THE EFFECT OF GUNFIRE ON THE LONGITUDINAL MOTION OF AN AIRPLANE

Thesis

by

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SUMMARY

The effect produced by the reaction forces and moments resulting from the fire of light and heavy calibre rapid fire guns on the path of motion of a modern military aircraft is discussed. This discussion is made on the basis of the "exact" equations of motion as developed by Bryan, Routh, and others, using the forms given in Dr. C. B. Millikan's notes on the dynamic stability of the airplane. The formal solution is made by the use of Heaviside operational calculus.

For general use there is developed a nomogram based on a series solution to the differential equations of motion. The application of this nomogram is illustrated and it is shown to give an easy and rapid approximation to the disturbance, and a sufficiently accurate one within the time ranges required. Since the exact solution requires several hours and the use of an experienced computer on a computing machine while the nomogram permits slide rule computations and results in a solution within a few minutes, its usefulness is readily apparent. The extension of the methods used in this paper to disturbed lateral motions is briefly discussed.

INTRODUCTION

The present war has modified the design of military aircraft in many particulars. Outstanding among these is the increase of armament over anything heretofore considered necessary. There has been an increase in the number of machine guns from the usual two of the last war to eight or twelve on modern pursuit and interceptor airplanes. The calibre has been increased from size .30 to .50, and rates of fire for all calibres including cannon have been raised. Heavier types of craft such as attack and patrol ships mount .50 calibre guns in batteries of four, and turrets with batteries of two 20 mm. cannon, or one 37 mm. or larger cannon are in existance or are being seriously contemplated.

The reaction forces arising from the firing of these weapons are considerable, and are of a magnitude sufficient to cause a disturbance in the flight path of the airplane. It is then necessary to know the magnitude of the disturbance for two purposes: whether a correction can be made by the pilot in sufficient time so that the aiming accuracy will not be impaired too greatly; for fire prediction devices that will take into account the disturbance in the flight path.

It is generally agreed that the reaction time of the average pilot is something of the order of six-tenths of a second. Hence the range of interest insofar as this problem is concerned is from the instant at which firing begins to a time 0.6 seconds later at which time it may be presumed that the pilot corrects the motion of the airplane. It is evident that the quantities which are to be investigated must be the angle through which the aircraft has been rotated and the time rate of change of the angular motion, since these quantities will determine the corrective control on the part of the pilot and the error produced in the aim. In the case of an airplane firing guns forward only, it may be of interest to determine the deceleration and final velocity. This is of academic interest only since the aim will not be impaired if the reaction forces pass through the airplane center of gravity.

In order to calculate the motion of the airplane under the disturbance produced by the gunfire, it is then necessary to know the airplane dynamic characteristics and the forces produced by the fire and the time duration of the latter.

REACTION FORCES DUE TO GUNFIRE

A comparative table of characteristics of aircraft machinesguns and aircraft cannon is given in reference 4. While the forces resulting from cannon fire are given, those corresponding to machine gún fire are not. Sufficient data is available, however, so that these forces can be computed. Thus, for the 37 mm. cannon the observed data are:

bullet weight	1.10 pounds
muzzle velocity	1250 feet per second
reaction force	1000 pounds
rate of fire	125 per minute

From the relations for impulse and momentum, the time duration of the explosion may be calculated:

$$t = \frac{mv}{f} = \frac{\frac{1.1}{32.2} \times 1250}{1000} = .0426 \text{ seconds}$$

$$a = \frac{v}{t} = \frac{1250}{.0426} = 29300 \text{ ft. per second}^2 \text{ assuming uniform acceleration.}$$

The gross weight of the 37 mm. shell is 1.4#, hence the case and powder weigh 0.3#. From the size of the shell case it is estimated that the case is 0.381 cubic inches of brass of .308 pounds per cubic inch density = .117# for the case leaving .183# powder. The powder to bullet weight ratio is .183 ÷ 1.1 = .166.

The .30 calibre machine gun fires a bullet weighing 180 grains from a cartridge of 395 grains gross. Assuming the same charge weight to cartridge weight as for the cannon shell, there is then 138 grains of powder, and the charge to bullet weight ratio is $138 \div 180 = .765$. Assuming the same rate of burning, the acceleration produced on the .30 calibre bullet is $\frac{.765}{.166} \times 29300 = 4.6 \times 29300$ ft/sec.^{2}

Since 180 grains = .0258#, the force on the gun is

$$f = ma = \frac{.0258}{32.2} \times (4.6 \times 29300) = 108 \text{ pounds per shot.}$$

$$t = \frac{mv}{f} = \frac{.0258}{32.2} \times \frac{2660}{108} = .0197 \text{ seconds per shot.}$$

$$1250 \text{ shots per second} = \frac{1250}{60} = 20.83$$

Average force from continuous fire = $108 \times 20.83 \times .0197$

$= 44.4\#$

From transfer of energy:

$$\frac{m}{t} = 1250 \times \frac{.0258}{32.2} = 1.000 \text{ slugs per minute} = .0167 \text{ slugs per second.}$$

$$f_{avg} = \frac{mv}{t} = .0167 \times 2660 = 44.3\#$$ which is a good check!

Since machine guns of .30 calibre will be considered in groups of 4 (battery fire), and since all guns will probably not fire simultaneously, the assumption of an average force acting continuously at the gun will be a reasonable one.

Similary, for the case of the .50 calibre guns, since they also will be considered in batteries of four, the assumption of an average force acting continuously instead of considering finite impulses will not be too drastic. Although in this case, however, it is more severe than in the .30 calibre case since the rate of fire is half that of the lighter calibre.

By means of an identical analysis the average force for the .50 calibre machine gun, firing continuously is found to be f_{avg} = 91.4#. Table I gives the pertinent data for the various types of guns considered here, taken or computed from data in reference 4.

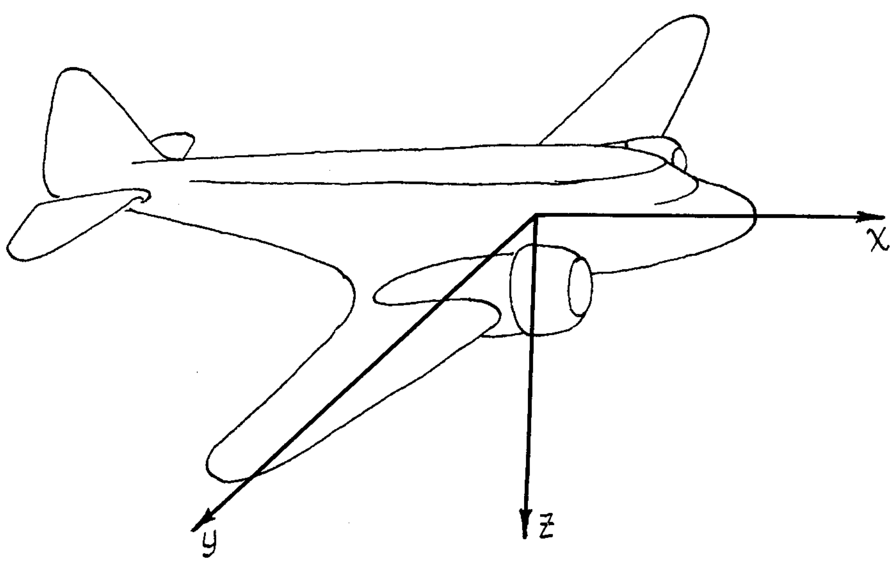
TABLE I

Gun Calibre	.30	.50	20 mm.	37 mm.
Rate of fire rds/min.	1250	650	600	125
Bullet weight #	.0258	.1062	.290	1.1
Muzzle velocity ft./ses.	2660	2550	2920	1250
Avg. reaction force, #	44.4	91.4	(439 (484*	1000*

*Only during explosion, i.e., actual and not average forces.

<u>NOMENCLATURE</u>

D	$\frac{d()}{dt}$, differential operator
X_u	$\frac{dX}{du}$, resistance derivative
X_w	$\frac{dX}{dw}$, " "
X_q	$\frac{dX}{dq}$, rotary "
Z_u	dZ/du resistance "
Z_w	dZ/dw " "
Z_q	dZ/dq rotary "
M_u	dM/du resistance "
M_w	dM/dw " "
M_q	dM/dq rotary "
X	component of external force ÷ mass in x-direction.
Z	" " " " in z direction ÷ mass.
M	" " " moment about y-axis ÷ mass.



X_g reaction force due to gunfire ÷ mass, in x-direction.

Z_g " " component in z-direction due to gunfire ÷ mass

M_g " moment about y axis ÷ mass

u dx/dt velocity in x-direction (disturbance velocity)

w dZ/dt " " z-direction

q dθ/ dt angular velocity about y axis

θ angular displacement from x-axis

U velocity in x-direction of airplane

K_B radius of gyration about the y axis (at the c.g.)

$$X_u = -2 \frac{q'S}{mU} C_D, \text{ where } q' = \frac{\rho}{2} U^2$$

$$Z_u = -2 \frac{q'S}{mU} C_L$$

$$M_u = 0 \quad \text{(power-off)}$$

$$X_w = \frac{q'S}{mU} \frac{1 - \frac{a_o}{\pi AR}\left(\frac{2}{e} - 1\right)}{1 + \frac{a_o}{\pi AR}} C_L$$

$$Z_w = -\frac{q'S}{mU} \left(\frac{a_o}{1 + \frac{a_o}{\pi AR}} + C_D \right)$$

$$M'_w = \frac{q'S}{mU} \frac{t}{K_B^2} \frac{a_o}{1 + \frac{a_o}{\pi AR}} \frac{dC_M}{dC_L}$$

$$X_q = 0$$

$$Z_q = 0$$

$$M'_q = -\eta_H \frac{q'S}{mU} K \frac{S_t}{S} \frac{l^2}{K_B^2} \frac{a_o}{1 + \frac{a_o}{\pi AR_t}}$$

EQUATIONS OF LONGITUDINAL MOTION

The general equations of motion for an airplane subjected to small disturbances is given by

$$
\begin{aligned}
&(D-X_u)\,u - X_w w - (X_q D - g\cos\theta_o)\,\theta = 0 && \text{(a)}\\
&-Z_u u + (D-Z_w)\,w - (Z_q D + UD - g\sin\theta_o)\,\theta = 0 && \text{(b)}\\
&-M_u u - M_w w + (K_B^2 D^2 - M_q D)\,\theta = 0 && \text{(c)}
\end{aligned}
\qquad (1)
$$

In the development of these equations* aerodynamic forces only were considered. For the case in which the forces and moments produced by the gunfire reaction are considered, the equations are modified to:

$$
\begin{aligned}
&(D - X_u)\,u - X_w w - (X_q D - g\cos\theta_o)\,\theta = X_g && \text{(a)}\\
&-Z_u u + (D - Z_w)\,w - (Z_g D + UD - g\sin\theta_o)\,\theta = Z_g && \text{(b)}\\
&-M_u u - M_w w + (K_B^2 D^2 - M_q D)\,\theta = M_g && \text{(c)}
\end{aligned}
\qquad (2)
$$

Solving the simultaneous equations for θ:

$$
\theta = \frac{\begin{vmatrix} X_g & D-X_u & -X_w \\ Z_g & -Z_u & D-Z_w \\ M_g & -M_u & -M_w \end{vmatrix}}{\begin{vmatrix} D-X_u & -X_w & (-X_q D - g\cos\theta_o) \\ -Z_u & D-Z_w & -(Z_q D + UD - g\sin\theta_o) \\ -M_u & -M_w & (K_B^2 D^2 - M_q D) \end{vmatrix}} = \frac{f(D)}{F(D)} \qquad (3)
$$

*Millikan, C. B., Aerodynamics of the Airplane, p. 16.

