# SOME FLOW PROBLEMS IN RAREFIED GAS DYNAMICS

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## ABSTRACT

This thesis discusses three rather loosely connected problems in free molecule and nearly free molecule flow.  First the expansion of a gas cloud into perfect vacuum is considered on the basis of the collision-less Boltzmann equation, and it is shown that if the initial distribution is an isothermal Maxwellian, the density obeys a diffusion equation with a diffusion coefficient proportional to the time.  This leads to the description of the free expansion of symmetric clouds in terms of a thick 'diffusion front' traveling at the initial isothermal speed of sound.  The expansion of asymmetric clouds and the flow due to sources and jets are also studied.

Second, a method of iteration proposed by Willis for calculating nearly free molecular flow is extended to general unsteady flows; it is then applied to the flow through an orifice to show that the correction to the mass flow is of the first order in the inverse Knudsen number. The coefficient, estimated by making some reasonable assumptions about the three-dimensional nature of the flow, is found to agree quite well with Liepmann's measurements.

Finally a physical basis is suggested for Krook's collision model used in the above calculations.  Several consequences of the model are then derived, including the important one that, in the Navier-Stokes limit, the model implies a Stokesian gas with a Prandtl number of unity.  The value to be given to the parameter in the model is also discussed at some length.

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# LIST OF PRINCIPAL SYMBOLS

| | |
|---|---|
| a | speed of sound |
| A | parameter in Krook's model for collisions |
| B | $\beta^{o}/\beta_1$ |
| $\underset{\sim}{c}$ | thermal or peculiar velocity |
| $\underset{\sim}{C}$ | $\underset{\sim}{c}\sqrt{\beta}$ |
| $D\underset{\sim}{z}$ | element of volume in z-space |
| f | molecular velocity distribution function |
| F | Maxwellian distribution |
| $\underset{\sim}{F}$ | external force |
| g | relative velocity between molecules |
| $\mathcal{G}$ | the 'gain' functional |
| $\mathcal{H}$ | the Heaviside step function |
| I | the differential scattering cross-section |
| K | the Knudsen number |
| $L_1$ | characteristic length |
| $\mathcal{L}$ | the 'loss' functional |
| m | mass of molecule |
| $\dot{m}$ | mass flow rate per unit area |
| n | number density of molecules |
| N | total number of molecules; $n^{o}/n_1$ |
| $p_{ij}$ | $\rho\overline{c_i c_j}$ |
| p | $p_{ii}/3$ |
| $\underset{\sim}{q}$ | heat flow vector |

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## LIST OF PRINCIPAL SYMBOLS (Cont'd)

| | |
|---|---|
| $Q$ | source function |
| $r$ | spherical radius |
| $R$ | cylindrical radius; radius of orifice |
| $\mathcal{R}$ | gas constant |
| $s$ | parameter along characteristics |
| $t$ | time |
| $T$ | temperature |
| $\underset{\sim}{u}$ | gas velocity |
| $\underset{\sim}{U}$ | $\underset{\sim}{u}\sqrt{\beta}$ |
| $\underset{\sim}{v},\ \underset{\sim}{w}$ | molecular velocities |
| $\underset{\sim}{V}$ | $\underset{\sim}{v}\sqrt{\beta}$ |
| $\underset{\sim}{x}$ | position vector |
| | |
| $\beta$ | $1/2\,\mathcal{R}\,T$ |
| $\gamma$ | ratio of specific heats |
| $\delta$ | Dirac delta function; thickness of 'diffusion front' |
| $\epsilon$ | small parameter |
| $\vartheta$ | polar angle |
| $\lambda$ | mean free path |
| $\mu$ | viscosity |
| $\nu$ | kinematic viscosity; number of collisions per unit volume and unit time |
| $\rho$ | density |
| $\varphi$ | cylindrical angle |

## LIST OF PRINCIPAL SYMBOLS (Cont'd)

$\Omega$      solid angle

$Y_0$      initial value of $Y$ ; value at origin, or in orifice-plane

$Y_1$      value of $Y$ at a standard state 1 (upstream infinity in orifice flow)

$Y_2$      value of $Y$ at downstream infinity (in orifice flow)

$\bar{Y}$      mean value of $Y$, $= \int Y f D\underline{v}$

$Y^0$      value of $Y$ in free molecule flow, or in a zeroth order

approximation

$Y^n$      value of $Y$ in nth order approximation

# I. INTRODUCTION

## 1.1 General remarks

This thesis is chiefly concerned with some problems in free molecular and nearly free molecular flow of gases, a regime which may be called 'rarefied gas dynamics'. This class of flows can be somewhat more precisely described in terms of a Knudsen number K, which we define as the ratio of a typical mean free path $\lambda_1$ in the gas to a typical 'scale', say $L_1$, which characterizes the variations in the flow. The number of collisions among molecules in a volume $L_1^3$ is of order $n\bar{c} \ L_1^3/\lambda_1$ where $n$ is the number density of molecules and $\bar{c}$ is their average speed; and as the flux into the same volume is of order $n\bar{c} \ L_1^2$, it is seen that the inverse Knudsen number is a measure of the number of collisions per molecule in a given volume.

Thus when the Knudsen number is infinity, the molecules in the region $L_1^3$ do not collide among themselves at all; this is what we call free molecule flow. At high Knudsen numbers in general the collisions are few, and it appears feasible to take account of them in an approximate way.

The difficulties of the general field of rarefied gas dynamics can be attributed to two main reasons: (i) the structure of the Boltzmann equation, particularly the collision integrals in it, which has made it impossible (till now) to find any solutions spanning the whole range from gasdynamic ($K \rightarrow 0$) to free molecule ($K \rightarrow \infty$) flows; (ii) the complexity of surface interactions which are not completely

understood, and so (perhaps) not even properly formulated. Most problems studied till now seem to have tried to attack both aspects simultaneously, and as the generally scarce experimental information has often carried the uncertainties of surface interaction parameters, it becomes difficult to judge the measure of success achieved in tackling the first aspect of the general problem (i.e. collisions).

In this report we consider mostly those problems in which the effect of surface interactions (if present) is relatively unimportant. This would then include problems involving free expansion, jets, flow through an orifice, etc. The last will be discussed in some detail as Liepmann (Ref. 1) has recently made a comprehensive set of measurements of the mass flow through an orifice, essentially covering the whole Knudsen number range.

As the chances of finding a theoretical solution valid throughout the transition from gasdynamic to free molecule flow seem rather small at present, the question naturally arises as to how much of the range can be 'covered' starting from the two limits $K = 0$ and $K = \infty$. The limits themselves are relatively simple (though not trivial), chiefly because the collision terms vanish from the Boltzmann equation in both limits - for different reasons, of course. In the gasdynamic limit ($K = 0$) there are so many collisions (in a volume of order $L_1^3$ again) that there is a local static thermodynamic equilibrium at every point. In the free molecule limit there are so few collisions that the molecules are in a sense 'frozen', and their distribution is constant on trajectories.

In the following we shall always try to center the discussion around fluid mechanical quantities, as the main aim is to understand the deviations exhibited by rarefied gas flows from the familiar fluid mechanics. After a brief general discussion of the Boltzmann equation in the next section, we consider completely free molecular flow in the second chapter. This includes a general discussion of free expansion of symmetric and asymmetric clouds, and of the mechanism of a peculiar kind of (collisionless) diffusion, whose validity and limitations are studied. The flow field of free molecule sources and jets is then investigation in section 2.3. Finally a detailed study of the flow through an orifice is made.

Chapter III is devoted to nearly free molecular flow. A method of iteration proposed by Willis is first extended to unsteady flows, and applied to the problem of the flow through an orifice. The results obtained using Krook's model for the approximate evaluation of the collision integrals show reasonable agreement with the experiments of Liepmann.

Finally a brief study is made, in Chapter IV, of Krook's model for collisions. It is shown that the model corresponds to a Stokesian gas with Prandtl number unity in the Navier-Stokes limit, and it is concluded that the model is a fairly realistic and simple representation of the effects of the collision terms in the Boltzmann equation throughout the Knudsen number range.

## 1.2  The Boltzmann equation

Any flow quantity can be calculated as an appropriate moment of the distribution function, which is the basic unknown in rarefied gas dynamics. This function, which we denote by $f(\underline{x}, t; \underline{v})$, is the number density of molecules in physical $(\underline{x})$ and velocity $(\underline{v})$ space, and may in general be a function of $\underline{x}$ and the time t. The evolution of f in $\underline{x}\,\underline{v}$ t space describes the flow, and is governed by the Boltzmann equation. For a monatomic gas this equation can be written (Chapman & Cowling, Ref. 2)

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\frac{\partial f}{\partial \underline{x}} + \underline{F}\cdot\frac{\partial f}{\partial \underline{v}} = \int f(\underline{v}')\, f(\underline{w}')\, g I \, d\Omega \, D\underline{w}$$

$$- f(\underline{v}) \int f(\underline{w})\, g I \, d\Omega \, D\underline{w} \qquad (1-1)$$

where $\underline{F}$ is an external force (per unit mass) acting on the molecules, g is the relative velocity between two molecules having velocities $\underline{v}$, $\underline{w}$ before collision and $\underline{v}'$, $\underline{w}'$ after collision, and I is the (differential) cross-section for scattering into the elementary solid angle $d\Omega$. $D\underline{w}$ denotes an element of volume in $\underline{w}$-space, and the integrals over $\underline{w}$ on the right are the collision terms: the first term gives the number of particles 'gained' by a unit $\underline{x}\,\underline{v}$ volume due to collisions, and we denote it by the functional $\mathcal{G}(f)$; the second term similarly gives the number lost, and we denote this by $f\,\mathcal{L}(f)$, so that we can also write (1-1) as

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\frac{\partial f}{\partial \underline{x}} + \underline{F}\cdot\frac{\partial f}{\partial \underline{v}} = \mathcal{G}(f) - f\,\mathcal{L}(f). \qquad (1-1a)$$

It is supposed, in (1-1), that the (six) components of the velocities $\underset{\sim}{v}'$, $\underset{\sim}{w}'$ are expressed in terms of $\underset{\sim}{v}$ and $\underset{\sim}{w}$ and the 'configuration' of the encounter: this can be done by writing down the (six) equations for conservation of linear and angular momentum during each encounter. One can also interpret the integrations in (1-1) as effected over all possible values of $\underset{\sim}{w}$ and of the relative angular momentum for a given $\underset{\sim}{v}$.

The fluid mechanical quantities of the greatest interest to us can all be written down as the first few moments of f; thus

$$\rho(\underset{\sim}{x},t) = m\, n(\underset{\sim}{x},t) = m \int f(\underset{\sim}{x},t;\underset{\sim}{v})\, D\underset{\sim}{v},$$

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \frac{1}{n} \int \underset{\sim}{v}\, f(\underset{\sim}{x},t;\underset{\sim}{v})\, D\underset{\sim}{v},$$

$$p_{ij}(\underset{\sim}{x},t) = m \int c_i c_j\, f(\underset{\sim}{x},t;\underset{\sim}{v})\, D\underset{\sim}{v}$$

(1-2)

where $\rho = mn$ is the gas density, $\underset{\sim}{u}$ is the gas velocity and $p_{ij}$ is the pressure tensor. m is the mass of each molecule, and $\underset{\sim}{c} = \underset{\sim}{v} - \underset{\sim}{u}$ is the peculiar velocity of the molecule. We also define the temperature T by

$$2\mathcal{R}T = \frac{1}{\beta} = \frac{2}{3n} \int c^2 f(\underset{\sim}{x},t;\underset{\sim}{v})\, D\underset{\sim}{v} = \frac{2}{3}\frac{p_{ii}}{\rho} = \frac{2p}{\rho}$$

(1-2a)

and the heat flux vector $\underset{\sim}{q}$ by

$$\underset{\sim}{q} = \int \tfrac{1}{2} m c^2 \underset{\sim}{c}\, f(\underset{\sim}{x},t;\underset{\sim}{v})\, D\underset{\sim}{v}.$$

(1-2b)

We carry out a simple dimensional analysis of (1-1a) for steady flow in the absence of forces. Assuming $g \sim 1/\sqrt{\beta}$, the collision terms contain a parameter of the form $(n\, I/\sqrt{\beta})\, f$. If we take $v_1$ for a characteristic molecular velocity and $L_1$ (as before) for a characteristic scale of the flow, the left hand side contains a parameter like $(v/L_1)\, f$. Thus, in terms of dimensionless variables (1-1) can be written as

$$\underline{v} \cdot \frac{\partial f}{\partial \underline{x}} = \frac{n\, I\, L_1}{v_1 \sqrt{\beta}} \left\{ g(f) - f\, L(f) \right\}. \tag{1-3}$$

The quantity $1/nI$ is in the nature of a mean free path $\lambda_1$. It is not obvious what one should choose for $v_1$: if $v_1 \sim 1/\sqrt{\beta}$, the parameter in front of the collision terms is just $L_1/\lambda_1 = K^{-1}$, the inverse of the Knudsen number. It is also possible to put $v_1 \sim u_1$ (a characteristic gas velocity), when the parameter becomes $L_1/\lambda_1 \sqrt{\beta} \cdot u_1$, which is identical with the combination $\mathcal{R}e/M^2$ ( $\mathcal{R}e$ being the Reynolds number and M the Mach number) familiar in gasdynamics.

The main point is that the Boltzmann equation contains two characteristic velocities $v_1$ and $g_1$, and one characteristic length $\lambda_1$; the length $L_1$ can be looked upon as introduced from the boundary conditions. The dimensional analysis is not complete, because the boundary conditions have not been discussed in detail; in particular one would have to take account of accommodation coefficients, surface temperatures and so on; however, as stated earlier, we want

to confine ourselves to problems in which these are not important.

To illustrate this more specifically, we consider the problem of the flow through an orifice. The configuration is shown in figure 1. There is a hole of diameter D in a large, thin, insulated sheet which separates the upstream side from the downstream side. The problem is most naturally formulated in terms of D (as $L_1$), the pressure $p_1$ and density $\rho_1$ far upstream, and the pressure difference across the orifice, $\Delta p = p_1 - p_2$. (We assume there is a (large) enclosure at a constant temperature $T_1 = T_2$.) The boundary conditions in this case yield one non-dimensional parameter $\varpi = \Delta p / p_1$, a velocity $u_1 \sim \sqrt{\Delta p / \rho_1}$ and a length D. Combining these with the parameter obtained from the equation, namely $L_1 / \lambda_1 \, v_1 \, \sqrt{\beta}$, and noting that $1 / \sqrt{\beta} \sim \sqrt{\dfrac{\Delta p}{\rho_1 \varpi}}$ , it is seen that there are essentially two independent parameters in the problem, which we can choose to be

$$\varpi \sim \frac{\Delta p}{p_1} \, , \quad Re = \sqrt{\frac{\Delta p}{\rho_1}} \cdot \frac{D}{v_1} \sim \sqrt{\frac{\beta_1 \Delta p}{\rho_1}} \cdot \frac{D}{\lambda_1} \sim \sqrt{\varpi} \cdot \frac{D}{\lambda_1}$$

where $\nu \sim \lambda / \sqrt{\beta}$ is the kinematic viscosity. One can interpret

$$\varpi = \frac{\Delta p}{\rho_1} \cdot \frac{\rho_1}{p_1} \sim \frac{u_1^2}{a_1^2} \sim M_1^2$$

as the square of a Mach number, though it should be noted that $u_1$ is not necessarily of the order of the actual velocity of flow through the orifice.

The possible regimes of flow through an orifice are shown in figure 2. The transition from gasdynamic to free molecule flow can be made at various levels of $\varpi$: $\varpi \sim 1$ corresponds to the 'compressible'

transition, and $\varpi \sim 0$ to the 'incompressible' transition.

The quantity $\epsilon = n \, \mathrm{IL}_1 / v_1 \sqrt{\beta}$ largely determines the nature of the flow. The limit $\epsilon \to 0$ corresponds to free molecule flow and the other extreme $\epsilon \to \infty$ leads to the Euler and Navier-Stokes equations. The latter limit will be discussed in section 4.3 in connection with Krook's model, but it can be seen that the problem is a singular perturbation, with the small number $\epsilon^{-1}$ multiplying the derivative of f. One thus expects boundary layer type solutions for f, with a region (or interval, corresponding to the relaxation time, in initial value problems) of rapid change. The Euler and Navier-Stokes equations are the outer limits of the Boltzmann equation; and experience with other singular perturbation problems suggests that considerable care is required to obtain the higher approximations. It is perhaps due to this reason that it is so difficult to go beyond the Navier-Stokes equations.

The boundary condition at a surface can in general be expressed as

$$f(\mathbf{x}, t; \mathbf{v}_+) = \int X(\mathbf{v}_+, \mathbf{v}_-) \, f(\mathbf{x}, t; \mathbf{v}_-) \, D\mathbf{v}_- \tag{1-4}$$

where the $\mathbf{v}_+$ are the velocity vectors with the normal component $v_n = \mathbf{v} \cdot \mathbf{n} > 0$ ($\mathbf{n}$ is the outward normal to the surface), the $v_-$ are velocities such that $v_n < 0$, and X, in general, is a stochastic kernel. Equation (1-4) relates the distribution of particles emitted from the surface to that of the incident particles. The usual assumptions of diffuse and specular reflection at the surface can be written as simple assumptions on X.

The difficulties of handling a boundary condition like (1-4) are obvious: more so as there is no precise knowledge of the nature of X. As stated earlier, however, we will be mostly concerned, in the following, with problems in which surface conditions do not play any essential role.

## II.  THE FREE MOLECULE LIMIT

### 2.1  General considerations

This chapter is devoted to pure free molecule flow, obtained by putting the parameter $nIL_1/v_1\sqrt{\beta}$ in (1-3) equal to zero, that is, by neglecting all collisions.  Apart from being interesting in its own right, investigation of the limit is anyway a prerequisite to the study of the effects of collisions.  As we will later have occasion to discuss flow fields due to sources, we write the general collisionless Boltzmann equation, with sources, as

$$\left[\frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial \underline{x}} + \underline{F}\cdot\frac{\partial}{\partial \underline{v}}\right] f(\underline{x}, t; \underline{v}) = Q(\underline{x}, t; \underline{v}) \qquad (2\text{-}1)$$

where  $Q$  is the number of molecules, per unit $\underline{x}\ \underline{v}$ volume per unit time, introduced (or created) at $\underline{x}$, t.  The general initial value problem can now be posed as follows:  given f ($\underline{x}$, t = 0; $\underline{v}$) = $f_0$ ($\underline{x}$; $\underline{v}$), say, over all space, what is  f  at later times?

Problems involving free expansion of a gas cloud (described by $Q = 0$ in (2-1)) have attracted some attention in the last few years, though most of these studies seem to have been made from the gas-dynamic point of view (Refs. 3, 4, 5).  On the other hand, approaches to the problem from the free molecule limit, though of considerable interest in space sciences and in astrophysics, seem far fewer.  Molmud (Ref. 6) has, however, recently worked out the density field in free molecular free expansion of symmetric gas clouds, making use

of an intuitive 'analogy' with heat diffusion. We will examine the justification and limitation of this analogy later, starting rigorously from Boltzmann's equation.

This might be the place to make a brief remark about the mechanism of free molecule flow. The distribution function f in the free molecule limit is in general discontinuous in velocity space, which can often be divided into regions which are vacant (i. e. , there are no molecules with velocity vectors lying in them) and regions which are occupied by molecules of particular kinds, e.g. those coming from a solid surface. Though the distribution in the occupied regions may be like $e^{-\beta v^2}$, the $\beta$ does not necessarily correspond to the local temperature, and moreover the distribution is usually not isotropic (so that it is not a local Maxwellian). The flow is in fact due to the 'development' in time and (physical) space of these regions in velocity space.

This 'development' is very generally described by equation (2-1) which is a linear, inhomogeneous first order partial differential equation, with a vector parameter $\underset{\sim}{v}$ which takes all possible values. To solve the equation completely one has in general to be given f on a six-dimensional hypersurface in $\underset{\sim}{x}t\underset{\sim}{v}$ space, and in most practical problems this takes the form of an initial condition in physical space:

$$f\left(\underset{\sim}{x}, t = 0; \underset{\sim}{v}\right) = f_0\left(\underset{\sim}{x}; \underset{\sim}{v}\right). \tag{2-2}$$

$f_0$ is, of course, an arbitrary function of $\underset{\sim}{x}$ and $\underset{\sim}{v}$.

The most straightforward way of solving (2-1) is to write down

the characteristics, following Courant-Hilbert (Ref. 7); in terms of

a parameter s along the characteristics, they are governed by the

eight ordinary differential equations

$$\frac{dt}{ds} = 1 , \qquad \frac{d\underset{\sim}{x}}{ds} = \underset{\sim}{v} , \qquad \frac{d\underset{\sim}{v}}{ds} = \underset{\sim}{E} , \qquad \frac{df}{ds} = Q \qquad (2\text{-}3)$$

which have the solution

$$t = s , \qquad \underset{\sim}{v} = \underset{\sim}{E}s + \underset{\sim}{v} , \qquad \underset{\sim}{x} = \frac{\underset{\sim}{E}s^2}{2} + \underset{\sim}{v}s + \underset{\sim}{\xi} ,$$

$$f = \int_0^s Q \, ds + f_0 \qquad (2\text{-}4)$$

where Q is integrated with respect to s after being expressed as a

function of s and $(\underset{\sim}{\xi}, \underset{\sim}{v})$, which is a point on the initial surface. (See

Fig. 3.) However the six initial coordinates $\underset{\sim}{\xi}, \underset{\sim}{v}$ can be eliminated

from (2-4) and we can write f at any given time as

$$f(\underset{\sim}{x}, t; \underset{\sim}{v}) = f_0 \left( \underset{\sim}{x} - \underset{\sim}{v}t + \frac{\underset{\sim}{E}t^2}{2}; \underset{\sim}{v} - \underset{\sim}{E}t \right)$$

$$+ \int_0^t Q \left\{ \underset{\sim}{x} - \underset{\sim}{v}(t-s) + \frac{1}{2}\underset{\sim}{E}(t-s)^2, s; \underset{\sim}{v} - \underset{\sim}{E}(t-s) \right\} ds . \qquad (2\text{-}5)$$

This is the general solution of the initial value problem that was posed

earlier. If there are no forces on the molecules, (2-5) reduces to

$$f(\underset{\sim}{x}, t; \underset{\sim}{v}) = f_0(\underset{\sim}{x} - \underset{\sim}{v}t; \underset{\sim}{v}) + \int_0^t Q \{\underset{\sim}{x} - \underset{\sim}{v}(t-s), s; \underset{\sim}{v}\} ds . \qquad (2\text{-}6)$$

The density, gas velocity and all other flow quantities can now be

derived as appropriate moments of f.

