LONG DISTANCE ENERGY CORRELATIONS IN RANDOM MEDIA

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While at the California Institute of Technology, my life was enriched both personally and academically. There are many people who contributed to my life during my stay at Caltech; far too many to list them all. The students and faculty in the Applied Mathematics Department were a focal point, with whom I cherish many memories. This thesis would not have been possible, however, without the ceaseless concern of my advisor, Ben White.

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Everyone has stumbling blocks in their path, whether intrinsic or placed there by others, real or imagined. This thesis is dedicated to all those who never got as far as I did, because they could not overcome their stumbling blocks.

ABSTRACT

This thesis considers the long distance motion of waves in a random medium. Using the geometrical optics approximation and a stochastic limit theorem, we find evolution equations for rays and for energy correlations, in two and three dimensions.

Our equations are valid on a long distance scale, well after the focusing of rays has become significant. We construct asymptotic expansions of the two point energy correlation function in two and three dimensions.

In two dimensions we numerically solve the partial differential equation that determines the two point energy correlation function. We also perform Monte-Carlo simulations to calculate the same quantity. There is good agreement between the two solutions.

We present the solution for the two point energy correlation function obtained by regular perturbation techniques. This solution agrees with our solution until focusing becomes significant. Then our solution is valid (as shown by the Monte-Carlo simulations), while the regular perturbation solution becomes invalid.

Also presented are the equations that describe energy correlations after a wave has gone through a weakly stochastic plane layered medium.

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I.1 Rays in Random Media

In the thesis we deal with the problem of wave propagation through a random medium. Examples of wave propagation through random media are given by: microwave and optical propagation in atmospheric turbulence, acoustic wave propagation through a biological medium, and stress waves through the earth's crust.

When a wave is impingent upon a medium in which the local wave speed varies randomly, the amplitude and phase of the wave experience random fluctuations. For example, in transmitting an optical beam through the atmosphere, the random fluctuations of the refractive index cause a spreading of the beam, a decrease in the temporal and spatial coherence, a beam wandering, and scintillations of the received intensity.

The motion of waves in a weakly inhomogeneous medium has been extensively studied (see, for example, Ishimaru (1), Keller (2), Chernov (3), Uscinski (4), Tatarski (5)). Up until recently only regular perturbation methods have been employed to predict the motion and properties of these waves. The solutions that are obtained can not be valid for long distances, because an essential assumption in using a regular perturbation expansion is that large fluctuations have not occurred. Even before the fluctuations become very large, we expect a regular perturbation series to lose its validity.

Chernov (3) assumed that the position of a single ray was a Markov process, and obtained the position of a ray after it had travelled a long distance.

I.2 Previous Results Using A Stochastic Limit Theorem

Suppose a wave enters a medium with a random wave velocity. We assume the wave velocity is nearly deterministic, the random component being homogeneous, isotropic and small. If the standard deviation of the velocity variations is σ ($\sigma<<1$), then a typical velocity inhomogeneity will produce a small deflection.

If the wave length of the wave is much smaller than the characteristic size of the velocity inhomogeneities, then the approximation of geometric optics can be used. Geometrical optics has been used previously, together with linearization of the equations of motion (for small σ), to find fluctuations in ray direction, phase and wave intensity. It is known that these results are not valid for long distances.

Recently, Kulkarny and White (6) have used a stochastic limit theorem, in two dimensions, to determine the long distance motion and properties of high frequency waves in a random medium. They do this by considering the rays as continuous Markov processes. They have determined that large fluctuations occur on a distance scale of $\sigma^{-2/3}$. The technique they used does not require the linearization of any quantities, and it allows focusing of raytubes to occur (such focusing points are called caustics).

Let $\mu(\tau)$ be the probability density that a ray travels a distance τ before it focuses for the first time. Kulkarny and

White showed that if τ is measured in "universal time" then, for all sufficiently differentiable and mixing random media, there is a universal representation for $\mu(\tau)$.

A major result of (6) is the approximation of $\mu(\tau)$ given by:

$$\mu(\tau) = \left\{\frac{\alpha^2 \tau^{-5/2}}{\sqrt{2\pi}} + .314\right\} e^{\left(-\frac{\alpha^4}{6\tau^3} - .281\tau\right)} \qquad (I.1.1)$$

where $\alpha = K(1/2) = 1.854+$. A graph of $\mu(\tau)$ is given in Figure I.1.

The only way the actual medium under consideration changes the statistical location of the caustics, is by the scaling to "universal time". This scaling is given by

$$\tau = (\gamma_2 \sigma)^{2/3} s \qquad (I.2.2)$$

$$\gamma_2^2 = 6 \int_0^\infty \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 B(r)\, dr \qquad (I.2.3)$$

where s is the physical distance a ray travels and B(r) is the covariance function for the random velocity field. From (I.2.2) and (I.2.3) we see that if σ is made smaller, a ray must go further to have the same probability of focusing.

Other results for the evolution of a single ray and its associated raytube area, as well as the transformation to physical coordinates, are given in Kulkarny and White (6).

White (7) has extended the above results to a three dimensional random medium. Once again there is a "universal focusing curve". The scaling to this universal time also depends

on a single constant, derivable from the random field.

I.3 Results In Thesis

This thesis continues the investigation of the motion and properties of waves on the $\sigma^{-2/3}$ scale. We do this primarily by the construction of infinitesimal generators that describe the evolution of physical quantities. We consider the system of equations that describe the evolution of energy correlations in detail.

The salient features of the assumptions in our work are that:

1) The random medium that the wave enters is only weakly stochastic, and the randomness is homogeneous and isotropic.

2) When the wave enters the random medium, the wavefront is nearly flat (the deviation from a planar wavefront may only be of order $\sigma^{2/3}$).

3) We study the high frequency phenomena that occur, that is we use the geometrical optics approximation. Therefore, we can consider the wavefront as advancing by the motion of rays.

Our technique is an "honest" technique in the sense of Keller (2). In an "honest" technique, no assumptions are made that could be derived from previous assumptions.

The two major approximations in our work are due to the high frequency of the wave and the smallness of the departure from a deterministic velocity field. The stochasticity of the medium only enters when applying the Papanicolaou and Kohler theorem.

In chapter one, we derive the equations of motion for rays and their associated raytubes in a random medium using the geometrical optics approximations. Then we assume the medium is only weakly stochastic and scale the equations of motion to be on the $\sigma^{-2/3}$ scale.

Chapter two is concerned with the Papanicolaou and Kohler theorem. We describe the theorem, and then simplify its requirements by restricting its applicability to dynamical systems.

Using the theorem we then find $\mathfrak{L}_N^A$, the generator that describes the evolution of N rays and their associated raytubes in two dimensions. We also find $\mathfrak{L}_N$, the generator that describes the evolution of N rays in two dimensions.

To obtain these results we show that, on the $\sigma^{-2/3}$ scale, the vertical deviation of ray positions is O(1) while horizontal deviation of ray positions is $O(\sigma^{2/3})$. This suggests using the variation in vertical ray position with respect to the initial conditions as a definition of raytube area.

We cannot, however, assume a priori that the variation in a horizontal rays position, with respect to the initial conditions, is small. We have shown, though, that $\mathfrak{L}_N^A$ is derivable from $\mathfrak{L}_{2N}$ and the assumption that the raytube area is approximated by the variation of the vertical ray positions (see section 2.6). This

leads us to believe that we can use variations in vertical ray position as a definition of raytube area, in calculations performed on the $\sigma^{-2/3}$ scale.

In chapter three we study the g(z) function used in $\mathcal{E}_N^A$ and $\mathcal{E}_{2N}$. This function, which depends on the random medium, is the only way different random media change the evolution of rays (analogous to the γ_2 constant in (I.2.2)).