From the expansion of the determinants:

$$f(D) = X_g \left[Z_u M_w + M_u (D-Z_w) \right] + Z_g \left[M_u X_w + M_w (D-X_u) \right] + M_g \left[(D-X_u)(D-Z_w) - Z_u X_w \right] \quad (4)$$

$$F(D) = (D-X_u) \left[(D-Z_w)(K_B^2 D^2 - M_q D) - M_w (Z_q D + UD - g \sin \theta_o) \right] + Z_u \left[M_w(-X_q D - g \cos \theta_o) - X_w (K_B^2 D^2 - M_q D) \right] + M_u \left[(D-Z_w)(-X_q D + g \cos \theta_o) - X_w (Z_q D + UD - g \sin \theta_o) \right] \quad (5)$$

By means of the Heaviside formula, equation (3) may be put into the form

$$\theta = \frac{f(o)}{F(o)} + \sum_{\alpha_i} \frac{f(\alpha) e^{\alpha t}}{\alpha F'(\alpha)} \quad (6)$$ where α_i are the roots of $F(D) = 0$. Inspection of equation (5) shows that F(D) is a quartic, which is one of the reasons for the difficulty in obtaining a rapid solution because solving for the roots of the quartic is a long and tedious process. For the purposes of this investigation the approximate factorizations of Bairstow and others are insufficiently accurate. An accurate factorization, as will be seen later, is necessary so that the boundary conditions to the problem will be maintained in the solution of the differential equation of motion (6).

For use in conjunction with the Heaviside formula, the following statements simplify the actual calculations: if there are conjugate roots in the equation $F(D) = 0$, that is, roots of the form (a+ib) and (a-ib), then*

*Klemin & Ruffner, Operator Solutions in Airplane Dynamics, J. Ae. Sci., Vol. III, page 252.

$$\sum_{\alpha_i} \frac{f(\alpha)e^{\alpha t}}{\alpha F'(\alpha)} = \sum_{\substack{\alpha = a+ib, \\ a-ib}} \frac{f(\alpha)\ e^{\alpha t}}{\alpha F'(\alpha)} = \frac{2\ e^{a t}}{(a^2+b^2)(e^2+f^2)} \times \left(\left[a(ce-df)+ b(de-cf)\right] \cos bt + \left[b(ce+df)-a(de-cf)\right] \sin bt \right)$$

$$\text{where } c = f(a) - \frac{b^2}{2!} f''(a) + \frac{b^4}{4!} f^{IV}(a) - \frac{b^6}{6!} f^{VI}(a) + \dots \qquad (7)$$

$$d = bf'(a) - \frac{b^3}{3!} f'''(a) + \frac{b^5}{5!} f^{V}(a) + \dots$$

$$e = F'(a) - \frac{b^2}{2!} F'''(a) + \frac{b^4}{4!} F^{V}(a) - \frac{b^6}{6!} F^{VII}(a) + \dots$$

$$f = bF''(a) - \frac{b^3}{3!} F^{IV}(a) + \frac{b^5}{5!} F^{VI}(a) + \dots$$

The procedure followed in the solution of a given case by the method given above is as follows. The values of the aerodynamic derivatives are obtained from the relationships given on page 7. These values are then substituted into the equations for f(D) and F(D). The roots of the quartic F(D) = 0 are then obtained. In the example given in the appendix, the method of Zimmerman* was used. This method is essentially one of repeated approximations starting from an arbitrarily assumed value, and using a graph to obtain successively closer approximations. Having once obtained the roots of F(D) = 0, these can be introduced into the Heaviside formula and $\theta = \theta$ (t) can be obtained numerically. The solution obtained then holds for all values of time, or until conditions are changed.

*Zimmerman, C. H., N.A.C.A., Technical Report No. 589.

Now since the range of interest exists only till time t = 0.6 seconds it is feasible to investigate a solution in series form. Such a form, if the number of terms are few enough, would lend itself to a more rapid numerical solution. It would also be adaptable to some form of graphical solution.

Assuming, then, a solution of the form

$$\theta = \theta_o + \overset{I}{\theta}_o t + \overset{II}{\theta}_o \frac{t^2}{2!} + \overset{III}{\theta}_o \frac{t^3}{3!} + \overset{IV}{\theta}_o \frac{t^4}{4!} + \overset{V}{\theta}_o \frac{t^5}{5!} \tag{8}$$

where the subscript $()_o$ indicates conditions at time t = 0, there is obtained by differentiation

$$\overset{I}{\theta} = \overset{I}{\theta}_o + \overset{II}{\theta}_o t + \overset{III}{\theta}_o \frac{t^2}{2!} + \overset{IV}{\theta}_o \frac{t^3}{3!} + \overset{V}{\theta}_o \frac{t^4}{4!} \tag{9}$$

From equation (2a) there is obtained:

$$\frac{d\theta}{dt} = \frac{1}{X_q}\left\{ \frac{du}{dt} - X_u u - X_w w + g\theta - Xg \right) \qquad \text{(a)}$$

$$\frac{d^2\theta}{dt^2} = \frac{1}{X_q}\left\{ \frac{d^2u}{dt^2} - X_u \frac{du}{dt} - X_w \frac{dw}{dt} + g \frac{d\theta}{dt} \right\} \qquad \text{(b)} \quad (10)$$

$$\frac{d^3\theta}{dt^3} = \frac{1}{X_q}\left\{ \frac{d^3u}{dt^3} - X_u \frac{d^2u}{dt^2} - X_w \frac{d^2w}{dt^2} + g \frac{d^2\theta}{dt^2} \right\} \qquad \text{(c)}$$

From equation (2b), similarly:

$$\frac{d\theta}{dt} = \frac{1}{Z_q+U}\left\{ - Z_u u + \frac{dw}{dt} - Z_w w - Zg \right\} \qquad \text{(a)}$$

$$\frac{d^2\theta}{dt^2} = \frac{1}{Z_q+U}\left\{ - Z_u \frac{du}{dt} + \frac{d^2w}{dt^2} - Z_w \frac{dw}{dt} \right\} \qquad \text{(b)} \quad (11)$$

$$\frac{d^3\theta}{dt^3} = \frac{1}{Z_q+U}\left\{ - Z_u \frac{d^2u}{dt^2} + \frac{d^3w}{dt^3} - Z_w \frac{d^2w}{dt^2} \right) \qquad \text{(c)}$$

And from equation (2c):

$$\frac{d^2\theta}{dt^2} = \frac{1}{K_B^2}\left\{ M_g + M_q \frac{d\theta}{dt} + M_w w + M_u u \right\} \qquad (a)$$

$$\frac{d^3\theta}{dt^3} = \frac{1}{K_B^2}\left\{ M_q \frac{d^2\theta}{dt^2} + M_w \frac{dw}{dt} + M_u \frac{du}{dt} \right\} \qquad (b) \qquad (12)$$

$$\frac{d^4\theta}{dt^4} = \frac{1}{K_B^2}\left\{ M_q \frac{d^3\theta}{dt^3} + M_w \frac{d^2w}{dt^2} + M_u \frac{d^2u}{dt^2} \right\} \qquad (c)$$

$$\frac{d^5(\theta)}{dt^5} = \frac{1}{K_B^2}\left\{ M_q \frac{d^4\theta}{dt^4} + M_w \frac{d^3w}{dt^3} + M_u \frac{d^3u}{dt^3} \right\} \qquad (d)$$

Applying the initial conditions that at time $t = 0$, $u = w = \theta = \dot{\theta} = 0$, then the following relations obtain:

from (10a) $\left.\frac{d\theta}{dt}\right|_0 = 0 = \left.\frac{du}{dt}\right|_0 - X_g$

from (11a) $\left.\frac{d\theta}{dt}\right|_0 = 0 = \left.\frac{dw}{dt}\right|_0 - Z_g$

from (12a) $\left.\frac{d^2\theta}{dt^2}\right|_0 = \frac{M_g}{K_B^2}$

Then from (10b) $\left.\frac{d^2\theta}{dt^2}\right|_0 = \frac{M_g}{K_B^2} = \frac{1}{X_q}\left\{ \left.\frac{d^2u}{dt^2}\right|_0 - X_u X_g - X_w Z_g \right\}$

$$\text{or} \quad \left.\frac{d^2u}{dt^2}\right|_0 = \frac{X_q M_g}{K_B^2} + X_u X_g + X_w Z_g$$

from (11b) $\left.\frac{d^2\theta}{dt^2}\right|_0 = \frac{M_g}{K_B^2} = \frac{1}{Z_q+U}\left\{ - Z_u X_g + \left.\frac{d^2w}{dt^2}\right|_0 - Z_w Z_g \right\}$

$$\text{or} \quad \left.\frac{d^2w}{dt^2}\right|_0 = (Z_q+U)\frac{M_g}{K_B^2} + Z_u X_g + Z_w Z_g$$

from (12b) $$\left.\frac{d^3\theta}{dt^3}\right|_0 = \frac{1}{K_B^2}\left\{ M_q \frac{M_g}{K_B^2} + M_w Z_g + M_u X_g \right\}$$

from (12c) $$\left.\frac{d^4\theta}{dt^4}\right|_0 = \frac{1}{K_B^2}\left\{ M_q \left.\frac{d^3\theta}{dt^3}\right|_0 + M_w \left.\frac{d^2 w}{dt^2}\right|_0 + M_u \left.\frac{d^2 u}{dt^2}\right|_0 \right\}$$

Noting that $X_q = M_u = 0$, then:

$$\theta = \overset{''}{\theta}_0 \frac{t^2}{2} + \overset{'''}{\theta}_0 \frac{t^3}{6} + \overset{IV}{\theta}_0 \frac{t^4}{24} + \overset{V}{\theta}_0 \frac{t^5}{120} \qquad (8)$$

$$\dot{\theta} = \overset{''}{\theta}_0 t + \overset{'''}{\theta}_0 \frac{t^2}{2} + \theta_0 \frac{t^3}{6} + \theta_0 \frac{t^4}{24} \quad \text{where} \qquad (9)$$

$$\overset{''}{\theta}_0 = \frac{M_g}{K_B^2}$$

$$\overset{'''}{\theta}_0 = \frac{1}{K_B^2}\left\{ M_g \overset{''}{\theta}_0 + M_w Z_g \right\}$$

$$\overset{IV}{\theta}_0 = \frac{1}{K_B^2}\left\{ M_q \overset{'''}{\theta}_0 + M_w \overset{''}{W}_0 \right\}, \quad \overset{''}{W}_0 = U\overset{''}{\theta}_0 + Z_u X_g + Z_w Z_g$$

$$\overset{V}{\theta}_0 = \frac{1}{K_B^2}\left\{ M_q \overset{IV}{\theta}_0 + M_w \overset{'''}{w}_0 \right\}, \quad \overset{'''}{w}_0 = U\overset{'''}{\theta}_0 + Z_u \overset{''}{u}_0 + Z_w \overset{''}{w}_0$$

$$\overset{''}{u}_0 = X_u X_g + X_w Z_g$$

It is seen that each coefficient in the series solution is a function of the known stability derivatives of the airplane, and of the external forces and moments produced by the gunfire reactions. This form of the general solution readily lends itself to nomographical methods for the solution of particular cases.

