A MATHEMATICAL FRAMEWORK FOR DISCUSSING THE STATISTICAL DISTRIBUTION OF GALAXIES IN SPACE AND ITS COSMOLOGICAL IMPLICATIONS

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DEDICATION

To Psyche,

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For the Stars, which I have always loved, but never really knew before.

For a very special, very precious Understanding - - a much deeper one, I think, than you realize.

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ABSTRACT

The spectral-theoretic techniques of stationary time-series analysis are generalized and applied to the study of the statistical distribution of galaxies in space and the observed distribution on the sky. Sampling techniques and criteria are developed for the measurement of the Fourier transform of the autocovariance function, the so-called "power spectrum". The theory is extended to curved, nonstatic space-times and the possibility of using the spectral density obtained from counts of galaxies in the formulation of cosmological tests is discussed. A similar development is made for the statistical structure of the background light due to very faint galaxies, and the possibility of measurement of this structure and <u>its</u> application to cosmological tests is considered. It is shown that in both cases (counts and background) significant cosmological data can be obtained if our knowledge of the luminosity function, the spectra, and the evolution of galaxies is improved. Finally, application is made of the theory to the analysis of a small count problem in order to learn something about the general form of the spatial covariance.

A MATHEMATICAL FRAMEWORK FOR DISCUSSING THE STATISTICAL DISTRIBUTION OF GALAXIES IN SPACE AND ITS COSMOLOGICAL IMPLICATIONS

I. Introduction

There has been hope for many years that counts of galaxies could yield significant information on the cosmological problem, and theoretical results relating the mean number of galaxies per brightness interval to the apparent brightness have been obtained for all the simple cosmological models of current interest (Sandage, (1), Heckmann and Schücking, (2)). Sandage (1) has recently shown that the resulting relationships are probably too insensitive to be of practical use as criteria for deciding between models, since observational errors (primarily photometric ones) and as yet inaccurately known evolutionary effects mask the small differences from one model to another. It is clear, however, that much more statistical information than the mean number can be extracted from counts; several workers, notably Neyman and Scott (3) and Limber (4,5) have done work with the <u>second</u> moments (the autocorrelation function) in attempts to obtain quantitative information on the clustering tendency of galaxies. The application of these treatments to the formulation of cosmological tests has not been made, and indeed this application is difficult and cumbersome with the forms of the theory that have been used.

In this paper we develop a general framework for discussing the distribution of galaxies in space (and on the sphere); proper account is taken of the discrete nature of the distribution, and *ad hoc* clustering models are avoided. We proceed with the discussion of sampling statistics, and confidence criteria are derived for the sample second moments. To this end, the spectral-theoretic techniques of stationary time-series analysis are generalized to the more complicated stochastic processes represented by the distribution of galaxies. We find, just as in the simple one-dimensional, continuous, monovariate cases (Blackman and Tukey (6), Grenander and Rosenblatt (7)— hereafter referred to as GR) that the efficient statistic to use is *not* the autocorrelation function but its Fourier transform, the spectral function; and the sampling means and variances of this function are easily definable *in terms of, and are simply related to, the function itself*. We can thus make meaningful statements about the probable accuracy of the spectral function of one sample, and about the significance of differences between samples, or, more important, between a sample function and a theoretical one.

The above results are then extended to curved, non-static space-times, and specific treatments are given for the Friedmann universes (Heckmann and Schücking, (2)) with vanishing pressure and cosmological constant, and for the steady-state theory (Hoyle, (8)).

With the information obtainable from counts, another test of a slightly different kind can be investigated. Miller (9) has noted that the photometry of faint objects is limited in accuracy by an inherent "graininess" of the night sky, and he suggests that this is due to the light of unresolved galaxies and stars in the background. Using data from counts and a given cosmological model, the statistical structure of this light can be predicted. The second part of this work treats this problem; the techniques are similar to those used for the counts.

Finally, the theory is applied to a small pilot problem, and the results of numerical computations for various theoretical models are presented.

II. THE MATHEMATICAL-STATISTICAL FRAMEWORK

Throughout the remainder of this work, we take the point of view that the distribution of galaxies is a particular realization of a certain stochastic process in space-time. From the beginning we invoke the "cosmological principle" in the form given by Neyman and Scott (10); namely that the distribution of galaxies is a <u>stationary</u>[1] <u>stochastic</u> process in space at any instant of cosmic time. This point is really unnecessary to the development and indeed is to some extent testable, but simplifies the development greatly. The mathematical formulation of the problem is hampered on four counts, and we shall dispose of them in turn. First and most obvious is the fact that galaxies are discrete - i.e., in a given region either a galaxy is, present (its center is present, say) or not; most of the classical theory of stationary processes deals with <u>continuous</u> distributions. The next two difficulties are easier in principle to dispose of: the distribution of galaxies is three-dimensional and multivariate, or at least is most easily and informatively considered multivariate, since the different types of galaxies have quite different physical properties and probably are distributed quite differently.

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1. That is, a process all of whose probability distributions are invariant under translation; the term "statistically homogeneous universe" is often used in our application.

Finally, the three-dimensional manifold in which the process is embedded need not be Euclidean (though it must be homogeneous in order to speak of "stationary processes").

Discrete processes of the type encountered here are discussed very briefly by Bartlett (11) and are called by him "point processes". One can make the idea precise by the following definition.

Definition 2.1 A point process is an additive stochastic set function taking positive integer values.

The simplest example (and indeed, almost all interesting point processes are of this type) is the number (the integer value) of occurrences of a given kind of event in a given region (the set).

We shall be interested in processes which admit of a fairly simple mathematical description, as given below:

Definition 2.2 An n-th order regular point process is a point process $X(s)$ (The argument s is a set note.) for which differentiable functions $f_1(x_1)$, $f_2(x_1,x_2) \ldots f_n(x_1,x_2 \ldots x_n)$ exist such that

$$(2.1) \qquad E\Big(X(s)\Big) = \int_s f_1(x)\,d\mu$$

(2.2)

a) $$E\left(\overline{X(s_1)}\,\overline{X(s_2)}\right) = \int_{s_1}\int_{s_2} f_2(x_1,x_2)\,d\mu_1\,d\mu_2$$

b) $\vdots$

c) $$E\left(\overline{X(s_1)}\,\overline{X(s_2)}\ldots\overline{X(s_n)}\right) = \int_{s_1}\int_{s_2}\cdots\int_{s_n} f_n(x_1\ldots x_n)\,d\mu_1,d\mu_2\ldots d\mu_n \quad ,$$

where $\overline{X}(s) = X(s)-E\left(X(s)\right)$, and the s_j are mutually disjoint; furthermore, the probability that $X(s) > 1$ is $o\left(\mu(s)\right)$, where $\mu(s)$ is the measure of s, as $\mu(s)$ tends to zero. The function $f_1(x)$ shall be called the <u>mean density</u>; $f_j(x_1\ldots x_j)$, $j>1$, the centered <u>n-th moment density</u>. For $j = 2$, we shall often refer to the <u>covariance density</u>[1].

It follows easily from the definition that the probability that $X(s) = 1$ exactly is $f_1(x)\mu(s) + o\left(\rho(s)\right)$, where $\rho(s)$ is the diameter of s, for $X\in s$; and if $s_1 \cap s_2$ is void, the probability that $X(s) = 1$ and $X(s') = 1$ is $\left\{f_1(x_1)f_1(x_2)+f_2(x_1,x_2)\right\}\mu(s_1)\mu(s_2)+o\left(\rho(s_1)\rho(s_2)\right)$, for $x_1\in s_1$, $x_2\in s_2$. It is often convenient to think of the moment densities in this probabilistic sense (See Appendix

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1. Note that even though the moments can be given in terms of densities, no stochastic "density" exists for $X(s)$ itself; i.e., $X(s)$ is <u>not</u> stochastically differentiable.

I for further details).

Relations (2.2 a,b...c) are easily generalized to non-disjoint sets; we give here only the generalization for n = 2.

(2.3)

$$\operatorname{cov}\Big(X(s_1),X(s_2)\Big) = \int_{s_1}\int_{s_2} f_2(x_1,x_2)d\mu_1 d\mu_2 + \int_{s_1\cap s_2} f_1(x)d\mu .$$

Note that it is impossible to give this general covariance a density except in a symbolic fashion; we could write

(2.4)

$$\operatorname{cov}\Big(X(s_1),X(s_2)\Big) = \int_{s_1}\int_{s_2}\Big\{f_2(x_1,x_2) + f_1(x_1)\,\delta\,(x_1,x_2)\Big\}d\mu_1 d\mu_2 ,$$

with the proper generalization of the Dirac delta - and indeed, we shall at times use this formalism for simplicity. But the meaning <u>must</u> be kept clearly in mind, for the singular term does not behave at all like the proper density portion under many of the operations we shall perform on the parent process.

Let us now turn to the description of the distribution of galaxies in space in terms of such stochastic processes. We shall for the moment consider only Euclidean 3-space and ignore time variation altogether; these omissions will be rectified when cosmological effects are discussed.

We now partition the set of all galaxies into distinguishable classes (say spirals, ellipticals, dwarf irregulars; or, if more precision is desired, E0, E1, etc.) This subdivision can be as fine or as coarse as one wishes; a sharp criterion for the selection is missing, but we shall see that the subdivision should be as fine as possible. Let the index β, $\beta = 1,2,..\ n$ label the classes, and let M be the absolute bolometric magnitude. Then any galaxy can be distinguished by the coordinate vector $\underline{x}$, the brightness M, and the major class β . Thus the number of β-galaxies $N_\beta(s,\sigma)$ in the spatial region s and in the magnitude interval σ: $M_1<M<M_2$ is a 4-dimensional point process, which we shall assume is at least second-order regular, and because of the cosmological principle is stationary in the spatial coordinates (though of course not in the brightness). The distribution of all galaxies is then a multivariate process:

$$(2.5)\qquad \underline{N}\ (s,\sigma) = \begin{pmatrix} N_1(s,\sigma) \\ \cdot \\ \cdot \\ \cdot \\ N_n(s,\sigma) \end{pmatrix} .$$

Since the N_j are stationary in space, the mean densities have no spatial variation; let $\lambda_\beta(M)$ be the mean density for N_β . Similarly, the stationarity requires that the covariance densities be functions only of the coordinate <u>differences</u>. We postulate further that the distribution is invariant under rotation, so that it is statistically <u>isotropic</u> as well as homogeneous. This clearly requires that the covariance densities be functions only of the coordinate distance. Thus

(2.6)

a) $$E\left(N_\beta(s,\sigma)\right) = \mu(s) \int_\sigma \lambda_\beta(M)\,dM$$

b) $$E\left(\overline{N}_\beta(s_1,\sigma_1)\overline{N}\ (s_2,\sigma_2)\right) = \int_{s_1}\int_{s_2}\int_{\sigma_1}\int_{\sigma_2}\left\{f_\beta\ (|\underline{x}_1-\underline{x}_2|,M_1,M_2)\ \cdot\right.$$
$$\left.\cdot\ d^3x_1 d^3x_2 dM_1 dM_2\right\} + \delta_{\beta\gamma}\,\mu(s_1\cap s_2)\int_{\sigma_1\cap\sigma_2}\lambda\ _\beta(M)\,dM\ ,$$

where $\overline{N}_\beta(s,\sigma) = N_\beta(s,\sigma)-E\left(N_\beta(s,\sigma)\right)$. (This is a convention we shall use throughout this paper; i.e., if x is a stochastic variable, $\overline{x}=x-E(x)$. We shall not consider complex stochastic processes, so there will be no confusion with complex conjugation.) Note that $f_{\beta\gamma}(r,M_1,M_2) = f_{\gamma\beta}(r,M_2,M_1)$.

We cannot, of course, directly observe this distribution - at least not until we can obtain redshifts and hence distance with much greater facility than we can at present. What we do observe is the projection of this distribution on the celestial sphere. This projection is also a point process, now in three dimensions - the two spherical coordinates and the <u>apparent</u> brightness - and because of the isotropy condition is clearly stationary on the sphere.

In addition, we see on a given photographic plate only the galaxies brighter than a certain limiting magnitude - which probably depends on β - and this limit is itself not fixed for a given plate, but statistically variable (as was perhaps first pointed out in the literature by Neyman and Scott, (10)). All of this is easily incorporated into the picture of the distribution as a regular point process.

Let us consider at this time only the problem of the analysis of counts; the analysis of the background light is in many respects similar - albeit a bit more complicated - and will be deferred until we are ready to discuss that problem fully.

Let $d_M N_\beta(s,M)$ be the number of β-galaxies in S in dM at M; note that this number is either zero or one - its <u>expectation</u> is $\lambda_\beta(M)\mu(s)dM$, and <u>this</u> quantity varies smoothly with dM. Then if $\mathcal{Z}'_\beta(\Omega,\delta^o)$ is the number of β-galaxies in the solid angle Ω in the <u>apparent</u> bolometric magnitude interval $\delta^o: M_1^o < M^o < M_2^o$ (We will take the redshift into account later - forget it for now.), we have

$$(2.7) \quad d_{m^o}\mathcal{Z}'_\beta(\Omega,M^o) = \int_C d_r d_M N_\beta\Big(s(r),M(r)\Big) \quad ,$$

where $dm^o = dM$, $M(r) = m^o - 5\log_{10} r$, (distances will be measured in decaparsecs throughout this paper unless otherwise noted); c is the solid cone generated by Ω, and $s(r)$ is the intersection of c and the sphere of radius r.

One does not, of course, measure m^o; the arguments are *unchanged if some limited-band magnitude is* substituted for m^o - but one does not really measure this either, at least not at the present state of the art of nebular photometry. One does measure some approximation to m^o, and this "observed magnitude" we will call m. In addition, one cannot count to indefinitely large m's; as m increases, the probability that one will see the galaxy (as a galaxy) becomes smaller and smaller, and finally disappears. Let $\mathcal{J}_\beta(m^o)$ be the probability that a β-galaxy at m^o is observed and counted; <u>we assume that this selection is a process independent of all other factors</u>.[1] Let $P_\beta(m|m^o)dm$ be the probability that a β-galaxy at <u>real</u> (bolometric or otherwise) magnitude m^o be given an observed magnitude m in dm, assuming that it is counted. Thus the probability that a β-galaxy at m^o is (1) counted and (2) assigned a magnitude m in dm is $P_\beta(m|m^o)\mathcal{J}_\beta(m^o)dm$. (See Appendix II)

Now if $\mathcal{Z}_\beta(\Omega,\delta)$ is the number of β-galaxies counted in Ω and δ,

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1. This assumption is probably not entirely valid for human counters; the tendency is to identify galaxies more readily when the count density is high than when it is low.

(2.8) $$\mathcal{Z}_\beta(\Omega,\delta) = \int_\delta d_m \mathcal{Z}_\beta(\Omega,m) \quad ,$$

and

(2.9) $$E\left(\mathcal{Z}_\beta(\Omega,\delta)\right) = \int E\left(d_m \mathcal{Z}_\beta(\Omega,m)\right) .$$

But by the definition of the expectation value,

(2.10)

$$E\left(d_m \mathcal{Z}_\beta(\Omega,m)\right) = dm \int_{-\infty}^{\infty} P_\beta(m|m^o) \rho_\beta(m^o) E\left(d_{m^o} \mathcal{Z}'_\beta(\Omega,m^o)\right) dm^o$$

(recall that $E\left(d_{m^o} \mathcal{Z}'_\beta(\Omega,m^o)\right) dm^o$ is the probability that $\mathcal{Z}'_\beta(\Omega,\{dm^o \text{ at } m^o\})=1$ exactly), and from (2.7) we obtain

(2.11)
$$E\left(d_{m^o} \mathcal{Z}'_\beta(\Omega,m')\right) = \int_C E\left(d_r d_m N_\beta(s[r],M[r])\right) \quad \cdot$$

$$\cdot \quad = dm^o \mu(\Omega) \int_0^\infty r^2 dr \lambda_\beta(m^o - 5 \log r) \quad ,$$

so

(2.12)
$$E\left(\mathcal{Z}_\beta(\Omega,\ )\right) = \mu(\Omega) \int_\delta dm \int_{-\infty}^{\infty} dm^o P_\beta(m|m^o) \rho_\beta(m^o) \int_0^\infty dr r^2 \lambda_\beta(m^o - 5 \log r).$$

One normally counts (if one has magnitudes at all) in <u>intervals</u> of m; let $\delta_j = \{m | m_{j-1} < m \leq m_j\}$, and let $\mathcal{Z}_{\beta j}(\Omega) = \mathcal{Z}_{\beta}(\Omega, \delta_j)$. Note that we have now integrated the point process in one variable and the point nature of the process in that variable disappears (See the discussion on mollification in Appendix I.). Let $C_{\beta j}(m^o) =$

$\mathcal{J}_{\beta}(m^o) \int_{\delta_j} P_{\beta}(m | m^o) dm$. Then

(2.13)

$$E\left(\mathcal{Z}_{\beta j}(\Omega)\right) = \mu(\Omega) \int_{-\infty}^{\infty} dm^o \; C_{\beta j}(m^o) \int_0^{\infty} dr \; r^2 \lambda_{\beta}(m' - 5 \log r) \quad .$$

In a similar fashion (see Appendix I for the origin of the overlap term), we obtain

(2.14)

$$\mathrm{cov}\left(\mathcal{Z}_{\beta j}(\Omega), \mathcal{Z}_{\beta' j'}(\Omega')\right) = \iint_{\Omega\Omega'} d\omega d\omega' \left\{ \int_{-\infty}^{\infty} \int dm^o dm^{o\prime} C_{\beta j}(m^o) \quad \cdot \right.$$

$$C_{\beta' j'}(m^{o\prime}) \int_0^{\infty}\int_0^{\infty} dr dr' r^2 r'^2 f_{\beta\beta'}(|\underline{r} - \underline{r}'|, m^o - 5 \log r, \; m^{o\prime} - 5 \log r'$$

$$+ \quad \mu(\Omega \cap \Omega') \, \delta_{\beta\beta'} \delta_{jj'} \int_{-\infty}^{\infty} dm^o C_{\beta j}(m^o) \int_0^{\infty} r^2 dr \lambda_{\beta}(m^o - 5 \log r) \quad .$$

Let γ be the angle between two points on the sphere. Then let

(2.15)

a) $$M_{\beta j} = \int_{-\infty}^{\infty} dm^o C_{\beta j}(m^o) \int_0^{\infty} dr\ r^{2\prime}\ {}_{\beta}(m^o - 5 \log r)$$

b) $$G_{\beta j \beta' j'}(\gamma) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dm^o dm^{o\prime} \Big\{ C_{\beta j}(m^o) C_{\beta' j'}(m^{o\prime}) \cdot$$

$$\cdot \int_0^{\infty}\int_0^{\infty} dr\ dr' r^2 r'^2 f_{\beta\beta'}(\alpha, m^o - 5 \log r, m^{o\prime} - 5 \log r') \Big\},$$

where $\alpha = |\underline{r}' - \underline{r}| = \sqrt{(r^2 + r'^2 - 2rr' \cos \gamma)}$,

be the mean and covariance densities, respectively, of $\mathcal{Z}_{\beta j}(\Omega)$.

It is possible to perform what amounts to a spectral analysis of processes on the sphere, but the analysis is algebraically very involved and not terribly illuminating. Fortunately, we need not pursue this course; we are interested in small areas on the sphere, and shall assume henceforth a plane approximation. The validity of this assumption depends critically on the diameter within which $G_{\beta j, \beta' j'}(\gamma)$ is large. This in turn depends on the brightness levels j_j´ , but is probably not larger than four or five degrees at most for galaxies of about visual magnitude 16 or fainter (Neyman and Scott, (3); Limber,

(4)). We thus neglect terms which are $O(\gamma^2)$ compared to terms of order unity.

The next assumption we make is one of practical necessity, and is a real, physical assumption which for the moment seems impossible of direct verification, but nonetheless seems reasonable in the light of what is known about the distribution of galaxies. It is

Postulate A: All β-galaxies are distributed identically except for frequency. That is to say, the probability distributions of the process give the probability of the existence or non-existence of a β-galaxy in a given region, but the brightness of that β-galaxy is independent of those distributions. Mathematically, this means that

(2.16)

$$P(\beta\text{-galaxy in } dm \text{ at } m \text{ in } d^3x \text{ at } \underline{x} \mid \text{any data on neighbors})$$

$$= P(\beta\text{-galaxy in } dM \text{ at } M) \bullet P(\beta\text{-galaxy in } d^3x \text{ at } \underline{x} \mid \text{any data on neighbors})$$

, and from the probabalistic interpretation of the moment densities, it is clear that we can write

(2.17) $$f_{\beta\beta'}(\alpha, M, M') = \lambda_\beta(M)\lambda_{\beta'}(M')g_{\beta\beta'}(\alpha) \quad .$$

It must be stressed that by making the β-division fine enough, this can be realized to any desired accuracy; but practicality demands a quite coarse division. Previous discussions have sidetracked this problem entirely by the tacit assumption that <u>all</u> galaxies are distributed identically, but the observation that ellipticals tend to cluster by themselves and spirals likewise seem to invalidate this. In any case, we are forced to Postulate A by the fact that we can identify a galaxy as a β-galaxy but without further detailed knowledge can know nothing about its absolute magnitude.

Thus

$$(2.18) \quad G_{\beta j,\beta' j'}(\gamma) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dm^{o} dm^{o\prime} C_{\beta j}(m^{o}) C_{\beta' j'}(m^{o\prime}) \cdot$$

$$\cdot \int_{0}^{\infty}\int_{0}^{\infty} dr\, dr' r^{2} r'^{2} \lambda_{\beta}(m^{o} - 5 \log \underline{r}) \lambda_{\beta'}(m^{o\prime} - 5 \log r') g_{\beta\beta'}(\alpha)$$

is an integral equation relating the observable $G_{\beta j,\beta' j'}(\gamma)$ to the spatial covariance function $g_{\beta\beta'}(\alpha)$. After some manipulation this may be put into the classical form of a Fredholm equation of the first kind; it does not have a compact kernel for reasonable behavior for the C's and λ's. Little useful knowledge exists concerning the solution of such equations; a new linear-programming based method for solving such mathematically unstable problems has been developed by Douglas (13) and Cannon (14), but we shall use a simpler parametric approach to solve (approximately) a somewhat simplified version of (2.18). Let

(2.19)

$$G_{\beta\beta'}(m,m',\gamma) = \int_0^\infty\!\!\int_0^\infty dr\left\{r^2 r'^2 \lambda_\beta (m-5 \log r)\lambda_{\beta'} (m -5 \log r') g_{\beta\beta'}(\alpha)\right\} .$$

Now $\alpha^2 = (r-r')^2 + 4rr'\sin^2 \gamma/2 \cong (r-r')^2+rr'\gamma^2$ in our plane approximation. If we set $x = \sqrt{rr'}$, $\Delta = r-r'$, (2.19) becomes

(2.20)

$$G_{\beta\beta'}(m,m',\gamma) = \int_0^\infty x^4 dx \int_{-\infty}^{\infty} d\Delta\lambda_\beta (m-5 \log r_+)\lambda_{\beta'}(m -5 \log r_-) \cdot$$

$$\cdot\ g_{\beta\beta'}\left((\Delta^2+\gamma^2 x^2)^{1/2}\right) \left(1+ \frac{\Delta^2}{4x^2}\right)^{-1/2} ,$$

where $r_\pm = x \left(\sqrt{1+ \frac{\Delta^2}{4x^2}} \pm \frac{\Delta}{2x}\right)$.

Before we proceed to the form we shall use for computation, we consider an interesting limiting case for (2.20). Suppose that we are dealing with very distant objects; that is, β and β' represent classes of galaxies that are reasonably bright intrinsically and m and m' are large. Then the $x^4.\lambda_\beta.\lambda_{\beta'}$ term will be very small in the region where $\frac{\Delta}{x}$ is appreciable and Δ is still in the region where $G_{\beta\beta'}(\Delta)$ is large. We can thus neglect terms of the order of $\frac{\Delta^2}{x^2}$, and we get

(2.21)

$$G_{\beta\beta'}(m,m';\gamma) \cong \int_0^\infty dx\, x^4 \lambda_\beta(m-5\log x)\lambda_{\beta'}(m'-5\log x)\int_{-\infty}^{\infty} g_{\beta\beta'}(\sqrt{\Delta^2+\gamma^2x^2})\,d\Delta\ .$$

Note that in replacing $r\pm$ by x, we are really only neglecting a term quadratic in $\frac{\Delta}{x}$; the linear term vanishes, since the inner integral in (2.19) is differentiable and invariant under $\Delta \rightarrow -\Delta$. The asymptotic nature of (2.20) is easily verified if $g_{\beta\beta'}(\alpha)$ falls off sufficiently rapidly as α becomes large, and if the λ_β's cut off on the faint end. We say nothing about the general luminosity function cutting off at the faint end (as well it may not) - only that the functions for the given classes β , β' do so, and if may well be that (2.20) does not obtain at any m, m' for some classes. For those in which it does, however, we shall find (2.21) very useful.

Let

(2.22) $$A_{\beta\beta'}\left(\frac{\gamma x}{\ell}\right) = \int_{-\infty}^{\infty} g_{\beta\beta'}((\Delta^2+\gamma^2x^2)^{1/2})\frac{d\Delta}{\ell} \quad ,$$

where ℓ is some (at the moment arbitrary) characteristic length; we shall call $A_{\beta\beta'}(\xi)$ the auxiliary covariance function, and its two-dimensional Fourier transform, (regarding $A_{\beta\beta'}(\xi)$ as a circularly symmetric function of the radius ξ) $\mathcal{A}_{\beta\beta'}(\eta)$, the auxiliary spectral function. We now assume that $g_{\beta\beta'}(\alpha)$ has the form

$$(2.23) \qquad g_{\beta\beta'}(\alpha) = \sum_{i=1}^{N} a^{i}_{\beta\beta'} \left(\frac{\alpha}{\ell}\right)^{2i} e^{-\frac{\alpha^2}{2\ell^2}},$$

where ℓ is chosen appropriately. Any bounded, absolutely integrable, spherically symmetric function can be so expanded to within any accuracy (in the L_2 norm) if we take N large enough, so this is no real restriction; the method becomes unwieldy in the extreme, though, if we are driven to take N very large. It may be necessary (or at least useful) in practice to use the sum of two or more sums as in (2.23) with different ℓ's to better the approximation without resorting to many terms. This generalization is very straightforward and we shall not pursue it further. Inserting (2.23) into (2.20), we obtain

(2.24)

$$G_{\beta\beta'}(m,m',r) = \int_0^\infty dx . x^4 \left\{ \sum_{i=o}^{N} \sum_{k=o}^{i} a^{i}_{\beta\beta'} \left(\frac{x^2\alpha^2}{2}\right)^k \cdot e^{-\frac{x^2\alpha^2}{2\ell^2}} \binom{i}{k} \ell I^{\beta\beta}_{i-k}(m,m',x) \right\},$$

where

(2.25)

$$I_{i-k}(m,m',x) = \int_{-\infty}^{\infty} dy \; y^{2(i-k)} \frac{e^{-y^2/2}}{\sqrt{(1+y^2 \frac{\ell^2}{4x^2})}} \cdot$$

$$\left\{ \lambda_\beta \left(m - 5 \log r^+(y)\right) \lambda_{\beta'}\left(m-5 \log r^-(y)\right) \right\}$$

and

$$r^{\pm}(y) = x\left(\sqrt{(1+y^2\ell^2/4x^2)} \pm \frac{y\ell}{2x}\right) .$$

The functions $I^{\beta\beta'}_{i-k}(x)$ are asymptotic to

(2.26)

$$\tilde{I}^{\beta\beta'}_{i-k}(m,m',x) = \lambda_{\beta}(m-5 \log x)\lambda_{\beta'}(m'-5 \log x) .$$

$$\sqrt{2\pi}\,(2i-2k-1)\,!!$$

as x/ℓ becomes large, and if $\tilde{I}^{\beta\beta'}_{i-k}$ is substituted for $I^{\beta\beta'}_{i-k}$, the approximation of (2.21) is obtained. The error, again, is $O(\ell^2/x^2)$.

We could attempt to estimate $G_{\beta j\beta' j'}(y)$ from data taken from counts, but we shall see that it is much more efficient to estimate the spectral density. The only dependence on γ in (2.24) is through the

$\left(\frac{x^2\gamma^2}{\ell^2}\right)^k e^{-\frac{x^2\gamma^2}{2\ell^2}}$ term. Let

$$(2.27)\quad \frac{1}{(2\pi)^2}\int e^{-i\eta\cdot y}\, y^{2k}\, e^{-y^2/2}\, d^2y = Q_k(\eta)\, e^{-\eta^2/2} .$$

$Q_k(\eta)$ is an even polynomial, and is in fact $\frac{2^k k!}{2\pi} L_k(\eta^2/2)$, where $L_k(x)$ is the Laguerre polynomial of degree k. Then the two dimensional Fourier transform of $G_{\beta\beta'}(m,m',\gamma)$ is

(2.28)

$$\mathcal{G}_{\beta\beta'}(m,m',\eta) = \ell^3\int_0^\infty x^2dx \sum_{i=0}^{N}\sum_{\kappa=0}^{i} a^i_{\beta\beta'}\binom{i}{k} \cdot$$

$$Q_k\left(\frac{\eta\ell}{x}\right) e^{-\frac{\eta^2\ell^2}{2x^2}} \quad I^{\beta\beta'}_{i-k}(m,m',x) \quad ,$$

and if we define

(2.29)

$$I_k^{\beta j\beta' j'}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dm\ dm' \Big\{ C_{\beta j}(m)\ C_{\beta',j'}(m') \quad .$$

$$\cdot\ I^{\beta\beta'}_{i-k}(m,m',x)\Big\} = \int_{-\infty}^{\infty} dy\ y^{2k}\ \frac{e^{-y^2/2}}{\sqrt{(1+\frac{y^2\ell^2}{4x^2})}} \Big\{\Big[\Big[\int_{-\infty}^{\infty} dm\ C_{\beta j}(m) \quad \cdot$$

$$\cdot\lambda_\beta(m-5\ \log r^+(y)\Big)\Big]\Big[\int_{-\infty}^{\infty} dm' C_{\beta' j'}(m')\ \lambda_{\beta'}(m'-5\ \log r^-(y)\Big],$$

we can write the transform of $G_{\beta j\beta' j'}{}^{(\gamma)}$ as

(2.30)

$$\mathcal{G}_{\beta j\beta' j'}(\xi) = \ell^3\int_0^\infty x^2dx \sum_{i=0}^{N}\sum_{k=0}^{i} a^i_{\beta\beta'}\binom{i}{k} Q_k\left(\frac{\xi\ell}{x}\right) \quad \cdot$$

$$e^{-\frac{\xi^2\ell^2}{2x^2}} \quad I^{\beta j\beta' j'}_{i-k}(x) \quad .$$

We now superimpose a grid with cells of side h on the region which is to be counted; let Ω_{lp} be the square with coordinates lh, ph, and let $Z_{lp,\beta j} = Z_{\beta j}{}^{(\Omega_{lp})}$. $Z_{lp,\beta j}$ is then a triple stochastic series, stationary in the "space" indices ℓ and p. It has a spectral density on the square $-\pi < \eta_i \leq \pi$, i=1,2, and comparison with the one-dimensional development in Appendix I yields

(2.31)
$$\Lambda_{\beta j} = h^2 \int_0^\infty dr\ r^2 \left\{ \int_{-\infty}^{\infty} dm\, C_{\beta j}(m)\, \lambda_\beta(m-5 \log r) \right\}$$

and

$$\mathcal{H}_{\beta j,\beta' j'}(\eta) = h^2 \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \mathcal{G}_{\beta j,\beta' j'}\left(\frac{\eta+2\pi \underline{k}}{h}\right) \cdot$$

$$\cdot \frac{\sin^2(\eta_1/2+k_1\pi)}{(\eta_1/2+k_1\pi)^2} \; \frac{\sin^2(\eta_2/2+k_2\pi)}{(\eta_2/2+k_2\pi)^2}$$

$$+ \delta_{jj'} \delta_{\beta\beta'} \Lambda_{\beta j} (2\pi)^{-2} ,$$

$$\overset{\mathrm{Def}}{\equiv} \mathcal{K}_{\beta j,\beta' j'}(\eta) + \delta_{jj'} \delta_{\beta\beta'} \Lambda_{\beta j} (2\pi)^{-2}$$

where $\quad |\eta+2\pi \underline{k}| = \left[(\eta_1+2\pi k_1)^2 + (\eta_2+2\pi k_2)^2\right]^{1/2}$,

for the mean and spectral density, respectively. It is $\mathcal{K}_{\beta j,\beta' j'}$ that we shall estimate from count data, and from it ℓ and the $\{a^i_{\beta\beta'}\}$ in (2.23).

III. SAMPLING TO DETERMINE $\mathcal{H}_{\beta_j \beta_j}(\eta)$

We have seen how a 3-space dimensional point process projects into a two-dimensional one on the sphere, and have obtained relationships between the three-dimensional moments and the two-dimensional ones. The question now arises of how one determines the two-dimensional moments from the observed distribution. The problem in a much simpler form has been discussed at length by several authors (GR, (7) , Blackman and Tukey (6)), and most of the immediately interesting questions have been answered. The transition to multidimensional, multivariate processes is in principle simple, but involves much manipulation. The inclusion of the inherently point nature of the process is a good deal more difficult, and requires that we formulate a simple model which we can study in detail. Let us point out here, though, that the "adoption" of this model does not restrict in any way the foregoing general description of the way galaxies are distributed, but is analagous to the "normal approximation" in studying continuous random processes. It allows us to <u>describe approximately</u> the behavior of <u>samples</u>, and it will turn out that this is all we shall require in the instances when we are forced to use it.

The model we shall use is the Poisson process with stochastic parameter, or the <u>Quasi-Poisson process</u>. It is discussed in detail and several new results which we shall need are obtained in Appendix II. It should be noted that the application of such a model to the distribution of galaxies was first made by Layzer (15), though its strength lies in the subsequent facility with which sampling predictions can be made, and he did not develop the subject.

The investigations of one-dimensional, monovariate processes which have been made indicate that the covariance is not easy to estimate reliably, and is very difficult to obtain easily applied variance estimates for. There are, on the other hand, a family of estimators of the spectral density (or, rather, a smoothed version of the spectral density) which are very tractable, and for which simple variance estimates exist. We shall see shortly why the covariance is difficult to deal with; it is not surprising that the spectrum should be the efficient statistic here also.

For simplicity of notation, let us contract the double index βj into one index, β, and let the couple (ℓ ,p) identifying the square $\Omega_{\ell p}$ be $\underline{\ell}$. Thus $\Lambda_{\beta j}$ becomes Λ^{β}, $\mathcal{H}_{\beta j,\beta' j'}(\eta)$ becomes $\mathcal{H}^{\beta\beta'}(\eta)$, $\mathcal{Z}_{\ell p,\beta j}$ becomes $\mathcal{Z}^{\beta}_{\underline{\ell}}$. (The notation is now the same as that of Theorem 8, Appendix II.) We abandon for a moment the viewpoint of Chapter II and consider $\mathcal{Z}^{\beta}_{\underline{\ell}}$ as a process in two dimensions which we wish to analyze, quite forgetting the three-dimensional process which is "behind" it.

Let $\operatorname{cov}\left(\mathcal{Z}_{\underline{l}}^{\beta}, \mathcal{Z}_{\underline{k}}^{\alpha}\right) = \mathcal{R}_{\underline{l}-\underline{k}}^{\beta\alpha} + \delta_{\underline{l}-\underline{k}}^{\beta\alpha} \Lambda^{\beta}$. Since the parent point process is isotropic, $\mathcal{R}_{l_1, l_2}^{\beta\alpha} = \mathcal{R}_{l_1, -l_2}^{\beta\alpha}$,

and $\mathcal{R}_{\underline{l}}^{\beta\alpha} = \mathcal{R}_{-\underline{l}}^{\beta\alpha} = \mathcal{R}_{l}^{\alpha\beta}$.

Thus there is no "quadrature spectrum" (GR, 16), but only a "cospectrum" - that is, the spectral density is <u>real</u> for all β and α.

Suppose that we have a sample of $\mathcal{Z}_{\underline{l}}^{\beta}$ counted on an M by N rectangle. We can estimate the $\mathcal{R}$'s by

(3.1) $$\mathcal{R}_{\underline{l}}^{*\alpha\beta} = \frac{1}{MN} \sum_{\underline{v}=\underline{1}}^{\underline{N}-\underline{l}} \mathcal{Z}_{\underline{v}}^{\alpha} \mathcal{Z}_{\underline{v}+\underline{l}}^{\beta}$$

for $l_1, l_2 > 0$, and analogously for other values. (Here the sum $\sum_{\underline{v}=\underline{1}}^{\underline{N}-\underline{l}} = \sum_{v_1=1}^{N-l_1} \sum_{v_2=1}^{M-l_2}$.) We assume at present that we <u>know</u> Λ^{β}, so that we can form $\overline{\mathcal{Z}}_{\underline{v}}^{\alpha}$; this restriction will be lifted presently. Equation (3.1) does in fact provide an estimate for $\mathcal{R}_{\underline{l}}^{\alpha\beta}$, for

(3.2) $$E\left(\mathcal{R}_{\underline{l}}^{*\alpha\beta}\right) = \frac{1}{NM} \sum_{\underline{v}=\underline{1}}^{\underline{N}-\underline{l}} E\left(\mathcal{Z}_{\underline{v}}^{\alpha} \mathcal{Z}_{\underline{v}+\underline{l}}^{\beta}\right)$$

$$= \frac{N-|l_1|}{N} \frac{M-|l_2|}{M} \mathcal{R}_{\underline{l}}^{\alpha\beta} + \delta_{\underline{l}}^{\alpha\beta} \Lambda^{\alpha},$$

and it could be easily normalized. The term in Λ^{α} arises, of course, from the point nature of $\mathcal{Z}_{\underline{v}}^{\alpha}$, and can be removed if one has an estimate for Λ^{α}. Perhaps the most straightforward way of estimating the spectrum would be

$$(3.3)\qquad h^{*}_{\alpha\beta}(\underline{\eta}) = \frac{1}{(2\pi)^2}\sum_{\underline{\nu}=-(\underline{N}-\underline{1})}^{\underline{N}-\underline{1}} R_{\underline{\nu}}^{*\alpha\beta}\, e^{-i\underline{\nu}\cdot\underline{\eta}} .$$

The quantity $h^{*}_{\alpha\beta}(\underline{\eta})$ is called the periodogram and it can be shown (GR, (7), p. 145) that its variance does not even tend to zero with larger samples. Analysis indicates that values of $h^{*}_{\alpha\beta}(\underline{\eta})$ more than about π/N apart in the argument are very nearly uncorrelated, however, so that as we add more points to the sample we merely add to the number of "degrees of freedom" in the statistic $h^{*}_{\alpha\beta}$. To prevent this, we smooth the periodogram, making several of the elementary frequency bands contribute to the value of the new variable at each point. Consider the estimate

$$(3.4)\qquad \mathcal{H}^{*}_{\alpha\beta} = \frac{1}{(2\pi)^2}\sum_{\underline{\nu}=-(\underline{N}-\underline{1})}^{\underline{N}-\underline{1}} w_{\underline{\nu}}\, R_{\underline{\nu}}^{*\alpha\beta}\, e^{-i\underline{\nu}\cdot\underline{\eta}}$$

where $w_{0,0} = 1$, $w_{\underline{\nu}} = 0$ for $|\underline{\nu}| > m$, $w_{\nu_1,\nu_2} = w_{-\nu_1,\nu_2} = w_{-\nu_1,-\nu_2}$. choose $N, M > m$. Let $w(x)$ be the function whose Fourier series has coefficients $w_{\underline{\nu}}$. Then

$$(3.5)\qquad w_{\underline{\nu}} = \iint_{-\pi}^{\pi} w(x)\, e^{i\underline{\nu}\cdot\underline{x}}\, d^2x ,$$

and it is easy to show that

$$(3.6)\qquad \mathcal{H}^{*}_{\alpha\beta}(\eta) = \iint_{-\pi}^{\pi} w(\eta - y)\, h^{*}_{\alpha\beta}(y)\, d^2y$$

Using (3.2), (3.4), (3.5), one obtains

(3.7)
$$E(\mathcal{H}^*_{\alpha\beta}(\eta)) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d^2x \Big\{ \mathcal{H}_{\alpha\beta}(\underline{x}) \cdot \frac{1}{(2\pi)^2 NM} \cdot$$
$$\cdot \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\sin^2(\frac{N}{2}(y_1-x_1))\ \sin^2(\frac{M}{2}(y_2-x_2))}{\sin^2(\frac{y_1-x_1}{2})\ \sin^2(\frac{y_2-x_2}{2})} \quad .$$

The factors $\frac{1}{2\pi N}\ \frac{\sin^2(Nx/2)}{\sin^2(x/2)}$ will occur often in the work in this section, and we shall assume throughout that N and M are sufficiently large that their width is negligible, i.e., we approximate (3.7) by

$$(3.8) \qquad E(\mathcal{H}^*_{\alpha\beta}(\eta)) \cong \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d^2x\, \mathcal{H}_{\alpha\beta}(\underline{x})\, w(\eta - \underline{x}) .$$

The transformation is accomplished formally by the replacement of $\frac{1}{2\pi N}\ \frac{\sin^2(Nx/2)}{\sin^2(x/2)}$ by the Dirac delta (the correct replacement is the delta function and not a multiple thereof, since $\int_{-\pi}^{\pi} \frac{\sin^2(Nx/2)}{\sin^2(x/2)} = 2\pi N .$), and is strictly valid only asymtotically, as N and M tend to infinity. The error introduced is slight, however, and in all other cases in which it will be used, there will be larger uncertainties from other sources; this is not the case for (3.7), and one might prefer the exact relation. In any case, we see that $\mathcal{H}^*_{\alpha\beta}(\eta)$ does in fact form an estimator for $\mathcal{H}_{\alpha\beta}(\eta)$, and the bias decreases with the width of $w(\underline{x})$. In the limit $w(\underline{x}) = \delta^{(2)}(\underline{x})$, however, $\mathcal{H}^*_{\alpha\beta}(\eta)$ is just the periodogram, and so it is necessary to compromise somewhat on $w(\underline{x})$. To study the usefulness of the estimate we need to find its variance, and it is to this problem that most of the rest of this section will be devoted. We shall later perform circular averages over the sample spectra, so we in fact need covariances. From (3.4), we obtain

(3.9)

$$\mathrm{cov}\left(\mathcal{H}^{*}_{\alpha\beta}(\underline{\eta}),\ \mathcal{H}^{*}_{\alpha\beta}(\underline{\eta}')\right) = \frac{1}{(2\pi)^4}\sum_{\underline{\mu},\underline{\nu}=-(\underline{N}-\underline{1})}^{(\underline{N}-\underline{1})} w_{\underline{\mu}}\, w_{\underline{\nu}}\, e^{-i(\underline{\eta}\cdot\underline{\nu}+\underline{\eta}'\cdot\underline{\mu})}\,\mathrm{cov}\left(R^{*\alpha\beta}_{\underline{\nu}},\ R^{*\alpha\beta}_{\underline{\mu}}\right)$$

but

(3.10)

$$\mathrm{cov}\left(R^{*\alpha\beta}_{\underline{\nu}},\ R^{*\alpha\beta}_{\underline{\mu}}\right) = \frac{1}{N^2M^2}\sum_{\underline{n},\underline{m}} \mathrm{cov}\left(\overline{Z}^{\alpha}_{\underline{n}}\,\overline{Z}^{\beta}_{\underline{n}+\underline{\nu}},\ \overline{Z}^{\alpha}_{\underline{m}}\,\overline{Z}^{\beta}_{\underline{m}+\underline{\mu}}\right),$$

where the range of summation on $\underline{n}$ and $\underline{m}$ is such that the vectors $\underline{n}$, $\underline{m}+\boldsymbol{\nu}$, $\underline{m}$, $\underline{m}+\boldsymbol{\mu}$ lie in the rectangle of height N and width M. The covariance terms in the sum in (3.10) are given in Theorem 8, Appendix II, under the assumption that $Z^{\alpha}_{\underline{n}}$ is Quasi-Poisson and that the fundamental has vanishing third moments. In addition, we assume that the fundamental is approximately normal in the sense that the relation

(3.11)

$$\operatorname{cov}\left(\bar{\rho}^{\alpha}(\underline{x})\,\bar{\rho}^{\beta}(\underline{y}),\ \bar{\rho}^{\alpha}(\underline{z})\,\rho^{\beta}(\underline{w})\right)$$

$$= \operatorname{cov}\left(\bar{\rho}^{\alpha}(\underline{x})\,\bar{\rho}^{\alpha}(\underline{z})\right)\operatorname{cov}\left(\bar{\rho}^{\beta}(\underline{y})\,\bar{\rho}^{\beta}(\underline{w})\right)$$

$$+ \operatorname{cov}\left(\bar{\rho}^{\alpha}(x)\,\bar{\rho}^{\beta}(w)\right)\operatorname{cov}\left(\bar{\rho}^{\beta}(\underline{y}),\ \bar{\rho}^{\alpha}(\underline{z})\right)$$

which holds exactly for normal processes, is approximately satisfied. These assumptions are not likely to hold with any great precision, but almost certainly introduce no vast error and allow us to proceed where before we could not. The normal approximation for the fundamental allows us to use the extensive literature on spectral anlaysis of stationary continuous normal processes and stochastic series (at least as a guide) in the bewilderingly voluminous algebraic exercise required to evaluate cov $(\mathcal{H}^{*}_{\alpha\beta}(\eta),\mathcal{H}^{*}_{\alpha\beta}(\eta'))$. We can do little more than indicate the path of this analysis here; the details are for the most part straightforward, albeit extremely tedious.

