approach should also be used to deal with the problem of the size of the demand for every class of work. We expect a more general camp-on approach to the queueing and service strategy that takes advantage of the class and current size of the service scheduled to yield better performance than the one obtained through strategies such as superfluous broadcast delivery.

The above systems are just examples of the many potential applications of the camp-on model. We anticipate many more to surface in the next few years. This thesis is expected to help model their performance and even to suggest new useful service concepts.

APPENDIX I:

Transformed Equilibrium Equations: Derivation

In Chapter II, it was found that the equilibrium probability distribution of queue lengths in the camp-on model satisfies the equation

$$\begin{aligned}
\left[\sum_{i=1}^{n} \left[\lambda_{2r_i} + \nu_i + k_i\eta\right] \chi_{(n>0)} + \lambda_{1n}\chi_{(n<N)} + \mu\chi_{(n>0)}\right] p(\mathbf{x}_n) \; = \\
\gamma_n \; \chi_{(k_n=0)} \; p(\mathbf{x}_{n-1}) \\
+ \sum_{i=1}^{n} \lambda_{2r_i} \; \chi_{(k_i>0)} \; p(\mathbf{x}_n^{-i}) \\
+ \sum_{i=1}^{n} (k_i+1)\eta \; \chi_{(n>0)} \; p(\mathbf{x}_n^{+i}) \qquad (I.1) \\
+ \sum_{r_0=1}^{R} \chi_{(r_0 \neq r_i)} \sum_{k_0=0}^{\infty} \mu\chi_{(n<N)} \; p(\mathbf{x}_{n+1,1}) \\
+ \sum_{r_0=1}^{R} \chi_{(r_0 \neq r_i)} \sum_{i=1}^{n+1} \sum_{k_0=0}^{\infty} \nu_i \; \chi_{(n<N)} \; p(\mathbf{x}_{n+1,i}), \\
(0 \leq n \leq N).
\end{aligned}$$

We now wish to express this equilibrium equation in terms of the transformed function $P(\mathbf{Z}_n;\mathbf{R}_n)$, the generating function for the sizes of the second-level systems, conditioned on the size of the first-level queue. Here,

$$\begin{aligned}
P(\mathbf{Z}_n;\mathbf{R}_n) \; &= \; P(z_1, z_2, \ldots, z_n; \mathbf{R}_n) \\
&= \sum_{k_1,\ldots,kn=0}^{\infty} p(\mathbf{x}_n) \prod_{i=1}^{n} z_i^{k_i}. \qquad (I.2)
\end{aligned}$$

Let us start with the left side of Equation $(I.1)$. In using the transform definition, we can distinguish two distinct type of terms: i) the transformed form of $p(\mathbf{x}_n)$ times a constant C, merely $CP(\mathbf{Z}_n;\mathbf{R}_n)$, and ii) the transformed form of $p(\mathbf{x}_n)$ times k_i. From simple well-known results or a table of Z-transform pairs[18], we note that:

$$n\ f(n) \quad \longleftrightarrow \quad z\,\frac{d}{dz}F(z).$$

If one takes into account that $P(\mathbf{Z}_n;\mathbf{R}_n)$ is the n-dimensional Z-transform of the multivariable function $p(\mathbf{K}_n;\mathbf{R}_n)$, then one finds that the corresponding transform pair is

$$k_i\ p(\mathbf{x}_n) \quad \longleftrightarrow \quad z_i\frac{\partial}{\partial z_i}P(\mathbf{Z}_n;\mathbf{R}_n).$$

In the right side of Equation $(I.1)$, we have to consider three distinct transform types: i) the transformed form for $p(\mathbf{x}_n^{-i})$, ii) the transform form corresponding to $(k_i+1)p(\mathbf{x}_n^{+i})$, and iii) the transformed form for $\sum_{k_0=0}^{\infty} p(\mathbf{x}_{n+1,i})$.

Here again, one can trace a parallelism between the multidimensional $P(\mathbf{Z}_n;\mathbf{R}_n)$ and the one-dimensional Z-transform. In the first place, it is clear as above that

$$f(n-1) \quad \longleftrightarrow \quad zf(z).$$

The transform pair corresponding to $(n+1)f(n+1)$ is not different from the one corresponding to $nf(n)$ after a suitable change of variable. Similarly, we note that

$$\sum_{n=0}^{\infty} f(n) \ = \ F(z)\Big|_{z=1}.$$

Accordingly, the multidimensional transform pairs for the three cases under consideration for the right side of Equation $(I.1)$ are:

$$p(\mathbf{x}_n^{-i}) \quad \longleftrightarrow \quad z_iP(\mathbf{Z}_n;\mathbf{R}_n),$$

$$(k_i+1)p(\mathbf{x}_n^{+i}) \quad \longleftrightarrow \quad z_i \frac{\partial}{\partial z_i} P(\mathbf{Z}_n;\mathbf{R}_n),$$

$$\sum_{k_0=0}^{\infty} p(\mathbf{x}_{n+1,i}) \quad \longleftrightarrow \quad P(\mathbf{Z}_{n+1,i};\mathbf{R}_{n+1,i}).$$

Therefore, by direct inspection of Equation (I.1) and from the transform pairs here derived, one recognizes that the transformed equilibrium probability equation for the camp-on model comes down to

$$\begin{aligned}
\Big[\sum_{i=1}^{n}\big[\lambda_{2r_i}+\nu_i\big]+\lambda_{1n}\chi_{(n<N)}+\mu\chi_{(n>0)}\Big]P(\mathbf{Z}_n;\mathbf{R}_n)+\sum_{i=1}^{n} z_i\eta\,\frac{\partial}{\partial z_i}P(\mathbf{Z}_n;\mathbf{R}_n) = {}&\\
\gamma_n\; P(\mathbf{Z}_{n-1};\mathbf{R}_{n-1})&\\
+\sum_{i=1}^{n}\lambda_{2r_i}\;\chi_{(n>0)}\; z_i P(\mathbf{Z}_n;\mathbf{R}_n)&\\
+\sum_{i=1}^{n}\eta\;\chi_{(n>0)}\; z_i\;\frac{\partial}{\partial z_i}P(\mathbf{Z}_n;\mathbf{R}_n)&\\
+\sum_{r_0=1}^{R}\chi_{(r_0\neq r_i)}\;\mu\;\chi_{(n<N)}\;P(\mathbf{Z}_{n+1,1};\mathbf{R}_{n+1,1})&\\
+\sum_{r_0=1}^{R}\chi_{(r_0\neq r_i)}\sum_{i=1}^{n}\nu_i\;\chi_{(n<N)}P(\mathbf{Z}_{n+1,i};\mathbf{R}_{n+1,i}).&
\end{aligned}$$

After grouping together common terms, we get the more compact form:

$$\begin{aligned}
\sum_{i=1}^{n}(z_i-1)\left[\eta\frac{\partial}{\partial z_i}P(\mathbf{Z}_n;\mathbf{R}_n)-\lambda_{2r_i}\;P(\mathbf{Z}_n;\mathbf{R}_n)\right]+{}&\\
\left[\mu_n\chi_{(n>0)}+\sum_{i=1}^{n}\nu_i+\lambda_{1n}\chi_{(n<N)}\right]\;P(\mathbf{Z}_n;\mathbf{R}_n) = {}&\\
\gamma_n\;\chi_{(n>0)}\;P(\mathbf{Z}_{n-1};\mathbf{R}_{n-1})&\\
+\sum_{r_0=1}^{\infty}\chi_{(r_0\neq r_i)}\;\mu_{n+1}\;\chi_{(n<N)}\;P(\mathbf{Z}_{n+1,1};\mathbf{R}_{n+1,1})&\\
+\sum_{r_0=1}^{\infty}\chi_{(r_0\neq r_i)}\sum_{i=1}^{n+1}\nu_i\;\chi_{(n<N)}\;P(\mathbf{Z}_{n+1,i};\mathbf{R}_{n+1,i}),&
\end{aligned}$$

for all $0\le|z_i|\le 1$. This completes the derivation of the transformed global balance equation for the camp-on system.

APPENDIX II:

Proof of Theorem 1

In Chapter 3, Theorem 1, we claimed that the general form of the generalized Z-transform for the equilibrium joint probability distribution of queue lengths in a camp-on system with infinite or semi-infinite storage capacity is given by

$$P(\mathbf{Z}_n;\mathbf{R}_n) = p_0 \prod_{i=1}^{n} \Psi_i(\mathbf{x}_i)$$
$$= p_0 \prod_{i=1}^{n} \frac{\gamma_i}{\eta} \sum_{l_i=0}^{\infty} \frac{Y_i^{l_i}}{(\alpha_i)_{l_i+1}}, \qquad 1 \le n \le N \qquad (II.1)$$

where

$$Y_n = \sum_{i=1}^{n} \frac{\lambda_{2r_i}}{\eta}(z_i - 1), \qquad (II.2)$$

$$\alpha_n = \frac{\mu_n}{\eta} + \sum_{i=1}^{n-1} l_i, \qquad (II.3)$$

$$\mu_n = \mu + \sum_{i=1}^{n} \nu_i, \qquad (II.4)$$

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \qquad (II.5)$$

Here, we show by algebraic reduction that this proposed transformed state-probability distribution $P(\mathbf{Z}_n)$* is a solution to the set of transformed independent balance equations for the camp-on model:

$$\sum_{k=1}^{n} (z_k - 1)\left[\eta \frac{\partial}{\partial z_k} P(\mathbf{Z}_n) - \lambda_{2r_k} P(\mathbf{Z}_n)\right] + \mu_n P(\mathbf{Z}_n) = \gamma_n P(\mathbf{Z}_{n-1}), \qquad (II.6)$$

* In this appendix we will use the shorthand form $P(\mathbf{Z}_n)$ instead of the complete form $P(\mathbf{Z}_n;\mathbf{R}_n)$ for the transformed state-probability distribution of queue sizes.

$$\lambda_{1n}\ P(\mathbf{Z}_n) \ = \ \mu\ P(\mathbf{Z}_{n+1,1}) \ + \ \sum_{i=1}^{n+1} \nu_i\ P(\mathbf{Z}_{n+1,i}), \qquad 0 \le n \le N-1. \qquad (II.7)$$

These are the equations that state the detailed conditions for flow balance among neighboring states in the camp-on model.

Before proceeding with our exposition, we want to mention a common algebraic transformation that will be used quite often in this appendix and in subsequent appendices. For any two sequences $\{a_i\}$ and $\{b_i\}$

$$\sum_{j=k}^{N} \sum_{i=j-l}^{N} a_i\ b_j \ = \ \sum_{i=0}^{N} \sum_{j=0}^{i+l} a_i\ b_j. \qquad (II.8)$$

We start our proof with the transformed state Equation $(II.6)$. We observe that the partial derivative of $P(\mathbf{Z}_n)$ with respect to z_k yields

$$\begin{aligned} \frac{\partial}{\partial z_k} P(\mathbf{Z}_n) &= \sum_{i=k}^{n} P(\mathbf{Z}_n) \frac{\partial \Psi_i(\mathbf{Z}_i)}{\partial z_k} \\ &= \frac{\lambda_{2r_k}}{\eta} \sum_{i=k}^{n}\ P(\mathbf{Z}_n) \frac{l_i}{Y_i}, \end{aligned}$$

for $\Psi_i(\mathbf{Z}_i)$ is independent of z_k for $k > i$.

After taking the partial derivative of the transformed state probability distribution $P(\mathbf{Z}_n)$ with respect to z_k in Equation $(I.1)$, Equation $(II.6)$ becomes

$$\sum_{k=1}^{n} \lambda_{2r_k}(z_k - 1) P(\mathbf{Z}_n) \left[\sum_{i=k}^{n} \frac{l_i}{Y_i} \ - \ 1 \right] \ + \ \mu_n P(\mathbf{Z}_n) \ = \ \gamma_n P(\mathbf{Z}_{n-1}). \qquad (II.9)$$

Observe also that from Equation $(II.2)$, $\lambda_{2r_k}/\eta(z_k - 1)$ can be written in terms of Y_k as

$$\begin{aligned} \frac{\lambda_{2r_k}}{\eta}(z_k - 1) &= \frac{\lambda_{2r_k}}{\eta}(z_k - 1) \ + \ (Y_{k-1} \ - \ Y_{k-1}) \\ &= Y_k \ - \ Y_{k-1}. \end{aligned}$$

Thus, using the above relation, the first term on the right side of Equation $(II.9)$ reduces to

$$\sum_{k=1}^{n} \lambda_{2r_k}(z_k - 1)\ P(\mathbf{Z}_n) \left[\sum_{i=k}^{n} \frac{l_i}{Y_i} \ - \ 1\right]$$

$$= \sum_{k=1}^{n} P(\mathbf{Z}_n) \left[\sum_{i=k}^{n} \eta l_i \frac{(Y_k - Y_{k-1})}{Y_i} \ - \ \eta(Y_k - Y_{k-1})\right]$$

$$= \sum_{k=1}^{n} \eta P(\mathbf{Z}_n)[\beta_k \ - \ \delta_k]. \qquad (II.10)$$

Here, β_k and δ_k stand for

$$\beta_k = \sum_{i=k}^{n} l_i \frac{(Y_k - Y_{k-1})}{Y_i},$$

$$\delta_k = (Y_k - Y_{k-1}),$$

respectively.

Through simple manipulations of the above expressions, we first recognize that the summation of the δ_k's yields

$$\sum_{k=1}^{n} \delta_k = (Y_n - Y_0)$$

$$= Y_n.$$

In the same way, using also the algebraic relationship in Equation $(II.8)$, the summation of the β_k's yields

$$\sum_{k=1}^{n} \beta_k = \sum_{k=1}^{n} \sum_{i=k}^{n} l_i \frac{(Y_k - Y_{k-1})}{Y_i}$$

$$= \sum_{i=1}^{n} l_i \frac{(Y_i - Y_0)}{Y_i} \qquad = \sum_{i=1}^{n} \eta l_i.$$

In both cases, it is clear from the definition in Equation $(II.2)$ that $Y_0 = 0$. Inserting the above results in Equation $(II.10)$ and recalling the definition of α_n in

Equation $(II.3)$, we find that Equation $(II.10)$ reduces to

$$\gamma_n\, P(\mathbf{Z}_n) = P(\mathbf{Z}_n)\left[\mu_n + \sum_{i=1}^{n} \eta l_i + \eta Y_n\right]$$

$$= P(\mathbf{Z}_n)\left[\alpha_n + \eta(l_n + Y_n)\right]. \qquad (II.11)$$

On one hand, observe that the coefficient α_j in Equation $(II.3)$ is independent of the summation index l_i, if $j \leq i$; it is a linear function of l_i if $j > i$. Define the change of variable $l_i = m_i + 1$. Based on the above relationship between α_j and l_i, the function $(\alpha_j)_{l_j+1}$ can be reformulated to accommodate this change of variable as

$$(\alpha_j)_{l_j+1} = \begin{cases} (\alpha_j)_{m_j+2} = (\alpha_j)_{m_j+1}(\alpha_j + m_j + 1), & \text{if } j \leq i; \\ (\alpha_j + 1)_{m_j+1} = (\alpha_j)_{m_j+1}\frac{(\alpha_j+m_j+1)}{\alpha_j}, & \text{if } j > i. \end{cases} \qquad (II.12)$$

Define the change of variable $l_n = m_n + 1$ on the left side of Equation $(II.11)$. Using the relationship given in Equation $(II.12)$, it follows that

$$P(\mathbf{Z}_n)\, l_n = P(\mathbf{Z}_n)\,\frac{m_n + 1}{\alpha_n + m_n + 1},$$

where it is understood that we are also using m_n rather than l_n as the summation index for $\Psi_n(\mathbf{Z}_n)$ on the right-hand side of the expression for $P(\mathbf{Z}_n)$. From Equation $(II.11)$, it follows that

$$\begin{aligned} P(\mathbf{Z}_n)\,[l_n - Y_n] &= -\,P(\mathbf{Z}_n)\,Y_n\left[\frac{m_n + 1}{\alpha_n + m_n + 1} - 1\right] \\ &= -\,P(\mathbf{Z}_n)\,\frac{\alpha_n}{\alpha_n + m_n + 1}. \end{aligned}$$

If we look carefully into the definition of $(\alpha_j)_{m_j+1}$ from Equations $(II.5)$ and $(II.12)$, we notice that the above relation is no more than

$$P(\mathbf{Z}_n)[l_n - Y_n] = -P(\mathbf{Z}_{n-1})\Psi_n(\mathbf{Z}_n)\frac{\alpha_n}{\alpha_n + m_n + 1}\, Y_n$$
$$= -P(\mathbf{Z}_{n-1}) \sum_{m_n=0}^{\infty} \frac{\gamma_n}{\eta} \frac{Y_n^{m_n+1}}{(\alpha_n)_{m_n+2}}\, \alpha_n.$$

Reversing the change of variable $m_n = l_n - 1$ and keeping in mind the relationship between the α_n and l_n in Equation $(II.12)$, we get

$$P(\mathbf{Z}_n)[l_n - Y_n] = -\alpha_n P(\mathbf{Z}_{n-1}) \sum_{l_n=1}^{\infty} \frac{\gamma_n}{\eta} \frac{Y_n^{l_n}}{(\alpha_n)_{l_n+1}}.$$

The above expression is almost equivalent to $\alpha_n P(\mathbf{Z}_n)$ except for the missing term $l_n = 0$. Adding and subtracting this missing term into the above equation, we find

$$P(\mathbf{Z}_n)[l_n - Y_n] = -\alpha_n P(\mathbf{Z}_{n-1}) \sum_{l_n=0}^{\infty} \frac{\gamma_n}{\eta} \frac{Y_n^{l_n}}{(\alpha_n)_{l_n+1}} + \alpha_n P(\mathbf{Z}_{n-1}) \frac{\gamma_n}{\eta} \frac{1}{\alpha_n}$$
$$= -\alpha_n P(\mathbf{Z}_n) + \frac{\gamma_n}{\eta} P(\mathbf{Z}_{n-1}).$$

Substituting this expression for $P(\mathbf{Z}_n)[l_n + Y_n]$ back into Equation $(II.11)$, we finally obtain

$$\frac{\gamma_n}{\eta} P(\mathbf{Z}_{n-1}) = \alpha_n P(\mathbf{Z}_n) - \alpha_n P(\mathbf{Z}_n) + \frac{\gamma_n}{\eta} P(\mathbf{Z}_{n-1}).$$

We see that the first two terms on the right side of the above equation cancel out and the desired equality for Equation $(II.4)$ is achieved, as we asserted. This proves that the transformed equilibrium Equation $(II.6)$ is satisfied by $P(\mathbf{Z}_n)$ in Equation $(II.1)$.