Following the procedure of reference 8, equation (9), for example, can be put into a form such that a generalized nomogram may be constructed from it immediately. Thus, separating the equation into two parts ϕ_1 and ϕ_2 such that

$$\dot{\theta} = \phi_1 + \phi_2 \qquad \text{(a)} \qquad \text{(10)}$$

where $$\phi_1 = \overset{''}{\theta}_o t + \overset{'''}{\theta}_o \frac{t^2}{2} \qquad \text{(b)}$$

and $$\phi_2 = \overset{IV}{\theta}_o \frac{t^3}{6} + \overset{V}{\theta} \frac{t^4}{24} \qquad \text{(c)}$$

and by introducing new variables x and y, a determinant may be obtained for each part. This determinant gives the scale proportions of the nomogram. There is a nomogram for each part, and by adding the solutions obtained from each, $\dot{\theta} = \dot{\theta}(t)$ results. Now let $x = \overset{''}{\theta}_o$ and $y = -\phi_1$, from which $tx + y + \frac{1}{2}\overset{'''}{\theta}_o t^2 = 0$ (a)

$$x + o - \overset{''}{\theta}_o = 0 \qquad \text{(b)} \qquad \text{(11)}$$

$$0 + y + \phi_1 = 0 \qquad \text{(c)}$$

Writing the determinant:

$$\begin{vmatrix} t & 1 & \frac{1}{2}\overset{'''}{\theta}_o t^2 \\ 1 & 0 & -\overset{''}{\theta}_o \\ 0 & 1 & \phi_1 \end{vmatrix} = 0$$

The several operations required to put this determinant into the form from which the nomogram may be obtained directly, are indicated below:

$$\begin{vmatrix} t & 1 & \frac{1}{2}\overset{'''}{\theta}_0 t^2 \\ 1 & 0 & -\theta_0 \\ 0 & 1 & \phi_1 \end{vmatrix} = \begin{vmatrix} t & 1+t & \frac{1}{2}\overset{'''}{\theta}_0 t^2 \\ 1 & 1 & -\theta_0 \\ 0 & 1 & \phi_1 \end{vmatrix} = \begin{vmatrix} \frac{t}{1+t} & 1 & \frac{\frac{1}{2}\theta_0 t^2}{1+t} \\ 1 & 1 & -\theta_0 \\ 0 & 1 & \phi_1 \end{vmatrix} = 0$$

By rearranging the columns and shifting the rows downward by one row:

$$\begin{vmatrix} 0 & \phi_1 & 1 \\ \frac{t}{1+t} & \frac{\frac{1}{2}\theta_0 t^2}{1+t} & 1 \\ 1 & -\theta_0 & 1 \end{vmatrix} = 0 \qquad (12)$$

The form which includes the scale factors for the nomogram is given by:

$$\begin{vmatrix} 0 & \mu_1 \phi_1 & 1 \\ \dfrac{\delta_3 \cdot \mu_1 \frac{t}{1+t}}{(\mu_1 - \mu_3)\frac{t}{1+t} + \mu_3} & \dfrac{\mu_1 \cdot \mu_3 \frac{\frac{1}{2}\overset{'''}{\theta}_0 t^2}{1+t}}{(\mu_1 - \mu_3)\frac{t}{1+t} + \mu_3} & 1 \\ \delta_3 & -\mu_3 \overset{''}{\theta}_0 & 1 \end{vmatrix} = 0 \qquad (13)$$

From the choice of extensions of each scale and from the space limitations of the nomogram the scale factors μ_1, μ_3 and δ_3 are obtained. Since the quantities in the first column represent the coefficients of x and those in the second column the coefficients of y, and knowing the scale factors, the nomogram may now be drawn. The first and third rows give the equations for the lines of ϕ_1 and $\ddot{\theta}_0$ respectively. The second row gives the equation of $\dddot{\theta}_0$ as a function of x, y, and t. This equation results in a family of curves in which $\dddot{\theta}_0$ is a constant. These curves together with the lines of constant values of t form a grid. Figure 2 is the nomogram represented by the determinant of (13). Connecting a given value of $\ddot{\theta}_0$ with a given value of $\dddot{\theta}_0$ at a designated time by means of a line (isopleth) results in a value of ϕ_1 at the intersection of the isopleth and the ϕ_1-scale. This value of ϕ_1 satisfies the equation (10 c)

$$\phi_1 = \ddot{\theta}_0 t + \frac{\dddot{\theta}_0 t^2}{2}$$

The complete curve of $\phi_1 = \phi_1(t)$ is then obtained by reiteration of this process at different values of time t. Adding the solutions of ϕ_1 those obtained by using a similar nomogram for ϕ_2 results in $\dot{\theta} = \dot{\theta}(t)$. By a similar nomographical means the curve of $\theta = \theta(t)$ can also be obtained.

The method of obtaining the numerical values for the construction of these nomographical charts is illustrated in the Appendix.

INVESTIGATION OF A PARTICULAR CASE

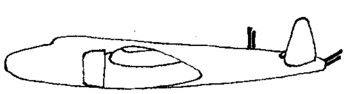
A twin engined military airplane of recent design, Figure 1, was investigated for longitudinal motion under fire. The investigation was made using the exact analysis and, later when the generalized nomogram had been developed, was checked by the second method given in the previous section. Three types of gun installations more or less typical of the latest practice were assumed. These installations were deliberately assumed in the worst probable position insofar as the effect on stability was concerned. Specifically, the conditions assumed were: 1) batteries of four .50 calibre machine guns in each turret, 2) battery of two 20 mm. cannon in each turret, 3) one 37 mm. cannon in each turret. In each case the aft turret guns were assumed directed downward while the tail turret guns were assumed deflected 30° downward from the horizontal. Presumably such a position could be attained if the aircraft were operating against ground targets. The data and sample calculations are included in the Appendix, the results being tabulated in table II.

In addition a small two seater airplane was investigated. The object was primarily to check the accuracy of the nomogram and to establish the scale limits of the diagram. The aerodynamic derivatives of the bomber were calculated from the relations given on page 7. Those of the two-seater were taken directly from reference 2. The derivatives are tabulated in table II, while the gunfire forces are given in table III.

Table II - Aerodynamic Derivatives

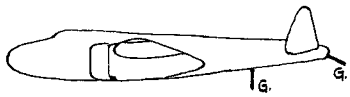
Airplane Derivatives	Bomber	Two Seater
X_u	-.01615	-.0739
Z_u	-.1720	-.3230
X_w	-.0460	-.0935
Z_w	-1.21	-.228
M_w	-.811	-1.40
M_q	-146.5	-160
U	375	252
K_B^2	77.3	38.6

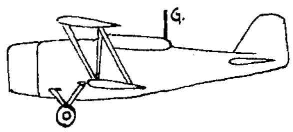
Table III - Gunfire Forces and Moments



Case	Calibre and Condition of Firing	X_g	Z_g	M_g
I	battery of four .50 calibre per turret firing upward-continuous fire	.414	.716	17.4
	battery of two .20 mm. per turret firing downward			

Table III Continued





II	a) simultaneous fire of both guns	1.128	-1.95	-46.10
III	b) staggered fire	.562	-.973	-23.03
IV	one .37 mm. per turret firing downward	1.162	-2.010	-47.60
V	one .37 mm. cannon 7 ft. aft of c.g. firing upward	0	1.37	9.56

Table IV - Calculated Equations for Angle

and Angular Velocity as Functions of Time

Case	$\theta = \theta(t)$
I	$\theta = .1168 - .0373\,e^{-1.5535\,t}\sin(112.04t^{\circ}+24.38^{\circ})$ $+ .4458\,e^{-.00695\,t}\sin(5.51t^{\circ}-13.23^{\circ})$
II	approximately same as IV
III	$\theta = -.1055 + .0488\,e^{-1.5535\,t}\sin(112.04t^{\circ}+24.35)$ $+ .5858\,e^{-.00695\,t}\cos(5.51t^{\circ}+81.1^{\circ})$
IV	$\theta = -.2484 + .1006\,e^{-1.5535\,t}\sin(112.04t^{\circ}+22.72^{\circ})$ $-1.2527\,e^{-.00695}\cos(5.51t^{\circ}+80.37^{\circ})$
V	$\theta = .0211 + .0058\,e^{-.0350\,t}\sin(10.88t^{\circ}+31^{\circ})$ $- .0337\,e^{-2.1891\,t}\sin(132.04t^{\circ}+46.2^{\circ})$

Table IV - Continued

Case	$\dot{\theta} = \dot{\theta}(t)$
I	$\dot{\theta} = .0931\, e^{-1.5535\, t} \sin(112.04t^{o} - 27.17^{o})$
	$-.0429\, e^{-.00695\, t} \sin(5.51t^{o} - 103.23^{o})$
II	approximately same as IV
III	$\dot{\theta} = .1222\, e^{-1.5535\, t} \cos(112.04t^{o} + 62.68^{o})$
	$-.0564\, e^{-.00695\, t} \sin(5.51^{o} + 85.21^{o})$
IV	$\dot{\theta} = .2512\, e^{-1.5535\, t} \cos(112.04t^{o} + 61.20^{o})$
	$-.1208\, e^{-.00695\, t^{o}} \sin(5.51t^{o} + 84.50^{o})$
V	$\dot{\theta} = .00111\, e^{-.0350 t} \cos(10.88t^{o} + 33^{o})$
	$-.1071\, e^{-2.1891 t} \cos(132.04t^{o} + 89.7^{o})$

The calculated equations for the angle and the angular velocity as functions of time are given in table III. It should be noted that these equations were obtained by the use of the Heaviside unit function. Hence, the equations give the reaction of the airplane to forces and moments which are zero until time zero at which point in time they assume finite values instantaneously, these values being continuously maintained. Thus, for rapid fire guns the equation gives the motion of the airplane. For guns of the 37 mm. type, in which there is a considerable period of time between each shot, the path of motion is obtained by repeated use of the equation in which the Heaviside function is alternately positive and negative. That is, at time zero a positive force is applied which lasts for the duration of the explosion time and at the end of that time an equal negative force is applied to the airplane. By combining the solutions, the true path is obtained. This is most easily done graphically and is illustrated in figure 5.

Figures 3 to 6 are the plots of the equations given in table III. The worst condition for the attack bomber exists for the case of two .20 mm cannon per turret fixed simultaneously. Figure 5 shows that at .6 seconds after firing has begun the airplane has rotated approximately .06 radius (integrating under curve of $\dot{\theta} = \dot{\theta}(t)$ and assuming linear variation of $\dot{\theta}$ with t) and and has an angular velocity of .10 radius per second. At a range of 200 yards and a firing rate of 500 rounds per minute this corresponds to 2.1 feet dispersion per round computed from the angular deflection of the airplane, but neglecting the angular velocity, and at the maximum effective practical range of the cannon, 1200 yards, the dispersion would be 12 feet. If it is assumed that effectively the target, say an airplane, presents a 6 foot diameter vulnerable area, and if satisfactory fire is defined as placing at least two successive rounds within the target, then the effective range of the 20 mm. cannon on this airplane is reduced to 600 yards. It is the opinion of several military pilots that the angular velocity of .10 radius per second, or about 6 degrees per second, could be overcome so that gunner could maintain satisfactory aim in the period following the pilot reaction time. Since the burst of fire rarely lasts more than a second or two it is seen that over a considerable period of the firing time the effectiveness of the weapon is seriously reduced at ranges over 600 yards. At shorter ranges a slight error in initial aim may result in missing the target altogether because of the rotation of the plane during fire.

Figure 4 shows that alternate firing of the 20 mm. cannon in a turret can be well approximated by use of an average force. Hence, for this case as well as that of the .50 calibre machine guns the nomogram can be used instead of the long "exact" method. In fact, the nomogram can be applied to any case in which an average force may be used to approximate the true conditions. Figures 3, 4, and 6 show the degree to which the true curve is approximated by the use of the nomogram. This approximation is seen to be quite satisfactory, up to the peak of the curve reached by the use of the nomogram. Thereafter, the divergence becomes quite large. It has been found in practice (on the cases investigated) that a better determination of the peak results from using the nomogram first as a four term approximation, then as a three term approximation, and splitting the difference. This may not be true for all cases. However, since the sizes of the third and fourth terms are always opposite, it will give a good indication of the trend. In the bomber case the peak occurs after .6 seconds and the approximation by nomogram is good until that time. In the two seater case the peak occurs before .6 seconds but the approximation at the peak is good. The nomogram, then, will give a good approximation for the peak value of $\dot{\theta}$ if it occurs before .6 second, and a good approximation to the value of $\dot{\theta}$ at .6 second if the peak occurs later than .6 second. Complete nomograms for use in determing θ and $\dot{\theta}$ as functions of time have been calculated and are included for general use. (figures 7 and 8). These should prove useful in giving a rapid indication as to the effect of various combinations of turrets.