## 2.2 Expansion of a gas cloud

Let us briefly study the case $Q = 0$. This includes the 'free expansion' class of problems, in which one considers a cloud of gas confined within a certain region, whose boundaries are suddenly removed at time $t = 0$. In such a problem $f_0 = 0$ outside the cloud. More generally an arbitrary initial distribution over all space may be given. In either case, the distribution at $t > 0$ is simply obtained from (2-6) by putting $Q = 0$:

$$f(\underset{\sim}{x}, t; \underset{\sim}{v}) = f_0(\underset{\sim}{x} - \underset{\sim}{v}t; \underset{\sim}{v}). \tag{2-7}$$

The corresponding flow quantities are most conveniently worked out by making the transformation

$$\underset{\sim}{x} - \underset{\sim}{v}t = \underset{\sim}{x}',$$
$$D\underset{\sim}{x}' = dx_1' \, dx_2' \, dx_3' = -t^3 \, dv_1 \, dv_2 \, dv_3 = -t^3 \, D\underset{\sim}{v}. \tag{2-8}$$

Thus

$$\varsigma(\underset{\sim}{x}, t) = \int f_0(\underset{\sim}{x} - \underset{\sim}{v}t, \underset{\sim}{v}) \, D\underset{\sim}{v}$$

$$= \frac{1}{t^3} \int f_0\left(\underset{\sim}{x}'; \frac{\underset{\sim}{x} - \underset{\sim}{x}'}{t}\right) D\underset{\sim}{x}', \tag{2-9}$$

where the integration is performed over all $\underset{\sim}{x}'$, of course. (Note that $Dx'/Dv$ is just the Jacobian of the transformation, and only its absolute value appears in (2-9).) The gas velocity $\underset{\sim}{u}$ is given by

$$\underline{u}(\underline{x}, t) = \frac{1}{\rho(\underline{x}, t)} \int \frac{\underline{x} - \underline{x}'}{t} \cdot f_0\left(\underline{x}'; \frac{\underline{x} - \underline{x}'}{t}\right) \frac{D\underline{x}'}{t^3}$$

$$= \frac{\underline{x}}{t} - \frac{1}{\rho t^4} \int \underline{x}' f_0\left(\underline{x}'; \frac{\underline{x} - \underline{x}'}{t}\right) D\underline{x}'. \qquad (2\text{-}10)$$

Any other flow quantity can be similarly calculated as some (transformed) moment of the <u>initial</u> distribution function.

Equation (2-10) shows what is apparently a general feature of 'free' gaskinetic flows, namely that the <u>gas</u> velocity can be split into two parts one of which (like x/t) is purely kinematic, in the sense that it does not depend on any dynamic variable (like temperature, e.g.), and another part which is 'thermal' and tends to have a characteristic value like $1/\sqrt{\beta} \sim \sqrt{RT}$. Physically the kinematic part arises from the presence at $\underline{x}$ of molecules which took exactly the time t to get there. It of course vanishes in steady free flow. In unsteady flows it seems to be the counterpart of the asymptotic 'similarity' component of the velocity that one encounters in gasdynamics.

Some interesting examples of free expansion are worked out below.

## 2.2.1  The 'point' cloud

At sufficiently large distances any cloud must look like a point.
Thus suppose there are  N  molecules all concentrated at the origin
at t = 0 and distributed according to a Maxwellian, so that

$$f_0(x;v) = \delta(x) N \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2}. \tag{2-11}$$

( $\delta(x)$ is the Dirac delta function such that $\int_{(0)} \delta(x) \, Dx = 1.$ )
Then from (2-7), (2-9) and (2-10),

$$f(x,t;v) = \delta(x - vt) N \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2},$$

$$\varrho(x,t) = \frac{1}{t^3} \int \delta(x') N \left(\frac{\beta}{\pi}\right)^{3/2} \exp\left\{-\beta\left(\frac{x-x'}{t}\right)^2\right\} \cdot Dx' \tag{2-12}$$

$$= \frac{N}{t^3} \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta x^2/t^2},$$

and $\qquad u = x/t.$

The result for the velocity is quite obvious, as only molecules with
velocity $v = x/t$ can reach $x$ at time  t, so $v = x/t = u$.  Also the ther-
mal or  peculiar velocity $c = v - u$ is then zero, so the temperature
T = 0.  At any point $x$ the velocity shoots up suddenly to infinity
at  t = 0+, then drops off as $t^{-1}$.  The density on the other hand, builds
up gradually from zero to a maximum (of $27 N/8 x^3 e^{3/2} = 0.07361$
$N/x^3$ ) at t = $x\sqrt{\frac{2}{3}\beta}$, and drops off like $t^{-3}$ at large times.  Thus
the decay in density is exponential in space and algebraic in time.

As may be expected (and as will be shown later) the same
solution will be obtained if $f_0 = 0$ but $Q = \delta(x) \delta(t) N \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2}$,
which represents a pulse source.  Incidentally, it may be worth noting

that as all molecules at $x$ have the single velocity $x/t$, the relative velocity g between any two of them is zero, and hence they will never collide. Thus, in so far as one can speak of a 'point' cloud with an initial Maxwellian (or any other)distribution expanding into 'perfect' vacuum, equations (2-12) give an exact fundamental solution of the full Boltzmann equation, because the collision integrals vanish identically for (2-12)! However, as the full Boltzmann equation is non-linear, it is not possible to superpose these fundamental solutions to derive solutions for more complicated problems.

## 2.2.2 Symmetric clouds

We discuss the one-dimensional case in some detail, especially because an exact solution of the problem in the gasdynamic limit has been obtained by Greifinger & Cole (Ref. 4). Suppose the gas is confined between two planes $x_1 = \pm l$ and is allowed to expand into vacuum at $t = 0$. (Note that an equivalent problem is to consider a wall at $x_1 = 0$ - when we will have a kind of free molecule shock tube - if the boundary condition at the wall corresponds to specular reflection of the molecules.) We can then write

$$f_0(x;v) = \left[\mathcal{H}(x_1+l) - \mathcal{H}(x_1-l)\right] n \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2}$$

where $\mathcal{H}$ is the Heaviside step function. Due to the symmetry the coordinates $x_2$, $x_3$ at the point of observation $x$ can be taken to be zero; then from (2-9) it is easily seen that

$$\rho(x,t) = \frac{1}{t^3} n \left(\frac{\beta}{\pi}\right)^{3/2} \int_{-l}^{+l} \exp - \frac{\beta(x_1-x_1')^2}{t^2} \cdot dx_1'$$
$$\times \int_{-\infty}^{+\infty} e^{-\beta x_2'^2} dx_2' \int_{-\infty}^{+\infty} e^{-\beta x_3'^2} dx_3',$$

or, introducing $\xi = x_1 / l$ and $\tau = t / l\sqrt{\beta}$,

$$\rho = \tfrac{1}{2}\rho_0 \left\{ \operatorname{erf} \frac{\xi+1}{\tau} - \operatorname{erf} \frac{\xi-1}{\tau} \right\} \tag{2-13}$$

where $\rho_0 = mn$ is the initial density. This result is the same as that given by Molmud (Ref. 6), but we want to give it a different interpretation here. A typical density profile is shown in figure 4.

Now for large times, i.e., for $t \gg \beta^{\frac{1}{2}}(x_1 \pm l)$, (2-13) reduces to

$$\frac{\rho}{\rho_0} \simeq \sqrt{\frac{10}{3\pi}} \cdot \frac{l}{a_0 t} \simeq 1.03 \frac{l}{a_0 t} \tag{2-13a}$$

where $a_0 = \sqrt{\frac{5}{3} RT_0}$ is the initial speed of sound in the cloud (with $\gamma = 5/3$ for the monatomic gas, of course). The density is roughly constant in x (provided x is not too large) and drops off as $1/t$.

As might be expected from general similarity considerations, Greifinger & Cole find an asymptotic relation identical in form with (2-13a), but with a different numerical constant. Their calculation gives, when $\gamma = 5/3$,

$$\frac{\rho}{\rho_0} \simeq \frac{1}{2} \frac{l}{a_0 t}, \tag{2-13b}$$

i.e.; about half of what one expects in free molecule flow.

Differentiating (2-13) with respect to $\xi$, one obtains

$$\frac{1}{\rho_0} \frac{\partial \rho}{\partial \xi} = -\frac{2}{\tau \sqrt{\pi}} \sinh \frac{2\xi}{\tau^2} \cdot \exp - \frac{\xi^2 + l}{\tau^2}, \tag{2-14}$$

which is always negative (considering only $x_1 > 0$, as the cloud is always symmetric), so $\rho$ decreases monotonically; but differentiating once more it is easily verified that $\partial \rho / \partial \xi$ is a maximum at a point $\overline{\xi} = \overline{\xi}(\tau)$ such that

$$\bar{\xi} = \coth \frac{2\bar{\xi}}{\tau^2} . \tag{2-15}$$

$\bar{\xi}$ is plotted versus $\tau$ in figure 5. For large times (which means, as it will turn out, $t \gg l\sqrt{\beta}$), equation (2-15) can be approximated by

$$\bar{\xi} = \frac{1}{2}\tau, \quad \text{or} \quad \frac{\bar{x}_1}{t} = \frac{1}{\sqrt{2\beta}} . \tag{2-16}$$

That is, the region of most rapid change in density travels asymptotically at a definite velocity $1/\sqrt{2\beta} = \sqrt{\mathcal{R}T_0}$, which is equal to the isothermal speed of sound in the initial cloud! To find out how this region is spreading, one can define a thickness of the region by

$$\delta = \frac{\Delta\rho}{(\partial\rho/\partial x_1)_{\bar{x}_1}} \tag{2-17}$$

where $\Delta\rho$ is the difference in densities across the region, or (what amounts to the same thing) the density at $x_1 = 0$; this is given by (2-13a). Putting (2-16) into (2-14), one obtains

$$\delta = \sqrt{\frac{e}{2}} \cdot \frac{t}{\sqrt{\beta}} \simeq 1.17 \frac{t}{\sqrt{\beta}} . \tag{2-18}$$

Finally one can work out the gas velocity from (2-10); the result is

$$u_1 = \frac{2}{\sqrt{\pi\beta}} \frac{\sinh \frac{2\bar{\xi}}{\tau^2} \cdot \exp - \frac{\bar{\xi}^2+1}{\tau^2}}{\operatorname{erf} \frac{\bar{\xi}+1}{\tau} - \operatorname{erf} \frac{\bar{\xi}-1}{\tau}} ; \tag{2-19}$$

at large times $(t \gg (x_1 \pm l)\sqrt{\beta})$ this reduces to

$$u_1 \simeq \frac{x_1}{t} .$$

At small times $(t \ll (x_1 \pm l)\sqrt{\beta})$,

$$u_1 \simeq \frac{x_1 \mp l}{t} .$$

It may also be verified easily that at $x_1 = \bar{x}_1$, $u_1 \rightarrow \bar{u}_1 = 1/\sqrt{2\beta}$ for $t \gg l\sqrt{\beta}$; and that the mass flow $\rho u_1$ is a maximum at $\bar{x}_1$.

To summarize, the flow field can be roughly described as follows. At large times there is a growing region of practically uniform density near the center, but this density is falling in time (as $1/t$). Most of the density change occurs in a layer which is traveling with a definite velocity given by $\sqrt{RT_0}$ and whose thickness is increasing linearly with time. The gas velocity in this layer is also of order $\sqrt{RT_0}$, and the largest mass flow is taking place there.

One can carry out an exactly similar analysis of a spherically symmetric cloud, to obtain

$$\frac{\rho}{\rho_0} = \frac{1}{2} \left\{ \operatorname{erf} \frac{\xi+1}{\tau} - \operatorname{erf} \frac{\xi-1}{\tau} \right\}$$
$$- \frac{\tau}{\xi\sqrt{\pi}} \sinh \frac{2\xi}{\tau^2} \cdot \exp - \frac{\xi^2+1}{\tau^2} ,$$

$$u_1 = \frac{1}{\rho\sqrt{\pi\beta}} \sinh \frac{2\xi}{\tau^2} \cdot \exp - \frac{\xi^2+1}{\tau^2} . \qquad (2\text{-}20)$$
$$\times \left\{ \frac{1}{\xi} \coth \frac{2\xi}{\tau^2} - \frac{\tau^2}{\xi^2} \right\}$$

where $x$ is now the distance from the origin, and $l$ is the radius of the original cloud. The 'front' (i.e., the maximum density gradient) is located at the solution of

$$\tau^2(\bar{\xi}^2 + \tau^2) + 2\bar{\xi}^2 = 2\bar{\xi}(\bar{\xi}^2 + \tau^2) \cdot \coth \frac{2\bar{\xi}}{\tau^2} ; \qquad (2\text{-}21)$$

for large $\tau$ we obtain again, by expanding the coth (this time to two orders), exactly the same result as before: a velocity of $1/\sqrt{2\beta}$ and a thickness of $\sqrt{e/2} \cdot t/\sqrt{\beta}$.

We will return to a more basic discussion of these flows in section 2. 2. 4.

## 2.2.3 Asymmetric clouds

It is obvious, from equations (2-9) and (2-10), that the expansion of any arbitrary cloud can, at worst, be numerically computed. All the same it is desirable to have at least some simple model for asymmetric clouds so that one can get a rough idea of any tendencies toward symmetry, if in fact such tendencies exist. For this purpose we consider a distribution function of the form

$$f_0(x;v) = n_0 \left[ exp - (a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2) \right] \left( \frac{\beta}{\pi} \right)^{3/2} e^{-\beta v^2} \qquad (2\text{-}22)$$

as a model for an asymmetric isothermal cloud. The cloud is actually supposed infinite in extent, but the density falls off exponentially with distance. The density contours are the ellipsoids

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = const.$$

(assuming the a are all positive). The advantage of the form assumed in (2-22) is that the variables are separable.

From equation (2-9) we can work out the density at later times:

$$\rho = \frac{n_0 \beta^{3/2}}{[(\beta + a_1 t^2)(\beta + a_2 t^2)(\beta + a_3 t^2)]^{1/2}} \, exp - \left\{ \frac{\beta a_1 x_1^2}{\beta + a_1 t^2} \right.$$
$$\left. + \frac{\beta a_2 x_2^2}{\beta + a_2 t^2} + \frac{\beta a_3 x_3^2}{\beta + a_3 t^2} \right\}. \qquad (2\text{-}23)$$

It thus appears that the contours of ρ are always ellipsoids, but the eccentricity of the contours is a function of time. Equation (2-23) shows that as $t \rightarrow \infty$ the contours become spheres.

Consider an ellipsoid of revolution, given initially by

$$a_1 x_1^2 + a_2 R^2 = \text{const.} = \frac{x_1^2}{l_1^2} + \frac{R^2}{l_2^2}$$

where $l_1$ and $l_2$ are the semi-axes. From (2-23) the axes of the contours at later times are related by

$$l_1'^2 \propto \frac{\beta + a_1 t^2}{\beta a_1} , \qquad l_2'^2 \propto \frac{\beta + a_2 t^2}{\beta a_2},$$

so their ratio is

$$\frac{l_1'}{l_2'} = \left\{ \frac{l_1^2/l_2^2 + t^2/\beta l_2^2}{1 + t^2/\beta l_2^2} \right\}^{1/2}. \qquad (2\text{-}24)$$

This equation is plotted in figure 6, as $l_1'/l_2'$ vs. $t/l_2\sqrt{\beta}$ for some values of $l_1/l_2$. The limit $l_1/l_2 = 0$ corresponds to a flat cloud. For oblate spheroids (i.e., $l_1/l_2 < 1$) the shape of the expanded cloud is almost spherical at $t \simeq 2 l_2/\sqrt{\beta}$, and this time depends only slightly on the initial eccentricity. The narrow portions of the cloud expand out much faster than the other parts. It is also interesting to note that the flatter the initial cloud, the higher is the rate at which it tends to symmetry, though the actual time taken is longer.

For prolate spheroids ($l_1/l_2 > 1$), (2-24) can be written

$$\frac{l_2'}{l_1'} = \left\{ \frac{l_2^2/l_1^2 + t^2/\beta l_1^2}{1 + t^2/\beta l_1^2} \right\}^{1/2} \qquad (2\text{-}24a)$$

which has the same form as (2-24) except that $l_1 / l_2$ is replaced by the reciprocal, and t is non-dimensionalized with $l_1 \sqrt{\beta}$. With this interpretation, therefore, figure 6 also describes the expansion of prolate spheroids.

We may then generalize the above results as follows: an asymmetric cloud in free molecule expansion always tends to symmetry, which is roughly achieved at times of order $2 l \sqrt{\beta}$, where $l$ is the longest initial dimension of the cloud.

This result is somewhat different from the conclusion reached by Dyson (Ref. 5) after a gasdynamic analysis of the same problem. He found that an initially oblate spheroid actually becomes prolate after expansion, and vice versa.

## 2.2.4 Free molecule diffusion

It will be noticed that some of the integrals written down in previous sections show a superficial resemblance to those encountered in heat diffusion problems; and this fact has actually been used by Molmud to evaluate them. The purpose of this section is to show rigorously that, under certain conditions, there is in fact a peculiar diffusive mechanism in the flow.

We have already expressed the density in free expansion as the integral

$$\rho(\underset{\sim}{x}, t) = \frac{1}{t^3} \int f_0\left(\underset{\sim}{x}'; \frac{\underset{\sim}{x} - \underset{\sim}{x}'}{t}\right) D\underset{\sim}{x}'. \tag{2-9}$$

Differentiating this with respect to $\underset{\sim}{x}$,

$$\frac{\partial \rho(\underset{\sim}{x}, t)}{\partial \underset{\sim}{x}} = \frac{1}{t^3} \int \frac{1}{t} \frac{\partial f_0}{\partial \underset{\sim}{y}} D\underset{\sim}{x}' \tag{2-25}$$

where $\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{x}') = (\underset{\sim}{x} - x')/t$. Comparing (2-25) with the expression (2-10) for $\underset{\sim}{u}$, it is seen that $\partial \rho / \partial \underset{\sim}{x}$ is proportional to $\rho \underset{\sim}{u}$ if, and only if,

$$\frac{\partial f_0(\underset{\sim}{x}', \underset{\sim}{y})}{\partial \underset{\sim}{y}} = k \underset{\sim}{y} f_0 \tag{2-26}$$

where k is a constant, independent of $\underset{\sim}{x}$. Thus only if $f_0$ can be written as $g(\underset{\sim}{x}) e^{-\beta v^2}$ (and so is an 'isothermal' Maxwellian, with constant $\beta$) is (2-26) satisfied; and in this case the mass flux is

indeed proportional to the density gradient:

$$\rho u = -\frac{t}{2\beta} \frac{\partial \rho}{\partial x}.$$  (2-27)

Putting this relation in the equation of conservation of mass (obtained as a first integral of Boltzmann's equation), namely

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho u) = 0,$$

one obtains

$$\frac{\partial \rho}{\partial t} - \frac{t}{2\beta} \nabla^2 \rho = 0.$$  (2-28)

This is a diffusion equation for the density with a time-dependent diffusion coefficient. It can be transformed to a diffusion equation with a constant coefficient (equal to unity, in fact) if we put

$$\frac{t^2}{4\beta} = \tilde{t},$$

when (2-28) becomes

$$\frac{\partial \rho}{\partial \tilde{t}} - \nabla^2 \rho = 0.$$  (2-29)

It is thus no surprise to see integrals looking like those in heat diffusion.

The mechanism of the 'diffusion' exhibited in (2-28) and (2-29)has nothing in common with the other familiar phenomena, namely ordinary ('Navier-Stokes') diffusion which depends on intermolecular collisions, or Knudsen diffusion which depends on

collisions with surfaces (instead of molecules). The mechanism of free expansion can in fact be best described as a kind of 'collisionless' or 'kinematic' diffusion. All the peculiar features noted earlier, like the 'diffusive front' traveling at a definite velocity and growing linearly with time, can be explained now as simple consequences of the diffusion coefficient in (2-28) being proportional to time.

The limitations of this interpretation should be clearly understood. In particular, it should be noted that if $\beta$ is not constant, one does not get (as might be expected) a diffusion equation with a correspondingly variable diffusion coefficient; there simply is no diffusion equation in that case, as $\rho u$ and $\partial \rho / \partial x$ cannot be related. Also, if the Maxwellian is centered about a non-zero mean velocity, additional terms are introduced into the equations. Finally, as we shall see later, the solutions when there are time-dependent sources have no simple analogy.

It thus seems that while the resemblance to diffusion is useful in some problems, it is not very general; as the approach through the basic differential equations is at least as simple and vastly more general and fundamental, it will be adopted in the rest of the following work.