Chapter four begins by relating expectations in physical space to expectations taken raywise. On the $\sigma^{-2/3}$ scale the waves in a random medium have become very wrinkled and have developed loops in their wavefronts. However, the width of these loops is small. Because of the looped structure of the wavefront, a fixed physical recording device sees different pieces of the wavefront at different instances of time. This was observed experimentally by Hesselink (9).

The simple transformation that relates statistics in physical space to statistics taken ray-wise was used in (6) for the motion of a single ray. We generalize this transformation to account for N rays.

Then we find the system of equations that describes the N point energy correlation function (see (4.3.8)). When we refer to energy in physical space, we shall mean the total wave energy passing through a point in physical space.

When the wavefront has developed loops, the total energy is the sum of the energy from each of the pieces of the wavefront that passes through the physical point (we assume that the energy adds incoherently). This is the energy that a physical device

would record.

To find the energy correlation system, we had to assume that raytube area is well approximated by the variation of vertical ray positions. It is for this derivation that we belabored the discussion of raytube areas in chapter two.

Now we specialize to the two point energy correlation function (with a plane initial wavefront). If $R_2(t,M)$ represents this correlation function for two observing points separated by a distance M, at an upstream distance of $t\sigma^{-2/3}$ then we find (see (4.4.20)):

$$U_t = 2\,(g(0)-g(M))\,U_{WW} - W\,U_M$$

$$U(0,M,W) = \delta(W)$$

$$R_2(t,M) = \int_{-\infty}^{\infty} U(t,M,W)\,dW \qquad \text{(I.3.1)}$$

Lastly, we find a short distance approximation to $R_2(t,M)$ (using two different methods) and show that $R_2(t,M)$ has a logarithmic singularity in M as $M \downarrow 0$. A short distance, on our long distance scale, now means that large amplitude fluctuations have not yet occurred. This means we are in the regime where the $\mu(\tau)$ curve is nearly flat (see Figure I.1).

Chapter five derives a system of equations that determines the two point energy correlation function after a wave has travelled through N plane layered media. For a plane initial

wavefront the resulting equations are quite simple (see (5.3.8)).

We use these results to determine the equations for the two dimensional analogue of the shadowgraph problem (see Taylor (10)). We solve these equations for a "thin" stochastic medium, and are then able to explain the experimental results of Hesselink and White (11).

Chapter six finds the generator ${}_3\mathcal{L}_N$ which describes the evolution of N rays in three dimensions. In a manner exactly analogous to chapter four we relate expectations in space to expectations taken raywise and find a system that describes the N point energy correlation function in three dimensions.

Let ${}_3R_2(t,r)$ be the two point energy correlation function when the two observing points are at an upstream distance of $t\sigma^{-2/3}$ and have a separation of r. From (6.3.19) we have:

$$P_t = 2[G''(0)-G''(r)]\, P_{U_1U_1} + 2\left[G''(0)-\frac{G'(r)}{r}\right] P_{U_2U_2} - U_1 P_r - \frac{U_2^2}{r} P_{U_1} + \frac{U_1U_2}{r} P_{U_2}$$

$$P(0,r,U_1,U_2) = \delta(U_1)\delta(U_2)$$

$${}_3R_2(t,r) = \int dU_1 \int dU_2\, P(t,r,U_1,U_2) \qquad (I.3.2)$$

where G(r) is a function obtainable from the random medium (analogous to g(M) in (I.3.1)).

Finally we find a short distance approximation to ${}_3R_2(t,r)$ (not using (I.3.2), though); and then discuss the G(r) function

(section 6.5).

In chapter seven we solve the system (I.3.1) numerically. We determine from this calculation that the regular perturbation results for $R_2(t,M)$ become invalid at the "onset of focusing" (where $\mu(\tau)$ becomes significantly different from zero, at about $\tau=.6$ on Figure I.1).

In chapter eight we describe the Monte-Carlo calculations we performed to find $R_2(t,M)$. We found good agreement between the simulation results and the numerical solution to (I.3.1), for universal times up to $t=1.4$. For universal times after the onset of focusing ($\tau=.6$) the simulation results do not agree with the regular perturbation results. We conclude that (I.3.1) correctly describes the two point energy correlation function for far longer distances than the regular perturbation method does.

The appendix contains the regular perturbation technique used to find $R_2(t,r)$ and ${}_3R_2(t,r)$. We also use this technique to solve the shadow graph problem. The solutions we obtain here are the same solutions that we obtained by taking a short distance approximation to our systems in sections 4.5, 4.6, 5.5 and 6.4.

Since a short distance approximation to our (long distance) equations produces the same answer as the regular perturbation technique; our equations are truly an extension of the linearization technique.

Finally we derive a bound on σ, which limits the applicability of the regular perturbation method. From (A.3.4) we require that

$$\sigma < \frac{.04}{\gamma_2 \Sigma^{3/2}} \qquad (I.3.3)$$

where Σ is the correlation length of the random medium. If (I.3.3) is violated, then the regular perturbation solution for $R_2(t,M)$ can not be valid.

I.4 Review Of Ito Calculus

To read this thesis, we require some familiarity with the forward Kolmogorov equation. We will often use the Ito equations associated with a given forward Kolmogorov equation. We now give a quick review.

Consider the vector Ito equation

$$d\underline{X} = \underline{b}(\underline{X},t)\, dt + \underline{\sigma}(\underline{X},t)\, d\underline{\beta} \qquad (I.4.1)$$

where $\underline{\beta}$ is a vector of standard Brownian motions, so that $E[\beta_i]=0$ and $d\beta_i d\beta_j=\delta_{ij}dt$. Corresponding to (I.4.1) is the infinitesimal generator

$$\mathcal{L} = \sum_{i,j} a_{ij}(\underline{X},t) \frac{\partial^2}{\partial X_i \partial X_j} + \sum_k b_k(\underline{X},t) \frac{\partial}{\partial X_k} \qquad (I.4.2)$$

where $\underline{\underline{a}} = \frac{1}{2}\, \underline{\underline{\sigma}}\, \underline{\underline{\sigma}}^T$.

The forward Kolmogorov equation associated with (I.4.1) is

$$\mathcal{L}^* P(t,\underline{X}) = \frac{\partial}{\partial t} P(t,\underline{X}) \qquad (I.4.3)$$

where a star represents the adjoint operator. If $P(0,\underline{X})$ is the initial probability density of the variables $\{X_i\}$, then $P(t,\underline{X})$ will be the probability density of the variables $\{X_i\}$ at time t. If the intial values of $\underline{X}$ were known exactly (say $\underline{X}(0)=\underline{Y}$) then the initial conditon for (I.4.3) is:

$$P(0,\underline{X}) = \delta(\underline{X}-\underline{Y}) = \prod_i \delta(X_i-Y_i) \qquad (I.4.4)$$