While only the $\dot{\theta}$ nomogram was employed, the θ nomogram being the last part of the research, the same remarks as those made concerning the $\dot{\theta}$ nomogram are applicable to the diagram for θ.

Examination of the equations of lateral motion of an airplane under small disturbances shows that they are quite similar to those for the longitudinal motion. By a similar but not identical procedure, equivalent nomograms for use in connection with lateral motion could be developed. The effect of gunfire on the lateral motion of an airplane is of particular interest since broadside firing for fighter planes as well as attack and bomber craft is being discussed in contemporary British aeronautical journals.

REFERENCES

1. Aerodynamics of the Airplane - Dynamic Stability
   Clark B. Millikan, California Institute.

2. Operator Solutions in Airplane Dynamics
   Klemin and Ruffner, Journal of the Aeronautical Sciences, Vol.3, No.7, page 252.

3. Intermediate Calibre Aircraft Cannon
   Louis Bruchiss, Aviation, April 1941.

4. A Simplified Application of the Method of Operators to the Calculation of Disturbed Motions of an Airplane
   Robert T. Jones, N.A.C.A. Technical Report 560

5. An Analysis of Longitudinal Stability in Power Off Flight with Charts for Use in Design
   C. H. Zimmerman, N.A.C.A. Technical Report 521

6. The Construction of Nomographic Charts
   F.T.Mavis, International Textbook Co., 1939

7. Application of a Nomographic Method for a Graphic Determination of the Flight Path of an Airplane in the Case of Variable Aerodynamic Coefficients.
   W. Richter, Ingenieur-Archiv, Band X, Seite 292, 1939

8. The Nomogram, Allcock and Jones, Pitman & Sons, 1932

9. Application of Nomographic Methods for the Graphical Determination of the Flight Path in Longitudinal Motion of an Airplane.
   W. Richter, Ingenieur-Archiv, Band X, Seite 28, 1939

10. Applied Aerodynamics, L. Bairstow, Longmans, Green & Co. 1939

$$\phi_1 = \ddot{\theta}_0 t + \frac{1}{2}\dddot{\theta}_0 t^2$$

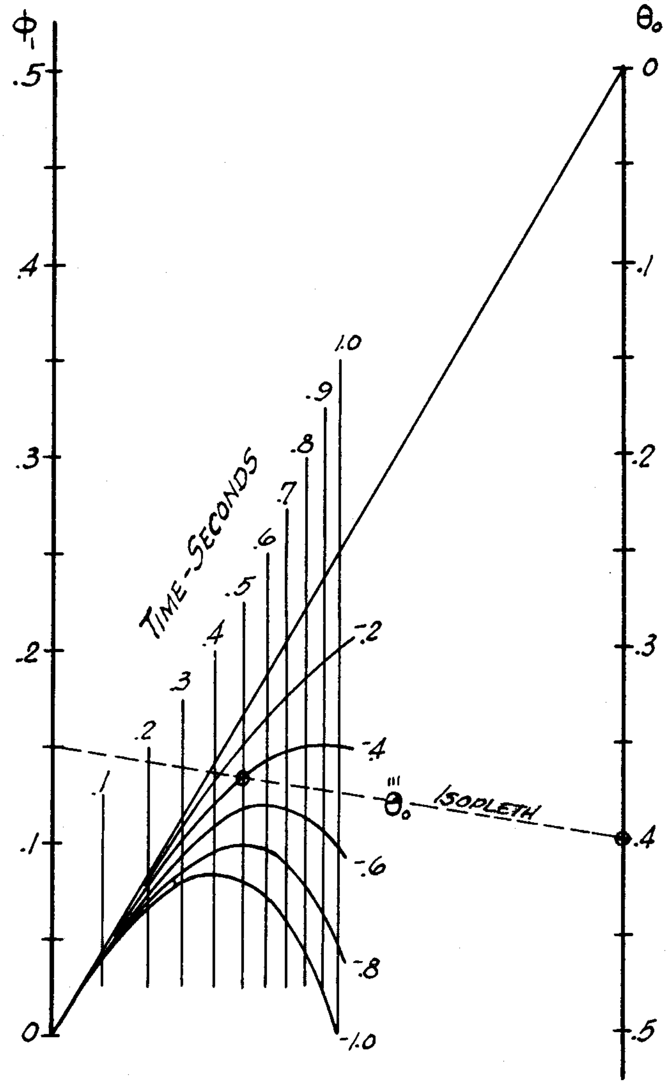
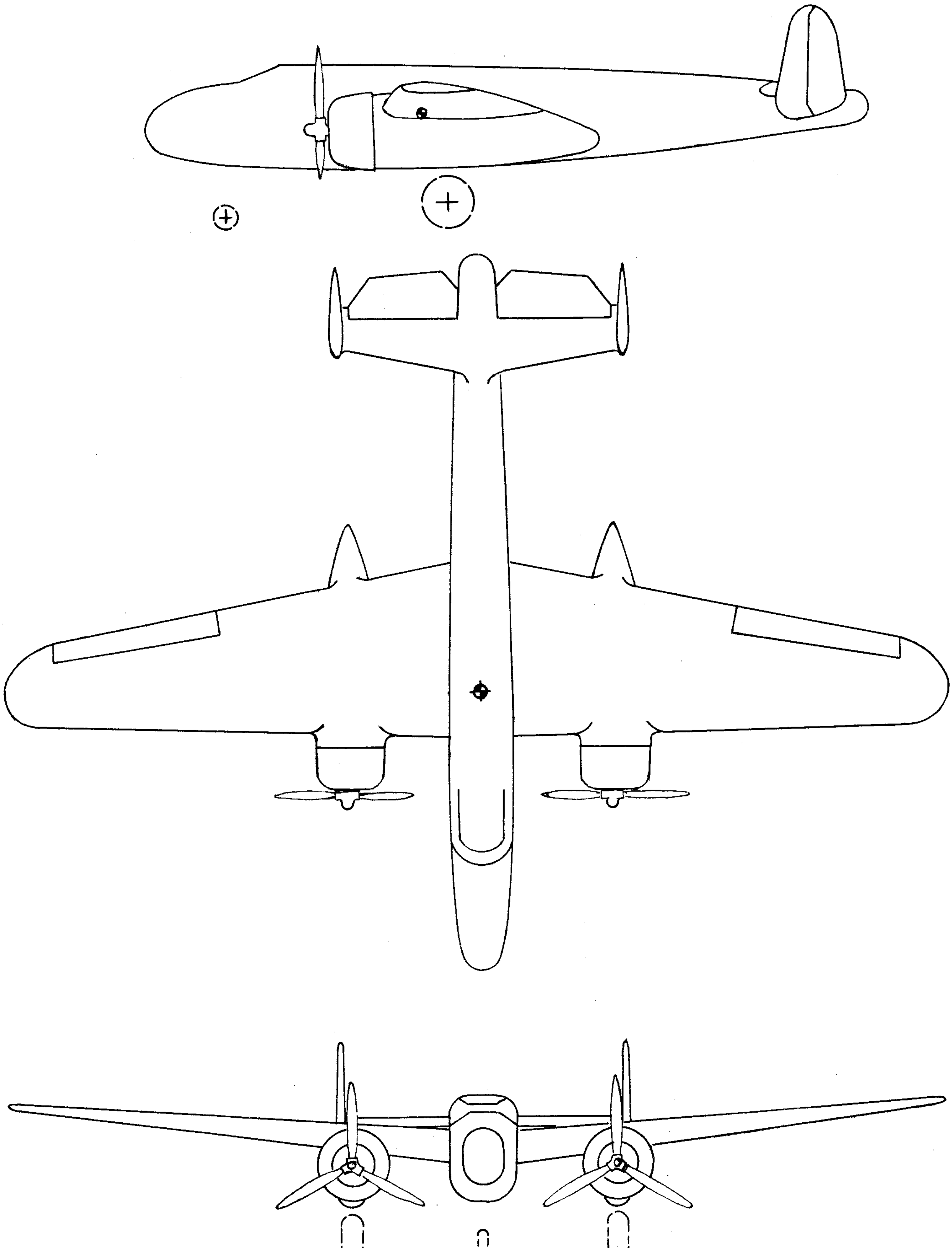


Figure 1 - Nomogram for the determination of $\phi_1 = \phi_1(t)$

Figure 2 - Three-view of Attack Bomber Investigated for Effect of Gunfire on Longitudinal Motion



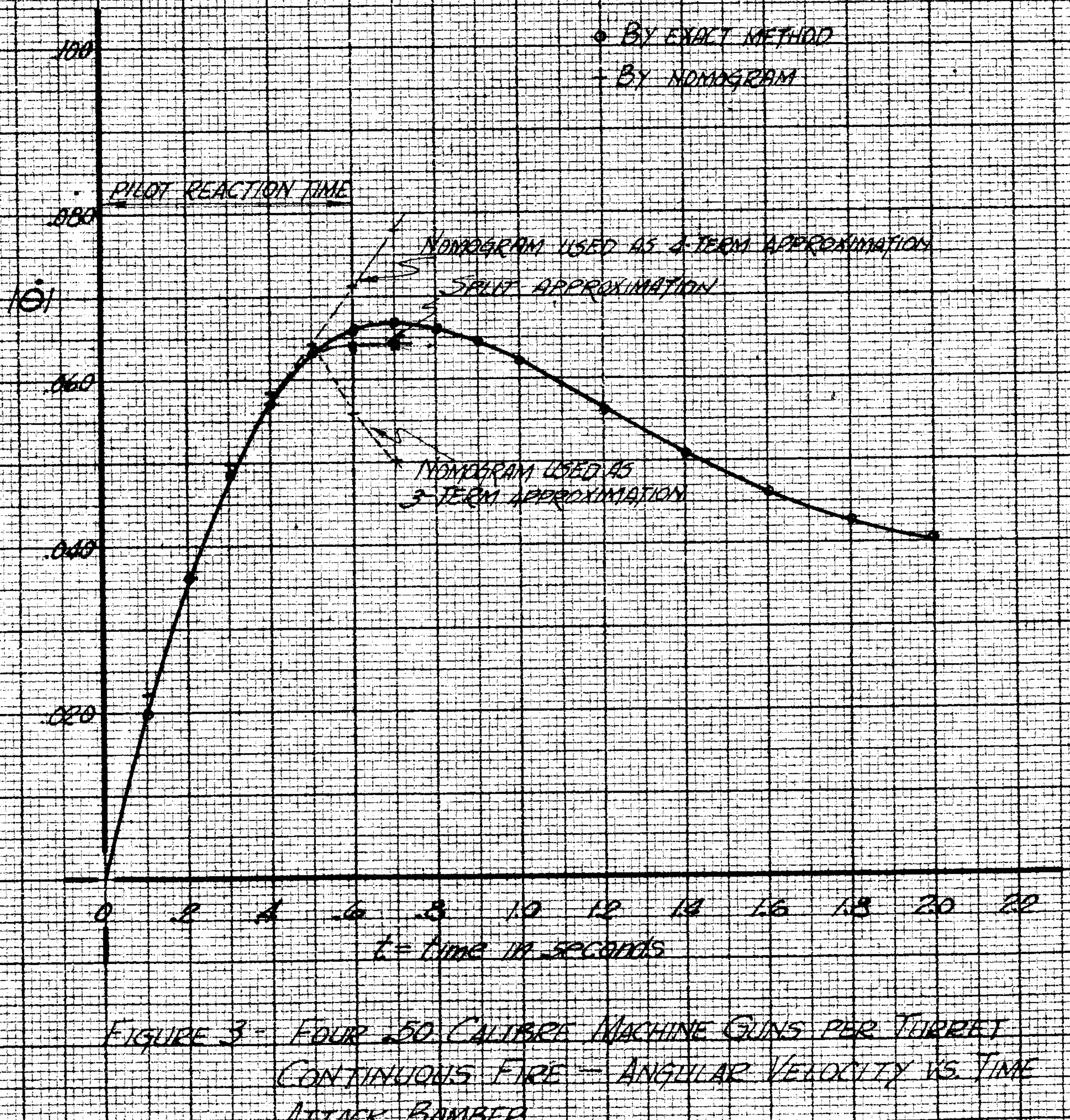


FIGURE 3 - FOUR .50 CALIBRE MACHINE GUNS PER TURRET
CONTINUOUS FIRE — ANGULAR VELOCITY VS. TIME
ATTACK BOMBER.