## 2.3 Flow due to sources

In this section we are concerned with the flow due to free-molecule sources, and so with the non-homogeneous equation (2-1), $Q \neq 0$. It will, however, be assumed that the external force $\underset{\sim}{F} = 0$. The solution for $f$ is then given by (2-6):

$$f(\underset{\sim}{x}, t; \underset{\sim}{v}) = f_0(\underset{\sim}{x} - \underset{\sim}{v}t; \underset{\sim}{v}) + \int_0^t Q\{\underset{\sim}{x} - \underset{\sim}{v}(t-s), s; \underset{\sim}{v}\}\, ds. \qquad (2-6)$$

If $f_0 = 0$ and we have a 'pulse source'

$$Q = Q(\underset{\sim}{x}, t; \underset{\sim}{v}) = \delta(\underset{\sim}{x})\,\delta(t)\,N\left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2},$$

we obtain again the point cloud solution

$$f(\underset{\sim}{x}, t; \underset{\sim}{v}) = \delta(\underset{\sim}{x} - \underset{\sim}{v}t)\,\mathcal{H}(t)\,N\left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2} \qquad (2-12)$$

One problem of particular interest is the continuous point source which emits molecules with a given mean velocity $\underset{\sim}{u}_0$ at a certain rate $N(t)$ (molecules per unit time). This would, for example, give the flow field due to a free molecule jet at large distances from the exit. The source in this case has the form

$$Q = \delta(\underset{\sim}{x})\,\dot{N}(t)\left(\frac{\beta}{\pi}\right)^{3/2} \exp{-\beta(\underset{\sim}{v} - \underset{\sim}{u}_0)^2}. \qquad (2-30)$$

Putting this in (2-6) one gets

$$f(\underset{\sim}{x}, t; \underset{\sim}{v}) = \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta(\underset{\sim}{v} - \underset{\sim}{u}_0)^2} \int_0^t \delta\{\underset{\sim}{x} - \underset{\sim}{v}(t-s)\}\,\dot{N}(s)\, ds. \qquad (2-31)$$

In order to integrate this over velocity space to get the density, etc., it is convenient to introduce the transformation

$$\sigma = \frac{1}{t - s} \qquad (2\text{-}32)$$

and note that

$$\delta\{x - v(t - s)\} = \delta\left\{x - \frac{v}{\sigma}\right\} = -\sigma^{-3}\,\delta(v - x\sigma).$$

Putting this in (2-31) and integrating over $v$, we get

$$\rho(x,t) = \int^{\infty}\left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta(x\sigma - u_0)^2}\,\dot{N}(t - 1/\sigma)\,\sigma\,d\sigma,$$

$$u(x,t) = \frac{x}{\rho}\int^{\infty}\left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta(x\sigma - u_0)^2}\,\dot{N}(t - 1/\sigma)\,\sigma^2\,d\sigma. \qquad (2\text{-}33)$$

The lower limit in these integrals is $\sigma = 1/t$ if the source starts emitting at $t = 0$, and $\sigma = 0$ if the source has been emitting since $t = -\infty$.

The simplest case is when $\dot{N}(t) = \dot{N}\mathcal{H}(t)$ is a step function in time ($\dot{N}$ being a constant). The integrals in (2-33) can then be evaluated explicitly, and give

$$\rho(x,t) = \frac{\dot{N}}{2\beta x^2}\left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2 \sin^2\vartheta}\left\{\exp-\beta\left(\frac{x}{t} - u_0\cos\vartheta\right)^2\right.$$

$$+ \sqrt{\pi\beta}\,u_0\cos\vartheta\cdot\operatorname{erfc}\sqrt{\beta}\left(\frac{x}{t} - u_0\cos\vartheta\right),$$

$$u(x,t) = \frac{2x}{x\sqrt{\beta}}\left\{\sqrt{\beta}\left(\frac{x}{t} + u_0\cos\vartheta\right)\exp\left[-\beta\left(\frac{x}{t} - u_0\cos\vartheta\right)^2\right]\right. \qquad (2\text{-}34)$$

$$+ \frac{\sqrt{\pi}}{4}(1 + 2\beta u_0^2\cos^2\vartheta)\operatorname{erfc}\sqrt{\beta}\left(\frac{x}{t} - u_0\cos\vartheta\right)\Big\}$$

$$\div\left\{\exp-\beta\left(\frac{x}{t} - u_0\cos\vartheta\right)^2 + \sqrt{\pi\beta}\,u_0\cos\vartheta\cdot\right.$$

$$\left.\times\operatorname{erfc}\sqrt{\beta}\left(\frac{x}{t} - u_0\cos\vartheta\right)\right\}.$$

Here $\vartheta$ is the angle between $\underset{\sim}{x}$ and $\underset{\sim}{u}_0$, and $x = |\underset{\sim}{x}|$ is the distance from the source.

Consider first the symmetric source, $u_0 = 0$. Equations (2-34) then simplify considerably:

$$\rho(\underset{\sim}{x},t) = \frac{N}{2\beta x^2}\left(\frac{\beta}{\pi}\right)^{3/2}e^{-\beta x^2/t^2},\qquad(2\text{-}35a)$$

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \frac{\underset{\sim}{x}}{t} + \frac{\underset{\sim}{x}}{2x}\sqrt{\frac{\pi}{\beta}}\,e^{\beta x^2/t^2}\,erfc\,\frac{x\sqrt{\beta}}{t}.\qquad(2\text{-}35b)$$

First of all it is seen that the gas velocity splits into two parts again, one part containing $\beta$ and the other not. At small times or large distances $(x\sqrt{\beta}/t \gg 1)$, using the asymptotic expansion for erfc,

$$\underset{\sim}{u} \simeq \frac{\underset{\sim}{x}}{t} + \frac{\underset{\sim}{x}t}{2\beta x^2} \simeq \frac{\underset{\sim}{x}}{t},$$

the same as for the pulse source, of course. Secondly, the time $t$ enters into (2-35a) only in the combination $x\sqrt{\beta}/t$, and disappears when $x\sqrt{\beta}/t \ll 1$. Thus at any time a steady state prevails at sufficiently small $x$, and an unsteady state at sufficiently large $x$.

Naturally, at any given point $\underset{\sim}{x}$ the flow becomes steady at very large times and the 'kinematic' part of the velocity then vanishes. The field of the steady, symmetric source is given by

$$\rho = \frac{N\sqrt{\beta}}{2\pi^{3/2}}\cdot\frac{1}{x^2},$$

$$\underset{\sim}{u} = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\cdot\frac{\underset{\sim}{x}}{x}, \quad u = |\underset{\sim}{u}| = \frac{1}{2}\sqrt{\frac{\pi}{\beta}} = \frac{\pi}{4}\,\bar{c}.\qquad(2\text{-}36)$$

The total mass flux at any radius is $4\pi x^2 \cdot \rho u = N$. It is very

interesting to note that the gas velocity is constant throughout, and it is the density that drops off as the inverse square of the distance - in contrast to the ordinary incompressible hydrodynamic source. The proper thing to compare with is actually the supersonic source, for which the velocity tends to a constant at infinity. A general comparison of the velocity fields of gasdynamic and free molecule sources is made in figure 7. *

Let us return now to the case $\underset{\sim}{u}_o \neq 0$. From (2-34) it is obvious that the streamlines still radiate in straight lines from the center, but now they tend to crowd around an axis in the direction $\underset{\sim}{u}_o$. The remarks made above concerning the unsteady character of the field still apply (they are quite general, in fact). The steady field is given by

$$\rho = \frac{1}{2} \frac{N \sqrt{\beta}}{\pi^{3/2} x^2} \left[ e^{-U_o^2} + \sqrt{\pi}\, U_o \cos\vartheta \cdot e^{-U_o^2 \sin^2\vartheta} (1 + \mathrm{erf}\, U_o \cos\vartheta) \right],$$

$$\underset{\sim}{U} = \underset{\sim}{u}\sqrt{\beta} = \frac{\underset{\sim}{x}}{x} \cdot \frac{U_o \cos\vartheta\, e^{-U_o^2 \cos^2\vartheta} + \sqrt{\pi}/2\,(1 + 2U_o^2 \cos^2\vartheta)(1 + \mathrm{erf}\, U_o \cos\vartheta)}{e^{-U_o^2} + \sqrt{\pi} \cdot U_o \cos\vartheta \cdot (1 + \mathrm{erf}\, U_o \cos\vartheta)}. \qquad (2-37)$$

Here $\underset{\sim}{U}_o = \underset{\sim}{u}_o \sqrt{\beta}$, and is proportional to the initial Mach number. At right angles to the axis the density is reduced by the factor $e^{-U_o^2}$ in comparison with the symmetric source, but the velocity remains the same.

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* $x_o$ in the figure is defined by $\tilde{Q} = 4\pi a_o \rho_o x_o^2$, where $\tilde{Q}$ is the total mass flux from the source. For the gasdynamic sources $\rho_o$, $a_o$ are the stagnation point values of density and sonic speed, and for the free molecule source $a_o$ is the sonic speed at the origin.

## 2.3.1  The 'high Mach number' limit

The case of $U_o \gg 1$ is of some practical interest, and an asymptotic expression for the integrals in (2-33) can be obtained using the method of steepest descents (see, e.g., Jeffreys, Ref. 8) in the limit $U_o \rightarrow \infty$. One then gets

$$\rho(x,t) \approx \frac{\sqrt{\beta}}{\pi x^2} U_o \cos\vartheta \cdot \dot{N}\left(t - \frac{x\sqrt{\beta}}{U_o \cos\vartheta}\right) e^{-U_o^2 \sin^2\vartheta},$$

$$u(x,t) \approx \frac{x}{x} u_o \cos\vartheta.$$

$$(2-38)$$

It will be noticed that $\rho$ at $(x, t)$ depends only on the strength of the source at the time ( $t - x\sqrt{\beta}/U_o \cos\vartheta$); obviously the expressions are valid only when $t > x/u_o \cos\vartheta$ and are thus not correct at $\vartheta \simeq \pi/2$. As we shall see below, however, the density is so small at $\vartheta \simeq \pi/2$ that (2-38) is practically good everywhere.

In the case when $\dot{N}$ is constant the density given by (2-38) has been plotted in figure 8, for a few values of $U_o$. The exact solution is also included for $U_o = 1$ and the closeness of the approximation for $\vartheta \lesssim \pi/2$ is quite impressive. The agreement is of course very much better at the higher $U_o$ for which the density falls off so rapidly with increasing $\vartheta$ that the error committed in using (2-22) for all angles is hardly noticeable.

It is obvious from (2-38) that the angle of spread of the jet is of order $1/U_o$.

More complicated problems can be solved by superposition of the source solutions given above, but we will not go into them here.

## 2.4 Steady flow through an orifice

Proceeding along the same lines as in previous sections, the free molecule flow past any body can be determined in the following way. In steady flow without any sources, the Boltzmann equation becomes simply

$$\underset{\sim}{v} \cdot \frac{\partial f(\underset{\sim}{x}; \underset{\sim}{v})}{\partial \underset{\sim}{x}} = 0 ; \tag{2-39}$$

the corresponding boundary condition on $f$ might be prescribed in the form

$$f(\underset{\sim}{x}; \underset{\sim}{v}) = f_1(\underset{\sim}{x}; \underset{\sim}{v}) \quad \text{on} \quad S(\underset{\sim}{x}) = 0$$

where $S(\underset{\sim}{x}) = 0$ is some surface(s). The solution of (2-39) can then be formally written

$$f(\underset{\sim}{x}; \underset{\sim}{v}) = f_1(\underset{\sim}{\xi}; \underset{\sim}{v}) \tag{2-40}$$

where $\underset{\sim}{\xi}$ is the 'root' of

$$\underset{\sim}{\xi} \times \underset{\sim}{v} = \underset{\sim}{x} \times \underset{\sim}{v}, \qquad S(\underset{\sim}{\xi}) = 0.$$

Stated in words, this means that as the distribution function is constant on trajectories (from (2-39)), its value at any point $\underset{\sim}{x}$ for any velocity $\underset{\sim}{v}$ is equal to that at the point $\underset{\sim}{\xi}$ where the vector $\underset{\sim}{v}$, produced backwards, meets the surface S.

Consider the flow through a circular orifice (Fig. 1). We

denote conditions far upstream and far downstream by subscripts 1
and 2 respectively. The infinite sheet (in which the hole of radius
R is cut) is assumed to be at temperature $T_1$. It is also assumed
that both pressure and density far downstream are very small: $p_2, \rho_2 \simeq 0$.

If we draw the velocity space at any point P ($\underset{\sim}{x}$), we can immedi-
ately distinguish a region C, which is the backward cone subtended at
P by the orifice. Molecules with velocity vectors lying in C can only
come from downstream infinity, where however $f = f_2 = 0$; hence
there are no molecules traveling upstream through the orifice, and
the cone C is 'vacant'. All the velocity vectors lying outside C can,
however, be produced either to the sheet or to upstream infinity, hence
the distribution outside C is just $f_1$. Assuming $f_1$ is a Maxwellian we
thus get

$$f = 0 \quad \text{in } C$$
$$= n_1 \left( \frac{\beta_1}{\pi} \right)^{3/2} e^{-\beta_1 v^2} \quad \text{everywhere else.} \tag{2-41}$$

f is similar downstream of the orifice too, except that the vacant
region is the major part of velocity space, and the backward cone is
now a 'full' Maxwellian. If the point P is in the plane of the orifice
the cone becomes the half-space, hence f, and all the mean quantities
associated with it, are constant all across the orifice.

The number density is relatively easy to calculate, being given
(in non-dimensional form) by

$$N(\underset{\sim}{x}) = \frac{n^0(\underset{\sim}{x})}{n_1} = \int_0^\infty \left( \frac{\beta_1}{\pi} \right)^{3/2} e^{-\beta_1 v^2} (4\pi - \Omega) v^2 dv$$
$$= 1 - \frac{\Omega(\underset{\sim}{x})}{4\pi}, \tag{2-42}$$

where $\Omega(x)$, the solid angle subtended by the orifice at $x$, is

$$\Omega = \int_0^{2\pi} z \, d\varphi \int_0^R \frac{\eta'^2 \, d\eta'}{(\eta^2 + z^2 - 2\eta\eta'\cos\varphi + \eta'^2)^{3/2}} \qquad (2\text{-}43)$$

using a cylindrical coordinate system (z, $\eta$, $\phi$). (The superscript

o is used in this section to denote free molecule flow.) The evaluation

of this integral is somewhat tedious but fairly straightforward (with the

assistance of Byrd & Friedman, Ref. 9), and leads to the expression

$$\frac{\Omega}{4\pi} = \frac{1}{4}\left\{ \Lambda_0(\alpha_1, \alpha_2) - \cos\alpha_1 \sin\alpha_2 \, F_0(\alpha_1) \right\} \qquad (2\text{-}44)$$

where $\Lambda_0$ is Heuman's lambda function and $(\pi/2) \, F_0(\alpha_1)$ is the

complete elliptic integral of the first kind, $K(\sin\alpha_1)$. Heuman

(Ref. 10) has discussed the properties of the function $\Lambda_0$, and has

also tabulated both $\Lambda_0$ and $F_0$. The arguments $\alpha_1$, $\alpha_2$ are given,

in terms of polar coordinates (r, $\theta$) from the center of the orifice, by

$$\sin\alpha_1 = \left(\frac{4r\sin\vartheta}{r^2 + 2r\sin\vartheta + 1}\right)^{1/2}, \qquad \tan\alpha_2 = \frac{r\cos\vartheta}{r\sin\vartheta - 1}.$$

(Distances have all been non-dimensionalized by dividing by $R$.) A

relation corresponding to equation (2-44) has been derived by Sadowsky

& Sternberg (Ref. 11) for the stream function of a source ring.

The density field calculated from (2-42) and (2-44) has been

plotted in figure 9 for the region upstream of the orifice. The field

downstream is obtained easily by making use of the symmetry

$$N(r, \pi - \vartheta) = 1 - N(r, \vartheta) \qquad (2\text{-}45)$$

which follows from the nature of $f^o$ discussed before. In the plane of the orifice $\Omega = 2\pi$, so N = 1/2. On the axis $\alpha_1 = 0$ and $\tan \alpha_2 = -r$, and (2-44) reduces to

$$\frac{\Omega}{4\pi} = \frac{1}{2}\left(1 - \sin \alpha_2\right);$$

hence

$$N = N(z) = \frac{1}{2}\frac{z + \sqrt{z^2 + 1}}{\sqrt{1 + z^2}} = \cos^2 \frac{\alpha}{2} \tag{2-46}$$

if $2\alpha$ is the included angle at the vertex of C.

The non-dimensional gas velocity $\underset{\sim}{U}$ can similarly be expressed as

$$\underset{\sim}{U}(\underset{\sim}{x}) = \sqrt{\beta_1}\; \underset{\sim}{u}(\underset{\sim}{x}) = -\frac{\sqrt{\beta_1}}{n^o(\underset{\sim}{x})} \int_C n_1 \left(\frac{\beta_1}{\pi}\right)^{3/2} e^{-\beta_1 v^2} \underset{\sim}{v}\; D\underset{\sim}{v} . \tag{2-47}$$

This integral unfortunately turns out to be too complicated to express analytically, and we must at present content ourselves with calculating it along $\theta = 0$ and $90°$. Along the axis we can show quite easily that

$$U(z) = -U_z(z) = \frac{1}{\sqrt{\pi}}\left(1 - \frac{z}{\sqrt{(1 + z^2)}}\right) = \frac{2}{\sqrt{\pi}} \sin^2 \frac{\alpha}{2}; \tag{2-48}$$

across the orifice U is constant and equal to $-U_z(0) = 1/\sqrt{\pi}$ (or $u = -u_z(0) = \frac{1}{2}\bar{c}_1$). However it is not difficult to guess the qualitative behavior of U for other values of $\theta$, using N as a guide.

It is interesting to note that we can immediately obtain, from the mean field, the free molecule mass flow through the orifice (per unit area) as

$$\dot{m}^0 = \rho(-u_z) = \tfrac{1}{2}\rho_1 \cdot \tfrac{1}{2}\bar{c}_1 = \tfrac{1}{4}\rho_1\bar{c}_1 \, ,$$

a well-known result derived usually by a different method (see e.g. Present, Ref. 12). A little thought shows that U, like N, is antisymmetric about the orifice and has the value

$$U(r, \pi - \vartheta) = \frac{2}{\sqrt{\pi}} - U(r, \vartheta) \, . \tag{2-49}$$

Thus, far downstream the gas velocity is finite and tends to $\bar{c}_1$, but the density of course drops off to zero. (Compare the results of section 2.3 on source flows.) It should be noticed that though the velocity field is antisymmetric about the plane $z = 0$ the stream line pattern will be perfectly symmetric as it depends only on the geometry of the cone C.

The temperature field is obtained from the relation

$$RT(x) = \tfrac{1}{3}\overline{c^2} = \tfrac{1}{3}(\overline{v^2} - u^2) \, .$$

But

$$\overline{v^2}(x) = \frac{1}{n^0(x)} \int_0^{\infty} n_1 \left(\frac{\beta_1}{\pi}\right)^{3/2} e^{-\beta_1 v^2} v^2 (4\pi - \Omega) \, v^2 dv,$$

$$= (\overline{v^2})_1 \, , \tag{2-50}$$

hence

$$\mathcal{R}\, T(\underline{x}) \;=\; \mathcal{R}\, T_1 \;-\; \tfrac{1}{3}\, u^2(\underline{x})$$

or

$$\frac{T(\underline{x})}{T_1} \;=\; \frac{\beta_1}{\beta(\underline{x})} \;=\; \frac{1}{B(\underline{x})} \;=\; 1 - \tfrac{2}{3}\, U^2(\underline{x}). \qquad\qquad (2\text{-}51)$$

Thus the temperature is very simply related to the velocity.* It drops slightly from $T_1$ to $(1 - 2/3\pi)\, T_1 = 0.7879\, T_1$ at the orifice and then sharply decreases to $T_2 = (1 - 8/3\pi)\, T_1 = 0.1512\, T_1$ far downstream. Along the axis, we get from (2-48)

$$B = B(z) = \left[\, 1 - \frac{2}{3\pi}\, \frac{1}{(z^2+1)\{ z + \sqrt{(z^2+1)}\}^2} \,\right]^{-1}$$

$$= \left( 1 - \frac{8}{3\pi}\, \sin^4 \frac{\alpha}{2} \right)^{-1}. \qquad\qquad (2\text{-}52)$$

It is also interesting to calculate the pressure tensor

$$p_{ij}(\underline{x}) \;=\; m \int c_i\, c_j\, f\, D\underline{v} \;=\; \varrho\, (\, \overline{v_i v_j} - u_i u_j \,).$$

On the axis one obtains

$$p_{zz} \;=\; p_1 \left[\, \frac{1 + \cos^3 \alpha}{2} - \frac{\sin^4 \alpha}{\pi (1 + \cos \alpha)} \,\right],$$

and it is easy to verify that this has no linear relation with the velocity

_____

*The inviscid energy equation is very similar to (2-51), and gives

$$\frac{T}{T_1} \;=\; 1 - \frac{\gamma - 1}{\gamma}\, U^2 \;=\; 1 - \frac{2}{5}\, U^2.$$

gradient whose relevant component is

$$\frac{\partial u_z}{\partial z} = \frac{1}{\sqrt{\pi}\beta} \sin^3 \alpha.$$

The relations between the stresses and velocity gradient in free molecule flow are of course not unique, and in fact vary with the geometry and boundary conditions of each problem.

What is even more interesting is that no such relation is shown by the above equations even at large distances from the orifice; for small $\alpha$ we get

$$p_{zz} \simeq p_1 \left(1 - \frac{3}{4}\alpha^2\right), \qquad \frac{\partial u_z}{\partial z} \simeq \frac{1}{\sqrt{\pi}\beta}\alpha^3.$$

On the other hand, at large distances (i.e., several mean free paths from the orifice) one expects the flow to be governed essentially by the Navier-Stokes equations again, collisions no longer being negligible. It is thus seen that even in the limit $R/\lambda_1 \to 0$ the free-molecule flow field discussed above cannot be valid at large distances from the orifice. We will return to this point briefly later.