The task is made somewhat less onerous by the use of the large M, N asymptotic results discussed above. It may easily be verified that products of highly packed trigonometric factors of width of the order N^{-1} or M^{-1} behave in precisely the same manner under integration with smooth functions as products of Dirac deltas; the correspondence again becomes good for very large N and M, and we shall use it in the heuristic discussion to follow. If $M_{\underline{n}}^{\alpha} = \int_{S_{\underline{n}}} \rho^{\alpha}(\underline{x})\, d^2x$, then (3.11) holds also for $M_{\underline{m}}^{\alpha}$, and it is easily verified that $\operatorname{cov}(\overline{M}_{\underline{n}}^{\alpha}, \overline{M}_{\underline{l}}^{\beta}) = R_{\underline{n}-\underline{l}}^{\alpha\beta}$. We then use the results of Theorem 8, Appendix II, to write

(3.12)

$$\begin{aligned}
\operatorname{cov}\left(R_{\underline{\nu}}^{*\alpha\beta}, R_{\underline{\mu}}^{*\alpha\beta}\right) = \frac{1}{N^2M^2} \sum_{\underline{n},\underline{m}} \Big\{ & \left(R_{\underline{n}-\underline{m}}^{\alpha\alpha}\, R_{\underline{n}-\underline{m}+\underline{\nu}-\underline{\mu}}^{\beta\beta} \right. \\
& \left. + R_{\underline{n}-\underline{m}-\underline{\mu}}^{\alpha\beta}\, R_{\underline{n}-\underline{m}+}^{\alpha\beta} \right) \\
& + \delta_{\underline{n},\underline{m}}\, \Lambda^{\alpha} R_{\underline{n}-\underline{m}+\underline{\nu}-\underline{\mu}}^{\beta\beta} + \delta_{\underline{n},\underline{m}+\underline{\mu}}^{\alpha\beta}\, \Lambda^{\alpha} R_{\underline{n}-\underline{m}+\underline{\nu}}^{\alpha\beta} \\
& + \delta_{\underline{n}+\underline{\nu},\underline{m}+\underline{\mu}}\, \Lambda^{\beta} R_{\underline{n}-\underline{m}}^{\alpha\alpha} + \delta_{\underline{n}+\underline{\nu},\underline{m}}^{\alpha\beta}\, \Lambda^{\beta} R_{\underline{n}-\underline{m}-\underline{\mu}}^{\alpha\beta} \\
& + \left(R_{\underline{\nu}}^{\alpha\beta} + \Lambda^{\alpha}\Lambda^{\beta}\right)\left(\delta_{\underline{n},\underline{m}}\, \delta_{\underline{n}+\underline{\nu},\underline{m}+\underline{\mu}} + \delta_{\underline{n},\underline{m}+\underline{\mu}}^{\alpha\beta}\, \delta_{\underline{n}+\underline{\nu},\underline{m}}^{\alpha\beta}\right) \\
& + R_{\underline{n}-\underline{m}}^{\alpha\alpha}\, \delta_{\underline{\nu},0}^{\alpha\beta}\, \delta_{\underline{\mu},0} \\
& + R_{\underline{\nu}}^{\alpha\beta}\left(\delta_{\underline{n}+\underline{\nu},\underline{m}}^{\alpha\beta}\, \delta_{\underline{\mu},0}^{\alpha\beta} + \delta_{\underline{n},\underline{m}}\, \delta_{\underline{\mu},0}^{\alpha\beta}\right) \\
& + R_{\underline{\mu}}^{\alpha\beta}\left(\delta_{\underline{\nu},0}^{\alpha\beta}\, \delta_{\underline{n}+\underline{\nu},\underline{m}+\underline{\mu}} + \delta_{\underline{\nu},0}^{\alpha\beta}\, \delta_{\underline{n}+\underline{\nu},\underline{m}}^{\alpha\beta}\right) \\
& + \Lambda^{\alpha}\, \delta_{\underline{\nu},0}^{\alpha\beta}\, \delta_{\underline{\mu},0}^{\alpha\beta}\, \delta_{\underline{n},\underline{m}} \Big\} .
\end{aligned}$$

The products of Kroenecker deltas can in several cases be simplified. We next replace the R's by their Fourier developments:

(3.13) $$R^{\alpha\beta}_{\underline{\nu}} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathcal{K}_{\alpha\beta}(\eta)\, e^{i\eta\cdot\underline{\nu}}\, d^2\eta .$$

The sums on n and m then can be taken inside the integrals, and one gets sums like

(3.14) $$\sum_{\underline{n},\underline{m}} e^{i(\eta+\xi)\cdot(\underline{n}-\underline{m})} = \sum_{\underline{n}} e^{i(\eta+\xi)\cdot\underline{n}} \sum_{\underline{m}} e^{-i(\eta+\xi)\cdot\underline{m}} .$$

Now the sums on $\underline{n}$ and $\underline{m}$ can have different ranges (and will, if $\underline{\nu} \neq \underline{\mu}$). We can, however, extend the sums to $-N \leq n_1$, $m_1 \leq N$, $-M \leq n_2$, $m_2 \leq M$, and make a fractional error like m/N or m/M, which goes to zero in the asymptotic approximation we are using. One could as well suppose that data are available on the NxM rectangle and on a strip of width m surrounding it. Then one can redefine R^* in (3.1) by letting the sum run to $\underline{N}$ instead of $\underline{N}-\underline{1}$. This changes the form of the trigonometric factor in the expectation value slightly and makes the sums in the covariance analysis more tractable. In practice, however, one uses (3.1) as is, and we shall keep it. Then

(3.15)
$$\sum_{\underline{n},\underline{m}} e^{i(\underline{\eta}+\underline{\xi})(\underline{n}-\underline{m})} \cong \frac{\sin^2\left(\frac{N}{2}(\eta_1+\xi_1)\right)\sin^2\left(\frac{M}{2}(\eta_2+\xi_2)\right)}{\sin^2\left(\frac{1}{2}(\eta_1+\xi_1)\right)\sin^2\left(\frac{1}{2}(\eta_2+\xi_2)\right)}$$

$$\sim (2\pi)^2 NM\,\delta^2(\underline{\eta}+\underline{\xi}) .$$

When these are inserted in (3.12), the result is (note that $\mathcal{K}_{\alpha\beta}(\underline{\eta}) = \mathcal{K}_{\alpha\beta}(-\underline{\eta})$)

(3.16)
$$\operatorname{cov}\left(R_{\underline{\nu}}^{*\alpha\beta}, R_{\underline{\mu}}^{*\alpha\beta}\right) =$$

$$\frac{1}{N^2M^2}\Big\{\iint_{-\pi}^{\pi}\left[\mathcal{K}_{\alpha\alpha}(\underline{\eta})\mathcal{K}_{\beta\beta}(\underline{\eta})e^{i\underline{\eta}\cdot(\underline{\nu}-\underline{\mu})} + \mathcal{K}_{\alpha\beta}^2(\underline{\eta})e^{i\underline{\eta}\cdot(\underline{\nu}+\underline{\mu})}\right](2\pi)^2NM\,d^2\eta$$

$$+ NM\Lambda^{\alpha}\iint_{-\pi}^{\pi}\left[\mathcal{K}_{\beta\beta}(\underline{\eta})e^{i\underline{\eta}\cdot(\underline{\nu}-\underline{\mu})} + \delta_{\alpha\beta}\mathcal{K}_{\alpha\beta}(\underline{\eta})e^{i\underline{\eta}\cdot(\underline{\nu}+\underline{\mu})}\right]d^2\eta$$

$$+ NM\Lambda^{\beta}\iint_{-\pi}^{\pi}\left[\mathcal{K}_{\alpha\alpha}(\underline{\eta})e^{i\underline{\eta}\cdot(\underline{\nu}-\underline{\mu})} + \delta_{\alpha\beta}\mathcal{K}_{\alpha\beta}(\underline{\eta})e^{i\underline{\eta}\cdot(\underline{\nu}+\underline{\mu})}\right]d^2\eta$$

$$+ \left[\iint_{-\pi}^{\pi}\mathcal{K}_{\alpha\beta}(\eta)e^{i\underline{\eta}\cdot\underline{\nu}}d^2\eta + \Lambda^{\alpha}\Lambda^{\beta}\right]\cdot\left[NM\delta_{\underline{\nu},\underline{\mu}} + NM\delta_{\alpha\beta}\delta_{\underline{\nu},-\underline{\mu}}\right]$$

$$+ \delta_{\alpha\beta}\,\delta_{\underline{\nu},0}\,\delta_{\underline{\mu},0}\,(2\pi)^2 NM\,\mathcal{K}_{\alpha\alpha}(0)$$

$$+ 2NM\Big[\delta_{\alpha\beta}\delta_{\underline{\mu},0}\iint_{-\pi}^{\pi}\mathcal{K}_{\alpha\beta}(\underline{\eta})\,e^{i\underline{\eta}\cdot\underline{\nu}}\,d^2\eta$$

$$+ \delta_{\alpha\beta}\,\delta_{\underline{\nu},0}\iint_{-\pi}^{\pi}\mathcal{K}_{\alpha\beta}(\eta)\,e^{i\underline{\eta}\cdot\underline{\mu}}\,d^2\eta\Big]$$

$$+ NM\,\Lambda^{\alpha}\delta_{\alpha\beta}\,\delta_{\underline{\nu},0}\,\delta_{\underline{\mu},0}\Big\}.$$

Inserting this expression in (3.9), one obtains

(3.17)

$$\mathrm{cov}\left(\mathcal{H}^*_{\alpha\beta}(\underline{\eta}),\mathcal{H}^*_{\alpha\beta}(\underline{\eta}')\right)\cong$$

$$\frac{(2\pi)^2}{2NM}\iint_{-\pi}^{\pi}\{(\mathcal{H}_{\alpha\alpha}(\underline{\xi})\mathcal{H}_{\beta\beta}(\underline{\xi})+\mathcal{H}^2_{\alpha\beta}(\underline{\xi}))$$

$$\cdot\left(w(\underline{\xi}+\underline{\eta})\,w(\underline{\xi}-\underline{\eta}')+w(\underline{\xi}-\underline{\eta})\,w(\underline{\xi}-\underline{\eta}')\right)\}d^2\xi$$

$$+\frac{1}{(2\pi)^2NM}\iiiint_{-\pi}^{\pi}\mathcal{K}_{\alpha\beta}(\underline{\xi})\,w(\underline{\eta}+\underline{\eta}'-\underline{\xi}-\underline{\kappa})\,w(\underline{\kappa})\,d^2\kappa\,d^2\xi$$

$$+\frac{\delta_{\alpha\beta}}{(2\pi)^2NM}\iiiint_{-\pi}^{\pi}\mathcal{K}_{\alpha\beta}(\underline{\xi})\,w(\underline{\eta}-\underline{\eta}'-\underline{\xi}-\underline{\kappa})\,w(\underline{\kappa})\,d^2\kappa\,d^2\xi$$

$$+ \frac{\delta_{\alpha\beta}}{(2\pi)^2 NM} \mathcal{K}_{\alpha\alpha}(0)$$

$$+ \frac{2\delta_{\alpha\beta}}{(2\pi)^2 NM} \iint \mathcal{K}_{\alpha\alpha}(\xi)\left[w(\underline{\xi}+\underline{\eta}) + w(\underline{\xi}+\underline{\eta}')\right] d^2\xi$$

$$+ \frac{\delta_{\alpha\beta}}{(2\pi)^4} \frac{\Lambda^\alpha}{NM} ,$$

Noting that $\mathcal{H}_{\alpha\beta}(\eta) = \mathcal{K}_{\alpha\beta}(\eta) + \frac{\delta_{\alpha\beta}\Lambda^\alpha}{(2\pi)^2}$, and again using the symmetry properties of $\mathcal{K}_{\alpha\beta}(\eta)$.

We now note that in order to keep the bias of the estimate within acceptable limits, the width of the function w must be small compared to the scale over which the spectrum changes appreciably. If this requirement is met, we can write

(3.18)

$$\mathrm{cov}\left(\mathcal{H}^*_{\alpha\beta}(\underline{\eta}), \mathcal{H}^*_{\alpha\beta}(\underline{\eta}')\right) \cong$$
$$\frac{(2\pi)^2}{2NM} \iint_{-\pi}^{\pi} \Big\{ \left(\mathcal{H}_{\alpha\alpha}(\underline{\xi})\mathcal{H}_{\beta\beta}(\underline{\xi}) + \mathcal{H}^2_{\alpha\beta}(\underline{\xi})\right) \cdot$$
$$\cdot \left(w(\underline{\xi}+\underline{\eta})\, w(\underline{\xi}-\underline{\eta}') + w(\underline{\xi}-\underline{\eta})\, w(\underline{\xi}-\underline{\eta}')\right)\Big\} d^2\xi$$

$$+\frac{1}{(2\pi)^2 NM}\left\{\mathcal{K}_{\alpha\beta}(\eta+\eta') + \delta_{\alpha\beta}\mathcal{K}_{\alpha\beta}(\eta-\eta')\right\}$$

$$+\frac{\delta_{\alpha\beta}}{(2\pi)^2 NM}\mathcal{K}_{\alpha\alpha}(0) + \frac{2\delta_{\alpha\beta}}{(2\pi)^2 NM}\left\{\mathcal{K}_{\alpha\alpha}(\eta) + \mathcal{K}_{\alpha\alpha}(\eta')\right\}$$

$$+\frac{\delta_{\alpha\beta}\Lambda^{\alpha}}{(2\pi)^4 NM} .$$

The first term is the same as that which would arise if we had analyzed the variance as if $Z_{\underline{\nu}}^{\alpha}$ were a normal process. The others arise from the point nature of $Z_{\underline{\nu}}^{\alpha}$, and behave quite differently from the first term. Note that the first term is quadratic in the spectra; the others, except the last, are linear.

The first correction to be made to the development above is that we do not, in fact, know Λ^{α}. The best we can do is to estimate it by

(3.19) $$\Lambda^{*\alpha} = \frac{1}{NM}\sum_{\underline{\nu}=\underline{1}}^{\underline{N}} Z_{\underline{\nu}}^{\alpha} .$$

Clearly,

(3.20) $$E\Lambda^{*\alpha} = \Lambda^{\alpha} ,$$

and it can be shown easily that (3.19) has the smallest variance of all linear estimators. We then form

(3.21) $$\tilde{Z}_{\underline{\nu}}^{\alpha} = Z_{\underline{\nu}}^{\alpha} - \Lambda^{*\alpha},$$

and let $\tilde{\mathcal{H}}^{*}_{\alpha\beta}$ be constructed as in (3.4), with $\tilde{R}_{\underline{\nu}}^{*\alpha\beta}$ replacing $R_{\underline{\nu}}^{*\alpha\beta}$ where

(3.22) $$\tilde{R}_{\underline{l}}^{*\alpha\beta} = \frac{1}{NM} \sum_{\underline{\nu}=\underline{1}}^{\underline{N}-\underline{1}} \tilde{Z}_{\underline{\nu}}^{\alpha} \tilde{Z}_{\underline{\nu}+\underline{l}}^{\beta} .$$

Analysis similar in all respects to that leading to the previous results shows that

(3.23) $$E\, \tilde{\mathcal{H}}^{*}_{\alpha\beta}(\underline{\eta}) = \iint_{-\pi}^{\pi} d^2x\, \mathcal{H}_{\alpha\beta}(\underline{x})\, w(\underline{\eta}-x) - \frac{(2\pi)^2}{NM} \mathcal{H}_{\alpha\beta}(0)\, w(\underline{\eta}) .$$

The covariance analysis this time is even more involved than before and produces no results of real interest. The correction terms are, as in the expectation, of the order $O(\frac{1}{NM})$ compared to the terms in (3.18) (which do not change), and are in such a direction as to <u>reduce</u> the variance. We shall ignore them, and take (3.18) as the expression for $\operatorname{cov}(\tilde{\mathcal{H}}^{*}_{\alpha\beta}(\underline{\eta}), \tilde{\mathcal{H}}^{*}_{\alpha\beta}(\underline{\eta}'))$.

We note that the covariance is substantially larger and more complicated when $\alpha = \beta$ than when α and β are different. We are interested in determining $\mathcal{H}_{\alpha\beta}(\underline{\eta})$, however, since we already have an optimal estimate for Λ^{α}

(and it is $\mathcal{K}_{\alpha\beta}$ that directly involves $\mathcal{G}_{\alpha\beta}$), and it would be fortuitous if we could find some way of subtracting off the Λ term which would simultaneously lower the variance appreciably. It is remarkable that this can in fact be done, and in the simplest of fashions. Let first

(3.24) $$\tilde{\mathcal{K}}^*_{\alpha\beta}(\eta) = \tilde{\mathcal{H}}^*_{\alpha\beta} - \frac{\delta_{\alpha\beta}\Lambda^*_\alpha}{(2\pi)^2} .$$

Then clearly

(3.25) $$E\,\tilde{\mathcal{K}}^*_{\alpha\beta}(\eta) = \iint_{-\pi}^{\pi} \mathcal{K}_{\alpha\beta}(\underline{x})\, w(\underline{\eta}-\underline{x})\, d^2x - \frac{(2\pi)^2}{NM}\mathcal{H}_{\alpha\beta}(0)\, w(\underline{\eta}) .$$

We should like to make the correction term as small as possible; we can make the $\mathcal{H}_{\alpha\beta}(0)$ a $\mathcal{K}_{\alpha\beta}(0)$ by replacing $\tilde{\mathcal{K}}^*_{\alpha\beta}(M)$ with

(3.26) $$\mathcal{K}^+_{\alpha\beta}(\eta) = \tilde{\mathcal{H}}^*_{\alpha\beta} - \delta_{\alpha\beta}\left\{\frac{\Lambda^{*\alpha}}{(2\pi)^2} + \frac{\Lambda^{*\alpha}}{NM} w(\eta)\right\} .$$

A moment's reflection will show that if we define

(3.27) $$R^{+\alpha\beta}_{\underline{\nu}} = \tilde{R}^{*\alpha\beta}_{\underline{\nu}} - \delta_{\alpha\beta}\Lambda^{*\alpha}\left\{\delta_{\underline{\nu},0} - \frac{(N-\nu_1)(M-\nu_2)}{N^2M^2}\right\},$$

then

(3.28) $$\mathcal{K}^+_{\alpha\beta}(\eta) = \frac{1}{(2\pi)^2}\sum_{\underline{\nu}=-(\underline{N}-\underline{1})}^{(\underline{N}-\underline{1})} w_{\underline{\nu}}\, R^{+\alpha\beta}_{\underline{\nu}}\, e^{-i\underline{\nu}\cdot\underline{\eta}} .$$

If we now calculate the covariance, we find (making use this time of the (a) part of Theorem 8 to treat the $R^* \Lambda^*$ term that arises),

(3.29)
$$\operatorname{cov}\left(\mathcal{K}^{+}_{\alpha\beta}(\eta), \mathcal{K}^{+}_{\alpha\beta}(\eta')\right) \cong \frac{(2\pi)^2}{2NM} \iint_{-\pi}^{\pi} \{(\mathcal{H}_{\alpha\alpha}(\underline{\xi})\mathcal{H}_{\beta\beta}(\underline{\xi}) + \mathcal{H}^2_{\alpha\beta}(\underline{\xi}))$$
$$\cdot \left(w(\underline{\xi}+\eta)w(\underline{\xi}'-\eta) + w(\underline{\xi}-\eta)w(\underline{\xi}'-\eta)\right)\} d^2\xi$$
$$+ \frac{1}{(2\pi)^2 NM}\left\{\mathcal{K}_{\alpha\beta}(\eta+\eta') + \delta_{\alpha\beta}\mathcal{K}_{\alpha\beta}(\eta-\eta')\right\}.$$

The distinction in form for the covariances in the cases $\alpha = \beta$ and $\alpha \neq \beta$ has disappeared; when we perform circular averages, the contribution from the two "heterodyne" (sum and difference frequency) terms will be equal.

We are not yet making the most efficient use of our data, however. We wish to estimate, finally, a circularly symmetric function $\mathcal{G}_{\alpha\beta}(\eta)$. The sampling spectrum $\mathcal{K}_{\alpha\beta}(\eta)$ and its estimate $\mathcal{K}^{+}_{\alpha\beta}(\eta)$ are not circularly symmetric, and we need to perform some manner of averaging to make best use of the known symmetry of $\mathcal{G}_{\alpha\beta}$.

Recall that, letting $Q(\underset{\sim}{\eta}) = \dfrac{\sin^2(\eta_1/2)\ \sin^2(\eta_2/2)}{(\eta_1/2)^2\ (\eta_2/2)^2}$,

(3.30) $$\mathcal{K}_{\alpha\beta}(\underset{\sim}{\eta}) = h^2 \sum_{\underset{\sim}{k}=-\infty}^{\infty} \mathcal{G}_{\alpha\beta}\left(\tfrac{1}{h}(\underset{\sim}{\eta}+2\pi\underset{\sim}{k})\right) Q(\underset{\sim}{\eta}+2\pi\underset{\sim}{k}),$$

so that, if $\mathcal{G}$ is sufficiently regular,

(3.31)
$$\iint_{-\pi}^{\pi} w(\underset{\sim}{\eta}-\underset{\sim}{x})\,\mathcal{K}_{\alpha\beta}(\underset{\sim}{x})\,d^2x$$
$$= h^2 \sum_{\underset{\sim}{k}} \iint_{-\pi}^{\pi} w(\underset{\sim}{\eta}-\underset{\sim}{x})\,\mathcal{G}_{\alpha\beta}\left(\tfrac{1}{h}(\underset{\sim}{x}+2\pi\underset{\sim}{k})\right) Q(\underset{\sim}{x}+2\pi\underset{\sim}{k})\,d^2x$$
$$= h^2 \iint_{-\infty}^{\infty} w(\underset{\sim}{\eta}-\underset{\sim}{x})\,\mathcal{G}_{\alpha\beta}\left(\frac{\underset{\sim}{x}}{h}\right) Q(\underset{\sim}{x})\,d^2x,$$

where W is now defined everywhere by its Fourier series.

Then

(3.32)
$$E\,\mathcal{K}^{\dagger}_{\alpha\beta}(\underset{\sim}{\eta}) = h^2 \iint_{-\infty} w(\underset{\sim}{\eta}-\underset{\sim}{x})\,\mathcal{G}_{\alpha\beta}\left(\frac{\underset{\sim}{x}}{h}\right) Q(\underset{\sim}{x})\,d^2x$$
$$-\frac{(2\pi)^2}{NM}\,w(\underset{\sim}{\eta})\,h^2 \sum_{\underset{\sim}{\nu}=-\infty}^{\infty} \mathcal{G}_{\alpha\beta}\left(\frac{2\pi\underset{\sim}{\nu}}{h}\right) Q(2\pi\underset{\sim}{\nu}).$$

Now let us perform a circular averaging on $\mathcal{K}^{+}$. Let

(3.33) $$\hat{\mathcal{K}}_{\alpha\beta}(\eta) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{K}^{+}_{\alpha\beta}(\eta\cos\theta, \eta\sin\theta)\, d\theta \stackrel{\text{Def.}}{\equiv} \mathcal{A}\, \mathcal{K}^{+}_{\alpha\beta}(\eta),$$

where $\mathcal{A}$ is the indicated linear operator. It is easily verified that $\mathcal{A}$ commutes with smoothing with a circularly symmetric kernel, with Fourier transformation, and with the expectation operator.

Since w($\underline{x}$) is doubly periodic in the plane, one can, of course, not choose it cirularly symmetric, but one can "almost" do so. Let $w_o(\underline{x})$ be a continuous, bounded, circularly symmetric function on the whole plane, with $\iint_{-\infty}^{\infty} w_o(x)\, d^2x = 1$. Suppose that $w_o(\underline{x})$ is bounded in absolute value by a positive strictly decreasing function of the radius which is also integrable. Then let

(3.34) $$w(\underline{x}) = \sum_{\underline{\nu}=-\infty}^{\infty} w_o(|\underline{x} + 2\pi\underline{\nu}|).$$

It is clear that w(x) is bounded, continuous, doubly periodic, and that

(3.35) $$\iint_{-\pi}^{\pi} w(x)\, d^2x = 1,$$

Let W(k) be the two-dimensional inverse Fourier transform of $w_o(x)$;

(3.36) $$W(k) = \iint w_o(x)\, e^{i k \cdot x}\, d^2x .$$

Then the Fourier coefficients in the series for $w(\underline{x})$ are, since $\exp(i\underline{n}\cdot(\underline{x}+2\pi\underline{\nu})) = \exp(i\underline{n}\cdot\underline{x})$ for n_1, n_2 integers,

(3.37) $$\begin{aligned} w_{\underline{n}} &= \iint_{-\pi}^{\pi} w(x)\, e^{i\underline{n}\cdot\underline{x}}\, d^2x \\ &= \iint_{-\pi}^{\pi} \sum_{\underline{\nu}} w_o(\underline{x}+2\pi\underline{\nu})\, e^{i\underline{n}\cdot\underline{x}}\, d^2x \\ &= \iint_{-\infty}^{\infty} w_o(x)\, e^{i\underline{n}\cdot\underline{x}}\, d^2x \\ &= W(\underline{n}). \end{aligned}$$

Also, if we take a circularly symmetric function $W(|\underline{x}|)$, with support radius m, and let the weights $w_{\underline{n}} = W(\underline{n})$, then $w(\underline{x})$ will have the form (3.34) with a circularly symmetric w_o; this can be easily seen by reversing the above argument. Let us suppose that the $w_{\underline{n}}$ have been so chosen.

Consider first the first term in the expectation of $\hat{\mathcal{K}}_{\alpha\beta}$, that corresponding to the first term on the right in (3.32). Let this term be $E_1^{(0)}\hat{\mathcal{K}}_{\alpha\beta}(\eta)$; the second (regression) term will be $E^{(0)}\mathcal{K}_{\alpha\beta}(\eta)$. Then, letting $\underline{e}(\theta) = (\cos\theta, \sin\theta)$, we have

(3.38) $$E^{(0)}\hat{\mathcal{K}}_{\alpha\beta}(\eta)$$
$$= \frac{h^2}{2\pi}\int_0^{2\pi} d\theta \int_0^\infty\int_0^{2\pi} x\,dx\,d\alpha\; w(\eta e(\theta) - x e(\alpha))\, Q(x e(\alpha))\, \mathcal{G}_{\alpha\beta}\left(\frac{x}{h}\right)$$
$$= 2\pi h^2 \int_0^\infty x\,dx\, \mathcal{G}_{\alpha\beta}\left(\frac{x}{h}\right)\left\{\frac{1}{2\pi}\int_0^{2\pi} Q(x\underline{e}(\alpha))\,d\alpha \,\cdot\right.$$
$$\left.\cdot\, \frac{1}{2\pi}\int_0^{2\pi} w(\eta\underline{e}(\theta) - x\underline{e}(\alpha))\,d\theta\right\}$$

But w($\underline{x}$) is given by (3.34), so we can write

(3.39)
$$E^{(0)}\hat{\mathcal{K}}_{\alpha\beta}(\eta) = 2\pi h^2\int_0^\infty x\,dx\,\mathcal{G}_{\alpha\beta}\left(\frac{x}{h}\right)\tilde{Q}(x)\,\tilde{w}(\eta, x)$$
$$+ 2\pi h^2 \sum_{\underline{\nu}\neq(0,0)} \int_0^\infty x\,dx\,\mathcal{G}_{\alpha\beta}\left(\frac{x}{h}\right)\left\{\frac{1}{2\pi}\int_0^{2\pi} Q(xe(\alpha))\,d\alpha\right.$$
$$\left.\cdot\,\frac{1}{2\pi}\int_0^{2\pi} w_0(|\eta\underline{e}(\theta) - x\underline{e}(\alpha) + 2\pi\underline{\nu}|)\,d\theta\right\}$$

where

$$\tilde{Q}(x) = \mathcal{A}Q(\underline{x})$$
$$\tilde{w}(\eta, x) = \frac{1}{2\pi}\int_0^{2\pi} w_0([x^2+\eta^2 - 2\eta x\cos\alpha]^{1/2})\,d\alpha$$

The second term contains all contributions from the aliased frequencies and will be small if we have chosen h correctly; thus we do not need to evaluate it too carefully. It will suffice for our purposes to note again that the width of w($\underline{x}$) must be small compared to the scale of variations in $\mathcal{K}_{\alpha\beta}(\underline{x})$ (and hence in $\mathcal{G}_{\alpha\beta}(\frac{x}{h})$.) It must also be small, of course, compared to 2π. Since

$w_o(x)$ has mass unity, we may set

(3.40)
$$2\pi\int_0^\infty x\,dx\,\mathcal{G}_{\alpha\beta}\left(\frac{x}{h}\right)\cdot\frac{1}{2\pi}\int_0^{2\pi} Q(x\underline{e}(\alpha))\,d\alpha\int_0^{2\pi} w_o(\eta\underline{e}(\theta)-x\underline{e}(\alpha)+2\pi\underline{\nu})\frac{d\theta}{2\pi}$$
$$\cong \frac{1}{2\pi}\int_0^{2\pi} Q(\eta\underline{e}(\theta)+2\pi\underline{\nu})\,\mathcal{G}_{\alpha\beta}\left(\frac{1}{h}[\![\eta\underline{e}(\theta)+2\pi\underline{\nu}]\!]\right)d\theta.$$

In the term $E^{(1)}\hat{\mathcal{K}}_{\alpha\beta}$, we have

(3.41)
$$\mathcal{A}w(\eta) = \sum_{\underline{\nu}}\frac{1}{2\pi}\int_0^{2\pi} d\alpha\; w_o(|\eta\underline{e}(\alpha)+2\pi\underline{\nu}|)$$
$$= \sum_{\underline{\nu}}\frac{1}{2\pi}\int_0^{2\pi} d\theta\; w_o\left([|2\pi\underline{\nu}|^2+\eta^2-4\pi|\underline{\nu}|\cos\theta]^{1/2}\right)$$
$$= \sum_{\underline{\nu}}\tilde{w}(\eta,|\underline{\nu}|)$$
$$\cong w_o(\eta)$$

for the functions w_o of interest (and, of course, for $\eta < 2\pi$.) Thus,

(3.42)
$$E(\hat{\mathcal{K}}_{\alpha\beta}(\eta)) = 2\pi h^2\int_0^\infty x\,dx\,\mathcal{G}_{\alpha\beta}\left(\frac{x}{h}\right)\tilde{Q}(x)\,\tilde{w}(\eta,x)$$
$$+ h^2\sum_{\underline{\nu}\neq(0,0)}\int_0^{2\pi} Q(\eta\underline{e}(\theta)+2\pi\underline{\nu})\,\mathcal{G}_{\alpha\beta}\left(\frac{1}{h}|\eta\underline{e}(\theta)+2\pi\underline{\nu}|\right)d\theta$$
$$-\frac{(2\pi)^2}{NM}w_o(\eta)\,h^2\,\mathcal{G}_{\alpha\beta}(0).$$

In order to calculate the variance, we ignore the difference between $\mathcal{K}_{\alpha\beta}(\eta)$ and $h^2\mathcal{G}_{\alpha\beta}(\frac{\eta}{h})$, an approximation which must be good to within a few percent for the procedure to work at all. Then

(3.42)
$$\begin{aligned}
&\mathrm{cov}\,(\hat{\mathcal{K}}_{\alpha\beta}(\eta), \hat{\mathcal{K}}_{\alpha\beta}(\eta')) \\
&= \frac{1}{(2\pi)^2}\int_0^{2\pi}\!\!\int_0^{2\pi} d\theta\, d\phi\ \mathrm{cov}\,(\mathcal{K}^{+}_{\alpha\beta}(\eta\underline{e}(\theta)), \mathcal{K}^{+}_{\alpha\beta}(\eta'\underline{e}(\phi))) \\
&\cong \frac{h^4}{NM}\int_0^{2\pi}\!\!\int_0^{2\pi} d\theta\, d\phi \iint_{-\infty}^{\infty} d^2\xi\, \mathcal{S}^2_{\alpha\beta}(\tfrac{\xi}{h})\, w_0(\eta\underline{e}(\theta)-\underline{\xi})\, w_0(\eta'\underline{e}(\phi)-\underline{\xi}) \\
&\quad + \frac{(1+\delta_{\alpha\beta})h^2}{NM\,(2\pi)^3}\int_0^{2\pi} d\theta\, \mathcal{G}_{\alpha\beta}\left(\tfrac{1}{h}(\eta^2+\eta'^2-2\eta\eta'\cos\theta)^{1/2}\right),
\end{aligned}$$

where

$$\mathcal{S}^2_{\alpha\beta}(\tfrac{\xi}{h}) = \left(\mathcal{G}_{\alpha\alpha}(\tfrac{\xi}{h}) + \frac{M_\alpha}{(2\pi)^2}\right)\left(\mathcal{G}_{\beta\beta}(\tfrac{\xi}{h}) + \frac{M_\beta}{(2\pi)^2}\right) + \left(\mathcal{G}_{\alpha\beta} + \frac{\delta_{\alpha\beta}M_\alpha}{(2\pi)^2}\right)^2$$

(Recall that $M_\alpha = h^{-2}\Lambda^\alpha$ is the mean density for the parent point process on the sphere.) Rearranging slightly and changing the variables of integration in the last integral, we have

(3.43)

$$\operatorname{cov}\left(\hat{\mathcal{K}}_{\alpha\beta}(\eta), \hat{\mathcal{K}}_{\alpha\beta}(\eta')\right)$$
$$\cong \frac{(2\pi)^3 h^4}{NM} \int_0^\infty \xi \, d\xi \, \mathcal{S}^2_{\alpha\beta}\left(\frac{\xi}{h}\right) \tilde{w}(\eta, \xi) \, \tilde{w}(\eta', \xi)$$
$$+ \frac{(1+\delta_{\alpha\beta}) h^2}{(2\pi)^2 NM} \cdot \frac{2}{\pi} \int_{|\eta-\eta'|}^{\eta+\eta'} \mathcal{G}_{\alpha\beta}\left(\frac{u}{h}\right) \frac{u \, du}{\sqrt{u^2+(\eta-\eta')^2}\sqrt{(\eta+\eta')^2-u^2}}$$

and

(3.44)
$$\operatorname{var}\left(\hat{\mathcal{K}}_{\alpha\beta}(\eta)\right) \cong \frac{(2\pi)^3 h^4}{NM} \int_0^\infty \xi \, d\xi \, \mathcal{S}^2_{\alpha\beta}\left(\frac{\xi}{h}\right) \tilde{w}^2(\eta, \xi)$$
$$+ \frac{(1+\delta_{\alpha\beta}) h^2}{(2\pi)^2 NM} \cdot \frac{2}{\pi} \int_0^{2\eta} \mathcal{G}_{\alpha\beta}\left(\frac{u}{h}\right) \frac{du}{\sqrt{(2\eta)^2-u^2}} .$$

We must next turn our attention to a discussion of the function w(x) and its Fourier coefficients $w_{\boldsymbol{\nu}}$. We have seen that it is analytically advantageous to choose the $w_{\boldsymbol{\nu}}$ as values $W(|\boldsymbol{\nu}|)$ of the Fourier transform of some smooth function $w_o(x)$. We should like $w_o(x)$ to be as highly peaked as is practical without increasing the variance unduly, and must remember that $W(\boldsymbol{\nu})$ must vanish for $|\boldsymbol{\nu}|>m$. Were it not for this last point, the use of Gaussian window pairs would be very convenient; the rate of decrease of the Gaussian at large values of the argument, however, suggests a suitable and very useful approximation. Consider the weights

(3.45)

$$w_{\underline{\nu}} = \begin{cases} W'(|\underline{\nu}|) = \left(1 - \frac{\nu^2}{9\sigma^2}\right)^2 e^{-\frac{\nu^2}{2\sigma^2}} & , \ \nu < 3\sigma \\ 0 & , \ \nu > 3\sigma \end{cases}$$

The function $W'(j)$ vanishes with its gradient on the circle $j = 3\sigma$, and does not exceed 2×10^{-4} outside this circle. The integral from 3σ to infinity does not exceed 10^{-3} of the total mass, and so the Fourier transform is essentially determined by the behavior within 3σ of the origin. The interesting fact is that the Gaussian $W(j) = \{e^{-j^2/2\sigma^2(.8225)^2}\}$ fits $W'(j)$ to within .007 everywhere, and has a total mass with .9% of that of $W'(j)$. This could doubtless even be improved somewhat; the point is that we can calculate with an almost-Gaussian set of weights which vanish outside $m=3\sigma$, but can do analysis with the simpler Gaussian set with a very small error. Suppose, then, that

(3.46)

$$w_0(x) = \frac{1}{2\pi\alpha^2} e^{-x^2/2\alpha^2} \quad , \quad \alpha \ll \pi .$$

Then

(3.47)
$$\tilde{w}(\xi,\eta) = \frac{1}{(2\pi)^2\alpha^2}\int_0^{2\pi} e^{-\frac{(\xi^2+\eta^2-2\xi\eta\cos\theta)}{2\alpha^2}}\,d\theta$$
$$= \frac{1}{2\pi\alpha^2} e^{-\frac{(\xi^2+\eta^2)}{2\alpha^2}} \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{\xi\eta\cos\theta/\alpha^2}\,d\theta$$
$$= \frac{1}{2\pi\alpha^2} e^{-\frac{(\xi^2+\eta^2)}{2\alpha^2}} I_0\left(\frac{\xi\eta}{\alpha}\right).$$

If ξ or η is large compared to α, $\xi^2+\eta^2-2\xi\eta\cos\theta \approx (\xi-\eta)^2+\xi\eta\theta^2$ over the range where the integrand is large enough to contribute significantly; this leads immediately to the large-argument asymptotic form

(3.48)
$$\tilde{w}(\xi,\eta) \sim \frac{1}{(2\pi)^{3/2}\alpha}\,\frac{1}{\sqrt{\xi\eta}}\, e^{-\frac{(\xi-\eta)^2}{2\alpha^2}}.$$

If either ξ or η is small, $\tilde{w}(\xi,\eta)$ differs little from w_o evaluated at the larger argument; in fact, as $\xi\eta/\alpha^2 \to 0$, we get, by expanding the exponential in the integral for I_o,

(3.49)
$$\tilde{w}(\xi,\eta) \sim \frac{1}{2\pi\alpha^2} e^{-\frac{(\xi-\eta)^2}{2\alpha^2}}\left(1-\frac{\xi\eta}{\alpha}+\frac{3}{4}\left(\frac{\xi\eta}{\alpha}\right)^2-\frac{5}{12}\left(\frac{\xi\eta}{\alpha}\right)^3+\dots\right).$$

Making use of a relation of Kummer (Watson(17), p. 104), we can write

(3.50)

$$\tilde{w}(\xi,\eta) = \frac{1}{2\pi\alpha^2} e^{-\frac{(\xi-\eta)^2}{2\alpha^2}} {}_1F_1\left(\tfrac{1}{2};1;-\frac{2\xi\eta}{\alpha^2}\right).$$

Since ${}_1F_1\left(\frac{1}{2}, 1, -\frac{2\xi\eta}{\alpha^2}\right)$ is a smooth function over the range where the exponential factor peaks strongly, we can now do the variance integral approximately by taking $\left\{\mathcal{S}_{\alpha\beta}(\eta/h)\,{}_1F_1\left(\frac{1}{2};1;-2\eta^2\alpha^{-2}\right)\right\}^2$ out of the integral and performing the integration over the exponential. We obtain

(3.51)

$$\mathrm{var}\left(\hat{\mathcal{K}}_{\alpha\beta}(\eta)\right)$$
$$\cong \frac{h^4\sqrt{\pi}}{\alpha\eta NM}\mathcal{S}^2_{\alpha\beta}(\eta/h) + \frac{(1+\delta_{\alpha\beta})h^2}{(2\pi)^2NM}\cdot\frac{2}{\pi}\int_0^{2\eta}\mathcal{G}_{\alpha\beta}\left(\frac{u}{h}\right)\frac{du}{\sqrt{(2\eta)^2-u^2}}$$

for $\eta \gg \alpha$. For η less than α, the asymptotic form used for w is very poor, and we can get a good estimate by using the form with $I_0\left(\frac{\xi\eta}{\alpha^2}\right)$:

(3.52)

$$\mathrm{var}\left(\hat{\mathcal{K}}_{\alpha\beta}(\eta)\right)$$
$$\cong \frac{h^4\pi}{NM\alpha^2}\mathcal{S}^2_{\alpha\beta}(\eta/h) + \frac{(1+\delta_{\alpha\beta})h^2}{(2\pi)^2NM}\cdot\frac{2}{\pi}\int_0^{2\eta}\mathcal{G}_{\alpha\beta}\left(\frac{u}{h}\right)\frac{du}{\sqrt{(2\eta)^2-u^2}}$$

for $\eta \ll \alpha$. The estimates agree for $\eta = \alpha/\sqrt{\pi}$, and the error is only a few percent in using (3.51) for $\eta \geq \alpha/\sqrt{\pi}$, (3.52) for $\eta < \alpha/\sqrt{\pi}$.

The covariance can be treated similarly. Note that for $|\xi - \eta|$ greater than 2α or so, the contribution from the first term is very small, but that from the second does not decrease appreciably. Thus the degree to which estimates for the spectral density in widely separated frequency bands are uncorrelated is limited by the point nature of the process, and this limitation can be serious. To see more clearly the various dependences on the parameters of the problem, we set

(3.53) $$\hat{\mathcal{G}}_{\alpha\beta}(\xi) = \frac{1}{h^2}\hat{\mathcal{K}}_{\alpha\beta}(h\xi),$$
$$L_1 = hN$$
$$L_2 = hM$$
$$L' = hM = \frac{3h}{.8225\alpha} = \frac{3.64h}{\alpha},$$

so that

(3.54) $$\operatorname{var}\left(\hat{\mathcal{G}}_{\alpha\beta}(\xi)\right) \cong \frac{1}{L_1 L_2}\min\left(\frac{.49L'}{\xi}, .24L'^2\right) \cdot \left\{\left(\mathcal{G}_{\alpha\alpha}(\xi) + \frac{M_\alpha}{(2\pi)^2}\right)\left(\mathcal{G}_{\beta\beta}(\xi) + \frac{M_\beta}{(2\pi)^2}\right) + \left(\mathcal{G}_{\alpha\beta}(\xi) + \frac{\delta_{\alpha\beta}M_\alpha}{(2\pi)^2}\right)\right\}$$
$$+ \frac{1+\delta_{\alpha\beta}}{(2\pi)^2 L_1 L_2}\frac{2}{\pi}\int_0^1 \mathcal{G}_{\alpha\beta}(2\xi u)\frac{du}{\sqrt{1-u^2}}.$$

Note that in this approximation the mesh size <u>does not appear</u>. Thus h must simply be sufficiently small that the attenuation of high frequencies through $\tilde{Q}(\eta)$ and aliasing are not too serious. This means simply that it must be small compared to the correlation length on the sphere. L', on the other hand, measures the length over which we consider the correlation nonvanishing, and so must be substantially greater than the correlation length. We must therefore expect most of the important details of the spectrum to occur at frequencies greater than $1/L$; so the first choice in the minimum will generally be the one of interest. The <u>sample size</u> L_1L_2 must then be as large as is necessary to bring the variance within the desired limits.

It is also of interest to note the effect of the magnitude interval Δm. The mean densities M_α go essentially as Δm; the spectra $\mathcal{G}_{\alpha\beta}(\xi)$, as $(\Delta m)^2$. For small Δm, then, the "point" terms (the $M_\alpha M_\beta$ product is the first term and the second term) dominate while for Δm large the variance is dominated by the terms in $\mathcal{G}_{\alpha\alpha}\mathcal{G}_{\beta\beta} + (\mathcal{G}_{\alpha\beta})^2$, which is the form for continuous normal processes. It is clear that this last condition is the optimal one, so the magnitude interval should be as large as possible consistent with satisfactory resolution.

It is perhaps expedient at this point to introduce a few conceptual devices to facilitate heuristic consideration of the theory we have developed, and to perform a few very crude approximate calculations to discover just what one is up against in applying the theory to real counts.