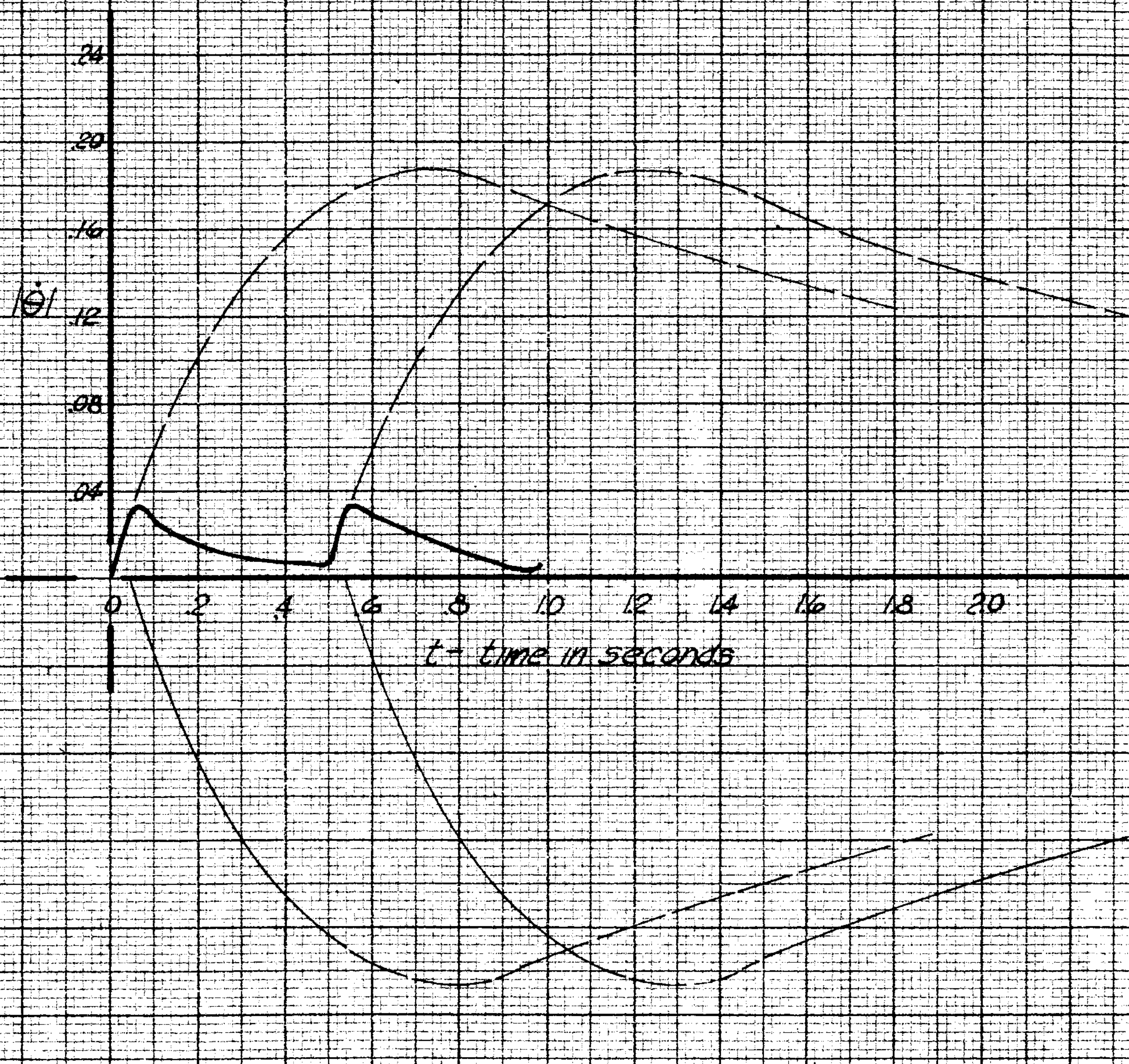


FIGURE 4 - ONE 37MM CANNON IN EACH TURRET - CONTINUOUS FIRE @ 120 ROUNDS/MINUTE = ANGULAR VELOCITY VS TIME - ATTACK BOMBER

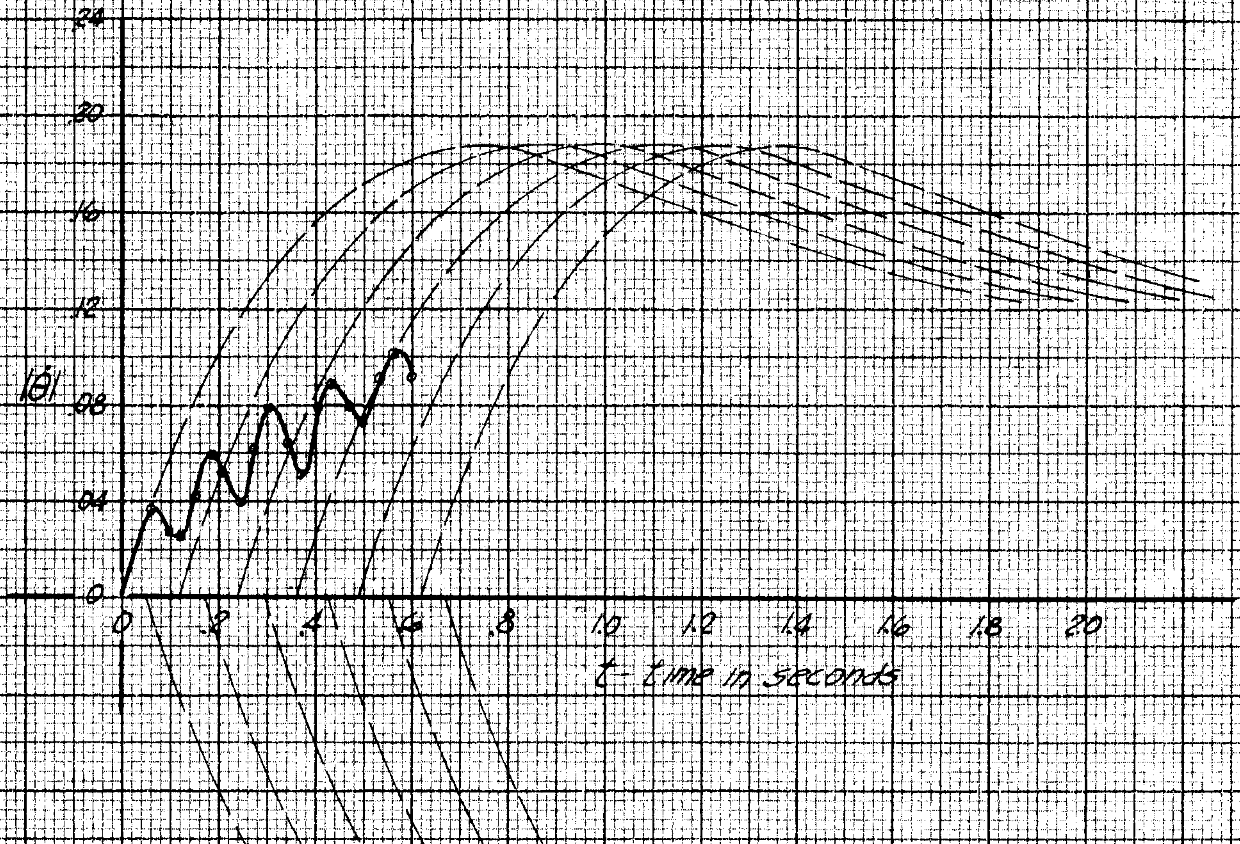


FIGURE 5 - BATTERIES OF TWO 20MM. CANNON FIRING SIMULTANEOUSLY - ANGULAR VELOCITY VS. TIME ATTACK BOMBER - FIRING RATE = 500 RNDS/MIN.