We now need to verify the consistency of Equation $(II.1)$ for $P(\mathbf{Z}_n)$ with respect to the transformed independent balance equation. First, we will rewrite the general term $\nu_k P(\mathbf{Z}_{n+1,k})$ in Equation $(II.7)$ in a more convenient way for this computation. In order to do so, we observe that

$$\nu_k P(\mathbf{Z}_{n+1,k}) = \nu_k P(z_1, \cdots, z_{i-1}, 1, z_i, \cdots, z_n)$$

$$= \nu_k \; p(\mathbf{Z}_{k-2}) \left[\sum_{l_{k-1}=0}^{\infty} \sum_{l_k=0}^{\infty} \frac{Y_{k-1}^{l_{k-1}+l_k}}{(\alpha_{k-1})_{l_{k-1}+1} (\alpha_k)_{l_k+1}} \right] \times \frac{\gamma_{k-1}\gamma_k}{\eta^2} \prod_{j=k+1}^{n+1} \Psi_j(\mathbf{Z}_{j-1}), \qquad (II.13)$$

where the function $\Psi_j(\mathbf{Z}_{j-1})$ stands for

$$\Psi_j(\mathbf{Z}_{j-1}) = \frac{\gamma_j}{\eta} \sum_{l_j=0}^{\infty} \frac{Y_{j-1}^{l_j}}{(\alpha_j)_{l_j+1}}. \qquad (II.13)$$

The notation $\Psi_j(\mathbf{Z}_{j-1})$ is used to reflect the fact that there is a first-level customer about to quit the the camp-on system at position $i < j$. Hence, all the second-level systems at positions $j \geq i$ must be shifted one place up with respect to the service center.

Define a temporary changes of variables $l'_k = l_k + l_{k-1}$ and $l'_j = l_j$ if $j \neq k$. Also, for ease of notation, define α_n^{k-1} as

$$\alpha_n^{k-1} = \mu_n + \sum_{\substack{i=1 \\ l_{k-1}=0}}^{n-1} l_i, \quad n > k, \qquad (II.14)$$

or, what is equivalent,

$$\alpha_n = \alpha_n^{k-1} + l_{k-1}.$$

Here we are taking advantage of the fact that l'_k is just a dummy variable that can be renamed l_k. Thus, Equation $(II.13)$ becomes

$$\nu_k P(\mathbf{Z}_{n+1,k}) = \nu_k P(\mathbf{Z}_{k-2}) \left[\sum_{l_{k-1}=0}^{\infty} \sum_{l_k=l_{k-1}}^{\infty} \frac{Y_{k-1}^{l_k}}{(\alpha_{k-1})_{l_{k-1}+1}(\alpha_k)_{l_k-l_{k-1}+1}} \right] \times \frac{\gamma_{k-1}\gamma_k}{\eta^2} \prod_{j=k+1}^{n+1} \Psi_j(\mathbf{Z}_{j-1}^{k-1}). \qquad (II.15)$$

Here, $\Psi_j(\mathbf{Z}_{j-1}^{k-1})$ reflects the effect that $l_{k-1} = 0$ for those α_n with $n > k$.

From the algebraic relationship in Equation $(II.8)$, the order of summation with respect to the indices l_k and l_{k-1} can be interchanged. Taking into account the correct upper bounds for this change, we conclude that the general term $\nu_k P(\mathbf{Z}_{n+1,k})$ in Equation $(II.15)$ can also be expressed as

$$\nu_k P(\mathbf{Z}_{n+1,k}) = P(\mathbf{Z}_{k-2}) \sum_{l_k=0}^{\infty} Y_{k-1}^{l_k} \frac{\gamma_{k-1}\gamma_k}{\eta^2} \times \qquad (II.16)$$
$$\left[\sum_{l_{k-1}=0}^{l_k} \frac{\nu_k}{(\alpha_{k-1})_{l_{k-1}+1}(\alpha_k)_{l_k-l_{k-1}+1}} \right] \prod_{j=k+1}^{n+1} \Psi_j(\mathbf{Z}_{j-1}^{k-1}).$$

Let S_k denote the accumulated sum up to the k^{th} step on the right side of Equation $(II.7)$:

$$S_k = \sum_{i=1}^{k} \nu_i \, P(\mathbf{Z}_{n+1,i}), \qquad 1 \le k \le n+1. \qquad (II.17)$$

Without loss of generality, assume ν_1 to include both the service and the reneging rate for the customer in service. In order to prove Equation $(II.7)$, we want to show that $S_{n+1} = \gamma_{n+1} \, P(\mathbf{Z}_n)$. We will show this by induction on S_k. First, from Equation $(II.16)$, we see that for $k = 2$, $\mu_1 = \nu_1$, and so S_2 reduces to

$$S_2 = \frac{\mu_1}{\alpha_1} \frac{\gamma_1}{\eta} p_0 \prod_{i=2}^{n+1} \Psi_j(\mathbf{Z}_{j-1}) \; +$$
$$\frac{\gamma_1\gamma_2}{\eta} p_0 \left[\sum_{l_2=0}^{\infty} Y_1^{l_2} \sum_{l_1=0}^{l_2} \frac{\nu_2}{(\alpha_1)_{l_1+1}(\alpha_2)_{l_2-l_1+1}} \right] \prod_{j=3}^{n+1} \Psi_j(\mathbf{Z}_{j-1}^{1}),$$

or

$$S_2 = \frac{\gamma_1\gamma_2}{\eta^2}\, p_0 \sum_{l_2=0}^{\infty} Y_1^{l_2} \left[\frac{\eta}{(\alpha_2)_{l_2+1}} + \sum_{l_1=0}^{l_2} \frac{\nu_2}{(\alpha_1)_{l_1+1}(\alpha_2)_{l_2-l_1+1}} \right] \times$$
$$\prod_{j=3}^{n+1} \Psi_j(\mathbf{Z}_{j-1}^1). \qquad (II.18)$$

At this point, it is convenient to generalize the expression inside the brackets in Equation $(II.18)$. This will be an important step in the inductive argument. We will call this expression $R(l_k)$. We begin by denoting $L_k(l_k)$ as

$$L_k(l_k) = \frac{\eta}{(\alpha_k)_{l_k}(\alpha_{k+1})_{l_{k+1}-l_k+1}}.$$

Let $s(l_{k-1})$ be defined by

$$\begin{aligned} s(l_{k-1}) &= L_{k-1}(l_{k-1}) + \frac{\nu_k}{(\alpha_{k-1})_{l_{k-1}+1}(\alpha_k)_{l_k-l_{k-1}+1}} \\ &= \frac{\eta}{(\alpha_{k-1})_{l_{k-1}}(\alpha_k)_{l_k-l_{k-1}+1}} + \frac{\nu_k}{(\alpha_{k-1})_{l_{k-1}+1}(\alpha_k)_{l_k-l_{k-1}+1}}. \end{aligned}$$

Notice that for $k = 2$, $s(l_2 = 0)$ corresponds to the expression inside brackets in Equation $(II.18)$, since $l_1 = 0$. Recalling the definitions of $(\alpha_j)_{l_j+1}$ in Equations $(II.3)$ to $(II.5)$ and factorizing common terms in Equation $(II.18)$, we find that

$$\begin{aligned} s(l_{k-1}) &= L_{k-1}(l_{k-1}) \left[1 + \frac{\nu_k}{\alpha_{k-1} + l_{k-1}} \right] \\ &= \frac{(\alpha_{k-1} + l_{k-1}) + \nu_k/\eta}{\alpha_{k-1} + l_{k-1}} L_{k-1}(l_{k-1}). \end{aligned}$$

Also, from Equations $(II.3)$ and $(II.5)$, we notice that

$$\begin{aligned} (\alpha_{k-1} + l_{k-1}) + \frac{\nu_k}{\eta} &= \alpha_k, \\ (\alpha_k + l_k)(\alpha_k)_{l_k} &= (\alpha_k)_{l_k+1}, \end{aligned}$$

and

$$\frac{\alpha_k}{(\alpha_k)_{l_k - l_{k-1}+1}} = \frac{1}{(\alpha_k + 1)_{l_k - l_{k-1}}}.$$

Therefore, the expression for $s(l_{k-1})$ boils down to

$$s(l_{k-1}) = L_{k-1}(l_{k-1})\frac{\alpha_k}{\alpha_{k-1} + l_{k-1}} = L_{k-1}(l_{k-1}+1). \qquad (II.19)$$

From this recursion on $L_{k-1}(l_{k-1})$, we note that the generalized summation $R(l_k)$ is of the form

$$\begin{aligned} R(l_k) &= L_{k-1}(0) + \sum_{l_{k-1}=0}^{l_k} \frac{\nu_k/\eta}{\alpha_k + \eta_k - 1} L_{k-1}(l_{k-1}) \\ &= L_{k-1}(m) + \sum_{l_{k-1}=m}^{l_k} \frac{\nu_k/\eta}{\alpha_k + \eta_k - 1} L_{k-1}(l_{k-1}) \\ &= L_{k-1}(l_k). \end{aligned}$$

Therefore, evaluating $L(l_{k-1})$ at $l_{k-1} = l_k$, we conclude that

$$\begin{aligned} R(l_k) &= \frac{\eta}{(\alpha_{k-1})_{l_k+1}(\alpha_k)_{l_k-l_k}} \\ &= \frac{\eta}{(\alpha_{k-1})_{l_k+1}}. \end{aligned} \qquad (II.20)$$

Going back to Equation $(II.18)$ and applying that equation with $k = 2$, we now have that $R(l_2) = \eta/(\alpha_1)_{l_2+1}$ so that S_2 reduces to

$$S_2 = \frac{\gamma_1\gamma_2}{\eta^2}\, p_0 \left[\sum_{l_2=0}^{\infty} Y_1^{l_2} \frac{\eta}{(\alpha_1)_{l_2+1}}\right] \prod_{j=3}^{n+1} \frac{\gamma_j}{\eta} \sum_{l_j=0}^{\infty} \frac{Y_{j-1}^{l_j}}{(\alpha_j^1)_{l_j+1}}.$$

Renaming indices such that $l_2 = l_1'$ and $l_1 = l_2'$, α_j^1 becomes α_j^2, and S_2 turns out to be

$$\begin{aligned} S_2 &= p_0 \sum_{l_1=0}^{\infty} \frac{\gamma_1}{\eta} \frac{Y_1^{l_1}}{(\alpha_1)_{l_1+1}}\, \gamma_2 \prod_{j=3}^{n+1} \frac{\gamma_j}{\eta} \sum_{l_j=0}^{\infty} \frac{Y_{j-1}^{l_j}}{(\alpha_j^2)_{l_j+1}} \\ &= p_0 \Psi_1(\mathbf{Z}_1)\gamma_2 \prod_{j=3}^{n+1} \frac{\gamma_j}{\eta} \sum_{l_j=0}^{\infty} \frac{Y_{j-1}^{l_j}}{(\alpha_j^2)_{l_j+1}}. \end{aligned}$$

We claim that, in general, S_k is of the form

$$S_k = P(\mathbf{Z}_{k-1})\,\gamma_k \prod_{j=k+1}^{n+1} \frac{\gamma_j}{\eta} \sum_{l_j=0}^{\infty} \frac{Y_{j-1}^{l_j}}{(\alpha_j^k)_{l_j+1}}. \qquad (II.21)$$

We will prove this claim by induction on k. If this expression for S_k is true, then from Equations $(II.17)$ and $(II.21)$ it must be that

$$\begin{aligned} S_k &= S_{k-1} + \nu_k P(\mathbf{Z}_{n+1,k}) \\ &= P(\mathbf{Z}_{k-2})\frac{\gamma_{k-1}\gamma_k}{\eta^2} \sum_{l_k=0}^{\infty} Y_{k-1}^{l_k} \times \\ &\left[\frac{\eta}{(\alpha_k^{k-1})_{l_k+1}} + \sum_{n_{k-1}=0}^{l_k} \frac{\nu_k}{(\alpha_{k-1})_{l_{k-1}+1}(\alpha_k^{k-1})_{l_k-l_{k-1}+1}} \right] \times \qquad (II.22) \\ &\prod_{j=k+1}^{n+1} \frac{\gamma_j}{\eta} \sum_{l_j=0}^{\infty} \frac{Y_{j-1}^{l_j}}{(\alpha_j^{k-1})_{l_j+1}}. \end{aligned}$$

Again, we recognize the expression inside brackets in Equation $(II.22)$ as $R(l_k)$. Thus, it follows from the relation in $(II.20)$ that

$$S_k = P(\mathbf{Z}_{k-2})\frac{\gamma_{k-1}}{\eta} \sum_{l_k=0}^{\infty} \frac{Y_{k-1}^{l_k}}{(\alpha_{k-1})_{l_k}}\, \gamma_k \prod_{j=k+1}^{n+1} \frac{\gamma_j}{\eta} \sum_{l_j=0}^{\infty} \frac{Y_{j-1}^{l_j}}{(\alpha_j^{k-1})_{l_j+1}}.$$

Finally, interchanging the name of the summation indices l_k and l_{k-1}, α_j^{k-1} becomes α_j^k, and the expression for S_k transforms into

$$S_k = P(\mathbf{Z}_{k-1})\gamma_k \prod_{j=k+1}^{n+1} \frac{\gamma_j}{\eta} \sum_{l_j=0}^{\infty} \frac{Y_{j-1}^{l_j}}{(\alpha_j^k)_{l_j+1}}.$$

This is the same expression as in Equation $(II.21)$, as promised. This proves the claim on S_k.

It now follows immediately that for $k = n + 1$,

$$S_{n+1} = p_0 \prod_{i=1}^{n} \frac{\gamma_i}{\eta} \sum_{l_i=0}^{\infty} \frac{Y_i^{l_i}}{(\alpha_i)_{l_i+1}} \gamma_n$$

$$= \gamma_n P(\mathbf{Z}_n).$$

This proves the consistency of $P(\mathbf{Z}_n)$ with the transformed independent balance Equation $(II.7)$. By uniqueness of the solution of the first-order partial differential equation, $P(\mathbf{Z}_n)$ as proposed in Equation $(II.1)$ is indeed the unique solution of the transformed equilibrium joint probability distribution of the queue lengths for the camp-on queueing model.

APPENDIX III:

Proof of Theorem 2

In this appendix we prove the statement made in Theorem 2, Chapter 3, that the joint probability distribution of queue lengths in a non-reneging camp-on system with infinite or semi-infinite storage capacity is given by

$$p(\mathbf{x}_n) \;=\; p_0\, \psi_1(\mathbf{x}_1)\cdots\psi_n(\mathbf{x}_n), \tag{III.1}$$

where

$$\psi_i(\mathbf{x}_i) \;=\; \frac{\gamma_i}{\xi_i} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \binom{k_i+l_i}{l_i} \left[\frac{\lambda_{2r_i}}{\xi_i}\right]^{k_i} \left[\frac{\xi_{i-1}}{\xi_i}\right]^{l_i}, \tag{III.2}$$

and

$$\xi_n \;=\; \xi_n(\mathbf{x}_n) \;=\; \mu_n + \sum_{i=1}^{n} \lambda_{2r_i}. \tag{III.3}$$

Essentially, we want to show that the state-probability distribution proposed in Equation $(III.1)$ is a solution to the independent balance equations for the non-reneging camp-on system:

$$\left[\sum_{i=1}^{n}(\lambda_1+\nu_i)+\mu\chi_{(n>0)}\right] p(\mathbf{x}_n) \;=\; \gamma_n\, \chi_{(k_n=0)}\, p(\mathbf{x}_{n-1}) + \sum_{i=1}^{n} \lambda_{2r_i}\, \chi_{(k_i>0)}\, p(\mathbf{x}_n^{-i}), \quad (1 \le n \le N) \tag{III.4}$$

and

$$\lambda_{1n}\, p(\mathbf{x}_n) \;=\; \sum_{r_0=1}^{R} \chi_{(r_0\neq r_i)} \sum_{k_0=0}^{\infty} \mu p(\mathbf{x}_{n+1,1}) + \sum_{r_0=0}^{R} \chi_{(r_0\neq r_i)} \sum_{i=1}^{n+1} \sum_{k_0=0}^{\infty} \nu_i\, p(\mathbf{x}_{n+1,i}), \qquad (0 \le n < N). \tag{III.5}$$

As in Appendix II, we start with the independent balance equation $(III.4)$. For $n \geq 1$, this balance equation $(III.2)$ could also have been written as

$$p(\mathbf{x}_n) = \sum_{j=1}^{n} \frac{\lambda_{r_j}}{\xi_n} \chi_{(k_i>0)}\, p(\mathbf{x}_n^{-j}). \qquad (III.6)$$

Let S_i be the accumulated sum of the first i terms in Equation $(II.6)$:

$$S_i = \sum_{j=1}^{i} \frac{\lambda_{2r_j}}{\xi_n} p(\mathbf{x}_n^{-j}).$$

We want first to show that

$$S_i = \frac{\lambda_{2r_i}}{\xi_n}\, p(\mathbf{x}_n^{-i}) \frac{\binom{k_i+l_i}{l_i}}{\binom{k_i-1+l_i}{l_i}}$$

$$= \frac{\lambda_{2r_i}}{\xi_n}\, p(\mathbf{x}_n^{-i}) \frac{k_i + l_i}{k_i}. \qquad (III.7)$$

Clearly, this statement is true for $i = 1$, since $l_0 = l_1 = 0$ in $\Psi_1(\mathbf{Z}_1)$. Using induction, assume that Equation $(III.7)$ also holds for some $i > 1$. We must then have

$$S_i = S_{i-1} + \frac{\lambda_{2r_i}}{\xi_n}\, p(\mathbf{x}_n^{-i})$$

$$= \frac{\lambda_{2r_{i-1}}}{\xi_n}\, p(\mathbf{x}_n^{-i+1}) \frac{k_{i-1} + l_{i-1}}{k_{i-1}} + \frac{\lambda_{2r_i}}{\xi_n}\, p(\mathbf{x}_n^{-i}).$$

Replace $p(\mathbf{x}_n^{-i})$ and $p(\mathbf{x}_n^{-i+1})$ by its corresponding expression based on Equation $(III.1)$. Then, S_i can be expressed as

$$S_i = \frac{\lambda_{2r_{i-1}}}{\xi_n}\, p(\mathbf{x}_{i-2})\, \psi_{i-1}(\mathbf{x}_{i-1}^{-i+1})\, \psi_i(\mathbf{x}_i^{-i+1}) \frac{k_{i-1} + l_{i-1}}{k_{i-1}}\, f_n(i+1)$$
$$+ \frac{\lambda_{2r_i}}{\xi_n}\, p(\mathbf{x}_{i-1})\, \psi_i(\mathbf{x}_i^{-i})\, \psi_{i+1}(\mathbf{x}_{i+1}^{-i})\, f_n(i+2)$$

$$= p(\mathbf{x}_{i-1})\, s_i\, f_n(i+2), \qquad (III.8)$$

where the functions s_i and $f_n(i)$ are of the form:

$$s_i \;=\; \frac{\xi_{i-1}}{\xi_n}\psi_i(\mathbf{x}_i^{-i+1})\;\psi_{i+1}(\mathbf{x}_{i+1}) \;+\; \frac{\lambda_{2r_i}}{\xi_n}\psi_i(\mathbf{x}_i^{-i})\;\psi_{i+1}(\mathbf{x}_{i+1}^{-i}), \quad (III.9)$$

$$f_n(j) \;=\; \prod_{i=j}^{n}\psi_j(\mathbf{x}_j). \quad (III.10)$$

Here we have used the fact that

$$\lambda_{2r_i}\psi_i(\mathbf{x}_i^{-i}) \;=\; \xi_i\psi_i(\mathbf{x}_i). \quad (III.11)$$