Consider first a monovariate, one-dimensional point process N(s). Let its mean density be λ and its covariance density $\lambda^2 f(\tau)$. Then the spectrum is

(3.55)
$$\phi(k) = \frac{\lambda^2}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx \, .$$

We form an estimate $\phi^{\dagger}$ of the spectrum in the manner developed in this chapter, and we find

(3.56)
$$\operatorname{var}\left(\phi^{\dagger}(k)\right) \cong C \frac{L'}{L} \left(\phi(k) + \frac{\lambda}{2\pi}\right)^2 + \frac{1}{2\pi L}\left(\phi(2k) + \phi(0)\right),$$

where C is of order unity (it depends on the shape of w(x)), L is the length of the sample, and $L' = mh$ is the length over which the weights w_ν are nonvanishing. Now

(3.57)
$$\phi(0) = \frac{\lambda}{2\pi} \left\{ \lambda \int_{-\infty}^{\infty} f(x)\, dx \right\} .$$

The integral $\int_{-\infty}^{\infty} f(x)\, dx$ (and its generalizations to higher dimensions) we shall call the "covariance measure". The product of this quantity with the mean density is a dimensionless quantity n_c which is a measure of the clustering tendency, and is at least vaguely related to the "mean cluster population". In an extremely näive model in which cluster centers are distributed in a uniform Poisson fashion, galaxies are distributed about the cluster center in a Poisson fashion with parameter density $Wg(x)$ (where $g(x)$ is a fixed function of distance from the cluster center and W is a stochastic variable which varies from cluster to cluster) the mean number of galaxies per cluster is $kE(W)$, and the quantity n_c is $kE(W^2)/E(W)$, where k is a constant.

If $f(x) > 0$ everywhere, then $\phi(k)$ is bounded by its value at zero, and we take $\phi(0)$ as a rough measure of the size of $\phi(k)$. But $\phi(0) = \frac{\lambda}{2\pi} n_c$, so

(3.58)
$$\mathrm{var}(\phi^+(k)) \propto C\frac{L'}{L}\left(\frac{\lambda}{2\pi}\right)^2(n_c+1)^2 + \frac{1}{2\pi L}\cdot\frac{\lambda}{2\pi}\cdot 2n_c$$

$$= \frac{\lambda}{(2\pi)^2 L}\left(C\lambda L'(n_c+1)^2 + 2n_c\right).$$

Now $(n_c+1)^2 > 2n_c$ always; the quantity $\lambda L'$ is just the mean number in the length L', so for <u>most</u> problems of moderate count density, the first term dominates easily. In the covariance, however, recall that the first term disappears for k appreciably different from k', and we must keep the second term. For galaxies, as we shall see,

n_c is of the order of fifty, so the point nature of the process will not play a dominant role for any but the rarest kinds of objects.

Let us now estimate the projected covariance and spectrum of the galactic distribution, taking very simple models for the spatial distribution. Limber (12) finds that the spatial covariance for all galaxies considered together is Gaussian to within his errors, and has the form

$$f(x) = \beta e^{-x^2/2\ell^2} \tag{3.59}$$

where β is about 20 and ℓ is of the order of 3 Mpc. He finds that the mean density is about .6/Mpc^3, using a Gaussian luminosity function with a standard deviation σ of 2 magnitudes. This luminosity function is almost certainly grossly in error, but it is simple to compute with and should give us at least order of magnitude estimates for the quantities involved. We choose a mean absolute magnitude M_o of -15. Letting $Z_m(\Omega)$ be the number of galaxies counted in solid angle between m-1/2 and m+1/2, we find, using (2.13), that

$$\begin{aligned} E(Z_m(\Omega)) &= \mu(\Omega)\cdot\tfrac{1}{2}\Lambda\cdot 10^{.6(m-M_o-25)}\, e^{\frac{1}{2}\left(\frac{3\sigma}{5c}\right)^2} \\ &\stackrel{D.f.}{\equiv} \mu(\Omega)\, M_m \end{aligned} \tag{3.60}$$

where Λ is in Mpc^{-3}, and $c=\log_{10} e=1/2.303\ldots$. We next do the projected covariance in the auxiliary function approximation, (2.21), and obtain the following results:

(3.61)
$$G_{mm'}(0) \cong \frac{\beta\Lambda^2 \ell}{5c\sigma\sqrt{2}}\, e^{-\frac{1}{4\sigma^2}(m-m')^2}\, 10^{\left(\frac{m+m'}{2}-M_0-25\right)}\, e^{\left(\frac{5}{2}\right)^2\left(\frac{\sigma}{5c}\right)^2},$$

$$I_n^{mm'} \overset{\mathrm{Def}}{\equiv} \int_0^\infty G_{mm'}(\gamma)\,\gamma^n\, d\gamma$$

$$\cong \frac{\Gamma_n \beta \ell^{n+2}\Lambda^2}{5c\sigma\sqrt{2}}\, e^{-\frac{1}{4\sigma^2}(m-m')^2}\, 10^{\frac{4-n}{5}\left(\frac{m+m'}{2}-M_0-25\right)}\cdot$$

$$\cdot\, e^{\left(\frac{4-n}{2}\right)^2\left(\frac{\sigma}{5c}\right)^2},$$

where $\Gamma_n = \int_0^\infty x^n e^{-\frac{x^2}{2}} dx = 2^{\frac{n-1}{2}}\,\Gamma\left(\frac{n+1}{2}\right).$

Let the quantity $10^{\frac{1}{5}\left(\frac{m+m'}{2}-M_0-25\right)} = \overline{r}\left(\frac{m+m'}{2}\right)$; this is the distance in Mpc that an object of absolute magnitude M_0 would have at apparent magnitude $\frac{m'+m}{2}$.

Then if $G_{mm'}(\gamma)$ were Gaussian, say

(3.62)
$$G_{mm'}(\gamma) = \kappa(m,m')\, e^{-\frac{\gamma^2}{2\gamma_0^2(m,m')}},$$

we would find

(3.63)
$$\frac{I_n^{mm'}}{G_{mm'}(0)} = \gamma_0^{n+1}(m,m')\,\Gamma_n .$$

From (3.61)

(3.64)
$$\frac{I_n^{mm'}}{G_{mm'}(0)} = \ell^{n+1}\,\Gamma_n\, \overline{r}^{-(n+1)}\left(\frac{m+m'}{2}\right) \exp\left\{-\left(\left(\frac{5}{2}\right)^2-\left(\frac{4-n}{2}\right)^2\left(\frac{\sigma}{5c}\right)^2\right\}.$$

Now $P_n = \left(\frac{5}{2}\right)^2 - \left(\frac{4-n}{2}\right)^2 = 9/4,\ \frac{16}{4},\ \frac{21}{4},\ \frac{24}{4} \ldots$ for $n = 0, 1, 2, 3, \ldots,$ and so we get a reasonable fit near the origin if we take $P_n = 2(n+1)$, and $\gamma_0 = \frac{\ell}{\overline{r}\left(\frac{m+m'}{2}\right)}\, e^{-2(\sigma/5c)^2} \approx \frac{\ell}{10\,\overline{r}\left(\frac{m+m'}{2}\right)}$ for $\sigma = 2$ magnitudes. The covariance is <u>not</u> Gaussian for large argument; the moments I_n indicate that it falls off at first <u>more</u> rapidly than a Gaussian, though ultimately it will fall off far less so. Moments through order 3, however, compare satisfactorily with a Gaussian form.

Then

(3.65)
$$G_{mm'}(\gamma) \cong \frac{\beta \ell \lambda^2}{5c\sigma\sqrt{2}}\, e^{-\frac{1}{4\sigma^2}(m-m')^2}\, \overline{\Upsilon}^{5}\left(\tfrac{m+m'}{2}\right) e^{\left(\frac{\sigma}{2c}\right)^2} e^{-\frac{\gamma^2}{2\gamma_0^2(m,m')}}$$
$$\cong M_m M_{m'} \cdot \frac{4\beta}{5c\sigma\sqrt{2}}\, e^{-\frac{1}{4\sigma^2}(m-m')^2} \cdot \gamma_0(m,m')\, e^{-\frac{\gamma^2}{2\gamma_0^2(m,m')}}.$$

For $\beta = 20$, $\sigma = 2$, the constant $\frac{4\beta}{5c\sigma\sqrt{2}}$ is about 15. Then

(3.66)
$$\begin{aligned} n_c &= \frac{1}{M_m}\int G_{mm'}(\gamma)\, d^2\gamma \\ &\cong 15\, M_m \cdot 2\pi \cdot \gamma_0^3 \\ &\cong 1500 \cdot \left(\tfrac{\ell}{10}\right)^3 \\ &\cong 40\,. \end{aligned}$$

If we thus work with, say, 5 classes of objects of roughly equal frequency, we expect n_c for each to be somewhere between 5 and 10, and the contribution to the variance from the point terms is small but perhaps not negligible. The correlation length on the sky is γ_o, which is $.3 \times 10^{-1/5(m-10)}$ radians, and the quantity $M_m \gamma_o^2$, which is the mean number in a square γ_o on a side, is $1.5 \times 10^{\frac{m-10}{5}}$, which is already 15 at m=15 (for which the correlation length is about 2 degrees). Since L' must be larger than γ_o, the quantity $CM_m L'^2$ - the two-dimensional analogue of $C\lambda L'$ in (3.58)-is very large for the cases of interest, and the heterodyne term is always small.

For the range m=18 to 20 (and perhaps to 21 with plate materials currently under test), the correlation length $\gamma_o(m,m)$ varies from 26 minutes (at 18) down to 10 minutes (at 20) and 6 minutes (at 21).

Analysis of counts at levels this faint will require much smaller cell sizes than have been used in the past; cells as small as one square minute can be used to advantage. If we consider the variance of the spectral estimate for $G_{mm}(\gamma)$ and neglect, for this argument, the point term $M_m/(2\pi)^2$, we find from (3.54) that for square regions

(3.67) $$\sigma^2_{m,m'}(\xi) = \frac{L'}{L_1^2\,\xi}$$

for frequencies of interest ($\xi \gtrsim 2/L'$); here $\sigma^2_{m,m'}(\xi) = \dfrac{\mathrm{var}\,\hat{\mathcal{G}}_{mm'}(\xi)}{(\mathcal{G}_{mm}(\xi))^2}$. Let us assume for the sake of argument that $\mathcal{G}_{mm'}(\xi)$ is approximately Gaussian;

(3.68) $$(2\pi)^2 \mathcal{G}_{mm'}(\xi) = M_m M_{m'} A \gamma_o^2(m,m')\, e^{-\frac{\xi^2\gamma_o^2}{2}},$$

with A in the vicinity of 1/2. If we choose $L'=10\gamma_o$, so that the weights $W_{\underline{v}}$ have a width about three times that of the covariance, then the spectrum is smoothed with a Gaussian of width $x_o = \frac{3.64}{L'} \approx \frac{1}{3\gamma_o}$. The aliasing frequency is π/h, and for $h=1'$ is large compared to relevant frequencies in $\mathcal{G}_{mm'}$. For a normalized standard deviation $\sigma_{mm'}(\xi)$ of .10 at the low end of the spectrum, we need $L_1 \; 7L' = 70\gamma_o$. At $m=18$, this is about thirty degrees; it decreases to 12 degrees at 20, and is seven degrees at 21. Multiplying by three will reduce the standard deviation to about .03. At $m=18$ this covers an appreciable part of the sphere, and the area must be subdivided in order to preserve any vestige of accuracy in the plane approximation. It is easy to see that this does not affect the accuracy, and provides an internal check on the variance estimate.

Let us also calculate roughly the variances in the number counts themselves. We have, from (2.14), (2.15),

(3.69)
$$var(Z_m(\Omega)) = \iint_{\Omega\,\Omega} G_{mm}(|\underline{\gamma}-\underline{\gamma}'|)\, d^2\gamma\, d^2\gamma' + \mu(\Omega) M_m .$$

Let us suppose that the diameter of Ω is large compared to γ_o. In this case, we can write approximately, from (3.65),

(3.70)

$$\begin{aligned}\text{var}\,(Z_m(\Omega)) &= \mu(\Omega)\left(M_m + \int_\Omega G_{mm}(r)\,d^2r\right)\\ &\cong \mu(\Omega)M_m\left(1 + 2\pi M_m \frac{4\beta}{5c\sigma\sqrt{2}}\,r_0^3\right)\\ &\cong \mu(\Omega)M_m\,(1+ N_c)\\ &\cong 40\,\mu(\Omega)M_m .\end{aligned}$$

The standard deviation is thus $\sqrt{40}$ or about 6.3 times larger than for a random distribution of galaxies. To get counts to two percent at m=20, which one would like in order to say something meaningful about the cosmology, one needs

(3.71)

$$\frac{(\text{var}\,Z_m(\Omega))^{1/2}}{\mu(\Omega)M_m} = 6.3\,\sqrt{1/\mu(\Omega)M_m}$$

$$= .02$$

Since $M_m \simeq 14\times10^{.6(m-10)}$, we need an area about .1 radians, or about 6°, on a side. At m=18, where we need 1% accuracy, one needs counts (with no systematic errors) over a 30° square.

IV. COUNT STATISTICS IN EXPANDING NON-EUCLIDEAN UNIVERSES

We are now ready to generalize the results of Chapter II to non-static universes in (possibly) curved space-times. We assume, again on the basis of the cosmological principle, that there is a universally defined cosmic time τ, and that surfaces of constant τ are three-dimensional manifolds with constant curvature. In order to apply the statistical description we have developed, it is necessary that we make one more seeingly ad hoc assumption, but this assumption is implicit in the usual interpretation of the cosmological principle.

(B). The correlation length is small compared to (1) the radius of curvature and (2) the Hubble length c/H.

The first assures us that within the effective support radius of the covariance density, space is essentially flat; the second, that little "evolution" occurs while light is traversing this distance. These two numbers (the radius and c/H) are comparable for the most promising simple models, except those for which the radius is infinite. The values of 3-5 Mpc obtained by, e.g., Limber (12) certainly satisfy this requirement.

With these assumptions, we can regard the gravitational potentials of the galaxies as small perturbations superposed on a homogeneous metric of the

Robertson type (Irvine, (18)). We shall use so-called "intrinsic" comoving coordinates, for which

(4.1) $$ds^2 = d\tau^2 - R^2(\tau)\left[\frac{d\rho^2}{1-k\rho^2} + \rho^2 d\gamma^2\right]$$

where $k = \pm 1$ or 0, according to whether the spatial sections have positive, negative, or vanishing curvature, and $d\gamma^2 = d\theta^2 + \sin^2\theta d\phi^2$. These coordinates possess the property that a sphere of coordinate radius ρ at time τ has proper area $4\pi R^2(\tau)\rho^2$.

Robertson (19) first showed rigorously that the flux at the origin from a source of intrinsic brightness L radiating isotropically at coordinate distance ρ is

(4.2) $$\mathcal{F} = \frac{L}{4\pi R_2^2\rho^2} \frac{R_1^2}{R_2^2} ,$$

where $R_1 = R(\tau)$ at emission; $R_2 = R(\tau)$ at absorption. Since the redshift $z = \frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{R_2 - R_1}{R_1}$,

$$\mathcal{F} = \frac{L}{4\pi R_2^2}(1+z)^{-2}.$$

The equation of a radial "light track" is given by $d\gamma = ds = 0$, or

(4.3) $$\frac{d\tau}{R(\tau)} = \frac{-d\rho}{\sqrt{1-k\rho^2}} .$$

This gives a one-to-one relationship between the "emission time" τ_e and the coordinate radius, once the reception time is given. If τ_0 is the present epoch, then θ, ϕ, and $t = \tau_o - \tau_e$ from a suitable set of coordinates for the light cone.

Consider first the statistics in an evolving relativistic universe. We make the assumption (perhaps incorrectly even in this context) that no galaxies are forming or are disappearing. Thus the stochastic process at one epoch is an evolved version of one preceding, but the evolution in time is deterministic in that the number of galaxies does not change - though the galaxies may move, and thus the covariance density may change with time.

Let $\lambda_\beta(M, \tau)$ be the proper mean density of β-galaxies of magnitude M at cosmic time τ. Suppose that evolutionary changes are the same for all β-galaxies, and that

(4.4) $$M = M_o - \epsilon_\beta(t) \quad ,$$

Where M_o is the magnitude at the present epoch (and so $\epsilon_\beta(0) = 0$; it is presumably positive for $t > 0$). Then, letting $\lambda^o_\beta(M) = \lambda_\beta(M, \tau_o)$,

$$\lambda_\beta(M, \tau) = \lambda^o_\beta(M + \epsilon_\beta(t)) \left\{ \frac{R(\tau_o)}{R(\tau)} \right\}^3$$

$$= \lambda^o_\beta(M + \epsilon_\beta(t)) (1+z)^3.$$

From (4.2) we obtain the relation between apparent and absolute magnitudes:

(4.5) $$m = M + 5 \log (\rho R(\tau_0)) + k_\beta(z) + 5 \log (1+z),$$

where $k_\beta(z)$ is the so-called "k-correction" which arises from the effect of redshift on finite-bandpass measurements (Humason, Mayall, and Sandage, (20)). Let

(4.6)
$$5 \log \mathcal{R}_\beta(t) = 5 \log (R_0 \rho(t)) + k_\beta(z(t)) + 5 \log (1+z) - \epsilon_\beta(t)$$

Then

(4.7) $$m = M_0 + 5 \log \mathcal{R}_\beta(t).$$

The proper volume element in $d\Omega$ and dt is

(4.8)
$$dV = R^3(\tau_0 - t) \frac{d\rho(t)\rho^2(t)}{\sqrt{1 - k\rho^2}} d\Omega = \frac{\rho^2(t)}{(1+z(t))^2} R_0^2 \, dt \, d\Omega$$

Thus the analogue of (2.11) for an evolving universe is

(4.9)

$$E(d \, Z'_\beta(\Omega, m^\circ)) = \mu(\Omega) \, dm^\circ \int_0^\infty \lambda_\beta(M_\beta(m^\circ; t) \, \tau_0 - t) \rho^2(t) R^2(\tau_0 - t) dt,$$

where

(4.10)

$$M_\beta(m, t) = m - 5 \log (\rho(t) R_o) - k_\beta(z) - 5 \log (1+z)$$

(We set $R(\tau) \equiv 0$ for all times previous to the singular "creation" if there is one in the model we are considering; if so, let this time be $\tau = 0$, and extend integrals in t to τ_o only, instead of infinity.) Using (4.4) and (4.6), we obtain

(4.11)

$$E(d_{m^o} Z(\Omega, m^o)) = \mu(\Omega) dm^o R_o^2 \int_0^\infty \{\lambda_\beta(m^o - 5 \log R_\beta(t)) \cdot (1+z(t)) \rho^2(t)\} dt ,$$

and, letting

(4.12)

$$\lambda_{\beta j}(x) = \int_{-\infty}^{\infty} C_{\beta j}(m) \lambda_\beta^o (m - 5 \log x) \, dm ,$$

relation (2.31) for the mean number of βj galaxies per counting square becomes

(4.13) $$\Lambda_{\beta j}^{(R)} = h^2 R_o^2 \int_0^\infty \lambda_{\beta j}(R_\beta(t)) (1+z(t)) \rho^2(t) dt$$

We run into a slight difficulty in forming the covariance density of $\mathcal{Z}_{\beta j}(\Omega)$, since we need the three-dimensional covariance density for points at <u>different times</u>. Since, however, the correlation length is presumed short compared to the Hubble distance, the metric changes little during time intervals comparable with the correlation length. The peculiar velocities of galaxies are so small compared with the velocity of light that the galaxies within one correlation radius cannot appreciably change their configuration during this time (of the order of 10^8 years), so we can use the three-dimensional covariance for some intermediate time (we shall use the geometric mean for analytical convenience) at the proper distance corresponding to the given coordinate differences at that mean time. With this approximation, and letting $\lambda_\beta(M,\tau)\,\lambda_{\beta'}(M',\tau)\,g_{\beta\beta'}(\alpha,\tau)$ be the covariance density at time τ (we retain Postulate A), we obtain

(4.14)

$$G_{\beta j\beta' j'}(\gamma) = \int_0^\infty\!\!\int_0^\infty \{\rho^2(t)\rho^2(t')\,R^2(\tau_0-t)\,R^2(\tau_0-t')\,dt\,dt'$$

$$\cdot\lambda_{\beta j}(\mathcal{R}_\beta(t))\,\lambda_{\beta' j'}(\mathcal{R}_\beta(t'))\,g_{\beta\beta'}(\alpha,\tau_0-x)\,(1+z(x))^6\}.$$

where $H\alpha$ is the proper distance between points on radial rays separated by an angle γ at coordinates $\rho(t)$, $\rho(t')$ at time x, and x is between t and t'. For small angles and distances much smaller than $R(\tau_0 - x)$, (4.1) and (4.9) yield

(4.15) $\alpha^2(t,t',\gamma) = (t-t')^2 + \rho(t)\rho(t')R^2(\tau_0 - x)\gamma^2$

For small t, $\rho(t) \propto t$; for larger t, if $(t-t')$ is small compared to t and t', then $\rho(t)\rho(t')$ is approximately equal to $\rho^2(x)$; in any case, then, if $x = \sqrt{tt'}$, $\rho(t)\rho(t') \cong \rho^2(x)$. Let $\Delta = (t-t')$; then

(4.16) $$\alpha^2(x, \Delta, \gamma^2) \cong \Delta^2 + \rho^2(x) R^2(\tau_0 - x)\gamma^2.$$

It is also clear that we can replace $R^2(\tau_0 - t)R^2(\tau_0 - t')$ by $R^4(\tau_0 - x)$ in (4.14). Thus, using (2.20),

(4.17)

$$G_{\beta j \beta' j'}(\gamma) = R_0^4 \int_0^\infty\int_0^\infty dt\,dt' \{ \rho^4(x)\,(1+z(x))^2 \cdot$$

$$\cdot \lambda_{\beta j}(\mathcal{R}_\beta(t))\lambda_{\beta' j'}(\mathcal{R}_{\beta'}(t'))\, g_{\beta\beta'}(\sqrt{\Delta^2 + \rho^2(x)R^2(\tau_0 - x)\gamma^2})\},$$

$$\cong R_0^4 \int_0^\infty dx \int_{-\infty}^\infty d\Delta \{ \frac{\rho^4(x)\,(1+z(x))^2}{(1+\Delta^2/4x^2)} \cdot$$

$$\cdot \lambda_{\beta j}(\mathcal{R}_\beta(t^+))\lambda_{\beta' j'}(\mathcal{R}_{\beta'}(t^-))\, g_{\beta\beta'}(\sqrt{\Delta^2 + \rho^2(x)R^2(\tau_0 - x)\gamma^2})\},$$

where $t^{\pm} = x\left(\sqrt{1+\Delta^2/4x^2} \pm \frac{\Delta}{2x}\right)$. We now run into the grave difficulty that we do not know how $g_{\beta\beta'}(\alpha, \tau)$ changes with time. The problem is currently being investigated by Layzer and others, but to the author's knowledge, no results have yet been obtained. We are thus driven for the present to assume a rather simple variation, for which at least some theory exists. Let

(4.18) $$g_{\beta\beta'}(\alpha, \tau) = K(\tau) \sum_{i=0}^{N} a^{i}_{\beta\beta'} \left(\frac{\alpha}{\ell(\tau)}\right)^{2i} e^{-\frac{\alpha^2}{2\ell^2(\tau)}} .$$

Thus we suppose that only the scale changes, and together with the scale the "contrast" $K(\tau)$, which measures the strength of the clustering tendency.

It is necessary to digress here for a moment on the relationship between $K(\tau)$ and $\ell(\tau)$. Layzer (21) has obtained a "cosmological virial theorem" from which he derived a relation linking the contrast and correlation length for the mass density field in the fluid approximation. It is at first glance not obvious how the statistical process we are considering is related to the "smeared" density field, and perhaps this point deserves closer attention.

Theorem 10, Appendix II, establishes the existence of such a (proper) density field $\mu(\underline{x})$, (provided $N_\beta(s)$ is quasi-Poisson), and gives

(4.19) $$\mu(\underline{x}, \tau) = \sum_{\beta} \int_{-\infty}^{\infty} \mu_\beta(M_0)\, \rho_\beta(\underline{x}, M_0, \tau)\, dM_0 ,$$

where $\mu_\beta(M_o)$ is the mass of a β-galaxy whose magnitude the present epoch is M_o, and where $\rho_\beta(\underline{x}, M_o, \tau)$ is the fundamental for $N_\beta(s, \sigma)$. (It can easily be shown using Postulate A that

$$\rho_\beta(\underline{x}, M_o, \tau) = \lambda^o_\beta(M_o)\,\rho_\beta(\underline{x}, \tau)\left\{\int_{-\infty}^{\infty}\lambda^o_\beta(M_o)\,dM_o\right\}^{-1}$$

where $\rho_\beta(x, \tau)$ is the fundamental for $N_\beta(s, \tau)$, the number of all β-galaxies in S at τ.) Then

(4.20)

$$\begin{aligned}\mathrm{cov}\,(\mu(\underline{x},\tau), \mu(\underline{x},\tau)) &= \sum_\beta\sum_\gamma\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dM_o\,dM_o'\{\mu_\beta(M_o)\mu_\beta(M_o')\lambda^o_\beta(M_o)\lambda^o_\gamma(M_o')\cdot \\ &\qquad R_o^6\,(R(\tau))^{-6}\,g_{\beta\gamma}(0,\tau)\}\end{aligned}$$

but $g_{\beta\gamma}(0,\tau) = \kappa(\tau)\sum_{\beta,\gamma} a^o_{\beta\gamma}$, and

(4.21)

$$E(\mu(\underline{x},\tau)) = \sum_\beta\int_{-\infty}^{\infty}\mu_\beta(M_o)\lambda^o_\beta(M_o)\,R_o^3\,(R(\tau))^{-3}\,dM_o$$

Thus Layzer's "density contrast" α^2 is

(4.22)

$$\alpha^2 = \frac{\mathrm{cov}\,(\mu(\underline{x},\tau), \mu(x,\tau))}{[E\,(\mu(x,\tau))]^2} = \kappa(\tau)\sum_{\beta,\gamma} a^o_{\beta\gamma}$$

It is also easy to verify that $\ell(\tau)$ is a constant times his correlation length λ; he shows that $\frac{\alpha^2\lambda^2}{R^3}$ is approximately constant, so that $\kappa(\tau)\,\ell^2(\tau)\,(1+z)^3$ is also. If we let $\kappa(\tau_o) = 1$, $\ell(\tau_o) = \ell_o$, then

$\kappa(\tau) = \ell_0^2 (\ell(\tau))^{-2} (1+z)^{-3}$ in this approximation. Though no theory is yet available for the variation of $\ell(\tau)$, this relation at least reduces the number of unknown evolutionary factors by one. When a detailed theory becomes available, one should replace the $a^{i}_{\beta\beta'}$ by functions $a^{i}_{\beta\beta'}(\tau)$; the changes this generalization makes in the theory to be developed are easily incorporated and will not be developed further. Thus

(4.23)

$$G^{(R)}_{\beta j,\beta' j'} =$$

$$R_0^4 \ell_0^2 \int_0^\infty dx \Big\{ \rho^4(x) (1+z(x))^{-1} \ell^{-1}(\tau_0 - x) \cdot$$

$$\cdot \sum_{i=0}^{N} a^{i}_{\beta\beta'} \sum_{k=0}^{i} \binom{i}{k} \left(\frac{\rho(x) R(\tau_0 - x) \gamma}{\ell(\tau_0 - x)} \right)^{2k} \cdot$$

$$\cdot \quad \exp\left(\frac{-\rho^2(x) R^2(\tau_0 - x) \gamma^2}{2 \ell^2(\tau_0 - x)} \right) \mathcal{I}^{\beta j \beta' j'}_{i-k}(x) \Big\},$$

where

(4.24)

$$\mathcal{J}_{\nu}^{\beta j \beta' j'}(x) = \int_{-\infty}^{\infty} dy \Big\{ \frac{y^{2\nu} e^{-y^2/2}}{\sqrt{1 + y^2 \ell^2(\tau_0 - x)/4x^2}} \cdot$$

$$\lambda_{\beta j}(\mathcal{R}_{\beta}(t^+)) \lambda_{\beta' j'}(\mathcal{R}_{\beta'}(t^-)) \Big\}$$

precisely as in Chapter 2. We can take the limits as $\pm\infty$ in (4.24) because the λ factors cut off for large $\mathcal{R}_{\beta}(t)$. With $G_{\beta j \beta' j'}(\gamma)$, we can form the spectral density $\mathcal{G}_{\beta j, \beta' j'}(\eta)$ as in (2.30), and obtain

(4.25)

$$\mathcal{G}^{(R)}_{\beta j, \beta' j'}(\xi) = R_0^2 \ell_0^2 \int_0^{\infty} dx \, \Big\{ \rho^2(x)(1 + z(x)) \ell(\tau_0 - x)$$

$$\cdot \sum_{i=0}^{N} a^i_{\beta\beta'} \sum_{k=0}^{i} \binom{i}{k} Q_k \left(\frac{\xi \, \ell(\tau_0 - x)}{R(\tau_0 - x)\rho(x)} \right) \cdot$$

$$\cdot \exp\left(\frac{-\xi^2 \ell^2(\tau_0 - x)}{2 R^2(\tau_0 - x) \rho^2(x)} \right) \mathcal{J}_{i-k}^{\beta j \beta' j'}(x) \Big\}.$$

The spectral function for the counts by squares, $\mathcal{H}_{\beta j \beta' j'}(\eta)$ is formed from (4.25) by replacing $\mathcal{G}_{\beta j \beta' j'}(\xi)$ in (2.31) by $\mathcal{G}^{(R)}$.

The above development holds for an arbitrary evolving model satisfying the cosmological principle and in which no creation or destruction of galaxies takes place. Given $R(\tau)$, k, and τ_0, one can compute $\rho(t)$ and $z(t)$. The theory for $\epsilon_\beta(t)$ is a bit more difficult; a decent attempt can be made for elliptical galaxies, since presumably star formation ceased in them long ago. Spirals and irregulars are at present almost hopeless, though estimates have been made for the dependence of star formation rate on gas density (Schmidt, (22)), and perhaps reasonable estimates for ϵ_β will be possible soon. As noted before, a theory for $\ell(\tau)$ is under development at present.

Especially interesting for their simplicity are those models with vanishing pressure and cosmological constant; for these models $R(\tau)$, τ, $\rho(t)$ and $z(t)$ can all be expressed as simple trigonometric or hyperbolic functions of a parameter Θ, called the "development angle" (see, e.g., Sandage (23)). Computations using these models will be discussed later.

We now turn our attention to the steady-state models of Bondi and Gold (24) and Hoyle (8). As far as the present development is concerned the two approaches are the same, since we do not use the field equations and so it is really immaterial whether or not there are any. Hoyle and Narlikar(25) have recently proposed a version of the steady-state which violates Postulate (B) and is really not in the "spirit" of the cosmological principle at all; the statistical theory we develop cannot be applied to this version without extensive revision.

The metric in the steady-state model is

(4.26) $$ds^2 = d\tau^2 - \frac{e^{2H\tau}}{H^2}(d\rho^2 + \rho^2 d\gamma^2) \quad ;$$

The light-track equation is

(4.27) $$d\rho = -e^{-H\tau} H d\tau,$$

or

(4.28) $$\rho(t) = e^{-H\tau_0}(e^{Ht} - 1).$$

Since τ_0 is arbitrary here, we can set the scale of the parameter ρ by taking $\tau_0 = 0$, and (4.28) becomes

(4.29) $$\rho(\tau) = e^{Ht} - 1.$$

Similarly,

(4.30) $$(1 + z(t)) = \frac{R(\tau_0)}{R(\tau_0 - t)} = e^{Ht} = 1 + \rho(t),$$

or

(4.31) $$\rho(t) = z(t).$$

We take here a slightly different view on galactic evolution; galaxies still evolve in the steady-state, of course, but the population as a whole does not, since new ones are forming continuously. Thus $\lambda_\beta(M,\tau) \equiv \lambda_\beta^{ss}(M)$ is not a function of τ, and we need know no evolutionary details. The modulus relation is

(4.32)
$$m(t) = M + 5 \log \frac{\rho(t)}{H} + k_\beta(z) + 5 \log (1+z(t))$$
$$\overset{\text{Def}}{\equiv} M + 5 \log \mathcal{R}_\beta^{ss}(t);$$

$k_\beta(z)$ is independent of the model, of course, and is the same as before. The proper volume element per unit solid angle is

(4.33) $$\frac{e^{-3Ht}\rho^2 d\rho}{H^3} = \frac{e^{-2Ht}\rho^2(t)\,dt}{H^2},$$

and so the count expectation value is

(4.34) $$\Lambda_{\beta j}^{ss} = \frac{h^2}{H^2}\int_0^\infty \lambda_{\beta j}^{ss}\left(\mathcal{R}_\beta^{ss}(t)\right)(1-e^{-Ht})^2\,dt,$$

where $\lambda_{\beta j}^{ss}(x)$ is formed as before, using $\lambda_\beta^{ss}(M)$.

Here

(4.35) $$\alpha^2(t\,,t'\,,\gamma) \cong \Delta^2 + \gamma^2(1-e^{-Hx})^2 H^{-2}\,,$$

so the covariance density becomes

(4.36)

$$G_{\beta j\beta' j'}(\gamma) \cong \int_0^\infty dx \int_{-\infty}^{\infty} \frac{d\Delta}{\sqrt{1+\Delta^2/4x^2}} \Big\{ \lambda^{ss}_{\beta j}(R^{ss}_{\beta}(t^+))\cdot$$

$$\cdot\, \lambda^{ss}_{\beta' j'}(R^{ss}_{\beta'}(t^-))\, H^{-4}(1-e^{-Hx})^4 \cdot$$

$$\cdot\, g^{ss}_{\beta\beta'}\left(\sqrt{(\Delta^2+\gamma^2(1-e^{-Hx})^2/H^2)}\right)\Big\}\,.$$

Inserting (4.18) ($\kappa(\tau) \equiv 1$, $\ell(\tau) \equiv \ell$) for $g^{ss}_{\beta\beta'}(\alpha)$,

we obtain

(4.37) $$G_{\beta j\beta' j'}(\gamma) = \int_0^\infty dx \Big\{ e^{-\frac{\gamma^2(1-e^{-Hx})^2}{2H^2\ell^2}} \cdot$$

$$\cdot \sum_{i=0}^{N} a^{i}_{\beta\beta'} \sum_{k=0}^{i} \binom{i}{k} \left(\frac{\gamma(1-e^{-Hx})}{H\ell}\right)^{2k} \cdot$$

$$\cdot\, \ell H^{-4}(1-e^{-Hx})^4\, \tilde{I}^{\beta j\beta' j'}_{i-k}(x)\,,$$

where $\tilde{\mathcal{J}}_{i-k}^{\beta j \beta' j'}$ is defined as $\mathcal{J}_{i-k}^{\beta j \beta' j'}$ is in (4.24) except that ℓ is constant and the $\lambda_{\beta j}$'s and $\mathcal{R}_{\beta}(t)$'s are replaced by the steady-state ones. $\mathcal{H}^{(ss)}_{\beta j \beta' j'}(\eta)$ is formed as before.

If we limit our attention to distant galaxies (say more distant than a few times ℓ_o - 50 Mpc or so - the expressions can be rewritten in terms of the auxilliary functions introduced in Chapter 2 and become much simpler. These auxiliary functions now are time-dependent, of course, but in a very simple fashion;

(4.38)

$$A_{\beta\beta'}\left(\frac{d}{\ell(\tau)},\tau\right) = \int_{-\infty}^{\infty} g_{\beta\beta'}\left((x^2+d^2)^{1/2},\tau\right)\frac{dx}{\ell(\tau)}$$

$$= k(\tau)\int_{-\infty}^{\infty}\sum_{i=0}^{N} a^i_{\beta\beta'}\left(\frac{x^2+d^2}{\ell^2(\tau)}\right)^i e^{-\frac{x^2+d^2}{2\ell^2(\tau)}}\frac{dx}{\ell(\tau)}$$

$$= \frac{\ell_o^2}{\ell^2(\tau)(1+z)^3}\,A^o_{\beta\beta'}\left(\frac{d}{\ell(\tau)}\right),$$

where $A^{o}_{\beta\beta'}$ with no explicit time dependence is the function at the present epoch, and is also, of course, a polynomial times a Gaussian:

(4.39)

$$A^{o}_{\beta\beta'}(x) = \sqrt{2\pi}\, e^{-x^2/2} \sum_{k=0}^{N} x^{2k} \sum_{i=k}^{N} \qquad (2i-2k-1)!!$$

$$\stackrel{\text{Def.}}{\equiv} e^{-x^2/2} \sum_{k=0}^{N} b^{k}_{\beta\beta'} x^{2k}$$

It is not necessary to work in terms of the a's at all, but directly with the b's; it will turn out that in all the statistics we shall do, the function $A_{\beta\beta'}(x)$ enters directly, rather than the (seemingly) more fundamental spatial covariance. The spatial covariance can, of course, always be found by solving for the a's, but one cannot expect very high accuracy, especially for the higher orders. This approximation corresponds to taking the λ terms out of the $\mathcal{J}$ integrals as $\lambda_{\beta j}(\mathcal{R}_{\beta}(x))$, and supposing that $2x \gg \ell$, so that the square root in the denominator becomes unity. Then (4.23) goes over into

(4.40)

$$G^{(R)}_{\beta j\beta'j'}(\gamma) \cong R_0^4 \ell_0^2 \int_0^\infty dx \{\rho^4(x)(1+z(x))^{-1}\ell^{-1}(\tau_0 - x)$$

$$\cdot \overset{\circ}{A}_{\beta\beta'}\left(\frac{\rho(x)\,R(\tau_0-x)\gamma}{\ell(\tau_0-x)}\right)\lambda_{\beta j}(R_\beta(x))\lambda_{\beta' j'}(R_{\beta'}(x))\}$$

for the Friedmann universes. The spectral density is

(4.41)

$$\mathcal{G}_{\beta j\beta'j'}(\eta) \cong R_0^2 \ell_0^2 \int_0^\infty dx \{\rho^2(x)(1+z(x))\,\ell(\tau_0 - x)$$

$$\cdot \overset{\circ}{\mathcal{A}}_{\beta\beta'}\left(\frac{\ell(\tau_0-x)\eta}{\rho(x)\,R(\tau_0-x)}\right)\lambda_{\beta j}(R_\beta(x))\lambda_{\beta' j'}(R_{\beta'}(x))\},$$

where, as before,

(4.42)

$$\mathcal{A}^{o}_{\beta\beta'}(y) = \int A_{\beta\beta'}(x)\, e^{i\underline{x}\cdot\underline{y}} \frac{d^2x}{(2\pi)^2}$$

$$= \sum_{k=0}^{N} b^{k}_{\beta\beta'} \frac{2^k k!}{2\pi} L_k\left(\frac{y^2}{2}\right) e^{-y^2/2}$$

$$= \sum_{k=0}^{N} b^{k}_{\beta\beta'} Q_k(y) e^{-y^2/2} .$$

is the auxi liary spectral function.

For the steady-state, the covariance density in this limit is

(4.43)

$$G^{ss}_{\beta j \beta' j'} = \int_0^\infty dx \{ H^{-4}(1-e^{-Hx})^4 \ell_0 \cdot$$

$$\cdot A^{o}_{\beta\beta'}\left(\frac{\gamma(1-e^{-Hx})}{H\ell}\right) \lambda_{\beta j}(R^{ss}_{\beta}(x))\, \lambda_{\beta' j'}(R^{ss}_{\beta'}(x)) \},$$

and the spectral density

(4.44)

$$\mathcal{G}^{ss}_{\beta j \beta' j'}(\eta) = \int_0^\infty dx \{ H^{-2} (1 - e^{-Hx})^2 \ell_0^3 \cdot$$

$$\cdot \mathcal{A}^{\circ}_{\beta\beta'}\left(\frac{H \ell \eta}{1 - e^{-Hx}}\right) \lambda^{ss}_{\beta j}(\mathcal{R}^{ss}_{\beta}(x)) \lambda^{ss}_{\beta' j'}(\mathcal{R}^{ss}_{\beta}(x)) .$$

V. THE COSMIC LIGHT I: STATISTICS

One quantity which is, at least in principle, easily obtained from a given cosmological model is the mean cosmic light; that is, the combined light from all the galaxies expressed as a mean flux over the celestial sphere. There are in practice a number of difficulties, chief among which are (as yet) unknown evolutionary corrections and, perhaps most important, the enormous uncertainty concerning the luminosity function. These points also affect our efforts here, of course, and one can only point out that no insurmountable difficulties seem to exist which would prohibit great improvement in the quality of the data in the future.

The measurement of this light is quite another question. Various estimates place it well below the night sky brightness, and competition from aurorae, the zodiacal light, and the light of faint stars in the galaxy makes the measurement very difficult if not impossible.

If one can determine anything about the spatial distribution of galaxies, however, one can say something about the structure of the fluctuations in this background light, and it is much easier to disentangle this structure from the other obscuring effects than to do so for an absolute mean level. We shall see, in fact, that knowledge of the spatial covariance density (or spectral density) is sufficient, with some other information obtainable from nearby galaxies which does not relate to their spatial distribution, to determine the second moments of the background fluctuations. We shall develop a sampling theory for this problem parallel to that developed for the counts, and finally, on the basis of the data of Limber (12), determine whether such a measurement is indeed feasible with presently existing instruments.

Let us first obtain an expression for the specific intensity of the background from the galaxies alone, as seen, say, from just outside the Galaxy. We assume that the orientation of the rotational axes of galaxies in space is random and independent of the spatial distribution. Let θ be the angle between the axis of the Galaxy and the line of sight, and ϕ the angle between the plane formed by the axis and line of sight and some reference direction. Let $dN_\beta(\underline{x}, M, \underline{n})$ be the number of β-galaxies in $d^3\underline{x}$ at $\underline{x}$ which lie in dM at M and whose axial direction is in $d\omega$ at $\underline{n}$. Then clearly

(5.1)

$$E(dN_\beta(\underline{x}, M, \underline{n}) = d^3x \frac{d\Omega}{4\pi} \cdot dM \, \lambda_\beta(M)$$

and we can write symbolically

(5.2)

$$\mathrm{cov}\ \left(dN_{\beta}\ (\underline{x},M\underline{n}),\ dN_{\beta'}\ (\underline{x}',M\ ,\underline{n}\ )\right) = d^3\underline{x}d^3\underline{x}'dMdM'd\Omega\, d\Omega' \cdot$$

$$\cdot \Big\{ \lambda_{\beta}\ (M)\lambda_{\beta'}\ (M')g_{\beta\beta'}(|\underline{x}-\underline{x}'|)\ (4\pi)^{-2} + \delta^3(\underline{x}-\underline{x}\ )\delta(M-M') \cdot$$

$$\cdot\ \delta^{(2)}(\underline{n}-\underline{n}')\ \frac{\lambda_{\beta}\ (M)}{4\pi}\ \Big\} \quad .$$

Now suppose that the monochromatic intensity emergent from an element of area $dA=d^2y$ on the disc of a given β -galaxy (See Figure 5.1) is $I_{\beta}\ (\theta,\underline{y},\nu,M)dA$, where $\underline{y}$ is the (two-dimensional) coordinate vector of the point at which the line of sight intersects the plane of the galaxy, in a coordinate system one of whose axes is the line of nodes and the other of which lies in the plane of the Galaxy perpendicular to the line of nodes (See Figure 5.2). We suppose that the spectrum of all β-galaxies is the same and that the distribution of intensity simply scales in a uniform manner when we go from a bright β-galaxy to a faint one; that is, I_{β} has the form

(5.3)
$$I_{\beta}\ (\theta,y,\nu,M) = L_{\beta\nu}\ 10^{-.4M}\frac{1}{[S_{\beta}(M)]^2}\ \mathcal{J}_{\beta}\left(\theta,\ \underline{y}/S_{\beta}(M),\ \nu\right)$$

where S_{β} (M) scales the size of a β-galaxy appropriately with M, and $4\pi L_{\beta\nu}\ 10^{-.4M}$ is the total emission from the galaxy. Note that in general the dimensionless factor $\mathcal{J}_{\beta}$

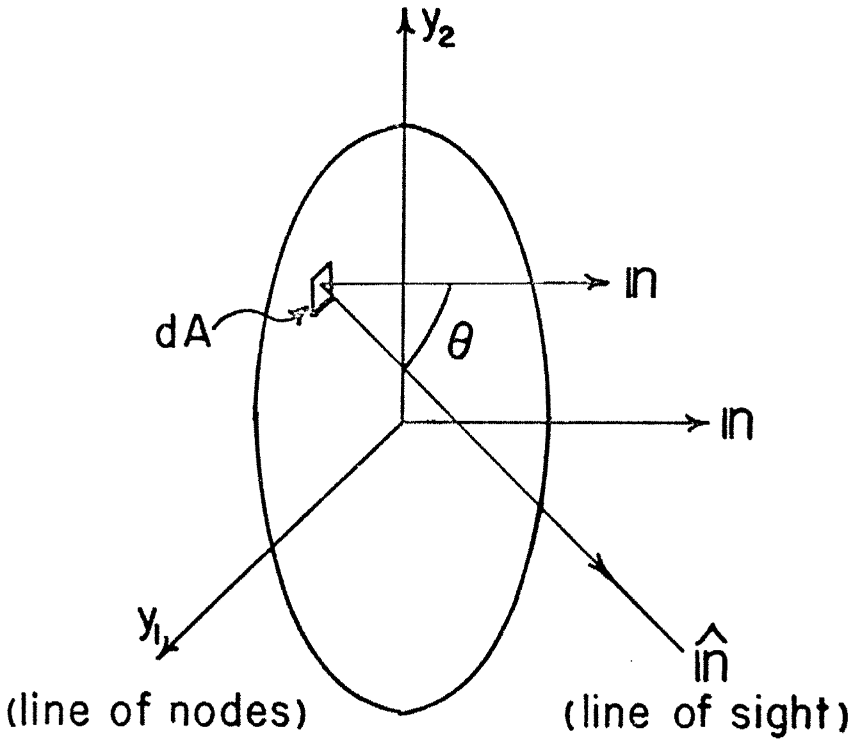
y_2
dA
θ
$\mathbb{n}$
$\mathbb{n}$
y_1
(line of nodes)
$\hat{\mathbb{n}}$
(line of sight)

FIG. 5.1

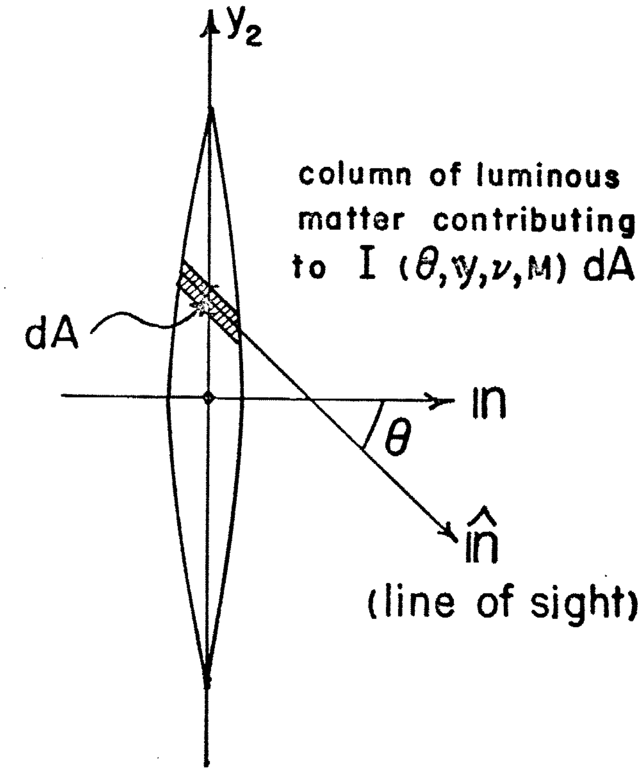
y_2
column of luminous
matter contributing
to $I\,(\theta, \psi, \nu, M)\, dA$
dA
$\mathbb{n}$
θ
$\hat{\mathbb{n}}$
(line of sight)

FIG. 5.2

does depend on ν , since galaxies with appreciable absorption appear redder when observed edge-on than when observed face-on. Orientation effects on intensity and color have been explored by Holmberg (26). Note also that

(5.4)
$$\iint d^2Z \, \frac{d\Omega}{4\pi} \, \mathcal{J}_\beta (\theta, \underline{Z}, \nu) \equiv 1,$$

or, letting $\int d^2Z \, \mathcal{J}_\beta(\theta, \underline{Z}, \nu) = \xi_{\beta\nu}(\theta)$,

(5.5)
$$\int \frac{d\Omega}{4\pi} \, \xi_{\beta\nu}(\theta) \equiv 1 \ .$$

Then the total intensity at θ from the galaxy is $L_{\beta\nu} \ \xi_{\beta\nu} \ (\theta) 10^{-.4M}$.