For the sake of completeness, we collect below the values of the stress tensor on the axis:

$$p_{zz} = p_1 \left[ \frac{1+\cos^3\alpha}{2} - \frac{\sin^4\alpha}{\pi(1+\cos\alpha)} \right], \qquad = \left(\frac{1}{2} - \frac{1}{\pi}\right)p_1 \quad \text{at orifice;}$$

$$p_{xx} = p_{yy} = p_1 \left[ \frac{1}{2} + \frac{(3-\cos^2\alpha)\cos\alpha}{4} \right], \qquad = \frac{1}{2}p_1 \quad \text{at orifice;}$$

$$p = \frac{1}{3}p_{ii} = p_1 \left[ \frac{1+\cos\alpha}{2} - \frac{1}{3\pi}\frac{\sin^4\alpha}{(1+\cos\alpha)^2} \right], \qquad = \left(\frac{1}{2} - \frac{1}{3\pi}\right)p_1 \quad \text{at orifice.}$$

Finally, consider the entropy. In gasdynamics it is given by

$$S = c_v \ln p \varrho^{-\gamma}$$

so that, using the values obtained for p and ρ, we get for the ratio of the entropy at the orifice to the entropy at upstream infinity:

$$\frac{S_0}{S_1} = \ln \frac{p_0}{p_1} \left( \frac{\varrho_1}{\varrho_0} \right)^{5/3} \simeq 0.223,$$

- apparently a considerable decrease. But if one calculates Boltzmann's H-function, defined by

$$H = \int f \ln f \, D\underset{\sim}{v},$$

then for the f given by equation (2-41) (remembering that, from (2-50), $\overline{v^2} = \overline{(v^2)}_1$),

$$H = n \left\{ \ln n_1 + \frac{3}{2} \ln \left( \frac{\beta_1}{\pi} \right) - \frac{3}{2} \right\}.$$

The same quantity per unit mass is $\widetilde{H} = H/nm$, and this obviously is a constant throughout, as might have been expected from the absence of collisions.

The quantities N, B, U and $U_r$ (the radial component of U, which will turn out to be of interest in the next chapter) are plotted in figures 9 and 10. As an illustration of the kind of flow pattern to be expected in free molecule flow, the streamlines have been worked out for a slit and are shown in figure 11. The calculations for the slit are very similar to those for the circular orifice, but have the

advantage that they lead to closed expressions for all quantities; in particular

$$N = N(r, \vartheta) = 1 - \alpha/\pi ,$$

$$U = U(r, \vartheta) = \frac{\sin \alpha}{2\sqrt{\pi}(1 - \alpha/\pi)}$$

where $2\alpha = \tan^{-1} 2 r \cos \theta / (r^2 - 1)$ is again the included angle at P. The temperature is still given by (2-51). It can be easily shown that the vector $U$ bisects the angle $2\alpha$ at P, and this suggests an obvious geometric construction for drawing the streamlines. As remarked earlier the streamlines are symmetric about the plane of the orifice. The effect of the wall in slowing down the flow is quite impressive.

One might make a few remarks here on the general validity of the above picture when the flow takes place from one finite reservoir into another. Conditions upstream are obviously not going to be affected too much if the linear size of the reservoir is sufficiently large compared to the orifice diameter and the mean free path. Downstream of the orifice, however, the presence of a wall tends to bring the gas temperature back to $T_1$, while we saw above that the temperature $T_2$ in the case of an infinite reservoir is much less than $T_1$. Thus heat transfer at the walls becomes important, and the mean field will presumably be rather different from the one discussed here. The point has been discussed by Liepmann (Ref. 1), and we will return to it briefly later.

## III. NEARLY FREE MOLECULAR FLOW

### 3.1 Introduction

The purpose of this chapter is to take account of collisions in an approximate way by perturbing on the results obtained with the collisionless Boltzmann equation in the last chapter. Particular emphasis is laid on getting a numerical estimate for the departure from the free molecule limit in the flow through an orifice, for which measurements are available, in the hope that in this way one may get an idea of the usefulness of the perturbation method, the validity of the molecular model chosen, etc.

The method used is a scheme of iteration proposed by Willis (Ref. 13). However this method is first generalized to unsteady problems and derived from a different approach in section 3.2 below, so that it can be applied directly to all the flows discussed in the last chapter. The method is then utilized (in Sec. 3.3) to study the orifice problem which is discussed in some detail.

## 3.2 General iteration scheme

Let us consider an unsteady flow without any external forces. The distribution function is then governed by the equation

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} = \mathcal{G}(f) - f \mathcal{L}(f) + Q. \tag{3-1}$$

With the approach through characteristics that is adopted here the extension to the case when there are forces is fairly obvious, and will not be given here.

The basic idea of the iteration scheme is to approximate f by successive iterates $f^0$, $f^1$, etc., where each $f^n$ is obtained from the previous one from the equation

$$\frac{\partial f^n}{\partial t} + \underline{v} \cdot \frac{\partial f^n}{\partial \underline{x}} = \mathcal{G}(f^{n-1}) - f^n \mathcal{L}(f^{n-1}) + Q \tag{3-2}$$

with $f^0$ being the free molecule solution

$$\frac{\partial f^0}{\partial t} + \underline{v} \cdot \frac{\partial f^0}{\partial \underline{x}} = Q. \tag{3-3}$$

It is of course hoped that the first iterate $f^1$ will already be a good approximation to the departure from free molecule flow. As we will mostly be concerned only with the first iterate, we will study the equation

$$\frac{\partial f^1}{\partial t} + \underline{v} \cdot \frac{\partial f^1}{\partial \underline{x}} = \mathcal{G}(f^0) - f^1 \mathcal{L}(f^0) + Q. \tag{3-4}$$

It will be noticed that $\mathcal{G}(f^0)$ essentially acts as a source distribution, so that it can be lumped with Q. In the following Q will be omitted for the sake of brevity, as all one has to do to take account of it is to replace $\mathcal{G}$ by $\mathcal{G} + Q$.

Writing $\mathcal{G}(f^0) = \mathcal{G}^0$ and $\mathcal{L}(f^0) = \mathcal{L}^0$, equation (3-4) becomes

$$\frac{\partial f'}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f'}{\partial \underset{\sim}{x}} + f' \mathcal{L}^0 = \mathcal{G}^0 , \qquad (3\text{-}4\text{a})$$

- a quasi-linear, first order differential equation. The characteristics of (3-4a) are given by

$$\frac{dt}{ds} = 1, \qquad \frac{d\underset{\sim}{x}}{ds} = \underset{\sim}{v}, \qquad \frac{df'}{ds} + f' \mathcal{L}^0 = \mathcal{G}^0;$$

using the initial condition

$$f(\underset{\sim}{x}, t=0; \underset{\sim}{v}) = f_0(\underset{\sim}{x}; \underset{\sim}{v}) \qquad (3\text{-}5)$$

their solution can be written

$$t = s, \qquad \underset{\sim}{x} = \underset{\sim}{v} s + \underset{\sim}{\xi},$$

$$f' = \exp\left\{-\int_0^s \mathcal{L}^0 ds'\right\} \cdot \left[\int_0^s \mathcal{G}^0 \left\{\exp\int_0^{s'} \mathcal{L}^0 ds''\right\} ds' + f^0\right].$$

Eliminating s and $\underset{\sim}{\xi}$ as was done with equation (2-4) and putting $\underset{\sim}{x}' = \underset{\sim}{x} - \underset{\sim}{v}t$, we get

$$f'(\underset{\sim}{x}, t; \underset{\sim}{v}) = f_0(\underset{\sim}{x}'; \underset{\sim}{v}) \exp - \int_0^t \mathcal{L}^0(\underset{\sim}{x}' + \underset{\sim}{v}t', t'; \underset{\sim}{v}) dt'$$

$$+ \int_0^t \mathcal{G}^0(\underset{\sim}{x}' + \underset{\sim}{v}t', t'; \underset{\sim}{v}) \cdot \qquad (3\text{-}6)$$

$$\times\left\{\exp - \int_{t'}^t \mathcal{L}^0(\underset{\sim}{x}' + \underset{\sim}{v}t'', t''; \underset{\sim}{v}) dt''\right\} dt'$$

where

$$\mathcal{L}^{\circ}(\underset{\sim}{x}, t; \underset{\sim}{v}) \equiv \mathcal{L}\{f^{\circ}(\underset{\sim}{x}, t; \underset{\sim}{v})\}$$

etc. Note that $f_o(\underset{\sim}{x}'; \underset{\sim}{v})$ is just the free-molecule solution $f^o(\underset{\sim}{x}, \underset{\sim}{v})$.

To proceed further from (3-6) one has to postulate some molecular model. The simplest to use is perhaps the model for scattering proposed by Bhatnagar, Gross & Krook (Ref. 14), in which $\mathcal{L}(f)$ and $\mathcal{G}(f)$ are assumed to have the form

$$\mathcal{L}(f) = A n,$$

$$\mathcal{G}(f) = A n F = A n^2 \left(\frac{\beta}{\pi}\right)^{3/2} \exp\{-\beta(\underset{\sim}{v}-\underset{\sim}{u})^2\} \tag{3-7}$$

where n, $\underset{\sim}{u}$ and $\beta$ are the corresponding moments of f. A discussion of this model is reserved for the next chapter, but it may be remarked here that the model obeys the conservation laws (of mass, energy and momentum) at each point and instant, but not for each collision. The gain function is a local Maxwellian normalized to the local collision density, while the loss function is assumed to be independent of molecular velocity. The value to be given to A will also be discussed later; it can in general be a function of the local thermodynamic state. In nearly free molecule flow it would seem best to choose A such that the number of collisions (which is the most significant quantity) is given as accurately as possible. We know that if the distribution is a complete Maxwellian ( - a standard state which will be denoted by the subscript 1), the number of collisions per unit volume and unit

time is $n_1 \bar{c}_1 / 2\lambda_1$, where $\lambda_1$ is the mean free path; equating this with the number given by Krook's model, which is $\frac{1}{2}A_1 n_1^2$, one gets

$$A_1 = \frac{\bar{c}_1}{n_1 \lambda_1} . \tag{3-7a}$$

As $n_1 \lambda_1$ is inversely proportional to the scattering cross-section and tends to be a constant for a given kind of molecule, it is also seen that a fairly general form for A would be

$$A = A(x, t) = \frac{\bar{c}(x,t)}{n_1 \lambda_1} . \tag{3-7b}$$

It is however often convenient to assume A = const. = $A_1$, at least when temperature variations in the flow are small.

The choice of A is by no means a trivial matter, as is easily shown by an application of (3-6) to the flow due to the point cloud (discussed in Sec. 2.2.1). The integrals in (3-6) can be evaluated in this case, and one gets

$$f'(x,t;v) = f_0(x';v) \exp\left[-AN\left(\frac{\beta}{\pi}\right)^{3/2}\frac{1}{2t^2}\left\{\frac{1}{v}\sqrt{\frac{\pi}{\beta}} + \frac{1}{\beta v^2}\right\}\right] + \cdots$$

But, as was pointed out earlier, no collisions can possibly occur in this flow at t > 0 if they do not at t = 0; thus we expect

$$f'(x,t;v) = f_0(x';v)$$

which is possible only if A = 0. If A is taken to be given by (3-7b)

this is obvious, as the temperature (and so $\bar{c}$) is zero in this particular case. But if, as is ordinarily done, A had been assumed to be a constant number, a meaningless result would have been obtained.

This example suggests that in problems in which there are wide variations in flow quantities one should perhaps iterate also on the value of A as a series $A^o$, $A^1$, ... to obtain reasonable approximations.

## 4.3 Nearly free molecular flow through an orifice

In this section we study specifically the flow through an orifice at small values of an inverse Knudsen number $\epsilon$, defined as the ratio of the radius R of the orifice to the mean free path $\lambda_1$ in the gas far upstream. The free molecule (or gaskinetic) limit corresponds to $\epsilon \rightarrow 0$, and the mass flow rate $\dot{m}$ in this limit has already been shown (Sec. 2.4) to be

$$\dot{m}^0 = \frac{1}{4} \rho_1 \bar{c}_1 . \tag{3-8}$$

In the following we will primarily be interested in the mass flow (at non-zero $\epsilon$), as this is the quantity on which measurements are available, though all other flow quantities can also be calculated from the results.

The basic physical mechanisms operating to change $\dot{m}$ from its free molecule value are the following. First there is a decrease in the density near the orifice (as we have seen earlier), which results in fewer collisions, and so tends to decrease the mass flow. But on the other hand the decrease in $\rho$ causes an increase in the mean free path, so the probability of a molecule reaching the orifice is higher. Finally, due to the persistence of velocities, molecules coming out of a collision retain a part of the local gas velocity, which again tends to increase the mass flow. There is no obvious reason for neglecting any of these mechanisms in comparision with the others; in fact the calculation below, which takes account of all these factors, shows them to be of roughly equal importance.

For steady flow, equation (3-4a) for the first iterate $f^1$ can be written

$$\underset{\sim}{v} \cdot \frac{\partial f^1}{\partial \underset{\sim}{x}} + f^1 \mathcal{L}^0 = g^0 \, ,$$

whose characteristics are

$$\frac{d\underset{\sim}{x}}{ds} = \underset{\sim}{v} \, , \qquad \frac{df^1}{ds} + f^1 \mathcal{L}^0 = g^0$$

with the solution

$$\underset{\sim}{x} = \underset{\sim}{v} s + \underset{\sim}{\xi} \, ,$$

$$f^1 = \exp \left\{ -\int^s \mathcal{L}^0 ds' \right\} \cdot \left[ \int^s g^0 \left\{ \exp \int^{s'} \mathcal{L}^0 ds'' \right\} ds' + const. \right] .$$

(3-9)

First consider only the molecules traveling towards the orifice from upstream infinity. The integral above will look simpler in terms of

$$\zeta = \frac{|\underset{\sim}{x} - \underset{\sim}{\xi}|}{-v} \, ,$$

which is now just the distance measured from the point $\underset{\sim}{\xi}$ towards $+\infty$; also the boundary condition can be written

$$f \longrightarrow f_1 = n_1 \left( \frac{\beta_1}{\pi} \right)^{3/2} e^{-\beta_1 v^2} \quad as \quad \zeta \longrightarrow \infty,$$

and this determines the constant in (3-9). One thus gets

$$f^1(\underset{\sim}{x} ; \underset{\sim}{v}) = f_1 \exp - \int_\zeta^\infty \frac{\mathcal{L}^0}{v} d\zeta'$$
$$+ \int_\zeta^\infty \frac{g^0}{v} \left\{ \exp - \int_\zeta^{\zeta'} \frac{\mathcal{L}^0}{v} d\zeta'' \right\} d\zeta'.$$

(3-9a)

The quantity $\mathcal{L}(f)/v$ is essentially the reciprocal of the mean free path, hence the first term above vanishes for any finite $\zeta$, leaving

us finally with

$$f'(\xi; \underset{\sim}{v}) = \int_0^\infty \frac{g(f^0)}{v} \left\{ \exp - \int_0^{\zeta'} \frac{L(f^0)}{v} \, d\zeta'' \right\} d\zeta'. \tag{3-10}$$

An exactly similar integral can be written for molecules traveling upstream, using the appropriate boundary condition (at downstream infinity) - their number is no longer zero when collisions are allowed though it will turn out to be quite small (see p.58).

Equation (3-10) has an immediate physical interpretation, for $g(f^0)/v$ is just the number of molecules which, after collision in unit distance, are traveling with velocity $v$, and the exponential multiplying it is the probability of such molecules reaching the orifice (or $\zeta \neq 0$). The first term in (3-9a) represents the number of molecules starting from upstream infinity which reach the point $\zeta$, and this of course is zero, as the probability that they will not suffer a collision in an infinite distance is zero.

Using Krook's model, as given by (3-7) and (3-7a)*, equation (3-10) becomes

$$f'(\xi; \underset{\sim}{v}) = f_1 \cdot 2\epsilon' \int_0^\infty N^2 B^{3/2} \exp \left\{ V^2 - B(\underset{\sim}{V} - \underset{\sim}{U})^2 \right\} \cdot$$
$$\times \exp \left\{ -\frac{\epsilon'}{V} \int_0^{\xi'} 2N(\xi') \, d\xi' \right\} \cdot \frac{d\xi'}{V} \tag{3-11}$$

---

*The assumption that $A$ is a constant equal to $\overline{c}_1 / n_1 \lambda_1$ is justified in the present case as $\overline{c}$ varies by only about 10 per cent upstream of the orifice - and, as will be shown later, the upstream side accounts for most of the correction to free molecule flow.

where we have used the non-dimensional quantities

$$V = \underset{\sim}{v}\sqrt{\beta_1}, \qquad U = \underset{\sim}{u}\sqrt{\beta_1}, \qquad N = n^0/n_1, \qquad B = \beta^0/\beta_1,$$

$$\underset{\sim}{\xi} = \zeta/R, \qquad \epsilon' = \epsilon/\sqrt{\pi} = R/\lambda_1\sqrt{\pi}.$$

(The superscript 0 always denotes free-molecule conditions.) N, B and U are of course the same quantities that were computed in section 2.4, and so the integral in (3-11) can be evaluated, in principle at least, for each velocity vector $\underset{\sim}{V}$ at each point $\underset{\sim}{\xi}$ in the plane of the orifice.

An explicit and analytical calculation of (3-11) is, however, practically impossible. Fortunately, though, one can still get a good estimate of the result by making a few simple approximations. First, it will be assumed that the flow is uniform across the orifice; this is strictly true in free molecule flow (as was shown earlier), and should be a good approximation in nearly free molecule flow too. This means that we evaluate (3-10) only at the center of the orifice, and assume it is the same all across it; and $\zeta$ can therefore be replaced by r, the radial distance measured from the center of the orifice.

Next we observe, from figures 9 and 10 and from the discussion in section 2.4, that the values of N, B and U - but not of $U_r$, which appears in the term $(\underset{\sim}{V} - \underset{\sim}{U})^2$ in (3-11) - are independent of $\vartheta$ at r = 0 and $\infty$, though not in-between. It is obvious, therefore, that the integral in (3-11) is similar for all rays, though its actual numerical value will vary somewhat. This fact leads to considerable

simplification in what follows.

It is convenient for purposes of analysis to rewrite (3-11) as

$$f'(r=0; \underset{\sim}{V}) = f_1 \cdot 2\epsilon' \int_0^\infty g_1(r') \, g_2(r',V) \, H(r,V; \epsilon') \frac{dr}{V} \qquad (3-12)$$

where

$$g_1(r) = N^2 B^{3/2} \, e^{-BU^2},$$

$$g_2(r,V) = \exp\left\{-2BU_r V + (1-B)V^2\right\}$$

$$= \exp\left\{h_1(r)V + h_2(r)V^2\right\}, \qquad (3-12a)$$

$$H(r,V; \epsilon') = \exp - \frac{\epsilon'}{V} \int_0^r 2N(r')dr'$$

$$= \exp - \frac{\epsilon'}{V} h(r).$$

It is particularly instructive to work out the integral in (3-11) along the axis, $\theta = 0$, where N, B, U and $U_r = -U$ are all analytically known from equations (2-46), (2-48) and (2-52); also

$$h(z) = \int_0^z 2N(z)\,dz = z + \sqrt{(z^2+1)} - 1. \qquad (3-13)$$

We have therefore

$$f'(-V\underset{\sim}{\hat{z}}, z=0) = f_1 \cdot 2\epsilon' \int_0^\infty g_1(z) \, g_2(z,V) \, H(z,V; \epsilon') \frac{dz}{V} \qquad (3-14)$$

where $\hat{z}$ is the unit vector along the z-axis; and we want the lowest

order term in $\epsilon'$ from this expression. Now it is shown in

Appendix 3B that the integral for the mass flow (or, for that matter,

for any moment of $f^1$) can be split in a way which corresponds exactly

to splitting the distribution function in (3-14) as follows:

$$ f'(z=0; -V\hat{z}) = f_1 \, 2\epsilon' \left[ \int_0^\infty (g_1 g_2 - 1) \frac{dz}{V} + \int_0^\infty H \frac{dz}{V} \right]. \qquad (3-15) $$

The error committed in this splitting is $o(\epsilon')$; the lowest order term

in $\epsilon'$ turns out to be $O(\epsilon')$ and (3-15) reduces, correct to $O(\epsilon')$, to

$$ f'(z=0; -V\hat{z}) = f_1 \left[ 1 + \frac{2\epsilon'}{V} \left\{ \frac{1}{2} + \int_0^\infty (g_1 g_2 - 1) \, dz \right\} \right] \qquad (3-16) $$

using the integral evaluated in Appendix 3A. The function $g_2$ can be

expanded as a series in V,

$$ g_2 = 1 + V h_1 + V^2 \left( \frac{h_1^2}{2!} + h_2 \right) + V^3 \left( \frac{h_1^3}{3!} + h_1 h_2 \right) $$
$$ + V^4 \left( \frac{h_1^4}{4!} + \frac{h_1^2 h_2}{2} + \frac{h_2^2}{2} \right) + \cdots, $$

and putting this in (3-16) we can write

$$ f'(z=0; -V\hat{z}) = f_1 \left\{ 1 + \frac{2\epsilon'}{V} \phi_0(V) \right\} \qquad (3-17) $$

where

$$\phi_0(V) = \sum_{n=0}^{\infty} A_n V^n ,$$

$$A_0 = \frac{1}{2} + \int_0^{\infty} (g_1 - 1) \, dz ,$$

$$A_1 = \int_0^{\infty} g_1 h_1 \, dz , \tag{3-18}$$

$$A_2 = \int_0^{\infty} \left( \frac{g_1 h_1^2}{2!} + g_1 h_2 \right) dz , \ldots$$

From arguments given previously the distribution function should look similar on other rays (i.e., for other $\underline{V}$), so we may write

$$f'(r=0; \underline{V}) = f_1 \left\{ 1 + \frac{2\epsilon'}{V} \phi(\underline{V}) \right\} . \tag{3-19}$$

Now, inferring the variation of the integrand in (3-12) with $\theta$ from figures 9 and 10 and recalling that N, B and U vary between the same limits at r = 0 and r = $\infty$ for all $\theta$, we conclude that the effect on the <u>integral</u> of their dependence on $\theta$ must be small. (For instance $\int_0^{\infty} (1 - N(r)) \, dr$ is exactly the same for $\theta$ = 0 and $\theta$ = 90°.) However the variation of $U_r$ (and hence of $h_1$) with $\theta$ cannot be neglected. Near the origin we obviously have $U_r = U_z(z = r) \cos \theta$ as $|U_z| \simeq U$. For r $\gg$ 1 it can be easily shown that

$$-U_r \simeq U \simeq \frac{2}{\sqrt{\pi}} \frac{\Omega}{4\pi} ;$$

but $\Omega \sim \pi \cos \theta / r^2$, so again $U_r \simeq U_z (z = r) . \cos \theta$. Thus, it is

a very good approximation to take

$$U_r(r, \vartheta) \simeq U_z(z = r) \cdot \cos \vartheta. \qquad (3\text{-}20)$$

Finally, it will be noticed that $h_2$ accounts for a relatively small contribution to the function $\phi_o$ in comparison with $h_1$, as it varies from about -0.27 to zero as $r$ goes from 0 to $\infty$, and appears only in $V^2$ and higher order terms.