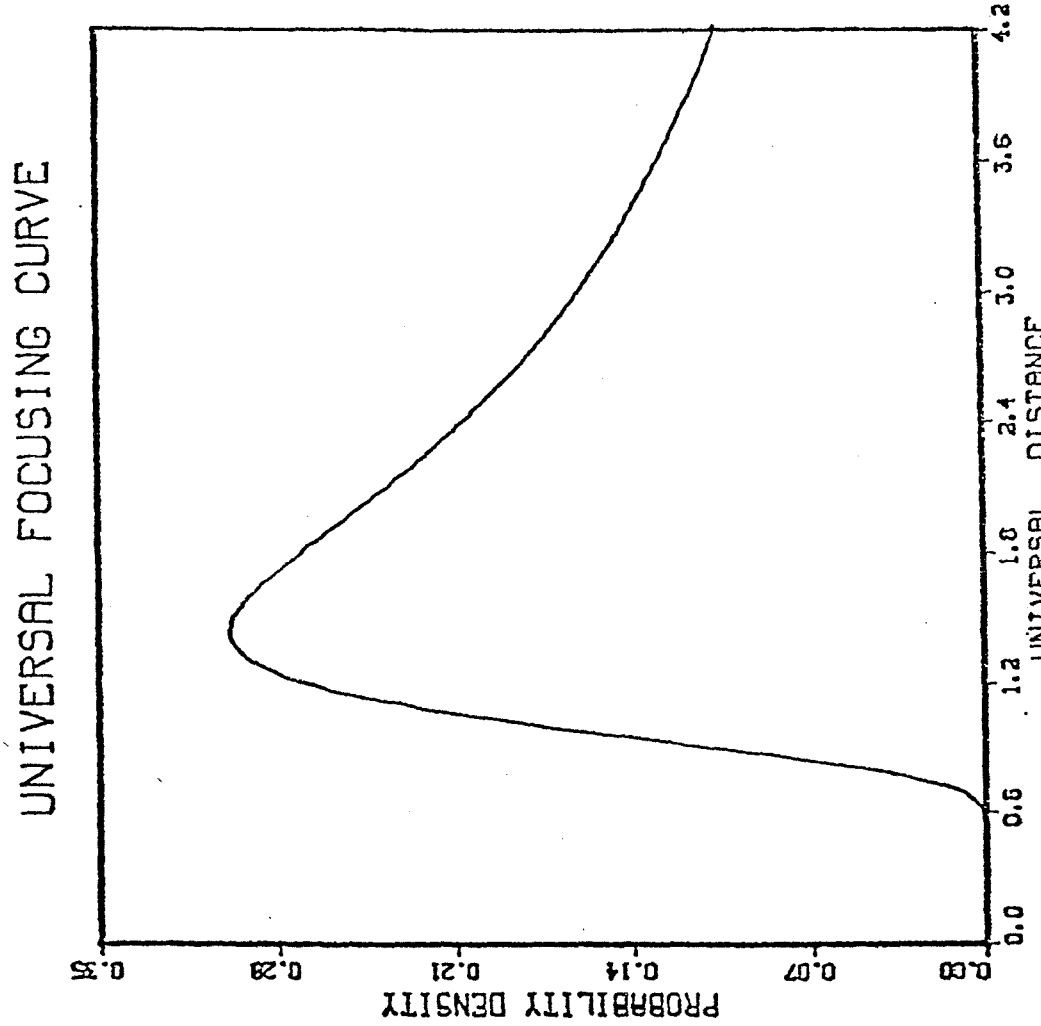
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Figure I.1

1.1 Equation Of Motion For Rays

In this section we obtain the equations of motion for rays in a random medium using the geometrical optics approximation.

We start with the wave equation

$$Q_{TT} = C^2(\underline{X})\ \Delta Q \tag{1.1.1}$$

where $C(\underline{X})$ is the local wave velocity. We represent a solution of (1.1.1) as

$$Q(\underline{X},T) = e^{iK[T+\phi(\underline{X})]}\psi(\underline{X},K) \tag{1.1.2}$$

A wavefront is a surface of constant phase, or $\phi(\underline{X})$=constant. The geometric optics assumption is to take K large and assume that Ψ has an expansion of the form:

$$\psi(\underline{X},K) \sim \sum_{m=0}^{\infty} \Phi_m(\underline{X})\ (iK)^{-m} \tag{1.1.3}$$

Using (1.1.2), (1.1.3) in (1.1.1) and equating the coefficients of powers of K yields:

$$O(K^2): \quad \Phi_0 = C^2\ \Phi_0 |\nabla\phi|^2 \tag{1.1.4}$$

$$O(K): \quad \Phi_1 = C^2[\Phi_1 |\nabla\phi|^2 + \{2\nabla\phi\cdot\nabla\Phi_0 + \Phi_0\Delta\phi\}] \tag{1.1.5}$$

We now assume that:

$$\Phi_0 \neq 0 \tag{1.1.6}$$

$$C(\underline{X}) \neq 0 \tag{1.1.7}$$

Using (1.1.6), (1.1.7) in (1.1.4), (1.1.5) we find the eikonal equation:

$$C^2(X) \ |\nabla\phi|^2 = 1 \tag{1.1.8}$$

and the transport equation:

$$2\nabla\phi \cdot \nabla\Phi_0 + \Phi_0 \Delta\phi = 0 \tag{1.1.9}$$

we define

$$P_i = \frac{\partial \phi}{\partial X_i} \tag{1.1.10}$$

and rewrite (1.1.8) using (1.1.10) as

$$\frac{1}{2} C(X) \ P_i P_i - \frac{1}{2} \ \frac{1}{C(\underline{X})} = 0 \tag{1.1.11}$$

We solve (1.1.11) by the method of characteristics. The characteristic equations corresponding to (1.1.11) are:

$$\frac{dX_i}{ds} = C(\underline{X}) \ P_i$$

$$\frac{dP_i}{ds} = - \frac{C_{X_i}(\underline{X})}{C(\underline{X})}$$

$$\frac{d\phi}{ds} = \frac{1}{C(\underline{X})} \tag{1.1.12}$$

We define $\underline{X}(s,\underline{\alpha})$ to be the position of a ray which, at s=0, started from a position $\underline{X}(0,\underline{\alpha})$. The variable $\underline{\alpha}$ is used to parametrize the inital wavefront.

We consider a continuum of rays leaving the initial wavefront. Later we will consider only N rays, and specify $\underline{\alpha}_1$, $\underline{\alpha}_2, \cdots, \underline{\alpha}_N$.

We assume we are working in a three dimensional rectangular coordinate system. The projection of $\underline{X}(s,\underline{\alpha})$ along the three coordinate axes is given by

$$\underline{X}(s,\underline{\alpha}) = (\underline{X}_1(s,\underline{\alpha}), \underline{X}_2(s,\underline{\alpha}), \underline{X}_3(s,\underline{\alpha}))^T \tag{1.1.13}$$

We also define

$$\underline{P}(s,\underline{\alpha}) = (\underline{P}_1(s,\underline{\alpha}), \underline{P}_2(s,\underline{\alpha}), \underline{P}_3(s,\underline{\alpha}))^T \tag{1.1.14}$$

$$\begin{aligned} \underline{V}(s,\underline{\alpha}) &= C(\underline{X}(s,\underline{\alpha}))\ \underline{P}(s,\underline{\alpha}) \\ &= (\underline{V}_1(s,\underline{\alpha}), \underline{V}_2(s,\underline{\alpha}), \underline{V}_3(s,\underline{\alpha}))^T \end{aligned} \tag{1.1.15}$$

From (1.1.12), (1.1.13), (1.1.14), (1.1.15) we find

$$\frac{d}{ds}\ \underline{X}(s,\underline{\alpha}) = C(\underline{X})\ \underline{P}(s,\underline{\alpha}) = \underline{V}(s,\underline{\alpha})$$

$$\frac{d}{ds}\ \underline{P}(s,\underline{\alpha}) = -\frac{\nabla C(\underline{X})}{C^2(\underline{X})}$$

$$\frac{d}{ds}\,\phi(s,\underline{\alpha}) = \frac{1}{C(\underline{X})} \tag{1.1.16}$$

From (1.1.8), (1.1.10), (1.1.15a) we have

$$1=C^2|\nabla\phi|^2= C^2|\underline{P}|^2=|\underline{V}|^2 \tag{1.1.17}$$

So we see that s represents arc-length along a ray and $\underline{V}(s,\underline{\alpha})$ is the unit tangent to $\underline{X}(s,\underline{\alpha})$.

From (1.1.16a) we have

$$\frac{d}{ds}\,C(\underline{X}(s,\underline{\alpha})) = \frac{d\underline{X}}{ds}\cdot\nabla C(\underline{X}) = \underline{V}\cdot\nabla C(\underline{X}) \tag{1.1.18}$$

Using (1.1.15a), (1.1.16b), (1.1.18) we compute

$$\frac{d}{ds}\,\underline{V} = \frac{-\nabla C+\underline{V}(\underline{V}\cdot\nabla C)}{C} \tag{1.1.19}$$

Equations (1.1.16a), (1.1.16c), (1.1.19) are the equations of motion for the position, velocity and phase of a ray in a random medium.