Let us now consider the stochastic surface brightness $J^\beta_{j\nu}(r_j \underline{n})$ of a shell S_j of radius r_j and (small) thickness h, as seen in the direction of the radius vector;

(5.6)
$$J^\beta_{j\nu} \ (r_j \underline{n}) = \iiint\limits_{S_j \, \Omega \, M} I_\beta \ (\theta, y(\underline{x}-\hat{\underline{n}}r), \nu, M) \ \sec\theta \ dN_\beta \ (\underline{x}, M, n).$$

Here $\underline{y}(\underline{x}-\underline{n}r)$ is that position on the face of the Galaxy (whose axis is $\underline{n}$ and whose center is at $\underline{x}$) which we see when we look along $\hat{\underline{n}}$, the line of sight. Explicitly, if $\underline{Z}$ is the projection of $\underline{x}-\underline{n}r$ on the tangent plane of the shell, then

(5.7) $$\underset{\sim}{y}(\underset{\sim}{Z}) = \left(-\sqrt{(Z^2 - \underline{Z}\cdot\underline{n})\csc^2\theta}\ ,\ -\underline{Z}\cdot\underline{n}\csc\theta\sec\theta\right).$$

The factor of sec θ in (5.6) arises, of course, from the projection, since $r^2 d^2\hat{\underline{n}} = \cos\theta\, d^2 y$. We then have, if r_j is much larger than the correlation distance,

(5.8)

$$E\left(J^{\beta}_{j\nu}(r_j\hat{\underline{n}})\right) = \iint_{S_j\,\Omega\,M}\!\!\int \{I_\beta(\theta, \underset{\sim}{y}(\underset{\sim}{x} - \hat{\underline{n}} r_j), \nu, M)\cdot \lambda_\beta(M)\,(4\pi)^{-1}\sec\theta\}\, d^3x\, d\Omega\, dM$$

$$\cong h \iiint I_\beta(\theta, y, \nu, M)\,\lambda_\beta(M)\, d^2y\, \frac{d\Omega}{4\pi}\, dM$$

$$= h\, L_{\beta\nu} \int 10^{-.4M}\,\lambda_\beta(M)\, dM\, .$$

Similarly,

(5.9) $$\mathrm{cov}\left(J^{\beta}_{j\nu}(r_j\underline{\hat{n}}),\ J^{\beta'}_{k\nu}(r_k\underline{\hat{n}}')\right)$$

$$\cong h^2 L_{\beta\nu} L_{\beta'\nu}\Big\{\int 10^{-.4M}\lambda_\beta(M)\,dM\int 10^{-.4M'}\lambda_\beta(M')\,dM'\cdot$$

$$\cdot\, g_{\beta\beta'}(|r_j\underline{\hat{n}} - r_k\underline{\hat{n}}'|)\Big\}$$

$$+\ \delta_{jk}\,\delta_{\beta\beta'}\int_{S_j}\int_{\Omega}\int_{M}\Big\{I_\beta(\theta,\not{y}(\underline{x}-\underline{\hat{n}}r_j,\nu,M)\cdot$$

$$\cdot\, I_\beta(\theta,\not{y}(\underline{x}-\underline{\hat{n}}'r_j),\nu,M)\ \lambda_\beta(M)$$

$$\cdot\ \sec^2\theta\ \frac{d\Omega}{4\pi}\ d^3x\ dM\Big\},$$

under the assumption that the galaxies are small compared with the scale of changes in $g_{\beta\beta'}(\alpha)$; this is almost certainly valid. If one integrates over $\underline{\hat{n}}$ or $\underline{\hat{n}}'$ in (5.9), it becomes clear that we can write the second term as

(5.10)

$$\iiint \Big\{ I_\beta(\theta, \underline{y}(\underline{x} - \hat{n} r_j), \nu, M)\; I_\beta(\theta, \underline{y}(\underline{x} - \hat{n}' r_j), \nu, M) \cdot$$

$$\cdot\, \lambda_\beta(M) \sec^2\theta \,\frac{d\Omega}{4\pi} \Big\} d^3x \, dM$$

$$= h\, L^2_{\beta\nu} \int \frac{d\Omega}{4\pi} \int dM \Big\{ \xi^2_{\beta\nu}(\theta)\, \lambda_\beta(M)\, 10^{-.8M} \cdot \frac{1}{s_\beta(M)}$$

$$\cdot\, F_{\beta\nu}\left(\frac{|\hat{\underline{n}} - \hat{\underline{n}}'| r_j}{s_\beta(M)} \right) \Big\},$$

where

(5.11)

$$F_{\beta\nu}(x) = \int_0^{\pi} \int_0^{2\pi} \Big\{ \mathcal{I}_\beta(\theta, \underline{y}, \nu)\, \mathcal{I}_\beta(\theta, \underline{y} + (x\cos\phi, x \sin\phi \sec\theta), \nu) \Big\}$$

$$\cdot\, \tan\theta \, d\theta \, d\phi \, d^2y \cdot \Big\{ \int \xi_{\beta\nu}(\theta)\, d\Omega \Big\}^{-1}$$

is a form factor of width approximately the mean galactic diameter, and with unit mass, i.e., $\int F_{\beta\nu}(x)\,d^2x = 1$.

If it happens that we can ignore the galactic size and hence the width of $F_{\beta\nu}(x)$, it is evident that the effect of orientation, at least at a single wavelength (or fixed wavelength band) is to redistribute the β-galaxies in brightness. It should thus be possible to find a new luminosity function which takes this into account and thenceforth to ignore orientation effects altogether. This is indeed possible. It is clear that a galaxy whose axis is at angle θ to the line of sight appears to be of brightness $\tilde{M} = M - 2.5 \log \mathcal{S}_{\beta\nu}(\theta)$. The number of galaxies so inclined is proportional to the solid angle, so if $\tilde{\lambda}_\beta(\tilde{M})$ is the mean number per unit volume per interval in $\tilde{M}$, then

(5.12)
$$\tilde{\lambda}_\beta(\tilde{M}) = \int \frac{d\Omega}{4\pi} \lambda_\beta\left(\tilde{M} + 2.5 \log \mathcal{S}_{\beta\nu}(\theta)\right)$$

We have suppressed the frequency dependence, but note that it is present; it is also possible to define $\tilde{M}$ and $\tilde{\lambda}$ for finite or infinite bandpasses. Then we find that

(5.13)
$$E\left(J^{\beta}_{j\nu}(v_j, \underline{\hat{n}})\right) = h\, L_{\beta\nu} \int 10^{-.4\tilde{M}}\, \tilde{\lambda}_\beta(\tilde{M})\, d\tilde{M}$$
$$= h\, \lambda_\beta \langle \mathcal{L}_{\beta\nu} \rangle ,$$

where $\lambda_\beta = \int \lambda_\beta(M)\,dM = \int \tilde{\lambda}_\beta(\tilde{M})\,d\tilde{M}$, and $\langle \mathcal{L}_{\beta\nu} \rangle$ is the average luminosity of β-galaxies (averaged with $\tilde{\lambda}_\beta$, note). The covariance becomes

(5.14)
$$\mathrm{cov}\left(J^{\beta}_{j\nu}(r_j \hat{\underline{n}}),\, J^{\beta'}_{k\nu}(r_k \hat{\underline{n}}')\right) = \lambda_\beta \lambda_{\beta'} \langle \mathcal{L}_{\beta\nu} \rangle \langle \mathcal{L}_{\beta'\nu} \rangle \cdot g_{\beta\beta'}\left(|r_j \hat{\underline{n}} - r_k \hat{\underline{n}}'|\right) h^2 + \delta_{jk}\, \delta_{\beta\beta'}\, \delta^{(2)}\left(r_j(\hat{\underline{n}} - \hat{\underline{n}}')\right) \langle \mathcal{L}^2_{\beta\nu} \rangle \lambda_\beta h .$$

One can replace the $\delta^2(r_j(\hat{\underline{n}}-\hat{\underline{n}}'))$ with some appropriate average form factor if one chooses to take account approximately of the fact that galaxies have size; in the sampling technique to be described in Chapter VII, however, this is unnecessary.

Note that in counting, the luminosity function is effectively $\tilde{\lambda}_\beta(\tilde{M})$, and so the $\lambda_\beta(M)$ in the preceding chapters should strictly be replaced by $\tilde{\lambda}_\beta(\tilde{M})$. The distinction between the two luminosity functions tends to be small, since we have chosen $f_{\beta\nu}(\theta)$ so that its average is unity.

The intensity reaching us is not necessarily just the sum of $J^{\beta}_{j\nu}$ over all shells, however; there may be obscuring matter in the way. Zwicky (27) has long maintained that there is intergalactic obscuration, especially associated with large clusters. Such obscuration, for example, might be taken proportional to the fundamental, under the assumption that the distribution of galaxies in quasi-Poisson (see Appendix II). Until its nature and distribution is clarified, however, such ad hoc models are the only available course. We shall therefore ignore intergalactic absorption in the treatment to follow, but the possibility of its importance must be kept in mind.

We can treat with some certainty another phenomenon whose importance was first pointed out by Bonnor (28); *namely the absorption of the light of distant* galaxies by nearer ones.

We again assume circular symmetry for the galaxies (It should be noted that we do so only for convenience - certainly one more parameter would not essentially complicate things. But the reason is in reality a little deeper; the details of spiral structure, etc., constitute the main part of any deviation from circular symmetry, and the statistics of such detail is inherently very complicated. Since we will not begin to be able to resolve it in the cosmic light, it seems of little use to attempt to include it.)

So let $a_{\beta\nu}(\underline{Z}, \underline{n}, M)$ be the fraction of incident light absorbed by a β-galaxy of magnitude M whose axis is at $\underline{n}$ at a point $\underline{Z}$ on the projected image of the galaxy. Let

(5.15) $$\sigma_{\beta\nu}(M, \underline{n}) = \int a_{\beta\nu}(\underline{Z}, \underline{n}, M)\, d^2Z \;,$$

$$\langle \sigma_{\beta\nu} \rangle = \iint \frac{d\Omega}{4\pi} \frac{\lambda_\beta(M)}{\lambda_\beta} \sigma_{\beta\nu}(M, \underline{n})\, dM$$

be the absorption cross-section and the averaged absorption cross-section. Let $\tau(\underline{r})$ be the optical depth along the line of sight from the origin to $\underline{r}$, so that the apparent surface brightness of an object of real surface brightness $J_\nu(\underline{r})$ is $J_\nu(\underline{r}) \exp(-\tau_\nu(\underline{r}))$. Let $\exp(-\Delta\tau_{\nu j}(\hat{n}))$ be the fraction of incident intensity along the radius vector $\underline{\hat{n}}$ that is transmitted by the shell S_j.

We now need to investigate the statistical structure of this obscuration, and to do this without great algebraic complexity we need a result which we state here without proof.

"Theorem": Let $N_\beta(s)$ be a multivariate point process in any number of dimensions, and let $S(N_\beta(s))$ be a functional on $N_\beta(s)$. Let $\rho_\beta(x)$ be a fictitious stochastic process having for centered moments the corresponding centered moment densities of $N_\beta(s)$.

(Such a process need not exist, and in general does not, but that does not matter in the present context.) Let $N_\beta(s, \rho_\beta)$ be a Poisson process with parameter density $\rho_\beta(x)$. Then formally

(5.16)

$$E\left[S(N_\beta(s))\right] = E\left[E(S(N_\beta(s, \rho_\beta)) \mid \rho_\beta)\right],$$

which is true when it makes sense. This means, of course, the following: to evaluate the expectation value on the left, we evaluate the expectation value of the same functional on a Poisson process with unspecified parameter density $\rho_\beta(x)$. The result will, of course, be a functional of ρ_β. We then take a formal expectation value over all ρ_β's, replacing the central moments of ρ_β in this expression by the central moment densities of $N_\beta(s)$ wherever they occur. In cases where all moment densities of $N_\beta(s)$ exist, and the functional is, say, an entire function of linear functionals, this operation can be carried out, and, subject to convergence requirements being met, yields the desired result. In cases where there is no "power series" expansion, or there are singularities, we make no claim. The theorem is trivially true for quasi-Poisson processes by virtue of their definition in terms of a real stochastic density $\rho_\beta(x)$. The extension to general processes is in the nature of a conjecture, but it has been laboriously borne out in several cases and the author is confident of its general validity. It is, in any case, valid for the uses for which it will be put here.

If the distribution of galaxies is Poisson, it is clear that the expected value of the function of light absorbed in traversing shell j is

(5.17)

$$E\left(1-e^{-\Delta\tau_j(r)}\right)=\sum_{\beta}\langle\sigma_{\beta\nu}\rangle\rho_\beta\left(\underline{r}\,\frac{r_j}{r}\right)h+O(h^2),$$

where ρ_β is the parameter density for β-galaxies, and r_j is the radius of the shell (assuming that the galaxies are distributed in brightness by $\lambda_\beta{}^{-1}\lambda_\beta(M)$, as before.) If we consider products, $\left(1-e^{-\Delta\tau_{\nu j}(\underline{r})}\right)\left(1-e^{-\Delta\tau_{\nu j'}(\underline{r}')}\right)$, the expectation value is just the product of the separate expectation values unless $j=j'$ and the directions of $\underline{r}$ and $\underline{r}'$ are sufficiently proximate that the <u>same galaxy</u> can contribute to both terms. We shall see later that we need not consider lines of slight at very small angles, so we neglect this singular contribution.

Let us now obtain the expression for the total stochastic surface brightness for a non-stationary, non-Euclidean universe with Robertson-Walker metric. From (4.2), one obtains for the flux in the frequency interval $d\nu$ from the solid angle $d\Omega$ and emitted in the j^{th} shell,

(5.18)

$$F_{j\nu}(\hat{\underline{n}})\,d\nu\,d\Omega = \sum_{\beta} \frac{J^{\beta}_{j\nu(1+z(t_j))}\,dA_o\,d\nu}{\rho^2(t_j)\,R_o^2\,(1+z(t_j))}\, e^{-\tau_\nu(t_j\hat{\underline{n}})},$$

where t is again the radial (cosmic time) coordinate as in Chapter IV, and dA_o is the element of proper area corresponding to solid angle $d\Omega$.

The absorption factor must take into account the change in frequency of the absorbed light along the path. The light emitted at t with frequency $\nu(1+z(t))$ has frequency $\nu(1+z(t'))$ when it traverses an absorber at t', so

(5.19)

$$e^{-\tau_\nu(t_j\hat{\underline{n}})} = \prod_{k=1}^{j-1} e^{-\Delta\tau_{\nu(1+z(t_k)),k}(\hat{\underline{n}})}$$

assuming that the light cone is divided into spherical zones with interval h in the t-coordinate, with h small. We shall, of course, let h tend to zero before we are done. Then, using (4.8) for the proper area element, the total flux becomes

(5.20)

$$F_\nu(\underline{\hat{n}}) = \sum_{\beta, j} \frac{J^\beta_{j\nu(1+z(t_j))}\, e^{-\tau_\nu(t_j\underline{\hat{n}})}}{(1+z(t_j))^3}$$

with expectation value

(5.21)

$$E\,F_\nu(\underline{\hat{n}}) = \sum_{\beta, j} \frac{E\,J^\beta_{j\nu(1+z(t_j))}\; E\, e^{-\tau_\nu(t_j\underline{\hat{n}})}}{(1+z(t_j))^3}$$

$$+ \sum_{\beta, j} \frac{E\,(\bar{J}^\beta_{j\nu(1+z(t_j))}\, e^{-\tau_\nu(t_j\underline{\hat{n}})})}{(1+z(t_j))^3}$$

Consider the term $E\, \exp[-\tau_\nu(t_j\underline{\hat{n}})]$. Supposing that the distribution of β -galaxies is Poisson with parameter density ρ_β $(\underline{x})$,

(5.22)

$$E(e^{-\tau_\nu(t_j\underline{\hat{n}})} \mid \rho_\beta(\underline{x})) = \prod_{k=1}^{j-1} E(e^{-\Delta\tau_{\nu(1+z(t_k)),k}(\underline{\hat{n}})})$$

$$= \prod_{k=1}^{j-1}\left(1 - h\sum_\beta \langle\sigma_{\beta\nu_k}\rangle \rho_\beta(\underline{\hat{n}}\, t_k) + O(h^2)\right),$$

and so, using the result stated above,

(5.23)

$$E\, e^{-\tau_\nu(t_j \underline{\hat{n}})} = E\left(E\left(e^{-\tau_\nu(t_j\underline{\hat{n}})} \mid \rho_\beta(\underline{x})\right)\right)$$

$$= E \prod_{k=1}^{j-1} \left(1 - h \sum_\beta \langle \sigma_{\beta \nu_k} \rangle \rho_\beta(\underline{\hat{n}} t_k) + O(h^2)\right)$$

$$= \exp\left\{ - \sum_\beta \int_0^{t_j} \lambda_\beta(t) \langle \sigma_{\beta \nu_k} \rangle \, dt \right\} \cdot$$

$$\cdot\, E \prod_{k=1}^{j-1} \left(1 - h \sum_\beta \langle \sigma_{\beta\nu_k} \rangle \bar{\rho}_\beta(\underline{\hat{n}} t_k)\right) + O(h)$$

Here $\nu_k = \nu(1+z(t_k))$, $\nu(t) = \nu(1+z(t))$. But, letting

$\tau = \tau_0 - t$, $\tau_k = \tau_0 - t_k$,

(5.24)

$$E\left(\prod_{k=1}^{j-1}\left(1 - h \sum_\beta \langle \sigma_{\beta \nu_k} \rangle \bar{\rho}_\beta(\underline{\hat{n}} t_k)\right)\right)$$

$$= 1 + \frac{h^2}{2} \sum_{\beta_1, \beta_2} \sum_{k=1}^{j-1} \sum_{\ell=1}^{j-1} \langle \sigma_{\beta_1 \nu_k} \rangle \langle \sigma_{\beta_2 \nu_\ell} \rangle E\left(\bar{\rho}_{\beta_1}(\underline{\hat{n}} t_k) \bar{\rho}_{\beta_2}(\underline{\hat{n}} t_\ell)\right)$$

$$- \frac{h^3}{3!} \sum_{\beta_1, \beta_2, \beta_3} \sum_{k, \ell, m = 1}^{j-1} \langle \sigma_{\beta_1 \nu_k} \rangle \langle \sigma_{\beta_2 \nu_\ell} \rangle \langle \sigma_{\beta_2 \nu_m} \rangle \cdot$$

$$\cdot\, E\left(\bar{\rho}_{\beta_1}(\underline{\hat{n}} t_k) \bar{\rho}_{\beta_2}(\underline{\hat{n}} t_\ell) \bar{\rho}_{\beta_3}(\underline{\hat{n}} t_m)\right) + \ldots$$

$$= 1 + \frac{h^2}{2} \sum_{\beta_1\beta_2} \sum_{k,\ell=1}^{j-1} \{ \langle \sigma_{\beta_1} \nu_k \rangle \langle \sigma_{\beta_2} \nu_\ell \rangle \lambda_{\beta_1}(\tau_k) \lambda_{\beta_2}(\tau_\ell) \cdot$$

$$\cdot\, g_{\beta_1\beta_2}(t_k - t_\ell, \tau_k) \} + \ldots + O(h)$$

$$= 1 + \frac{1}{2} \int_0^{t_j} \int_0^{t_j} \Big\{ \sum_{\beta_1} \langle \sigma_{\beta_1} \nu(t_1) \rangle \lambda_{\beta_1}(\tau_1) \sum_{\beta_2} \langle \sigma_{\beta_2} \nu(t_2) \rangle \lambda_{\beta_2}(\tau_2) \cdot$$

$$\cdot\, g_{\beta_1\beta_2}(t_1 - t_2, \tau_1) \Big\} \, dt_1 \, dt_2 + \text{higher order terms}$$

We again assume that the covariance density is non-zero only over a distance small compared with the Hubble distance, so that it really does not matter whether we use τ_1 or τ_2 (τ_j or τ_k) for its time argument. The "higher order terms" in 5.24 are indeed much smaller than the second order term which we kept; the n'th such term is of order

(5.25)

$$\frac{\langle \sigma\lambda \rangle^n}{n!} \int_0^{t_j} dt_1 \cdots \int_0^{t_j} dt_{n-1} \; \langle G_n(t_1 - t_2, t_1 - t_3, \ldots, t_1 - t_n) \rangle$$

where $\langle \sigma\lambda \rangle$ is some mean value of the cross-section times the spatial density of galaxies, and $\langle G_n(\cdot) \rangle$ is an average dimensionless centered n-th moment density. One integration can be carried out and yields a factor t_j: the others give some average value of $\langle G_n \rangle$ multiplied by the correlation length to the (n-1)st power. Thus the term is like $C \langle \sigma\lambda \rangle t_j (\ell \langle \sigma\lambda \rangle)^{n-1}$ but ℓ is of the order of 4 Mpc or so, C is of order 10^{n-1} or so if the fundamental is approximately normal, and $\langle \sigma\lambda \rangle \sim 10^{-4} \mathrm{Mpc}^{-1}$ at the most pessimistic, so $\langle \sigma\lambda \rangle \ell \lesssim 4\mathrm{x}10^{-3}$, and this is the ratio of one term to the next. The value of $\langle \sigma\lambda \rangle t_j$ grows with t_j, but even at 10^4Mpc is only of order unity. There are complications of course, because at such distances the spatial density will be much higher, but one cannot expect even the second-order term to be very important - though the first-order one (which we removed in (5.23) can be. It is to be noted that the result is very much as if there were no correlations; that is, as if the distribution were Poisson. The reason for this behavior is not difficult to see; we expect correlation effects only if there is an appreciable probability per photon that, given one absorption, another occurs within a correlation distance. This conditional probability is of the order $C \langle \sigma\lambda \rangle \ell$, which we have seen is very small. Thus, though the probability of an absorption somewhere may be quite high, the galaxies are spread so thinly that the probability of absorption in any given correlation distance is tiny.

The next term, $E\left(\bar{J}^{\beta}_{j\nu(1+z_j)}\, e^{-\tau_\nu(t_j\hat{\underline{n}})}\right)$, is treated similarly, and one obtains

(5.26)

$$E\left(\bar{J}^{\beta}_{j\nu(1+z_j)}\, e^{-\tau_\nu(t_j\hat{\underline{n}})}\right) = -h \langle \mathcal{L}_{\beta\nu}(\tau_j)\rangle \lambda_{\beta j} \cdot e^{-\sum_\beta \int_0^{t_j} \lambda_\beta(\tau)\langle\sigma_{\beta\nu(t)}\rangle dt} \cdot \sum_{\beta'} \int_0^{t_j} \langle \sigma_{\beta'\nu(t)}\rangle \lambda_{\beta'}(\tau)\, f_{\beta\beta'}(t_j - t,\, \tau_j)\, dt + (\text{higher order terms}).$$

If we let the time-dependent auxil iary covariance function be

(5.27)

$$A_{\beta\beta'}\left(\frac{\delta}{\ell(t)},\, \tau\right) = \int_{-\infty}^{\infty} g_{\beta\beta'}\left((\delta^2 + y^2)^{1/2},\, t\right) \frac{dy}{\ell(t)},$$

we can write (5.13) as

(5.28)

$$E(F_\nu(\hat{\underline{n}})) = \sum_\beta \int_0^\infty dt \Big[\langle \mathcal{L}_{\beta\nu(t)}(\tau) \rangle \lambda_\beta(\tau) \cdot$$

$$\cdot\ e^{-\sum_\beta \int \lambda_\beta(\tau') \langle \sigma_{\beta\nu(t')} \rangle dt'} \; (1+z(t))^{-3} \cdot$$

$$\cdot \Big\{ 1 - \frac{1}{2} \sum_{\beta'} \lambda_{\beta'}(t) \langle \sigma_{\beta'\nu(t)} \rangle \ell(\tau) A_{\beta\beta'}(0,\tau) \cdot$$

$$+ \frac{1}{2} \sum_{\beta',\beta''} \Big(\ell(\tau) \int_0^t dt' \lambda_{\beta'}(\tau') \lambda_{\beta''}(\tau') \langle \sigma_{\beta'\nu(t')} \rangle \cdot$$

$$\cdot \langle \sigma_{\beta''\nu(t')} \rangle A_{\beta'\beta''}(0,\tau') \Big) \Big\} \Big]$$

According to our earlier estimates, the second term in the bracket is larger than the third, and the second does not exceed 10^{-3}. We are therefore justified in neglecting the correlation terms, but if we know $g_{\beta\beta'}(x,\tau)$ we <u>can include them</u> if it ever becomes necessary to do so.

If we look at the covariance of a pair of $F_\nu(\hat{\underline{n}})$'s, the only term that contributes significantly is the term which contains the covariance of the J's. Thus

(5.29)

$$\operatorname{cov}\left(F_\nu(\hat{\underline{n}}), F_\nu(\hat{\underline{n}}')\right) = \int_0^\infty dt \int_0^\infty dt' \left\{ \frac{1}{(1+z(t))^3} \frac{1}{(1+z(t'))^3} \cdot \right.$$

$$\cdot \exp\left(-\sum_\beta \left\{\int_0^t + \int_0^{t'}\right\} \lambda_\beta(\tau'') \langle \sigma_{\beta\nu(t'')} \rangle dt''\right) \cdot$$

$$\cdot \sum_{\beta,\beta'} \left[\lambda_\beta(\tau) \lambda_{\beta'}(\tau') \langle \mathcal{L}_{\beta\nu(t)}(\tau) \rangle \langle \mathcal{L}_{\beta'\nu(t')}(\tau') \rangle \cdot \right.$$

$$\left. \cdot\, g_{\beta\beta'}(\alpha) + \mathcal{F}_\beta(\alpha) \delta_{\beta\beta'} \langle \mathcal{L}^2_{\beta\nu(t)}(\tau) \rangle \lambda_\beta(\tau) \right],$$

where α is the proper distance between the points $t\hat{\underline{n}}$ and $t'\hat{\underline{n}}'$, and $\mathcal{F}_\beta(\alpha)$ is the average form factor for β-galaxies. We now make use of the fact that the correlation distance ℓ is always short compared to the distance of the objects contributing significantly to the background light, so that we can write

(5.30)

$$\operatorname{cov}\left(F_\nu(\hat{\underline{n}}),\, F_\nu(\hat{\underline{n}}')\right) \cong \int_0^\infty \frac{dt}{(1+z(t))^6}\left\{ e^{-2\sum_\beta \int_0^t \lambda_\beta(\tau')\langle\sigma_{\beta\nu(t')}\rangle dt'} \cdot \right.$$

$$\cdot \sum_{\beta,\beta'}\Big[\lambda_\beta(\tau)\lambda_{\beta'}(\tau)\langle\mathcal{L}_{\beta\nu(t)}(\tau)\rangle\langle\mathcal{L}_{\beta'\nu(t)}(\tau)\rangle\, \ell A_{\beta\beta'}(\tfrac{\delta}{\ell},\tau)$$

$$\left. + \int_{-\infty}^{\infty}\mathcal{F}_\beta\left((\delta^2+t'^2)^{1/2}\right) dt'\, \delta_{\beta\beta'}\langle\mathcal{L}^2_{\beta\nu(t)}(\tau)\rangle\lambda_\beta(\tau)\Big]\right\},$$

where $\delta = \rho(t)\, R(\tau_0 - t)\, |\hat{\underline{n}} - \hat{\underline{n}}'|$.

There are now several quite important effects which we have not included; first, we must take into account the fact that when we measure the background, we shall wish to exclude bright objects from our measurements. The expressions so far have been for the <u>total</u> light. So suppose that we reject objects with a probability ϕ_β (m), where m is the apparent magnitude. We can clearly include this by changing the definition of $\langle\mathcal{L}_{\beta\nu(t)}(\tau)\rangle$ to

(5.31)

$$\langle\mathcal{L}_{\beta\nu}(\tau)\rangle' = L_{\beta\nu}(\tau)\int_{-\infty}^{\infty} 10^{-.4\tilde{M}_0}\,\frac{\tilde{\lambda}_\beta(\tilde{M},\tau)}{\lambda_\beta(\tau)}\,\phi(m(\tilde{M}_0,t))\, d\tilde{M}_0 .$$

Note that the ratio $\lambda_\beta(\tilde{M}_0, \tau)/\lambda_\beta(\tau)$ is independent of τ if we define M_o and $\tilde{M}_o$, as in Chapter IV, to be the present luminosities. (We assume that $\xi_{\beta\nu}(\theta)$ is time-independent; its effects are quite small anyhow, and no evolutionary theory for it seems probable in the near future.) The evolutionary corrections are included in $L_{\beta\nu}(\tau)$; the quantity $m(M_o, T)$ is that defined in (4.7) with $M_o = \tilde{M}_o$. We define $\langle \mathcal{L}^2_{\beta\nu}(\tau)\rangle'$ analagously; note that here also only one power of $\phi(m)$ appears in the integral.

The author (29) has recently shown that there is an uncertainty in the relation between $\underline{\hat{n}} - \underline{\hat{n}}'$ and the coordinate angle between the sources of the incoming rays due to differential gravitational scattering by nearer galaxies. The effect amounts to a few percent for the distances important here and should be included.

Thus δ becomes itself a stochastic variable, with mean $\rho(t)R(\tau_0 - t)|\underline{\hat{n}}-\underline{\hat{n}}'|$, and with standard deviation $\rho(t)R(\tau_0 - t)\left(f(t, \underline{n}-\underline{n})\right)^{1/2}$. The normalized variance $f(t, \gamma)$ grows at first quadratically with γ and then goes over to a logarithmic form as γ becomes large. Since the deflections are caused by a large number of almost-independent interactions, we are justified in assuming that the distribution of δ is normal with the indicated mean and variance.

It is not difficult to see how this affects the covariance. The flux $F_\nu(\hat{\underline{n}})$ is the sum of contributions along the path, and for distances apart greater than a few times ℓ these contributions are independent, quite apart from any angular fluctutations. The covariance of two F's, then, can be represented with high accuracy by an integral something like (5.30). Now suppose we know the distribution of galaxies for $t < t_o$. Galaxies slightly more distant than this are distributed independently of this known distribution, but the nearer distribution determines the error in δ. Since the covariance is built up of a sum (an integral, actually), of "local" covariances, we can, by the general rules for conditional moments, evaluate each "local" covariance for fixed δ, and then average over the distribution of δ's. This clearly "smears" the distribution independently at each distance.

Let $D_\beta(\delta) = \int_{-\infty}^{\infty} \mathcal{F}((\delta^2+t)^{1/2})\, dt$. Then we replace $A_{\beta\beta'}(\delta,\tau)$ and $D_\beta(\delta)$ in (5.30) by

(5.32)

$$\langle A_{\beta\beta'}(\delta/\ell(\tau), \tau)\rangle = \int A_{\beta\beta'}\left(\frac{\delta}{\ell(\tau)}, \tau\right) dP(\delta)$$

$$\langle D_\beta(\delta_o)\rangle = \int D_\beta(\delta)\, dP(\delta)$$

where $\delta_o = \rho(t)R(\tau_o - t)|\underline{\hat{n}}-\underline{\hat{n}}'| = E(\delta)$, and $P(\delta)$ is the distribution function for δ . Over the distances (approximately a galactic diameter) where $D(\delta)$ is appreciable, the standard deviation is linear in δ , and, as remarked before, is an effect of a few percent. Since $D_\beta(\delta)$ is an average form factor anyway, with (as we shall see) quite negligible effect on the observed distribution, we shall merely keep the old form. Since we are interested in observing structure characterized by $A_{\beta\beta'}$, however, we must investigate the effect on it more closely. Since the expected deflections are small, we can expand:

(5.33)

$$\left\langle A_{\beta\beta'}\left(\frac{\delta_o}{\ell},\tau\right)\right\rangle = \int\left\{ A_{\beta\beta'}\left(\frac{\delta_o}{\ell},\tau\right) + \frac{\partial A_{\beta\beta'}}{\partial(\delta_o/\ell)}\Big]_o \frac{\delta-\delta_o}{\ell} + \frac{1}{2}\frac{\partial^2 A_{\beta\beta'}}{\partial(\delta_o/\ell)^2}\Big]_o\left(\frac{\delta-\delta_o}{\ell}\right)^2 + \ldots\right\} dP(\delta)$$

$$\cong A_{\beta\beta'}\left(\frac{\delta_o}{\ell},\tau\right) + \frac{1}{2}\frac{\partial^2 A_{\beta\beta'}}{\partial(\delta_o/\ell)^2}\Big]_o \frac{f(t,|\underline{\hat{n}}-\underline{\hat{n}}'|)}{|\underline{\hat{n}}-\underline{\hat{n}}'|^2}\frac{\delta_o^2}{\ell^2} ,$$

where $\Big]_o$ indicates that the preceding quantity is to be evaluated at δ_o .

For large δ_o, the error becomes large, but here $A_{\beta\beta'}$ and $\frac{\partial^2 A_{\beta\beta'}}{\partial(\delta_o/\ell)^2}$ are small, as is $f(t, |\hat{\underline{n}}-\hat{\underline{n}}'|)/(|\hat{\underline{n}}-\hat{\underline{n}}'|)^2$, so the net contribution is also small. We thus adopt (5.33). The covariance then becomes

(5.34)

$$\mathrm{cov}\,(F_\nu(\hat{\underline{n}}), F_\nu(\hat{\underline{n}}+\underline{\gamma})) = \int_0^\infty \frac{dt}{(1+z(t))^4}\Big\{ e^{-2\int_0^t \sum_\beta \lambda_\beta(\tau')\langle\sigma_{\beta\nu}(t')\rangle dt'}$$

$$\cdot \sum_{\beta,\beta'}\Big[\lambda_\beta(\tau)\lambda_{\beta'}(\tau)\langle\mathcal{L}_{\beta\nu(t)}(\tau)\rangle'\langle\mathcal{L}_{\beta'\nu(t)}(\tau)\rangle'\,\ell(\tau)\,\cdot$$

$$\Big(A_{\beta\beta'}\Big(\frac{\delta_o}{\ell},\tau\Big) + \frac{1}{2}\,\frac{\partial^2 A_{\beta\beta'}}{\partial(\delta_o/\ell)^2}\Big]_o f(t,\gamma)\,\frac{\rho^2(t)\,R^2(\tau_o-t)}{\ell^2(t)}\Big)$$

$$+\; D_\beta(\delta_o)\,\delta_{\beta\beta'}\,\langle\mathcal{L}^2_{\beta\nu(t)}(\tau)\rangle'\,\lambda_\beta(\tau)\Big]\Big\}.$$

Let us pause for a moment and investigate the significance of the various terms in this expression. The $(1+z(t))^6$ is geometrical; two powers of the quantity come from the flux-luminosity relation, the other four produce proper comoving volume elements. The exponential is the square of the expected value of the obscuration to distance t. The first term in the sum gives the two-particle contribution to the covariance of the emissivity J_ν, averaged along the line of sight. The factors $\langle \mathcal{L}_{\beta\nu}(t)\rangle' \lambda_\beta(t)$ are the average luminosities from β-galaxies at t, taking into account the fact that we do not include the brighter ones at short distances; $\lambda_\beta(t)$ is the proper mean density of β-galaxies, so $\langle \mathcal{L}_{\beta\nu}(t)\rangle' \lambda_\beta(t)$ is the expected proper volume emissivity from β-galaxies at t. The term in $A_{\beta\beta'}$, of course, is the projected normalized covariance, which can be associated with the enhancement in probability of finding a galaxy at projected distance δ from another that is present, associated with the presence of this first galaxy. The term in $D(\delta_o)$ is the one-particle contribution and corresponds to the possibility of receiving light along $\underline{\hat{n}}$ and $\underline{\hat{n}} + \underline{\gamma}$ from the <u>same</u> galaxy, with small enough $\underline{\gamma}$.

For the relativistic evolving universes,

$\lambda_\beta(t) = (1+z(t))^3 \lambda_\beta^0$, and ℓ is in general a function of t, as are $A_{\beta\beta'}$ and $L_{\beta\nu}$ (which appears as a factor in the mean luminosities.) With the assumption made in the preceding chapter, however, the time dependence of $A_{\beta\beta'}$ is simple and is given in (4.37). For these models the covariance is

(5.36)

$$\operatorname{cov}\left(F_\nu(\hat{\underline{n}}), F_\nu(\hat{\underline{n}}+\underline{\gamma})\right) = \int_0^\infty \frac{dt}{(1+z(t))^3}\left\{ e^{-2\int_0^t (1+z)^3 \sum_\beta \lambda_\beta^\circ \langle \sigma_{\beta\nu(t')}\rangle dt'} \cdot \right.$$

$$\cdot \sum_{\beta,\beta'} \left[\lambda_\beta^\circ \lambda_{\beta'}^\circ \langle \mathcal{L}_{\beta\nu(t)}(\tau)\rangle' \langle \mathcal{L}_{\beta\nu(t)}(\tau)\rangle' \frac{\ell_0^2}{\ell(\tau)} \cdot \right.$$

$$\cdot \left(A_{\beta\beta'}^\circ \left(\frac{\delta_0}{\ell}\right) + \frac{1}{2} A_{\beta\beta'}^{\circ\prime\prime}\left(\frac{\delta_0}{\ell}\right) f(t,\gamma) \frac{\rho^2(t) R_0^2}{(1+z)^2 \ell^2} \right) \cdot$$

$$\left. \left. + \; D_\beta(\delta_0)\, \delta_{\beta\beta'} \langle \mathcal{L}^2_{\beta\nu(t)}(\tau)\rangle' \lambda_\beta^\circ \right] \right\}.$$

To obtain the spectral density it is necessary to find the Fourier transform of the term in $A_{\beta\beta'}$. This can be done exactly if one knows the function f, but must in general be done numerically. Since the effect of the scattering is small, and is linear (i.e., f is quadratic) in the angle over most of the range where A has appreciable size, we can suppose that the linear relation holds for all angles. Then $f(t,\gamma) = a(t)\gamma^2$. The function a(t) depends somewhat on the model and on the statistics, but is given with sufficient accuracy by

(5.36)

$$a(t) = \frac{8\pi^2}{15}(4\,G\rho_0)^2 \ell_0 \sum_{\beta,\beta'} A_{\beta\beta'}(0)\, H_0^{-3}\left(1-\frac{1}{\sqrt{1+z(t)}}\right)^3 \Psi .$$

The factor Ψ is of order unity; for the $q_0=1/2$ Friedmann model with ℓ constant it is unity; for the steady-state it is about 1.5.

The exact expression for $\left(1-\frac{1}{\sqrt{1+z}}\right)^3\Psi$, from which any case can be computed, is given by the author (29). Then

(5.37)

$$\frac{1}{(2\pi)^2}\int\left(A^{0}_{\beta\beta'}(x) + \frac{x^2 a(t)}{2} A''_{\beta\beta'}(x)\right) e^{i\underline{\eta}\cdot\underline{x}}\, d^2x$$

$$= \mathcal{A}^{0}_{\beta\beta'}(\eta) + \frac{a(t)}{2(2\pi)^2}\int x^2 A''_{\beta\beta'}(x)\, e^{i\underline{\eta}\cdot\underline{x}}\, d^2x .$$

Now $A''_{\beta\beta'} = \nabla_{\underline{x}}^{2} A - \frac{1}{x}\frac{dA}{dx}$, and $x^2 e^{i\underline{\eta}\cdot\underline{x}} = -\nabla_{\eta}^{2}\, e^{i\underline{\eta}\cdot\underline{x}}$.

Thus we get

(5.38)

$$\mathcal{A}^{0}_{\beta\beta'}(\eta) + \frac{a(t)}{2}\nabla_{\eta}^{2}\left\{\eta^2 \mathcal{A}_{\beta\beta'}(\eta) + \frac{\eta}{2\pi}\int_0^\infty A_{\beta\beta'}(x) J_1(\eta x)\, dx\right\},$$

$$\overset{\mathrm{Def}}{\equiv} \mathcal{A}^{0}_{\beta\beta'}(\eta) + a(t)\, \mathcal{A}^{1}_{\beta\beta'}(\eta) .$$

The spectral density $\mathcal{S}_\nu(\eta)$ for the process $F_\nu(\hat{n})$ is then given by (5.35) with the term $[A^o_{\beta\beta'}(\delta_o/\ell) + \frac{1}{2} A^{o\,''}_{\beta\beta'}(\delta_o/\ell) f(t,\gamma)\,\delta_o^2/(\gamma^2\ell^2)]$ replaced by

$$\frac{\ell^2(\tau)(1+z)^2}{\rho^2(t) R_o^2}\left\{\mathcal{A}^o_{\beta\beta'}\left(\frac{\ell(\tau)(1+z(t))\eta}{\rho(t) R_o}\right) + a(t)\mathcal{A}^1_{\beta\beta'}\left(\frac{\ell(\tau)(1+z(t))\eta}{\rho(t) R_o}\right)\right\},$$

and the form factor D_β replaced by its Fourier transform. We will find that this term adds a constant component to the spectrum of the sampled process; detailed discussion will be deferred until sampling is considered.

The development for the steady-state model is similar; the covariance is given by

(5.39)

$$cov\,(F_\nu(\hat{n}), F_\nu(\hat{n}+\gamma)) = \int_0^\infty dt\left\{e^{-6Ht - 2\sum_\beta \lambda_\beta \int_0^t \langle\sigma_{\beta\nu(t')}\rangle dt'}\right.$$

$$\cdot \sum_{\beta,\beta'}[\lambda_\beta\lambda_{\beta'}\langle\mathcal{L}_{\beta\nu(t)}\rangle'\langle\mathcal{L}_{\beta'\nu(t)}\rangle'\ell\left(A_{\beta\beta'}\left(\frac{\delta_o}{\ell}\right) + \frac{a(t)}{2}A''_{\beta\beta'}\left(\frac{\delta_o}{\ell}\right)\frac{\delta_o^2}{\ell^2}\right)$$

$$\left. + D_\beta(\delta_o)\,\delta_{\beta\beta'}\langle\mathcal{L}^2_{\beta\nu(t)}\rangle'\lambda_\beta]\right\},$$

where $\delta_o = \dfrac{(1-e^{-Ht})\gamma}{H_o}$. The spectral density is obtained in the same fashion as before.