Let us concentrate on the term s_i in Equation (III.8). Inserting $\psi_i(\mathbf{x}_i)$ from Equation (III.2) into Equation (III.9), it follows that

$$\begin{aligned} s_i \;=\;& \frac{\xi_{i-1}}{\xi_n}\;\psi_i(\mathbf{x}_i^{-i+1})\psi_{i+1}(\mathbf{x}_{i+1}) \;+\; \frac{\lambda_{2r_i}}{\xi_n}\;\psi_i(\mathbf{x}_i^{-i})\psi_{i+1}(\mathbf{x}_{i+1}^{-i}) \\ =\;& \frac{\gamma_i\gamma_{i+1}}{\xi_i\xi_{i+1}}\sum_{l_i=0}^{k_{i-1}-1+l_{i-1}}\;\sum_{l_{i+1}=0}^{k_i+l_i}\binom{k_i+l_i}{l_i}\binom{k_{i+1}+l_{i+1}}{l_{i+1}}x_i^{k_i}\,x_{i+1}^{k_{i+1}}\,y_i^{l_i}\,y_{i+1}^{l_{i+1}}\frac{\xi_{i-1}}{\xi_n} \;+ \\ & \frac{\gamma_i\gamma_{i+1}}{\xi_i\xi_{i+1}}\sum_{l_i=0}^{k_{i-1}+l_{i-1}}\;\sum_{l_{i+1}=0}^{k_i+l_i-1}\binom{k_i-1+l_i}{l_i}\binom{k_{i+1}+l_{i+1}}{l_{i+1}}x_i^{k_i}\,x_{i+1}^{k_{i+1}}\,y_i^{l_i}\,y_{i+1}^{l_{i+1}}\frac{\xi_i}{\xi_n}. \end{aligned}$$

Define the change of variable $j = l_i + 1$ for the first term of s_i above. The factor ξ_{i-1} cancels out and in turn gets replaced by ξ_i. If we also define the change of variable $l_i = j$ for the second term of s_i, then

$$\begin{aligned} s_i \;=\;& \frac{\gamma_i\gamma_{i+1}}{\xi_i\xi_{i+1}}\sum_{j=1}^{k_{i-1}+l_{i-1}}\;\sum_{l_{i+1}=0}^{k_i+j-1}\binom{k_i+j-1}{j-1}\binom{k_{i+1}+l_{i+1}}{l_{i+1}}x_i^{k_i}\,x_{i+1}^{k_{i+1}}\,y_i^{j}\,y_{i+1}^{l_{i+1}}\frac{\xi_i}{\xi_n} \;+ \\ & \frac{\gamma_i\gamma_{i+1}}{\xi_i\xi_{i+1}}\sum_{j=0}^{k_{i-1}+l_{i-1}}\;\sum_{l_{i+1}=0}^{k_i+j-1}\binom{k_i+j-1}{j}\binom{k_{i+1}+l_{i+1}}{l_{i+1}}x_i^{k_i}\,x_{i+1}^{k_{i+1}}\,y_i^{j}\,y_{i+1}^{l_{i+1}}\frac{\xi_i}{\xi_n}. \end{aligned}$$

We observe that both terms in s_i above are almost identical except for the combinatorial coefficient involving the summation index j. Using the combinatorial identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \qquad (III.12)$$

this allows us to write s_i as a single expression of the form

$$s_i = \frac{\gamma_i\gamma_{i+1}}{\xi_i\xi_{i+1}} \sum_{j=0}^{k_{i-1}+l_{i-1}} \sum_{l_{i+1}=0}^{k_i+j-1} \binom{k_i+j}{j}\binom{k_{i+1}+l_{i+1}}{l_{i+1}} x_i^{k_i}\, x_{i+1}^{k_{i+1}}\, y_i^j\, y_{i+1}^{l_{i+1}} \frac{\xi_i}{\xi_n}. \qquad (III.13)$$

Moreover, after renaming j as l_i and recognizing the terms $\psi_i(\mathbf{x}_i)$ and $\psi_{i+1}(\mathbf{x}_{i+1}^{-i})$ and then using the relationship in Equation $(III.11)$, it is found that the expression for s_i is equivalent to

$$\begin{aligned} s_i &= \frac{\gamma_i}{\xi_i} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \binom{k_i+l_i-1}{l_i} \frac{k_i+l_i}{k_i} x_i^{k_i}\, y_i^{l_i} \times \\ &\qquad \frac{\gamma_{i+1}}{\xi_{i+1}} \sum_{l_{i+1}=0}^{k_i+l_i-1} \binom{k_{i+1}+l_{i+1}}{l_{i+1}} x_{i+1}^{k_{i+1}}\, y_{i+1}^{l_{i+1}} \frac{\xi_i}{\xi_n} \\ &= \frac{\lambda_{2r_i}}{\xi_n}\, \psi_i(\mathbf{x}_i^{-i})\, \psi_{i+1}(\mathbf{x}_{i+1}^{-i}) \frac{k_i+l_i}{k_i}. \end{aligned} \qquad (III.14)$$

Substituting the above result for s_i back into Equation $(III.8)$, we get

$$S_i = \frac{\lambda_{2r_i}}{\xi_n}\, p(\mathbf{x}_{i-1})\, \psi_i(\mathbf{x}_i^{-i})\, \psi_{i+1}(\mathbf{x}_{i+1}^{-i})\, \frac{k_i+l_i}{k_i} f_n(i+2).$$

Note that this expression for S_i is identical to the first term of Equation $(III.8)$ if we use i instead of $i-1$ as the index. Therefore,

$$S_i = \frac{\lambda_{2r_i}}{\xi_n}\, p(\mathbf{x}_n^{-i}) \frac{k_i+l_i}{k_i}.$$

But this is the same expression as given for S_i in Equation ($III.7$). We conclude that the inductive hypothesis in Equation ($III.7$) is true for all $i \geq 1$.

In particular, for $i = n$, it follows from this last result that

$$S_n = \sum_{i=1}^{n} \frac{\lambda_{2r_i}}{\xi_n} p(\mathbf{x}_n^{-i}) = \frac{\lambda_{2r_n}}{\xi_n} p(\mathbf{x}_n^{-n}) \frac{k_n + l_n}{k_n}.$$

From the definitions of the state probability $p(\mathbf{x}_n^{-n})$ and $\psi_n(\mathbf{x}_n)$,

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\lambda_{2r_i}}{\xi_n} p(\mathbf{x}_n^{-i}) &= \frac{\lambda_{2r_i}}{\xi_n} p(\mathbf{x}_{n-1}) \frac{\gamma_n}{\xi_n} \sum_{l_n=0}^{k_{n-1}+l_{n-1}} \binom{k_n - 1 + l_n}{l_n} \frac{k_n + l_n}{k_n} x_n^{k_n - 1} y_n^{l_n} \\
&= p(\mathbf{x}_{n-1}) \frac{\gamma_n}{\xi_n} \sum_{l_n=0}^{k_{n-1}+l_{n-1}} \binom{k_n + l_n}{l_n} x_n^{k_n} y_n^{l_n} \\
&= p(\mathbf{x}_n).
\end{aligned}$$

This proves that $p(\mathbf{x}_n)$ satisfies the independent balance equation in ($III.4$) for $k_i > 0$. However, we still have to show that Equation ($III.1$) holds, if $k_i = 0$ for some i. We will have to distinguish between two possible situations: i) $k_i = 0$, but $i \neq n$, and ii) $k_i = k_n = 0$.

i) For $k_1 = 0$, Equation ($III.4$) reduces trivially to the one shown in Equation ($III.7$) for $l_1 = l_2 = 0$. So we need only to consider those cases where $k_i = 0$ and $i > 1$. Using similar arguments as those used in the derivation of Equation ($III.7$), we note that S_i will now be of the form $p(\mathbf{x}_{i-2}) s_i f_n(i+2)$, and in Equation ($III.8$), where

$$s_i = \frac{\xi_{i-2}}{\xi_n} \psi_{i-1}(\mathbf{x}_{i-1}^{-i+2}) \psi_i(\mathbf{x}_i) \psi_{i+1}(\mathbf{x}_{i+1}) + \frac{\lambda_{2r_i}}{\xi_n} \psi_{i-1}(\mathbf{x}_{i-1}) \psi_i(\mathbf{x}_i^{-i}) \psi_{i+1}(\mathbf{x}_{i+1}^{-i}).$$

We have omitted the contribution from the term $p(\mathbf{x}_n^{-i+1})$, since we have assumed $k_{i-1} = 0$ $(i > 1)$.

Now replace $\psi_{i-1}(\mathbf{x}_{i-1},\ \psi_i(\mathbf{x}_i)$ and $\psi_{i+1}(\mathbf{x}_{i+1})$ by their corresponding expressions according to Equation $(III.2)$. Then s_i turns out to be of the form

$$s_i = \frac{\gamma_{i-1}\gamma_i\gamma_{i+1}}{\xi_{i-1}\xi_i\xi_{i+1}} \sum_{l_{i-1}=0}^{k_{i-2}-1+l_{i-2}} \sum_{l_i=0}^{l_{i-1}} \sum_{l_{i+1}=0}^{k_i+l_i} \binom{k_i+l_i}{l_i}\binom{k_{i+1}+l_{i+1}}{l_{i+1}} \times$$
$$x_i^{k_i}\, x_{i+1}^{k_{i+1}}\, y_{i-1}^{l_{i-1}}\, y_i^{l_i}\, y_{i+1}^{l_{i+1}}\, \frac{\xi_{i-2}}{\xi_n} \qquad +$$
$$\frac{\gamma_{i-1}\gamma_i\gamma_{i+1}}{\xi_{i-1}\xi_i\xi_{i+1}} \sum_{l_{i-1}=0}^{k_{i-2}+l_{i-2}} \sum_{l_i=0}^{l_{i-1}} \sum_{l_{i+1}=0}^{k_i-1+l_i} \binom{k_i-1+l_i}{l_i}\binom{k_{i+1}+l_{i+1}}{l_{i+1}} \times$$
$$x_i^{k_i}\, x_{i+1}^{k_{i+1}}\, y_{i-1}^{l_{i-1}}\, y_i^{l_i}\, y_{i+1}^{l_{i+1}}\, \frac{\xi_i}{\xi_n}.$$

Define the changes of variables $j = l_{i-1} + 1$ and $m = l_i + 1$ for the first term of s_i. Then the factor ξ_{i-2} cancels out and in turn is replaced by ξ_i.

If we now define the changes of variables $j = l_{i-1}$ and $m = l_i$ for the second term of s_i, it follows that

$$s_i = \frac{\gamma_{i-1}\gamma_i\gamma_{i+1}}{\xi_{i-1}\xi_i\xi_{i+1}} \sum_{j=1}^{k_{i-2}+l_{i-2}} \sum_{m=1}^{j} \sum_{l_{i+1}=0}^{k_i+m-1} \binom{k_i+m-1}{m-1}\binom{k_{i+1}+l_{i+1}}{l_{i+1}} \times$$
$$x_i^{k_i}\, x_{i+1}^{k_{i+1}}\, y_{i-1}^{j}\, y_i^{m}\, y_{i+1}^{l_{i+1}}\, \frac{\xi_i}{\xi_n} \qquad +$$
$$\frac{\gamma_{i-1}\gamma_i\gamma_{i+1}}{\xi_{i-1}\xi_i\xi_{i+1}} \sum_{j=0}^{k_{i-2}+l_{i-1}} \sum_{m=0}^{j} \sum_{l_{i+1}=0}^{k_i+m-1} \binom{k_i+m-1}{m}\binom{k_{i+1}+l_{i+1}}{l_{i+1}} \times$$
$$x_i^{k_i}\, x_{i+1}^{k_{i+1}}\, y_{i-1}^{j}\, y_i^{m}\, y_{i+1}^{l_{i+1}}\, \frac{\xi_i}{\xi_n}.$$

As before, both terms in s_i are almost identical except for the combinatorial coefficient involving the variable m.

Using the combinatorial identity of Equation $(III.12)$, we find that

$$s_i = \frac{\gamma_{i-1}\gamma_i\gamma_{i+1}}{\xi_{i-1}\xi_i\xi_{i+1}} \sum_{j=0}^{k_{i-2}+l_{i-2}} \sum_{m=0}^{j} \sum_{l_{i+1}=0}^{k_i+m-1} \binom{k_i+m}{m}\binom{k_{i+1}+l_{i+1}}{l_{i+1}}$$
$$x_i^{k_i}\, x_{i+1}^{k_{i+1}}\, y_{i-1}^{j}\, y_i^{m}\, y_{i+1}^{l_{i+1}}\, \frac{\xi_i}{\xi_n}.$$

Reversing the changes of variables by taking $l_i = j$ and $l_{i+1} = m$, we obtain

$$s_i = \frac{\gamma_{i-1}\gamma_i\gamma_{i+1}}{\xi_{i-1}\xi_i\xi_{i+1}} \sum_{l_{i-1}=0}^{k_{i-2}+l_{i-2}} \sum_{l_i=0}^{l_{i-1}} \sum_{l_{i+1}=0}^{k_i+l_i-1} \binom{k_i+l_i}{l_i}\binom{k_{i+1}+l_{i+1}}{l_{i+1}} x_i^{k_i}\, x_{i+1}^{k_{i+1}}\, y_{i-1}^{l_{i-1}}\, y_i^{l_i}\, y_{i+1}^{l_{i+1}}\, \frac{\xi_i}{\xi_n}.$$

We recognize $\psi_{i-1}(\mathbf{x}_{i-1})$, $\psi_i(\mathbf{x}_i^{-i})$ and $\psi_{i+1}(\mathbf{x}_{i+1}^{-i})$ in the above expressions for s_i. Recalling the relationship in Equation $(III.11)$, we have

$$s_i = \frac{\lambda_{2r_i}}{\xi_n}\psi_{i-1}(\mathbf{x}_{i-1})\ \psi_i(\mathbf{x}_i^{-i})\ \psi_{i+1}(\mathbf{x}_{i+1}^{-i})\ \frac{k_i+l_i}{l_i}. \qquad (III.15)$$

This expression is an obvious generalization of s_i in Equation $(III.14)$ for the special case $k_{i-1} = 0$. If more than one of the k_i's are zero, this procedure can be trivially generalized as a recursive application of this latest case.

ii) For $k_n = 0$, there is no contribution from $p(\mathbf{x}_n^{-n})$ but the second term in the expression for S_n is now replaced by $p(\mathbf{x}_{n-1})$. Consequently, s_n now turns out to be

$$s_n = \frac{\gamma_{n-1}\gamma_n}{\xi_{n-1}\xi_n} \sum_{l_{n-1}=0}^{k_{n-2}+l_{n-2}} \sum_{l_n=0}^{k_{n-1}+l_{n-1}-1} \binom{k_{n-1}+l_{n-1}}{l_{n-1}} x_{n-1}^{k_{n-1}}\, y_{n-1}^{l_{n-1}}\, y_n^{l_n}\, \frac{\xi_{n-1}}{\xi_n} + \frac{\gamma_{n-1}}{\xi_{n-1}} \sum_{l_{n-1}=0}^{k_{n-2}+l_{n-2}} \binom{k_{n-1}+l_{n-1}}{l_{n-1}} x_{n-1}^{k_{n-1}}\, y_{n-1}^{l_{n-1}}\, \frac{\gamma_n}{\xi_n}.$$

However, we observe that the second term of s_n can be thought as an extension of the summation with respect to l_n, if we allow l_n to take the value -1. That is,

$$\psi_{n-1}(\mathbf{x}_{n-1}) = \frac{\gamma_{n-1}}{\xi_{n-1}} \sum_{l_{n-1}=0}^{k_{n-2}+l_{n-2}} \binom{k_{n-1}+l_{n-1}}{l_{n-1}} x_{n-1}^{k_{n-1}}\, y_{n-1}^{l_{n-1}}\, \frac{\gamma_n}{\xi_n}$$

$$= \frac{\gamma_{n-1}}{\xi_{n-1}} \sum_{l_{n-1}=0}^{k_{n-2}+l_{n-2}} \binom{k_{n-1}+l_{n-1}}{l_{n-1}} x_{n-1}^{k_{n-1}}\, y_{n-1}^{l_{n-1}}\, y_n^{-1}.$$

It follows from the above result that s_n reduces to

$$s_n = \frac{\gamma_{n-1}\gamma_n}{\xi_{n-1}\xi_n} \sum_{l_{n-1}=0}^{k_{n-2}+l_{n-2}} \sum_{l_n=-1}^{k_{n-1}+l_{n-1}-1} \binom{k_{n-1}+l_{n-1}}{l_{n-1}} x_{n-1}^{k_{n-1}} y_{n-1}^{l_{n-1}} y_n^{l_n} \frac{\xi_{n-1}}{\xi_n}.$$

If we define the change of variable $j = l_n + 1$, make the corresponding transformation on s_n, and then rename j as l_n, we obtain

$$s_n = \frac{\gamma_{n-1}\gamma_n}{\xi_{n-1}\xi_n} \sum_{l_{n-1}=0}^{k_{n-2}+l_{n-2}} \sum_{j=0}^{k_{n-1}+l_{n-1}} \binom{k_{n-1}+l_{n-1}}{l_{n-1}} x_{n-1}^{k_{n-1}} y_{n-1}^{l_{n-1}} y_n^{j} \frac{\xi_n}{\xi_{n-1}} \frac{\xi_{n-1}}{\xi_n}$$

$$= \frac{\gamma_{n-1}\gamma_n}{\xi_{n-1}\xi_n} \sum_{l_{n-1}=0}^{k_{n-2}+l_{n-2}} \sum_{l_n=0}^{k_{n-1}+l_{n-1}} \binom{k_{n-1}+l_{n-1}}{l_{n-1}} x_{n-1}^{k_{n-1}} y_{n-1}^{l_{n-1}} y_n^{l_n}.$$

Finally, recognizing $\psi_{n-1}(\mathbf{x}_{n-1})$ and $\psi_n(\mathbf{x}_n)$ in the above expression for s_n , we conclude that

$$s_n = \psi_{n-1}(\mathbf{x}_{n-1})\, \psi_n(\mathbf{x}_n).$$

This is again in obvious agreement with the definition of s_n according to Equation $(III.15)$ for $\psi_{n+1} \equiv 0$. This completes the proof that the state probability distribution $p(\mathbf{x}_n)$ proposed in Equation $(III.1)$ is a solution to the independent balance equation in $(III.4)$.

Equation $(III.5)$, as mentioned before, deals with the effect that departing first-level customers have on the camp-on model. In order to prove the consistency of the proposed state-probability distribution $p(\mathbf{x}_n)$ with the independent balance equation in $(III.5)$, we have to redefine some of the camp-on parameters in Equation $(III.1)$ to accommodate the prevailing conditions at the second-level systems when a first-level customer is about to quit the camp-on system because of service completion or defection.