Putting these inferences together, we conclude that, approximately, the coefficients of $V^n$ in (3-18) are multiplied by corresponding powers of $\cos \theta$ on rays other than the axis, or that

$$\phi(\underline{V}) \simeq \phi_o(V \cos \vartheta) \qquad (3\text{-}21)$$

which also amounts to putting

$$g_2(r, V) \simeq g_2(z, V \cos \vartheta). \qquad (3\text{-}21a)$$

Using this in (3-19) the final result becomes

$$f'(r = 0; \underline{V}) = f_1 \left\{ 1 + \frac{2\epsilon'}{V} \sum_{n=0}^{\infty} A_n V^n \cos^n \vartheta \right\}. \qquad (3\text{-}22)$$

Results:

The coefficients $A_n$ have been obtained by numerical integration for n = 0 to 5, and have the following values:

$$A_o = -0.3944, \qquad A_1 = +0.7549, \qquad A_2 = +0.1055, \qquad (3-23)$$

$$A_3 = +0.0502, \qquad A_4 = -0.0045, \qquad A_5 = -0.0013.$$

The mass flow through the orifice is easily obtained as

$$\dot{m} = m \int (-v_z) f(\underline{v}) D\underline{v}. \qquad (3-24)$$

Substituting from (3-22)

$$\dot{m} = \frac{m}{\beta_1^2} \int_0^\infty \int_0^{\pi/2} f_1 \left\{ V\cos\vartheta + 2\epsilon'\cos\vartheta \cdot \sum_{n=0}^\infty A_n V^n \cos^n\vartheta \right\}$$

$$2\pi V^2 \sin\vartheta \, d\vartheta \, dV \qquad (3-25)$$

$$= \frac{1}{4} \varrho_1 \bar{c}_1 \left[ 1 + 4\epsilon' \sum_{n=0}^\infty \frac{A_n}{n+2} \left( \frac{n+1}{2} \right)! \right].$$

Using the values of $A_n$ given by (3-23) we get*

$$\dot{m} \simeq \frac{1}{4} \varrho_1 \bar{c}_1 (1 + 0.25\epsilon). \qquad (3-26)$$

The coefficient 0.25 obtained here is somewhat less than the value 0.26 given in a preliminary report (Ref. 15) where the calculations were based on the assumption

$$\Phi(\underline{V}) \equiv \frac{1}{V} \phi(\underline{V}) = \Phi_o(V\cos\vartheta),$$

---

*I have to thank Dr. Willis for pointing out a numerical mistake at this point.

which is slightly different from the one used here in (3-21); also $h_2$ was ignored. The approximation (3-21) is somewhat more consistent, but apparently the result obtained is not significantly different. No claim of great accuracy is made for the coefficient; nor indeed would it be justified, considering the approximate nature of the calculations.

Correction for backflow:

The backflow through the orifice is nil under free molecule conditions but one may in general expect some backflow when $\epsilon$ is finite, even though the pressure ratio $p_1/p_2$ is still infinite. Using exactly the same procedure as before, one can write down the distribution function for molecules traveling upstream as

$$f'(z=0; V\hat{z}) = f_1 2\epsilon' \int_0^{-\infty} g_1(z)\, g_2(z,V)\, H(z,V; \epsilon')\, \frac{dz}{V},$$

where $g_1$, $g_2$ and $H$ are still given by (3-12) but of course one now has to use the downstream field of N, B and U, obtained as shown in section 2.4. Both $h_1$ and $h_2$ are then negative, and $h(z) \rightarrow 1$ as $z \rightarrow -\infty$. By repeating the analysis made for the upstream integral one concludes that the highest order term in $f^1(z=0; V\hat{z})$ is $O(\epsilon')$, and is given by

$$f'(z=0; V\hat{z}) = f_1 2\epsilon' \int_0^{-\infty} g_1(z)\, g_2(z,V)\, \frac{dz}{V}.$$

The approximation (3-20) for the radial component $U_r$ is now rather crude, but conservative in the sense that it overestimates the backflow. The correction to the coefficient in (3-26) can again be expressed as

$$a' = 4 \sum_{n=0}^{\infty} \frac{A'_n}{n+2} \left(\frac{n+1}{2}\right)!$$

where

$$A'_0 = \int_0^{-\infty} g_1\, dz, \qquad A'_1 = \int_0^{-\infty} g_1 h_1\, dz,$$

$$A'_2 = \int_0^{-\infty} \left(\frac{g_1 h_1^2}{2!} + g_1 h_2\right) dz, \ \dots$$

The coefficients alternate in sign, and using the Euler-Maclaurin technique one arrives at an upper bound of -0.03 for $a'$, i.e., the coefficient in (3-26) would be reduced at most to 0.22 on account of backflow.

Comparison with experiment:

Figure 12 shows the experimental data of Liepmann, and also the result (3-26) which has been plotted assuming that the viscosity $\mu_1 = \frac{1}{2} \rho_1 \bar{c}_1 \lambda_1$. The backflow correction has not been included for two reasons: first, it is small and would not make much difference; secondly, its exact value is rather uncertain in the case of flow into a finite reservoir, where the flow field is affected somewhat by the presence of walls as discussed in section 2.4.

Figure 12 shows reasonable agreement between theory and experiment (notice that a small backflow correction would make the agreement even better); and it seems safe to conclude that: (i) the prediction of the theory that the departure from free molecule flow is linear in $\epsilon$ with a coefficient of the order of 0.25 is borne out by the experiments; and (ii) 'nearly' free molecule conditions prevail up to $\epsilon \sim 1.0$. The satisfactory nature of these conclusions also gives one some confidence in Krook's model for collisions, and a brief study of it is made in the next chapter.

Appendix 3A

We evaluate here an integral which we will need, namely

$$\chi(a) = a \int_0^\infty \exp\{-a(z + \sqrt{(1+z^2)})\}\, dz, \; = \int_0^\infty \exp\{-(\zeta + \sqrt{a^2 + \zeta^2})\}\, d\zeta.$$

Transform to $\xi = \zeta + \sqrt{(\zeta^2 + a^2)}$, $d\zeta/d\xi = (\xi^2 + a^2)/2\xi^2$ ; then

$$\chi(a) = \int_a^\infty \frac{\xi^2 + a^2}{2\xi^2}\, e^{-\xi}\, d\xi.$$

Integrating successively by parts, this can be reduced to

$$\chi(a) = \tfrac{1}{2}\left\{(1+a)\, e^{-a} - a^2\, Ei(a)\right\}$$

where Ei(a) is the exponential integral with the following expansion for small a (Bromwich, Ref. 16):

$$Ei(a) = -0.5772\ldots - \ln a + a + \ldots$$

Thus $\chi$ for small a is

$$\chi(a) = \tfrac{1}{2} + O(a^2 \ln a) = \chi(0) + o(a). \qquad (3-27)$$

Appendix 3B

Using the approximation (3-21a) in equation (3-11) for the distribution function before it is split, one gets

$$f'(r=0; \underline{V}) = f_1 2\epsilon' \int_0^\infty g_1(z)\, g_2(z, V\cos\vartheta)\, H(z, V; \epsilon')\frac{dz}{V}\,. \qquad (3-28)$$

Hence, from (3-24), the mass flow can be expressed as the integral

$$\dot{m} = 2\epsilon'\frac{g_1}{\beta_1^2}\left(\frac{\beta_1}{\pi}\right)^{3/2}\iiint e^{-V^2} g_1(z)\, g_2(z, V\cos\vartheta)\, H(z, V; \epsilon')$$

$$2\pi V^2 \cos\vartheta \sin\vartheta\, d\vartheta\, dz\, dV$$

$$= 2\varrho_1 \bar{c}_1 \epsilon'\int_0^\infty \int_0^\infty \int_0^{\pi/2} V^2 e^{-V^2} g_1(z)\, g_2(z, V\cos\vartheta)\, H(z, V; \epsilon') \qquad (3-29)$$

$$\cos\vartheta \sin\vartheta\, d\vartheta\, dz\, dV.$$

Consider first the integrations with respect to z and V. Putting $g_1(z)\, g_2(z, V\cos\theta) = G(z, V\cos\theta)$, we write

$$J(\epsilon') = \epsilon'\iint_0 V^2 e^{-V^2}(G-1)H\, dz\, dV + \epsilon'\iint V^2 e^{-V^2}H\, dz\, dV$$

$$= J_1(\epsilon') + J_2(\epsilon'),$$

the reason for the splitting being that (G - 1) is integrable in z. We want to extract from J ( $\epsilon'$) the lowest order terms in $\epsilon'$.

First writing $H = \exp - (\epsilon'/V) h(z)$ where $h(z) = z - 1 + (z^2 + 1)^{1/2}$, > 0 for all z > 0,

$$J_1(\epsilon') = \epsilon'\int_{V=0}^{\delta(\epsilon')}\int_{z=0}^\infty V^2 e^{-V^2}(G-1)\left\{\exp-\frac{\epsilon'}{V}h(z)\right\}dz\, dV$$

$$+ \epsilon'\int_{V=\delta}^\infty \int_{z=0}^\infty V^2 e^{-V^2}(G-1)\left\{\exp-\frac{\epsilon'}{V}h(z)\right\}dz\, dV.$$

Replacing (G - 1) by an upper bound and using (3-27) we see that the first term on the right is of order $\epsilon'(\delta^4/\epsilon')$ . In the second term the coefficient of h(z) is $\epsilon'/V, \leq \epsilon'/\delta$. So if we can choose $\delta(\epsilon')$ such that $\delta^4/\epsilon' \to 0$ and $\epsilon'/\delta \to 0$ as $\epsilon' \to 0$, which is possible with $\delta = \epsilon'^{m}$, $\frac{1}{4} < m < 1$, we can write

$$J_1(\epsilon') = \epsilon' \iint\limits_{0}^{\infty} V^2 e^{-V^2}(G-1)\, dz\, dV + o(\epsilon').$$

Similarly

$$J_2(\epsilon') = \epsilon' \Bigg[ \int\limits_{V=0}^{\delta} \int\limits_{z=0}^{\infty} V^2 e^{-V^2} \left\{ \exp - \frac{\epsilon'}{V} h(z) \right\} dz\, dV$$

$$+ \int\limits_{V=\delta}^{\infty} \int\limits_{z=0}^{\infty} V^2 e^{-V^2} e^{\epsilon'/V} \left\{ \exp - \frac{\epsilon'}{V} (z + \sqrt{z^2+1}) \right\} dz\, dV \Bigg]$$

$$= \epsilon' \, O\left(\frac{\delta^4}{\epsilon'}\right) + \epsilon' \int\limits_{\delta}^{\infty} V^2 e^{-V^2} \left(1 + \frac{\epsilon'}{V} + \cdots \right) \cdot$$

$$\times \int\limits_{z=0}^{\infty} \left\{ \exp - \frac{\epsilon'}{V} (z + \sqrt{z^2+1}) \right\} \frac{dz}{V} \cdot V dV,$$

and using the result (3-27) once more

$$J_2(\epsilon') = O(\delta^4) + \int\limits_{\delta}^{\infty} V^3 e^{-V^2} \left(1 + \frac{\epsilon'}{V} + \cdots \right) \cdot$$

$$\left\{ \frac{1}{2} + O\left(\frac{\epsilon'^2}{V^2} \ln \frac{\epsilon'}{V}\right) \right\} dV.$$

It is again clear that we can choose $\delta$ such that both $\delta^4$ and $\epsilon'^2/\delta^2$ are o ($\epsilon'$), by taking $\delta = \epsilon'^{m}$, $\frac{1}{4} < m < \frac{1}{2}$. Thus

$$J_2(\epsilon') = \frac{1}{4} + \frac{\sqrt{\pi}}{8} \epsilon' + o(\epsilon').$$

Putting these results in (3-29)

$$\dot{m} = 2 \rho_1 \overline{c}_1 \left[ \int_0^{\pi/2} \left\{ \frac{1}{4} + \frac{\sqrt{\pi}}{8} \epsilon' + \epsilon' \iint_0^{\infty} V^2 e^{-V^2}(G-1) \, dz \, dV \right\} \cdot \right.$$
$$\left. \cdot \cos \vartheta \sin \vartheta \, d\vartheta \right]$$
$$= \frac{1}{4} \rho_1 \overline{c}_1 \left[ 1 + \epsilon \left\{ \frac{1}{2} + \frac{8}{\sqrt{\pi}} \iint_0^{\infty} \int_0^{\pi/2} V^2 e^{-V^2}(G-1) \cdot \right. \right.$$
$$\left. \left. \cdot \cos \vartheta \sin \vartheta \, d\vartheta \, dz \, dV \right\} \right],$$

which, after expanding $g_2 = \exp(h_1 V + h_2 V^2)$ as before, leads to exactly the same result as (3-25), thus justifying the splitting of the integral used in equation (3-15).

A similar analysis will show that (3-15) also gives the correct result for any higher moment. The backflow integral can also be handled along the same general lines.

## IV. A STUDY OF KROOK'S COLLISION MODEL

### 4.1 Introduction

The collision integrals $\mathcal{G}(f)$ and $\mathcal{L}(f)$ in the Boltzmann equation can, in principle, be evaluated for any given  f  provided the inter-molecular force field is known, though in actual practice the evaluation is often extremely difficult.  (Not to mention the basic problem that  f  is <u>not</u> known before-hand!)  The force field itself is in turn too complicated to be derived rigorously from other 'first principles', so one is compelled to postulate various models for it, of the kind discussed by Chapman  &  Cowling, for example.

The basic idea behind Krook's treatment of the collision integral is that a prescription of the force field is much too fine for many purposes, and that it is often convenient to postulate (at a grosser level) a statistical model for the whole scattering process. Krook (Ref. 17) has recently made various proposals for such models, without, however, applying or discussing them in any detail.  Among the simplest of these is the one we have used in the last chapter, namely

$$\mathcal{L}(f) = A n,$$
$$\mathcal{G}(f) = A n F = A n^2 \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta(\underset{\sim}{v}-\underset{\sim}{u})^2} \tag{4-1}$$

where n, $\underset{\sim}{u}$ and $\beta$ are corresponding moments of f:

$$n = \int f D\underset{\sim}{v}, \qquad n\underset{\sim}{u} = \int \underset{\sim}{v} f D\underset{\sim}{v},$$
$$\beta^{-1} = \frac{2}{3n} \int c^2 f D\underset{\sim}{v}. \tag{4-1a}$$

This is the model we will be mostly concerned with in this chapter,

with A above taken not to depend on the velocity $\underset{\sim}{v}$. An even simpler version of (4-1), which has been used several times, assumes An is a constant $\bar{\nu}$, not varying in any way even over the flow field:

$$\underset{t}{\mathcal{G}}(f) - f \mathcal{L}(f) = \bar{\nu}\,(F-f). \tag{4-1b}$$

As this amounts to assuming a constant collision frequency (for a given flow), (4-1b) may properly be called the 'single relaxation time' model. If the flow quantities vary too widely over the field, this model would probably be a little oversimplified.

What is required of a collision model will naturally depend on the particular problem being studied, but anything claiming general validity will have to satisfy the following conditions: (i) Mass, momentum and energy should be conserved at the level of the model; i.e., the model should possess the same five invariants as the exact collision integral. As individual encounters between molecules are not considered at all, the conservations laws will not necessarily be satisfied in detail. But the equations of motion 'in the bulk' will not be affected, i.e., the same macroscopic equations will be obtained as from the complete Boltzmann equation. (ii) The distribution function should tend to a Maxwellian in equilibrium. (iii) The results derived from the model for a Maxwellian distribution should agree, as far as possible, with the exact results known for that case.

The purpose of the rest of this chapter is to see how far the

model (4-1) satisfies these requirements. Somewhat surprisingly, the physical basis of Krook's model does not seem to have been discussed till now*; nor, apparently, have some simple consequences of it been deduced. In section 4.2 one particular approach to the model is presented; in section 4.3 the Navier-Stokes limit is discussed. Of course a 'model' should not be taken too literally, but its implications should be worked out, if only to understand its short-comings.

*After this account had been written up, the work of Kogan (Prikl. Mat. Mekh. 22:597, 1958) was brought to the author's attention by Dr. R. F. Probstein. This paper gives the same interpretation for the loss term as given here, and also derives the viscosity and conductivity of the gas(but assuming A to be a constant number). However, the rest of the present discussion is different, and the whole study here is also felt to be more complete.

## 4.2 A physical interpretation

One way of understanding the basis of a model like (4-1) is the following. Consider first the loss term

$$\mathcal{L}(f) = \int f(\underset{\sim}{w})\, g\, I\, d\Omega\, D\underset{\sim}{w}.$$

It is well known (e.g. Ref. 2, p. 173) that for a Maxwellian molecule (force $\propto r^{-5}$) the combination gI in the integral is independent of the velocity; so $\mathcal{L}(f)$ is just proportional to $\int f D\underset{\sim}{w}$, i.e., to the local number density n. So we may put

$$\mathcal{L}(f) = A\,n, \qquad\qquad\qquad (4\text{-}2)$$

where A is a number independent of the molecular velocity (it might obviously depend on any mean quantity).

The rate at which molecules are 'lost' at $\underset{\sim}{v}$ per unit $\underset{\sim}{xv}$ volume is $f\mathcal{L}(f)$, hence the total number of molecules involved in collisions, per unit volume and unit time, is

$$\nu = \int f\,\mathcal{L}(f)\, D\underset{\sim}{v} = A\,n\int f\, D\underset{\sim}{v} = A\,n^2. \qquad\qquad (4\text{-}3)$$

This is the justification for the interpretation of A given in equation (3-7a) where it was put equal to $\overline{c}_1 / n_1 \lambda_1$ after comparing $\nu = A n^2$ with its value when f is a complete Maxwellian. It is thus seen that A is proportional to the scattering cross-section of the molecule, times $\overline{c}_1$, i.e., the constant A in Krook's model is a measure of

the volume swept out by a molecule per unit time, as observed relative to the local gas velocity.

The gain term is actually the more difficult to handle. As the total 'gain' must equal the total 'loss' (to conserve the number of molecules), we must have

$$\int \mathcal{G}(f)\, D\underset{\sim}{v} = \nu = An^2 \tag{4-3a}$$

too. Now consider rigid elastic spheres colliding among themselves. It can be easily shown that for two colliding spheres, the scattering is spherically symmetric in their center-of-mass system (see e.g. Present, Ref. 12, p. 143). If the result can be extended to any number of spheres, the scattering would be spherically symmetric in a system moving with the velocity of the CM of all the spheres, which is just the local gas velocity. Thus one may write

$$\mathcal{G}(f) = An\, \phi(c) \tag{4-4}$$

where $\phi$ is some function of the peculiar speed $c, = |\underset{\sim}{v} - \underset{\sim}{u}|$, such that $\int \phi D\underset{\sim}{v} = n$ (to satisfy (4-3a)). In general $\phi(c)$ would depend on the initial state of motion of the spheres, i.e., on f; as a first approximation, we assume it is universal. Then the Boltzmann equation(for steady flow) would become

$$\underset{\sim}{v} \cdot \frac{\partial f}{\partial x} = A n \{ \phi (c) - f \}.$$

But it is a general property of the Boltzmann equation that the collision integral is identically zero when the distribution is Maxwellian; and to satisfy this requirement $\phi$ must be a Maxwellian $F(c)$, and $\mathcal{G}$ must therefore by given by

$$\mathcal{G} (f) = A n F = A n^2 \left( \frac{\beta}{\pi} \right)^{3/2} \exp \{ - \beta (\underset{\sim}{v} - \underset{\sim}{u})^2 \}. \qquad (4\text{-}4a)$$

The mass and momentum conservation laws have already been used; for the energy to be conserved locally we must have

$$\int v^2 \mathcal{G} (f) D \underset{\sim}{v} = \int v^2 f \mathcal{L} (f) D \underset{\sim}{v},$$

which, after substituting from (4-2) and (4-4a) and noting that

$$\int f v^2 D \underset{\sim}{v} \equiv \int c^2 f D \underset{\sim}{v} + n u^2,$$

reduces to the condition that $\beta$ in $F$ must correspond to the local temperature. In other words, whatever the local distribution function may be, the gain term is taken proportional to the Maxwellian whose parameters are given by the actual local conditions (Eq. (4-1a)).