VI. THE COSMIC LIGHT II: CONTAMINATION

As remarked in the beginning of Chapter V, the cosmic light can be observed only as a small component of a general illumination of the night sky. This contamination has one part the statistics of which are of interest to us; this, of course, is the light from stars. One finds that, at visual magnitude 18, the number of stars near the galactic pole is comparable with the number of galaxies (of the order of 100 per square degree per unit magnitude interval - Hubble (29), van Rhijn, et al (30)), and that for fainter magnitudes the galaxies predominate. We shall see that most of the cosmic light comes from galaxies which are much fainter than 18, and probably one can efficiently use a cut off at around 19 in discriminating against bright objects.

We shall assume that the space distribution of stars is Poisson with a variable (but <u>not</u> stochastic) parameter density. We justify this seemingly <u>ad hoc</u> assumption by consideration of the kinematics of stars in the halo. First of all, no large-scale structure has ever been observed in the distribution of halo objects; this is largely due to the fact that orbits of objects in the halo tend to be almost radial, leading to perigalacticon distances of the order of 1 kiloparsec or less. Tidal forces at this distance are sufficient to disrupt structures whose mean density is less than about 10 solar masses per cubic parsec, which is enormously larger then typical densities in the halo (and indeed in the disc). Typical of structures which are observed and can persist are the globular clusters, which are quite small and compact, and typically have densities of 100 solar masses per cubic parsec and larger. Smaller and looser globular clusters can be expected, of course, down to the density limit; but such associations should be visible on a survey of the sort contemplated, since with any reasonable luminosity function several of its members will be bright enough to see and identify as an association on plates with a limiting magnitude around 20. Possible exceptions to this rule would include preferential clusterings of very <u>faint</u> stars, but no such clusters are known and it is difficult to see how they would come about. We can thus assume with some confidence that if we exclude globular clusters, the remaining halo objects show very little statistical structure. This conjecture can and should be confirmed by analysis of faint star counts.

Thus the contribution of the faint stars to the background covariance function contains a singular term only. If $\Phi(M,\underline{x})$ is the luminosity function for the halo stars, $dN(M,\underline{x})$ the number in dMd^3x, and $\mathcal{E}_\nu(M)$ the energy distribution for a star at M, we have

(6.1) $$\mathcal{J}_\nu^*(\Omega)=\int_{-\infty}^{\infty}dM\iint_{\Omega}\int_0^{\infty}dr\frac{\mathcal{E}_\nu(M)}{4\pi r^2}\,d\eta(M,r\underline{n})$$

for the total flux of starlight received from the solid angle Ω. We also wish to introduce a selection against bright stars. Let $\int_*(m)$ be the probability that a star at apparent magnitude m is successfully rejected. Then

(6.2) $$E\mathcal{J}_\nu^*(\Omega)=\mu(\Omega)\int_{-\infty}^{\infty}dM\int_0^{\infty}dr\,\frac{\mathcal{E}_\nu(M)}{4\pi}\,\Phi(M,r\underline{n})\int_*(M+5\log r)$$

is the expectation value of the flux from stars not resolved and rejected, and the covariance is

(6.3) $$\mathrm{cov}\left(\mathcal{J}_\nu^*(\Omega),\mathcal{J}_\nu^*(\Omega')\right)=\mu(\Omega\cap\Omega')\int_{-\infty}^{\infty}dM\int_0^{\infty}dr\left(\frac{\mathcal{E}_\nu(M)}{4\pi}\right)^2\cdot$$

$$\frac{1}{r^2}\,\Phi(M,r\underline{n})\int_*(M+5\log r)\,.$$

While the distribution of stars is not a stationary process, one expects that $\Phi(M,\underline{x})$ will vary only very slowly over distances of the order of a few degrees near the galactic poles. Since preliminary estimates place the total flux from faint stars close to that from the galaxies and so much smaller than other disturbing influences such as the night sky emission and the zodiacal light, and since there is no extended component of the starlight auto-correlation function, the "disturbance" from the stars is small.

The zodiacal light is immensely more troublesome, simply because it swamps the cosmic light [Roach, (31)] by a factor of about 50 in the most favorable areas of the sky in visual wavelengths. (Roach almost certainly underestimated the cosmic light, but the factor is still large.) Since the spectrum closely mimics that of the sun, the situation can be expected to improve in the infrared, but detector difficulties are enormous there.

More serious is that the final disturbing element in the light path, the atmosphere of the earth, is a strong emitter through most of the infrared, mostly in the light of OH. In addition, through the region 4000-7000Å, (and perhaps beyond) there is a continuum component in the airglow, supposedly arising from many unresolved faint emission features. The brightness of this continuum is, according to Roach (31), of the same order as the zodiacal light near the ecliptic pole, again about 50 times his estimate of the cosmic light. The airglow also varies markedly with time over short periods (hours or less) and so imposes the necessity for constant monitoring of the overall level during measurement.

The primary statistical effect of all this is the necessity of making very accurate measurements; if only one out of every hundred photons received is a cosmic light photon, one must clearly obtain large counts to get reliable data. Just how large this number is is easily calculable from the spectral theory we have developed here; this problem will be considered in the next chapter.

Also present is the problem of removing the regression component (airglow plus zodiacal light) from the measurements before spectral analysis can be applied; this is much more difficult. The zodiacal light measurements of Roach (31) and others fit an $A + B \sec \gamma$ law quite well, where γ is the ecliptic latitude. One would expect A to vanish, but it does not if the measures are correct; the zero, however, is very difficult to establish, and it is well within the realm of possibility that A does indeed vanish. The residuals should be of the order of the cosmic light itself and will be more amenable to standard regression removal techniques.

The temporally variable airglow admits of no such simple technique. It will almost certainly be necessary to use equipment capable of excluding the bright emission features of the night sky, either by interferencefilter techniques or simple masking in a dispersive spectrophotometer. It is tempting to dismiss the problem as impossible for ground-based astronomy because of the airglow, but with reasonable care and proper instrumentation it should be possible to deal with it without too much greater difficulty than the zodiacal light; constant monitoring of several fixed reference areas should suffice for a "first-order" regression removal. The remaining components of zodiacal light and airglow will then be in the form of slowly varying terms of moderate amplitude which must be separated from the small scale (about 20 minutes of arc) variations we seek in the cosmic light. We can then apply high-pass filtration techniques to suppress the rest of the background, and look at only the component due to galaxies (and stars, though the stars clearly add only a constant component to the spectrum, which we shall see is rather small).

VII. THE COSMIC LIGHT III: SAMPLING

In view of the difficulties discussed in the previous chapter, the sampling problem for the cosmic light can be expected to be an extremely complicated one. Fortunately, however, we can make a few very important simplifications which will render the problem more tractable.

We shall assume that the measurements are made with circular diaphragms whose centers are located on a square net on the sphere with cell side h; the diameters of the cells will be assumed smaller than h, so that there is no overlap. We assume further that no galaxy contributing to the cosmic light contributes to more than one diaphragm area.

It is clear that, at each level of brightness, the measurement of the background is essentially a counting procedure, though brightness levels are then added and brightness discrimination is lost. We see, though, that the kind of statistics one has here is of a nature very similar to that of the count problem; we deal here with a <u>scaled</u> point process (See Appendix II). In Chapter III it was shown that the "point terms" in the variance were not of crucial importance even in counting, and here they must be less so because there is no "thinning" of the data by brightness or class discrimination, and because we deal with very faint, distant, and therefore very numerous (as number per correlation area) objects. Thus we can safely neglect the point terms in the variance of the spectral estimates and deal with the process as if it were continuous, except for the singular contribution to the spectrum.

It will be assumed throughout the analysis that the measurements are made by pulse-counting with photomultipliers or some similar technique in which all recorded photons contribute equal amounts to the record. There is an additional contribution to the variance in the case in which DC or charge-integration (or photographic) techniques are used.

Let the overall mean intensity (stars and galaxies sufficiently faint, plus zodiacal light, plus airglow) be $L(\underline{n})$ (we assume that we are working in a fixed wavelength region, and shall suppress explicit wavelength dependence), which may vary slowly with the direction $\underline{n}$. Label the diaphragm centers with a couple of integers $\underline{j} = (j_1, j_2)$ as we did with the counting squares, so that the coordinates of the center are $(j_1 h, j_2 h)$. Let $F_{\underline{j}}$ be the measured flux from position $\underline{j}$, and let $S_{\underline{j}}$ be the disc of area γh^2 centered at $(j_1 h, j_2 h)$, which we will take to be the area covered by the diaphragm. (Note that $\gamma < \pi/4$.)

Then we can write

$$(7.1) \quad F_{\underline{j}} = \int_{S} L(\underline{n})d^2\underline{n} + \mathcal{F}_{\underline{j}}$$

where $\mathcal{F}_{\underline{j}}$ is the cosmic light plus star contribution with mean removed; $E\,\mathcal{F}_{\underline{j}} = 0$. We will take $\mathcal{F}_{\underline{j}}$ to be a stationary stochastic series, even though the star component is not strictly so; the covariance structure in which we are interested arises from the galaxies only and is characteristic of a stationary process.

Suppose first that we could measure the $\mathcal{F}_{\underline{j}}$ exactly. We cannot, because of inherent limitations in our photometry and, more important, because we do not know $L_{\underline{j}} = \int_{S_{\underline{j}}} L(\underline{n})d^2\underline{n}$. Let us determine the spectrum of $\mathcal{F}_{\underline{j}}$; let $\mathcal{R}(\underline{n})$ be the covariance of the intensity at $\underline{m}$ with that at $\underline{m}+\underline{n}$. Then from (5.34), (5.38), and (6.3) we find that $\mathcal{R}(\underline{m})$ has the form

$$(7.2) \quad \mathcal{R}(\underline{n}) = \hat{\mathcal{R}}(\underline{n}) + D_1(\underline{n}) + D_2\,\delta^2(\underline{n}) \ ,$$

where $\mathcal{R}(\underline{n})$ is the non-singular contribution from the galaxies, $D_1(\underline{n})$ is the singular component (and recall that we assume that $D_1(\underline{n})$ has sufficiently small width that if it is non-zero within one $S_{\underline{j}}$, it is zero in all the others) and $D_2\,\delta^2(\underline{n})$ is the singular component from the stars. Then if $\chi_{\underline{j}}(\underline{x})$ is the characteristic function for the region $S_{\underline{j}}$, we have

(7.3)

$$R_{\underline{\nu}} = \operatorname{cov}(\mathcal{J}_{\underline{j}}, \mathcal{J}_{\underline{j}+\underline{\nu}})$$
$$= \iint_{S_{\underline{j}}\, S_{\underline{j}+\underline{\nu}}} \mathcal{R}(\underline{n}-\underline{m})\, d^2n\, d^2m$$
$$= \iint \mathcal{X}_{\underline{j}}(\underline{n})\, \mathcal{X}_{\underline{j}+\underline{\nu}}(\underline{m})\, \mathcal{R}(\underline{n}-\underline{m})\, d^2n\, d^2m$$
$$= \iint \mathcal{X}_{0,0}(\underline{n})\, \mathcal{X}_{0,0}(\underline{m})\, \mathcal{R}(\underline{n}-\underline{m}+\underline{\nu}h)\, d^2n\, d^2m$$

Then the spectrum $\Phi(\underline{\eta})$ is

(7.4)

$$\Phi(\eta) = \frac{1}{(2\pi)^2} \sum_{\underline{\nu}} R_{\underline{\nu}}\, e^{i\underline{\eta}\cdot\underline{\nu}}$$
$$= \frac{1}{(2\pi)^2} \sum_{\underline{\nu}} \iint \Big\{ \mathcal{X}_{0,0}(\underline{n})\, \mathcal{X}_{0,0}(\underline{m})\, \hat{\mathcal{R}}(\underline{n}-\underline{m}-\underline{\nu}h) \cdot e^{i\underline{\eta}\cdot\underline{\nu}} \Big\} d^2n\, d^2m$$
$$+ \frac{1}{(2\pi)^2} \iint \mathcal{X}_{0,0}(\underline{n})\, \mathcal{X}_{0,0}(m) \left\{ D_1(\underline{n}-\underline{m}) + D_2\, \delta^2(\underline{n}-\underline{m}) \right\} d^2n\, d^2m$$

The second term in (7.4) is a constant, which we shall call $\lambda^2(2\pi)^{-2}$; the first term can be handled in the same manner as the one-dimensional expression in (Al.20) in Appendix I, and we obtain

(7.5)

$$\Phi(\eta) = \sum_{\underline{p}} \frac{1}{h^2} \left\{ \frac{2\pi a h}{|\underline{\eta}-2\pi\underline{p}|}\, J_1\!\left(a \left|\frac{\underline{\eta}-2\pi\underline{p}}{h}\right|\right) \right\}^2 \mathcal{S}^1\!\left(\frac{\underline{\eta}-2\pi\underline{p}}{h}\right) + \lambda^2(2\pi)^{-2}$$
$$\overset{\text{Def}}{\equiv} \psi(\underline{\eta}) + \lambda^2(2\pi)^{-2}$$

Here a is the radius of the diaphragm, (so $\pi a^2 = \gamma h^2$), and $\mathcal{S}^1$ is the non-singular contribution to the spectral density of the cosmic light, i.e.

(7.6) $$\mathcal{S}^1(k) = \frac{1}{(2\pi)^2}\int \hat{\mathcal{R}}(\underline{n})\, e^{i\underline{k}\cdot\underline{n}}\, d^2n .$$

We have discussed this quantity in Chapter V. We observe the aliasing of frequencies here also, of course, though again we have attenuation of higher frequencies through the J_1 term.

Given, therefore, a cosmological model and the auxilliary spectrum $\mathcal{A}_{\beta\beta'}(\underline{x})$, we can construct a theoretical $\phi(\eta)$, using (7.5).

Now suppose that the mean photon counting rate per unit flux from any $S_{\underline{j}}$ is $\varkappa$, so that if we count at each diaphragm position for time t and receive a count $N_{\underline{j}}$, we have (Note that $N_{\underline{j}}$ is a quasi-Poisson process with fundamental $\mathcal{F}_{\underline{j}}$) ,

(7.7)
$$\begin{aligned} E(N_{\underline{j}}) &= E\left(E_{\mathcal{F}_{\underline{j}}}(N_{\underline{j}})\right) \\ &= E(\varkappa t\, \mathcal{F}_{\underline{j}}) \\ &= \varkappa t\, L_{\underline{j}} . \end{aligned}$$

Furthermore,

$$(7.8)\quad E(N_{\underline{j}}\, N_{\underline{j}+\underline{\nu}}) = E(E(N_{\underline{j}}\, N_{\underline{j}+\underline{\nu}}) \mid \mathcal{F}_{\underline{h}})$$

$$= E(\kappa^2 t^2 F_{\underline{j}} F_{\underline{j}+\underline{\nu}} + \kappa t\, \delta_{\underline{\nu},0} F_{\underline{j}})$$

$$= \kappa^2 t^2 (R_{\underline{\nu}} + L_{\underline{j}} L_{\underline{j}+\underline{\nu}}) + \kappa t\, \delta_{\underline{\nu},0} L_{\underline{j}}$$

The covariance is, of course, missing the $L_{\underline{j}} L_{\underline{j}+\underline{\nu}}$ term, but we do not know how to separate it from our data except in a very approximate manner. We can, however, use our knowledge that the structure we seek is on a much smaller scale than the scale of changes in the background. In frequency terms, this means that the background is a low-frequency disturbance, so we need to subject the data to high-pass filtration.

Let

$$(7.9)\quad M_{\underline{j}} = \sum_{\underline{n}} g_{\underline{n}} N_{\underline{j}+\underline{n}}$$

where

$$(7.10)\quad \sum_{\underline{n}} g_{\underline{n}} = 0$$

$$g_{n_1,n_2} = g_{n_1,-n_2} = g_{-n_1,-n_2}$$

$$g_{\underline{n}} = 0 \quad \text{for } |\underline{n}| > K$$

It is convenient to pick the $g_{\underline{n}}$ in the following fashion: Let $g'_{\underline{n}} = G(|\underline{n}|)$, where $G(x)$ is a smooth function with all odd derivatives vanishing at the origin, and vanishing itself for $x > k$. Then let

$$g_{\underline{n}} = \delta_{\underline{n},0} - \frac{g'_{\underline{n}}}{\sum_{\underline{k}} g'_{\underline{k}}} \tag{7.11}$$

Let the Fourier sum of the $g_{\underline{n}}$ be

$$\mathcal{U}(x) = \sum_{\underline{n}} g_{\underline{n}} e^{i\underline{n}\cdot\underline{x}} \tag{7.12}$$

Then $\mathcal{U}(0) = \mathcal{U}'(0) = 0$, and it is easily shown that

$$\mathcal{U}(\underline{x}) = 1 - \sum_{\underline{n}} u(\underline{x} - 2\pi\underline{n}) \tag{7.13}$$

where

$$u(\underline{x}) = \frac{1}{\sum_{\underline{k}} g'_{\underline{k}}} \int G(|\underline{k}|)\, e^{i\underline{k}\cdot\underline{x}}\, d^2k \tag{7.14}$$

Thus by manipulating the weights $g_{\underline{n}}$, we can make the filter funtion $\mathcal{U}(\underline{x})$ cut out as much of the low-frequency and of the spectrum as we desire.

We estimate the covariance of the $M_{\underline{j}}$'s by

$$P^*_\nu = \frac{1}{NM} \sum_{\underline{j}} (M_{\underline{j}} M_{\underline{j}+\nu}) \tag{7.15}$$

Strictly speaking, it would be better to use an expression symmetric in $\underline{\nu}$ and $-\underline{\nu}$, but we shall perform circular averages later anyway, so we avoid this unnecessary complication. We assume here, as before, that we sample over an N- by M rectangle. The $N_{\underline{j}}$'s will supposedly have had the gross part of the sky and zodiacal light removed, but we shall see that this fact does not enter our considerations here. We find from (7.15), (7.9), and (7.8) that

(7.16)

$$E\,(P^*_{\underline{\nu}}) = \frac{1}{NM} \sum_{j} \sum_{\underline{n}} \sum_{\underline{m}} g_{\underline{n}}\, g_{\underline{m}} \left\{ \kappa^2 t^2 \left(R_{\underline{\nu}+\underline{m}-\underline{n}} + L_{\underline{j}+\underline{n}} L_{\underline{j}+\underline{\nu}+\underline{m}} \right) + \kappa t\, \delta_{\underline{\nu}+\underline{m}-\underline{n}}\, L_{\underline{j}+\underline{n}} \right\}$$

We estimate the spectrum in the usual way, i.e.

(7.17)

$$\Phi^*(\eta) = \frac{1}{(2\pi)^2 \kappa^2 t^2} \sum_{\underline{\nu}} w_{\underline{\nu}}\, P^*_{\underline{\nu}}\, e^{i \eta \cdot \underline{\nu}}$$

where the $w_{\underline{\nu}}$ are weights chosen in the same fashion as those used in Chapter III. Inserting (7.16) into (7.17), we find after a tedious bit of algebra that

(7.18)

$$E\,(\Phi^*(\eta)) = \iint_{-\pi}^{\pi} \mathcal{U}^2(\underline{x}) \left\{ \Psi(\underline{x}) + \frac{\lambda^2}{(2\pi)^2} + \frac{\langle L \rangle}{(2\pi)^2 \kappa t} + (2\pi)^2 |\ell(x)|^2 \right\} w(\eta - x)\, d^2x .$$

where

(7.19)
$$\langle L \rangle = \frac{1}{NM} \sum_{\underline{j}} L_{\underline{j}}$$
$$\ell(x) = \frac{1}{(2\pi)^2} \sum_{\underline{j}} L_{\underline{j}}\, e^{i\underline{j}\cdot \underline{x}}$$
$$w(x) = \frac{1}{(2\pi)^2} \sum_{\underline{j}} w_{\underline{j}}\, e^{i\underline{j}\cdot \underline{x}}$$

and we have made the assumption that K is small compared to M and N; if this is not so, $\langle L \rangle$ is altered slightly. Since $\ell(\underline{x})$ is concentrated at small $\underline{x}$ and $\mathfrak{U}^2(\underline{x})$ is proportional to $|\underline{x}|^4$ for small $\underline{x}$, we shall in the following neglect the $(2\pi)^2 |\ell(\underline{x})|^2$ term, even though it can be expected to be large. We shall return to this point later.

One can estimate the relative sizes of the $\psi(\underline{x})$ and the $\frac{\lambda^2}{(2\pi)^2}$ terms only with some difficulty, since even evaluation of the mean background (See Appendix III) requires explicit consideration of the redshift and the k- correction. We can obtain a crude estimate for the steady-state model, however, and it is to this that we now turn our attention.

Suppose that all galaxies have the same luminosity, say L_o, and all have the same Wien-law spectrum,

(7.20)

$$L(\nu) = \frac{L_0}{2\nu_0} (\nu/\nu_0)^2 e^{-\nu/\nu_0} .$$

Suppose that the expected number of such objects per cubic megaparsec is Λ . If $\mathcal{F}_\nu(\Omega)$ is the cosmic light flux from the solid angle Ω, we can write

(7.21)

$$\mathrm{cov}\,(\mathcal{F}_\nu(\Omega), \mathcal{F}_\nu(\Omega')) = \iint_{C\,C'} \Big\{ \frac{L(\nu(1+z))L(\nu(1+z'))}{(4\pi)^2\rho^2\rho'^2 R_0^4(1+z)(1+z')}$$

$$\cdot \rho^2\rho'^2 R^2(t)R^2(t') \left[\Lambda^2 g\left(((t-t')^2+\rho^2R^2\gamma^2)^{1/2}\right) \right.$$

$$\left. + \Lambda\,\delta(t-t')\,\delta^{(2)}(\rho R\underline{\gamma}) \right] \Big\} d\Omega\, d\Omega'\, dt\, dt'.$$

Here $\Lambda^2 g(\alpha)$ is the covariance density, $\underline{\gamma}$ is the angle between the lines of sight, C and C´ the cones generated by Ω and Ω'. We assume that $g(\alpha)$ has the form

(7.22) $\quad g(\alpha) = \beta e^{-\alpha^2/2\ell^2}$

If we insert (7.22) and (7.20) in (7.21), replace ρ and R by their explicit forms in terms of the radial coordinate

t, and pass to the auxil iary function approximation, we get (for the covariance density)

(7.23)

$$\operatorname{cov}\left(F_\nu(\underline{n}), F_\nu(\underline{n}+\underline{\gamma})\right) = \frac{L_0^2 \nu^4}{(8\pi)^2 \nu_0^6} \left\{ \int_{t_0}^{\infty} \frac{\beta \Lambda^2 \ell \sqrt{2\pi}}{H} \left[e^{-2Ht} e^{-\frac{2\nu e^{Ht}}{\nu_0}} \cdot \right.\right.$$

$$\left. \cdot\, e^{-\frac{\gamma^2}{2\ell^2 H^2}(1-e^{-Ht})^2} \right] H\,dt$$

$$\left. + \; H \Lambda \delta^{(2)}(\underline{\gamma}) \int_{t_0}^{\infty} \frac{e^{-2\nu e^{Ht}/\nu_0}}{(e^{Ht}-1)^2} \, H\,dt \right\}$$

The lower limit t_0 corresponds to the radial coordinate of objects closer (brighter) than which we do not include in the background measurement. If we let $u = 1+z = e^{Ht}$, we obtain

(7.24)

$$\operatorname{cov}\left(F_\nu(\underline{n}), F_\nu(\underline{n}+\underline{\gamma})\right) = \frac{L_0^2 \nu^4}{(8\pi)^2 \nu_0^6} \left\{ \left[\frac{\beta \Lambda^2 \ell \sqrt{2\pi}}{H} \cdot \right.\right.$$

$$\left. \cdot \int_{1+z_0}^{\infty} \frac{1}{u^3} e^{-\frac{2\nu u}{\nu_0}} e^{-\frac{\gamma^2}{2\ell^2 H^2}\left(1-\frac{1}{u}\right)^2} du \right]$$

$$\left. + \; H \Lambda \delta^{(2)}(\underline{\gamma}) \int_{1+z_0}^{\infty} \frac{e^{-\frac{2\nu u}{\nu_0}}}{u(u-1)^2} \, du \right\}.$$

Now the spectral density $\mathcal{S}(\underline{k})$ is

(7.25) $$\mathcal{S}(\underline{k}) = \frac{1}{(2\pi)^2}\int \operatorname{cov}\left(F_\nu(\underline{n}), F_\nu(\underline{n}+\underline{\gamma})\right) e^{-i\underline{\gamma}\cdot\underline{k}}\, d^2\gamma$$

so

(7.26) $$\mathcal{S}(0) = \frac{L_0^2\nu^4}{(8\pi)^2\nu_0^6(2\pi)^2}\left(\beta\Lambda \ell^3(2\pi)^{3/2}H + H\Lambda\right)\int_{1+z_0}^{\infty}\frac{e^{-\frac{2\nu u}{\nu_0}}}{u(u-1)^2}\,du$$

$$= \frac{L_0^2\nu^4\Lambda H}{(8\pi)^2\nu_0^6(2\pi)^2}\left(\beta\Lambda \ell^3(2\pi)^{3/2}+1\right)\int_{1+z_0}^{\infty}\frac{e^{-2\nu u/\nu_0}}{u(u-1)^2}\,du$$

Thus the ratio of singular to non-singular part is typically of the order of $[\beta\Lambda\ell^3(2\pi)^{3/2}]^{-1} \sim \frac{1}{5000}$ for $\beta=20$, $\Lambda=.6$, $\ell=3\text{Mpc}$, and thus is, as expected, very small. The integral in (7.26) can be evaluated; in fact,

(7.27) $$\int_{1+z_0}^{\infty}\frac{e^{-2\nu u/\nu_0}}{u(u-1)^2} = E_1(2\nu(1+z_0)) - e^{-2\nu/\nu_0}E_1(2\nu z_0) + \frac{e^{-2\nu/\nu_0}}{z_0}E_2(2\nu z_0).$$

If we fit the covariance (7.24) with a Gaussian by comparing moments as we did for the counts, we find, using (7.27), that

(7.28)
$$\mathrm{cov}\left(F_{\nu_0}(\underline{n}),\, F_{\nu_0}(\underline{n}+\underline{\gamma})\right) \cong$$
$$\frac{L_0^2}{(8\pi)^2\nu_0^2}\left\{.0176\,\frac{\beta\Lambda\ell\sqrt{2\pi}}{H}\,e^{-\gamma^2/\gamma_0^2} + 1.70\,H\delta^2(\underline{\gamma})\right\}$$

at $\nu = \nu_0$, $z_0 = .08$ (this corresponds to rejecting objects brighter than M=20 if the mean luminosity is M=-17, and a wavelength of about two microns, and $\gamma_0 \cong 6\ell H \cong 20'$ for ℓ =3Mpc.

One sees easily that the ratio of singular to non-singular part in the spectrum is essentially unaltered in passing to the discretized process considered in sampling; we can expect a singular contribution from stars of the same order as that from the galaxies, so this too will be negligible compared to the non-singular galaxy part. We may now neglect the $\lambda^2/(2\pi)^2$ term in (7.18).

We can write, as we did in Chapter III, that

(7.29)
$$E(\Phi^*(\underline{\eta})) \cong \iint_{-\pi}^{\pi} u^2(\underline{x})\left\{\psi(\underline{x}) + \frac{\langle L\rangle}{(2\pi)^2\kappa t}\right\} w(\underline{\eta}-\underline{x})\,dx$$
$$\cong \iint_{-\infty}^{\infty} \mathcal{U}^2(\underline{x})\, w(\underline{\eta}-x)\, Q^2(x)\, S^1\!\left(\frac{x}{h}\right) d^2x$$
$$+ \frac{\langle L\rangle}{(2\pi)^2\kappa t}\iint_{-\infty}^{\infty} \mathcal{U}^2(\underline{x})\, w_0(\underline{\eta}-\underline{x})\, d^2x\ ;$$

here

$$Q(\underline{x}) = \frac{2\pi a}{x} J_1\left(\frac{ax}{h}\right) .$$

and $w(\underline{x})$ and $U(\underline{x})$ are defined everywhere by their Fourier series. If we now perform circular averaging on $\Phi^*(\eta)$, we produce

$$\hat{\Phi}(\eta) = \frac{1}{2\pi} \int_0^{2\pi} \Phi^*(\eta \cos\theta, \eta \sin\theta)\, d\theta . \tag{7.30}$$

One can now proceed exactly as in Chapter III; in the notation introduced there, and at the level of approximation considered there,

$$E(\hat{\Phi}(\eta)) \cong 2\pi \int_0^\infty \tilde{\mathcal{U}}^2(x) Q^2(x) \mathcal{S}^1\left(\frac{x}{h}\right) \tilde{w}(\eta, x)\, x\, dx \tag{7.31}$$

$$+ \frac{1}{2\pi} \sum_{\underline{\nu} \neq 0} \int_0^{2\pi} \mathcal{U}^2(\eta \underline{e}(\theta) + 2\pi\underline{\nu})\, Q^2(|\eta \underline{e}(\theta) + 2\pi\underline{\nu}|)\, \mathcal{S}'\left(\frac{|\eta \underline{e}(\theta) + 2\pi\underline{\nu}|}{h}\right)$$

$$+ \frac{\langle L \rangle}{2\pi \varkappa t} \int_0^\infty \tilde{w}(\eta, x)\, \mathcal{U}^2(x)\, x\, dx .$$

The covariance of a pair of $\hat{\Phi}(\eta)$'s can also be calculated in the same fashion as performed there, and we obtain

(7.32)

$$\operatorname{cov}(\hat{\Phi}(\eta),\hat{\Phi}(\eta')) \cong \frac{(2\pi)^3}{NM}\int_0^\infty \xi d\xi \left\{\mathcal{U}^2(\xi)\left(\frac{(\pi a^2)^2}{h^2}\mathcal{S}^1\left(\frac{\xi}{h}\right) + \frac{\langle L\rangle}{(2\pi)^2\kappa t}\right)\right\}^2 \tilde{w}(\eta,\xi)\tilde{w}(\eta',\xi) ,$$

and

(7.33)

$$\operatorname{var}(\hat{\Phi}(\eta)) \cong \frac{(2\pi)^3}{NM}\int_0^\infty \xi d\xi \left\{\mathcal{U}^2(\xi)\left(\frac{(\pi a^2)^2}{h^2}\mathcal{S}'\left(\frac{\xi}{h}\right) + \frac{\langle L\rangle}{(2\pi)^2\kappa t}\right)\right\}^2 \tilde{w}^2(\eta,\xi)$$

If we choose $w_{\underline{j}}$ as in (3.45), we find

(7.34)

$$\operatorname{var}(\hat{\Phi}(\eta)) \cong \frac{\sqrt{\pi}}{\alpha\eta NM}\tilde{\mathcal{U}}^2(\eta)\left(\frac{(\pi a^2)^2}{h^2}\mathcal{S}'(\eta/h) + \frac{\langle L\rangle}{(2\pi)^2\kappa t}\right)^2$$

for $\eta \geq \alpha$; recall that α is the characteristic scale in the gaussian form for w(x), and is approximately $2.58m^{-1}$, where m is the radius of the record used in estimating the covariance. If we return to a notation in which the nature of the original process is more transparent, we have, setting

(7.35)
$$\mathcal{S}^*\left(\frac{\eta}{h}\right) = \frac{h^2}{(\pi a^2)^2} \hat{\Phi}(\eta),$$
$$L^2 = NMh^2,$$
$$L' = mh = \frac{2.58h}{\alpha},$$

(7.36)
$$\operatorname{var}(\mathcal{S}^*(k)) = \frac{0.49L'}{kL^2} \tilde{u}^2(hk)\Big[\mathcal{S}'(k) + \frac{\langle L\rangle h^2/(\pi a^2)^2}{(2\pi)^2 \kappa t}\Big]^2,$$

which is a form similar to (3.54) for the counts.

We wish now to investigate the problem of optimizing the parameters in some sense in order to reduce the sampling time to a reasonable value. Let us return for a moment to (7.34). If we count at each diaphragm position for time t, then the total sampling time will be $T = NMt$. For a given grid spacing, then, we can write

(7.37)
$$T = \frac{4.56L't}{k\sigma_0^2 h^2} \tilde{u}^2(hk)\Big[1 + \frac{\langle L\rangle h^2/(\pi a^2)^2}{(2\pi)^2 \mathcal{S}^1(k)\kappa t}\Big]^2$$

and for a given value of the error $\sigma_0^2 = \operatorname{var}\mathcal{S}^*(k)/(\mathcal{S}'(k))^2$, the total time is minimized for the choice of t such that

(7.38)
$$\frac{\langle L\rangle h^2/(\pi a^2)^2}{(2\pi)^2 \mathcal{S}^1(k)\kappa t} = 1$$

This is, of course, impossible to realize for all k; note, however, that the variance for a given sample has a factor

1/k, so that for large k, where we can expect $\mathcal{S}^1(k)$ to be small, the variance also drops. We can therefore expect to maintain at least reasonably constant fractional error over a small range in k by adjusting t so that (7.37) holds at the lower end of the k-interval. In this case, let us take $\mathcal{S}^1(0)$ as a reasonable estimate of the size of $\mathcal{S}(k)$; we can then make an effort to cast (7.37) into a more transparent form.

If F_g is the mean intensity of the extra-galactic light, and F_t is the local total mean intensity, then we can write

(7.39)
$$\langle L \rangle = \pi a^2 F_t$$
$$(2\pi)^2 \mathcal{S}'(0) = F_g^2 \int G(\gamma)\, d^2\gamma$$
$$= F_g^2 \mathcal{A}_{corr}$$

Where $G(\gamma)$ is the normalized covariance (non-singular part) of the cosmic light; the integral is a quantity which is analogous to the covariance measure (area, in this case) introduced in Chapter III, and represents (roughly) the area over which the presence of one galaxy affects the probability distributions of neighbors. Then (7.37) becomes

(7.40) $$\overline{\mathcal{N}} = \left(\frac{F_t}{F_g}\right)^2 \frac{h^2}{\mathcal{A}_{corr}},$$

where $\bar{\mathcal{N}} = \pi a^2 F_t \kappa T$ is the optimum mean photon count. For the steady-state calculation performed earlier, $\mathcal{A}_{corr} \simeq .2\,\gamma_o^2$.

Several pertinent remarks can be made already. The time required to accumulate $\bar{\mathcal{N}}$ counts is inversely proportional to the diaphragm area πa^2, so the larger the diaphragm (keeping $a < h$, recall) the better. The time required for a single measurement (supposing a to be some fixed fraction of h) is independent of h, since t would then be proportional to $\bar{\mathcal{N}} h^{-2}$, which depends only on the flux. The total time, though, goes, from (7.36), (7.37), as h^{-2}, so the total observing time required goes up linearly with the number of sample points.

Since γ_o is of the order of twenty minutes of arc, we could not reasonably expect to be able to use diaphragms larger than, say, four or five minutes. With $h = 1/5\,\gamma_o$ (about 4 minutes), and the approximate figure

$F_g \cong F_{t/100}$ (This is taking F_t of $150 S_{10}$[1], F_g about $1.5 S_{10}$ - this last figure is the approximate result for the steady-state model, using the result that the count relation of Holmberg (26) predicts about $.25 S_{10}$ from galaxies brighter than m=18 when aperture effects are included; the relativistic models for $q_o < 5$ predict somewhat higher cosmic fluxes.), we require 2000.

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1. The unit S^{10} is the flux from one 10-th magnitude object per square degree.

The total brightness from this area (say 10 square minutes) is about equivalent to one eleventh-magnitude object, so that, with the 200-inch telescope, the time required to make <u>one</u> count with, say, a 1000Å bandpass, a telescope-plus-filter-plus-phototube efficiency of 1 percent, is about five milliseconds. This efficiency figure is appropriate to trialkali (S20) tubes at about 7000Å , where, as we shall see, cosmological effects are appreciable. Thus each measurement requires about 10 seconds. It thus seems likely that the <u>actual measuring time</u> will be only a small part of the overall operating time, unless an extremely efficient means of moving from one diaphragm site to another is provided for. The possibility of multi-channel techniques for making many measurements simultaneously is quite attractive; though they do not improve the measuring-time to setting-time ratio, they would obviously cut the overall time. Considering this, it would be advantageous to increase t, thereby increasing accuracy at essentially no expense to total operating time.

Let us assume for the sake of simplicity that $\mathcal{S}^1(k)$ is Gaussian in form; i.e. $\mathcal{S}^1(k) = F_g{}^2 \mathcal{A}_{corr.} e^{-\frac{k^2\gamma_0^2}{2}}$, corresponding to the Gaussian form for the autocovariance. We need data covering the major run of $\mathcal{S}^1(k)$, say out to $k_2 \sim 2/\gamma_o$. If we take $L' = 6\gamma_o$, a reasonable choice, and limit k at the low end to $k_1 = 2/L' = \frac{.33}{\gamma_0}$ (recall that

(7.36) does not hold for $k \lesssim \sqrt{\pi}/L'$), we find at the lower end of the range in k that $\sigma_0 \sim \frac{3.5\gamma_0}{L}(1+\frac{1}{p})$ and, at the high end, $\sigma_0 \sim \frac{\gamma_0}{L}(1+\frac{7}{p})$ where p is the factor by which the observing time per reading exceeds the "optimum" defined by (7.37). In actuality, the spectrum does not fall off so rapidly as a Gaussian, and the problem for large k is not so acute; in any case the variance at the two ends can be made to agree by choosing a p of about 2, which results in an integration time of twenty seconds or so, (using, as before, a circular diaphragm of diameter three minutes on a net with four-minute spacings.) With this choice, a ten percent expected error at the ends of the k-range can be obtained with $L \cong 30\gamma_0$, or about ten degrees. The standard deviation in the center of the range is of the order of half this, and the standard deviation of the quantity

(7.40)
$$\mathcal{E} = \left\{ \int_{k_1}^{k_2} \frac{\operatorname{var} S^*(k)}{(S^1(k))^2} \frac{dk}{k_2 - k_1} \right\}^{1/2}$$

is about .08. Since $\mathcal{E}$ is effectively the sum of several almost-uncorrelated quantities, it should, in addition, be nearly normally distributed. If we have at our disposal a photometer with twenty independent channels (for which the relevant electronic instrumentation is currently being built for the Hale telescope), one would obtain measurements for one square degree (225 points) in about 200 seconds, neglecting setting time. Thus the entire measurement could be completed in <u>less than a night</u> if one devises efficient ways to get about in the sky. In practice, the time may be an order of magnitude larger than this, but is by no means prohibitive - given the necessary equipment.

We need still to discuss the high-pass filter function $\mathfrak{U}(x)$. Variations with scales of about a degree contribute to $S^1(k_1)$, and the relevant scales for variation in the airglow and zodiacal light is of the order of ten degrees (for changes of ten percent or so), which correspond to frequencies of the order of $k_{1/10}$ and amplitudes of some ten times the mean cosmic light. If we use the same Gaussian set for both the $w_{\underline{j}}$ and the $g_{\underline{n}}$, then

(7.41)
$$\mathfrak{U}(hk) \cong 1 - e^{-\frac{k^2(1.645\gamma_0)^2}{2}}.$$

At $k_1 = \frac{1}{3\gamma_0}$, this has the value .162; at $k_{1/10}$, .0016. The slow variations in the contaminating light thus should not appear at all in the processed data. At first glance, it may seem that the filter function serves no useful puspose; one could do as well by simply ignoring the derived low-frequency end of the spectrum. That this is in fact not true may be seen from (7.31); it is clear that without such a filter, the large contribution at low frequency from the slow variations would be mixed into the region of interest by the smoothing kernel $w(x)$, which has a width of about $5k_1$.

VIII. OBSCURATION IN THE GALAXY

So far in our discussion we have neglected entirely the effects of obscuring material in the Galaxy through which we must observe. Ideally, one would like to be able to treat the obscuration in a non-statistical manner; this requires a detailed map of the reddening of halo objects over the region in which we are counting or observing the background. Such a map can, at least in principle, be constructed from three-color (or more detailed) observations of horizontal-branch A stars (Sandage, (32)), but no such project has been undertaken over a large region. Considering the care which must go into the preparation of faint counts, it seems quite worthwhile to prepare such a map. On the other hand, we shall see that a minimum of knowledge concerning the obscuration is sufficient to determine its effect on the spectra of the processes we observe.

Let us consider first the background light, because its treatment is by far the simpler. Let, as before, $F(\underline{\gamma})$ be the specific intensity from direction $\underline{\gamma}$ (in the wavelength region of interest), and let $A(\underline{\gamma})$ be the transmission in this direction in this band; we assume that the band is sufficiently narrow that $1-A(\underline{x})$ does not depend significantly on the radiation spectrum of the objects which compose the cosmic light. Let $\xi(\underline{x})$ be the normalized autocovariance function for $A(\underline{\gamma})$, and let $\alpha(\underline{\gamma})$ be the mean transmission. In this picture, the large-scale run of $A(\underline{\gamma})$, say that due to the cosecant b term from the plane model, is absorbed into $\alpha(\gamma)$. The remaining fluctuations are regarded as stochastic, and we <u>assume</u> that the covariance has the form

$$(8.1) \quad \mathrm{Cov}(A(\underline{x}),A(\underline{y})) = \alpha(\underline{x})\alpha(\underline{y})\,\xi(|\underline{x}-\underline{y}|)$$

That is, ξ does not depend on direction or absolute position. This is clearly not justifiable <u>a priori</u>, but at high galactic latitudes where $1-A(\underline{x})$ is small anyway, deviations from the form (8.1) are not important. If $\hat{F}(\underline{\gamma})$ is the observed specific cosmic light intensity, we have

$$(8.2) \qquad \hat{F}(\underline{\gamma}) = A(\underline{\gamma})F(\underline{\gamma}),$$

and we can clearly assume $A(\underline{\gamma})$ and $F(\underline{\gamma})$ independent. Thus

$$(8.3) \qquad E\hat{F}(\underline{\gamma}) = \alpha(\underline{\gamma})EF(\underline{\gamma}) \quad ,$$

and we can define a corrected intensity $F_c(\underline{\gamma}) = \dfrac{\hat{F}(\underline{\gamma})}{\alpha(\underline{\gamma})}$. In practice, since we cannot separate the cosmic component from the contamination, we multiply the <u>total</u> flux by $\dfrac{1}{\alpha(\underline{\gamma})}$; the changes in $\alpha(\underline{\gamma})$ are very slow, and will be removed by the low-frequency filtration along with the changes in the contaminating light. Then

(8.4)

$$E\Big(F_c(\underline{\Upsilon})\ F_c(\underline{\Upsilon}+\underline{x})\Big) = \Big\{1+\xi(\underline{x})\Big\} \cdot E\Big(F(\underline{\Upsilon})\ F(\underline{\Upsilon}+\underline{x})\Big) ,$$

and

(8.5)

$$\mathrm{cov}\Big(F_c(\underline{\Upsilon}),\ F_c(\underline{\Upsilon}+\underline{x})\Big) = \mathrm{cov}\Big(F(\underline{\Upsilon}),\ F(\underline{\Upsilon}+\underline{x})\Big) +$$
$$\xi(\underline{x}) \cdot \Big\{\mathrm{cov}\Big(F(\underline{\Upsilon}),\ F(\underline{\Upsilon}+\underline{x})\Big) + \Big(EF(\underline{\Upsilon})\Big)^2\Big\}$$

Thus the spectrum of F_c becomes

(8.6)

$$\mathcal{S}_c(\underline{k}) = \mathcal{S}(\underline{k}) + \langle F\rangle^2\, \Xi(\underline{k}) + \iint_{-\infty}^{\infty} \mathcal{S}(\underline{y})\, \Xi(\underline{k}-\underline{y})\, d^2\underline{y} ,$$

where $\Xi(\underline{k})$ is the transform of $\xi(\underline{x})$; i.e., the spectrum of $A(\underline{\Upsilon})$. We expect the obscuration to be a fairly large-scale phenomenon; if this is not so, we must have detailed knowledge of the structure of $\xi(\underline{x})$ or $\Xi(\underline{k})$, and it is <u>essentially no more difficult</u> to construct a detailed map of $A(\Upsilon)$. If so, however, Ξ will be of appreciable size only for low frequencies, and is small in any case; if the mean absorption is in the neighborhood of 10%, then $\xi(\underline{x})$ will be of the order of .02 or .03 at maximum, and Ξ will be correspondingly small. Thus the high-pass filtration will remove the $\langle F\rangle^2\Xi(\underline{k})$ component, and we can approximate the other term to yield

$$(8.7)\quad \mathcal{S}_c(\underline{k}) \cong \mathcal{S}(\underline{k})\left[1+\iint_{-\infty}^{\infty} \Xi(\underline{y})\,d^2y\right]$$

$$= \mathcal{S}(\underline{k})\bigl(1+\xi(0)\bigr).$$

Thus we need only to know the mean transmission $\alpha(\gamma)$, and the normalized variance $\xi(0)$.