Let λ_{2r_0} denote the arrival rate to the second-level system associated with the departing first-level customer at position i of the queue $(\lambda_{2r_0} \neq \lambda_{2r_i},\ 1 \leq i \leq n)$. Define

$$x_{oi} = \frac{\lambda_{2r_{i-1}}}{\xi_{oi}}, \tag{III.16}$$

$$y_{oi} = \frac{\xi_{oi-1}}{\xi_{oi}}, \tag{III.17}$$

$$\xi_{oi} = \mu_i + \sum_{k=0}^{i-1} \lambda_{2r_i}, \tag{III.18}$$

$$\theta_i = \mu_i + \sum_{k=1}^{i-1} \lambda_{2r_i}. \tag{III.19}$$

These parameters are the equivalents to x_i, y_i, and ξ_i in Equations (3.3) and (3.5). As a consequence of inserting an arbitrary first-level customer and its associated second-level system with queue length k_0 at position i in the first-level queue, all the second-level systems above position $i-1$ are shifted one place up with respect to their distance from the service center. So we have to use k_{j-1} instead of k_j for all $j > i$ to indicate the proper queueing condition prevailing at the second-level queues when we refer to the function ψ_i in Equation $(III.1)$. Here,we will write $\psi_i(\mathbf{x}_{i-1})$ instead of $\psi_i(\mathbf{x}_i)$ to indicate the presence of a first-level customer at position $j < i$ that is about to leave this queueing system.

Define s_i as the i^{th} contribution to the right side of the independent balance Equation $(III.5)$. Then, from the definitions from $(III.16)$ to $(III.19)$ for a departing customer, we have

$$\begin{aligned} s_i &= \nu_i \sum_{k_0=0}^{\infty} p(\mathbf{x}_{n+1,i}) \\ &= \nu_i \sum_{k_0=0}^{\infty} p(\mathbf{x}_{i-1}) \prod_{\substack{j=i \\ k_{i-1}=k_0}}^{n+1} \psi_j(\mathbf{x}_{j-1}), \end{aligned} \tag{III.20}$$

with the function $\psi_i(\mathbf{x}_{i-1})$ given by

$$\psi_i(\mathbf{x}_{i-1}) \;=\; \frac{\gamma_{i-1}}{\xi_{oi}} \sum_{l_i=0}^{k_{i-2}+l_{i-1}} \binom{k_{i-1}+l_i}{l_i} x_{oi}^{k_{i-1}} y_{oi}^{l_i}, \qquad (III.21)$$

and $\gamma_{i-1} = \gamma_0$ for $j = i$. Without lost of generality, ν_1 is assumed to include both the service and reneging rates for the customer in service.

First, we want to express s_i in terms of the state probability distribution $p(\mathbf{x}_n)$. For $i = 1$, we observe from Equation $(III.19)$ that $\nu_1 = \mu_1 = \theta_1$. The above expression for s_1 reduces to

$$\begin{aligned} s_1 &= \theta_1 p_0 \sum_{k_0=0}^{\infty} \frac{\gamma_0}{\xi_{o1}} x_{o1}^{k_0} \prod_{i=2}^{n+1} \psi_i(\mathbf{x}_{i-1}) \\ &= \theta_1 p_0 \sum_{k_0=0}^{\infty} \frac{\gamma_0}{\xi_{o1}} x_{o1}^{k_0} \prod_{i=2}^{n+1} \sum_{l_i=0}^{k_{i-2}+l_{i-1}} \binom{k_{i-1}+l_i}{l_i} x_{oi-1}^{k_{i-1}} y_{oi}^{l_i}. \end{aligned}$$

Clearly, the summation index l_1 inside ψ_1 is always zero, since $k_{-1} = l_0 = 0$. Thus, s_1 could also have been written as

$$s_1 \;=\; \theta_1 p_0 \frac{\gamma_0}{\xi_{o1}} \sum_{k_0=0}^{\infty} x_{o1}^{k_0+l_1} \frac{\gamma_1}{\xi_{o2}} \sum_{l_2=0}^{k_0+l_1} \binom{k_1+l_2}{k_1+l_1} x_{o2}^{k_1} y_{o2}^{l_2} \prod_{i=3}^{n+1} \psi_i(\mathbf{x}_{i-1}).$$

If we interchange the order of the summations with respect to the indices k_0 and l_2 according to the algebraic relationship in $(II.8)$, it follows that

$$s_1 \;=\; \theta_1 p_0 \frac{\gamma_0\gamma_1}{\xi_{o1}\xi_{o2}} \sum_{l_2=0}^{\infty} \sum_{k_0=l_2-l_1}^{\infty} \binom{k_1+l_2}{k_1+l_1} x_{o1}^{k_0+l_1} x_{o2}^{k_1} y_{o2}^{l_2} \prod_{i=3}^{n+1} \psi_i(\mathbf{x}_{i-1}).$$

However, the summation with respect to k_0 is no more than a geometric series with ratio $x_{o1} < 1$, so the above expression yields

$$s_1 \;=\; \theta_1 p_0 \frac{\gamma_0\gamma_1}{\xi_{o1}\xi_{o2}} \sum_{l_2=0}^{\infty} \frac{x_{o1}^{l_2}}{1-x_{o1}} \binom{k_1+l_2}{k_1+l_1} x_{o2}^{k_1} y_{o2}^{l_2} \prod_{i=2}^{n+1} \psi_i(\mathbf{x}_{i-1}).$$

Moreover, from the definition of ξ_{oi} in Equation (III.18), we find that

$$1 - \frac{\lambda_{2r_0}}{\xi_{oi}} = \frac{\mu_i + \sum_{k=1}^{i-1} \lambda_{2r_k}}{\mu_i + \sum_{k=0}^{i-1} \lambda_{2r_k}} = \frac{\theta_i}{\xi_{oi}}. \qquad (III.22)$$

Therefore, after evaluating $1 - x_{o1}$ in s_1 and canceling out common factors, we get

$$s_1 = p_0 \frac{\gamma_0 \gamma_1}{\xi_{o2}} \sum_{l_2=0}^{\infty} \binom{k_1 + l_2}{k_1 + l_1} x_{o2}^{k_1} \left(\frac{\lambda_{2r_0}}{\xi_{o2}}\right)^{l_2} \prod_{i=3}^{n+1} \psi_i(\mathbf{x}_{i-1}). \qquad (III.23)$$

Before going through with the rest of the reformulation of s_1 as well as the other s_i's, it will be appropriate to carry out a side computation that will help simplify the upcoming exposition. For this, consider the relation

$$R_i(m) = \gamma_0 \prod_{j=m+1}^{i} \phi_j(\mathbf{x}_j) r_{i+1} \prod_{j=i+2}^{n+1} \psi_j(\mathbf{x}_{j-1}),$$

where the functions $\phi_i(\mathbf{x}_i)$ and r_i are defined as

$$\phi_i(\mathbf{x}_i) = \frac{\gamma_{i-1}}{\theta_i} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \binom{k_i + l_i}{l_i} \left(\frac{\lambda_{2r_{i-1}}}{\theta_i}\right)^{k_{i-1}} \left(\frac{\lambda_{2r_0}}{\theta_i}\right)^{l_{i-1}-l_i} \qquad (III.24)$$

$$r_i = \frac{\gamma_{i-1}}{\xi_{oi}} \sum_{l_i=0}^{\infty} \binom{k_{i-1} + l_i}{k_{i-1} + l_{i-1}} x_{oi}^{k_{i-1}} \left(\frac{\lambda_{2r_0}}{\xi_{oi}}\right)^{l_i} \psi_{i+1}(\mathbf{x}_i). \qquad (III.25)$$

Observe that for $m = i = 1$, we have $R_1(1) = r_2 \prod \psi_i(\mathbf{x}_{i-1})$, so that $s_1 = p_0 R_1(1)$ from Equation (III.23). In Appendix IV we will show that for $1 \le i \le n$,

$$\begin{aligned} R_i(m) &= R_n(m) \\ &= \gamma_0 \frac{\gamma_i}{\theta_{n+1}} \left(\frac{\lambda_{2r_m}}{\theta_{n+1}}\right)^{k_m} \left(\frac{\lambda_{2r_0}}{\theta_{n+1}}\right)^{l_m} \prod_{i=m+1}^{n} \psi_i(\mathbf{x}_i) \frac{\xi_i}{\theta_i} \left(\frac{\xi_i}{\theta_{i+1}}\right)^{k_i} \left(\frac{\theta_i \xi_i}{\theta_{i+1} \xi_{i-1}}\right)^{l_i} \\ &= \gamma_0 \frac{\gamma_i}{\theta_{n+1}} f_n(m) = R_n(m), \end{aligned} \qquad (III.26)$$

with the function $f_n(m)$ given by

$$f_n(m) = (\frac{\xi_i}{\theta_{i+1}})^{k_i+l_i} \prod_{i=m+1}^{n} \phi_i(\mathbf{x}_i)\frac{\xi_i}{\theta_i}(\frac{\xi_i}{\theta_{i+1}})^{k_i}(\frac{\theta_i\xi_i}{i\theta_{i+1}\xi_{i-1}})^{l_i}. \qquad (III.27)$$

The Equation $(III.26)$ and the fact that $l_1 \equiv 0$ give

$$\begin{aligned} s_1 &= \gamma_0\frac{\xi_1}{\theta_{n+1}}(\frac{\xi_1}{\theta_2})^{k_1}\prod_{i=2}^{n}\phi_i(\mathbf{x}_i)\frac{\xi_i}{\theta_i}(\frac{\xi_i}{\theta_{i+1}})^{k_i}(\frac{\theta_i\xi_i}{\theta_{i+1}\xi_{i-1}})^{l_i} \\ &= \gamma_0\frac{\xi_1}{\theta_{n+1}}f_n(1). \qquad (III.28) \end{aligned}$$

For $2 \le i \le n$, we must generalize the expression found in Equation $(III.26)$ for $i = 1$. From Equation $(III.20)$, we have

$$\begin{aligned} s_i = \nu_i\, p(\mathbf{x}_{i-1})\frac{\gamma_0}{\xi_{oi}}\sum_{k_0=0}^{\infty}\sum_{l_i=0}^{k_{i-1}+l_{i-1}}\binom{k_0+l_i}{l_i}(\frac{\lambda_{2r_0}}{\xi_{oi}})^{k_0}(\frac{\xi_{i-1}}{\xi_{oi}})^{l_i}\times \\ \frac{\gamma_i}{\xi_{oi+1}}\sum_{l_{i+1}=0}^{k_0+l_i}\binom{k_i+l_{i+1}}{l_{i+1}}x_{oi+1}^{k_i}\,y_{oi+1}^{l_{i+1}}\prod_{j=i+2}^{n+1}\psi_j(\mathbf{x}_{j-1}). \end{aligned}$$

First, we take care of the coefficient k_0 that apppears with the combinatorial coefficient associated with the summation index l_i. If we interchange the order of the summations with respect to the indices k_0 and l_{i+1} as in $(II.8)$ and then define a temporary change of variable $k_0' = k_0 - l_{i+1} + l_i$, it follows that

$$\begin{aligned} s_i &= \nu_i\, p(\mathbf{x}_{i-1})\frac{\gamma_0\gamma_i}{\xi_{oi}\xi_{oi+1}}\sum_{l_i=0}^{k_{i-1}+l_{i-1}}\sum_{l_{i+1}=0}^{\infty}\binom{k_i+l_{i+1}}{l_{i+1}}x_{oi+1}^{k_i}(\frac{\xi_{i-1}}{\xi_{oi}})^{l_i}\times \\ &\quad \sum_{k_0=l_{i+1}-l_i}^{\infty}\binom{k_0+l_i}{l_i}(\frac{\lambda_{2r_0}}{\xi_{oi}})^{k_0}y_{oi+1}^{l_{i+1}}\prod_{j=i+2}^{n+1}\psi_j(\mathbf{x}_{j-1}) \\ &= \nu_i\, p(\mathbf{x}_{i-1})\frac{\gamma_0\gamma_i}{\xi_{oi}\xi_{oi+1}}\sum_{l_i=0}^{k_{i-1}+l_{i-1}}\sum_{l_{i+1}=0}^{\infty}\binom{k_i+l_{i+1}}{l_{i+1}}x_{oi+1}^{k_i}(\frac{\xi_{i-1}}{\xi_{oi}})^{l_i}\times \\ &\quad \sum_{k_0=0}^{\infty}\binom{k_0+l_{i+1}}{l_i}(\frac{\lambda_{2r_0}}{\xi_{oi}})^{k_0+l_{i+1}-l_i}y_{oi+1}^{l_{i+1}}\prod_{j=i+2}^{n+1}\psi_j(\mathbf{x}_{j-1}). \qquad (III.29) \end{aligned}$$

At this point it is convenient to recall some useful combinatorial identities[35]. First, a common series expansion for combinatorial coefficients is

$$\binom{l+k}{j} = \sum_{n=0} \binom{l}{n}\binom{k}{j-n}. \qquad (III.30)$$

Taking $l = l_{i+1}$, $k = k_0$ and $j = l_i$, we find that the second combinatorial coefficient in Equation $(III.29)$ becomes

$$\binom{k_0 + l_{i+1}}{l_i} = \sum_{n=0}^{l_i} \binom{l_{i+1}}{n}\binom{k_0}{l_i - n}. \qquad (III.31)$$

Secondly, another common the combinatorial identity that will come handy at this point is

$$\binom{k+l}{l}\binom{l}{n} = \binom{k+n}{n}\binom{k+l}{k+n}. \qquad (III.32)$$

Thus, if we take $k = k_{i+1}$, $l = l_{i+1}$ and $n = n$, the first combinatorial coefficient in Equations $(III.29)$ and $(III.31)$ transform into

$$\binom{k_i + l_{i+1}}{l_{i+1}}\binom{l_{i+1}}{n} = \binom{k_i + n}{n}\binom{k_i + l_{i+1}}{k_i + n}. \qquad (III.33)$$

Therefore, substituting both of the results in Equations $(III.31)$ and $(III.33)$ back into Equation $(III.39)$ and regrouping common factors, we have

$$\begin{aligned} s_i &= \nu_i\, p(\mathbf{x}_{i-1}) \frac{\gamma_0 \gamma_i}{\xi_{oi}\xi_{oi+1}} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \sum_{n=0}^{l_i} \binom{k_i+n}{n} x_{oi+1}^{k_i} (\frac{\xi_{i-1}}{\xi_{oi}})^{l_i} \times \\ &\quad \sum_{k_0=0}^{\infty} \sum_{l_{i+1}=0}^{\infty} \binom{k_i + l_{i+1}}{k_i + n}\binom{k_0}{l_i - n} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{k_0 + l_{i+1} - l_i} y_{oi+1}^{l_{i+1}} \prod_{j=i+2}^{n+1} \psi_j(\mathbf{x}_{j-1}) \\ &= \nu_i\, p(\mathbf{x}_{i-1}) \frac{\gamma_0 \gamma_i}{\xi_{oi}\xi_{oi+1}} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \sum_{n=0}^{l_i} \binom{k_i+n}{n} x_{oi+1}^{k_i} (\frac{\xi_{i-1}}{\xi_{oi}})^{l_i} \times \\ &\quad \sum_{l_{i+1}=0}^{\infty} \binom{k_i + l_{i+1}}{k_i + n} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{l_{i+1} - l_i} y_{oi+1}^{l_{i+1}} \sum_{k_0=0}^{\infty} \binom{k_0}{l_i - n} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{k_0} \prod_{j=i+2}^{n+1} \psi_j(\mathbf{x}_{j-1}). \end{aligned}$$

By definition, $\binom{k}{j} = 0$ for all $j < k$ and $j < 0$. Summing with respect to k_0 on the line above, we notice that the contribution from the first $l_i - n$ terms in $\binom{k_0}{l_i - n}$ is null; i.e., $0 \leq k_0 \leq l_i - n - 1$. Thus, the expression for s_i reduces to

$$\begin{aligned} s_i \;=\; & \nu_i\, p(\mathbf{x}_{i-1}) \frac{\gamma_0 \gamma_i}{\xi_{oi}\xi_{oi+1}} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \sum_{n=0}^{l_i} \binom{k_i+n}{n} x_{oi+1}^{k_i} (\frac{\xi_{i-1}}{\xi_{oi}})^{l_i} \times \\ & \sum_{l_{i+1}=0}^{\infty} \binom{k_i+l_{i+1}}{k_i+n} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{l_{i+1}-l_i} y_{oi+1}^{l_{i+1}} \times \\ & \sum_{k_0=0}^{\infty} \binom{k_0+l_i-n}{k_0} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{k_0+l_i-n} \prod_{j=i+2}^{n+1} \psi_j(\mathbf{x}_{j-1}). \end{aligned}$$

From the binomial theorem, we know that[35]

$$\sum_{k=0}^{\infty} \binom{k+l}{k} t^k = (1-t)^{-(l+1)}. \qquad (III.34)$$

Hence, taking $l = l_i - n$ in the above expression, we have

$$\begin{aligned} s_i \;=\; & \nu_i\, p(\mathbf{x}_{i-1}) \frac{\gamma_0 \gamma_i}{\xi_{oi}\xi_{oi+1}} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \sum_{n=0}^{l_i} \binom{k_i+n}{n} x_{oi+1}^{k_i} (\frac{\xi_{i-1}}{\xi_{oi}})^{l_i} \times \\ & \sum_{l_{i+1}=0}^{\infty} \binom{k_i+l_{i+1}}{k_i+n} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{l_{i+1}-l_i} y_{oi+1}^{l_{i+1}} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{l_i-n} \times \\ & (1 - \frac{\lambda_{2r_0}}{\xi_{oi}})^{l_i-n+1} \prod_{j=i+2}^{n+1} \psi_j(\mathbf{x}_{j-1}). \end{aligned}$$

Using the equality in equation $(III.22)$ and regrouping those factors associated with the summation indices l_i, n and l_{i+1}, we get

$$\begin{aligned} s_i \;=\; & \nu_i\, p(\mathbf{x}_{i-1}) \frac{\gamma_0 \gamma_i}{\xi_{oi}\xi_{oi+1}} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \sum_{n=0}^{l_i} \binom{k_i+n}{n} x_{oi+1}^{k_i} (\frac{\xi_{i-1}}{\xi_{oi}})^{l_i} \times \\ & \sum_{l_{i+1}=0}^{\infty} \binom{k_i+l_{i+1}}{k_i+n} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{l_{i+1}-l_i} y_{oi+1}^{l_{i+1}} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{l_i-n} (\frac{\theta_i}{\xi_{oi}})^{l_i-n+1} \prod_{j=i+2}^{n+1} \psi_j(\mathbf{x}_{j-1}) \end{aligned}$$

$$= \nu_i \, p(\mathbf{x}_{i-1}) \frac{\gamma_0 \gamma_i}{\xi_{oi}\xi_{oi+1}} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \sum_{n=0}^{l_i} \binom{k_i+n}{n} \left(\frac{\xi_{i-1}}{\theta_i}\right)^{l_i} \left(\frac{\lambda_{2r_0}}{\theta_i}\right)^{-n} \times$$
$$\sum_{l_{i+1}=0}^{\infty} \binom{k_i+l_{i+1}}{k_i+n} x_{oi+1}^{k_i} \left(\frac{\lambda_{2r_0}}{\xi_{oi+1}}\right)^{l_{i+1}} \prod_{j=i+2}^{n+1} \psi_j(\mathbf{x}_{j-1}).$$