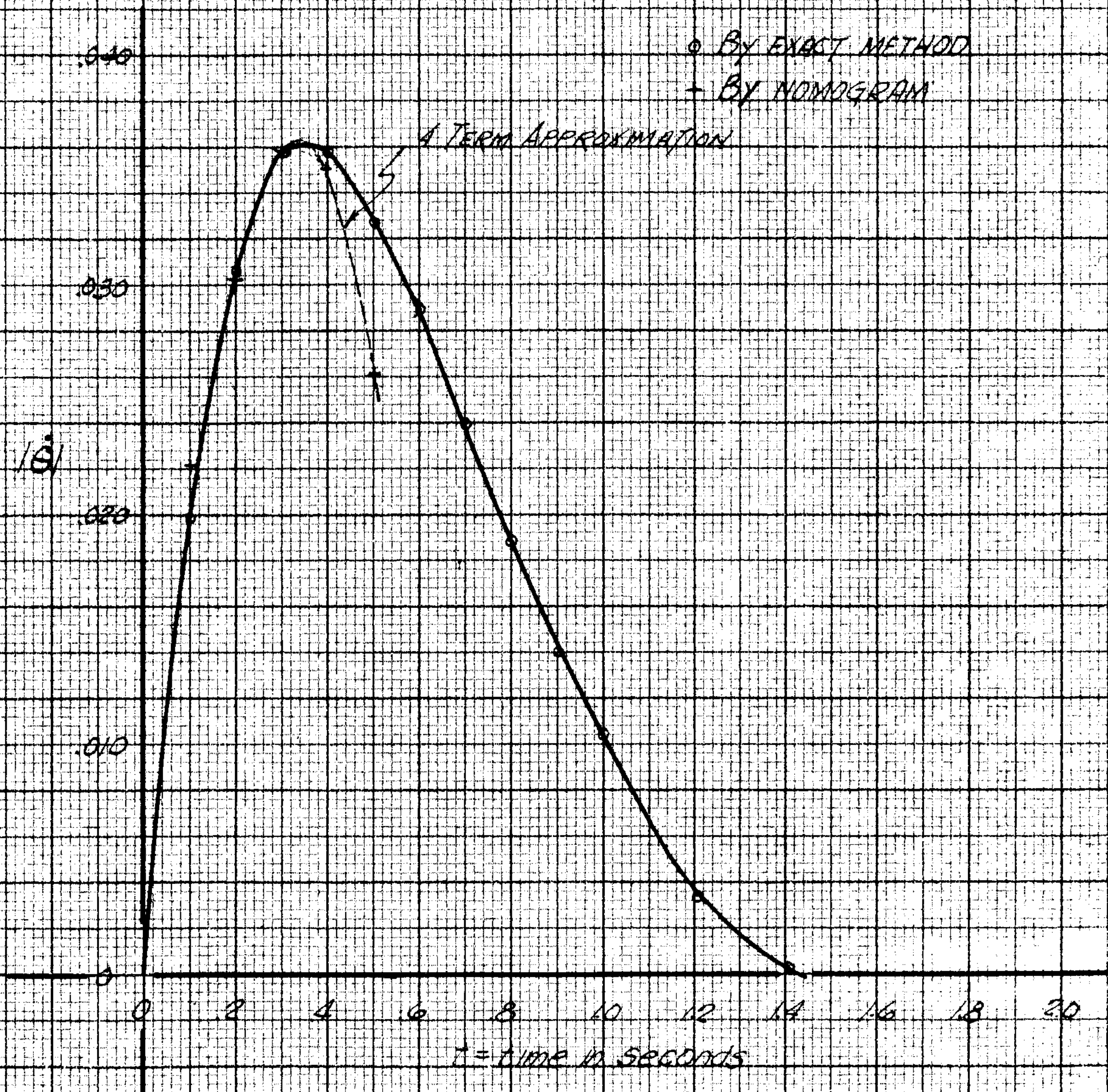


FIGURE 6 - CONSTANT FORCE OF 200# @ 7 FT. AFT OF C.G. - TWO SEATER AIRPLANE

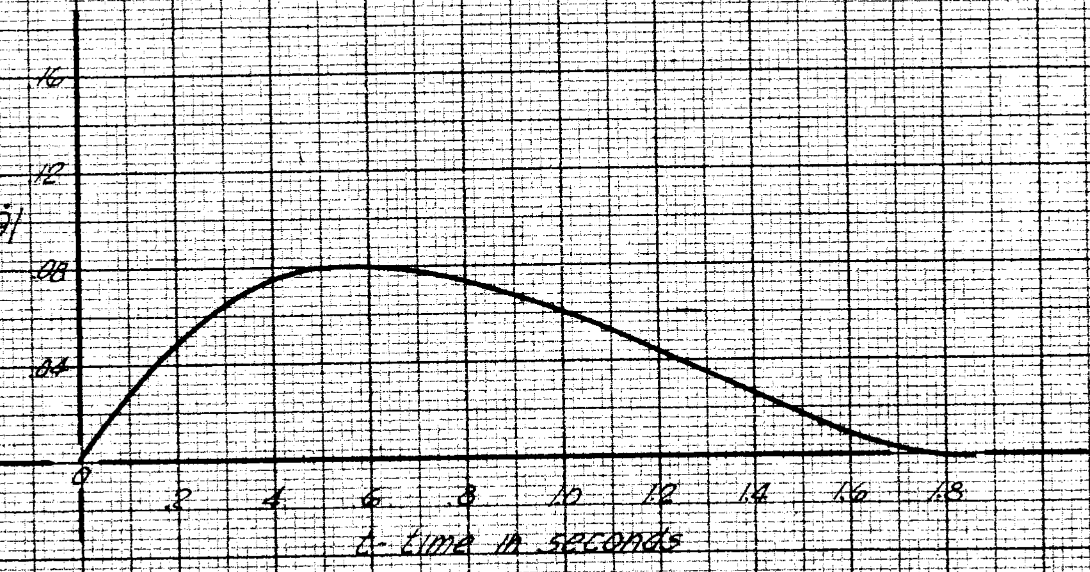


FIGURE 7. BATTERIES OF TWO 20 MM. CANNON PER TURRET FIRING ALTERNATELY – ANGULAR VELOCITY VS. TIME ATTACK BOMBER – FIRING RATE = 500 RNDS./MIN./GUN

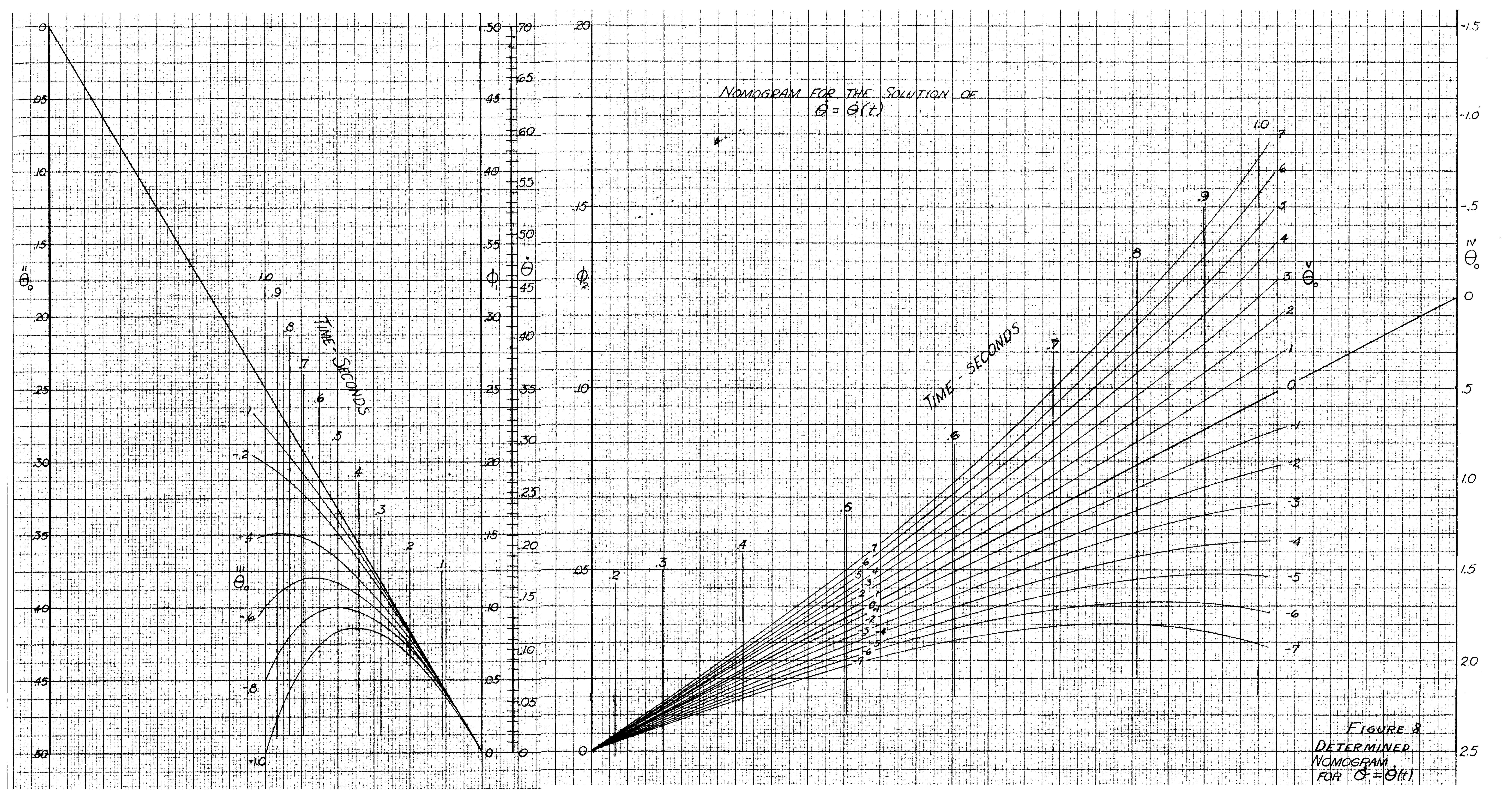


FIGURE 8
DETERMINED NOMOGRAM FOR $\dot{\theta} = \dot{\theta}(t)$

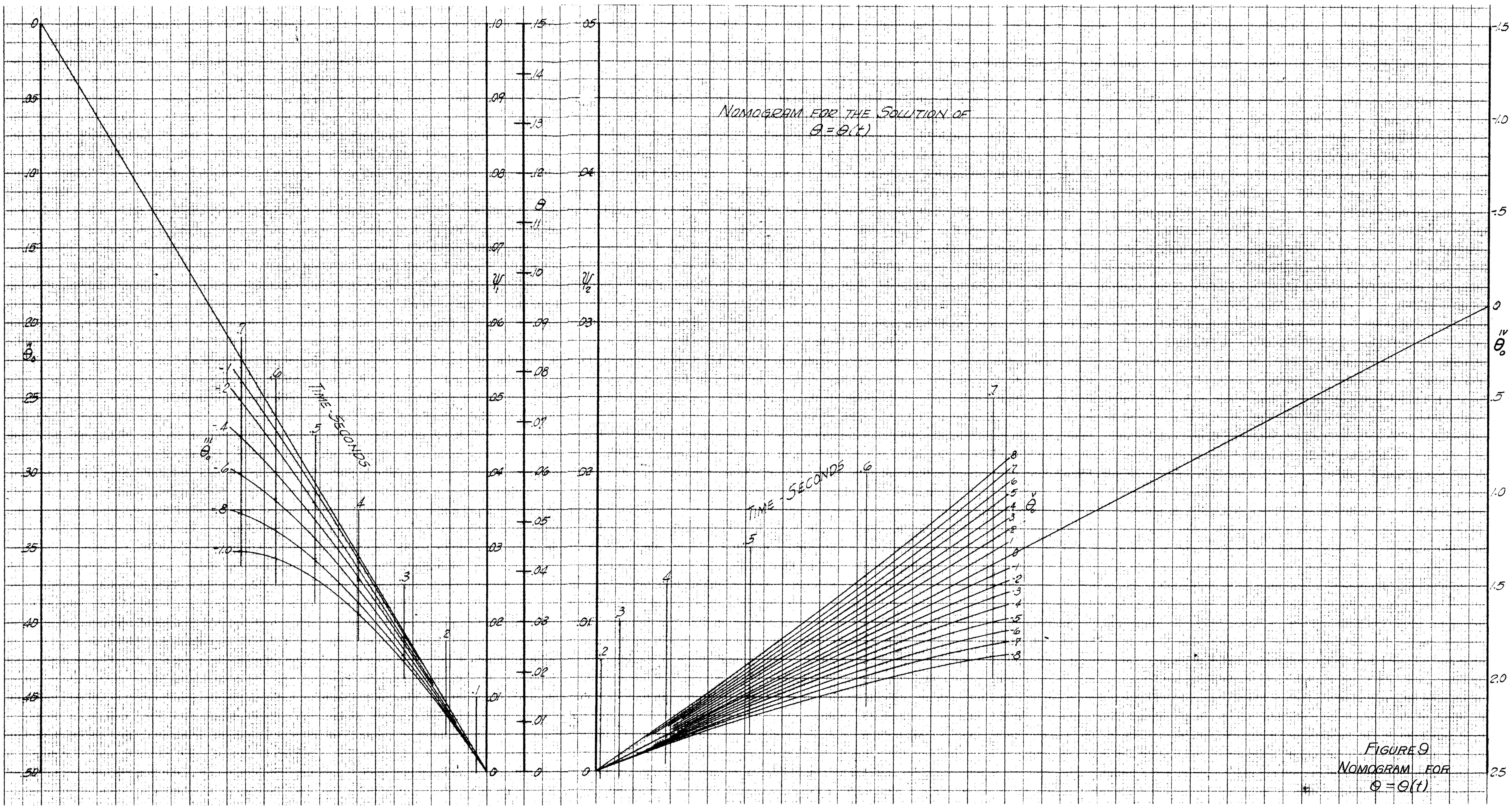


FIGURE 9
NOMOGRAM FOR
θ = θ(t)

Appendix

CALCULATION OF A PARTICULAR CASE BY MEANS OF THE EXACT METHOD

Using the airplane aerodynamic derivatives as given in table II for the attack bomber, and the gunfire forces and moments as given by table III for case IV, there is obtained by direct sunstitution into equations (4) and (5) :

$$f(D) = X_g\left[Z_u M_w + M_u(D-Z_w)\right] + Z_g\left[M_u X_w + M_w(D-X_u)\right] + M_g\left[(D-Z_w)(D-X_u) - Z_u X_w\right]$$
$$= \Big\{1.162\left[-.1720(-.0105x77.3)\right] - 2.010\left[-.0105x77.3(D+.0162)\right]$$
$$-47.60\left[(D+1.21)(D+.0162)+.1720x.0460\right]\Big\}$$

$$f(D) = -(47.60D^2 + 58.6396D + 1.1156)$$

$$F(D) = \Big\{(D-X_u)\left[(D-Z_w)(K_B{}^2D^2 - M_qD) - M_w(Z_qD + UD - g\sin\theta_o)\right]$$
$$Z_u\left[M_w(-X_qD + g\cos\theta_o) - X_w(K_B{}^2D^2 - M_qD)\right]\Big\}$$
$$= \Big\{(D-.0162)\left[(D+1.21)(77.3D^2+1.895x77.3D)+.0105x77.3(375.5D)\right]$$
$$-.172\left[-.0105x77.3(32.2)-.046(77.3D^2+1.895x77.3D)\right]\Big\}$$

$$F(D) = 77.3(D^4 + 3.121D^3 + 6.293D^2 + .1159D + .0581)$$

θ_o has been assumed equal to zero

Obtaining the roots of the quartic by means of Zimmerman's method:

$F(\lambda) = \lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = (\lambda^2 + a_1\lambda - b_1)(\lambda^2 + a_2\lambda - b_2)$ where

$B = a_1 + a_2$, $C = a_1a_2 + b_1 + b_2$, $D = a_1b_2 + a_2b_1$, $E = b_1b_2$. The values of the quartic coefficients obtained from inspection from the above equation for F(D) are:

$B = 3.121$, $C = 6.290$, $D = 0.1159$, $E = 0.0581$.