The effect on the counts is a little more difficult to analyze, because the absorption causes a shift in the observed magnitude rather than a simple multiplication of the number by a factor. For small $1-A(\underline{\gamma})$, however, one can use a simple linearization approach. The number $d_{m^o}\tilde{\mathcal{Z}}_\beta(m^o,\Omega)$ of β-galaxies in Ω in an interval m^o of observed (after absorption but before selection) apparent magnitude m^o is

$$(8.8)\quad d_{m^o}\tilde{\mathcal{Z}}_\beta(m^o,\Omega) = \int_\Omega d_m\, d_\gamma\, \mathcal{Z}_\beta\bigl(m^o - \mathcal{A}(\underline{\gamma})\bigr),$$

where $\mathcal{A}(\underline{\gamma}) = -2.5\log A(\underline{\gamma})$, and

$$(8.9)\quad E\,d_{m^o}\bigl(\tilde{\mathcal{Z}}(m^o,\Omega)\bigr) = d_{m^o}\int_\Omega d\omega \int_0^\infty r^2\,dr\; E\bigl(\lambda_\beta(m^o - \mathcal{A}(\gamma) - 5\log r)\bigr).$$

But

$$(8.10)\quad \lambda_\beta(m^o - \mathcal{A}(\underline{\gamma}) - 5\log r) \cong \lambda_\beta(m^o - E\mathcal{A}(\gamma) - 5\log r)$$

$$-\bigl(\mathcal{A}(\underline{\gamma}) - E\mathcal{A}(\gamma)\bigr)\frac{d\lambda_\beta}{dM}\bigl(m^o - E\mathcal{A}(\gamma) - 5\log r\bigr)$$

and

(8.11)

$$E\lambda_\beta(m^\circ - \mathcal{A}(\underline{\gamma}) - 5\log r) = \lambda_\beta(m^\circ - E\mathcal{A}(\underline{\gamma}) - 5\log r)$$

$$+ \frac{1}{2}E(\bar{\mathcal{A}}^2(\underline{\gamma}))\frac{d^2\lambda_\beta}{dM^2}(m^\circ - E\mathcal{A}(\gamma) - 5\log r) + \ldots$$

To this order,

(8.12)

$$\mathcal{A}(\underline{\gamma}) = \frac{-2.5}{\ln 10}\ln A(\gamma)$$

$$= \frac{2.5}{\ln 10}(1 - A(\underline{\gamma}))\left\{1 - \frac{1 - A(\underline{\gamma})}{2} + O(1 - A(\gamma))^2\right\};$$

$$E\mathcal{A}(\underline{\gamma}) = \frac{2.5}{\ln 10}\left\{1 - \alpha(\underline{\gamma}) - \frac{1}{2}\alpha^2(\underline{\gamma})\xi(0) + O(E(\bar{A}^3))\right)$$

These calculations were carried out in the nonrelativistic limit; exactly analogous results exist for the relativistic models.

One finds in a similar manner that the count covariance density becomes

(8.13)

$$G^{c}_{\beta\beta'}(m, m'; \gamma) \cong G_{\beta\beta'}(m + E\mathcal{A}, m' + E\mathcal{A}, \gamma)$$

$$+ \frac{1}{2}\left(\frac{2.5}{\ln 10}\right)^2 \left\{ \frac{\partial^2 G_{\beta\beta'}}{\partial m^2}\alpha^2 \xi(0) + 2\frac{\partial^2 G_{\beta\beta'}}{\partial m \partial m'}\alpha^2 \xi(\gamma) \right.$$

$$\left. + \frac{\partial^2 G_{\beta\beta'}}{\partial m'^2}\alpha^2 \xi(0) \right\}$$

with α (and hence $E\mathcal{A}$) evaluated at the approporiate points. Since α varies slowly over the sky, the new process is no longer strictly stationary, nor can it be made "second order stationary" by renormalizing. In any one reasonably small region, however, α is very nearly constant, and we can replace it by its mean value in that region. The correction terms are very small, and any approximate means of arriving at the second derivatives should suffice. The count covariance written for the j-intervals of measuring magnitude is

(8.14)

$$G^{c}_{\beta j \beta' j'}(\gamma) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{\beta j}(m)\, C_{\beta' j'}(m')\, dm\, dm'\, G^{c}_{\beta\beta'}(m, m'; \gamma)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dm\, dm' \Big\{ C_{\beta j}(m)\, C_{\beta' j'}(m')$$

$$+ \frac{1}{2}\left(\frac{2.5}{\ln 10}\right)^2 \alpha^2 \Big[\frac{d^2 C_{\beta j}(m)}{dm^2}\, C_{\beta' j'}(m')\, \xi(0)$$

$$+ 2\, \frac{dC_{\beta j}(m)}{dm}\, \frac{dC_{\beta' j'}(m')}{dm'}\, \xi(\gamma) + C_{\beta j}(m)\, \frac{d^2 C_{\beta' j'}(m')}{dm'^2}\, \xi(0) \Big] \Big\} \cdot$$

$$\cdot\; G_{\beta\beta'}(m + E\mathcal{A},\; m' + E\mathcal{A},\; \gamma)$$

after appropriate parts integration. The spectrum is obtained as before; and we have

(8.15)

$$\mathcal{G}_{\beta j \beta' j'}(\eta) \cong \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dm\, dm' \Big\{ C_{\beta j}(m)\, C_{\beta' j'}(m')$$

$$+ \frac{1}{2}\left(\frac{2.5}{\ln 10}\right)^2 \alpha^2\, \xi(0) \Big[\frac{d^2 C_{\beta j}}{dm^2} C_{\beta' j'}\; \frac{dC_{\beta j}}{dm}\, \frac{dC_{\beta' j'}}{dm'^2} + C_{\beta j}\, \frac{d^2 C_{\beta' j'}}{dm'^2} \Big] \Big\} \cdot$$

$$\cdot\; \mathcal{G}_{\beta\beta'}(m + E\mathcal{A},\; m' + E\mathcal{A},\; \eta)$$

In this form, we need only know the approximate derivatives of the selection function $c_{\beta j}(m)$. The non-stationarity of the real counts will become appreciable for large areas, but we have already seen that the counts must be broken down if large areas are to be covered, in order that the validity of the plane approximation be maintained. Thus one can use values for the absorption appropriate for each region treated as a sample.

IX. APPLICATION OF THE THEORY TO COSMOLOGICAL TESTS

As has been pointed out in the previous chapters, much that is known at best imprecisely must be clarified before the theory developed here can be used for effective cosmological tests. The object of the work outlined in this chapter is to show that with the best available data, there is indeed reasonably strong discrimination between the cosmological models of current interest; and that cosmological information can be obtained from the theory as soon as uncertainties in the luminosity function and total galactic spectral energy distributions are reduced somewhat, and perhaps some insight is gained into at least the gross features of the evolution of galaxies.

There exist reasonably detailed measures of the energy distributions of the nuclei of spirals and of elliptical galaxies (Oke, 33, 34) but very little work has been done on the bluer, low-surface brightness outer regions of spirals and irregular nebulae. Work on this is in progress, and instrumentation now under construction for the Hale telescope will make the task much easier than it is at present. The measurements which are available for the redder systems fit a Wien-law spectrum

(9.1) $$L(\nu) = \frac{L_0}{2} \frac{\nu^2}{\nu_0^3} e^{-\nu/\nu_0}$$

quite well (to within about 10 percent for the cases tested: two giant ellipticals, NGC 4472 and NGC 4374, and the dwarf elliptical M32), and it was decided to use Wien spectra in the computations for the other types in lieu of something better. The fit in any case should be quite good to the red of the Balmer limit at 3646Å. Systems most of whose light comes from early-type stars with appreciable Balmer discontinuities will have rather less flux in the ultraviolet than predicted by a fit of a Wien law to the visible part of the spectrum.

Six β-classes of objects were used in the calculations; the properties chosen for them are given in Table 9.1; ν_0 is the parameter in the Wien spectrum (9.1). The luminosity function of each type is chosen to be a Gaussian:

$$\lambda_\beta(M) = \frac{\lambda_\beta}{\sqrt{2\pi\sigma_\beta^2}} e^{-\frac{(M-\langle M\rangle_\beta)^2}{2\sigma^2}}, \tag{9.2}$$

with the indicated scale, mean, and variance. The total luminosity function $\lambda(M) = \sum_\beta \lambda_\beta(M)$ fits the Holmberg (26) total luminosity function (except for a change in brightness level corresponding to a decrease in the value of the Hubble constant from 140 to 100, the value used here), to within 20 percent, which is well within the limits of error for the Holmberg function. The individual means and variances are close to those given by Holmberg (35) and deVaucouleurs (36), but have been adjusted slightly to give a better fit to the total function of Holmberg. The extension to faint magnitudes ($M_V > -15$) is very uncertain, but even the exponential increase ($\lambda \propto 10^{.2M}$) favored by Zwicky (27) has little effect on the results given here; essentially all the light and number counts come from the bright end of the luminosity function, and the (relatively) small uncertainty here has much more effect on our results than the enormous uncertainty at the faint end.

The values of ν_o for each class were chosen to fit the mean color indices given by Holmberg (26) for each type, using as a reference the fit for the ellipticals obtained from the scans. The color indices were computed with the effective filter-sensitivity combinations of Stebbins and Whitford (Humason, Mayall, and Sandage (20)); with the values of ν_o obtained from this fit, B-V colors and bolometric corrections were computed using the relations of Mathews and Sandage (37). These integrations, as well as the computations to follow, were done numerically using the C.I.T. 7040-7094.

Use of the Wien-law spectrum makes computation of the k-correction for redshift of the spectrum through the instrumental bandpass extremely simple; if we define the effective wavelength and bandpass as

(9.3) $$\bar{\nu} = \frac{\int_0^\infty \nu\, S(\nu)\, d\nu}{\int_0^\infty S(\nu)\, d\nu}$$

and

(9.4) $$(\Delta\nu)^2 = \frac{\int_0^\infty (\nu-\bar{\nu})^2 S(\nu)\, d\nu}{\int S(\nu)\, d\nu}$$

Where $S(\nu)$ is the effective response function (of the plate, photomultiplier, or other receiving device), then to second order in $\alpha = \Delta\nu/\nu_0$, we have

(9.5) $$k(z,\nu_0) = 2.5 \log \left\{ \frac{\int_0^\infty S(\nu) L(\nu)\, d\nu}{\int_0^\infty S(\nu) L(\nu(1+z))\, d\nu (1+z)} \right\}$$

$$= \frac{5}{2\ln 10} \left\{ z\bar{u}\left(1 + \frac{z\alpha^2}{\bar{u}^2}\right) - \frac{\alpha^2}{2}(z^2 + 2z) - 3\ln(1+z) \right\},$$

where $\bar{u} = \bar{\nu}/\nu_0$. Even for rectangular bandpasses of 1000Å at 5000Å, α^2 is .16, and the dropped fourth-order terms are of the order of two or three percent. The k-corrections for a 1000Å bandpass at 6500Å (typical of red filter - 103aE plate combinations) are plotted in Figure 9.1.

The mean numbers $\Lambda_{\beta m}$, computed using (4.9)-(4.12), are given in Table 9.2 for all the cases calculated. Quadrature was performed using Simpson's rule on 500 points from Z=0 to Z=1.7; the contribution from more distant objects is less than one part in 10^4 for all cases. The limiting magnitude is taken to be 20 (photored), with a counting dispersion of .5 magnitude; the form of the cutoff used was $f(m) = \frac{1}{2}\left(1-\tanh\left(\frac{m-m_o}{\sigma_m}\right)\right)$, but the results are quite independent of the form chosen. Thus the counts at $m_R = 20$ are down about a factor of 2 from the "total" theoretical count, while the others are almost unaffected.

To compute autocovariances or spectra, we need information concerning the spatial autocovariance function. I have chosen the form proposed by Limber (12) for the computations here;

(9.6) $$g_{\beta\beta'}(\alpha) = \kappa C_{\beta\beta'} e^{-\alpha^2/2\ell^2}$$

where ℓ at the present epoch is chosen to be 4 Mpc; Limber obtained values around 2.5 to 3., but the change in the Hubble length brings this up to about 4. If ℓ changes with time, κ changes also; this is discussed in Chapter 4. The matrix $C_{\beta\beta'}$ has entries of the order of 25, if Limber's analysis is correct, but little more is known about it, except that ellipticals tend to cluster more markedly than spirals. To preserve the positive-definiteness of the autocovariance function, C must be a positive-definite matrix (it is, of course, symmetric), and the ad hoc form

$$(9.7) \qquad C = \begin{pmatrix} 30 & 25 & 20 & 20 & 25 & 20 \\ 25 & 25 & 20 & 20 & 25 & 25 \\ 20 & 20 & 16 & 16 & 20 & 20 \\ 20 & 20 & 16 & 16 & 20 & 20 \\ 25 & 25 & 20 & 20 & 25 & 25 \\ 20 & 25 & 20 & 20 & 25 & 30 \end{pmatrix}$$

was chosen, to exhibit something like a factor of 2 more "clustering tendency" for the giant E's than for the late-type spirals. Better values must await counts of brighter nebulae made by type. One has, of course, no *a priori* knowledge that the correlation length ℓ is independent of β, or that the Gaussian form is necessarily correct; evidence that the function is indeed *not* Gaussian will be introduced later.

Computations of spectral densities for the total count ((4.40), (4.43)) using this form for $g_{\beta\beta'}$, the came cutoff parameters as for the counts, and in the auxil iary-function approximation (which is excellent for the brightness levels considered) are given in Figures 9.2-9.4. The abscissa is $\log\left((2\pi)^2 \mathcal{G}(\eta) \Lambda^{-1}{}_{tot}\right)$; when this quantity is less than zero, the point terms in the variance dominate, and it becomes very difficult to get good estimates with reasonable sample sizes, as was discussed in Chapter III.

The curves in Figure 9.2 are for constant $\mathcal{l}$, an interval of one magnitude around $m_R = 20$, and for values of $q_0 = -(\ddot{R}R/\dot{R}^2)_0$ of 0 to 5 (for the Friedmann universes with vanishing cosmological constant and pressure), and for the steady-state. One can easily obtain accuracy of 10 percent in the spectral density estimates; this gives a "resolution" of about .08 in the ordinate. (These spectra are "raw" - that is, unsmoothed and unaliased, but aliasing is unimportant for these spectra with reasonable --10' or less-- cell sizes, and the smoothing does not introduce any qualitative differences in the spectra.). Thus one should be able to separate, say, the $q_0 = .25$ model from the $q_0 = 1$ one with no appreciable difficulty, <u>provided one knows the spatial correlation from counts of brighter nebulae</u>.

The spectra for m=18, 19, and 20 for the $q_0=1/2$ case are plotted in Figure 9.3, again assuming constant $\mathcal{l}$. The variation with m is, as expected, fairly large, and accuracy of the order of .1 magnitude or so must be obtained in order to keep the uncertainty from this source from becoming an appreciable fraction of the statistical error in $\mathcal{G}(\eta)$. Figure (9.4) shows the effect of a variable correlation length on the spectrum. The variation considered, $\ell \propto (1+z)^{-1/3}$, masks the difference in Figure (9.1) between the $q_o=1/2$ and the $q_o=0$ models; it thus becomes imperative to formulate some theoretical method to determine the time development of $g_{\beta\beta'}$, or at least of $\ell(\tau)$. This is the really weak point in the use of these tests; the other uncertainties need only better data that new instruments and emulsions should be able to provide shortly, but the time-dependence of the spatial co-variance poses a difficult theoretical problem, on which, to the best of the author's knowledge, there has been little progress.

The mean background intensity due to galaxies brighter than m=19.0 (photored) is plotted in Figure 9.5. The curves are labeled with the value of q_o, and the units are erg/sec-cm^2-str-c/s . A factor of roughly five separates the $q_o=0$ model from the $q_o=5$ one. Absorption by obscuring matter in galaxies was included, but is an extremely small effect (about 1 percent), and might as well have been neglected in this study. The spectral density $\mathcal{S}(\eta)$ for λ=7000Å and λ=10,000Å is plotted in Figures 9.6 and 9.7, this time smoothed and averaged for comparison with observations with a diaphragm 4' in diameter with centers on a 5' grid, and with covariances taken over a circle twenty grid spacings in radius (i.e., m=20; see (7.31) for the expression in terms of the raw spectrum.) The quantity plotted is $E(\mathcal{S}^*(\eta))$ with the term in $\langle L \rangle$ removed. (See 7.35.) The discrimination between models here is quite large at both wavelengths; considerations of detector efficiency strongly favor observation at 7000Å over those at 10,000Å.

The size of the spectrum at zero was somewhat overestimated in the rough calculations of Chapter VII, and this, together with the effect of the very broad smoothing kernel, brings the value of the optimum photon count up a factor of 4. On the other hand, little or no increase needs be made to maintain the variance at a suitable low level out to η=500 or so, so the net increase is only a factor of 2 above the value obtained before. These spectra are quite insensitive to variations in the rejection magnitude, but again (expectedly) change rather markedly if ℓ is allowed to vary with time. The 7000Å spectra for the $q_o=1/2$ model for constant ℓ and $\ell \propto (R(\tau)/R_o)^{1/3}$ are plotted in Figure 7.8. The interesting fact is that the variation seems to be in the opposite sense to that for the counts; this is not really the case, since the primary effect in both cases is to raise the high-frequency end of the spectrum relative to the low frequencies. The low frequencies, however, are relatively unaffected by the change in the count spectra, but are depressed in the spectra of the cosmic light. It might be possible to construct a crude check on the constancy of ℓ over the range of interest in this fashion, but this can be no substitute for a satisfactory theory. Note, though, that there is no real possibility of confusion with another value of q_o, since the two curves in Figure 9.8 intersect at about η =450, but the variable-ℓ curve is almost indistinguishable from the steady-state one. The statistics of the cosmic light thus appears quite promising for use as a cosmological test, again given better "input" data and a satisfactory theory for the time variation of ℓ , but perhaps can yield information of some interest even without the latter.

Before we pass to a discussion of the pilot study, a few remarks might be made on the usefulness of the data on second moments of the spatial distribution of galaxies for other than direct tests applied to the cosmological problem. The author (29) has shown that the measures of angular sizes of distant objects are disturbed by the inhomogeneous distribution of mass in the universe, and the quantity needed to predict the size of the effect is precisely the auxil iary covariance function (See the brief discussion in Chapter V.) The second moments are of primary interest to those interested in the clustering of galaxies *per se*, since any clustering model predicts second moments which must fit the observed values. Inasmuch as the clustering of galaxies probably reflects the fragmentation of an originally nearly homogeneous medium (if the universe is a general relativistic one with a singular origin), the second moments and their time development will doubtless play a role in any successful cosmogonical theory for the formation of galaxies and clusters of galaxies. We have already seen that a knowledge of $\mathcal{G}(0)$ allows the variance of total counts to be predicted, and thus accuracy limits to be placed on the number-magnitude cosmological test (Sandage, (1)) for any sample size. It thus seems worthwhile pursuing the subject even if the second-moment data itself were not of interest directly for cosmology.

We turn now to an application of the count-statistics theory to a set of galaxy counts made by the author on a region $5°\times6°$ centered at 14h31m, +31°48', near the distant Boötes cluster of galaxies (Humason, Mayall, and Sandage, (20)). Counts were made to the limit of an Eastman 103aE plate taken with the 48-inch Schmidt camera at Palomar; the plate was used in combination with a red plexiglass filter cutting off at about 6000Å. Galaxies were counted in 10' squares, using a counting reticle kindly lent by Professor Fritz Zwicky. A magnitude sequence both for stars and nebulae had already been set up in the region in the course of the Humason, Mayall, and Sandage (20) study of the Boötes cluster, and it was hoped that this sequence could be used to calibrate the cutoff statistics. That it could not be so used was only the last of several difficulties. Most serious was that the plate is not of uniform quality; the focus is poor along the extreme southern edge, and there appears to have been a very slight rotational shift of the plate about a point in the northwest corner (the latter is not serious.) Thus the region counted was not the full $6.5^{\circ} \times 6.5^{\circ}$ field covered by the 14x14-inch plate, but only the northwestern 5° (N-S) x 6° (E-W) portion of it. There was a noticeable focus change over this portion of the plate so the uniformity of count statistics over the region is difficult to assess; there at least were no significant trends across the plate. Counting was performed first by rows from east to west, and then repeated by columns from north to south. There was sufficient variation in the two sets of counts to make the author rather distrustful of the results, although a third check count from east to west showed good agreement with the second count. It is very difficult at the limit to distinguish stars from galaxies, particularly in the regions where the focus was less than excellent, and there is doubtless a random sprinkling of faint stars included in the count. These should affect only the mean, however, and should not affect the covariance structure except for its scale. The circularly averaged correlation and spectrum obtained from the counts are shown in Figures 9.9 and 9.10.

A fit with a theoretical spectrum requires that one know the limiting magnitude and dispersion, but the count was essentially complete to the limit of the available magnitude sequence ($M \sim 18.4$ photored). A series of out-of-focus plates taken to calibrate the plate on which the counts were made served to indicate only that most of the count was from objects too faint to register as out-of-focus images, although the first such plate was only .62 mm out of focus, yielding images .156 mm in diameter. (All of the nebulae in the magnitude sequence gave visible discs on the first out-of-focus plate.) It was disturbing however, to note that a comparison star at M=18.0 had been counted as a galaxy; one of the sequence objects (out of 25) had been missed, and it was a very condensed elliptical at M=17.8. Thus it was clear that a fairly large dispersion in the cutoff existed, but there was no information as to the value of the cutoff itself. It was not expected that the cutoff would be much fainter than 19, so cosmological effects should not be loo large if q_o is indeed between zero and one (or if the steady-state model is correct); so a fit to the limiting magnitude was made by comparison with the observed mean number count. This procedure has little to recommend itself, but it allowed one to proceed.

Since a change in the limiting magnitude primarily causes a change in scale of the normalized spectral density (See Figure 7.3), and a change in the dispersion has little effect at all, one should still be able to make meaningful remarks on the basis of such a very approximately determined cutoff. A cutoff at m_R=19.2 with a dispersion of 1.5 magnitudes (the latter may be a bit large) was obtained by this procedure, after an arbitrary 20 percent allowance was made for the inclusion of faint stars. The overall horizontal scale of the spectrum is reliable to perhaps 20 percent, the vertical scale to perhaps 30 percent. The theoretical dispersion of the observed spectral density is between 12 percent and 15 percent throughout the range plotted in Figure 9.9. The shape of the curve, however, is very nearly invariant to uncertainties in the cutoff parameters, and the spectrum <u>cannot</u> be fitted with a Gaussian spatial covariance density. The falloff at high frequencies is much too slow, and indicates the existence of a short-range "core". The covariance also falls off too slowly, and indicates, perhaps unreliably, an extended "wing". It was thus decided to attempt to fit the spectrum with a superposition of Gaussians, and this procedure met with some success. The curve labeled "fitted" in Figure 9.9 is the smoothed and averaged spectrum obtained from the following spatial covariance function:

(9.8)

$$g_{\beta\beta'}(x) = C_{\beta\beta'}\left(1.43\, e^{-\frac{1}{2}(x/1.5)^2} + .089\, e^{-\frac{1}{2}(x/4.0)^2} + .107\, e^{-\frac{1}{2}(x/10.0)^2}\right).$$

The fit to the covariance, illustrated in Figure 9.10, is very good (and is probably only fortuitous.) The spectrum (Figure 9.9) looks as if it contains an appreciable contribution from still smaller distances, but some of this may be due to short-range variations in the quality of the counts. Noise of this sort seems impossible to eliminate with human counters, and there is some doubt that counting accuracy will ever be sufficiently high to use effectively in cosmological tests until machine counting can be done. There seems to be no real obstacle to machine counting (and classification) now, but no such machines have been built as yet.

Considering the highly uncertain features of this study, the scale of the form 9.8 must be considered highly provisional; its accuracy is impossible to assess realistically. Limber (12) noted no deviation from a pure Gaussian form, but his counting squares were 1° on a side, and he would have missed most of the short-range detail - though his limiting magnitude was considerably brighter, and so the scale of his covariance should have been expanded. It is heartening to find that a superposition of Gaussians seems to work quite well; the scheme proposed before of polynomials-times-Gaussians would not work well over the whole range, but it seems likely that any spectral function one is likely to meet will be accurately expressible by combining the two, and it is quite possible that the simpler Gaussian-only scheme will be satisfactory.

The fact that the spatial covariance is not accurately representable by a Gaussian should cause no difficulty in the application of the theory to cosmological tests; since the observed spectra are linear in the spatial covariance function, and since the observed spectrum closely mimics the shape of the auxilliary spectral function, the same behavior with changes in the cosmology will be exhibited with a superposition of Gaussians as with a "pure" Gaussian. Detailed computations, however, should await better data. It cannot be excluded that the "wings" of the covariance are due to loose clustering of spirals, say, and the core to a tighter clustering of ellipticals, both with roughly Gaussian form. This indeed seems not unlikely from a cursory inspection of the plates; but again, only detailed counts by type (and hopefully, by machine) will answer this question.

TABLE 9.1 - Adopted Properties of the β-Classes Used in The Computations in This Chapter

β	Type	M_{Bol}	M_V	λ (Mpc^{-3})	σ(Mag)	ν_0 ($10^{14} sec^{-1}$)	B-V
1	gE, SO	-19.4	-19.0	.0123	1.2	1.10	.95
2	Sa	-20.2	-19.9	.00198	.8	1.20	.84
3	Sb	-19.8	-19.6	.00126	.8	1.40	.67
4	Sc	-17.8	-17.8	.0114	1.2	1.70	.49
5	Ir	-13.7	-13.7	.0765	1.8	1.95	.38
6	dE	-15.6	-15.3	.0595	1.4	1.15	.90

TABLE 9.2 - Mean numbers of nebulae per square degree in classes 1-6 in a range of one magnitude about the given m_R, assuming a cutoff at m_R = 20.0 with a dispersion of .5 magnitude.

q_o	m_R	Λ_{1m}	Λ_{2m}	Λ_{3m}	Λ_{4m}	Λ_{5m}	Λ_{6m}	Λ_{Totm}
0	18	63.9	19.8	22.6	18.9	2.9	7.0	135
	19	153	47.0	57.6	56.6	9.9	22.4	346
	20	170	51.0	67.2	80.1	17.0	34.5	420
.25	20	159	47.1	62.7	77.4	16.8	34.0	397
.5	18	60.6	18.8	21.7	18.5	2.9	7.0	129
	19	140	42.8	53.3	54.6	9.8	22.0	323
	20	149	43.8	58.7	74.9	16.6	33.5	376
1	18	57.8	17.8	20.8	18.2	2.8	6.9	124
	19	130	39.2	49.6	52.7	9.8	21.7	303
	20	132	38.2	52.0	70.3	16.3	32.6	342
2	20	108	30.1	42.0	62.8	15.7	31.1	290
5	20	68.8	17.5	25.4	47.6	14.3	27.4	201
SS	18	44.7	13.4	15.3	13.9	2.4	6.0	96
	19	99.2	29.1	35.5	38.2	7.8	18.1	228
	20	102	28.9	37.4	49.0	12.4	26.2	256

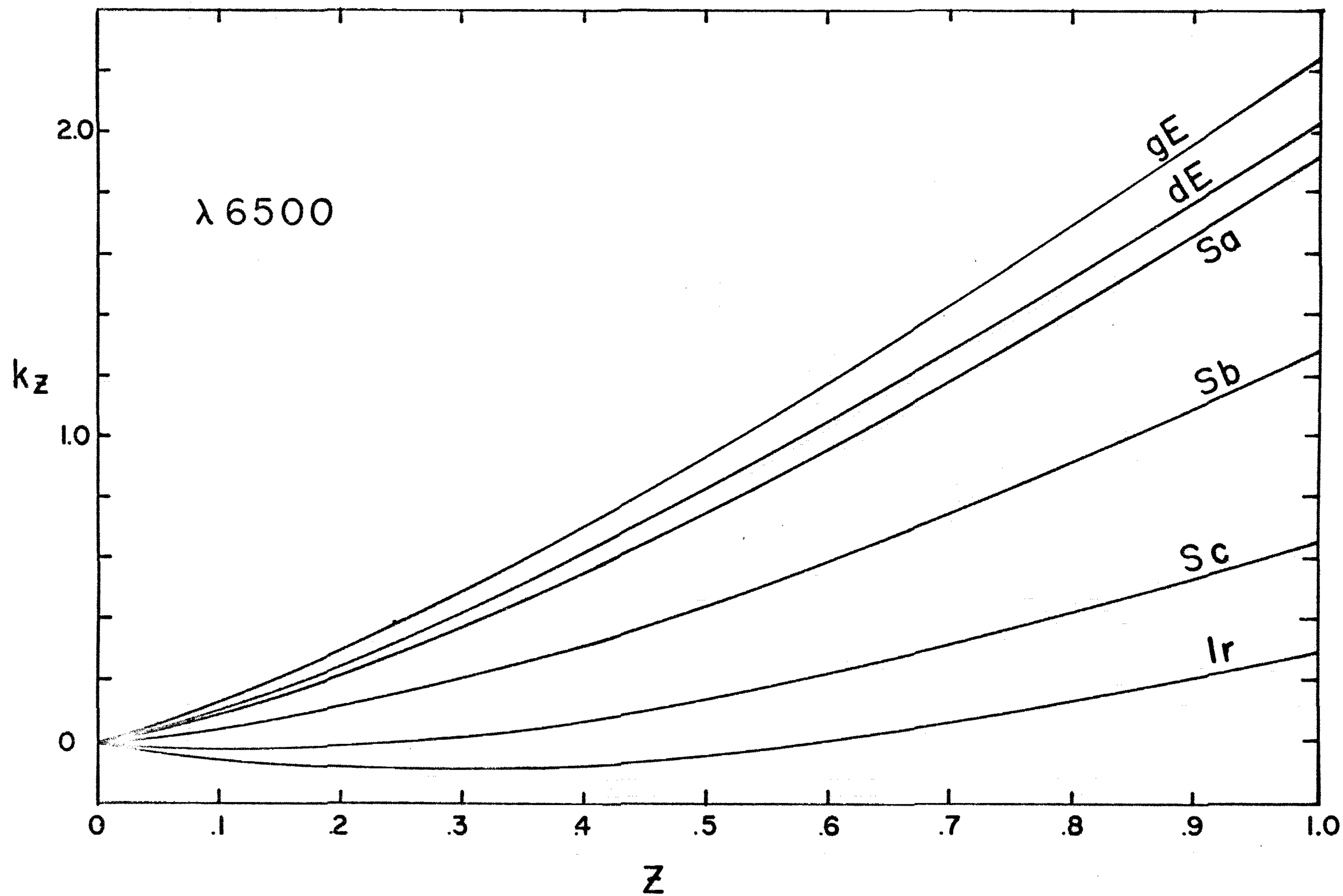

Figure 9.1 Computed k-corrections for the photored, using the spectral parameters given in Table 9.1.

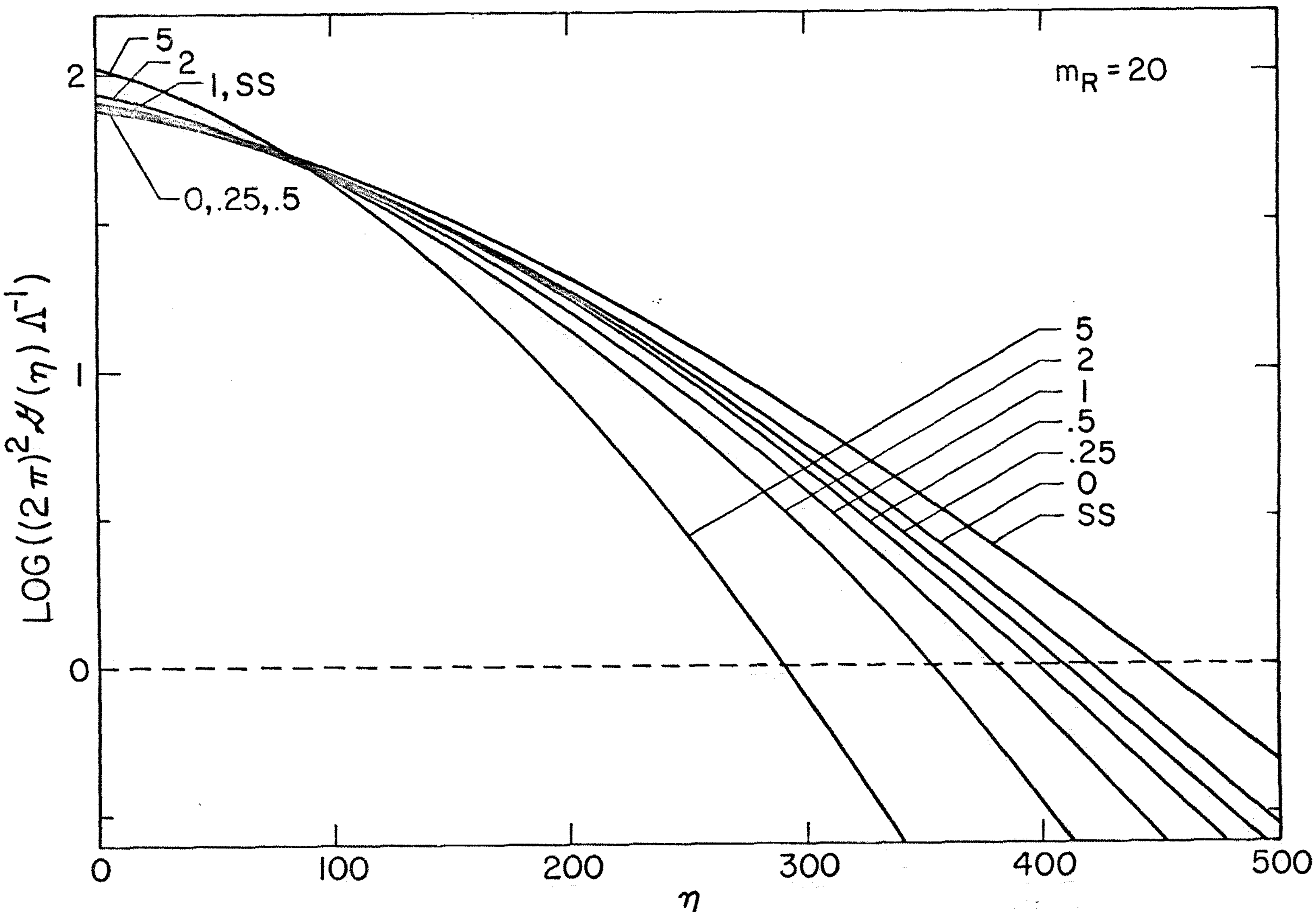

Figure 9.2 Raw count spectral density for $M_R=20$, $q_0=0, .25, .5, 1, 2, 5$, and the steady-state, for a constant correlation length of 4.0 Mpc. The abscissa is in inverse radians.

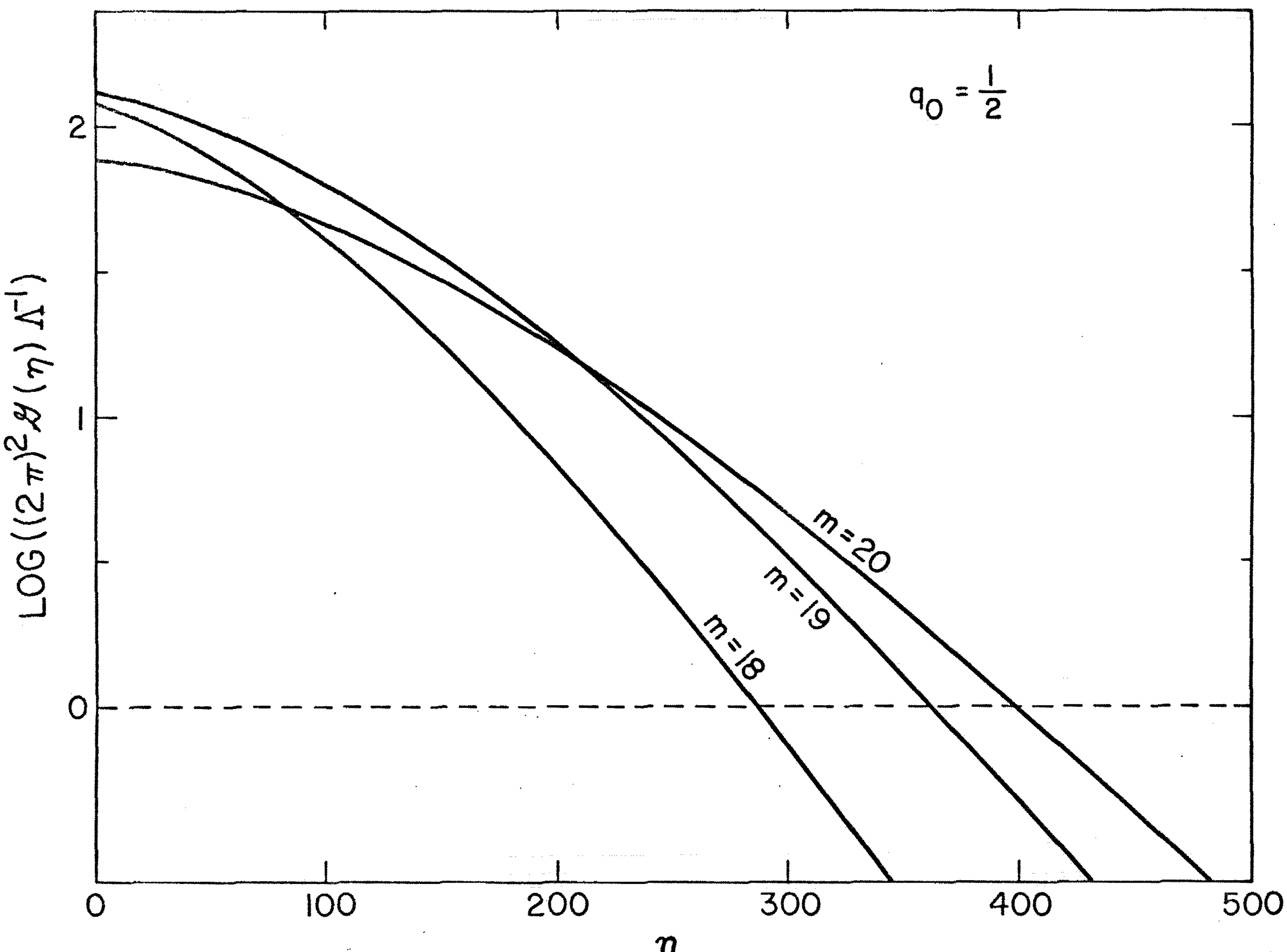

Figure 9.3 Raw count spectral density for q_o=1/2 and photored magnitudes of 18, 19, and 20 for a constant correlation length of 4.0 Mpc.

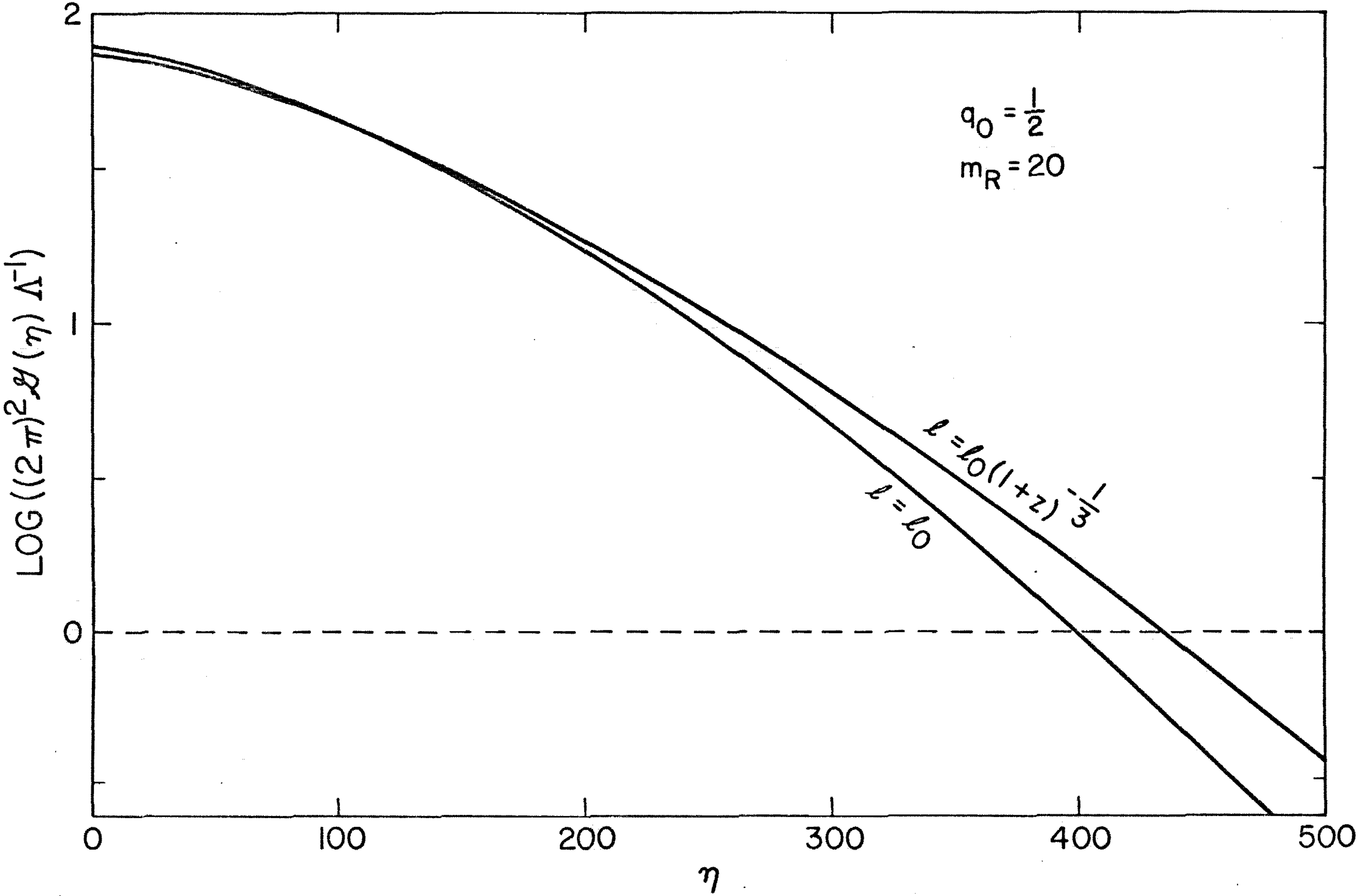

Figure 9.4 Raw count spectral density for $q_o=1/2$, $M_R=20$ for a constant correlation length of 4.0 Mpc, and for ℓ proportional to $(1+Z)^{-1/3}$.

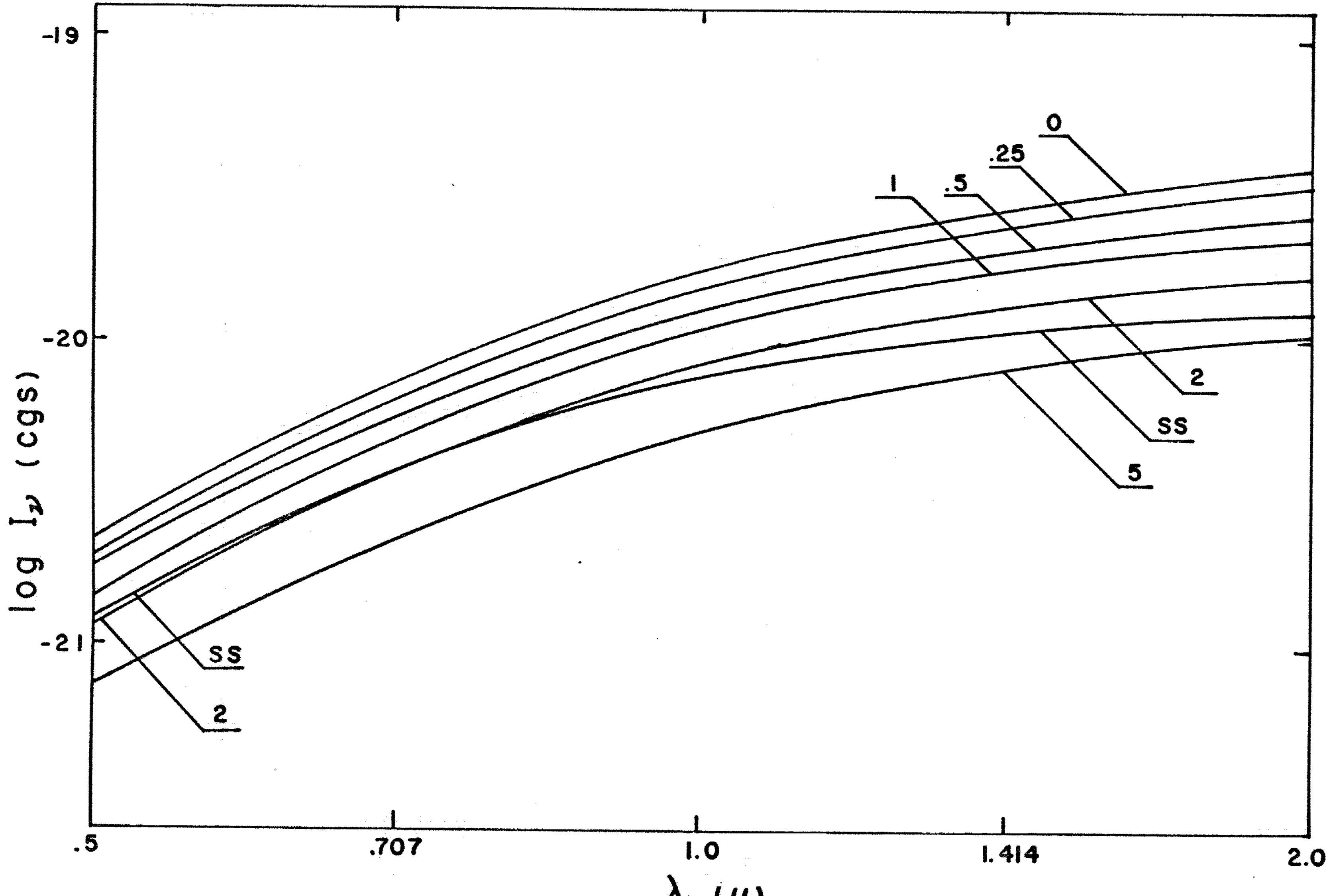

Figure 9.5 Cosmic light intensity in erg/cm^2-sec-str-cps for q_o=0, .25, .5, 1, 2, 5, and the steady-state (SS) for the spectral region 5000Å-2μ.