We now want to relate s_i to $R_n(i)$. Interchanging the order of summation with respect to the indices l_i and n as in $(II.8)$, we have

$$s_i = \nu_i \, p(\mathbf{x}_{i-1}) \frac{\gamma_0 \gamma_i}{\theta_i \xi_{oi+1}} \sum_{n=0}^{k_{i-1}+l_{i-1}} \binom{k_i+n}{n} \left(\frac{\lambda_{2r_0}}{\theta_i}\right)^{-n} \sum_{l_i=n}^{k_{i-1}+l_{i-1}} \left(\frac{\xi_{i-1}}{\theta_i}\right)^{l_i} \times$$
$$\sum_{l_{i+1}=0}^{\infty} \binom{k_i+l_{i+1}}{k_i+n} x_{oi+1}^{k_i} \left(\frac{\lambda_{2r_0}}{\xi_{oi+1}}\right)^{l_{i+1}} \prod_{j=i+2}^{n+1} \psi_j(\mathbf{x}_{j-1}).$$

In the above expression for s_i we recognize both $R_n(i)$ and a truncated geometric series with ratio $\xi_{i-1}/\theta_i < 1$. Summing with respect to l_i in the above expression for s_i, we have

$$s_i = \nu_i \, p(\mathbf{x}_{i-1}) \frac{\gamma_0 \gamma_i}{\theta_i \xi_{oi+1}} \sum_{n=0}^{k_{i-1}+l_{i-1}} \binom{k_i+n}{n} \left(\frac{\lambda_{2r_0}}{\theta_i}\right)^{-n} \times$$
$$\sum_{l_{i+1}=0}^{\infty} \binom{k_i+l_{i+1}}{k_i+n} x_{oi+1}^{k_i} \left(\frac{\lambda_{2r_0}}{\xi_{oi+1}}\right)^{l_{i+1}} \times$$
$$\left(\frac{\lambda_{2r_0}}{\theta_i}\right)^{n} \frac{1-\left(\frac{\xi_{i-1}}{\theta_i}\right)^{k_{i-1}+l_{i-1}-n+1}}{1-\frac{\xi_{i-1}}{\theta_i}} \prod_{j=i+2}^{n+1} \psi_j(\mathbf{x}_{j-1}).$$

But, from the definitions of ξ_i and θ_i in Equations $(III.18)$ and $(III.19)$, it follows that

$$1 - \frac{\xi_{i-1}}{\theta_i} = \frac{\nu_i}{\theta_i}.$$

Thus, after renaming n as l_i and grouping common factors with respect to the index l_i, s_i becomes

$$s_i = p(\mathbf{x}_{i-1}) \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \binom{k_i+l_i}{l_i} \left(\frac{\xi_{i-1}}{\lambda_{2r_0}}\right)^{l_i} \left[1 - \left(\frac{\xi_{i-1}}{\theta_i}\right)^{k_{i-1}+l_{i-1}-l_i+1}\right] \times$$

$$\gamma_0 \frac{\gamma_i}{\xi_{oi+1}} \sum_{l_{i+1}=0}^{\infty} \binom{k_i + l_{i+1}}{k_i + l_i} x_{oi+1}^{k_i} (\frac{\lambda_{2r_0}}{\xi_{oi+1}})^{l_{i+1}} \prod_{j=i+2}^{n+1} \psi_j(\mathbf{x}_{j-1}), \quad (III.35)$$

where the bottom line of Equation ($III.35$) is merely $R_n(i) = r_{i+1} \prod \phi_i(\mathbf{x}_{i-1})$. It then follows from the expression for $R_n(i)$ in Equation ($III.26$) that

$$\begin{aligned} s_i &= p(\mathbf{x}_{i-1}) \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \binom{k_i + l_i}{l_i} (\frac{\xi_{i-1}}{\lambda_{2r_0}})^{l_i} \left[1 - (\frac{\xi_{i-1}}{\theta_i})^{k_{i-1}+l_{i-1}-l_i+1}\right] \times \\ & \gamma_0 \frac{\xi_i}{\theta_{n+1}} (\frac{\lambda_{2r_i}}{\theta_{i+1}})^{k_i} (\frac{\lambda_{2r_0}}{\theta_{i+1}})^{l_i} \prod_{j=i+1}^{n} \psi_i(\mathbf{x}_i) \frac{\xi_i}{\theta_i} (\frac{\xi_i}{\theta_{i+1}})^{k_i} (\frac{\theta_i \xi_i}{\theta_{i+1} \xi_{i-1}})^{l_i}. \end{aligned}$$

Expressing s_i as a fuction of $p(\mathbf{x}_n)$ and recalling $f_n(i)$ from Equation ($III.27$), we obtain

$$s_i = \gamma_0 \, p(\mathbf{x}_n) \left[\frac{\xi_i}{\theta_{n+1}} f_n(i) - \frac{\xi_{i-1}}{\theta_{n+1}} f_n(i-1)\right]. \quad (III.36)$$

The above result covers the cases $2 \leq i \leq n$. For $i = n+1$, we must evaluate s_{n+1} directly from Equation ($III.23$). In this case, we have

$$s_{n+1} = \nu_{n+1} \, p(\mathbf{x}_n) \frac{\gamma_0}{\xi_{on+1}} \sum_{k_0=0}^{\infty} \sum_{l_{n+1}=0}^{k_n+l_n} \binom{k_0 + l_{n+1}}{l_{n+1}} (\frac{\lambda_{2r_0}}{\xi_{on+1}})^{k_0} (\frac{\xi_n}{\xi_{n+1}})^{l_{n+1}}.$$

From the binomial theorem in Equation ($III.34$) and the fact that

$$1 - \lambda_{2r_0}/\xi_{on+1} = \theta_{n+1}/\xi_{on+1},$$

we have

$$\begin{aligned} s_{n+1} &= \nu_{n+1} \, p(\mathbf{x}_n) \frac{\gamma_0}{\xi_{on+1}} \sum_{l_{n+1}=0}^{k_n+l_n} (\frac{\xi_{on+1}}{\theta_{n+1}})^{l_{n+1}+1} (\frac{\xi_n}{\xi_{on+1}})^{l_{n+1}} \\ &= \nu_{n+1} \, p(\mathbf{x}_n) \frac{\gamma_0}{\theta_{n+1}} \sum_{l_{n+1}=0}^{k_n+l_n} (\frac{\xi_n}{\theta_{n+1}})^{l_{n+1}}. \end{aligned}$$

Again, this last summation is a truncated geometric series with ratio ξ_n/θ_{n+1}. Since $1-(\xi_n/\theta_{n+1}) = \nu_{n+1}/\theta_{n+1}$, the expression for s_{n+1} becomes

$$
\begin{aligned}
s_{n+1} &= \nu_{n+1}\, p(\mathbf{x}_n) \frac{\gamma_0}{\theta_{n+1}} \left[\frac{1 - (\frac{\xi_n}{\theta_{n+1}})^{k_n+l_n+1}}{\nu_{n=1}/\theta_{n+1}} \right] \\
&= \gamma_0\, p(\mathbf{x}_n) \left[1 - (\frac{\xi_n}{\theta_{n+1}})^{k_n+l_n+1} \right] \\
&= \gamma_0\, p(\mathbf{x}_n)\, [1 - f_n(n)]. \qquad (III.37)
\end{aligned}
$$

Putting together the results in Equations ($III.28$), ($III.36$) and ($III.37$), we conclude that

$$
\begin{aligned}
\sum_{i=1}^{n+1} s_i = {} & \gamma_0\, p(\mathbf{x}_n) \frac{\xi_1}{\theta_{n+1}} f_n(1) + \gamma_0\, p(\mathbf{x}_n)\, [1 - f_n(n)] + \\
& \sum_{i=2}^{n} \gamma_0\, p(\mathbf{x}_n) \left[\frac{\xi_i}{\theta_{n+1}} f_n(i) - \frac{\xi_{i-1}}{\theta_{n+1}} f_n(i-1) \right].
\end{aligned}
$$

All the terms inside the brackets cancel out except for $f_n(1)$ and $f_n(n)$. Then, it follows that

$$
\begin{aligned}
\sum_{i=1}^{n+1} s_i = {} & \gamma_0\, p(\mathbf{x}_n) \frac{\xi_1}{\theta_{n+1}} f_n(1) + \gamma_0\, p(\mathbf{x}_n)\, [1 - f_n(n)] + \\
& \gamma_0\, p(\mathbf{x}_n)\, [f_n(n) - f_n(1)] \\
= {} & \gamma_0\, p(\mathbf{x}_n).
\end{aligned}
$$

This shows that $p(\mathbf{x}_n)$ as proposed in Equation ($III.17$) also satisfies the independent balance equation ($III.5$). Therefore, we conclude that the state probability distribution $p(\mathbf{x}_n)$ proposed in ($III.1$) is indeed a solution to the set of independent balance equations for the camp-on system. This at last proves Theorem 2.

APPENDIX IV:

Proof of $\mathbf{R_i(m) = R_n(m)}$

In Appendix III we claimed that the following recursion holds for $R_i(m)$, $(1 \leq i \leq n)$:

$$R_i(m) \;=\; \gamma_0 \prod_{j=m+1}^{i} \phi_j(\mathbf{x}_j) r_{i+1} \prod_{j=i+2}^{n+1} \psi_j(\mathbf{x}_{j-1}) = \; R_n(m), \qquad (IV.1)$$

where the functions $\phi_i(\mathbf{x}_i)$ and r_i were defined as

$$\phi_i(\mathbf{x}_i) \;=\; \frac{\gamma_{i-1}}{\theta_i} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \binom{k_i+l_i}{l_i} (\frac{\lambda_{2r_{i-1}}}{\theta_i})^{k_{i-1}} (\frac{\lambda_{2r_0}}{\theta_i})^{l_{i-1}-l_i}, \qquad (IV.2)$$

$$r_i \;=\; \frac{\gamma_{i-1}}{\xi_{oi}} \sum_{l_i=0}^{\infty} \binom{k_{i-1}+l_i}{k_{i-1}+l_{i-1}} x_{oi}^{k_{i-1}} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{l_i}\; \psi_{i+1}(\mathbf{x}_i). \qquad (IV.3)$$

In order to prove this claim, we focus on Equation $(IV.3)$. This can be expressed as

$$r_i \;=\; \frac{\gamma_{i-1}}{\xi_{oi}} \sum_{l_i=0}^{\infty} \binom{k_{i-1}+l_i}{k_{i-1}+l_{i-1}} x_{oi}^{k_{i-1}} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{l_i} \frac{\gamma_i}{\xi_{oi+1}} \sum_{l_{i+1}=0}^{k_{i-1}+l_i} \binom{k_i+l_{i+1}}{l_{i+1}} x_{oi+1}^{k_i}\; y_{oi+1}^{l_{i+1}}.$$

We start by interchaging the order of the summations with respect to the indices l_{i+1} and l_i in the above expression for r_i, using the algebraic relationship in $(II.8)$:

$$r_i \;=\; \frac{\gamma_{i-1}}{\xi_{oi}} \sum_{l_{i+1}=0}^{\infty} \binom{k_i+l_{i+1}}{l_{i+1}} x_{oi}^{k_{i-1}}\; y_{oi+1}^{l_{i+1}} \times$$
$$\frac{\gamma_i}{\xi_{oi+1}} \sum_{l_i=l_{i+1}-k_{i-1}}^{\infty} \binom{k_{i-1}+l_i}{k_{i-1}+l_{i-1}} x_{oi+1}^{k_i} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{l_i}. \qquad (IV.4)$$

Define the change of variable $m = l_i - l_{i+1} + k_{i-1}$. Substituting for l_i by m in Equation $(IV.4)$ and using the simple combinatorial relation

$$\binom{k}{n} = \binom{k}{k-n}, \qquad (IV.5)$$

we can rewrite the last expression for r_i in Equation $(IV.4)$ as

$$r_i = \frac{\gamma_{i-1}}{\xi_{oi}} \sum_{l_{i+1}=0}^{\infty} \binom{k_i + l_{i+1}}{l_{i+1}} x_{oi}^{k_{i-1}} \, y_{oi+1}^{l_{i+1}} \times$$
$$\frac{\gamma_i}{\xi_{oi+1}} \sum_{m=0}^{\infty} \binom{m + l_{i+1}}{k_{i-1} + l_{i-1}} x_{oi+1}^{k_i} \, (\frac{\lambda_{2r_0}}{\xi_{oi}})^{m+l_{i+1}-k_{i-1}}. \qquad (IV.6)$$

Recalling the combinatorial expansion in $(III.30)$, this time taking $j = k_{i-1} + l_{i-1}$, $l = l_{i+1}$, and $k = m$, we find that the second combinatorial coefficient in Equation $(IV.6)$ becomes:

$$\binom{m + l_{i+1}}{k_{i-1} + l_{i-1}} = \sum_{j=0}^{k_{i-1}+l_{i-1}} \binom{l_{i+1}}{j} \binom{m}{k_{i-1} + l_{i-1} - j}. \qquad (IV.7)$$

Similarly, recalling the combinatorial identity in $(III.32)$ with $l = l_{i+1}$, $n = j$, and $k = k_i$, for the first combinatorial coefficients in Equation $(IV.6)$ and $(IV.7)$, we have the identity

$$\binom{k_i + l_{i+1}}{l_{i+1}} \binom{l_{i+1}}{j} = \binom{k_i + j}{j} \binom{k_i + l_{i+1}}{k_i + j}. \qquad (IV.8)$$

Substituting both $(IV.7)$ and $(IV.8)$ back into Equation $(IV.6)$ and regrouping common factors, we end up with the following relation for r_i:

$$r_i = \frac{\gamma_{i-1}}{\xi_{oi}} \sum_{j=0}^{k_{i-1}+l_{i-1}} \binom{k_i + j}{j} x_{oi}^{k_{i-1}} \sum_{l_{i+1}=0}^{\infty} \binom{k_i + l_{i+1}}{k_i + j} x_{oi+1}^{k_i} \, y_{oi+1}^{l_{i+1}} \times$$
$$\frac{\gamma_i}{\xi_{oi+1}} \sum_{m=0}^{\infty} \binom{m}{k_{i-1} + l_{i-1} - j} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{m+l_{i+1}-k_{i-1}}$$

$$= \frac{\gamma_{i-1}}{\xi_{oi}} \sum_{j=0}^{k_{i-1}+l_{i-1}} \binom{k_i+j}{j} x_{oi}^{k_{i-1}} \frac{\gamma_i}{\xi_{oi+1}} \sum_{l_{i+1}=0}^{\infty} \binom{k_i+l_{i+1}}{k_i+j} x_{oi+1}^{k_i} y_{oi+1}^{l_{i+1}} \times$$

$$(\frac{\lambda_{2r_0}}{\xi_{oi}})^{l_{i+1}-k_{i-1}} \left[\sum_{m=0}^{\infty} \binom{m}{k_{i-1}+l_{i-1}-j} (\frac{\lambda_{2r_0}}{\xi_{oi}})^m \right]. \qquad (IV.9)$$

Let s denote the term inside brackets in Equation (IV.9). Since $\binom{k}{j} = 0$ for all $j > k$ and $j < 0$, the first $k_{i-1} + l_{i-1} - j$ terms in the summation with respect to m are all zero, and they therefore offer no contribution to s. It follows that

$$s = \sum_{m=0}^{\infty} \binom{m}{k_{i-1}+l_{i-1}-j} (\frac{\lambda_{2r_0}}{\xi_{oi}})^m$$

$$= \sum_{m=k_i+l_{i-1}-j}^{\infty} \binom{m}{k_{i-1}+l_{i-1}-j} (\frac{\lambda_{2r_0}}{\xi_{oi}})^m.$$

After the temporery change of variable $m' = m - k_i - l_{i-1} - j$ and using the combinatorial identity in Equation (IV.5), we have

$$s = \sum_{m=0}^{\infty} \binom{m+k_{i-1}+l_{i-1}-j}{k_{i-1}+l_{i-1}-j} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{m+k_{i-1}+l_{i-1}-j}$$

$$= \sum_{m=0}^{\infty} \binom{m+k_{i-1}+l_{i-1}-j}{m} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{m+k_{i-1}+l_{i-1}-j}.$$

Also, from the extension of the binomial expansion in (III.34), taking $k = k_{i-1} + l_{i-1} - j$, the above series expansion for s reduces to the following simple expression:

$$s = \left[1 - \frac{\lambda_{2r_0}}{\xi_{oi}}\right]^{-(k_{i-1}+l_{i-1}-j+1)} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{k_{i-1}+l_{i-1}-j}.$$

But we have already found in Equation (III.22) that $[1 - \lambda_{2r_0}/\xi_{oi}] = \theta_i/\xi_{oi}$. So the final expression for s is given by

$$s = (\frac{\theta_i}{\xi_{oi}})^{-(k_{i-1}+l_{i-1}-j+1)} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{k_{i-1}+l_{i-1}-j}.$$

Substituting this expression for s back into Equation $(IV.9)$, we obtain

$$r_i = \frac{\gamma_{i-1}}{\xi_{oi}} \sum_{j=0}^{k_{i-1}+l_{i-1}} \binom{k_i+j}{j} x_{oi}^{k_{i-1}} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{l_{i+1}+l_{i-1}-j} \times$$
$$\frac{\gamma_i}{\xi_{oi+1}} \sum_{l_{i+1}=0}^{\infty} \binom{k_i+l_{i+1}}{k_i+j} x_{oi+1}^{k_i} y_{oi+1}^{l_{i+1}} (\frac{\xi_{oi}}{\theta_1})^{k_{i-1}+l_{i-1}-j+1}.$$

If we rename the summation index j as l_i, interchange the order of summation with respect to l_{i+1}, and then regroup common factors, we find that r_i becomes

$$r_i = \frac{\gamma_{i-1}}{\theta_i} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \binom{k_i+l_i}{l_i} x_{oi}^{k_{i-1}} (\frac{\xi_{oi}}{\theta_i})^{k_{i-1}} (\frac{\lambda_{2r_0}}{\theta_i})^{l_{i-1}-l_i} \times$$
$$\frac{\gamma_i}{\xi_{oi+1}} \sum_{l_{i+1}=0}^{\infty} \binom{k_i+l_{i+1}}{k_i+l_i} x_{oi+1}^{k_i} y_{oi+1}^{l_{i+1}} (\frac{\lambda_{2r_0}}{\xi_{oi}})^{l_{i+1}} (\frac{\xi_{oi}}{\theta_i})^{k_{i-1}+l_{i-1}-l_i}. \quad (IV.10)$$

Comparing this result with the initial expressions for r_i and $\phi_i(\mathbf{x}_i)$ in Equations $(IV.2)$ and $(IV.3)$, we immediately recognize the first term in Equation $(IV.10)$ as $\phi_i(\mathbf{x}_i)$, while the second term is almost identical to r_{i+1} except for the missing factor $\psi_i(\mathbf{x}_i)$. Therefore, it follows that

$$\begin{aligned} R_i(m) &= \gamma_0 \prod_{j=m+1}^{i} \phi_j(\mathbf{x}_j) r_{i+1} \prod_{j=i+2}^{n+1} \psi_j(\mathbf{x}_{j-1}) \\ &= \gamma_0 \prod_{j=m+1}^{i} \phi_j(\mathbf{x}_j) \phi_{i+1}(\mathbf{x}_{i+1}) r_{i+2} \prod_{j=i+3}^{n+1} \psi_j(\mathbf{x}_{j-1}) \\ &= R_{i+1}(m). \end{aligned}$$

Applying this recursion $n-i$ consecutive times to itself, we end up with $R_i(m) = R_n(m)$. This proves the first part of the claim.