It should be emphasized that the assumption that $\phi$ is a universal function of $c$ is of a very fundamental character; from our point of view it is the strongest assumption underlying Krook's model.

A slightly different way of justifying (4-4a) is to say that as
the distribution (roughly) reaches a Maxwellian in one collision time,
so the molecules coming out of the collisions must be (roughly)
distributed according to a Maxwellian. In a sense this is an extension
of the idea of diffuse reflection at a wall to collisions within the gas,
taking the accommodation coefficient equal to unity. Thus: molecules
which have collided among themselves (instead of with a wall) come to
local equilibrium with the 'gas', and are emitted with a Maxwellian
corresponding to local 'gas' conditions, retaining no memory whatever
of their previous state. In other words, the scattered molecules behave
exactly as if they had collided with a wall which is locally at equilibrium
with the gas.

Suppose (out of curiosity!) that the 'accommodation coefficient'
above is not unity, but that a fraction $\alpha$ of the colliding molecules are
emitted with a Maxwellian, and the rest with their original distribution
f. Then

$$g(f) = An \left[ \alpha F + (1-\alpha) f \right],$$

but
$$g(f) - f \mathcal{L}(f) = An\alpha \left( F - f \right). \tag{4-5}$$

Thus an accommodation different from unity leaves the model unchanged
except for the constant. Note however that the number of colliding
molecules is still $An^2$, and is unaffected by $\alpha$.

It would seem reasonable that $g(f)$ should be like a Maxwellian
if f itself is a slight departure from a Maxwellian, as it is in the
Navier-Stokes limit (the implications of Krook's model in this limit

will be worked out in the next section). The more doubtful limit is
evidently the case of large departure from equilibrium. But in
nearly free molecule flow - which is one example of large 'disequilibrium'-
the model apparently works reasonably well, judging from the results
obtained for the orifice in the last chapter. It seems therefore that the
assumption that $\phi(c)$ in equation (4-4) is universal may not be as
restrictive as might appear at first sight; and on account of this it should
be very instructive to calculate shock structures using Krook's model.

In the arguments we have used here, the loss term has been
inspired by its form for Maxwell molecules, and the gain term by its
form for rigid spheres.* The justification for this obvious inconsistency
is the belief (or hope!) expressed earlier, that the force model is not
so extremely important, at least for an approximate description of
several phenomena. The usefulness of the model in fact depends on how
tolerable this inconsistency is.

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*It may be recalled that if one is to judge from the viscosity of
gases, most of them seem to be represented by force models lying
somewhere between the rigid sphere and the Maxwell molecule, which
therefore represent in some sense the two extremes for a monatomic gas.

## 4.3 The Navier-Stokes limit

When the collision integral in the Boltzmann equation is replaced by the model assumed in equation (4-1), we get one version of what has been called the 'Krooked' Boltzmann equation:

$$\mathcal{D}f \equiv \frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f}{\partial x} = An\,(F - f).$$

(4-6)

($\mathcal{D}$ here stands for the differential operator on f.) This is still a non-linear integro-differential equation as it stands, for though the form of the collision integral has been assumed to be known, the parameters in it (n, $\underset{\sim}{u}$ and $\beta$) are only defined in terms of the unknown f (cf.(4-1a)).

Before proceeding further with (4-6) it is useful to derive its moment equations. If we multiply it by any quantity $Z = Z(\underset{\sim}{v})$ which is a function of $\underset{\sim}{v}$ only, and integrate over all $\underset{\sim}{v}$, we obtain the corresponding Maxwell equation of transport

$$\frac{\partial}{\partial t}(n\bar{Z}) + \frac{\partial}{\partial x} \cdot (n\overline{Z\underset{\sim}{v}}) = An^2\,(\bar{Z}^0 - \bar{Z}),$$

(4-7)

where

$$n\bar{Z} = \int f(\underset{\sim}{v})\,Z(\underset{\sim}{v})\,\mathcal{D}\underset{\sim}{v},$$
$$n\bar{Z}^0 = \int F(\underset{\sim}{s})\,Z(\underset{\sim}{v})\,\mathcal{D}\underset{\sim}{v}.$$

(4-8)

It is easy to verify that the definition of the parameters in F

by (4-1a) ensures that

$$\int Z \, \mathcal{G}(f) \, D\underset{\sim}{v} = \int Z \, f \, \mathcal{L}(f) \, D\underset{\sim}{v} \qquad (4\text{-}9)$$

for

$$Z(\underset{\sim}{v}) = m, \quad m\underset{\sim}{v}, \quad \tfrac{1}{2} m v^2 ; \qquad (4\text{-}9a)$$

hence that these quantities are indeed conserved at the level of the model and are its collisional invariants. For the Z of (4-9a) the right hand side of the Maxwell equation (4-7) vanishes identically, and so we can derive, exactly as with the true Boltzmann equation, the familiar macroscopic equations of motion

$$\frac{d\rho}{dt} + \rho \frac{\partial u_i}{\partial x_i} = 0,$$

$$\rho \frac{du_j}{dt} = -\frac{\partial}{\partial x_i} p_{ij} , \qquad (4\text{-}10)$$

$$\rho \frac{d}{dt} \left( \tfrac{1}{2} u^2 + \frac{3p}{2\rho} \right) = -\frac{\partial}{\partial x_i} \left( p_{ij} u_j + q_i \right)$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}$$

and $p_{ij}$ and $q_i$ are the pressure tensor and heat flux vector defined in section 1.1. If we define an internal energy by $\mathcal{E} = 3p/2\rho = 3 \, \mathcal{R} \, T/2$, the energy equation in (4-10) can also be written as

$$\rho \frac{d\mathcal{E}}{dt} = -\frac{\partial q_i}{\partial x_i} + p_{ij} \frac{\partial u_j}{\partial x_i} . \qquad (4\text{-}10a)$$

Equations (4-10) are very general as they stand; the usual equations of gasdynamics can be derived from them if $p_{ij}$ and $q_i$ are related to the lower moments of f in a particular way. To study this one has to go back to (4-6).

Suppose there is given, in the problem, a length $L_1$ and velocity $v_1$ with which equation (4-6) can be non-dimensionalized; then, writing

$$x = x' L_1, \qquad v = v' v_1, \qquad t = t' L_1/v_1,$$

(4-6) becomes

$$D'f \equiv \frac{\partial f}{\partial t'} + v' \cdot \frac{\partial f}{\partial x'} = \frac{AnL}{v_1} (F - f). \qquad (4-11)$$

(f has been left undisturbed as both sides of the equation have the same dimension in f.) The non-dimensional parameter on the right has the form

$$A^* = \frac{AnL}{v_1} \sim \frac{\bar{c}}{v_1} \cdot \frac{L_1}{\lambda_1}$$

putting $A \sim \bar{c}/n_1 \lambda_1$. If $v_1 \sim \bar{c}$, $A^*$ is just the Knudsen number.

The Navier-Stokes limit corresponds to letting $A^*$ become very large, or, more conveniently, to letting $\epsilon = 1/A^*$ become very small. Thus (4-11), which can be written

$$\epsilon D'f = \epsilon \left( \frac{\partial f}{\partial t'} + v \cdot \frac{\partial f}{\partial x'} \right) = F - f, \qquad (4-11a)$$

is in the nature of a singular perturbation problem for small $\epsilon$.

This means that for small times ( - considering an unsteady problem) there will be a boundary-layer type solution for the description of which one needs to take account of the initial distribution. Similarly for small distances from a solid surface.

Normally, however, one wants to find a type of 'outer' solution of (4-11a) which does not require conditions on f for its determination, and, also, perturbations on this outer solution for small $\epsilon$ . It is by no means obvious how far one can go in obtaining such higher order perturbations without taking account of initial (or boundary) conditions on f.

It is not the purpose to go into these questions here, but only to show what results equation (4-6) yields when one employs the conventional perturbation methods for its analysis.

For a study of the Navier-Stokes limit one can either follow Chapman (Ref. 18) and use the Maxwell equation (4-7) or, more fundamentally, adopt Enskog's scheme (Ref. 2) of expanding f as a series in a small parameter $\epsilon_1$:

$$ f = f^o + \epsilon_1 f^1 + \epsilon_1^2 f^2 + \cdots \tag{4-12} $$

This $\epsilon_1$ can be identified as the value of $\epsilon = 1/A^* = \epsilon(\underset{\sim}{x})$ at some reference point, and we may write $\epsilon(\underset{\sim}{x}) = \epsilon_1 b(\underset{\sim}{x})$ where $b = O(1)$. Substituting (4-12) in equation (4-11a) and collecting terms of the same order in $\epsilon_1$, we get

$$ f^o = F, \quad f^1 = -b \mathcal{D} f^o, \quad \cdots \quad f^n = -b \mathcal{D} f^{n-1}. \tag{4-13} $$

Thus the zeroth order term is just the local Maxwellian. As all odd moments of the Maxwellian are zero, we immediately have

$$p_{ij}^0 = p\,\delta_{ij}, \qquad q_i^0 = 0,$$

and putting these in (4-10) we can easily get the Eulerian equations of gasdynamics,

$$\frac{d\rho}{dt} + \rho\,\frac{\partial u_i}{\partial x_i} = 0,$$
$$\rho\,\frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} \tag{4-14}$$

and

$$\frac{d}{dt}\left(\rho^{-\frac{5}{3}}p\right) = 0,$$

the last being of course the isentropic law for a monatomic gas.

The first order term is more interesting, and is simply (from (4-13))

$$f^1 = -b\,\mathcal{D}'f^0 = -b\,\mathcal{D}'F. \tag{4-15}$$

The enormous simplfication arising from the use of Krook's model is very striking at this point, if we compare (4-15) with the integral equation obtained when the exact collision term is used. The solution of (4-15) (or its counterpart) essentially supplies us with the transport properties of the gas.

Let us restrict ourselves to steady flow from now on (though this is actually unnecessary). Then (4-15) can be written

$$f' = -\frac{bL}{v_1} v_i \frac{\partial}{\partial x_i} \left\{ n \left(\frac{\beta}{\pi}\right)^{3/2} exp - \beta(\underline{v}-\underline{u})^2 \right\}$$

where n, $\beta$ and $\underline{u}$ are functions of $\underline{x}$. Carrying out the differentiation (and using the Eulerian equations (4-14)), we get

$$f' = \frac{bL}{v_1} F \left[ \frac{c_k}{\beta} \frac{\partial \beta}{\partial x_k} \left( C^2 - \frac{5}{2} \right) \right.$$
$$\left. - 2 \frac{\partial u_k}{\partial x_l} \left( C_k C_l - \frac{1}{3} C^2 \delta_{kl} \right) \right] \tag{4-16}$$

where

$$C_k = c_k \sqrt{\beta}.$$

By taking the appropriate moments of $f = f^0 + \epsilon_1 f^1$ one can now get expressions for the pressure tensor and heat flux vector. For instance $p'_{ij}$ can be written, from (4-16), as

$$p'_{ij} = p_{ij} = \rho \overline{c_i c_j} = \rho \left[ \overline{c_i c_j}^0 + \frac{bL}{v_1} \left\{ \frac{1}{\beta} \frac{\partial \beta}{\partial x_k} \left( \beta \overline{c_i c_j c_k c_m c_m}^0 \right. \right. \right.$$
$$\left. - \frac{5}{2} \overline{c_i c_j c_k}^0 \right) - 2\beta \frac{\partial u_k}{\partial x_l} \left( \overline{c_i c_j c_k c_l}^0 \right. \tag{4-17}$$
$$\left. \left. \left. - \frac{1}{3} \overline{c_i c_j c_m c_m}^0 \delta_{kl} \right) \right\} \right].$$

The moments of the Maxwellian which appear here (and later) are most conveniently found using a general formula given by Ikenberry & Truesdell (Ref. 19):

$$p^0_{i_1 i_2 \dots i_n} = \rho \overline{c_{i_1} c_{i_2} \dots c_{i_n}}^0$$
$$= \frac{n-1}{2\beta} p^0_{(i_1 i_2 \dots i_{n-2}} \delta_{i_{n-1} i_n)}$$

where the parentheses around the n indices indicate the sum over all combinations (into two groups of n-2 and 2) of the indices, divided by the number of such combinations, $\binom{n}{2}$. Thus,

$$p_i = 0, \qquad p_{ij}^0 = \frac{\varrho}{2\beta}\, \delta_{ij}, \qquad p_{ijk}^0 = 0,$$

$$p_{ijkl}^0 = \frac{3}{2\beta} \cdot \frac{\varrho}{2\beta} \left( \delta_{ij}\, \delta_{kl} + \delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk} \right),$$

and so on. Putting these results in (4-17) we get

$$p_{ij}^1 = p\delta_{ij} + \frac{1}{3A\beta} \frac{\partial u_k}{\partial x_k}\, \delta_{ij} - \frac{1}{2A\beta} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad (4\text{-}18)$$

from which we see that the 'viscous' stress can be written as

$$\underset{\sim}{\mathcal{T}} = -\underset{\sim}{p} + p\,\underset{\sim}{1}$$

$$= (\mu'\, div\, \underset{\sim}{u})\,\underset{\sim}{1} + \mu\, def\, \underset{\sim}{u} \qquad (4\text{-}18a)$$

with

$$\mu = \frac{1}{2A\beta}, \qquad \mu' = -\frac{1}{3A\beta} \qquad (4\text{-}18b)$$

being the familiar viscosity coefficients. It is immediately seen that the 'Stokesian' assumption that $3\mu' + 2\mu = 0$ is satisfied, as it should be for a monatomic gas. The corresponding expressions obtained by Chapman & Cowling for the viscosity coefficients can be written

$$\mu = \jmath\, \frac{5}{16}\, \frac{1}{\beta\, \Omega_1^{(2)}(2)}, \qquad \mu' = -\frac{2}{3}\, \mu, \qquad (4\text{-}19)$$

where $j$ is a numerical factor close to unity representing the sum of an infinite series, and $\Omega_1^{(2)}(2)$ is a certain integral depending on the inter-molecular force model.

As no assumption has till now been made on the dependence of A on the local thermodynamic state, it is legitimate to do so now, basing ourselves on (4-19) or on measured viscosity coefficients for gases. It may thus be concluded that A is a function only of the temperature; and we may write, from (4-19), the correspondence

$$A = A_\mu \cong \frac{1 \cdot 6}{j} \, \Omega_1^{(2)}(2) . \qquad (4\text{-}20)$$

For a Maxwellian molecule $\Omega_1^{(2)}(2)$ is independent of the temperature, hence A can be taken to be a constant. For rigid spheres

$$\Omega_1^{(2)}(2) = 2\pi\sigma^{-2} \sqrt{\frac{\mathcal{R}T}{\pi}} \qquad (4\text{-}21)$$

where $\sigma$ is the diameter of the molecule; and introducing the mean free path $\lambda = 1/n\,\pi\,\sigma^2\,\sqrt{2}$ we get

$$A_\mu = 0 \cdot 8 \, \frac{\bar{c}}{n\lambda} \qquad (4\text{-}21\text{a})$$

where $n\lambda$ is a constant. In section 3.2 we chose $A = A_\nu \cong \bar{c}/n\lambda$ in order to match the number of collisions, which is more significant than viscosity in nearly free molecular flows. To match viscosities, the

choice $A_\mu = 0.8 A_\nu$ is more appropriate. Incidentally, we <u>can</u> reconcile both numbers if we interpret the coefficient 0.8 in (4-21a) as the accommodation coefficient (for gas collisions) as was discussed in section 4.2.

From (4-16) the heat flux vector can also be calculated; this gives

$$q_i = \tfrac{1}{2} \rho \, \overline{c_i c_j c_j^o} = \frac{1}{A\beta} \left\{ \frac{\partial \beta}{\partial x_i} \left( \beta \, \overline{c_i c_j c_k c_k c_m c_m} - \tfrac{5}{2} \overline{c_i c_j c_k c_k^o} \right) \right\},$$

from which we get, after evaluating the moments as before,

$$q_i = -\frac{5\mathcal{R}}{4A\beta} \frac{\partial T}{\partial x_i}, \quad \text{or} \quad k = \frac{5\mathcal{R}}{4A\beta} \tag{4-22}$$

for the conductivity. Putting $c_p = 5\,\mathcal{R}/2$ for the specific heat at constant pressure, we obtain for the Prandtl number

$$Pr \equiv \frac{\mu c_p}{k} = 1. \tag{4-23}$$

This does not agree with the Prandtl number of a monatomic gas, which is roughly 2/3; hence it is not possible to match the constant in Krook's model for both viscosity and conductivity.

By making use of the relations (4-18) and (4-22) in the general macroscopic equations of motion (4-10), we obtain essentially the Navier-Stokes equations for a gas with zero bulk viscosity ($2\mu + 3\mu' = 0$)

and a Prandtl number of unity. This shows that, on the one hand, the chances of extending the model to gases which are not monatomic are not high, and that on the other hand the description afforded by the model of the behavior of monatomic gases in, and near, the gas-dynamic limit is likely to be quite realistic.

## 4.4 Conclusions about Krook's model

We collect and summarize here the results obtained in the present brief study of Krook's model.

First, some of its implications. (i) From the way the term $\mathcal{L}(f^o)$ has appeared in the solution of nearly free molecular flow problems, e.g. in equation (3-9a), it is obvious that $v/\mathcal{L}(f)$ represents the mean free path $\lambda'$ (as a function of velocity). As $\mathcal{L}(f)$ is assumed independent of the molecular velocity, $\lambda' = \lambda'(v)$ is thus taken to be proportional to the molecular speed in the model. For a completely Maxwellian distribution $\lambda'$ is given, for small velocities, by

$$\frac{\lambda'(v)}{\lambda} = \sqrt{2} \cdot \frac{v}{\bar{c}} = V\sqrt{\frac{\pi}{2}}$$

where

$$\lambda = \frac{1}{n\pi\sigma^{-2}\sqrt{2}} \;;$$

and if in Krook's model we take $A = \alpha\bar{c}/n\lambda$, we will have $\lambda'(v) = v\lambda/\alpha\bar{c}$, which is comparable to the exact result, and suggests again $\alpha < 1$. However, as $v \rightarrow \infty$, the exact result gives $\lambda' \rightarrow \sqrt{2}\,\lambda$, whereas Krook's model gives $\lambda' \rightarrow \infty$.

(ii) The relaxation time, in Krook's model, is

$$\tau = \frac{1}{An} = \frac{n\lambda}{\alpha n\bar{c}}$$

and is thus inversely proportional to $n\bar{c}$ if $n\lambda$ is taken to be constant.

This is the same dependence as one gets for rigid elastic spheres.

(iii) It has been shown in the previous section that the Navier-Stokes equations can be derived from Krook's model, but to be consistent one has to make the Stokesian assumption about the viscosity coefficients, and furthermore take a Prandtl number of unity.

A few remarks may be made here about the value to be assigned to A. If the problem being considered is in the continuum or near-continuum regime, A can be chosen as an arbitrary function of the local temperature, to reproduce the (known) behavior of the viscosity or the conductivity of the gas, but not both (as Pr = 1). A general 'psuedo-rigid sphere' value for A would be

$$A = \frac{\alpha \bar{c}}{n \lambda} \qquad (4\text{-}24)$$

where $\alpha = 0.8$(interpreted as an accommodation coefficient) will give the same viscosity for the Krook gas as for a gas composed of rigid spheres of cross-section $\pi \sigma^2/4 = A/\alpha \bar{c} 4 \sqrt{2}$. With this interpretation the number of collisions in the gas would also be correctly given as $n\bar{c}/2\lambda$ per unit volume and unit time. It is conceivable that in problems where heat conduction is more important than viscous effects one may wish to choose A to give the correct conductivity rather than viscosity.

In nearly free molecular flows, the choice of A given by (4-24)

apparently gives very good results. There is not enough information, however, to tell whether $\alpha = 0.8$ or $1.0$ corresponds better with reality.

It may be concluded, therefore, that the Krook model, in spite of its simplicity, describes the real behavior of gases fairly well both near the continuum limit and near the free molecule limit. Use of the model throughout the range between gasdynamics and gaskinetics may therefore be reasonably expected to be justified and worthwhile.