1.2 Raytube Area in Two Dimensions

Here we obtain the equations of motion for raytube area in two dimensions.

In this section we restrict ourselves to a two dimensional geometry. We define $\underline{\alpha}=\alpha$ to be arc-length along the initial wavefront. We define the unit vector perpendicular to $\underline{V}(s,\underline{\alpha})$,

$\underline{V}^{\perp}(s,\underline{\alpha})$, by

$$\underline{V}^{\perp}(s,\alpha) = \underline{\underline{M}}\ \underline{V}(s,\alpha) \tag{1.2.1}$$

where

$$\underline{\underline{M}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.2.2}$$

From (1.2.1), (1.2.2) we find

$$\underline{V}^{\perp} \cdot \underline{V}^{\perp} = 1$$

$$\underline{V}^{\perp} \cdot \underline{V} = 0 \tag{1.2.3}$$

Because α is arc-length we have

$$\frac{\partial}{\partial \alpha}\ \underline{X}(0,\alpha) = \underline{V}^{\perp}(0,\alpha) \tag{1.2.4}$$

We define τ by:

$$\frac{d}{ds} = \frac{1}{C}\ \frac{d}{d\tau} \tag{1.2.5}$$

Using (1.2.5) in (1.1.16c)

$$\frac{d}{d\tau}\ \phi(s,\alpha) = 1 \tag{1.2.6}$$

Therefore a value of τ specifies the location of a

wavefront. Since α is arc-length along the wavefront, α and τ are independent variables.

We write (1.1.16a), (1.1.19) using (1.2.5) as

$$\frac{d}{d\tau}\,\underline{X} = C\,\underline{V} \tag{1.2.7}$$

$$\frac{d}{d\tau}\,\underline{V} = -\nabla C+\underline{V}(\underline{V}\cdot\nabla C) = -\underline{V}^{\perp}(\underline{V}^{\perp}\cdot\nabla C) \tag{1.2.8}$$

From (1.2.2) and (1.2.8) we have

$$\frac{d}{d\tau}\,\underline{V}^{\perp} = \underline{V}\,(\underline{V}^{\perp}\cdot\nabla C) \tag{1.2.9}$$

We differentiate (1.2.7), (1.2.8) with respect to α for

$$\frac{d}{d\tau}\,\underline{X}_{\alpha} = (\nabla C\cdot\underline{X}_{\alpha})\underline{V} + C\underline{V}_{\alpha} \tag{1.2.10}$$

$$\frac{d}{d\tau}\,\underline{V}_{\alpha} = -\nabla\nabla C\underline{X}_{\alpha}+\underline{V}_{\alpha}(\underline{V}\cdot\nabla C)+\underline{V}[\underline{V}\cdot\nabla C]_{\alpha} \tag{1.2.11}$$

Now we compute (using (1.2.10), (1.2.8), (1.1.17))

$$\frac{d}{d\tau}\,(\underline{X}_{\alpha}\cdot\underline{V}) = (\underline{X}_{\alpha}\cdot\underline{V})\,(\underline{V}\cdot\nabla C) \tag{1.2.12}$$

From (1.2.3), (1.2.4) we have

$$\underline{X}_{\alpha}(0,\alpha)\cdot\underline{V}(0,\alpha) = 0 \tag{1.2.13}$$

The solution to (1.2.12) with (1.2.13) is

$$\underline{X}_\alpha(s,\alpha)\cdot\underline{V}(s,\alpha) = 0 \qquad (1.2.14)$$

So we see that $\underline{X}_\alpha$ has a component only in the direction of $\underline{V}^\perp$. To find this component we compute (using (1.2.10), (1.2.8), (1.1.17))

$$\frac{d}{d\tau}\left(\underline{X}_\alpha\cdot\underline{V}^\perp\right) = C\left(\underline{V}_\alpha\cdot\underline{V}^\perp\right) \qquad (1.2.15)$$

We define

$$A(s,\alpha) = \underline{X}_\alpha(s,\alpha)\cdot\underline{V}^\perp(s,\alpha)$$

$$B(s,\alpha) = \underline{V}_\alpha(s,\alpha)\cdot\underline{V}^\perp(s,\alpha) \qquad (1.2.16)$$

In two dimensions, the distance between two rays, divided by their initial separation, in the limit of an infinitesimal initial separation, is defined to be the area of the raytube associated with the bounding rays. Hence, the area is defined to be $|\underline{X}_\alpha(s,\alpha)|$. From (1.2.13) and (1.2.16) we have

$$\underline{X}_\alpha = A\,\underline{V}^\perp$$

$$\underline{V}_\alpha = B\,\underline{V}^\perp \qquad (1.2.17)$$

and so

$$|A(s,\alpha)| = |\underline{X}_\alpha(s,\alpha)| \qquad (1.2.18)$$

We therefore recognize $A(s,\alpha)$ as the signed raytube area corresponding to the ray $\underline{X}(s,\alpha)$. When a focus occurs, and adjacent rays intersect, the sign of the area changes. This is how $A(s,\alpha)$ can be negative as well as positive. This corresponds to the phase jump described by Sommerfeld ((12), page 318).

We find from (1.2.3), (1.2.4), (1.2.16)

$$A(0,\alpha) = 1 \tag{1.2.19}$$

From (1.2.5), (1.2.7) and (1.2.16) we have

$$\frac{d}{ds} A(s,\alpha) = B(s,\alpha) \tag{1.2.20}$$

A propagation equation for $B(s,\alpha)$ can be found by differentiating (1.2.16b) with respect to τ and using (1.2.3), (1.2.8), (1.2.9), (1.1.17). The result is

$$\frac{dB}{d\tau} = \underline{V}^{\perp} \{-\nabla\nabla C \, \underline{X}_{\alpha} + \underline{V}_{\alpha}(\underline{V}\cdot\nabla C)\} \tag{1.2.21}$$

Using (1.2.5), (1.2.17) in (1.2.21) we obtain

$$\frac{dB}{ds} = -\frac{A}{C} (\underline{V}^{\perp})^{T} \nabla\nabla C \underline{V} + \frac{B}{C} (\underline{V}\cdot\nabla C) \tag{1.2.22}$$

This is the final form of the propagation equation for B.

We now interpret $B(0,\alpha)$ as the initial curvature of the

wavefront. The first of the Frenet-Serret formulae is

$$\underline{V}_{\alpha}(0,\alpha) = \kappa(\alpha)\, \underline{V}^{\perp}(0,\alpha) \qquad (1.2.23)$$

where $\kappa(\alpha)$ is the curvature of the initial wavefront, at a position of $\underline{X}(0,\alpha)$. Comparing (1.2.17) (at s=0) with (1.2.23) we conclude

$$B(0,\alpha) = \kappa(\alpha) \qquad (1.2.24)$$

So the initial value of $B(s,\alpha)$ is just the curvature of the initial wavefront.

We have found, in (1.2.20) and (1.2.22), the equations of motion for raytube area. The initial conditions for these equations are (1.2.19) and (1.2.24).

1.3 Energy Associated With a Ray

Here we will find an expression for energy along a ray.

We define the energy, $\mathcal{E}(\underline{X})$ of the wave to be the squared magnitude of the field intensity. From (1.1.1) therefore

$$\mathcal{E}(\underline{X}) = |Q^2(\underline{X},T)| \qquad (1.3.1)$$

To leading order in K (see (1.1.2), (1.1.3)) we have

$$\mathcal{E}(\underline{X}) = \Phi_0(\underline{X})^2 \qquad (1.3.2)$$

If we multiply (1.1.9) by Φ_0 we find

$$\nabla\cdot(\Phi_0{}^2\nabla\phi)=0 \tag{1.3.3}$$

Now consider a tube of rays going from the wavefront at $\tau=0$ to the wavefront at $\tau=T$ (recall (1.2.5)). We define the intersection of the raytube with the wavefront at $\tau=0$ ($\tau=T$) to be S_0 (S_T). The sides of the raytube are defined to be Σ.