We then compute $R_n(m)$ from r_{n+1} in Equation $(IV.1)$. In this case, $\psi_{n+2} \equiv 1$, so

$$R_n(m) = \gamma_0 \prod_{i=m+1}^{n} \phi_i(\mathbf{x}_i) \frac{\gamma_n}{\xi_{on+1}} \sum_{l_{n+1}=0}^{\infty} \binom{k_n+l_{n+1}}{k_n+l_n} x_{on+1}^{k_n} (\frac{\lambda_{2r_0}}{\xi_{on+1}})^{l_{n+1}}. \quad (IV.11)$$

Again, the contribution of the first l_n terms to the summation with respect to the index l_{n+1} is null. Hence, from the combinatorial expansion in Equation $(III.34)$ with $k = k_n + l_n$, we get

$$R_n(m) = \gamma_0 \prod_{i=m+1}^{n} \phi_i(\mathbf{x}_i) \frac{\gamma_n}{\xi_{on+1}} x_{on+1}^{k_n} \left(\frac{\lambda_{2r_0}}{\xi_{on+1}}\right)^{l_n} \left(\frac{\xi_{on+1}}{\theta_{n+1}}\right)^{k_n+l_n+1}$$

$$= \gamma_0 \prod_{i=m+1}^{n} \phi_i(\mathbf{x}_i) \frac{\gamma_n}{\theta_{n+1}} \left(\frac{\lambda_{2r_0}}{\theta_{n+1}}\right)^{l_n} \left(\frac{\lambda_n}{\theta_{n+1}}\right)^{k_n}.$$

Shifting the factors $(\lambda_{2r_{i-1}}/\theta_i)^{k_{i-1}}$ and $(\lambda_{2r_0}/\theta_i)^{l_{i-1}}$ to the next lower term in the product form of $R_n(m)$ and canceling out common terms, we obtain

$$R_n(m) = \gamma_0 \frac{\gamma_i}{\theta_{n+1}} \left(\frac{\lambda_{2r_m}}{\theta_{m+1}}\right)^{k_m} \left(\frac{\lambda_{2r_0}}{\theta_{m+1}}\right)^{l_m} \prod_{i=m+1}^{n} \frac{\gamma_i}{\theta_i} \sum_{l_i=0}^{k_{i-1}+l_{i-1}} \binom{k_i + l_i}{l_i} \left(\frac{\lambda_{2r_i}}{\theta_{i+1}}\right)^{k_i} \left(\frac{\theta_i}{\theta_{i+1}}\right)^{l_i}.$$

If we compare this expression with Equation $(III.2)$ and add the missing terms in order to write $R_n(m)$ as a function of $\psi_i(\mathbf{x}_i)$, we have

$$R_n(m) = \gamma_0 \frac{\gamma_i}{\theta_{n+1}} \left(\frac{\lambda_{2r_m}}{\theta_{m+1}}\right)^{k_m} \left(\frac{\lambda_{2r_0}}{\theta_{m+1}}\right)^{l_m} \prod_{i=m+1}^{n} \psi_i(\mathbf{x}_i) \frac{\xi_i}{\theta_i} \left(\frac{\xi_i}{\theta_{i+1}}\right)^{k_i} \left(\frac{\theta_i \xi_i}{\theta_{i+1} \xi_{i-1}}\right)^{l_i}.$$

This completes the derivation of $R_n(m)$.

APPENDIX V:

Proof of Theorem 3

In this appendix we prove the statement made in Theorem 3, Chapter 3, relating the joint probability distribution of queue lengths for a size-constrained non-reneging camp-on system with the state probability distribution $p(\mathbf{x}_n)$ for the unconstrained model. In the non-reneging camp-on model with unconstrained queue sizes, we found that the equilibrium equation for the queueing systems reduced to

$$\Big[\sum_{i=1}^{n}(\lambda_{2r_i} + \nu_i) + \lambda_{1n}\, \chi_{(n<N)} + \mu\chi_{(n>0)}\Big]\; p(\mathbf{x}_n) \;=$$
$$\gamma_n\, \chi_{(k_n=0)}\; p(\mathbf{x}_{n-1})$$
$$+ \sum_{i=1}^{n} \lambda_{2r_i}\, \chi_{(k_i>0)}\; p(\mathbf{x}_n^{-i})$$
$$+ \sum_{r_0=1}^{R} \chi_{(r_0\neq r_i)} \sum_{k_0=0}^{\infty} \mu\chi_{(n<N)}\; p(\mathbf{x}_{n+1,1}) \qquad (V.1)$$
$$+ \sum_{r_0=1}^{R} \chi_{(r_0\neq r_i)} \sum_{i=1}^{n+1} \sum_{k_0=0}^{\infty} \nu_i\, \chi_{(n<N)}\; p(\mathbf{x}_{n+1,i}),$$

$$0 \leq n \leq N.$$

Let $(l_1, \cdots, l_s)$ be an ordered s-tuple, with $1 \leq l_1 < \cdots < l_s \leq n$. Let the l_i's identify the second-level systems with a full waiting line; e.g., the l_i^{th} second-level system size is $k_{l_i} = N_{r_{l_i}}$, where N_{r_l} is the storage capacity for class-r_l customers associated with the l_i^{th} second-level system. Denote by $p^*(\mathbf{x}_n)$ the joint probability distribution of queue lengths for the finite-storage camp-on model defined in

Theorem 3; that is,

$$p^*(\mathbf{K}_n;\mathbf{R}_n) = \begin{cases} p(\mathbf{K}_n;\mathbf{R}_n), & \text{if } k_i < N_{r_i}, \quad (0 \le n \le N); \\ \displaystyle\sum_{j_{l_1}=N_{r_{l_1}}}^{\infty} \cdots \sum_{j_{l_s}=N_{r_{l_s}}}^{\infty} p(\mathbf{J}_n;\mathbf{R}_n), & \text{if } k_l = N_{r_l}, \quad (1 \le s \le n). \end{cases} \tag{V.2}$$

Adding on both sides of Equation $(V.1)$ from $k_l = N_{r_l}$ to $k_l = \infty$ for each one of the l_i, Equation $(V.1)$ becomes

$$\begin{aligned} \Big[\sum_{i=1}^{n} (\lambda_{2r_i} + \nu_i) + \lambda_{1n}\, \chi_{(n<N)} + \mu \chi_{(n>0)} \Big] \, p^*(\mathbf{x}_n) \; = \\ \sum_{k_{l_1} \ge N_{r_{l_1}}} \cdots \sum_{k_{l_s} \ge N_{r_{l_s}}} \gamma_n \, \chi_{(k_n=0)} \, p(\mathbf{x}_{n-1}) \\ + \sum_{k_{l_1} \ge N_{r_{l_1}}} \cdots \sum_{k_{l_l} \ge N_{r_{l_s}}} \sum_{i=1}^{n} \lambda_{2r_i} \, \chi_{(k_i>0)} \, p(\mathbf{x}_n^{-i}) \\ + \sum_{k_{l_1} \ge N_{r_{l_1}}} \cdots \sum_{k_{l_s} \ge N_{r_{l_s}}} \sum_{r_0=1}^{R} \chi_{(r_0 \ne r_i)} \sum_{k_0=0}^{\infty} \mu \chi_{(n<N)} \, p(\mathbf{x}_{n+1,1}) \\ + \sum_{k_{l_1} \ge N_{r_{l_1}}} \cdots \sum_{k_{l_s} \ge N_{r_{l_s}}} \sum_{r_0=0}^{R} \chi_{(r_0 \ne r_i)} \sum_{i=1}^{n+1} \sum_{k_0=0}^{\infty} \nu_i \, \chi_{(n<N)} \, p(\mathbf{x}_{n+1,i}), \\ 0 \le n \le N. \end{aligned} \tag{V.3}$$

We take a closer look at the different terms on the right side of Equation $(V.3)$. First, in the term involving $p(\mathbf{x}_{n-1})$, we note that either $l_s < n$ and the last second-level system is not full, or else $l_s = n$ and so $\chi_{(k_n=0)} = 0$. Thus, we can write

$$\sum_{k_{l_1} \ge N_{r_{l_1}}} \cdots \sum_{k_{l_l} \ge N_{r_{l_s}}} \gamma_n \, \chi_{(k_n=0)} p(\mathbf{x}_{n-1}) \;=\; \gamma_n \, \chi_{(k_n=0)} p^*(\mathbf{x}_{n-1}). \tag{V.4}$$

With respect to the second term on the right side of $(V.3)$ involving $p(\mathbf{x}_n^{-i})$, we notice that either $k_{l_j} \ne i$ and so the i^{th} second-level system is not one of the full subsystems, or else $k_{l_j} = i$. If in the latter case we introduce the temporary change of variable $k_i' = k_i - 1$, it follows that

$$\sum_{k_{l_j} \geq N_{r_{l_j}}} p(\mathbf{x}_n^{-i}) = \begin{cases} \sum_{k_{l_j} \geq N_{r_{l_j}}} p(\mathbf{x}_n^{-i}) & \text{if } i \neq l_j; \\ p(\mathbf{x}_n)\Big|_{k_{l_j} = N_{r_{l_j}} - 1} + \sum_{k_{l_j} \geq N_{r_{l_j}}} p(\mathbf{x}_n) & \text{if } i = l_j. \end{cases}$$

Consequently, it follows from from the above result that the contribution from the newly arriving second-level customers reduces to

$$\sum_{k_{l_1} \geq N_{r_{l_1}}} \cdots \sum_{k_{l_s} \geq N_{r_{l_s}}} p(\mathbf{x}_n^{-i}) = p^*(\mathbf{x}_n^{-i}) + \chi_{i=l_j} p^*(\mathbf{x}_n).$$

With respect to the third and fourth terms in $(V.1)$, which involve the contribution from the departing customers $p(\mathbf{x}_{n+1,i})$, we notice that

$$\sum_{k_0 \geq 0} p(\mathbf{x}_{n+1,i}) = \sum_{k_0=0}^{N_{r_0}-1} p(\mathbf{x}_{n+1,i}) + p(\mathbf{x}_{n+1,i})\Big|_{k_0 = N_{r_0}}.$$

Therefore, it also follows that this contribution can be expressed as

$$\begin{aligned} \sum_{k_{l_1} \geq N_{r_{l_1}}} \cdots \sum_{k_{l_s} \geq N_{r_{l_s}}} \sum_{k_0 \geq 0} p(\mathbf{x}_{n+1,i}) &= \sum_{k_0=0}^{N_{r_0}-1} p^*(\mathbf{x}_{n+1,i}) + p^*(\mathbf{x}_{n+1,i})\Big|_{k_0 = N_{r_0}} \\ &= \sum_{k_0=0}^{N_{r_0}} p^*(\mathbf{x}_{n+1,i}). \end{aligned} \qquad (V.6)$$

Here, N_{r_0} is the storage capacity for the class of second-level customers currently in position i that are about to leave the camp-on system.

Inserting the findings from Equations $(V.4)$ to $(V.6)$ into Equation $(V.3)$, the equilibrium equation satisfied by $p^*(\mathbf{x}_n)$ reduces to

$$\Big[\sum_{i=1}^{n}(\lambda_{2r_i}+\nu_i)+\lambda_{1n}\,\chi_{(n<N)}+\mu\chi_{(n>0)}\Big]\,p^*(\mathbf{x}_n) \;=$$
$$\begin{aligned}
&\gamma_n\,\chi_{(k_n=0)}\,p^*(\mathbf{x}_{n-1})\\
&+\sum_{i=1}^{n}\lambda_{2r_i}\,\chi_{(k_i>0)}\,\Big[p^*(\mathbf{x}_n^{-i})\;+\;\chi_{(i=l_j)}p^*(\mathbf{x}_n)\Big]\\
&+\sum_{r_0=1}^{R}\chi_{(r_0\neq r_i)}\sum_{k_0=0}^{N_{r_0}}\mu\chi_{(n<N)}\,p^*(\mathbf{x}_{n+1,1}) \qquad (V.7)\\
&+\sum_{r_0=1}^{R}\chi_{(r_0\neq r_i)}\sum_{i=1}^{n+1}\sum_{k_0=0}^{N_{r_0}}\nu_i\,\chi_{(n<N)}\,p^*(\mathbf{x}_{n+1,i}),\\
&0\le k_i\le N_{r_i}\qquad 0\le n\le N.
\end{aligned}$$

Regrouping common terms with respect to $p^*(\mathbf{x}_n)$ in Equation $(V.7)$, and recalling

$$1-\chi_{(i=l_j)}=\chi_{(i\neq l_j)},$$

we conclude that

$$\Big[\sum_{i=1}^{n}(\lambda_{2r_i}\,\chi_{(i\neq j_l)}+\nu_i)+\lambda_{1n}\,\chi_{(n<N)}+\mu\chi_{(n>0)}\Big]\,p^*(\mathbf{x}_n) \;=$$
$$\begin{aligned}
&\gamma_n\,\chi_{(k_n=0)}\,p^*(\mathbf{x}_{n-1})\\
&+\sum_{i=1}^{n}\lambda_{2r_i}\,\chi_{(k_i>0)}\,p(\mathbf{x}_n^{-i})\\
&+\sum_{r_0=1}^{R}\chi_{(r_0\neq r_i)}\sum_{k_0=0}^{N_{r_0}}\mu\chi_{(n<N)}\,p^*(\mathbf{x}_{n+1,1}) \qquad (V.8)\\
&+\sum_{r_0=1}^{R}\chi_{(r_0\neq r_i)}\sum_{i=1}^{n+1}\sum_{k_0=0}^{N_{r_0}}\nu_i\,\chi_{(n<N)}\,p^*(\mathbf{x}_{n+1,i}),
\end{aligned}$$

$$0\le k_i\le N_{r_i},\qquad 0\le n\le N.$$

This equilibrium equation for $p^*(\mathbf{x}_n)$ is seen to be identical to the equilibrium equation derived for the camp-on model with finite storage capacity given in Section 1, Chapter III, i.e., Equation (3.20), since clearly, $\chi_{(i \neq l_j)} = \chi_{(k_i < N_{r_i})}$. This proves Theorem 3.

APPENDIX VI:

Workload Distribution Among the Queueing Stages

Here we proceed to carry out the intermediate steps in the derivation of the equilibrium workload distribution in each of the two stages of queues given in Chapter 3, Section II. From Equation (3.27) we know that

$$\begin{aligned} P(z;\mathbf{R}_n) &= P(\mathbf{Z}_n;\mathbf{R}_n)\Big|_{\mathbf{Z}_n=\mathbf{z}} \\ &= p_0 \prod_{i=1}^{n} \frac{\gamma_i}{\mu + \sum_{j=1}^{i} \lambda_{2r_j}(1-z)}, \end{aligned} \qquad (VI.1)$$

where $\mathbf{z}$ is a vector with all components $z_i = z$.

First, observe that $P(z;\mathbf{R}_n)$ is an all-pole function. All the poles z_i of $P(z;\mathbf{R}_n)$ are distinct and given by

$$z_i = \frac{\mu + \hat{\lambda}_i}{\hat{\lambda}_i}, \qquad (VI.2)$$

where we have written $\hat{\lambda}_i = \sum_{j=1}^{i} \lambda_{2r_j}$ as the total arrival rate to the second-level stage for the particular class assigment $\mathbf{R}_n$.

Since all the poles are distinct, $P(z;\mathbf{R}_n)$ can be expressed as a sum of single-pole functions through the method of partial-fraction expansion[23]. It follows from $(VI.1)$ that

$$P(z;\mathbf{R}_n) = \sum_{i=1}^{n} \frac{a_i}{\mu + \sum_{j=1}^{i} \lambda_{2r_j}(1-z)} \pi(\mathbf{R}_n). \qquad (VI.3)$$

From $(VI.2)$, we see that all the poles of $P(\mathbf{Z}_n)$ are located outside the unit circle. From a table of Z-transform pairs[18], we find that the distribution of the total workload among the two queueing stages is of the form

$$p(k; \mathbf{R}_n) = \sum_{i=1}^{n} \frac{a_i}{\mu + \hat{\lambda}_i} \left(\frac{\hat{\lambda}_i}{\mu + \hat{\lambda}_i} \right)^k \pi(\mathbf{R}_n). \qquad (VI.4)$$

The coefficients $a_i, \ldots, a_n$ in the method of partial-fraction expansion are found from $P(z; \mathbf{R}_n)$ by canceling the pole at $z = z_i$ and then evaluating the resulting function at $z = z_i^{-1}$ i.e.

$$a_i = \frac{P(z; \mathbf{R}_n)}{\pi(\mathbf{R}_n)} (1 - z_i^{-1} z) \Big|_{z=z_i}. \qquad (VI.5)$$

Evaluating a_i from $P(z; \mathbf{R}_n)$ in Equation $(V.2)$ and z_i in Equation $(VI.2)$, we have

$$\begin{aligned} a_i &= \mu \prod_{j \neq i}^{n} \left[\frac{\hat{\lambda}_i \mu}{\mu \hat{\lambda}_i + \hat{\lambda}_i \hat{\lambda}_j - \mu \hat{\lambda}_j - \hat{\lambda}_i \hat{\lambda}_j} \right] \\ &= \mu \prod_{j \neq i}^{n} \frac{\hat{\lambda}_i}{\hat{\lambda}_i - \hat{\lambda}_j}. \end{aligned} \qquad (VI.6)$$

For convenience, in Section 3, Chapter III, we defined the parameters $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ (they are not to be confused with α_{r_i} and β_{r_i} in Section 2, Chapter III) as:

$$\tilde{\alpha}_i = \frac{\hat{\lambda}_i}{\mu + \hat{\lambda}_i}, \qquad (VI.7)$$

$$\tilde{\beta}_i = \frac{\mu}{\mu + \hat{\lambda}_i}, \qquad (VI.8)$$

where $\tilde{\alpha}_i$ gives the average rate of arrival to the second-level stage when the customer class assignment for the second-level systems is $\mathbf{R}_n$ while $\tilde{\beta}_i$ gives the average rate of departure from the second-level stage under the same conditions. Observe that

$$\tilde{\alpha}_i - \tilde{\alpha}_j = \frac{\mu(\hat{\lambda}_i - \hat{\lambda}_j)}{(\mu + \hat{\lambda}_i)(\mu + \hat{\lambda}_j)}.$$

Writing the right side of this expression as a function of $\tilde{\alpha}_i$ and $\tilde{\beta}_i$, we have

$$\prod_{j\neq i}^{n}(\tilde{\alpha}_i - \tilde{\alpha}_j) = \prod_{j\neq i}^{n} \frac{\mu(\hat{\lambda}_i - \hat{\lambda}_j)}{(\mu + \hat{\lambda}_i)(\mu + \hat{\lambda}_j)}$$
$$= \prod_{j\neq i}^{n} \frac{\hat{\lambda}_i - \hat{\lambda}_j}{\hat{\lambda}_i}\tilde{\alpha}_i\tilde{\beta}_j;$$

we find that

$$\prod_{j\neq i}^{n} \frac{\hat{\lambda}_j}{\hat{\lambda}_i - \hat{\lambda}_j} = \prod_{j\neq i}^{n} \frac{\tilde{\alpha}_i\tilde{\beta}_j}{\tilde{\alpha}_i - \tilde{\alpha}_j}. \qquad (VI.9)$$

Replacing this last expression into the expression for a_i in Equation $(VI.6)$, we have

$$a_i = \mu \prod_{j\neq i}^{n} \frac{\tilde{\alpha}_i\tilde{\beta}_j}{\tilde{\alpha}_i - \tilde{\alpha}_j}. \qquad (VI.10)$$

Finally, using the expression for a_i in Equation $(VI.4)$, and recalling from $(VI.8)$ that $\mu/(\mu + \hat{\lambda}_i) = \tilde{\beta}_i$, we conclude that the joint probability distribution of the size of the first-level and second-level queues when the class assignment for the second-level systems is $\mathbf{R}_n$ has the form

$$p(k;\mathbf{R}_n) = \sum_{i=1}^{n} \tilde{\beta}_i\tilde{\alpha}_i^k\pi(\mathbf{R}_n) \prod_{j\neq i}^{n} \frac{\alpha_i\beta_j}{(\tilde{\alpha}_i - \tilde{\alpha}_j)}, \qquad n \geq 1,\ k \geq 0,$$

as we claimed in Equation (3.40). This completes the derivation of $p(k;\mathbf{R}_n)$.