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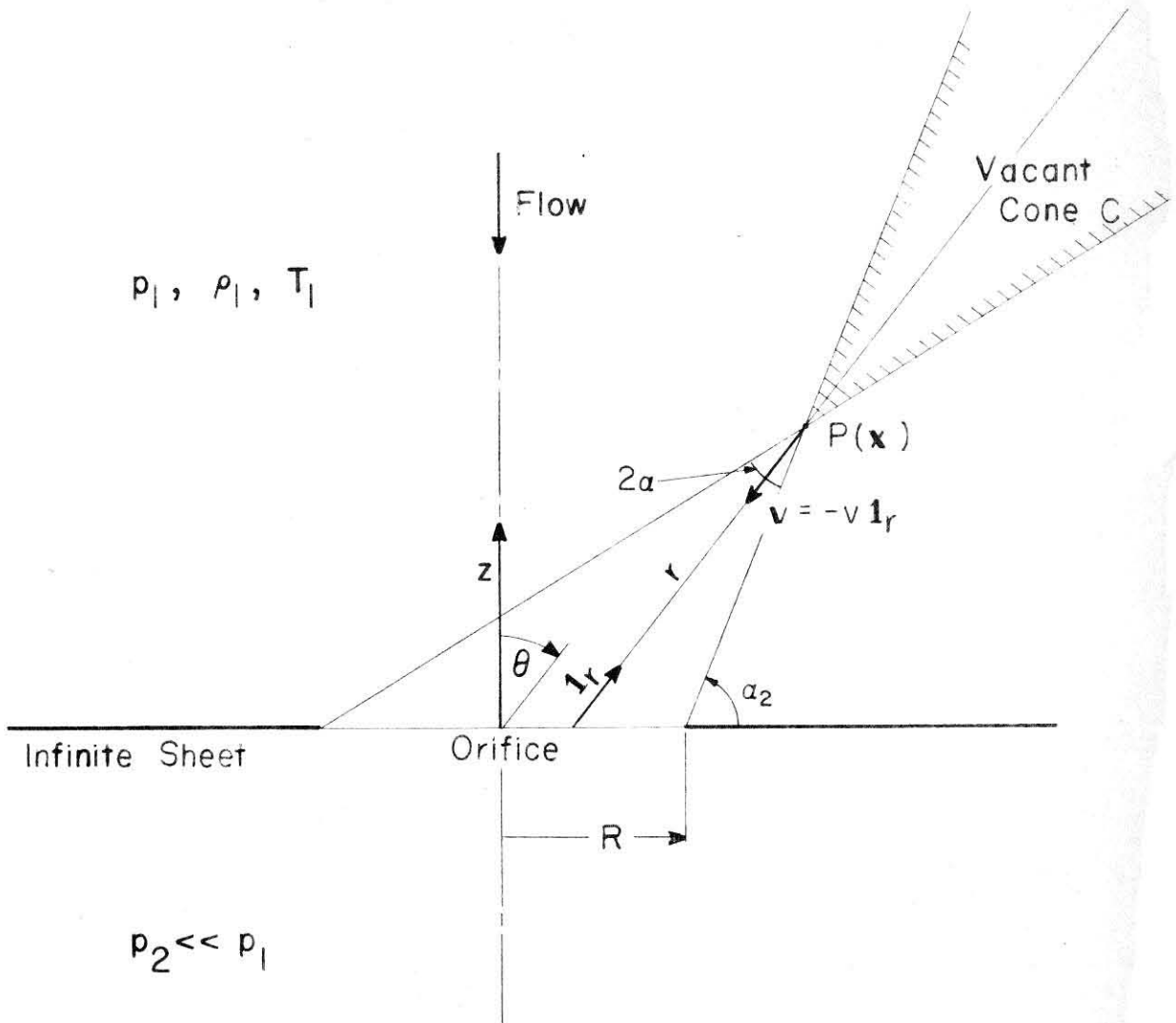19.   Ikenberry, E., & C. Truesdell.  On the pressures and the flux of energy in a gas according to Maxwell's kinetic theory, I. J. Ratl. Mech. Analysis 5:1, 1956.

Fig. I    Notation    for    Orifice    Flow
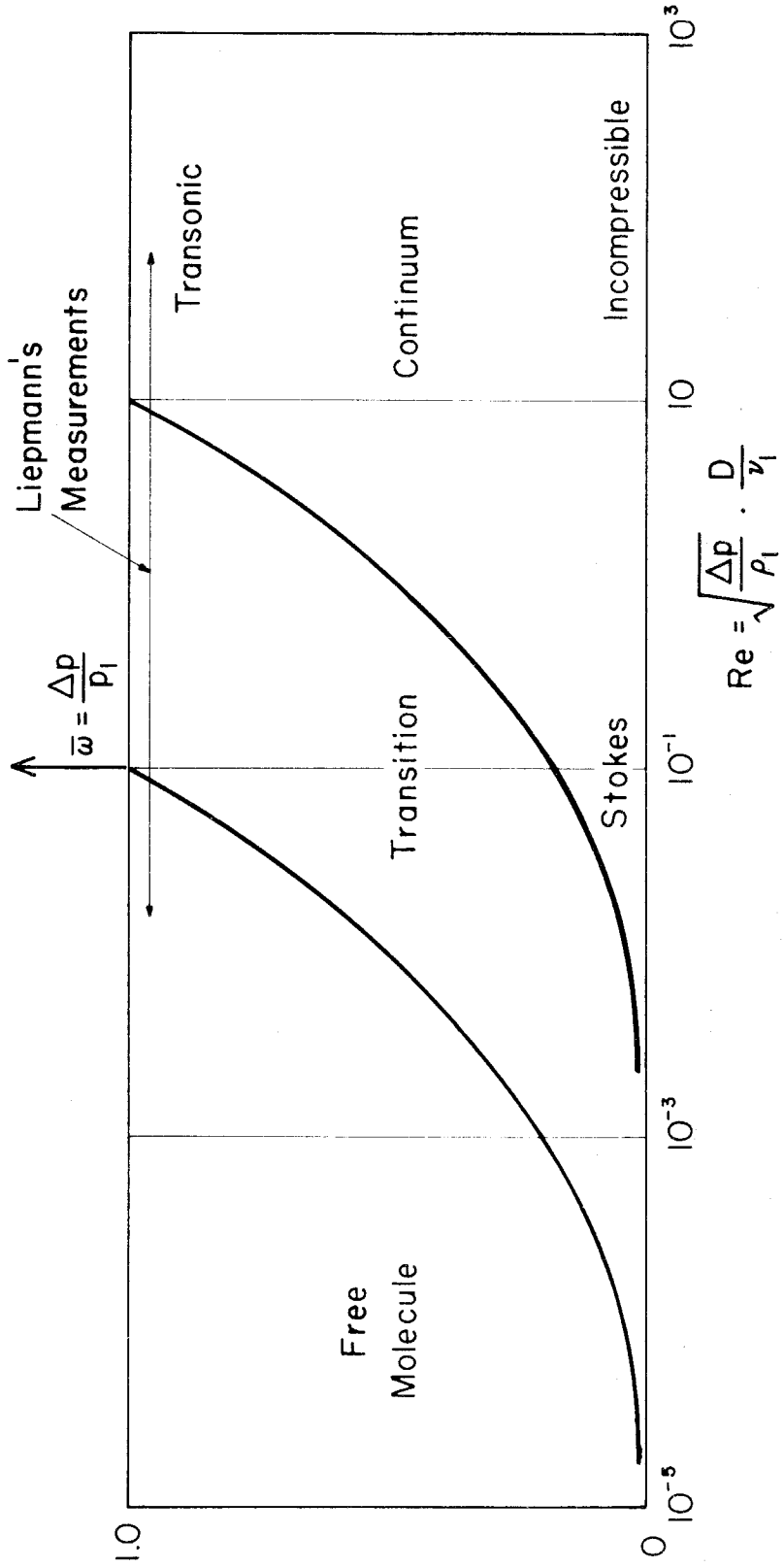
Fig. 2    Possible    Flows    through    an    Orifice
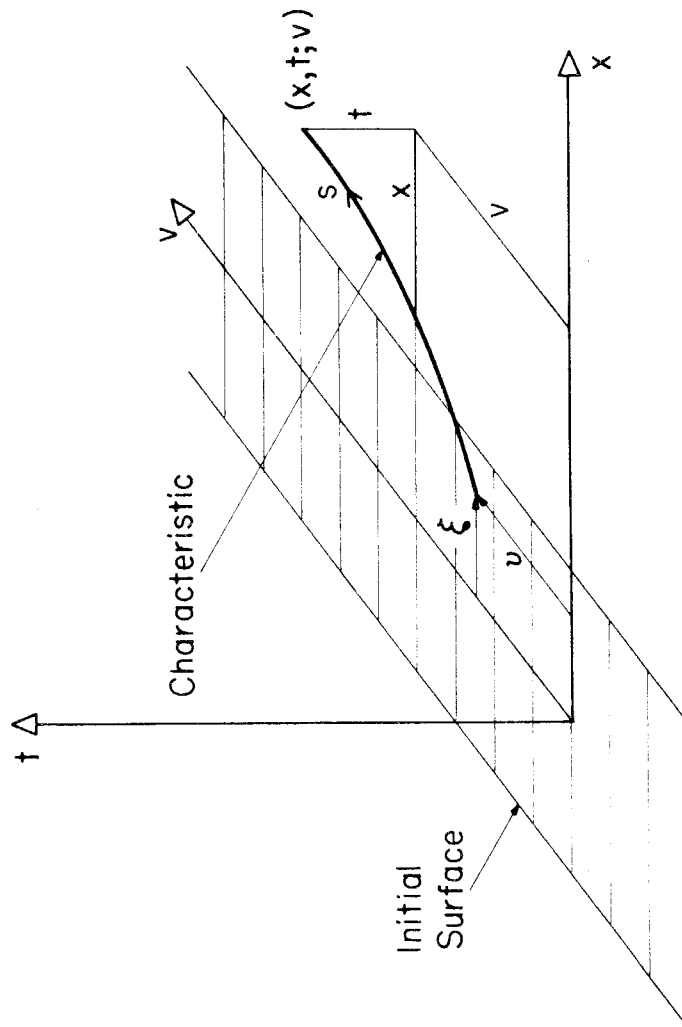
Fig. 3    Characteristics   in   $\underset{\sim}{x}$ $\underset{\sim}{v}$ t    Space

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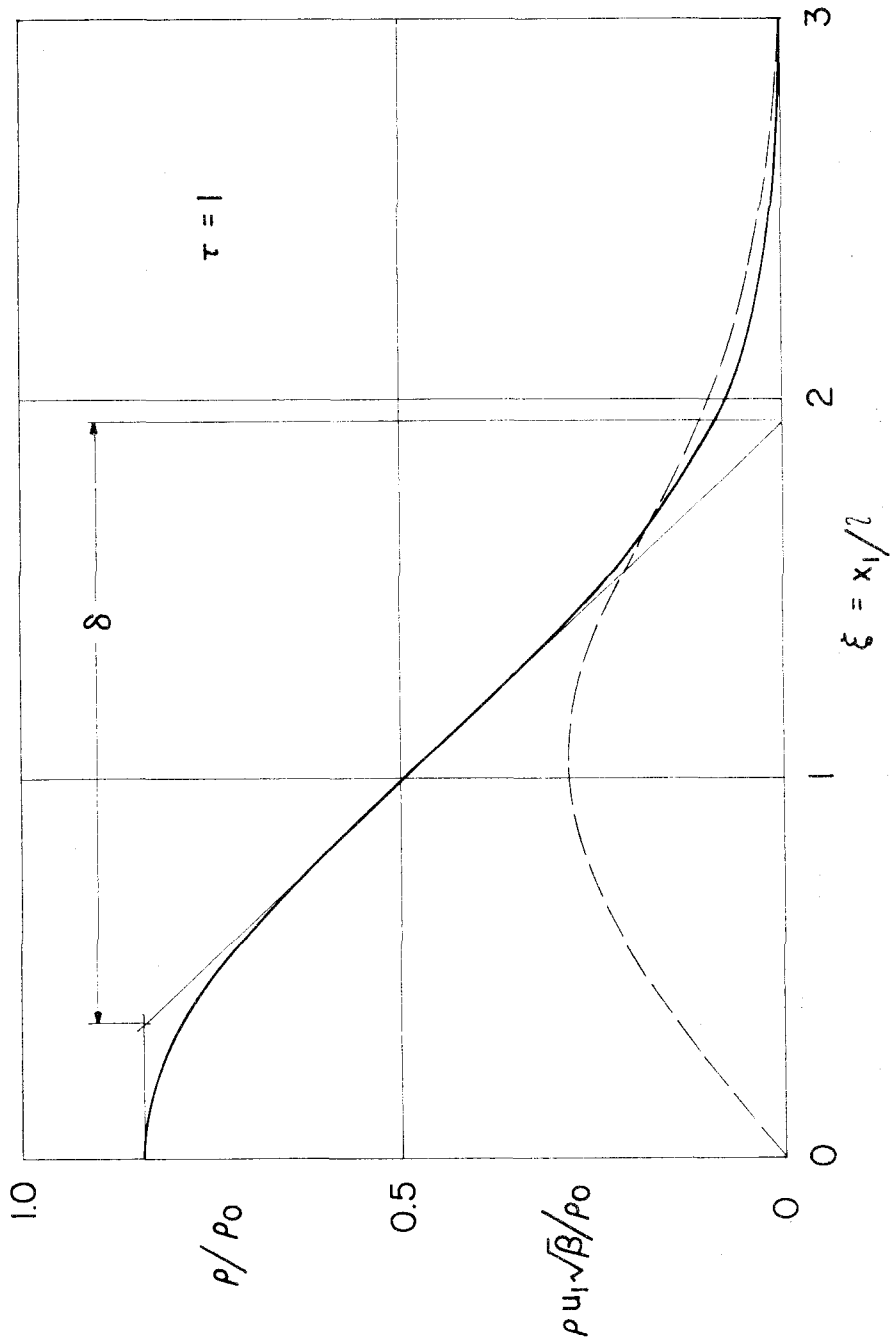


Fig. 4  Typical Density Profile in One-Dimensional Free Expansion
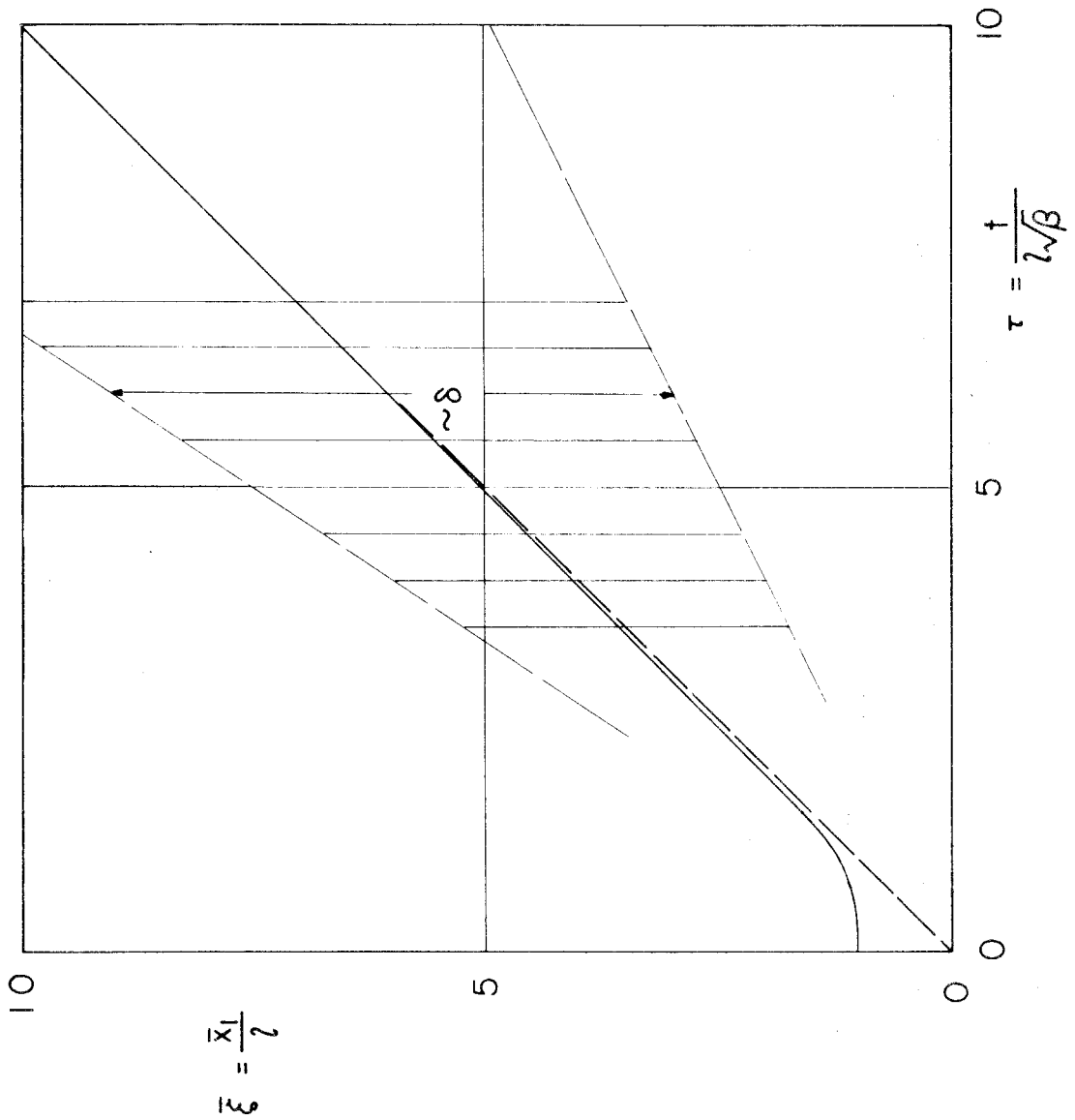
Fig. 5  Propagation of the "Diffusion Front" in One-Dimensional
Free Expansion
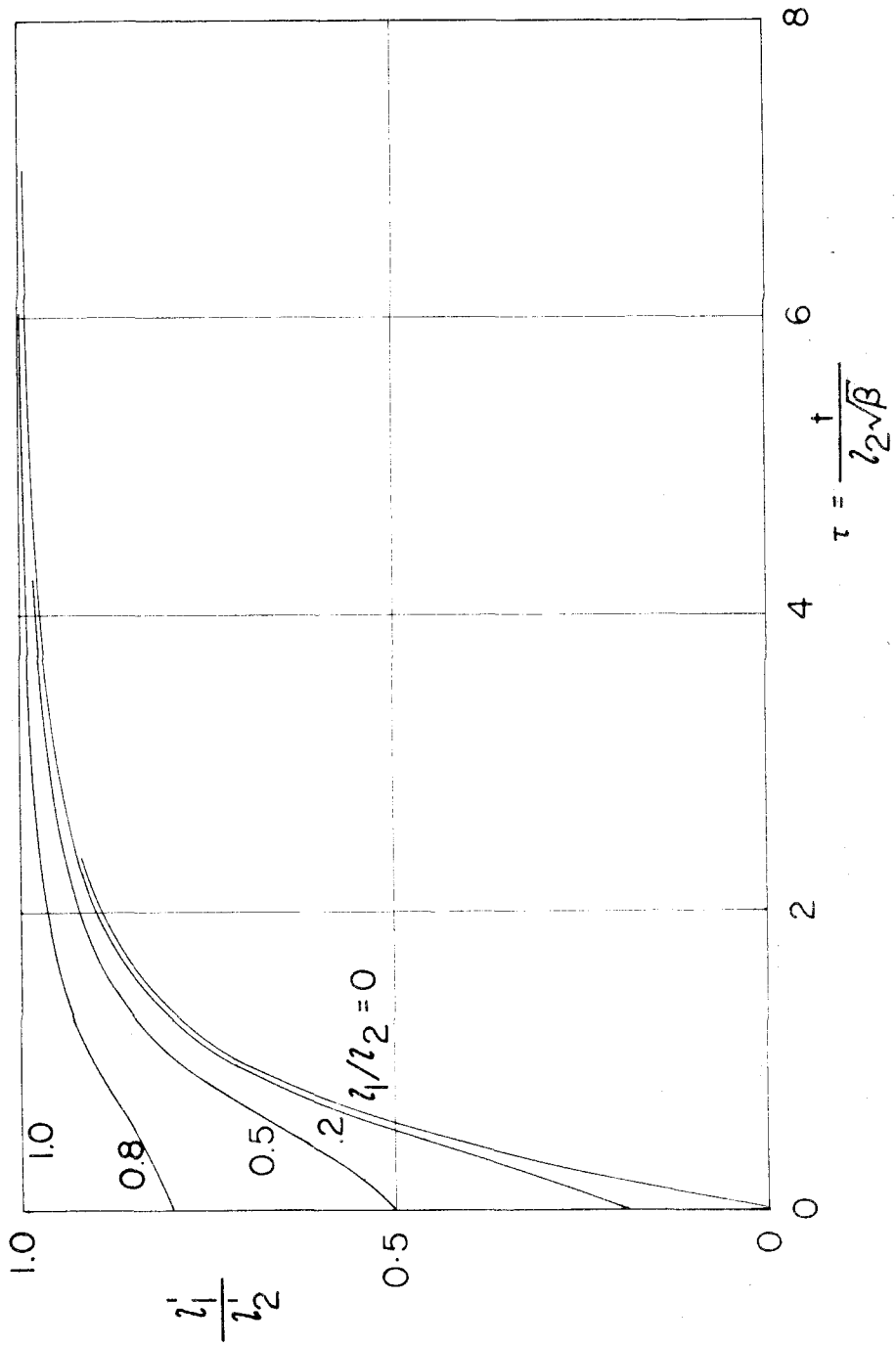
Fig. 6   Eccentricity   of   Expanding   Asymmetric   Clouds
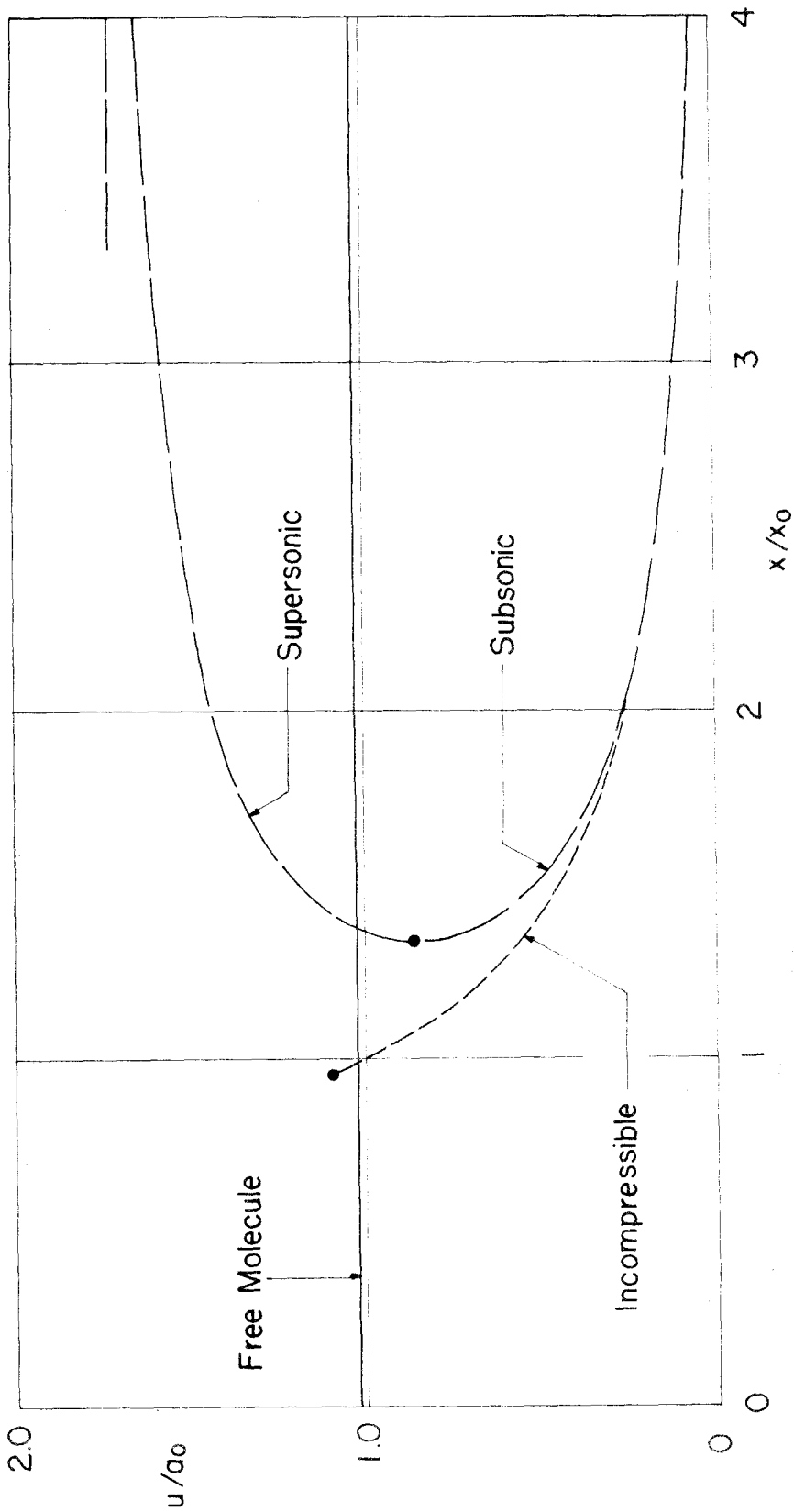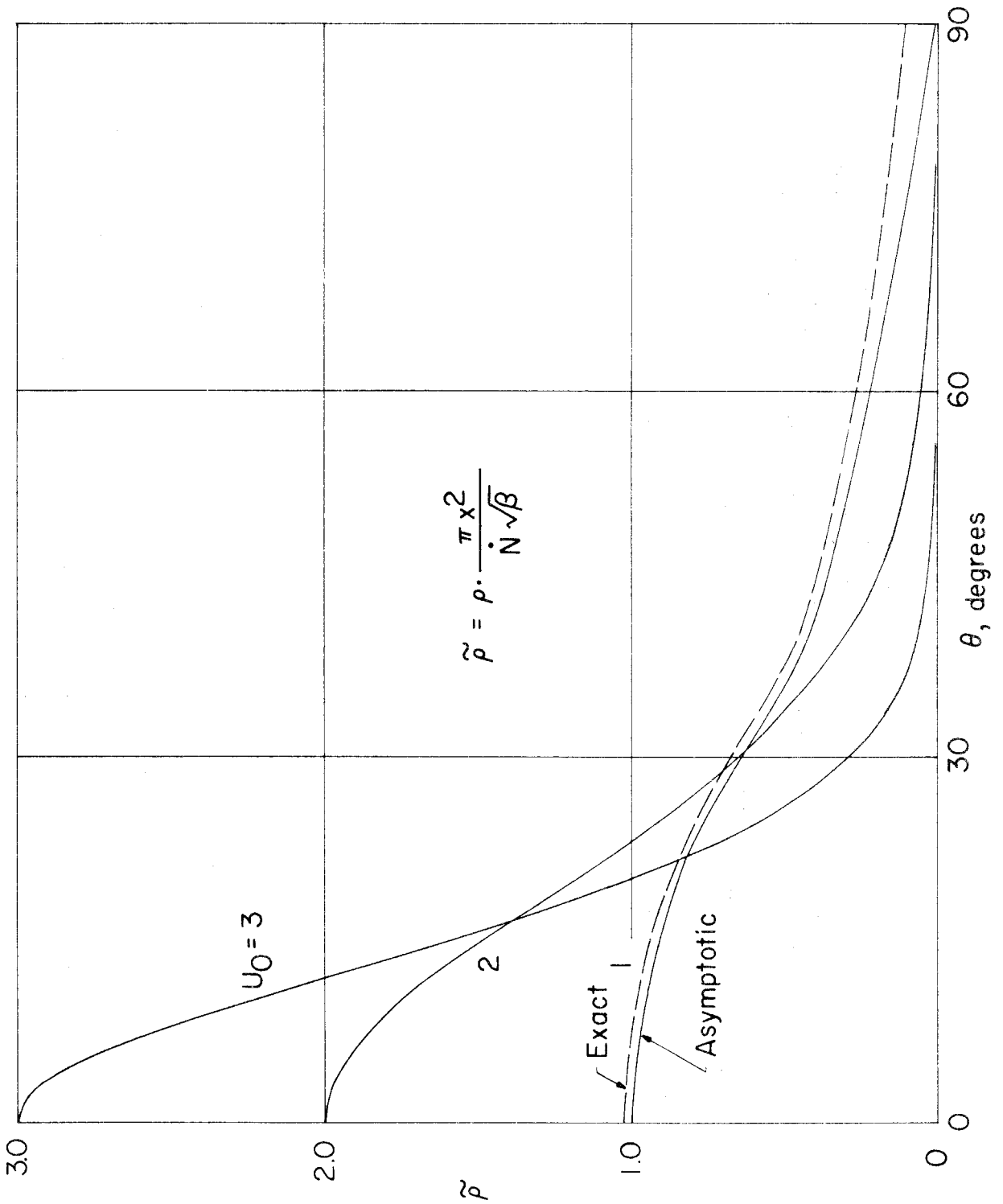
Fig. 7   Comparison of Source Flows