The roots of the quadratic factors in the equation for F(D) are the roots of the quartic. Numerical values for a_1 and b_1

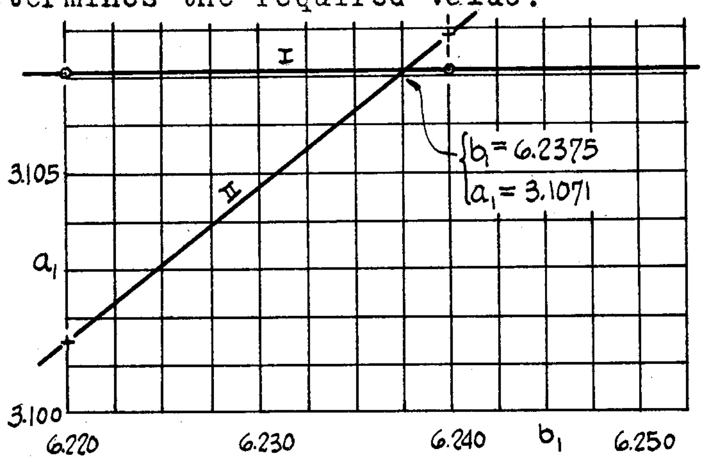
are obtained by using the relationships:

$$_{I}a_1 = \frac{b_1^2 B - b_1 D}{b_1^2 - E} \quad , \quad _{II}a_1 = \frac{B}{2} \pm \sqrt{\frac{B^2}{4} - C + b_1 + \frac{E}{b_1}}$$

Values of b_1 are assumed and substituted into the equations for a_1. When the value of b_1 such that the values of a_1 obtained from both equations are identical is reached, those values correspond to the true values for the quadratic factors in the equation for F(D).

b_1	$_{I}a_1$	$_{II}a_1$
6.2800	3.1071	3.1207
6.2400	3.1071	3.1079
6.2200	3.1070	3.1015

Plotting the values of a_1 against b_1 obtained from the two equations, the point of intersection of the two curves thus obtained determines the required value:



From $E = b_1 b_2$

$.0581 = 6.2375 b_2$

$b_2 = .0093$

$B = a_1 + a_2$

$3.121 = 3.1071 + a_2$

$a_2 = .0139$

check

$D = a_1 b_2 + a_2 b_1$

$= .0289 + .0867$

$.1159 \doteq .1156$

The check is sufficiently close, being accurate to .3%.

Substituting the values of a_1, a_2, b_1, and b_2 into the equation for F(D) there is obtained:

$$F(D) = 77.3(D^2 + 3.1070D + 6.2375)(D^2 + .0193D + .0093)$$

the roots of which are:

$$\alpha_{1,2} = a_1 \pm ib_1 = -1.5535 \pm i\ 1.9555 \qquad (A)$$

$$\alpha_{3,4} = a_2 \pm ib_2 = -.00695 \pm i\ .0962 \qquad (B)$$

From the roots defined by equation (A) and by means of the relationships of equation (7), the following result:

$$c = f(a) - \frac{b^2 f''(a)}{2} + \frac{b^4 f^{IV}(a)}{4!} - \ldots.$$

$$= -\left(47.60(-1.5535)^2 + 58.6396(-1.5535) + 1.1156\right) - \frac{(1.9555)^2}{2}\left(-2x47.60\right)$$

$$c = 157.12$$

$$d = bf'(a) - \frac{b^3 f'''(a)}{3!} + \ldots$$

$$= -1.9555\left(2x47.60(-1.5535) + 58.6396\right)$$

$$d = 174.51$$

$$e = F'(a) - \frac{b^2 F'''(a)}{2!} + \frac{b^4}{4!} F^{V}(a) - \ldots$$

$$= 77.3\left(4(-1.5535)^3 + 3x3.121(-15535)^2 + 2x6.293(-1.5535) + .1159\right) - \frac{(1.9555)^2}{2} x77.3\left(4x3x2(-1.5535) - 3x2x1x3.121\right)$$

$$e = 1826.66$$

$$f = bF''(a) - \frac{b^3}{3!} F^{IV}(a) + \frac{b^5}{5!} F^{VI}(a) - \ldots$$

$$77.3\left(1.9555\left[12(-1.5535) + 6x3.121(-1.5535) + 2x6293\right] - \frac{(1.9555)^3}{6}(24)\right)$$

$$f = -429.76$$

$a(ce + df) + b(de - cf) = 426050$

$b(ce + df) - a(de - cf) = 1014690$

$(a^2 + b^2)(e^2 + f^2) = 21948000$

From equation (7), then:

$$2e^{at}\left(\frac{a(ce+df)+b(de-cf)}{(a^2+b^2)(e^2+f^2)} \cos bt + \frac{b(ce+df) - a(de-cf)}{(a^2+b^2)(e^2+f^2)} \sin bt \right)$$

$$= 2e^{-1.5535t}\left(\frac{426050}{21948000} \cos 1.9555t + \frac{1014690}{21948000} \sin 1.9555t \right)$$

$$= .1006\ e^{-1.5535t} \sin(112.02t^{o} + 22.72^{o})$$

By operating on part (B) with the same relations as were used with part (A), the following result:

$a = -.00695$	$d = -5.5770$	$a(ce+df)+b(de-cf) = 8.3331$
$b = .0962$	$e = -4.4015$	$b(ce+df)-a(de-cf) = -49.0882$
$c = -.2702$	$f = 92.3503$	$(a^2+b^2)(e^2+f^2) = 79.4959$

$$2e^{-.00695t}\left(\frac{8.3331}{79.4959} \cos .0962t - \frac{49.0882}{79.4959} \sin .0962t \right)$$

$$1.2527\ e^{-.00695t}(\cos (5.5123t^{o} + 80.37^{o})$$

Substituting the value of zero into equations (4) and (5):

$$\frac{f(0)}{F(0)}\ \frac{\&1.1156}{4.4911}\ -.2484$$

The final equation for θ $\theta(t)$ is now:

$$\theta = -.2484 + .1006\ e^{-1.5535t}\sin(112.02t^{o} + 22.72^{o})$$
$$+ 1.2527\ e^{-.00695t}\cos(5.51t^{o} + 80.37^{o})$$

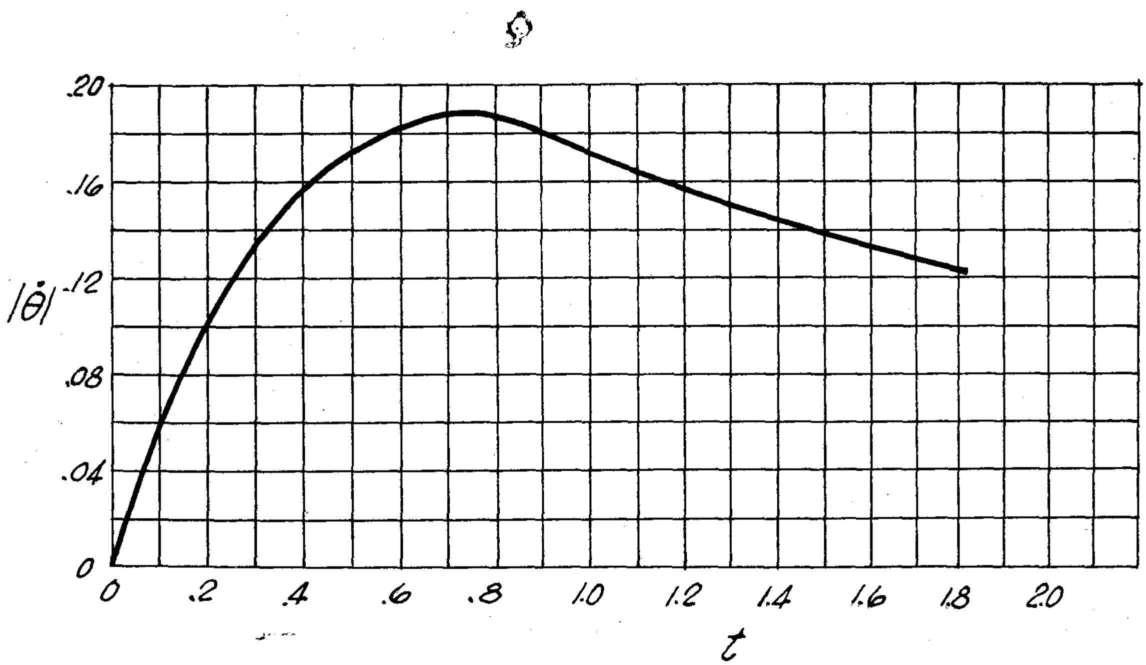
By differentiating with respect to t and combining terms having common factors:

$$\theta' = .2512\ e^{-1.5535t}\cos(112.02t^{o} + 61.20^{o})$$
$$-.1208\ e^{-.00695t}\sin(5.51t^{o} + 84.50^{o})$$

Tabulated values for θ and θ' are given in the following tables; the curves corresponding to these values are shown also.