APPENDIX VII:

Proof of Theorem 4

In this appendix, we prove the statement made in Theorem 4, Chapter IV, that the joint probability distribution of queue sizes along a service path with $(n+1)$ levels in a multilevel camp-on system is given by

$$p_M(\mathbf{J}_n;\mathbf{K}_n) = \prod_{i=0}^{n} \Psi_i(\mathbf{J}_i;\mathbf{K}_i), \qquad (VII.7)$$

where the function $\Psi_i(\mathbf{J}_i;\mathbf{K}_i)$ is of the form

$$\Psi_i(\mathbf{J}_i;\mathbf{K}_i) = \begin{cases} \pi(k_0), & \text{if } i=0 \\ 1, & \text{if } k_i=0,\ 0<i<n; \\ \sum_{l_i=0}^{M_i} \binom{M_i+k_i-j_{i+1}-l_i}{k_i-j_{i+1}} \times \binom{j_{i+1}-1+l_i}{l_i} \beta_i^{l_i} \alpha_{in}^{k_i-j_{i+1}} \alpha_i^{j_{i+1}}, & \text{if } k_i>0,\ 0<i<n; \\ \binom{M_n+k_n}{k_n} \beta_n^{1+k_0-j_1-\sum_{i=1}^{n-1} l_i} \times \prod_{i=1}^{n} \alpha_{in}^{k_i-j_{i+1}}, & \text{if } 0<i=n. \end{cases} \qquad (VII.2)$$

Here the parameters M_i, β_i, α_i and α_{in} stand for:

$$M_i = \sum_{m=1}^{i} [k_{m-1} - j_m - l_m], \qquad k_i \geq j_{i+1}, \qquad (VII.3)$$

$$\beta_i = \frac{\mu}{\mu_1 + \sum_{k=2}^{i+1} \lambda_k}, \qquad (VII.4)$$

$$\alpha_i = \frac{\lambda_{i+1}}{\mu_1 + \sum_{k=2}^{i+1} \lambda_k}, \qquad (VII.5)$$

$$\alpha_{in} = \frac{\lambda_{i+1}}{\mu_1 + \sum_{k=2}^{n+1} \lambda_k}. \qquad (VII.6)$$

Before proceeding to derive Equation $(VII.1)$, it will be useful to introduce some new notation that will help in simplifying the upcoming exposition:

i) In any relation involving state vectors such as $\Psi_i(\mathbf{J}_i; \mathbf{K}_i)$, define

$$\Psi_i^{-l}(\mathbf{J}_i; \mathbf{K}_i) = \Psi_i(\mathbf{J}_i; \mathbf{K}_i^{-l})$$

to mean decreasing the size of the level-$(l+1)$ queue by one. We will make use of a similar notation every time we need to increase $(+l)$ by one, or decrease $(-l)$ by one, the size of the l-component of a given vector.

ii) For $0 \le i \le n$ and $k_i > 0$, we will write the function $\Psi_i(\mathbf{J}_i; \mathbf{K}_i)$ as:

$$\Psi_i(\mathbf{J}_i; \mathbf{K}_i) = \begin{cases} \sum_{l_i=0}^{M_i} F_i(l_i)\ G_i(l_i)\ H_{in}(l_i), & \text{if } 0 \le i < n; \\ F_n(0)\ H_{nn}(0), & \text{if } i = n, \end{cases} \qquad (VII.7)$$

where $F_i(l_i)$, $G_i(l_i)$ and $H_{in}(l_i)$ are defined as

$$F_i(l_i) = \binom{M_i + k_i - j_{i+1} - l_i}{k_i - j_{i+1}}, \qquad (VII.8)$$

$$G_i(l_i) = \binom{j_{i+1} - 1 + l_i}{l_i}, \qquad (VII.9)$$

$$H_{in}(l_i) = \begin{cases} \pi(k_0), & \text{if } i = 0; \\ \beta_i^{l_i} \alpha_i^{j_{i+1}}, & \text{if } i < n; \\ \beta_n^{1+k_0-j_1-\sum_{i=1}^{n-1} l_i} \prod_{i=1}^{n} \alpha_{in}^{k_i - j_{i+1}}, & \text{if } i = n. \end{cases} \qquad (VII.10)$$

Also, whenever the meaning is clear, we will drop the explicit reference to the summation index in any of the above expressions; e.g., $F_i = F_i(l_i)$.

iii) From the definition of M_i in Equation $(VII.3)$, one can derive the following relationship between M_i and M_{i+1}:

$$M_i + k_i - j_{i+1} - l_i \;=\; M_{i+1}. \qquad (VII.11)$$

Therefore, one could also have expressed $F_i(l_i)$ as

$$F_i(l_i) \;=\; \binom{M_i + k_i - j_{i+1} - l_i}{k_i - j_{i+1}} \;=\; \binom{M_{i+1}}{k_i - j_{i+1}}. \qquad (VII.12)$$

We will use the former notation, Equation $(VII.8)$, whenever an explicit reference to the relation between l_i and $F_i(l_i)$ is in order.

Now that the notation is clear, we will proceed with the proof of Theorem 4. First, observe that in the distribution for $p_M(\mathbf{x}_n)$, the first-level queue size, k_0, and the first-level customer identifier, j_1, appear together in the form $k_0 - j_1$, except for $\pi(k_0) = (1-\rho_1)\rho_1^{k_0}$. Thus, if k_0 and j_1 are simultaneously increased by one, $p_M(\mathbf{x}_n)$ changes only by the multiplicative factor $\rho_1 = \lambda_1/\mu$. That is, the increase in the size of the first-level system is compensated by a departure of a first-level customer.

From the above observation, it follows that

$$p_M(\mathbf{J}_n^{+1};\mathbf{K}_n^{+0}) \;=\; \frac{\lambda_1}{\mu_1}\, p_M(\mathbf{J}_n;\mathbf{K}_n). \qquad (VII.13)$$

This is, in fact, a *partial* balance equation for the multilevel camp-on system. From Equation $(VII.1)$, it is clear that a *second* partial balance equation holds for the multilevel camp-on system. This equation is

$$\left[\mu_1 + \sum_{i=1}^{n} \lambda_{i+1}\right] p_M(\mathbf{x}_n) \;=\; \sum_{i=0}^{n} \lambda_{i+1}\, \chi_{(j_{i+1}<k_i)}\, p_M(\mathbf{x}_n^{-i}) + \lambda_n\, \chi_{(j_n=k_{n-1},\ k_n=0)}\, p_M(\mathbf{x}_{n-1}^{-n+1}). \qquad (VII.14)$$

Both partial balance equations together are equivalent to the equilibrium global balance Equation (4.6).

In order to verify that the proposed state probability distribution for the global balance equation in Equation ($VII.1$) satisfies the above partial balance equation, we must consider three different situations arising in multilevel camp-on systems:

<u>Case I:</u> All $j_{i+1} \neq k_i$ and $k_i > 0$ $(0 \leq i \leq n)$.

This case corresponds to a situation wherein none of the $n+1$ queues is empty and the state vector does not refer to a service path including one of the last customers in any subsystem. Hence, we can decrease the size of any queueing subsystem without going below j_i, the path index at the i^{th} queueing stage for the next subsystem.

Since $j_{i+1} < k_i$ for all queues, the second part of the right side of Equation ($VII.15$) does not contribute to $p_M(\mathbf{x}_n)$. In fact, we have the simpler relation

$$\begin{aligned} p_M(\mathbf{x}_n) &= \sum_{m=0}^{n} \alpha_{mn}\, p_M(\mathbf{x}_n^{-m}) \\ &= \sum_{m=0}^{n} \alpha_{mn} \prod_{i=0}^{n} \Psi_i(\mathbf{x}_i^{-m}). \end{aligned} \qquad (VII.15)$$

Because of the decreased size of the level-$(m+1)$ queue, the summation with respect to the index l_i in $\Psi_i(\mathbf{x}_i^{-m})$ with $m \geq i$ will have $M_i - 1$ as its upper limit rather than M_i. This comes from the observation that M_i is directly proportional to the sum of those k_j's with $j \leq i$, as indicated in Equation ($VII.3$). However, we can actually extend all these summations in $\Psi_i(\mathbf{x}_i^{-m})$, $m \geq i$, to include M_i. In doing so, we note that as one evaluates $F_i^{-m}(l_i)$, $m \geq i$, at $l_i = M_i$, then

$$\begin{aligned} F_i^{-m}(M_i) &= \binom{M_i - 1 + k_i - j_{i+1} - M_i}{k_i - j_{i+1}} \\ &= \binom{k_i - j_{i+1} - 1}{k_i - j_{i+1}} \\ &= 0. \end{aligned}$$

Therefore, if all summations are extended up to M_i, neither $\Psi_i(\mathbf{J}_i;\mathbf{K}_i)$ nor $p_M(\mathbf{x}_n)$ is altered by this change.

Moreover, the two functions in $\Psi_i(\mathbf{J}_i;\mathbf{K}_i)$, that is $F_i(l_i)$ and $H_{in}(l_i)$, depend only on the subsystem sizes k_j with $j \le i$, e.g.,

$$F_i^{-m} = \begin{cases} F_i, & \text{if } i < m; \\ F_i^{-m}, & \text{if } i \ge m. \end{cases}$$

Knowing this, and using the notation for $\Psi_i(\mathbf{J}_i;\mathbf{K}_i)$ in Equation $(VII.7)$ and the above observation on F_i^{-m}, we can now rewrite Equation $(VII.15)$ as

$$p_M(\mathbf{x}_n) = \sum_{m=0}^{n} \alpha_{mn} \prod_{i=0}^{m-1} \sum_{l_i=0}^{M_i} F_i G_i H_{in} \left[\prod_{i=m}^{n} \sum_{l_i=0}^{M_i} F_i^{-m} G_i H_{in}^{-m} \right],$$

since the function $G_i(l_i)$ depends only on the subsystem index j_i, not on the subsystem sizes. Also, without loss of generality and for simplicity of notation, we have assumed $l_n = 0$ and $G_n(0) = 1$.

From the definition of $H_{in}(l_i)$ in Equation $(VII.10)$, the following relationship can now be derived between H_{in} and H_{in}^{-m}:

$$H_{in}^{-m} = \begin{cases} H_{in}\rho_1^{-1}, & \text{if } i = m = 0; \\ H_{in}\beta_n^{-1}, & \text{if } i = n,\ m = 0; \\ H_{in}\alpha_{mn}^{-1}, & \text{if } 0 < i = m = n; \\ H_{in}, & \text{otherwise.} \end{cases} \qquad (VII.16)$$

However, from Equations $(VII.14)$ and $(VII.16)$, we note that $\rho_1\beta_n = \alpha_{0n}$. Therefore, Equation $(VII.15)$ for $p_M(\mathbf{x}_n)$ boils down to

$$\begin{aligned} p_M(\mathbf{x}_n) &= \sum_{m=0}^{n} \alpha_{mn} \prod_{i=0}^{m-1} \sum_{l_i=0}^{M_i} F_i G_i H_{in} \alpha_{mn}^{-1} \left[\prod_{i=m}^{n} \sum_{l_i=0}^{M_i} F_i^{-m} G_i H_{in} \right] \\ &= \sum_{m=0}^{n} \prod_{i=0}^{m-1} \sum_{l_i=0}^{M_i} F_i G_i H_{in} \left[\prod_{i=m}^{n} \sum_{l_i=0}^{M_i} F_i^{-m} G_i H_{in} \right]. \qquad (VII.17) \end{aligned}$$

If we look closely at Equation $(VII.17)$, we realize that, essentially, we have been left with the task of computing the expression

$$s_j = \sum_{m=0}^{j} \prod_{i=0}^{m-1} F_i \prod_{i=m}^{n} F_i^{-m}. \qquad (VII.18)$$

For the case $j = 1$, Equation $(VII.18)$ gives

$$s_1 = \prod_{i=0}^{n} F_i^{-0} + F_0 \prod_{i=1}^{n} F_i^{-1}.$$

But from Equations $(VII.3)$ and $(VII.8)$, we note that $F_i^{-0} = F_i^{-1}$ for $i > 1$, while $F_0 = 1$. Hence, recalling the definitions of $F_i(l_i)$ in Equation $(VII.8)$ or $(VII.12)$,

$$\begin{aligned} s_1 &= F_0[F_1^{-0} + F_1^{-1}] \prod_{i=2}^{n} F_i^{-1} \\ &= F_0 \left[\binom{M_2 - 1}{k_1 - j_2} + \binom{M_2 - 1}{k_1 - j_2 - 1} \right] \prod_{i=2}^{n} F_i^{-1} \\ &= F_0 \binom{M_2}{k_1 - j_2} \prod_{i=2}^{n} F_i^{-1} \\ &= F_0 F_1 \prod_{i=2}^{n} F_i^{-1}, \end{aligned}$$

where we have used the combinatorial identity for the sum of combinatorial coefficients in $(III.12)$.

At this point, a reasonable conjecture for s_j, based on the form of s_1, is

$$s_j = \prod_{i=0}^{j} F_i \prod_{i=j+1}^{n} F_i^{-j}. \qquad (VII.19)$$

It turns out that this conjecture for s_j can be proven. By using induction from the already proven case of $j = 1$, we find that

$$\begin{aligned} s_{j+1} &= s_j + \prod_{i=0}^{j} F_i \prod_{i=j+1}^{n} F_i^{-(j+1)} \\ &= \prod_{i=0}^{j} F_i \prod_{i=j+1}^{n} F_i^{-j} + \prod_{i=0}^{j} F_i \prod_{m=j+1}^{n} F_m^{-(j+1)}. \end{aligned} \qquad (VII.20)$$

Again, from Equations $(VII.3)$ and $(VII.8)$, we note that $F_i^{-j} = F_i^{-(j+1)}$ for $i > j+1$. On the other hand, for $i = j+1$,

$$\begin{aligned} F_{j+1}^{-j} + F_{j+1}^{-(j+1)} &= \binom{M_{j+2}-1}{k_{j+1}-j_{j+2}} + \binom{M_{j+2}-1}{k_{j+1}-j_{j+2}-1} \\ &= \binom{M_{j+2}}{k_{j+1}-j_{j+2}} \\ &= F_{j+1}. \end{aligned}$$

Therefore, substituting this last result into Equation $(VII.20)$ for s_{j+1}, it follows that

$$\begin{aligned} s_{j+1} &= \prod_{i=0}^{j} F_i [F_{j+1}^{-j} + F_{j+1}^{-(j+1)}] \prod_{i=j+2}^{n} F_i^{-(j-1)} \\ &= \prod_{i=0}^{j} F_i F_{j+1} \prod_{i=j+2}^{n} F_i^{-(j-1)} \\ &= \prod_{i=0}^{j+1} F_i \prod_{i=j+2}^{n} F_i^{-(j+1)}, \end{aligned}$$

as in Equation $(VII.19)$ with j replaced by $j+1$. This proves the conjecture for s_j.

In particular, evaluating s_j at $j = n$, we find

$$\begin{aligned} s_n &= \prod_{i=0}^{n} F_i \prod_{i=n+1}^{n} F_i^{-n}, \\ &= \prod_{i=0}^{n} F_i. \end{aligned}$$

Thus, substituting this result back into the expression for $p_M(\mathrm{x}_n)$ in Equation $(VII.17)$, we finally obtain

$$\begin{aligned} p_M(\mathbf{x}_n) &= \prod_{i=0}^{n} \sum_{l_i=0}^{M_i} F_i G_i H_{in} \\ &= \prod_{i=0}^{n} \Psi_i(\mathbf{J}_i; \mathbf{K}_i), \end{aligned}$$

the proposed state probability distribution in equilibrium for the multilevel camp-on system. This proves that $p_M(\mathbf{x}_n)$ in Equation $(VII.1)$ satifies the partial balance Equation $(VII.14)$, when $k_i > j_{i+1}$.

Case II: $k_m = j_{m+1}$ and $k_{m+1} \neq 0$, $0 \leq m < n$.

Since the level-$(m+2)$ subsystem is not empty, the contribution to $p_M(\mathbf{x}_n)$ because of $p_M(\mathbf{x}_n^{-m})$ is zero. For, if we were to decrease the size of the subsystem at the $(m+1)^{st}$ queueing stage by one customer, a bulk arrival of $k_{m+1}+1$ customers would be required to get back to state $\mathbf{x}_n$, and such a transition is forbidden. Also, since $k_m = j_{m+1}$, we will have $F_m = 1$ and $F_m^{-i} = 1 (0 \leq i \leq n)$. Hence, even though the term $p_M(\mathbf{x}_n^{-m})$s missing in Equation $(VII.14)$, the result presented in Equation $(VII.19)$ remains true for $F_m^{-m} = F_m = 1$. The same conclusion holds if there is more than one k_i with $k_i = j_{i+1}$. Thus, Case II reduces, in fact, to a special instance of Case I.