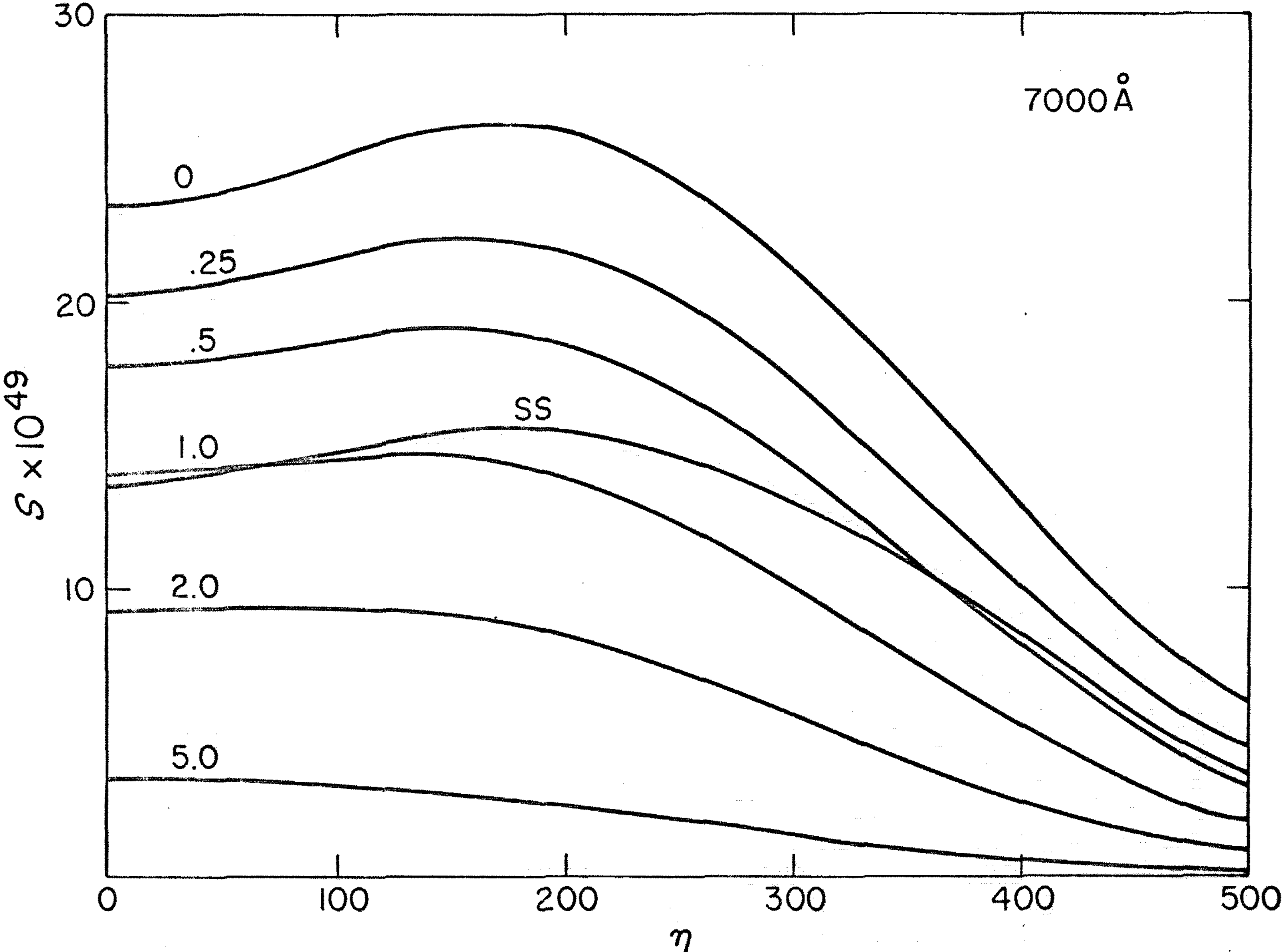

Figure 9.6 Smoothed cosmic light spectral density at 7000Å for q_o=0, .25, .5, 1, 2, and 5, and the steady-state model (SS), for a constant spatial correlation length of 4.0 Mpc.

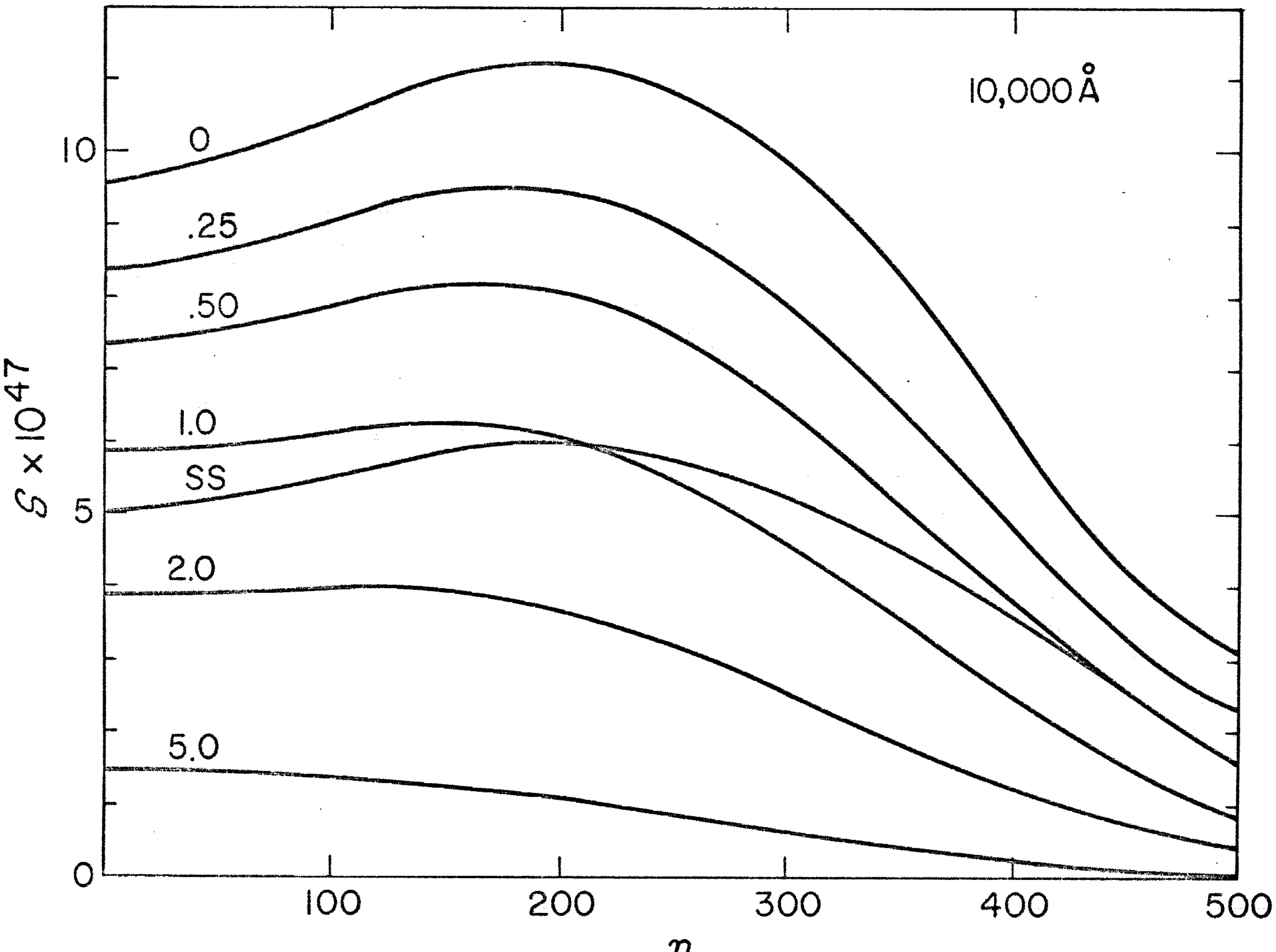

Figure 9.7 Smoothed cosmic light spectral density at 10,000Å for q_o=0,.25,.5,1,2,5, and the steady-state (SS), for a constant spatial correlation length of 4.0 Mpc.

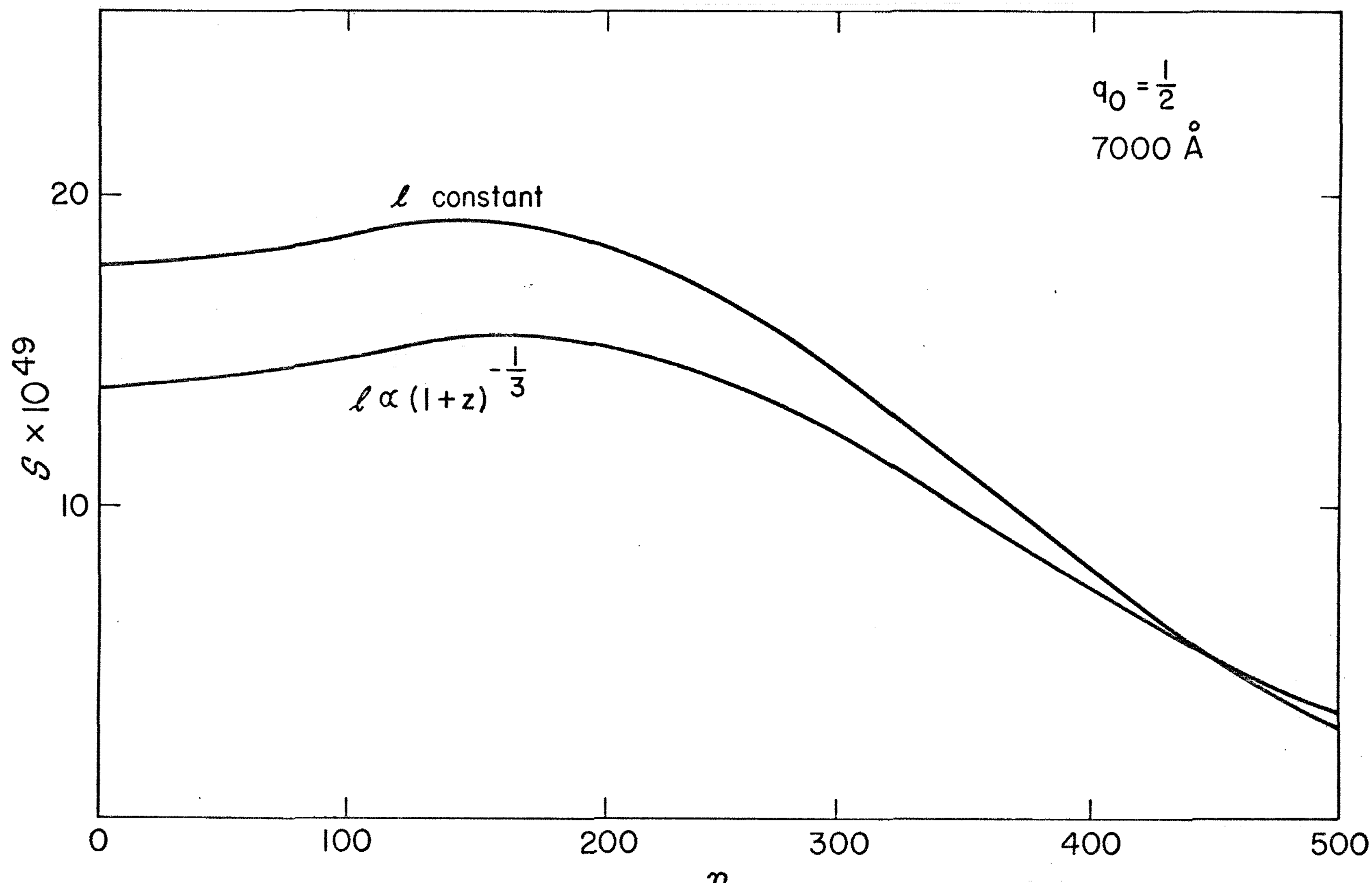

Figure 9.8 Smoothed cosmic light spectral density for $q_o=1/2$ at 7000Å, for a constant spatial covariance length of 4 Mpc, and for the case $\ell=4.0(1+Z)^{-1/3}$.

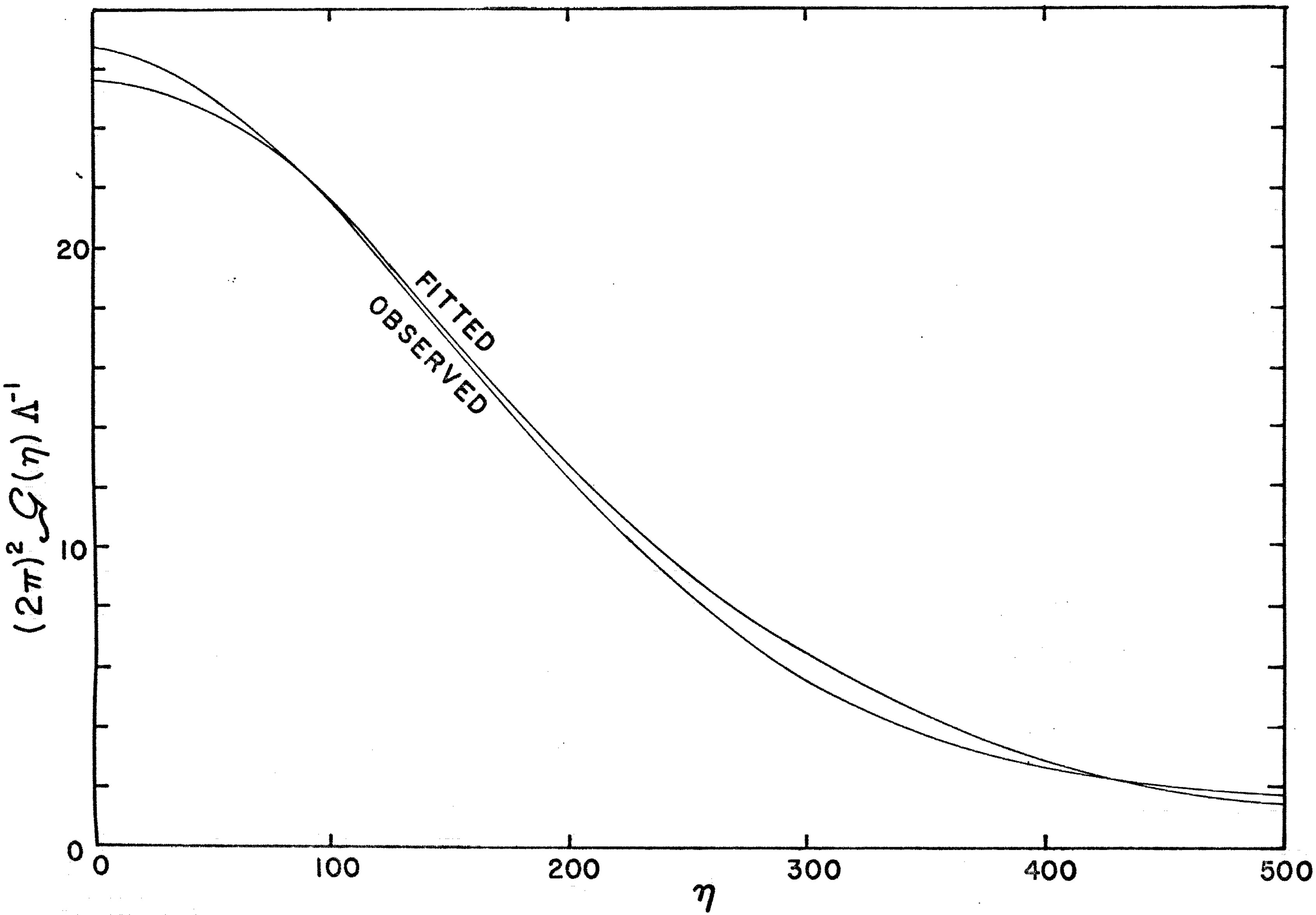

Figure 9.9 Observed count spectrum and three-parameter best fit (Equation 9.8) for cutoff parameters given in text.

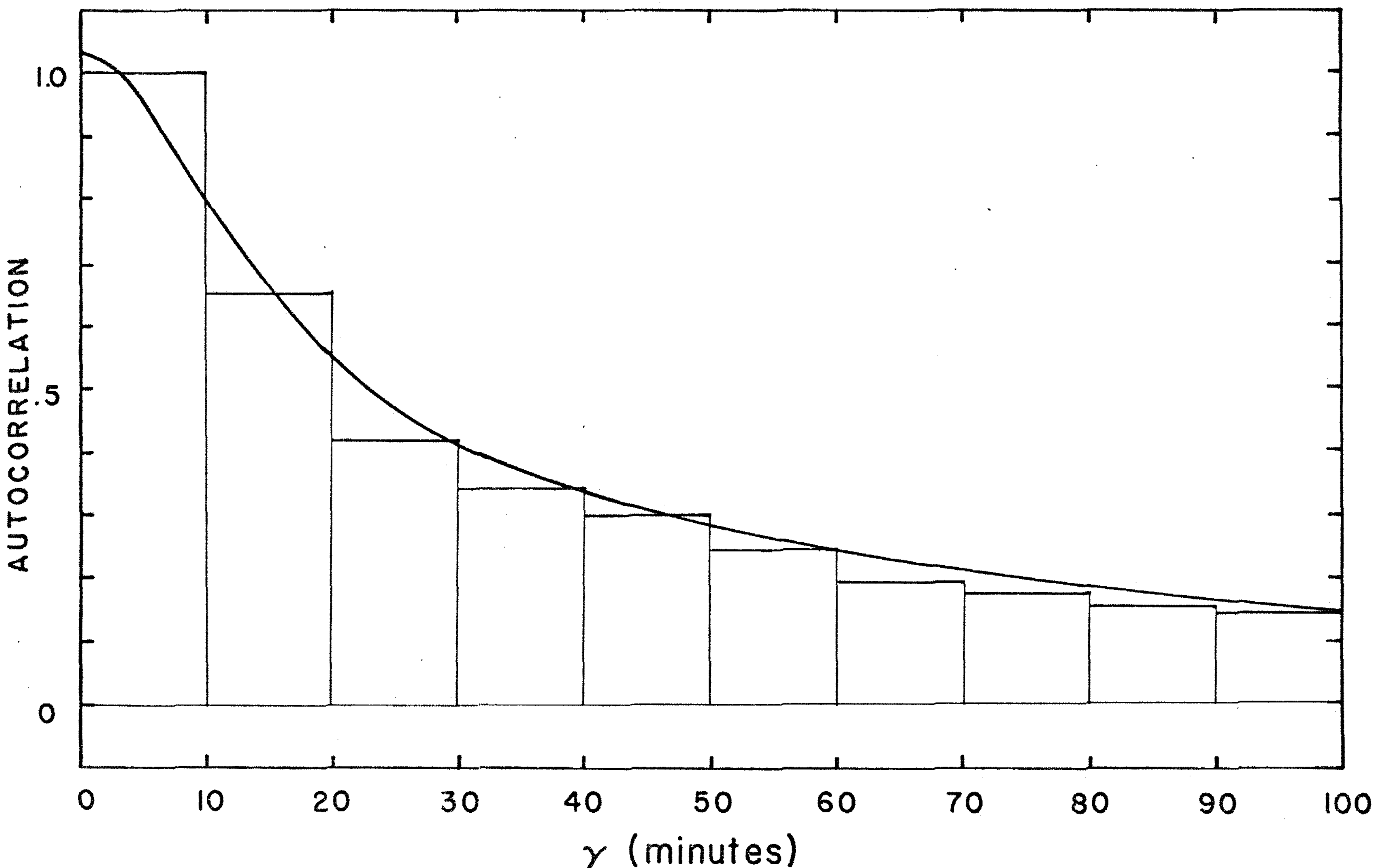

Figure 9.10 Observed covariance (averaged over 10' radial rings). The smooth curve is the covariance corresponding to the spatial covariance in (9.8).

APPENDIX I: STOCHASTIC PROCESSES; POINT PROCESSES

In this appendix we outline some of the results pertinent to the treatment of the distribution of galaxies as a stochastic point process. Proofs, in varying degrees of rigor, are to be found in the literature (see the Bibliography); the purpose here is to form a short introduction for those unfamiliar with the subject, as well as develop heuristically some new needed results on spectral analysis of point processes.

Succinctly stated, a stochastic process in n *dimensions is an n-parameter family of random variables*, or a "random function" in n variables. The ranges of these variables can be either continuous or discrete; if the latter, one usually refers to a stochastic series.

Ideas of continuity, differentiability (and even analyticity) can be introduced for such "functions" in both the "almost certain" (a.c.) sense, which means that the probability is unity that any sample function should have the specified property, and in the "mean square" (m.s.) sense, which is somewhat more general. One says that a stochastic process $X(x)$, say, is m.s. differentiable at X if there exists a stochastic process $Y(x)$ (called the m.s. derivative of $X(x)$, such that the expectation value of $\left|\frac{1}{h}\left(X(x+h) - X(x)\right) - Y(x)\right|^2$

tends to zero with h. We shall denote the expectation value (ensemble or population average) of a stochastic variable X by E(X). It is known that in a certain symbolic sense, the operation $E(\cdot)$ commutes with differentation and integration (GR, 7), and we shall make full use of this formalism, noting as we go any restrictive assumptions which must be made to insure its validity.

We shall use a consistent notation throughout this paper for a stochastic variable with mean removed; if X is any stochastic variable, we let $\overline{X} = X - E(X)$. We give the covariance a special symbol: $\mathrm{cov}\ (XY) = E(\overline{X}\cdot\overline{Y}) = E(XY) - E(X)E(Y)$.

Set intersection $\cap$ and union $\cup$ will have their usual meanings if s is a (Lebesgue) measurable set, we denote its measure by $\mu(s)$ and its diameter by $\rho(s)$.

Let us confine our attention to monovariate stochastic processes in one dimension; generalization to multivariate, multidimensional processes is usually immediate. Let X(x) be such a one-dimensional process.

The complete probabilistic description of a stochastic process under study is, of course, desirable, but is generally too complex to be of much value. Too, it can never by reconstructed from data, however copious. We must generally rely, then, on <u>moments</u> to form a useful, but necessarily incomplete, description.

Thus we have quantities like

(A1.1) $E\big(X(x)\big) = m(x)$, the mean function, and

$\operatorname{cov}\big(X(x), X(y)\big) = f(x,y)$, the autocovariance function.
We are mostly interested here in <u>stationary</u> processes; that is, processes all of whose probability distributions are translation-invariant. For such a process,

$$E\big(X(x)\big) = m \text{ , a constant, and}$$

(A1.2)
$$\operatorname{cov}\big(X(x), X(y)\big) = f(x-y) \quad .$$

A process is said to be <u>stationary to the second order</u> if it is not strictly stationary but (A1.2) holds. This is actually sufficient for most of what we do, but the cosmological principle implies strict stationarity, so we may as well assume it. Stationary processes have many very simple properties; for example, the process is m.s. continuous (differentiable n times, analytic, etc.) if $f(\tau)$ is continuous (differentiable n times, analytic, etc.) at $\tau = 0$. In addition, $f(\tau)$ is continuous (differentiable, etc.) everywhere if it is at $\tau = 0$. If $X(x)$ is m.s. continuous, it can be shown that $f(\tau)$ has a Fourier-Stieltjes transform,

(A1.3) $$f(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau}\, d\mathcal{F}(\omega) .$$

If, in addition, $\int_{-\infty}^{\infty} |f(\tau)|\, d\tau$ is finite, then there is a uniformly continuous function $F(\omega)$, the <u>spectral density</u>, such that $d\mathcal{F}(\omega) = F(\omega)d\omega$, and

(A1.4) $$f(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} F(\omega)\, d\omega$$

We shall always assume that this is the case. In addition, there exists a stochastic process $\mathcal{Z}(\omega)$, such that

(A1.5) $$X(x) = \int_{-\infty}^{\infty} e^{i\omega x} d\mathcal{Z}(\omega) \quad , \text{ and}$$

$\mathcal{F}(\omega) = E(|\mathcal{Z}(\omega)|^2)$; A1.4 is called the <u>spectral representation</u> of W(X) and is interesting largely because it can be shown that if X(x) is stationary to second order, then $\mathcal{Z}(w)$ is <u>orthogonal</u>; that is, $\mathcal{Z}(\omega_2) - \mathcal{Z}(\omega_1)$, the <u>spectral mass</u> in the interval (ω_1, ω_2), is uncorrelated with the mass in any non-overlapping interval. This property is shared by certain estimates for $\mathcal{F}(w)$, as we shall see, and it is this fact which makes spectral theory the powerful tool that it is.

Now consider a slightly different kind of stochastic process. Let N(s) be the number of occurrences of a certain event in the set s. By definition, N takes on only positive integral values, and its <u>value at a point</u> can be only zero or one (We generally exclude the possibility of two or more occurrences at one point - see the discussion of regular point processes in Chapter II) $N(s)$ is a prototype point process. It is certainly not continuous, and the spectral theory does not apply. Consider, however, a new stochastic variable

(A1.6) $$\mathcal{N}_\delta(x) = \frac{1}{\delta} N\left([x-\delta/2, x+\delta/2]\right) = \frac{1}{\delta}\int_{x-\delta/2}^{x+\delta/2} dN(x).$$

$\mathcal{N}_\delta(x)$ is a stochastic function of a continuous variable now, and it can be shown to be m.s. continuous (though no sample function is continuous, note!) and to possess a spectral form. Note that in the integral definition of $\mathcal{N}_\delta(x)$, we could write

(A1.7) $$\mathcal{N}_\delta(x) = \int_{-\infty}^{\infty} C_\delta(x-u)\, dN(u)$$

where $$C_\delta(u) = \begin{cases} 1/\delta, & -\delta/2 \le u \le \delta/2 \\ 0, & \text{otherwise.} \end{cases}$$

We can clearly generalize this. Let $g(x)$ be a smooth, positive, symmetric function with its mass concentrated near the origin, and $\int_{-\infty}^{\infty} g(x)\,dx = 1$; define $g_\epsilon(x) = 1/\epsilon\{g(x/\epsilon)\}$, and

(A1.8) $$\mathcal{N}_\epsilon(x) = \int_{-\infty}^{\infty} g_\epsilon(x-u)\, dN(u) \quad ,$$

which is as smooth (at least) as g. We will call such a process a mollified point process; g_ϵ is the mollifier. Note that as $\epsilon \to 0$, g_ϵ becomes more and more concentrated about $x = 0$. It is through the introduction of such "almost-point" processes that we shall formulate a spectral theory for point processes.

Suppose now that $N(s)$ is a regular stationary point process, with mean density m and covariance density (see Chapter II) $f(\tau)$. Then by definition,

$$\text{(A1.9)} \qquad \operatorname{cov}\left(N(s_1)N(s_2)\right) = \int_{s_1}\int_{s_2} f(x_1-x_2)\,dx_1\,dx_2 \, ,$$

if s_1 and s_2 have no points in common. If they do, another term appears, and it is to the heuristic derivation of this term that we now turn. Let $s_1 = [a_1, b_1]$, $s_2 = [a_2, b_2]$, and suppose that the intersection is not void. For sake of argument, let $a_1 < a_2$, $b_1 < b_2$, $a_2 < b_1$. We then have a situation like that shown in Figure A1.1.

Define

$$\sigma_1 = s_1 - s_2$$
$$\sigma_2 = s_2 - s_1$$
$$\sigma_3 = s_1 \cap s_2 \, .$$

Since $N(s)$ is additive, $N(s_1) = N(\sigma_1) + N(\sigma_3)$,

$N(s_2) = N(\sigma_2) + N(\sigma_3)$. Thus

(A1.10)

$$\begin{aligned}\operatorname{cov}\left(N(s_1),N(s_2)\right) &= \operatorname{cov}\left(N(\sigma_1)+N(\sigma_3),N(\sigma_2)+N(\sigma_3)\right)\\ &= \operatorname{cov}\left(N(\sigma_1),N(\sigma_3)\right)+\operatorname{cov}\left(N(\sigma_3),N(\sigma_2)\right)\\ &\quad + \operatorname{cov}\left(N(\sigma_1),N(\sigma_2)+\operatorname{cov}\left(N(\sigma_3),N(\sigma_3)\right)\right. .\end{aligned}$$

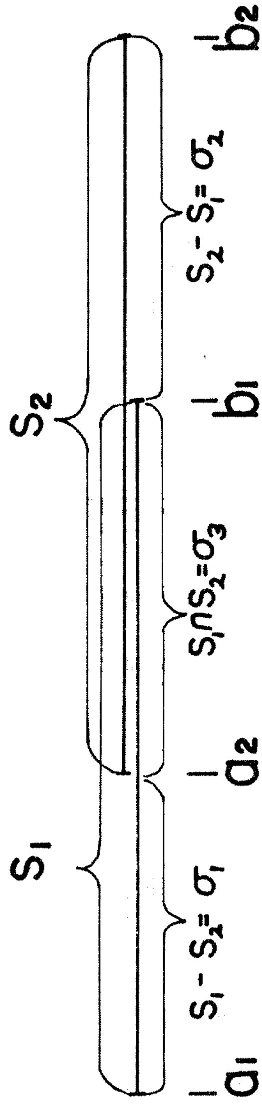

FIG. A1.1

By (A1.8), we can write the first three terms as integrals over f, since the σ_i are disjoint. Only the last term is troublesome. Let $x_j = a_2 + j\delta$, $\delta = \frac{b_1 - a_2}{M}$, $j = 0, 1, 2, \cdots M$; then $N(\sigma_3) = \sum_{j=1}^{M} N([x_{j-1}, x_j))$, and we can make the subdivision as small as we like. Let $N_j = N([x_{j-1}, x_j))$; then

(A1.11)

$$\operatorname{cov}\left(N(\sigma_3), N(\sigma_3)\right) = \sum_{i=1}^{M}\sum_{j=1}^{M} \operatorname{cov}(N_i, N_j)$$

$$= \sum_{\substack{i=1 \\ i \neq j}}^{M} \sum_{j=1}^{M} \operatorname{cov}(N_i, N_j) + \sum_{j=1}^{M} \operatorname{cov}(N_i, N_i) .$$

It is easy to see that the first term tends to

$\iint_{\sigma_3 \sigma_3} f(x_1 - x_2)\, dx_1 dx_2$, the normal expression. The second term does not, however, vanish as δ becomes small. By the definition of covariance,

(A1.12)

$$\operatorname{cov}(N_i, N_j) = E\left[(N_i - E(N_i)) \cdot (N_j - E(N_j))\right] = E(N_i N_j) - E(N_i)E(N_j) ;$$

and $E(N_i)E(N_j) = m^2 \delta^2$, since the mean density is m . But what is $E(N_i N_j)$? By definition, it is

$$\text{(A1.13)} \quad E(N_i\ N_j) = \sum_{k=0}^{\infty} k \cdot (\text{Prob. that } N_i N_j = k)$$

$$= \sum_{k=1}^{\infty} k \cdot (\text{Prob. that } N_i N_j = k).$$

Now for $N_i N_j$ to be non-zero, it is necessary that <u>both</u> N_i and N_j be non-zero. Since $N(s)$ is regular, the probability that N_i or N_j is one is $O(\delta)$; more than one, $O(\delta^2)$. (We assume second-order regularity, though first is all that is required.) If $i \neq j$,

(A1.14) $E(N_i N_j) = \delta^2 \left(m^2 + f(x_i - x_j)\right) + o(\delta^2)$ by the integral representation, and by (A1.12) is

(A1.15) $E(N_i N_j) = 1 \cdot (\text{Prob. that } N_i = 1,\ N_j = 1) + O(\delta^3)$,

so the probability that $N_i = 1$ <u>and</u> $N_j = 1$ is $[m^2 + f(x_i - x_j)]\delta^2$ plus a quantity small compared to δ^2. Similarly, the probability that $N_i = 1$ is $m\delta + O(\delta^2)$; so if $i = j$, the dominant term in (A1.12) is the $k=1$ term again, <u>which is now of order δ</u> , since if $N_i = 1$, $N_i N_i = 1$. Then

(A1.16)

$$
\begin{aligned}
E(N_i N_i) &= \cdot (\text{Prob. that } N_i^2 = 1) + O(\delta^2) \\
&= \cdot (\text{Prob. that } N_i = 1) + O(\delta^2) \\
&= m\delta + O(\delta^2) \quad .
\end{aligned}
$$

The second term in (A1.10) thus tends to $m M \delta = m(b_1 - a_2) = m\mu(\sigma_3)$; the regular terms combine easily, and the result is

(A1.17)

$$
\operatorname{cov}(N(s_1), N(s_2)) = \iint_{s_1\, s_2} f(x_1 - x_2)\,dx_1\, dx_2 + m\mu(s_1 \cap s_2) \quad .
$$

Though we derived this result for stationary processes, the argument goes through unchanged for non-stationary ones; the additional term is easily seen to be $\int_{\sigma_3} m(x)dx$, if m(x) is the mean density.

We have gone to such detail because, to the author's knowledge at least, no satisfactory development exists in the literature, though the result is stated and discussed briefly in Bartlett's book (11). This argument, in essentially this form, can easily be made completely rigorous.

Let us now assume that f(x) is uniformly continuous and absolutely integrable. It thus has a uniformly continuous Fourier transform

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx ,$$

since f(x) (and hence F) is even from its definition. Consider now the mollified process $\eta_\epsilon(x)$, defined in (A1.7). Let $(2\pi)^{-1}Q_\epsilon(\omega)$ be the Fourier transform of g_ϵ;

$$Q_\epsilon(\omega) = \int_{-\infty}^{\infty} g_\epsilon(x)\, e^{-i\omega x}dx .$$

It is then easily shown that

(A1.18)

$$E(\mathcal{N}_\epsilon(x)) = m$$

$$\operatorname{cov}(\mathcal{N}_\epsilon(x), \mathcal{N}_\epsilon(y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_\epsilon(x-u)\, g_\epsilon(y-v)\, f(u-v)\, du\, dv + m \int_{-\infty}^{\infty} g_\epsilon(x-u)\, g_\epsilon(y-u)\, du \quad ;$$

the latter tends to $f(x-y)$ as the functions g_ϵ become more and more concentrated about $x=0$ if $x \neq y$, but tends to infinity (it acquires a "delta-function" kind of singularity) when $x=y$. Thus the spectral density $F_\epsilon(\omega)$ of N (x) is, following a bit of algebra,

(A1.19)
$$F\ (w) = (Q_\epsilon(\omega))^2 \left\{ F(\omega) + \frac{m}{2\pi} \right\} \quad .$$

As $\epsilon \to 0$, $Q_\epsilon(\omega)$ tends to 1 (uniformly in any neighborhood of the origin), so the spectral density of the mollified process is closely related to the transform of the covariance density. It approaches the latter more and more closely as the mollification disappears, except that the discrete nature of the point process contributes a term which tends to a constant (this one would expect from a naive application of spectral theory to the point process, since this term is the transform of a delta function in the covariance density.) This constant term masks small differences in F(w) which we shall be looking for when we estimate the spectral density from samples, and techniques to "filter" the constant component must be devised.

We can also make the point process tractable by converting it into a stochastic series; it is this technique we shall employ for the analysis of counts. For this purpose, we divide the real line into intervals of length h (say) and number the intervals consecutively. Let $S_j = [(j-1/2)h, (j+1/2)h]$, and let $N_j = N(S_j)$. Then N_j is an integer-valued stochastic series - the number of occurrences in cell j. From (A1.18) it is clear that

(A1.20)
$$E(N_j) = hm$$

$$\operatorname{cov}(N_j, N_k) = r_{j-k} = \iint_{S_j S_k} f(x_1 - x_2)\, dx_1\, dx_2 + hm\,\delta_{jk}.$$

For a stochastic <u>series</u>, the appropriate spectral density is not the Fourier transform of the covariance function (which is now defined only for integer values - note that it is a function of j-k only), but the function the coefficients of whose Fourier <u>series</u> are $\{r_j\}$. There remains a close correspondence, however, with the function $F(\omega)$ since

(A1.21)

$$r_j = \int_{-h/2}^{h/2} dx_1 \int_{jh-h/2}^{jh+h/2} dx_2 \int_{-\infty}^{\infty} e^{i\omega(x_1-x_2)} F(\omega)\, d\omega + hm\,\delta_{j,0}$$

$$= h^2 \int_{-\infty}^{\infty} \frac{\sin^2(\omega h/2)}{(\omega h/2)^2} F(\omega) e^{i\omega jh}\, d\omega + h \int_{-\pi}^{\pi} e^{i\eta j} \cdot \frac{m}{2\pi}\, d\eta$$

$$= h \int_{-\infty}^{\infty} \frac{\sin^2 \eta/2}{(\eta/2)^2} F(\eta/h)\, e^{ij\eta}\, d\eta + h \int_{-\pi}^{\pi} e^{i\eta j} \frac{m}{2\pi}\, d\eta$$

$$= h \int_{-\pi}^{\pi} \left\{ \sum_{k=-\infty}^{\infty} F\left(\frac{\eta + 2\pi k}{h}\right) \frac{\sin^2(\eta/2 + k\pi)}{(\eta/2 + k\pi)^2} + \frac{m}{2\pi} \right\} e^{i\eta j}\, d\eta\ .$$

Thus the spectral density is h times the quantity in brackets. Since $\sum_{k=-\infty}^{\infty} \frac{\sin^2(\eta/2 + k\pi)}{(\eta/2 + k\pi)^2} \equiv 1$, this can be rewritten

(A1.22)

$$v_j = h \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} \left\{ F\left(\frac{\eta + 2\pi k}{h}\right) + \frac{m}{2\pi} \right\} \frac{\sin^2(\eta/2 + k\pi)}{(\eta/2 + k\pi)^2} e^{i\eta j} \, d\eta .$$

The phenomenon of the combination of higher frequencies with the fundamental (k=0) term is called "aliasing" by Tukey (6), and is a result of the indistinguishability of higher harmonies from the fundamental when measurements at equal intervals are taken. It is clear that if we wish to determine $F(\omega)$ from the spectrum of N_j, it is necessary that h be sufficiently small that all terms in the sum be small compared with the k=0 term. We are helped significantly here by the "window" term, $\left(\sin^2(\eta/2 + k\pi)\right)\left(\eta/2 + k\pi\right)^{-2}$; we can clearly learn nothing about $F(\omega)$ for arguments of the order π/h or larger. Note that one can arrive at these results slightly differently; we could mollify N(s) with a rectangular mollifier of width h centered on x, as in (A1.5), and then particularize to observations at discrete points only. The form (A1.22) arises naturally from this approach.

APPENDIX II: QUASI-POISSON PROCESSES

Point processes, because of their highly singular nature, are difficult to treat mathematically and do not appear to have been considered systematically, so a mathematically tractable model of such a process is valuable. It happens that a fairly general model is provided by what we shall call quasi-Poisson (QP) processes.

Let $\rho(\underline{x})$ be an m.s. continuous, a.c. positive stochastic process in n-space; let

(A2.1)
$$E(\rho(\underline{x})) = f_1(\underline{x})$$
$$E(\bar{\rho}(\underline{x})\,\bar{\rho}(\underline{y})) = f_2(\underline{x},\underline{y})$$
$$E(\bar{\rho}(\underline{x})\,\bar{\rho}(\underline{y})\,\bar{\rho}(\underline{z})) = f_3(\underline{x},\underline{y},\underline{z}),$$

etc., and let

(A2.2)
$$M(s) = \int_s \rho(\underline{x})\,d\mu$$

for any measurable set s. $M(s)$ is clearly a.c. positive. Let $N(s)$ be an integer-valued stochastic variable with a Poisson distribution with parameter $M(s)$. Then

Def. 1 : $N(s)$ is a <u>Quasi-Poisson process</u> with <u>fundamental</u> $\rho(\underline{x})$.

Such processes have been discussed in the literature; Layzer (15) proposed that the distribution of galaxies could be described by one.

Bartlett, in the discussion of a paper by Cox (38), purported to show that the description was general; i.e., that all point processes are of this type. His proof, involving the so-called characteristic functional, has formal validity but is in fact fallacious - the theorem is false, as we shall see presently. Let us first prove

Theorem 1: Let $\rho(\underline{x})$ possess bounded moments of orders 1, 2,..p. Then the moment densities of N(s) exist to order p and are identical with the moments of $\rho(\underline{x})$.

Proof: Let $s_1, s_2 .. s_j$, $j \leq p$ be disjoint measurable sets in n-space, and let $M(s_i) = M_i$, $N(s_i) = N_i$.

(A2.2)
$$E(N_i) = \sum_{n=0}^{\infty} nP(n)$$
$$= \sum_{n=0}^{\infty} n \int P(n|M_i)\,dP(M_i).$$

From the definition, $P(n|M_i) = \frac{M_i^n}{n!} e^{-M_i}$, so

(A2.3)
$$E(N_i) = \sum_{n=0}^{\infty} n \int \frac{M_i^n e^{-M_i}}{n!} dP(M_i)$$
$$= \sum_{n=0}^{\infty} \int \frac{M_i^{n+1} e^{-M_i}}{n!} dP(M_i)$$

Now since $M_i \geqslant 0$ a.c., $\frac{M_i^{n+1} e^{-M_i}}{n!} \leq M_i$, and the integral exists; indeed

(A2.4)
$$\sum_{n=0}^{N} \int \frac{M_i^{n+1} e^{-M_i}}{n!} \, dP(M\ ) = \int \sum_{n=0}^{N} \frac{M_i^{n+1} e^{-M_i}}{n!} \, dP(M_i)$$

$$\leq \int M_i \, dP(M_i) = E(M_i).$$

Thus $E(N_i)$ exists by dominated convergence, and since $\frac{M_i^{n+1} e^{-M_i}}{n!} \rightarrow M_i$, is equal to $E(M_i)$. But

(A2.5)
$$\begin{aligned} E(M_i) &= E \int_{S_i} \rho(\underline{x}) \, d\mu \\ &= \int_{S_i} E(\rho(\underline{x})) \, d\mu \\ &= \int_{S_i} f_1(\underline{x}) \, d\mu \\ &= E(N_i), \end{aligned}$$

so the mean density of $N(s)$ is $f_1(x)$. Let $E(N_i) = \lambda_i$; then

(A2.6)

$$E(\bar{N}_1\bar{N}_2\ldots\bar{N}_j) = \sum_{n_i=0}^{\infty} \prod_{i=1}^{j} (n_i-\lambda_i)\, P(n_1, \ldots n_j)$$

$$= \sum_{n_i=0}^{\infty} \left\{\prod_{i=1}^{j} (n_i-\lambda_i)\right\} \iint\cdots\int \prod_{i=1}^{j} P(n_i|M_i)\, dP(M_1,\ldots,M_j)$$

$$= \sum_{n_i=0}^{\infty} \left\{\prod_{i=1}^{j} (n_i-\lambda_i)\right\} \iint\cdots\int \prod_{i=1}^{j} \frac{M_i^{n_i} e^{-M_i}}{n_i!}\, dP(M_1,\ldots,M_j)$$

$$= \sum_{n_i=0}^{\infty} \iint\cdots\int \prod_{i=1}^{j} \left\{(M_i-\lambda_i)\frac{M_i^{n_i} e^{-M_i}}{n_i!}\right. dP(M_1,\ldots,M_j).$$

The integrals exist, since the moments of $\rho(\underline{x})$ exist through order j and are bounded. Again by dominated convergence, the sum exists and is equal to

$$\iint\cdots\int \prod_{i=1}^{j} (M_i-\lambda_i)\, dP(M_1\ldots M_j) = E(\bar{M}_1\bar{M}_2\ldots\bar{M}_j).$$ But again,

(A2.7) $$E(\bar{M}_1\bar{M}_2\cdots\bar{M}_j) = E\int_{S_1}\cdots\int_{S_j} \prod_{i=1}^{j} (\rho(\bar{x}_i)-f(\bar{x}_i))\, d\mu_1\ldots d\mu_j$$

$$= \int_{S_1}\cdots\int_{S_j} f_j(\bar{x}_1\ldots\bar{x}_j)\, d\mu_1\ldots d\mu_j\ ;$$

and the theorem is proved.

Theorem 2: Let $\rho(\underline{x})$ be as in Theorem 1, with $p \geq 2$. Then $N(s)$ is a regular point process of order p.