From (1.2.6), ϕ=constant is the equation of a wavefront, so $\nabla\phi$ is normal to the wavefronts. Now we integrate (1.3.3) over the volume of the raytube and use the divergence theorem to find:

$$0=\int_V \nabla\cdot(\Phi_0{}^2\,\nabla\phi)\,dV = \int_S \Phi_0{}^2(\underline{n}\cdot\nabla\phi)\,dS \tag{1.3.4}$$

where $\underline{n}$ is the outward normal to the surface of the tube of rays. Because the rays are parallel to $\nabla\phi$:

$$\underline{n}\cdot\nabla\phi=0 \text{ on } \Sigma \tag{1.3.5}$$

On S_0 (S_T) the rays and $\underline{n}$ are in opposite (the same) directions so

$$\underline{n}\cdot\nabla\phi = -|\nabla\phi| \text{ on } S_0$$

$$\underline{n}\cdot\nabla\phi = |\nabla\phi| \text{ on } S_T \tag{1.3.6}$$

From (1.1.8) we can write (1.3.6) as

$$\underline{n}\cdot\nabla\phi = -1/C \text{ on } S_0$$

$$\underline{n}\cdot\nabla\phi = 1/C \text{ on } S_T \qquad (1.3.7)$$

Using (1.3.5) and (1.3.7) in (1.3.4)

$$-\int_{S_0} \frac{1}{C}\,\Phi_0^{\,2}\,dS + \int_{S_T} \frac{1}{C}\,\Phi_0^{\,2}\,dS = 0 \qquad (1.3.8)$$

For a very narrow tube of rays, (1.3.8) becomes

$$\frac{\Phi_0^{\,2}(\underline{X}_0)}{C(\underline{X}_0)}\,|S_0| = \frac{\Phi_0^{\,2}(\underline{X}_T)}{C(\underline{X}_T)}\,|S_T| \qquad (1.3.9)$$

where $|S_0|$ ($|S_T|$) is the area of S_0 (S_T). In the limit of $|S_0|$, $|S_T| \to 0$ we have

$$\frac{d\,|S(\underline{X})|}{d\,|S(\underline{X}_0)|} = \frac{\Phi_0^{\,2}(\underline{X}_0)}{\Phi_0^{\,2}(\underline{X})}\,\frac{C(\underline{X})}{C(\underline{X}_0)} = \frac{\mathcal{E}(\underline{X}_0)\,C(\underline{X})}{\mathcal{E}(\underline{X})\,C(\underline{X}_0)} \qquad (1.3.10)$$

where we have used (1.3.2). Equation (1.3.10) is valid in two or three dimensions. We now restrict ourselves to two dimensions.

Let β represent arc-length on S_T. The ratio $d|S|/d|S_0|$ now becomes $d\beta/d\alpha$ (recall α is arc-length on S_0). We compute from (1.2.18)

$$A^2 = \left|\frac{\partial \underline{X}}{\partial \alpha}\right|^2 = \left|\frac{\partial \underline{X}}{\partial \beta}\,\frac{\partial \beta}{\partial \alpha}\right|^2 = \left(\frac{\partial \beta}{\partial \alpha}\right)^2 \qquad (1.3.11)$$

where $\left|\frac{\partial \underline{X}}{\partial \beta}\right| = 1$ because β is arc-length. Therefore (1.3.10) becomes

$$\pm A = \frac{\mathcal{E}(\underline{X}_0)}{\mathcal{E}(\underline{X})} \frac{C(\underline{X})}{C(\underline{X}_0)}$$

Since $\mathcal{E} > 0$ and $C > 0$ we invert (1.3.11) and write it as

$$\frac{\mathcal{E}(\underline{X})}{\mathcal{E}(\underline{X}_0)} = \frac{C(\underline{X})}{C(\underline{X}_0)} \frac{1}{|A|} \qquad (1.3.12)$$

At the occurrence of a focus, A changes sign, hence the absolute value of A is needed in (1.3.12). As A passes through zero, $\mathcal{E}$ becomes infinite. This is due to the breakdown of geometric optics at a focus. What really happens at the focal points is that diffraction effects become important, and keep the energy finite.

Equation (1.3.12) describes energy propagation along a ray as a function of A and C.

1.4 Scaling of Two Dimensional Equations

We now restrict ourselves to a two dimensional coordinate system with basis vectors $(\underline{i}, \underline{j})$. We define $f(X_1, X_2) = f(X_1 \underline{i} + X_2 \underline{j})$ for any space-varying function f.

We assume that the velocity field C(X) is weakly inhomogeneous. That is, we assume C(X) can be represented as

$$C(\underline{X}) = C_0 (1+\sigma \hat{C}(\underline{X})) \qquad (1.4.1)$$

where $\hat{C}(\underline{X})$ is mean zero and

$$0<\sigma<<1 \tag{1.4.2}$$

$\hat{C}$ and all of its derivatives are bounded (1.4.3)

We define the initial wavefront by

$$\underline{X}(0,\alpha) = \alpha\, \underline{j} - \sigma^{2/3} h(\alpha)\, \underline{i} + O(\sigma^{4/3}) \tag{1.4.4}$$

where $h(\alpha)$ is arbitrary but bounded, and has bounded derivatives. We assume that the derivatives of the $O(\sigma^{4/3})$ term in (1.4.4) are also of order $O(\sigma^{4/3})$.

What (1.4.4) represents is a slighty wavy initial wavefront. If $h(\alpha)=0$, then the wavefront defined by (1.4.4) is planar.

From $\underline{X}(0,\alpha)$ we can compute $\underline{V}(0,\alpha)$, $\underline{V}^{\perp}(0,\alpha)$ and $B(0,\alpha)$ using (1.2.4), (1.2.1), (1.2.16). We find

$$\underline{V}^{\perp}(0,\alpha) = \underline{X}_{\alpha}(0,\alpha) = \underline{j} - \sigma^{2/3} h'(\alpha)\, \underline{i} + O(\sigma^{4/3}) \tag{1.4.5}$$

$$\underline{V}(0,\alpha) = -\underline{\underline{M}}\, \underline{V}\,(0,\alpha) = \underline{i} + \sigma^{2/3} h'(\alpha)\, \underline{j} + O(\sigma^{4/3}) \tag{1.4.6}$$

$$B(0,\alpha) = \underline{V}_{\alpha}(0,\alpha)\cdot\underline{V}^{\perp}(0,\alpha) = \sigma^{2/3} h''(\alpha) + O(\sigma^{4/3}) \tag{1.4.7}$$

From (1.2.24), (1.4.7) we see that the initial wavefront has

a curvature of order $O(\sigma^{2/3})$.

Now we investigate the two dimensional equations on a long length scale, t, defined by

$$t=\sigma^{2/3}s \tag{1.4.8}$$

We scale $\underline{X}$, $\underline{V}$, A, B to form the new variables

$$\underline{X}^{\sigma}(t,\alpha) = \underline{X}(s,\alpha) - s\,\underline{i}$$

$$\underline{V}^{\sigma}(t,\alpha) = \frac{\underline{V}(s,\alpha) - \underline{i}}{\sigma^{2/3}}$$

$$A^{\sigma}(t,\alpha) = A(s,\alpha)$$

$$B^{\sigma}(t,\alpha) = \frac{B(s,\alpha)}{\sigma^{2/3}} \tag{1.4.9}$$

If $\sigma=0$ in (1.4.1) so the medium is uniform, then we would find from (1.1.16a), (1.1.19) and (1.4.4), (1.4.6):

$$\underline{X}(s,\alpha) = s\,\underline{i}$$

$$\underline{V}(s,\alpha) = \underline{i} \tag{1.4.10}$$

That is, a ray would travel in a straight line along the $\underline{i}$ axis. The change of variable in (1.4.9) centers $\underline{X}$, $\underline{V}$ by removing this leading order value.