$\theta' = .2512\ e^{-1.5535t}\cos(112.02t^{o} + 61.20^{o})$
$\quad - .1208\ e^{-.00695t}\sin(5.51t^{o} + 84.50^{o})$ which can be written as $\theta' = .2512\ e^{-at}\cos\alpha - .1208\ e^{-bt}\sin\beta$

t	$\cos\alpha^{o}$	.2512 Cos	$e^{-1.5535t}$	A	β^{o}	$\sin\beta$	$-.1208\sin\beta$	$e^{-.00695t}$	B	$\dot{\theta}$
0	.481	.121	1.000	.121	84.50	.995	-.120	1.00	-.1202	.0006
.1	.302	.076	.855	.065	85.05	.996	-.120	1.00	-.1204	-.0445
.2	.111	.028	.733	.021	85.60	.997	-.120	.999	-.1203	-.0998
.3	-.084	-.021	.628	-.0133	86.15	.998	-.121	.998	-.1203	-.1336
.4	-.276	-.069	.537	-.0372	86.70	.998	-.121	.998	-.1203	-.1575
.5	-.456	-.115	.461	-.0528	87.26	.999	"	.997	-.1203	-.1731
.6	-.621	-.156	.393	-.0613	87.81	.999	"	.996	-.1202	-.1815
.7	-.761	-.191	.337	-.0644	88.36	1.0	"	.995	-.1202	1.1846
.8	-.873	-.219	.289	-.0634	88.91	1.0	"	.994	-.1201	-.1835
.9	-.951	-.239	.247	-.0590	89.46	1.0	"	.993	-.1200	-.1790
1.0	-.994	-.250	.212	-.0529	90.01	1.0	"	.992	-.1198	-.1727
1.2	-.963	-.242	.155	-.0375	91.11	1.0	"	.991	-.1197	-.1572
1.4	-.787	-.198	.113	-.0223	92121	.999	"	.990	-.1195	-.1418
1.6	-.499	-.125	.083	-.0105	93.32	.998	"	.989	-.1193	-.1298
1.8	-.125	-.o31	.065	-.0020	94.42	.997	"	.988	-.1189	-.1209
2.0	.262	.066	.045	.0029	95.52	.995	-.120	.986	-.1185	-.1156



CALCULATION OF THE NOMOGRAM FOR THE $\theta = \theta(t)$ EQUATION

The determinant given by (13) is in the form from which the nomogram for the ϕ_1 function may be constructed directly.

$$\begin{vmatrix} 0 & \mu_1 \phi_1 & 1 \\ \dfrac{\delta_3 \cdot \mu_1 \frac{t}{1+t}}{(\mu_1-\mu_3)\frac{t}{1+t}+\mu_3} & \dfrac{\mu_1 \cdot \mu_3 \frac{\frac{1}{2}\dddot{\theta}_0 t^2}{1+t}}{(\mu_1-\mu_3)\frac{t}{1+t}+\mu_3} & 1 \\ \delta_3 & -\mu_3 \ddot{\theta}_0 & 1 \end{vmatrix} = 0 \qquad (13)$$

Assuming the limitations on the ϕ_1 nomgram as:

a) $0 < \phi_1 < .5$

b) $0 < \theta_0'' < .5$

c) $0 < t < 1.0$

d) $0 > \theta_0'''> -1.0$

e) dimensions 10" high x 6" wide

From (a) and (e) $\quad 10 = \mu_1(.5-0)$, or $\mu_1 = 20$

From (b) and (e) $\quad 10 = \mu_3(.5-0)$, or $\mu_3 = 20$

From (e) $\quad \delta_3 = 6$

Substituting the numerical values for the scale factors, the determinant becomes:

$$\begin{vmatrix} 0 & 20\phi_1 & 1 \\ \dfrac{6t}{t+1} & \dfrac{10\dddot{\theta}_0 t^2}{1+t} & 1 \\ 6 & -20\ddot{\theta}_0 & 1 \end{vmatrix} = 0 \qquad (13a)$$

The determinant given by (13a) dictates the construction of the ϕ_1 function nomogram, as described on page 145 of reference 8. The method is as follows (see figure):

A ϕ_1 scale is drawn vertically, extending between the values of zero and 0.50 in a length of 10 inches. This information is given by the first row of the determinant. Six inches to the left of the ϕ_1 scale another scale is drawn parallel to the former, its scale extensions being between zero and 0.50. The latter is the scaleefor θ_0''. The direction of the scale is the reverse of that adopted for ϕ_1. This information is obtained from the third row of the determinant. The first and third rows of the determinant really give the equations of parallel lines separated by six inches. The equation for the grid is given by the second row of the determinant. At various values of the time t, the values of the x coefficient and the y coefficient may be obtained from the quantities in the first and second columns respectively. For the ϕ_1 function these coefficients are:

$$x = 6t/t-1 \qquad y = 10\,\theta_0'''t^2/t-1$$

t	x	y/θ_0'''	$\theta_0'''=.2$	$\theta_0'''=.4$	$\theta_0'''=.6$	$\theta_0'''=.8$	$\theta_0'''=1.0$
0	0	0	0	0	0	0	0
.1	.546	.0908	.018	.036	.054	.072	.090
.2	1.000	.3336	.066	.134	.200	.266	.334
.3	1.286	.692	.138	.276	.416	.554	.692
.4	1.712	1.144	.228	.458	.686	.916	1.144
.5	2.000	1.664	.332	.666	.998	1.332	1.664
.6	2.250	2.248	.450	.900	1.348	1.798	2.248
.7	2.470	2.880	.576	1.152	1.728	2.304	2.880
.8	2.664	3.556	.712	1.422	2.134	2.844	3.556
.9	2.840	4.260	.952	1.704	2.556	3.408	4.260
1.0	3.000	5.000	1.000	2.000	3.000	4.000	5.000

By an algebraic manipulation identical to that given on page 15 the determinant for the ϕ_2 function can be put into the form:

$$\begin{vmatrix} 0 & \mu_1 \phi_2 & 1 \\ \dfrac{\delta_3 \mu_1 \dfrac{t^3}{t^3+6}}{(\mu_1-\mu_3)\dfrac{t^3}{t^3+6}+\mu_3} & \dfrac{\mu_1 \mu_3 \dfrac{\overset{V}{\theta}_0 t^4}{4(t^3+6)}}{(\mu_1-\mu_3)\dfrac{t^3}{t^3+6}+\mu_3} & 1 \\ \delta_3 & -\mu_3 \overset{IV}{\theta}_0 & 1 \end{vmatrix} = 0$$

The method of construction is identical to that given in page (vii).

NOMOGRAM LIMITATIONS

a) $0 < \phi_2 < .20$ $\qquad 10 = \mu_1 (.2-0)$ or $\mu_1 = 50$

b) $-1.5 < \overset{IV}{\theta}_0 < 2.5$ $\qquad 10 = \mu_3 (2.5+1.5)$ or $\mu_3 = 25$

c) $0 < t < .8$ $\qquad \delta_3 = 12$

d) $-8 < \overset{V}{\theta}_0 < +8$

e) dimensions 10" high x 12" wide

$$\begin{vmatrix} 0 & 50 \phi_2 & 1 \\ \dfrac{600 \dfrac{t^3}{t^3+6}}{47.5 \dfrac{t^3}{t^3+6}+25} & \dfrac{\dfrac{125}{4} \dfrac{\overset{V}{\theta}_0 t^4}{t^3+6}}{47.5 \dfrac{t^3}{t^3+6}+2.5} & 1 \\ 12 & -2.5 \overset{IV}{\theta}_0 & 1 \end{vmatrix} = 0$$

For the grid of $\overset{V}{\theta}_0$ & t :

$$x = \frac{240 \dfrac{t^3}{t^3+6}}{19 \dfrac{t^3}{t^3+6}+1} \qquad y = \frac{12.5 \overset{V}{\theta}_0 \dfrac{t^4}{t^3+6}}{19 \dfrac{t^3}{t^3+6}+1}$$

Table for construction of grid of ψ_2 function nomogram

t	χ	$\psi/\overset{v}{\theta}_0$	$\overset{v}{\theta}_0=2$	$\overset{v}{\theta}_0=3$	$\overset{v}{\theta}_0=4$	$\overset{v}{\theta}_0=5$	$\overset{v}{\theta}_0=6$	$\overset{v}{\theta}_0=7$
.1	.040	.000208	0	.001	.001	.001	.001	.001
.2	.314	.003263	.007	.010	.013	.016	.020	.023
.3	.989	.01548	.031	.046	.062	.077	.093	.108
.4	2.109	.0577	.115	.173	.231	.289	.346	.404
.5	3.530	.0919	.184	.276	.368	.460	.551	.643
.6	5.020	.1573	.315	.472	.629	.787	.944	1.101
.7	6.405	.2338	.468	.701	.935	1.169	1.403	1.637
.8	7.565	3153	.631	.946	1.261	1.577	1.892	2.207
.9	8.500	.3982	.796	1.195	1.593	1.991	2389	2.787
1.0	9.240	.4810	.962	1.443	1.924	2405	2.886	3.367

In like manner a nomogram can be constructed for the equation of $\theta = \theta(t)$. The determinants for the ψ_1 and ψ_2 functions which were used are given below as are also the determinants with the scale factors (numerical values) inserted. It should be noted that while data for the construction is given up to a value of time equal to 1 second, the series approximation used is good only till approximately ½ to .6 seconds.

$$\theta = \theta(t) = \psi_1(t) + \psi_2(t)$$

$$\begin{vmatrix} 0 & \mu_1\psi_1 & 1 \\ \dfrac{\delta_3\mu_1 \dfrac{t^2}{2+t^2}}{(\mu_1-\mu_3)\dfrac{t^2}{t^2+2}+\mu_3} & \dfrac{\mu_1\mu_3 \dfrac{\overset{'''}{\theta}_0 t^3}{3(t^2+2)}}{(\mu_1-\mu_3)\dfrac{t^2}{t^2+2}+\mu_3} & 1 \\ \delta_3 & -\mu_3\overset{''}{\theta}_0 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & \mu_1\psi_2 & 1 \\ \dfrac{\delta_3\mu_1 \dfrac{t^4}{t^4+24}}{(\mu_1-\mu_3)\dfrac{t^4}{t^4+24}+\mu_3} & \dfrac{\mu_1\mu_3 \dfrac{\overset{v}{\theta}_0 t^5}{5(t^4+24)}}{(\mu_1-\mu_3)\dfrac{t^4}{t^4+24}+\mu_3} & 1 \\ \delta_3 & -\mu_3\overset{iv}{\theta}_0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & 100\,\psi_1 & 1 \\ \dfrac{30\dfrac{t^2}{t^2+2}}{4\dfrac{t^2}{t^2+2}+1} & \dfrac{3333\,\overset{'''}{\theta}_0 \dfrac{t^3}{t^2+1}}{4\dfrac{t^2}{t^2+2}+1} & 1 \\ 6 & -20\,\overset{''}{\theta}_0 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & 200\,\psi_2 & 1 \\ \dfrac{960\dfrac{t^4}{t^4+24}}{79\dfrac{t^4}{t^4+24}+1} & \dfrac{40\dfrac{\overset{v}{\theta}_0 t^5}{t^4+24}}{79\dfrac{t^4}{t^4+24}+1} & 1 \\ 12 & -25\,\overset{iv}{\theta}_0 & 1 \end{vmatrix} = 0$$

FORM FOR COMPUTING COEFFICIENTS FOR USE WITH NOMOGRAMS

$$\theta' = \theta_o'' t + \theta_o''' \frac{t^2}{2!} + \theta_o \frac{t^3}{6} + \theta_o \frac{t^4}{24}$$

$M_g \div K_B^2 = (\quad\quad) = \ddot{\theta}_o$

$M_w \cdot Z_g = (\quad\quad) =$
$M_q \cdot \ddot{\theta}_o = (\quad\quad)$
$(\quad\quad) \div K_B^2 = \theta_o'''$

$U \cdot \theta_o'' = (\quad\quad)$
$Z_u \cdot X_g = (\quad\quad)$
$Z_w \cdot Z_g = (\quad\quad)$
$(\quad\quad) = \ddot{w}_o$

$M_w \cdot w_o'' = (\quad\quad)$
$M_q \cdot \dddot{\theta}_o = (\quad\quad)$
$(\quad\quad) \div K_B^2 = \overset{iv}{\theta}_o$

$X_u \cdot X_g = (\quad\quad)$
$X_w \cdot Z_g = (\quad\quad)$
$(\quad\quad) = u_o''$

$U \cdot \theta_o''' = (\quad\quad)$
$Z_u \cdot \ddot{u}_o = (\quad\quad)$
$Z_w \cdot \ddot{w}_o = (\quad\quad)$
$(\quad\quad) = w_o'''$

$M_w \cdot w_o''' = (\quad\quad)$
$M_q \cdot \overset{v}{\theta}_o = (\quad\quad)$
$(\quad\quad) \div K_B^2 = \overset{v}{\theta}$

Note: The nomogram as ordinarily employed is a four term approximation to the true curve. By neglecting the grid formed by $\overset{v}{\theta}_o$, a three term approximation can be obtained. In the first few tenths of a second it is more rapid and almost as accurate to use a three term approximation.