$$\tilde{\rho} = \rho \cdot \frac{\pi x^2}{\dot{N} \sqrt{\beta}}$$

Fig. 8    Density    Distribution    in    Free    Molecule    Jet;
Comparison    of    Exact    and    Asymptotic    Expressions
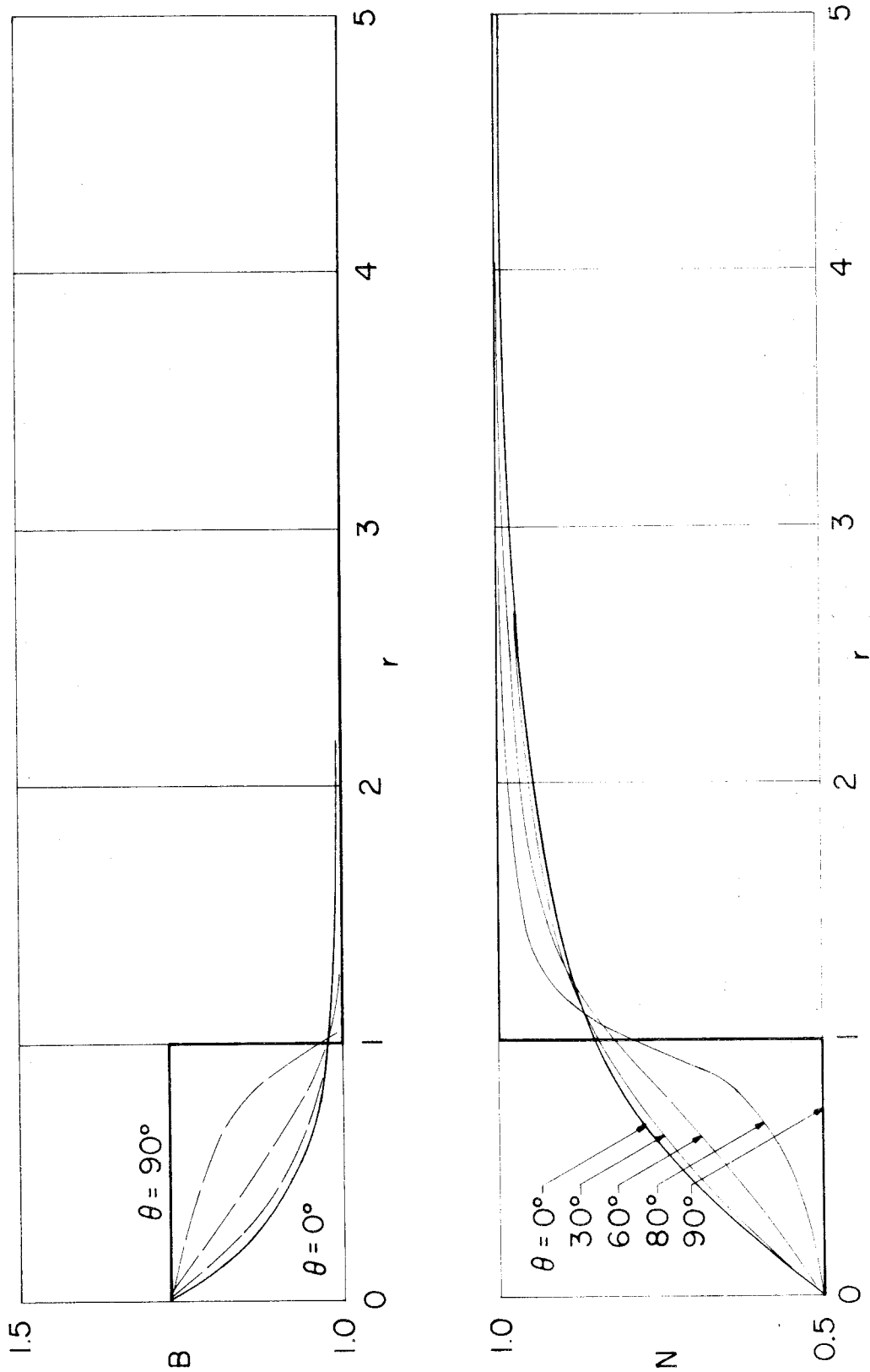
Fig. 9　Plots of Density and Temperature Fields in Free Molecule Flow through Orifice
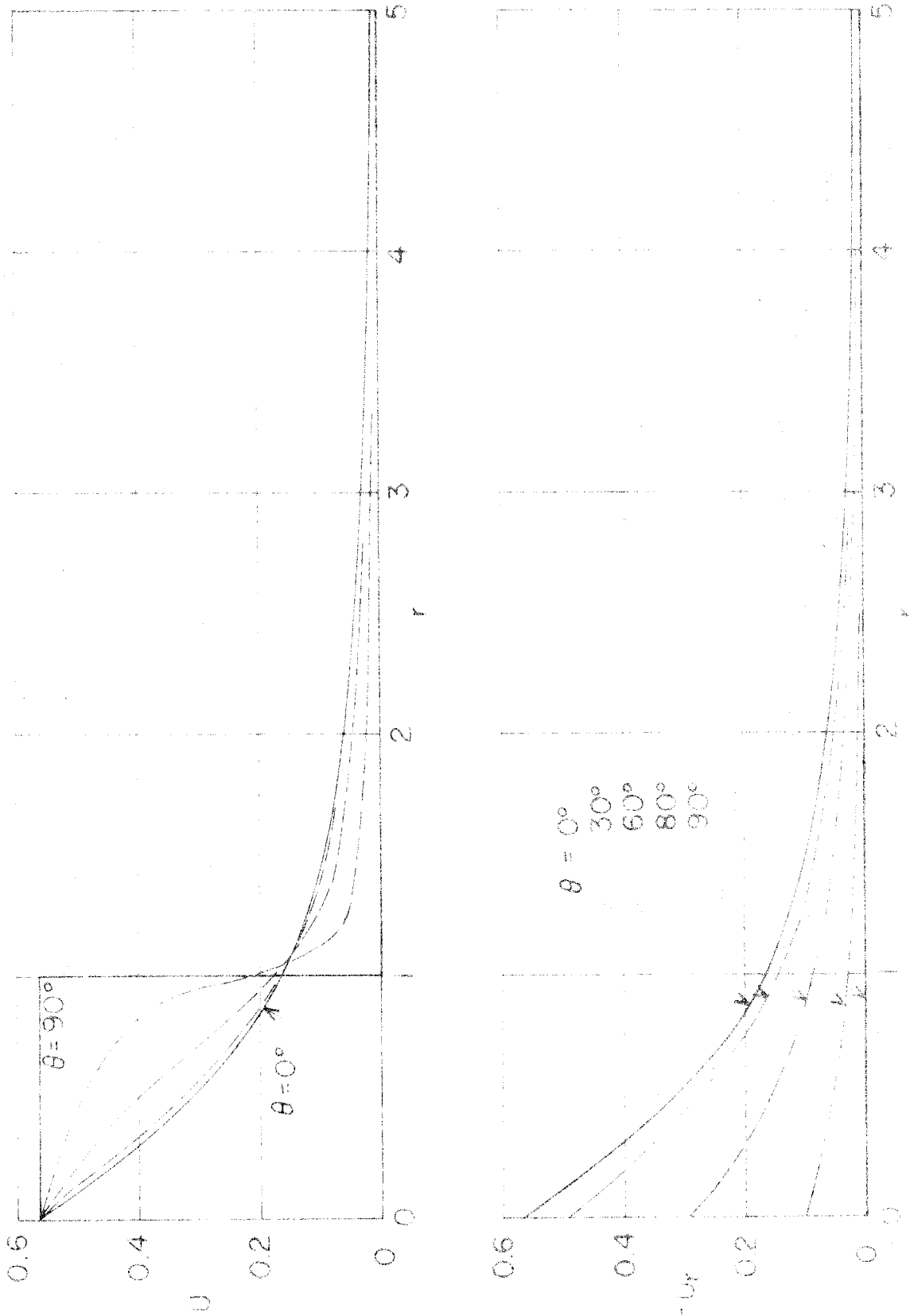
Fig. 10 Plots of Magnitude and Radial Component of Gas
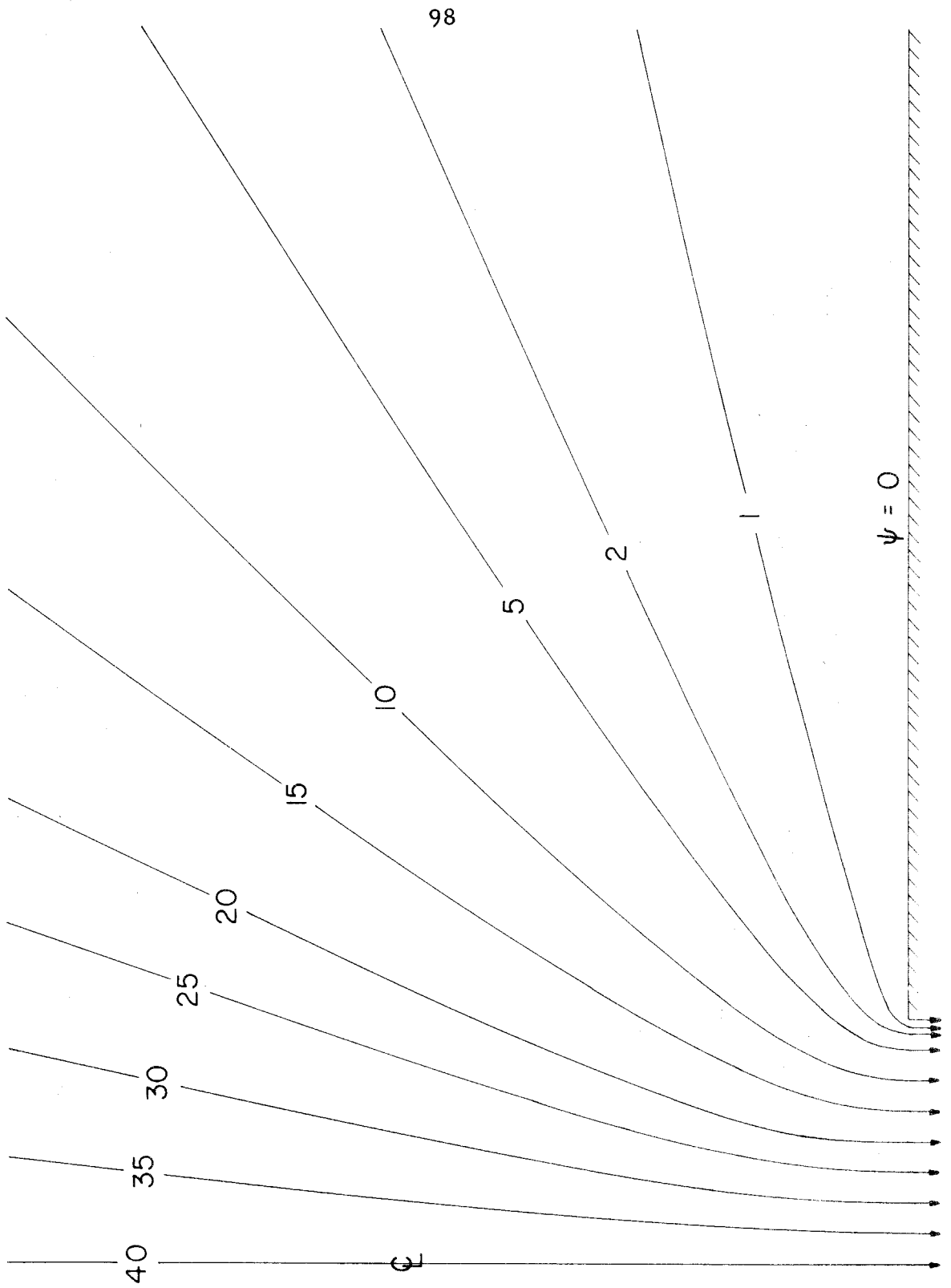Velocity in Free Molecule Flow through Orifice

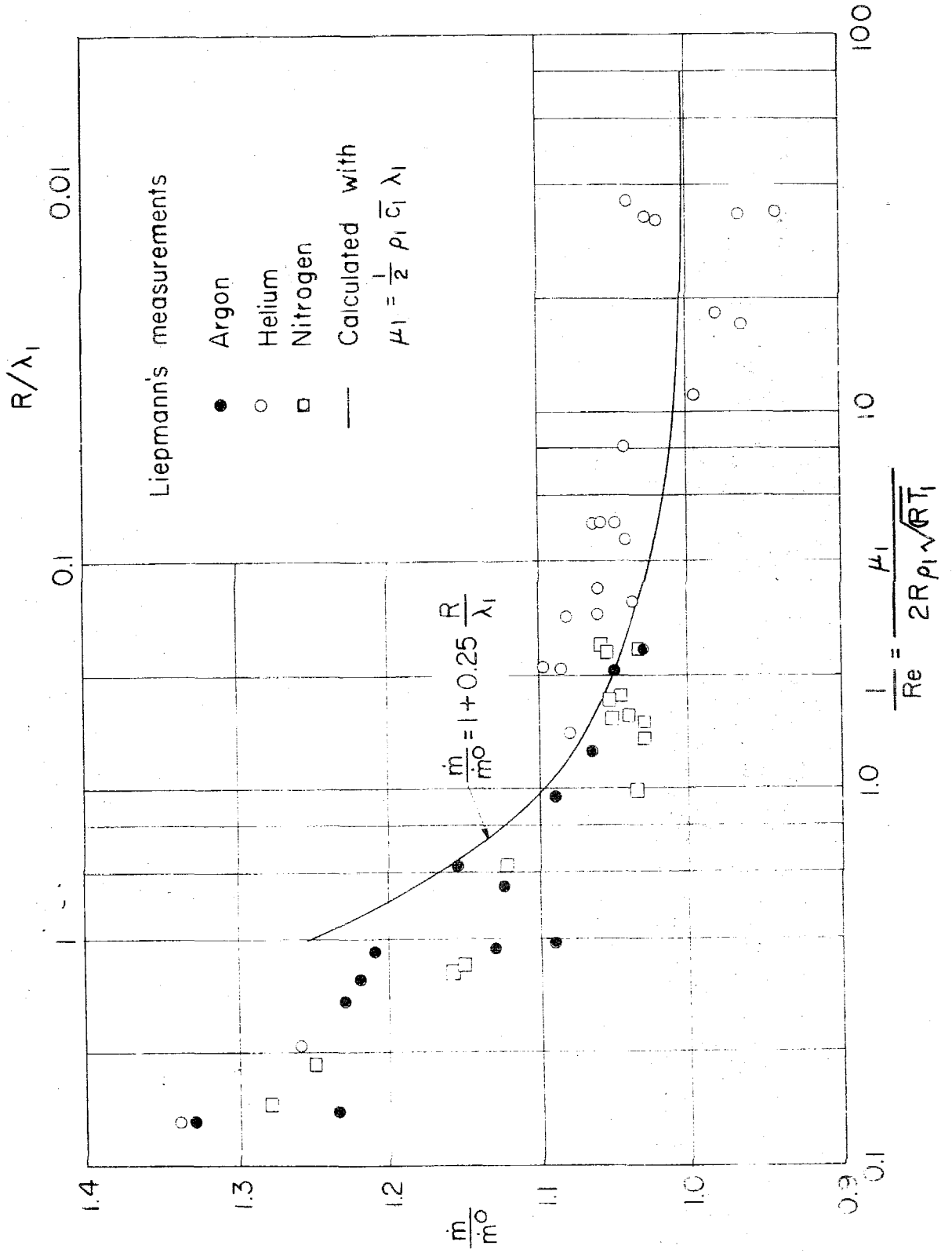Fig. 11  Streamlines  in  Free  Molecule  Flow  through  Slit

Fig. 12 Comparison of Expeiment and Calculation in Nearly Free Molecular Flow through Orifice