We note from (1.4.9b)

$$|\underline{v}^{\sigma}| < \frac{2}{\sigma^{2/3}} \tag{1.4.11}$$

We now use (1.4.1), (1.4.8), (1.4.9), (1.4.11) on our two dimensional equations of motion: (1.1.16a), (1.1.19), (1.2.20), (1.2.22).

From (1.1.16b), (1.2.20) we find

$$\frac{d}{dt}\,\underline{x}^{\sigma} = \underline{v}^{\sigma} \tag{1.4.12}$$

$$\frac{d}{dt}\,A^{\sigma} = B^{\sigma} \tag{1.4.13}$$

From (1.1.19) we have

$$\sigma^{2/3}\,\frac{d}{dt}\,(\sigma^{2/3}\underline{v}^{\sigma}+\underline{i}) = \frac{1}{C}\{\,-\nabla C+(\underline{i}+\sigma^{2/3}\underline{v}^{\sigma})[\underline{i}+\sigma^{2/3}\underline{v}^{\sigma}]\cdot\nabla C\,\}$$

$$\frac{d}{dt}\,\underline{v}^{\sigma} = \frac{1}{\sigma^{1/3}(1+\sigma\hat{C})}\,[\,-\underline{j}\,(\underline{j}\cdot\nabla\hat{C}\,)$$

$$+O(\sigma^{2/3}|\underline{v}^{\sigma}|+\sigma^{4/3}|\underline{v}^{\sigma}|^{2})\,]$$

$$\frac{d}{dt}\,\underline{v}^{\sigma} = -\,\frac{\underline{j}}{\sigma^{1/3}}\,\hat{C}_{x_2} + O(\sigma^{2/3}|\underline{v}^{\sigma}|) \tag{1.4.14}$$

From (1.2.22) we obtain:

$$\sigma^{4/3}\,\frac{dB^{\sigma}}{dt} = -\,\frac{1}{C}\,(\underline{M}\{\sigma^{2/3}\underline{v}^{\sigma}+\underline{i}\}^{T}\nabla\nabla C(\underline{M}\,\{\sigma^{2/3}\underline{v}^{\sigma}+\underline{i}\}\,)$$

$$+\,\frac{\sigma^{2/3}B^{\sigma}}{C}\,(\sigma^{2/3}\underline{v}^{\sigma}+\underline{i})\cdot\nabla C$$

$$\frac{dB^\sigma}{dt} = -\frac{A^\sigma}{\sigma^{1/3}}\hat{C}_{X_2X_2} + O(\sigma^{1/3}[|B^\sigma|+|A^\sigma||V^\sigma|]+\sigma^{2/3}|A^\sigma|] \qquad (1.4.15)$$

Now we decompose $\underline{X}^\sigma$ and $\underline{V}^\sigma$ into scalar functions along the coordinate axes by

$$\underline{X}^\sigma(s,\alpha) = X_1^\sigma(s,\alpha)\,\underline{i} + X_2^\sigma(s,\alpha)\,\underline{j}$$

$$\underline{V}^\sigma(s,\alpha) = V_1^\sigma(s,\alpha)\,\underline{i} + V_2^\sigma(s,\alpha)\,\underline{j} \qquad (1.4.16)$$

The argument to $\hat{C}$ in (1.4.14), (1.4.15) is $\underline{X}(s,\alpha)$. Using (1.4.9a), (1.4.16) we can write

$$\hat{C}(\underline{X}) = \hat{C}(\underline{X}^\sigma(t) + \frac{t}{\sigma^{2/3}}\,\underline{i}) = \hat{C}(\,X_1^\sigma(t) + \frac{t}{\sigma^{2/3}},\, X_2^\sigma(t)\,) \qquad (1.4.17)$$

Now we use (1.4.16), (1.4.17) in (1.4.12)-(1.4.15) to find

$$\frac{d}{ds}\begin{pmatrix} X_1^\sigma \\ X_2^\sigma \\ V_1^\sigma \\ V_2^\sigma \\ A^\sigma \\ B^\sigma \end{pmatrix} = \begin{pmatrix} V_1^\sigma \\ V_2^\sigma \\ 0 + O(\sigma^{2/3}) \\ -\frac{1}{\sigma^{1/3}}\hat{C}_{X_2}\left(\frac{t}{\sigma^{2/3}} + X_1^\sigma,\ X_2^\sigma\right) + O(\sigma^{2/3}) \\ B^\sigma \\ -\frac{A^\sigma}{\sigma^{1/3}}\hat{C}_{X_2X_2}\left(\frac{t}{\sigma^{2/3}} + X_1^\sigma,\ X_2^\sigma\right) + O(\sigma^{2/3}) \end{pmatrix} \tag{1.4.18}$$

Equation (1.4.18) has the final form of the two dimensional equations of motion. To find the initial conditions for (1.4.18) we use (1.4.9), (1.4.16) in (1.2.19), (1.4.4), (1.4.6), (1.4.7) to find

$$\begin{pmatrix} X_1^\sigma \\ X_2^\sigma \\ V_1^\sigma \\ V_2^\sigma \\ A^\sigma \\ B^\sigma \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} 0 + O(\sigma^{2/3}) \\ \alpha + O(\sigma^{2/3}) \\ 0 + O(\sigma^{2/3}) \\ h'(\alpha) + O(\sigma^{2/3}) \\ 1 \\ h''(\alpha) + O(\sigma^{2/3}) \end{pmatrix} \tag{1.4.19}$$

1.5 Scaling of Three Dimensional Equations

In this section we scale the three dimensional equations of motion (1.1.16a), (1.1.19).

We take a three dimensional rectangular coordinate system with basis vectors ($\underline{i}$, $\underline{j}$, $\underline{k}$). We define $f(X_1, X_2, X_3) = f(X_1\underline{i} + X_2\underline{j} + X_3\underline{k})$ for any space varying function f.

We now need two scalars to parametrize the initial wavefront: $\underline{\alpha}=(\alpha_1, \alpha_2)$. We define the initial wavefront by

$$\underline{X}(0,\underline{\alpha}) = \alpha_2 \underline{j} + \alpha_3 \underline{k} - \sigma^{2/3}h(\alpha_2,\alpha_3)\underline{i} + O(\sigma^{4/3}) \qquad (1.5.1)$$

where $h(\alpha_2,\alpha_3)$ is arbitrary but bounded, and has bounded derivatives. We assume that the $\underline{\alpha}$ derivatives of the $O(\sigma^{4/3})$ term in (1.5.1) are also of order $O(\sigma^{4/3})$.

These conditions insure that the initial wavefront has only $O(\sigma^{2/3})$ deviations from a plane wave. If $h(\alpha_2, \alpha_3)=0$ then the initial wavefront is planar.

The vector $\underline{V}(0,\underline{\alpha}) = a\underline{i}+b\underline{j}+c\underline{k}$ is of unit magnitude and is normal to the wavefront.