Case III: $k_{n-1} = j_n$ and $k_n = 0$.

In this situation, the customer in position j_n has just joined the level-n queue. For him, the multilevel camp-on system looks like a system with $(n-1)$ queueing stages rather than like a system with n queueing stages. This condition is indicated by the second term on the right side of the equilibrium balance Equation (4.6) or in the partial balance equation in $(VII.14)$. Therefore, in this situation,

$$p_M(\mathbf{x}_n) = \sum_{i=0}^{n-2} \alpha_{in}\, p_M(\mathbf{x}_n^{-i}) + \alpha_{n-1,n}\, p_M(\mathbf{x}_{n-1}^{-n+1}). \qquad (VII.21)$$

Consider the contributions to $p_M(\mathbf{x}_n)$ from the $p_M(\mathbf{x}_n^{-i})$'s in the above equation. This equation has the same form as that given in Equation $(VII.15)$ for Case I. There, we were able to extend the summation inside $\Psi_i(\mathbf{x}_i)$ up to M_i because $F_i(M_i) = 0$. In this instance, however, we cannot extend the upper limit of the summation involving the function $\Psi_{n-1}(\mathbf{x}_{n-1})$ up to M_{n-1} without altering $p_M(\mathbf{x}_n)$, since $F_{n-1} = 1i$, not 0. Nonetheless, Equation $(VII.19)$ will still apply if we account for the correct upper limit of $\Psi_{n-1}(\mathbf{x}_{n-1})$. Doing this, Equation $(VII.21)$ reduces to

$$\begin{aligned} p_M(\mathbf{x}_n) &= s_{n-2} + \alpha_{n-1n}\, p_M(\mathbf{x}_{n-1}^{-n+1}) \\ &= \left[\prod_{i=0}^{n-2}\sum_{l_i=0}^{M_i} F_i G_i H_{in}\right] \times \\ &\quad \left[\sum_{l_{n-1}=0}^{M_{n-1}-1} F_{n-1}(l_{n-1})\; G_{n-1}(l_{n-1})\; H_{n-1n}(l_{n-1})\right] H_{nn}(0) \\ &\quad + \alpha_{n-1n}\left[\prod_{i=0}^{n-2}\sum_{l_i=0}^{M_i} F_i G_i H_{i,n-1}\right] F_{n-1}^{-(n-1)}(0)\; H_{n-1,n-1}^{-(n-1)}(0). \end{aligned} \qquad (VII.22)$$

Here we have chosen to note the explicit dependence on the summation index l_{n-1}, e.g., $G_{n-1} = G_{n-1}(l_{n-1})$, and the fact that $F_n(0) = 1$.

Let us focus on the contribution from $p_M(\mathbf{x}_{n-1}^{-n+1})$ to $p_M(\mathbf{x}_n)$. From Equations $(VII.8)$ and $(VII.9)$, this contribution could also have been written as

$$\begin{aligned} p_M(\mathbf{x}_{n-1}^{-n+1}) = \left[\prod_{i=0}^{n-2}\sum_{n_i=0}^{M_i} F_i G_i H_{i,n-1}\right] \binom{M_{n-1} + j_n - 1}{M_{n-1}} \times \\ \alpha_{n-1n}\, \alpha_{n-1}^{j_n-1}\, \beta_{n-1}^{1+k_0-j_1-\sum_{i=1}^{n-2} l_i} \prod_{i=1}^{n-1} \alpha_{in-1}^{k_i-j_{i+1}}. \end{aligned} \qquad (VII.23)$$

But from Equations $(VII.4)$ to $(VII.6)$, we notice that $\alpha_{n-1n}/\alpha_{n-1} = \beta_n/\beta_{n-1}$. Similarly, from Equation $(VII.10)$, the following relationship can be derived between $H_{m,n-1}(l_m)$ and $H_{mn}(l_m)$ for $m < n$:

$$H_{m,n-1}(l_m) = \begin{cases} H_{mn}, & \text{if } 0 \le m < n-1; \\ \left(\frac{\beta_{n-1}}{\beta_n}\right)^{k_0 - j_1} \times \\ \prod_{i=1}^{n} \left(\frac{\alpha_{mn-1}}{\alpha_{mn}}\right)^{k_m - j_{m+1} H_{mn}(l_m)}, & \text{if } 0 \le m \le n-1. \end{cases}$$

Thus, completing the expression for $H_{nn}(0)$ as indicated by the above relation, the expression for $p_M(\mathbf{x}_{n-1}^{-n+1})$ in Equation $(VII.19)$ becomes

$$p_M(\mathbf{x}_{n-1}^{-n+1}) = \left[\prod_{i=0}^{n-2}\sum_{l_i=0}^{M_i} F_i G_i H_{in}\right] \binom{M_{n-1} + j_n - 1}{M_{n-1}} \alpha_{n-1}^{j_n} \beta_n \times$$
$$\beta_{n-1}^{k_0 - j_1 - \sum_{i=1}^{n-2} l_i} \prod_{i=1}^{n-2} \left(\frac{\alpha_{in-1}}{\alpha_{in}}\right)^{k_i - j_{i+1}} \prod_{i=1}^{n-2} \alpha_{in}^{k_i - j_{i+1}}.$$

However, $\alpha_{in-1}/\alpha_{in} = \beta_{n-1}/\beta_n$ from Equation $(VII.4)$. Accordingly, the above expression for $p_M(\mathbf{x}_{n-1}^{-n+1})$ reduces to

$$p_M(\mathbf{x}_{n-1}^{-n+1}) = \left[\prod_{i=0}^{n-2}\sum_{l_i=0}^{M_i} F_i G_i H_{in}\right] \binom{M_{n-1} + j_n - 1}{M_{n-1}} \alpha_{n-1}^{j_n} \times$$
$$\beta_{n-1}^{\sum_{i=1}^{n-1}(k_{i-1} - j_i) - l_i} \beta_n^{1 - \sum_{i=1}^{n-1}(k_{i-1} - j_i)} \prod_{i=1}^{n-2} \alpha_{in}^{k_i - j_{i+1}}.$$

From Equation $(VII.3)$, one recognizes the exponent of β_{n-1} as M_{n-1}. In the same way, if we rewrite the exponent for β_n as $\sum_{i=1}^{n-1}(k_{i-1} - j_i) = M_{n-1} + \sum_{i=1}^{n-2} l_i$, then $p_M(\mathbf{x}_{n-1}^{-n+1})$ boils down to

$$p_M(\mathbf{x}_{n-1}^{-n+1}) = \left[\prod_{i=0}^{n-2}\sum_{l_i=0}^{M_i} F_i G_i H_{in}\right] \binom{M_{n-1} + j_n - 1}{M_{n-1}} \alpha_{n-1}^{j_n} \beta_{n-1}^{M_{n-1}} \times$$
$$\beta_n^{1 + k_0 - j_i - \sum_{i=1}^{n-2} l_i - M_{n-1}} \prod_{i=1}^{n-2} \alpha_{in}^{k_i - j_{i+1}}.$$

At this point, it is convenient to relate the top line of the above expression $p_M(\mathbf{x}_{n-1}^{-n+1})$ to terms in $p_M(\mathbf{x}_n)$. For this, we observe that

$$G_{n-1}(M_{n-1})\Big|_{l_{n-1}=M_{n-1}} = \binom{M_{n-1}+j_n-1}{M_{n-1}},$$

$$H_{n-1,n}(M_{n-1})\Big|_{l_{n-1}=M_{n-1}} = \alpha_{n-1}^{j_n}\beta_{n-1}^{M_{n-1}},$$

$$H_{nn}(0) = \beta_n^{1+k_0-j_1-\sum_{i=1}^{n-2} l_i} \prod_{i=1}^{n-2} \alpha_{in}^{k_i-j_{i+1}}.$$

Hence, $p_M(\mathbf{x}_{n-1}^{-n+1})$ is equal to the missing term in $p_M(\mathbf{x}_{n-1}^{-n+1})$, resulting from evaluating the first summand on the right side of Equation $(VII.22)$ at $l_{n-1} = M_{n-1}$. This means that

$$\begin{aligned} p_M(\mathbf{x}_n) &= \left[\prod_{i=0}^{n-2}\sum_{l_i=0}^{M_i} F_iG_iH_{in}\right]\sum_{l_{n-1}=0}^{M_{n-1}-1} G_{n-1}(l_{n-1})\ H_{n-1,n}(n-1)\ H_{nn}(0) \\ &+ \alpha_{n-1n}\left[\prod_{i=0}^{n-2}\sum_{l_i=0}^{M_i} F_iG_iH_{i,n-1}\right]\times \\ &\qquad G_{n-1}(M_{n-1})\ H_{n-1,n-1}(M_{n-1})\ H_{nn}(0) \\ &= \left[\prod_{i=0}^{n-2}\sum_{l_i=0}^{M_i} F_iG_iH_{in}\right]\sum_{l_{n-1}=0}^{M_{n-1}} G_{n-1}(l_{n-1})H_{n-1,n}(l_{n-1})H_{nn}(0) \\ &= \prod_{i=0}^{n} \Psi_i(\mathbf{J}_i;\mathbf{K}_i). \end{aligned}$$

This is the desired result at last. The state probability distribution proposed in Equation $(VII.1)$ is indeed the solution of the global balance equation for multilevel camp-on systems. The proof of Theorem 4 is complete.

APPENDIX VIII:

Performance Derivations for PBX-like Communication Services

In Chapter V, Section 1, we stated that the waiting time distribution for class-r customers in the second-level stage of a two-level camp-on system with PBX-like communication service and infinite storage capacity is given by

$$W_{2r}(t) \;=\; \mu(1-\rho_1)e^{-\mu t}\int_0^{\rho_1} \frac{\beta_r}{x(1-\beta_r x)-\alpha_r}\left[xe^{\mu x t} - \gamma_r(x)e^{\mu\gamma_r(x)t}\right]\,dx, \quad (VIII.1)$$

with

$$\gamma_r(x) \;=\; \frac{\alpha_r}{1-\beta_r x}. \quad (VIII.2)$$

The general assumptions were that the queueing system is balanced; i.e., the service processes for first-level and second-level customers are independent and identically distributed, and no defections are allowed from any of the camp-on queues. In this appendix, we carry out the intermediate steps needed to obtain this result.

The probability $W_{2r}(t)$ that a new class-r customer arriving at the second queueing stage at time t_0 will have to wait for t units of time, before his service period starts, can be expressed as the sum of products of various independent events. The first probability is that the new customer arrives at his second-level system when it is located at position j in the second-level stage, its current queue size is k_j customers, and the size of the first-level system is n customers. This is given by $p_n(k_{jr})$ in Equation (3.36). Then we must multiply by the probability that such an arrival takes place at that second-level stage when the first-level system size is n.

The third factor is the correponding waiting time function for the newly incoming customer given the last two events. This is a $(j+k_j)$- phase Erlang distribution, for the service time has a negative exponential distribution with mean μ. This product has to be evaluated for all possible sizes n and k_j for the first-level and second-level systems and summed. Thus,

$$\begin{aligned} W_{2r}(t) &= \sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\frac{W_2^n(j;t)}{n} \\ &= \sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\sum_{k=0}^{\infty}\frac{p_n(k_{jr})\;E_{j+k_j}(t)}{n}. \end{aligned} \qquad (VIII.4)$$

Here, Prob$(j^{th}/n) = 1/n$ is the probability that this incoming class-r customer finds his second-level system in position j with respect to the service center when the first-level system is composed of n users. This is because no priority scheme is being implemented, just a plain FCFS service strategy.

From the expression for $p_n(k_{jr})$ in Equation (3.36) and Equation $(VIII.4)$, it follows that

$$W_{2r}(t) = \sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\sum_{k_j=0}^{\infty}\binom{n-j+k_j}{k_j}\frac{\beta_r^{n-j+1}\alpha_r^{k_j}(1-\rho_1)\rho_1^n}{n}\frac{(\mu t)^{j+k_j-1}}{(j+k_j-1)!}\mu e^{-\mu t}. \qquad (VIII.5)$$

Defining the temporary changes of variable $n' = n-j$ and $j' = j-1$ in Equation $(VIII.5)$, we find

$$W_{2r}(t) = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{\beta_r^{n+1}\alpha_r^{k}\;(1-\rho_1)\rho_1^{n+j+1}}{n+j+1}\frac{(\mu t)^{j+k}}{(j+k)!}\mu e^{-\mu t}. \qquad (VIII.6)$$

In deriving the last expression, we dropped the subscript j from k_j, it being unnecessary for this computation.

Since $\rho_1^n/n = \int_0^{\rho_1} x^{n-1}\, dx$ for all $n \geq 1$, we can rewrite this last equation for the waiting time distribution $W_{2r}(t)$ as an integral of the form

$$W_{2r}(t) = \mu(1-\rho_1)e^{-\mu t}\int_0^{\rho_1}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k}{k}\beta_r^{n+1}\alpha_r^k x^{n+j}\frac{(\mu t)^{j+k}}{(j+k)!}\, dx, \tag{VIII.7}$$

where we have interchanged the order of summation and integration. This is permissible because β, α, ρ_1, and $E_n(t)$ are probabilities, and so each of the summations in Equation $(VIII.6)$ is bounded by a convergent sequence for[36] $\rho_1 < 1$.

Using the the binomial expansion in Equation $(III.34)$ on the series expansion with respect to the index n, we have

$$W_{2r}(t) = \mu(1-\rho_1)e^{-\mu t}\int_0^{\rho_1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\beta_r\alpha_r^k x^j}{(1-\beta_r x)^{k+1}}\frac{(\mu t)^{j+k}}{(j+k)!}\, dx.$$

Exchange the order of summation between the sequences on the indices k and j (both sequences in the above equation being convergent), and consider the change of variables $j = n - k$. The resulting expression for $W_{2r}(t)$ will then involve two summations as in the left side of the algebraic relationship in Equation $(II.8)$. Thus, after reversing the summation order, the waiting time distribution for class-r second-level customers ends up being

$$W_{2r}(t) = \mu(1-\rho_1)e^{-\mu t}\int_0^{\rho_1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\beta_r\alpha_r^k x^{n-k}}{(1-\beta_r x)^{k+1}}\frac{(\mu t)^n}{n!}\, dx. \tag{VIII.8}$$

Here the summation over the index k is finite.

Using the formula for a truncated geometric series, Equation $(VIII.8)$ yields

$$W_{2r}(t) = \mu(1-\rho_1)e^{-\mu t}\int_0^{\rho_1}\sum_{n=0}^{\infty}\frac{\beta_r x^n}{1-\beta_r x}\frac{(\mu t)^n}{n!}\frac{1-\left[\frac{\alpha_r}{x(1-\beta_r x)}\right]^{n+1}}{1-\frac{\alpha_r}{x(1-\beta_r x)}}\, dx.$$

Reordering common terms and noting that $x(1-\beta_r x) - \alpha_r = (\beta_r x - \alpha_r)(1-x)$, it follows that

$$W_{2r}(t) = \mu(1-\rho_1)e^{-\mu t}\int_0^{\rho_1}\sum_{n=0}^{\infty}\frac{\beta x^{n+1}}{x(1-\beta_r x)-\alpha_r}\frac{(\mu t)^n}{n!}\left[1-\left[\frac{\alpha_r}{x(1-\beta_r x)}\right]^{n+1}\right]dx \qquad (VIII.9)$$

$$= \mu(1-\rho_1)e^{-\mu t}\int_0^{\rho_1}\sum_{n=0}^{\infty}\frac{\beta_r}{(\beta_r x-\alpha_r)(1-x)}\frac{(\mu t)^n}{n!}\left[x^{n+1}-\gamma_r(x)^{n+1}\right]dx,$$

where we have used the expression for $\gamma_r(x)$ given by Equation $(VIII.2)$.

If we now sum over the index n in Equation $(VIII.9)$ and use the series expansion for e^x, we finally get that the probability distribution for the waiting time period of class-r second-level customers in a camp-on queueing system with PBX-like communication services and infinite storage capacity is

$$W_{2r}(t) = \mu(1-\rho_1)e^{-\mu t}\int_0^{\rho_1}\frac{\beta_r}{(\beta_r x-\alpha_r)(1-x)}\left[xe^{\mu xt}-\gamma_r(x)e^{\mu\gamma_r(x)t}\right]dx. \qquad (VIII.10)$$

From this waiting time distribution $W_{2r}(t)$, we can also compute the cumulative probability distribution $W_{2r}(>t)$, that an incoming class-r customer will have to wait for more than t units of times before his service period begins:

$$W_{2r}(>t) = \int_t^{\infty} W_2(\tau)d\tau.$$

Exchanging the order of integration between τ and x and integrating with respect to τ, it follows that the cumulative distribution $W_{2r}(>t)$ is given by

$$W_{2r}(>t) = (1-\rho_1)\left[\int_0^{\rho_1} f(x)\frac{x}{1-x}e^{-\mu(1-x)t}dx - \int_0^{\rho_1} f(x)\frac{\gamma_r(x)}{1-\gamma_r(x)}e^{-\mu(1-\gamma_r(x))t}dx\right], \qquad (VIII.11)$$

where

$$f_r(x) = \frac{\beta_r}{(\beta_r x - \alpha_r)(1-x)}. \qquad (VIII.12)$$

Similarly, consider the mean waiting time for this class-r second-level customer in a PBX-like communication system. If we compute this from Equation $(VIII.10)$ and using the fact that $\int_0^\infty xe^{-ax}dx = 1/a^2$ for $a > 0$, we get

$$\overline{W}_{2r} = \int_0^\infty \tau \, W_{2r}(\tau) d\tau \qquad (VIII.13)$$

$$= \frac{(1-\rho_1)}{\mu} \int_0^{\rho_1} \frac{\beta_r}{(\beta_r x - \alpha_r)(1-x)} \left[\frac{x}{(1-x)^2} - \frac{\gamma_r(x)}{(1-\gamma_r(x))^2} \right] dx.$$

Finally, recalling again the definition of $\gamma_r(x)$ in Equation $(VIII.2)$ and evaluating the expression inside brackets in Equation $(VIII.13)$, we find that the mean waiting time $\overline{W}_{2r}$ is given by

$$\overline{W}_{2r} = \frac{(1-\rho_1)}{\mu} \int_0^{\rho_1} \frac{\beta_r}{(\beta_r x - \alpha_r)(1-x)} \left[\frac{x}{(1-x)^2} - \frac{\alpha_r(1-\beta_r x)}{\beta_r^2(1-x))^2} \right] dx$$

$$= \frac{(1-\rho_1)}{\beta_r \mu} \int_0^{\rho_1} \frac{1}{(1-x)^3} \, dx$$

$$= \frac{(2-\rho_1)}{2\beta_r \mu (1-\rho_1)} \, \rho_1. \qquad (VIII.14)$$

This concludes the derivation of the waiting-time distributions for class-r second-level customers in a two-level camp-on system with infinite storage capability and with PBX-like communication services (FCFS overall service discipline.)

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