Proof: We need only show that $P(N(s) > 1) = o(\mu(s))$. Let $n \geq 2$. Then

$$P(N(s) = n) = \int P(n|M)\, dP(M)$$

$$= \int M^2 \frac{M^{n-2} e^{-M}}{n!}\, dP(M)$$

$$\leq E((M(s))^2),$$

but

$$E((M(s))^2) = \iint_{S\,S} f_2(x_1, x_2)\, d\mu_1\, d\mu_2 = O(\mu(s)^2) = o(\mu(s)).$$

Q.E.D.

Theorem 3: There exist regular point processes which are not Q-P processes.

Proof: Since the moment densities of a Q-P process coincide with the moments of its fundamental, they must satisfy the restrictions which exist for the moments of a stochastic process. In particular, the covariance density must be "positive-definite"; that is,

$\iint_{S\,S} a(\underline{x}) a(\underline{y})\ f(\underline{x}, \underline{y})\ dx\ dy \geq 0$ for all integrable $a(x)$.

(This can be easily seen from a consideration of $E(\{\int_S a(\underline{x})\, \bar{\rho}(\underline{x})\, d\mu\}^2))$. There exist many processes for which this is not true; all processes in which the occurrence of one event creates a "zone of avoidance" which inhibits or prohibits occurrences nearby have $f_2(\underline{x}, \underline{x}+\underline{y})$ negative for small $\underline{y}$, and if s is small, the integral will be negative. Bartlett himself gives such a process (in which $f_2 < 0$ everywhere) in his book (11 , p. 167) - perhaps the simplest example is a stationary random distribution of hard spheres of radius a, for which N(s) is the number of sphere centers in s. It is easy to see that if λ is the mean density, the covariance density $f_2(\underline{x}_1-\underline{x}_2) = -\lambda^2$ for $|\underline{x}_1-\underline{x}_2| < 2a$, zero otherwise.

It may be added in passing that even if the covariance density is positive-definite, a point process need not be Q-P, for relationships among higher moments may not be satisfied. One such process is the number of particles in a set which possess a very strongly attractive limited-range binary interaction, which is strongly repulsive over the same range for tertiary and higher interactions. Such particles are almost always found in pairs, almost never in triples or higher multiples. This particular example has, among other difficulties, a negative fourth moment density, which is also impossible for the fourth moment of a continuous stochastic process - even though the covariance density can be made positive-definite with a suitable choice of the attractive potential.

<u>Theorem 4</u>: Let $N'(s)$ be derived from $N(s)$ by random selection with a measurable selection function $p(\underline{x})$; i.e., given that $N(s)$ has an occurrence at $\underline{x}$, the probability that $N'(s)$ has an occurrence (the event is "counted") is $p(\underline{x})$, $0 \leq p(\underline{x}) \leq 1$, and this probability is independent of other occurrences. Then $N'(s)$ is also Q-P, and has fundamental $p(\underline{x})\,\rho(\underline{x})$.

<u>Proof</u>: Since $p(\underline{x})$ is measurable and bounded, there exist simple step functions $P_1(\underline{x})$ and $P_2(\underline{x})$ such that $p_1(\underline{x}) \leq p(\underline{x}) \leq p_2(\underline{x})$. We can then partition s into measurable sets s_j, $s=\bigcup_{j=1}^{n} s_j$, $s_j \cap s_k = \emptyset$ for $j \neq k$, such that P_1 and P_2 are constant on each s_j. Let $N_1'(s)$, $N_2'(s)$ be the processes derived from $N(s)$ with selection functions $p_1(x)$ and $p_2(x)$, respectively, and set $N(s_j)=N_j$, $N_1(s_j)=N_{1j}$, $N_2(s_j)=N_{2j}$, $M(s_j)=M_j$. Now

$$P(N_1'(s)=n) = \int P(n \mid \rho(\underline{x}))\, dP(\rho) \quad ;$$

<u>given</u> $\rho(\underline{x})$, $N(s)$ is a Poisson process, as is N_j for each j. Furthermore,

$$P(n_{1j} \mid \rho(\underline{x})) = \sum_{n_j=n_{1j}}^{\infty} P(n_{1j} \mid n_j)\, P(n_j \mid \rho(\underline{x})) \ ,$$

and the sum clearly converges. But $P(n_j \mid \rho(x)) = \dfrac{M_j^{n_j} e^{-M_j}}{n_j!}$; $P(n_{1j}' \mid n_j)$ is, by definition of the selection process, binomial, with success probability $P_{1j} = P_1(\underline{x})$, $\underline{x} \in S_j$.

Thus

$$P(n_{1j} \mid n_j) = \binom{n_j}{n_{1j}} P_{1j}^{\,n_{1j}} (1 - P_{1j})^{n_j - n_{1j}}$$

so

$$P(n_{1j} \mid \rho(x)) = e^{-M_j} \sum_{n_j = n_{1j}}^{\infty} \frac{(n_j)!}{(n_{1j})!\,(n_j - n_{1j})!} \; \frac{P_{1j}^{\,n_{1j}} (1 - P_{1j})^{n_j - n_{1j}}}{(n_j)!}$$

$$= \frac{e^{-M_j} P_{1j}^{\,n_{1j}} M_j^{\,n_{1j}}}{(n_{1j})!} \, e^{M_j (1 - P_{1j})}$$

$$= \frac{e^{-M_j P_{1j}} (M_j P_{1j})^{n_{1j}}}{(n_{1j})!}$$

so N_{ij}' is also a Poisson process. Furthermore, $N_1'(s) = \sum_{j=1}^{n} N_{1j}$, and the processes for each j conditional on $\rho(\underline{x})$ are mutually independent. Since the sum of Poisson variables is also a Poisson variable, $N_1(s)$ is Poisson, with parameter $\sum M_j P_{1j}$. But

$$\sum M_j P_{1j} = \int \rho(\underline{x}) P_1(\underline{x})\, d\mu = M_1' .$$

Exactly similar results hold for $N_2'(s)$. From the definition of the selection process, it is clear that $P(N_1(s) \leq n) \geq P(N'(s) \leq n) \geq P(N_2(s) \leq n)$ for each n.

But

$$P(N_1'(s) \leq n) = \int \sum_{i=1}^{n} \frac{e^{-M_1'}(M_1')^i}{i!} \, dP(\rho)$$

and

$$P(N_2'(s) \leq n) = \int \sum_{i=1}^{n} \frac{e^{-M_2'}(M_2')^i}{i!} \, dP(\rho).$$

The integrands in each case are less than unity. We now choose a sequence of such P_1's and P_2's , say $\{P_{1n}\}$, $\{P_{2n}\}$, such that $P_{jn}(\underline{x}) \to p(\underline{x})$ for j=1,2 . The above integrals exist for each n, and the inclusion relation for the probabilities holds for each n. Thus by bounded convergence, since

$$\int P_{jn}(\underline{x})\,\rho(\underline{x})\,dx \to \int p(\underline{x})\,\rho(\underline{x})\,dx \quad \text{for each}$$

$\rho(\underline{x})$,

$$P(N'(s) \leq n) = \int \sum_{i=1}^{n} \frac{e^{-M'}(M')^i}{i!} \, dP(\rho),$$

and

$$P(N(s) = n) = \int \frac{e^{-M'}(M')^n}{n!} \, dP(\rho).$$

$N'(s)$ is therefore Q-P , and the fundamental is the integrand in M , i.e. $p(\underline{x})\,\rho(\underline{x})$.

Q.E.D.

Theorem 5: Let N(s) be a three-dimensional Q-P process, and suppose that $M = \int_{\text{all space}} \rho(\underline{x})\,d^3x$ has moments through order

$q \leq p$. Then if O is an arbitrary point in space, the radial projection $\tilde{N}(\Omega)$ of $N(s)$ on the unit sphere about O ; i.e. $N(\Omega) = \tilde{N}(s(\Omega))$, where $s(\Omega) = \left\{ x \,\middle|\, \frac{x}{|x|} \in \Omega \right\}$, is a Q-P process on the sphere, with fundamental $\int_0^\infty r^2 \rho(r\underline{n})\,dr = \rho_2(\underline{n})$. If the probability distributions of $\rho(x)$ are invariant under rotation about O , the process $\tilde{N}(\Omega)$ is stationary.

Proof: The result is self-evident.

Corollary: Let $N(s)$ be a stationary, isotropic Q-P process in three dimensions, and let $p(x)$ be a selection function as in Theorem 4, such that $\int p(x)\,d^3x < \infty$. Then if O is an arbitrary point in space, the radial projection $\tilde{N}(\Omega)$ of $N'(s)$ as in Theorem 5 is Q-P, and has fundamental $\lambda \int_0^\infty r^2 p(\underline{n}r)\,dr$, where λ is the mean density of $N(s)$. Furthermore, $q = p$.

Proof: We need only show that $M = \int p(x)\,\rho(x)\,d^3x$ has moments through order p. But this is trivial; for $1 < n \leq p$,

$$E(|M' - E(M')|) = \left| \iint \cdots \int p(\underline{x}) \cdots p(\underline{x}_n)\, f_n(\underline{x}_1 \cdots \underline{x}_n)\, d^3x_1 \cdots d^3x_n) \right|$$

$$\leq C_n \left[\int p(\underline{x})\, d^3x \right]^n$$

where C_n is the bound on f_n (which exists, since $n \leq p$). But $\int p(x)\, d^3x$ is finite, by assumption. The corollary follows by a direct application of Theorem 5.

Remark: If $p(\underline{x}) = p(|\underline{x}|)$, $N(\Omega)$, as defined in the corollary, is stationary on the sphere.

Theorem 6: Let $N(s,\sigma)$ be an $n+1$ dimensional Q-P process, where S is a set in n-dimensional space, and σ is a set on the real line. Let the first *n* parameters be $x_1, .. x_n$, $\underline{x} = (x_1, .. x_n)$, and the last parameter be m_o. Construct a new process $N\ (s,\sigma')$ in the following manner: Each occurrence of $N(s,\sigma)$ is represented once and only once by an occurrence of $N'(s,\sigma')$. The probability density for an occurrence of $N'(s,\sigma')$ at $(\underline{x},m)$ corresponding to an occurrence of $N(s,\sigma)$ at $(\underline{x},m_o)$ is $P(m|m_o)$, and at $\underline{x}' \neq \underline{x}$, zero. (Thus $\int_{-\infty}^{\infty} P(m|m_o)\, dm = 1$). Assume also that $\int_{-\infty}^{\infty} P(m|m_o)\, dm_o$ exists for each m and is a measurable function of m, and that this "redistribution" in m is independent of the spatial distribution.

Then $N'(s,\sigma')$ is a Q-P process, and if $\rho(x,m_o)$ is the fundamental for $N(s,\sigma)$, $N'(s,\sigma')$ has the fundamental $\rho'(x,m) = \int P(m|m_o)\, \rho(x,m_o)\, dm_o$.

Note in interpretation: The transition from N to N' represents a process in which one measures N, obtaining the first n coordinates correctly, but with the $(n+1)^{st}$

coordinate being subject to random measuring errors whose distribution is given by $P(m_o | m)$. One can clearly generate multidimensional processes some or all of whose coordinates are subject to such errors by a repeated application of the theorem, and all will be Q-P if the parent process is.

We need first to prove a Lemma.

Lemma: Let $\mathcal{N}(\sigma)$ be a Poisson process on the real line, with density $\rho(x)$. Let $\mathcal{N}'(\sigma')$ be derived from $\mathcal{N}(\sigma)$ as N' is from N in the statement of the theorem, with conditional distribution $P(x'|x)$. Then $\mathcal{N}'(s')$ is a Poisson process with density $\rho'(x') = \int P(x'|x)\, \rho(x)\, dx$.

Proof of Lemma: To show that $\mathcal{N}'(\sigma')$ is Poisson, we must show that the probability distribution for disjoint sets are independent, and that the probability distributions for a single set is given by $P(n) = \frac{e^{-M} M^n}{n!}$, where M is the integral of $\rho'(x)$ over the set - actually the first is sufficient, but we need to evaluate M. That the numbers in disjoint sets are independent is clear - indeed, from the nature of the process it is easy to see that any occurrence is independent of any other.

Let s be a measurable set on the real line and let s' be its complement; i.e., $s \cup s' = (-\infty, \infty)$, $s \cap s' = \emptyset$. Then $P(\mathcal{N}'(s) = n') = \sum_{n=0}^{n'} P$ (n occurrences of $\mathcal{N}(s)$ remain in s and $(n' - n)$ occurrence of $\mathcal{N}(s')$ arrive in s.)

Since the occurrence and subsequent relocation of each event in $\mathcal{N}(s)$ is independent of all others, we can write this as

$$P(\mathcal{N}'(s) = n') = \sum_{n=0}^{n'} P \text{ (n occurrences of } \mathcal{N}(s) \text{ remain in s)} \cdot P \text{ (}n'-n \text{ occurrences of } \mathcal{N}(s') \text{ arrive in s.)}$$

Now for a given set A, an event of $\mathcal{N}(A)$ can either leave A or remain in A when $\mathcal{N}'(A)$ is formed. Let $\mathcal{N}''(A)$ be the number of events which remain in A in the formation of $\mathcal{N}'(A)$. This is clearly a random selection process with selection function (Theorem 4) $\Phi_A(x) = \int_A P(x'|x)\,dx'$, for it is precisely this quantity which expresses the probability that an event at x will be counted in $\mathcal{N}'(A)$. By the results of Theorem 5, $\mathcal{N}(A)$ is Poisson with parameter $\lambda_A = \int_A \Phi_A(x)\,\rho(x)\,dx$. Similarly, if $\mathcal{N}'''(A)$ is the number that arrive in A from A' (that is, the number which <u>do</u> <u>not</u> remain in A') when $\mathcal{N}'(A')$ is formed, this is also a random selection process on A', with selection function $1-\Phi_{A'}(x) = \Phi_A(x)$ also. But $\mathcal{N}''(A) + \mathcal{N}'''(A) = \mathcal{N}(A)$, and since $\mathcal{N}''$ and $\mathcal{N}'''$ are independent, their sum is also Poisson, with parameter

$$M' = \lambda_A + \lambda_{A'} = \left\{ \int_A + \int_{A'} \right\} \Phi_A(x)\,\rho(x)\,dx = \int_{-\infty}^{\infty} dx\,\rho(x) \int_A P(x'|x)\,dx'$$

We can reverse the order of integration by reason of our assumption on $P(x'|x)$, and obtain

$$M' = \int_A \left\{ \int_{-\infty}^{\infty} P(x'|x)\, \rho(x)\, dx \right\} dx' \quad .$$

Proof of Theorem: We have

$$P(N'(s,\sigma') = n') = \int P(n'|\rho(\underline{x},m))\, dP(\rho) \quad ,$$

but n', given $\rho(\underline{x},m)$, is clearly Poisson, and since events do not enter or leave the spatial set s in the formation of N' from N, this process of formation is clearly equivalent to the case covered by the lemma. Thus

$$P(n'|\rho(\underline{x},m)) = \frac{e^{-M'(s,\sigma')}[M'(s,\sigma')]^{n'}}{(n')!};$$

$$M'(s,\sigma') = \int_{s,\sigma'} \rho'(\underline{x},m)\, d^n x\, dm,$$

and the process is Q-P with the indicated fundamental.

Q.E.D.

Corollary: Let $N_i(s)$ be a multivariate Q-P process with fundamental $\rho_i(s)$, $i=1,2,..n$. Let N_i', $i=1,2,..n$, be formed from N_i in the following manner: To each event of $N_i'(s)$ at $\underline{x}$, there corresponds one event of $N_j(s)$ at $\underline{x}$ for some j. The probability that

$dN_j'(\underline{x}) = 1$, given that $dN_i(\underline{x}) = 1$, is $P_{ji}(\underline{x})$. (Clearly $\sum_{j=1}^{n} P_{ji}(\underline{x}) = 1$ for each i.) Let $P_{ji}(\underline{x})$ be measurable for each i and j. Then $N_j(s)$ is QP, with fundamental $\sum_{j=1}^{n} P_{ji}(\underline{x}) \, \rho_i(\underline{x})$.

Theorem 7: Let $N(s)$ be an n-dimensional Q-P process, and let $\underline{x} = \underline{g}(\underline{x}')$ be a differentiable coordinate transformation. Then the new process so generated is also Q-P, with fundamental

$$\rho'(\underline{x}') = \rho(\underline{g}(\underline{x}')) \frac{\partial(x_1, \ldots, x_n)}{\partial(x_1', \ldots x_n')} .$$

The result is obvious.

Theorem 8: Let $N^{\alpha}(s)$ be an n-variate QP process in m-space, and let $\rho^{\alpha}(\underline{x})$ be its fundamental. Suppose that the third centered moments of $\rho^{\alpha}(\underline{x})$ vanish, and that ρ^{α} has bounded moments through order 4.

Let $S_{\underline{k}}$, $\underline{k} = (k_1, \ldots k_m)$, k_i an integer, be the set $\{\underline{x} \mid h(k_i - \frac{1}{2}) \leq x_i < h(k_i + \frac{1}{2})\}$. Thus the $S_{\underline{k}}$ form a cubic network covering all space. Let $N_{\underline{k}}^{\alpha} = N^{\alpha}(S_{\underline{k}})$,

$$M_{\underline{k}}^{\alpha} = \int_{S_{\underline{k}}} \rho^{\alpha} d\mu, \qquad \delta_{\underline{k}\underline{l}}^{\alpha\beta} = \delta_{\alpha\beta} \prod_{i=1}^{m} \delta_{k_i l_i}, \qquad R_{\underline{j}\underline{k}}^{\alpha\beta} = \mathrm{cov}(M_{\underline{j}}^{\alpha}, M_{\underline{k}}^{\beta}),$$

and $$\Lambda_{\underline{j}}^{\alpha} = E(M_{\underline{j}}^{\alpha}) .$$

Then

a).

$$E(\bar{N}_{\underline{j}}^{\alpha}\,\bar{N}_{\underline{k}}^{\beta}\,\bar{N}_{\underline{l}}^{\gamma})$$

$$= \delta_{\underline{j}\underline{k}}^{\alpha\beta}\,R_{\underline{k}\underline{l}}^{\beta\gamma} + \delta_{\underline{j}\underline{l}}^{\alpha\gamma}\,R_{\underline{j}\underline{k}}^{\alpha\beta} + \delta_{\underline{k}\underline{l}}^{\beta\gamma}\,R_{\underline{j}\underline{k}}^{\alpha\beta}$$

$$+ \delta_{\underline{j}\underline{k}}^{\alpha\beta}\,\delta_{\underline{k}\underline{l}}^{\beta\gamma}\,\Lambda_{\underline{j}}^{\alpha}$$

b). $$\operatorname{cov}(\bar{N}_{\underline{j}}^{\alpha}\,\bar{N}_{\underline{k}}^{\beta},\,\bar{N}_{\underline{l}}^{\gamma}\,\bar{N}_{\underline{r}}^{\delta})$$

$$= \operatorname{cov}(\bar{M}_{j}^{\alpha}\,\bar{M}_{k}^{\beta},\,\bar{M}_{l}^{\gamma}\,\bar{M}_{r}^{\delta})$$

$$+ \delta_{\underline{j}\underline{l}}^{\alpha\gamma}\,\Lambda_{\underline{j}}^{\alpha}\,R_{\underline{k}\underline{r}}^{\beta\delta} + \delta_{\underline{j}\underline{r}}^{\alpha\delta}\,\Lambda_{\underline{j}}^{\alpha}\,R_{\underline{k}\underline{l}}^{\beta\gamma}$$

$$+ \delta_{\underline{k}\underline{l}}^{\beta\gamma}\,\Lambda_{\underline{k}}^{\beta}\,R_{\underline{j}\underline{r}}^{\alpha\delta} + \delta_{\underline{k}\underline{r}}^{\beta\delta}\,\Lambda_{\underline{k}}^{\beta}\,R_{\underline{j}\underline{l}}^{\alpha\gamma}$$

$$+ (R_{\underline{j}\underline{k}}^{\alpha\beta} + \Lambda_{j}^{\alpha}\Lambda_{k}^{\beta})\{\delta_{\underline{j}\underline{l}}^{\alpha\gamma}\,\delta_{\underline{k}\underline{r}}^{\beta\delta} + \delta_{\underline{j}\underline{r}}^{\alpha\delta}\,\delta_{\underline{k}\underline{l}}^{\beta\gamma}\}$$

$$+ R_{\underline{j}\underline{l}}^{\alpha\gamma}\,\delta_{\underline{j}\underline{k}}^{\alpha\beta}\,\delta_{\underline{l}\underline{r}}^{\gamma\delta}$$

$$+ R_{\underline{j}\underline{k}}^{\alpha\beta}\,(\delta_{\underline{k}\underline{l}}^{\beta\gamma}\,\delta_{\underline{l}\underline{r}}^{\gamma\delta} + \delta_{\underline{j}\underline{l}}^{\alpha\gamma}\,\delta_{\underline{l}\underline{r}}^{\gamma\delta})$$

$$+ R_{\underline{l}\underline{r}}^{\gamma\delta}\,(\delta_{\underline{j}\underline{k}}^{\alpha\beta}\,\delta_{\underline{k}\underline{l}}^{\beta\gamma} + \delta_{\underline{j}\underline{k}}^{\alpha\beta}\,\delta_{\underline{k}\underline{r}}^{\beta\delta})$$

$$+ \Lambda_{\underline{j}}^{\alpha}\,\delta_{\underline{j}\underline{k}}^{\alpha\beta}\,\delta_{\underline{k}\underline{l}}^{\beta\gamma}\,\delta_{\underline{l}\underline{r}}^{\gamma\delta}.$$

Proof: The proof is rather involved algebraically, but is not difficult; it is a simple consequence of the following elementary relations for moments of Poisson processes: If N is a Poisson variable with parameter Λ, then

$$E(N) = \Lambda$$
$$E(N^2) = \Lambda^2 + \Lambda$$
$$E(N^3) = \Lambda^3 + 3\Lambda^2 + \Lambda$$
$$E(N^4) = \Lambda^4 + 6\Lambda^3 + 7\Lambda^2 + \Lambda$$

Consider first the result (a). Since the third centered moments of $\rho^\alpha(\underline{x})$ vanish, it is easy to see that the third centered moments of $M^\alpha_{\underline{j}}$ do also. Thus there is no "fundamental" term as there is in (b). Then by Theorem 1, if $\underline{i}$, $\underline{k}$, and $\underline{j}$ are all different, or if α, β, γ are all different, the triple expectation vanishes. One clearly gets a result different from that predicted by Theorem 1 only in a case where the same variable appears twice or three times in the expectation. In this case, one expects singular terms to enter, just as in the covariance (See Appendix I). One proceeds case by case; suppose that $\underline{j} = \underline{k}$ and $\alpha = \beta$.

Then $$E((\bar{N}^\alpha_{\underline{j}})^2 \bar{N}^\gamma_{\underline{l}}) = \int \sum_{n^\alpha_{\underline{j}}=0}^{\infty} (n^\alpha_{\underline{j}} - \Lambda^\alpha_{\underline{j}})^2 P(n^\alpha_{\underline{j}} | M^\alpha_{\underline{j}}) \cdot$$
$$\cdot \sum_{n^\gamma_{\underline{l}}=0}^{\infty} (n^\gamma_{\underline{l}} - \Lambda^\gamma_{\underline{l}}) P(n^\gamma_{\underline{l}} | M^\gamma_{\underline{l}}) \, dP(M^\alpha_{\underline{j}}, M^\gamma_{\underline{l}}),$$

as in Theorem 1. But

$$\sum_{n_{\underline{j}}^{\alpha}=0}^{\infty}(n_{\underline{j}}^{\alpha}-\Lambda_{\underline{j}}^{\alpha})^2 P(n_{\underline{j}}^{\alpha}|M_{\underline{j}}^{\alpha}) = E((N_{\underline{j}}^{\alpha}-\Lambda_{\underline{j}}^{\alpha})^2|M_{\underline{j}}^{\alpha})$$

$$= E(\{(N_{\underline{j}}^{\alpha})^2 - 2\Lambda_{\underline{j}}^{\alpha}N_{\underline{j}}^{\alpha} + (\Lambda_{\underline{j}}^{\alpha})^2\}|M_{\underline{j}}^{\alpha})$$

$$= (M_{\underline{j}}^{\alpha})^2 + M_{\underline{j}}^{\alpha} - 2\Lambda_{\underline{j}}^{\alpha}M_{\underline{j}}^{\alpha} + (\Lambda_{\underline{j}}^{\alpha})^2$$

$$= (M_{\underline{j}}^{\alpha}-\Lambda_{\underline{j}}^{\alpha})^2 + M_{\underline{j}}^{\alpha}$$

by the Poisson moment relations, since N is Q.P. Then

$$E((\bar{N}_{\underline{j}}^{\alpha})^2 N_{\underline{l}}^{\gamma}) = E(\{(M_{\underline{j}}^{\alpha}-\Lambda_{\underline{j}}^{\alpha})^2 + M_{\underline{j}}^{\alpha}\}\cdot\{M_{\underline{l}}^{\gamma}-\Lambda_{\underline{l}}^{\gamma}\})$$

$$= E((M_{\underline{j}}^{\alpha}-\Lambda_{\underline{j}}^{\alpha})^2(M_{\underline{l}}^{\gamma}-\Lambda_{\underline{l}}^{\gamma}))$$

$$+ E((M_{\underline{j}}^{\alpha}-\Lambda_{\underline{j}}^{\alpha})(M_{\underline{l}}^{\gamma}-\Lambda_{\underline{l}}^{\gamma}))$$

$$+ \Lambda_{\underline{j}}^{\alpha} E(M - \Lambda_{\underline{l}}^{\gamma})$$

$$= R_{\underline{jl}}^{\alpha\gamma} \quad .$$

The third term in (a) arises (obviously) when $\alpha = \beta = \gamma$, $\underline{j} = \underline{k} = \underline{l}$. In this case,

$$E((\bar{N}_{\underline{j}}^{\alpha})^3) = \int \sum_{n_{\underline{j}}^{\alpha}=0}^{\infty} (n_{\underline{j}}^{\alpha} - \Lambda_{\underline{j}}^{\alpha})^3 P(n_{\underline{j}}^{\alpha} | M_{\underline{j}}^{\alpha}) dP(M_{\underline{j}}^{\alpha});$$

$$\sum_{n_{\underline{j}}^{\alpha}=0}^{\infty} (n_{\underline{j}}^{\alpha} - \Lambda_{\underline{j}}^{\alpha})^3 P(n_{\underline{j}}^{\alpha} | M_{\underline{j}}^{\alpha}) = E\ ((n_{\underline{j}}^{\alpha} - \Lambda_{\underline{j}}^{\alpha})^3 | M_{\underline{j}}^{\alpha})$$

$$= E\ ((n_{\underline{j}}^{\alpha})^3 - 3\Lambda_{\underline{j}}^{\alpha}(n_{\underline{j}}^{\alpha})^2 + 3(\Lambda_{\underline{j}}^{\alpha})^2 n_{\underline{j}}^{\alpha} - (\Lambda_{\underline{j}}^{\alpha})^3 | M_{\underline{j}}^{\alpha})$$

$$= (M_{\underline{j}}^{\alpha})^3 + 3(M_{\underline{j}}^{\alpha})^2 + M_{\underline{j}}^{\alpha} - 3\Lambda_{\underline{j}}^{\alpha}((M_{\underline{j}}^{\alpha})^2 + M_{\underline{j}}^{\alpha}) + 3(\Lambda_{\underline{j}}^{\alpha})^2 M_{\underline{j}}^{\alpha} - (\Lambda_{\underline{j}}^{\alpha})^3$$

$$= (M_{\underline{j}}^{\alpha} - \Lambda_{\underline{j}}^{\alpha})^3 + 3(M_{\underline{j}}^{\alpha} - \Lambda_{\underline{j}}^{\alpha})^2 + 3\Lambda(M_{\underline{j}}^{\alpha} - \Lambda_{\underline{j}}^{\alpha}) + M_{\underline{j}}^{\alpha},$$

and

$$E((N_{\underline{j}}^{\alpha})^3) = 3R_{\underline{jj}}^{\alpha\alpha} + \Lambda_{\underline{j}}^{\alpha}.$$

Relation (a) then clearly represents the general expression; (b) is derived similarly and we shall omit the details.

Note: If $\mathrm{cov}(\bar{M}_{\underline{j}}^{\alpha}\bar{M}_{\underline{k}}^{\beta}, \bar{M}_{\underline{l}}^{\gamma}\bar{M}_{\underline{r}}^{\delta})$ in (b) is replaced by

$$\int_{s_{\underline{j}}}\int_{s_{\underline{k}}}\int_{s_{\underline{l}}}\int_{s_{\underline{r}}} \{ f^{(4)\alpha\beta\gamma\delta}_{(\underline{x}_1,\underline{x}_2,\underline{x}_3,\underline{x}_4)} - f^{(2)\alpha\beta}_{(\underline{x}_1,\underline{x}_2)} f^{(2)\gamma\delta}_{(\underline{x}_3,\underline{x}_4)} \} d\mu_1 d\mu_2 d\mu_3 d\mu_4 ,$$

where $f^{(p)\alpha_1,\alpha_2,\cdots,\alpha_p}_{(\underline{x}_1,\underline{x}_2,\cdots,\underline{x}_p)}$ is the <u>moment density</u> of order p , and if $f^{(3)\alpha,\beta,\gamma}_{(\underline{x}_1,\underline{x}_2,\underline{x}_3)} \equiv 0$, then Theorem 8 holds for general point processes. The proof is much more involved.

<u>Theorem 9</u>: Let N(s), M(s) be QP with fundamentals $\rho(\underline{x})$, $\tau(\underline{x})$. Then N(s) + M(s) = L(s) is also QP with fundamental $\rho(\underline{x}) + \tau(\underline{x})$.

The proof is trivial and depends only on the fact that the sum of two Poisson processes is Poisson.

Theorem 10: Let N(s) by QP with fundamental $\rho(\underline{x})$. Let $\mu(\underline{x})$ be bounded and measurable, and let

$$\mathcal{M}(s) = \int \mu(\underline{x})\, dN(\underline{x}) .$$

We shall call $\mathcal{M}(s)$ a scaled <u>quasi-Poisson process with fundamental $\rho(\underline{x})$ and scale $\mu(\underline{x})$</u> $\{ SQP(\rho(\underline{x}), \mu(x) \}$.

Let $\rho(\underline{x})$ have bounded moments through order 2. Let

$$\tilde{\mathcal{M}}(s) = \int_S \mu(\underline{x})\, \rho(\underline{x})\, d\mu ;$$

We shall call $\tilde{\mathcal{M}}(s)$ the <u>fluid approximation</u> to $\mathcal{M}(s)$, and $\mu(\underline{x})\rho(\underline{x})$ the <u>fluid density</u> for $\mathcal{M}(s)$. Consider a sequence of processes $N_k(s)$, QP with fundamental $k\,\rho(\underline{x})$, ($\underline{k}$=1, 2, ...) , and

$$\mathcal{M}_k(s) = \int_S \frac{\mu(\underline{x})}{k}\, dN_k(\underline{x}) .$$

Then $\underset{k\to\infty}{\text{l.i.m.}}\ \mathcal{M}_k(s) = \tilde{\mathcal{M}}(s)$ for any bounded measurable s.

Proof: First, $E(\mathfrak{M}_k^2(s))$ exists, by Theorem 1 and the bounded measurability of $\mu(\underline{x})$. Then

$$E\left((\mathfrak{M}_k(s)-\mathfrak{M}(s))^2\right) = E\left(\mathfrak{M}_k^2(s)\right)-2E\left(\mathfrak{M}_k(s)\mathfrak{M}(s)\right) + E\left(\left(\mathfrak{M}^2(s)\right)\right).$$

It is sufficient to establish the theorem for the case where $\mu(\underline{x})$ is a step function, since if $\{\mu_n(\underline{x})\}$ is a sequence of step functions converging to $\mu(\underline{x})$ a.e., it is clear that

$$\text{l.i.m.}\ \frac{1}{k}\int \mu_n(\underline{x})\, dN_k(\underline{x}) = \frac{1}{k}\int \mu(\underline{x})\, dN_k(\underline{x})$$

for each k, by the boundedness of the first and second moment densities of $N_k(\underline{x})$, and, in fact,

$$E\left(\left\{\frac{1}{k}\int_S [\mu_n(\underline{x})-\mu(\underline{x})]\,dN_k(\underline{x})\right\}^2\right) = \iint_{S\,S} \frac{(\mu_n(\underline{x})-\mu(\underline{x}))}{k}\frac{(\mu_n(\underline{y})-\mu(\underline{y}))}{k} k^2 f_2(\underline{x},\underline{y})\,d\mu_{\underline{x}}\,d\mu_{\underline{y}}$$

$$+ \int_S \left|\frac{\mu_n(\underline{x})-\mu(\underline{x})}{k}\right|^2 k f_1(\underline{x})\,d\mu$$

$$\leq \text{Const} \int_S |\mu_n(\underline{x})-\mu(\underline{x})|^2\,d\mu .$$

Thus the limit is uniform in k; it is also clear that

$$\text{l.i.m.} \int_S \mu_n(\underline{x})\,\rho(\underline{x})\,d\mu = \int_S \mu(\underline{x})\,\rho(\underline{x})\,dr\,,$$ so if

$$\mathfrak{M}_{kn}(s) = \int_S \frac{\mu_n(x)}{k}\,dN_k(\underline{x})\,, \quad \mathfrak{M}_{,n}(s) = \int_S \mu_n(\underline{x})\,\rho(\underline{x})\,d\mu,$$

$$\left[E\left((\mathfrak{M}_k(s)-\mathfrak{M}(s))^2\right)\right]^{1/2} \leq \left[E\left((\mathfrak{M}_k(s)-\mathfrak{M}_{kn}(s))^2\right)\right]^{1/2}$$

$$+\left[E\left((\mathfrak{M}_{kn}-\mathfrak{M}_{,n})^2\right)\right]^{1/2} + \left[E\left((\mathfrak{M}_{,n}(s)-\mathfrak{M}(s))^2\right)\right]^{1/2}$$

The first term and the last can be made as small as we wish independent of k, so we need only consider the second term. Suppose for some n that $\mu_n(\underline{x})$ is equal to a constant μ_p on each of s_p, $\bigcup_{p=1}^{N} s_p = s$. Let B be a bound both for $\mu(\underline{x})$ and all $\mu_n(\underline{x})$.

Then

$$E\left(\left(\int_S \frac{\mu_n(\underline{x})}{k}\, dN_k(\underline{x}) - \int_S \mu_n(\underline{x})\rho(\underline{x})\, d\mu\right)^2\right)$$

$$= \sum_{p,q} E\left[\left(\frac{\mu^p}{k} N_k(s_p) - \mu^p m(s_p)\right)\left(\frac{\mu^q}{k} N_k(s_q) - \mu^q m(s_q)\right)\right]$$

$$= \sum_{p,q} \mu^p \mu^q \left[\frac{1}{k^2} E(N_k(s_p) N_k(s_q)) - \frac{2}{k} E\,(N_k(s_p) m(s_q)) + E\,(m(s_p) m(s_q))\right)$$

$$= \sum_{p,q} \mu^p \mu^q \Bigg[\iint_0^\infty \Bigg\{ \frac{1-\delta_{pq}}{k^2} \sum_{\substack{n_p, n_q = 0 \\ p \neq q}}^{\infty}{}' n_p P_p(n_p | k m_p) n_q P_p(n_q | k m_q)$$

$$+ \frac{\delta_{pq}}{k^2} \sum_{n_p} n_p^2 P(n_p | k m_p) - \frac{2}{k} \sum_{n_p = 0}^{\infty} n_p P_p(n_p | m_p) m_q$$

$$+ m_p m_q \Bigg\} dP(m_p, m_q)\Bigg] \quad ,$$

where $P_p(n|x) = \frac{e^{-x}x^n}{n!}$, the Poisson probability, and $M_p = M(s_p) = \int_{s_p} \rho(x)\,d\mu$. The term for $p = q$ in the first expectation is different from the rest and is so indicated.

This reduces to

$$\sum_{p,q} \mu^p \mu^q \left[\int_0^\infty \left\{ (1-\delta_{pq}) m_p m_q + \delta_{pq}\left(m_p^2 + \frac{m_p}{k}\right) - 2 m_p m_q + m_p m_q \right\} dP(m_p, m_q) \right]$$

$$= \sum_p (\mu^p)^2 \int_0^\infty \frac{m_p}{k}\, dP(m_p)$$

$$= \frac{1}{k} E\left(\int_S (\mu_n(\underline{x}))^2 \rho(\underline{x})\, d\mu \right)$$

$$\leq \frac{B^2}{k} E\left(m(s)\right).$$

Thus this term goes to zero with k, <u>independent of n</u>, and the theorem is proved.

<u>Remark</u>: The indefinite subdivision of $N(s)$ carried out in Theorem 10 to arrive at the fluid approximation is precisely the kind of process usually envisioned in the transition from discrete particles to a fluid. Theorem 10 is proved only for QP processes; it is not clear whether there exist processes that are not QP for which such a subdivision (which does not destroy the essential probabilistic features of the distribution) exists, but it is certainly <u>not in general possible</u> to form such a subdivision. The moment densities of each subdivision must agree with the moments of the fluid density; we have seen in Theorem 3 that no m.s. continuous density exists for general point processes, and thus <u>no consistent fluid approximation exists</u> for general point distribution.

APPENDIX III

AN APPROXIMATE CALCULATION FOR THE MEAN COSMIC LIGHT

Galaxies, since they are constructed of stars and stars radiate something like black bodies, also radiate approximately like black bodies, but with a broader energy distribution on account of the mixing of the light of stars of widely varying temperatures. One analytic form that closely mimics the black-body distribution but is somewhat broader is the Wien law,

(A3.1) $$L(\nu) = \frac{L}{2\nu_0}\left(\frac{\nu}{\nu_0}\right)^2 e^{-\left(\frac{\nu}{\nu_0}\right)} ,$$

where L is the bolometric luminosity. If the mean radiation temperature of the galaxy is T_r, then one finds that

(A3.2) $$\nu_0 \approx \frac{1.4\,kT_r}{h} .$$

Suppose that the total mean luminosity per cubic megaparsec at the present epoch is $\mathcal{L}\left(\frac{\nu}{\nu_0}\right)$ with the spectral distribution (A3.1); let $\xi = \nu/\nu_0$. Then if $\mathcal{F}_\nu(\Omega)$ is the flux at ν in the solid angle Ω for the relativistic universes,

(A3.3)
$$E\mathcal{F}_\nu(\Omega) = \int_0^{\tau_0} \frac{\mathcal{L}(\xi(1+z))}{4\pi\rho^2(t)R_0^2(1+z)} \left\{(1+z)^3 R^2(\tau_0-t)\rho^2(t)\,dt\,\mu(\Omega)\right\} .$$

The quantity in brackets is the volume element normalized so that the emission per <u>coordinate</u> volume is constant. We neglect evolutionary effects. If we introduce the "development angle" θ (Mattig, (39)),

(A3.4) $$\theta = \int_0^{\tau} \frac{d\tau'}{R(\tau')},$$

we get, using (A3.1),

(A3.5) $$E\,\mathcal{F}_{\nu}(\Omega) = \frac{\mathcal{L}\,\xi^2 R_0}{8\pi\nu_0}\,\mu(\Omega)\int_0^{\theta_0}(1+z)\,e^{-\xi(1+z)}\,d\theta\,.$$

For the case of vanishing pressure and cosmological constant, τ and $R(\tau)$ can be written explicitly in terms of θ (Mattig, (39));

(A3.6) $$\tau = \frac{a}{k}\left(\theta - S_k(\theta)\right)$$

$$R = \frac{a}{k}\left(1 - C_k(\theta)\right),$$

where

$$S_k(\theta) = \begin{cases} \sin\theta \\ \theta \\ \sinh\theta \end{cases} \quad \text{for} \quad k = \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

$$C_k(\theta) = \begin{cases} \cos\theta \\ 1 \\ \cosh\theta \end{cases} \quad \text{for} \quad k = \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

for the metric

(A3.7)

$$ds^2 = d\tau^2 - R^2(\tau)\left\{\frac{d\rho^2}{1-k\rho^2} + \rho^2 d\gamma^2\right\}.$$

(The case k=0 is degenerate and will be included in our general result.) We can parametrize the light cone, as before, with the angle variable and the value of θ on the intersection of the light cone with a particular θ = const. surface. We may also parametrize with the red-shift, and for all the cases of interest the relation between θ and z is one-to-one. The integral in (A3.5) is most tractable if we use $U = (1+z)$ as the variable of integration, so we must find $d\theta/dU$. But

(A3.8) $$U = (1+z) = \frac{R_o}{R(\theta)} = \frac{1 - C_k(\theta_o)}{1 - C_k(\theta)} ,$$

so

(A3.9) $$dU = \frac{[1 - C_k(\theta_o)]^2}{[1 - C_k(\theta)]^2} \cdot \frac{kS_k(\theta)}{[1 - C_k(\theta_o)]^2} \, d\theta .$$

If we define the deceleration parameter q_o in the usual way,

$$q_o = -\left.\frac{\ddot{R}R}{\dot{R}^2}\right)_o ,$$

we find

(A3.10) $$q_o = \frac{1}{1+C_k(\theta_o)} .$$

But from (A3.8),

(A3.11) $$C_k(\theta) = 1 - \frac{1-C_k(\theta_o)}{U}$$

$$= \frac{Uq_o + (1-2q_o)}{Uq_o} ;$$

so

(A3.12) $$S_k(\theta) = \frac{\sqrt{(2q_0-1)/k}}{u q_0}\sqrt{2uq_0+(1-2q_0)}\ ,$$

and

(A3.13)

$$d\theta = \left(\frac{2q_0-1}{2kq_0}\right)^{1/2}\frac{du}{u\sqrt{u-\left(1-\frac{1}{2q_0}\right)}}\ .$$

One also finds

(A3.14) $$H_0R_0 = \frac{k\,S_k(\theta_0)}{(1-C_k(\theta_0))}$$
$$= \left\{\frac{k}{(2q_0-1)}\right\}^{1/2},$$

so replacing $d\theta$ and R_0 in (A3.5), we obtain

(A3.15) $$\frac{E\mathcal{F}_\nu(\Omega)}{\mu(\Omega)} = \frac{\mathcal{L}\xi^2}{8\pi\nu_0H_0}\frac{1}{\sqrt{2q_0}}\int_1^\infty\frac{e^{-\xi u}\,du}{\sqrt{u-\left(1-\frac{1}{2q_0}\right)}}\ ;$$

and by a simple change of variable, this becomes

(A3.16)

$$\frac{E\mathcal{F}_\nu(\Omega)}{\mu(\Omega)} = \frac{\mathcal{L}\xi^2}{8\pi\nu_0H_0}\frac{1}{2q_0}e^{\xi\left[\frac{1}{2q_0}-1\right]}\int_1^\infty\frac{e^{-\frac{\xi x}{2q_0}}}{\sqrt{x}}\,dx\ .$$

The integral is just $E_{1/2}(\xi/2q_o)$, the exponential integral of order 1/2. We can rewrite this as

(A3.17)

$$\frac{E\mathcal{F}_\nu(\Omega)}{\mu(\Omega)} = \frac{\mathcal{L}\nu^2}{8\pi\nu_o^3 H_o} \frac{e^{-\nu/\nu_o}}{\nu/\nu_o} \left\{ \frac{\nu}{2q_o\nu_o} e^{\frac{\nu}{2q_o\nu_o}} E_{1/2}\left(\frac{\nu}{2q_o\nu_o}\right) \right\},$$

The factor in curly brackets tends to one as $\nu \to \infty$, and is always less than one. It decreases monotonically with q_o for a fixed ν, and tends to one as q_o tends to zero. For the steady-state, we pick $\tau_o = \theta_o = 0$, and find

(A3.18) $\quad \theta = 1 - e^{-H\tau} = -z.$

Since here the emissivity per unit proper volume is constant, (A3.5) becomes

(A.19)

$$E\mathcal{F}_\nu^{ss}(\Omega) = \frac{\mathcal{L}\xi^2\mu(\Omega)}{8\pi\nu_o H_o} \int_{-\infty}^{0} \frac{e^{-\xi\mu}}{u^2} d\theta$$

(A3.19)

$$= \frac{\mathcal{L}\xi^2\mu(\Omega)}{8\pi\nu_0 H_0}\int_1^\infty \frac{e^{-\xi u}}{u^2}\,du$$

$$= \frac{\mathcal{L}\xi^2\mu(\Omega)}{8\pi\nu_0 H_0}\,E_2(\xi) \quad ,$$

which can be written analogously to (A3.17) as

(A3.20)

$$\frac{E\mathcal{F}_\nu(\Omega)}{\mu(\Omega)} = \frac{\mathcal{L}\nu^2}{8\pi\nu_0^3 H_0}\,\frac{e^{-\nu/\nu_0}}{\nu/\nu_0}\left\{\frac{\nu}{\nu_0}e^{\nu/\nu_0}E_2(\nu/\nu_0)\right\},$$

which tends, as $\nu \to \infty$, to the same limit as all the relativistic models do. For the currently favored value of q_o of about 1/2 (Sandage, (23)), the steady-state values are well below the relativistic ones. The results of a more refined calculation are presented in Chapter IX.

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