If we define

$$\underline{T}_2 = \frac{\partial \underline{X}(0,\underline{\alpha})}{\partial \alpha_2} = \underline{j} - \sigma^{2/3}h_{\alpha_2}\underline{i} + O(\sigma^{4/3})$$

$$\underline{T}_3 = \frac{\partial \underline{X}(0,\underline{\alpha})}{\partial \alpha_3} = \underline{k} - \sigma^{2/3}h_{\alpha_3}\underline{i} + O(\sigma^{4/3}) \qquad (1.5.2)$$

then

$$\underline{V}(s,\underline{\alpha})\cdot\underline{T}_2 = 0 = b - \sigma^{2/3}ah_{\alpha_2} + 0(\sigma^{4/3})$$

$$\underline{V}(s,\underline{\alpha})\cdot\underline{T}_3 = 0 = c - \sigma^{2/3}ah_{\alpha_3} + 0(\sigma^{4/3}) \qquad (1.5.3)$$

If we take

$$a^2 + b^2 + c^2 = 1 \qquad (1.5.4)$$

so that $|\underline{V}(0,\underline{\alpha})|=1$ then we can solve (1.5.3), (1.5.4) for

$$a = 1 + 0(\sigma^{4/3})$$

$$b = \sigma^{2/3}h_{\alpha_2} + 0(\sigma^{4/3})$$

$$c = \sigma^{2/3}h_{\alpha_3} + 0(\sigma^{4/3}) \qquad (1.5.5)$$

So that

$$\underline{V}(0,\underline{\alpha}) = \underline{i} + \sigma^{2/3}(h_{\alpha_2}\,\underline{j} + h_{\alpha_3}\,\underline{k}) + 0(\sigma^{2/3}) \qquad (1.5.6)$$

We now take

$$C(\underline{X}) = C_0\,(1+\sigma\,\hat{C}(\underline{X})) \qquad (1.5.7)$$

and assume (1.4.2), (1.4.3). We define new variables by

$$t = \sigma^{2/3} s$$

$$\underline{X}^{\sigma}(t,\underline{\alpha}) = \underline{X}(s,\underline{\alpha}) - s\underline{i}$$

$$\underline{V}^{\sigma}(t,\underline{\alpha}) = \frac{\underline{V}(s,\underline{\alpha}) - \underline{i}}{\sigma^{2/3}} \qquad (1.5.8)$$

We now use (1.5.7), (1.5.8) in (1.1.16a), (1.1.19). Equation (1.1.16a) becomes

$$\frac{d}{dt}\underline{X}^{\sigma} = \underline{V}^{\sigma} \qquad (1.5.9)$$

Equation (1.1.19) becomes

$$\sigma^{4/3}\frac{d}{dt}\underline{V}^{\sigma} = \frac{1}{C}\left[-\nabla C + (\underline{V}^{\sigma}\sigma^{2/3} + \underline{i})\left|(\underline{V}^{\sigma}\sigma^{2/3} + \underline{i})\cdot\nabla C\right.\right]$$

$$\frac{d}{dt}V^{\sigma} = \frac{1}{\sigma^{1/3}}\left[-\hat{C}_{X_2}\,\underline{j} - \hat{C}_{X_3}\,\underline{k}\right] + O(\sigma^{1/3}|V^{\sigma}| + \sigma^{2/3}) \qquad (1.5.10)$$

We decompose $\underline{X}^{\sigma}$ and $\underline{V}^{\sigma}$ into scalar functions along the coordinate axes by:

$$\underline{X}^{\sigma}(t,\underline{\alpha}) = X_1^{\sigma}(t,\underline{\alpha})\,\underline{i} + X_2^{\sigma}(t,\underline{\alpha})\,\underline{j} + X_3^{\sigma}(t,\underline{\alpha})\,\underline{k}$$

$$\underline{V}^{\sigma}(t,\underline{\alpha}) = V_1^{\sigma}(t,\underline{\alpha})\,\underline{i} + V_2^{\sigma}(t,\underline{\alpha})\,\underline{j} + V_3^{\sigma}(t,\underline{\alpha})\,\underline{k} \qquad (1.5.11)$$

The $\hat{C}$ function in (1.5.10) is a function of $\underline{X}$. We can write this as:

$$\hat{C}(\underline{X}) = \hat{C}(\underline{X}^{\sigma} + \frac{t}{\sigma^{2/3}}\underline{i}) = \hat{C}(\frac{t}{\sigma^{2/3}} + X_1^{\sigma}, X_2^{\sigma}, X_3^{\sigma}) \qquad (1.5.12)$$

Using (1.5.11), (1.5.12) in (1.5.9), (1.5.10) we obtain:

$$\frac{d}{dt}\begin{pmatrix} X_1^{\sigma} \\ X_2^{\sigma} \\ X_3^{\sigma} \\ V_1^{\sigma} \\ V_2^{\sigma} \\ V_3^{\sigma} \end{pmatrix} = \begin{pmatrix} V_1^{\sigma} \\ V_2^{\sigma} \\ V_3^{\sigma} \\ 0 \\ -\sigma^{-1/3}\hat{C}_{X_2}(t\sigma^{-2/3}+X_1^{\sigma}, X_2^{\sigma}, X_3^{\sigma}) + \eta \\ -\sigma^{-1/3}\hat{C}_{X_3}(t\sigma^{-2/3}+X_1^{\sigma}, X_2^{\sigma}, X_3^{\sigma}) + \eta \end{pmatrix} \qquad (1.5.13)$$

where $\eta = O(\sigma^{1/3}|\underline{V}^{\sigma}| + \sigma^{2/3})$. Equation (1.5.13) has the final form of the three dimensional equations of motion. We obtain the initial conditions for (1.5.13) by using (1.5.11) in (1.5.1) and (1.5.6). We find

$$\begin{pmatrix} X_1^{\sigma} \\ X_2^{\sigma} \\ X_3^{\sigma} \\ V_1^{\sigma} \\ V_2^{\sigma} \\ V_3^{\sigma} \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} 0 + O(\sigma^{2/3}) \\ \alpha_2 + O(\sigma^{2/3}) \\ \alpha_3 + O(\sigma^{2/3}) \\ 0 + O(\sigma^{2/3}) \\ h_{\alpha_2}(\alpha_2,\alpha_3) + O(\sigma^{2/3}) \\ h_{\alpha 3}(\alpha_2,\alpha_3) + O(\sigma^{2/3}) \end{pmatrix} \qquad (1.5.14)$$

2.1 Papanicolaou and Kohler Limit Theorem

In this section we state the Papanicolaou and Kohler (8) limit theorem.

Define $C^{k,p}(R^N)$ to be the collection of functions on R^N with continuous derivatives up to order k for which there exists an integer $p \geq 0$ with

$$\left| \frac{\partial^{\alpha} f(\underline{X})}{\partial X_1^{\alpha_1} \ldots \partial X_N^{\alpha_N}} \right| \leq C \ (1+|\underline{X}|^p)$$

$$0 \leq \alpha_1 + \alpha_2 + \ldots + \alpha_N = \alpha \leq k \qquad (2.1.1)$$

Consider the vector ordinary differential equation:

$$\frac{d}{d\tau} \underline{X}^{(\varepsilon)}(\tau,\sigma,\underline{X}) = \frac{1}{\varepsilon} \underline{F}(\tau, \underline{X}^{(\varepsilon)}, \frac{\tau}{\varepsilon^2}) + \underline{G}(\tau, \underline{X}^{(\varepsilon)}, \frac{\tau}{\varepsilon^2})$$

$$0 \leq \sigma < \tau \leq T \ , \quad \varepsilon \in (0,1]$$

$$\underline{X}^{(\varepsilon)}(\sigma,\sigma,\underline{X}) = \underline{X} \in R^N \qquad (2.1.2)$$

If three conditions hold then the process $X^{(\varepsilon)}$ (the solution of (2.1.2)) converges weakly as $\varepsilon \to 0$ to a diffusion Markov process with infinitesimal generator (backward operator), $\mathcal{L}_\sigma$:

$$\mathcal{L}_\sigma = \sum_{i,j=1}^{N} a^{ij}(\sigma,\underline{X}) \frac{\partial^2}{\partial X_i \partial X_j} + \sum_{k=1}^{N} b^k(\sigma,\underline{X}) \frac{\partial}{\partial X_k}$$

$$\sigma \in [0,T] \qquad (2.1.3)$$

where $\mathcal{L}_\sigma$ is defined on $C^{2,P}(R^N)$.

We now specify the three conditions and give the definitions for a^{ij} and b^i.

Condition I: Mixing

Let $(\Omega, \ , P)$ be a probability space with

$$\Omega = \left\{ \begin{array}{l} \text{all continuous vector valued functions} \\ F(\tau,X,t): [0,T] \times R^N \times [0,\infty) \to R^N \end{array} \right\} \quad (2.1.4)$$

Let $\mathcal{F}_s^t$ be a family of sub-sigma algebras, contained in Ω and defined for $0 \le s \le t \le \infty$, such that

$$\mathcal{F}_{s_1}^{t_1} \subset \mathcal{F}_{s_2}^{t_2} \quad \text{for} \quad 0 \le s_2 \le s_1 \le t_1 \le t_2 \le \infty \quad (2.1.5)$$

The mixing rate $\rho(t)$ is defined by

$$\rho(t) = \sup_{s \ge 0} \sup_{\substack{A \in \mathcal{F}_{t+s}^{\infty} \\ B \in \mathcal{F}_0^s}} \left| P(A|B) - P(A) \right| \quad (2.1.6)$$

We require $\rho(t)$ to satisfy the rate condition:

$$\int_0^{\infty} \sqrt{\rho(t)}\, dt < \infty \quad (2.1.7)$$

We denote integration over Ω relative to P by $E[.]$.

The variables $\underline{F}$ and $\underline{G}$ are elements of Ω, however, we will

not always include ω in the argument lists of $\underline{F}$ and $\underline{G}$.

Condition II: Conditions on F and G

We require

$$E[\ \underline{F}(\tau,\underline{X},t)\ ] = 0 \tag{2.1.8}$$

The following conditions apply to $\underline{F}$ and $\underline{G}$, we state then just for $\underline{F}$.

$\underline{F}$ must be jointly measurable with respect to its arguments. For fixed τ, $\underline{X}$, t, $\underline{F}(\tau,\underline{X},t,\omega)$ is $\mathcal{F}_t^t$ measurable as a function $\omega \in \Omega$. (2.1.9)

$$\left| F_i(\tau,\underline{X},t,\omega) \right| \leq C\ (1+\ |\underline{X}|) \tag{2.1.10}$$

$$\left| \frac{\partial F_i(\tau,\underline{X},t,\omega)}{\partial X_j} \right| \leq C$$

$$i,j = 1,2,\ldots,N \tag{2.1.11}$$

There exists an integer $r \geq 0$ such that

$$\left| \frac{\partial^{\alpha} F_i(\tau,\underline{X},t,\omega)}{\partial X_1^{\alpha_1} \ldots \partial X_N^{\alpha_N}} \right| \leq C\ (\ 1+|\underline{X}|^r)$$

$$2 \leq \alpha_1+\alpha_2\ldots+\alpha_N = \alpha \leq 4 \tag{2.1.12}$$

and

$$E^{1/2}[\ \{F_i(s+h,\underline{X},t)-F_i(s,\underline{X},t)\}^2\ ] \leq C\ h\ (1+|\underline{X}|)$$

$$\text{for } s,\ s+h \in [0,T] \tag{2.1.13}$$

$$E^{1/2}\Big[\ \Big\{\frac{\partial F_i(s+h,\underline{X},t)}{\partial X_j} - \frac{\partial F_i(s,\underline{X},t)}{\partial X_j}\Big\}\ \Big] \leq C\ h \tag{2.1.14}$$

Condition III: Requirements of the Generator

Define

$$L[\cdot] = \frac{1}{\varepsilon^3} \int_{\tau}^{\tau+\varepsilon} d\sigma \int_{\tau}^{\sigma} ds\ \cdot \tag{2.1.15}$$

$$L'[\cdot] = \frac{1}{\varepsilon} \int_{\tau}^{\tau+\varepsilon} ds\ \cdot \tag{2.1.16}$$

$$a^{ij}(\tau,\underline{X}) = \lim_{\varepsilon\to 0} L\Big[\ E\Big[\ F_i\Big(\tau,\underline{X},\frac{s}{\varepsilon^2}\Big)\ F_j\Big(\tau,\underline{X},\frac{\sigma}{\varepsilon^2}\Big)\ \Big]\ \Big] \tag{2.1.17}$$

$$b^j(\tau,\underline{X}) = \lim_{\varepsilon\to 0} L\Big[\ \sum_{i=1}^{N} E\Big[\ \frac{\partial F_j\big(\tau,\underline{X},\frac{s}{\varepsilon^2}\big)}{\partial X_i}\ F_i\Big(\tau,\underline{X},\frac{\sigma}{\varepsilon^2}\Big)\ \Big]\ \Big] + \lim_{\varepsilon\to 0} L'\Big[\ E\Big[\ G_j\Big(\tau,\underline{X},\frac{s}{\varepsilon^2}\Big)\Big]\ \Big] \tag{2.1.18}$$

We require that:

$$\Big|a^{ij}(\tau,\underline{X}) - L\Big[\ E\Big\{\ F_i\Big(\tau,\underline{X},\frac{s}{\varepsilon^2}\Big)\ F_j\Big(\tau,\underline{X},\frac{\sigma}{\varepsilon^2}\Big)\Big\}\ \Big]\ \Big| \leq \varepsilon\ C\ (1+|\underline{X}|^2) \tag{2.1.19}$$

$$|b^j(\tau,\underline{X}) - L[\sum_{i=1}^{N} E[\frac{\partial F_j}{\partial X_i}(\tau, \underline{X}, \underline{s}_{\varepsilon^2}) F_i(\tau,\underline{X},\underline{\sigma}_{\varepsilon^2})]]$$

$$- L'[E[G_j(\tau,\underline{X},\underline{s}_{\varepsilon^2})]]| \leq \varepsilon C (1+|\underline{X}|) \quad (2.1.20)$$

$$a^{ij}(\tau,\underline{X}) \text{ is non-negative definite} \quad (2.1.21)$$

and that a^{ij} has a symmetric square root c^{ij}

$$a^{ij}(\tau,\underline{X}) = \sum_{k=1}^{N} c^{ik}(\tau,\underline{X})\, c^{jk}(\tau,\underline{X}) \quad (2.1.22)$$

such that:

$$\left| c^{ij}(\tau,\underline{X}) \right| \leq C (1+ |\underline{X}|) \quad (2.1.23)$$

$$\left|\frac{\partial c^{ij}(\tau,\underline{X})}{\partial X_k}\right| \leq C \quad (2.1.24)$$

$$\left|b^j(\tau,\underline{X})\right| \leq C (1+|\underline{X}|) \quad (2.1.25)$$

$$\left|\frac{\partial b^j(\tau,\underline{X})}{\partial X_k}\right| \leq C \quad (2.1.26)$$

and also

$$\left|a^{ij}(\tau,\underline{X})-a^{ij}(\tau+h,\underline{X})\right| \leq C\, h\, (1+|\underline{X}|) \quad (2.1.27)$$

$$\left|b^j(\tau,\underline{X})-b^j(\tau+h,\underline{X})\right| \leq C\, h\, (1+|\underline{X}|) \quad (2.1.28)$$

Lastly we need to assume there exists an integer r such that

$$c^{ij} \in C^{4,r}(R^N) \tag{2.1.29}$$

$$b^j \in C^{4,r}(R^N) \tag{2.1.30}$$

2.2 Applying the Papanicolaou and Kohler Theorem to Dynamical Systems

In this thesis we will only use the Papanicolaou and Kohler theorem on vector ordinary differential equations of the form:

$$\frac{d}{dt}\,\underline{p}(t) = \underline{q}(t)$$

$$\frac{d}{dt}\,\underline{q}(t) = \frac{1}{\varepsilon}\,\underline{K}(\underline{p}, \frac{t}{\varepsilon^2}) \tag{2.2.1}$$

where $\underline{p}$, $\underline{q}$ and $\underline{K}$ are vectors with M components.

Equation (2.2.1) can be obtained from dynamical systems of the form

$$\underline{Y}''(t) + \frac{1}{\varepsilon}\,\underline{K}(\underline{Y}(t), \frac{t}{\varepsilon^2}) = 0 \tag{2.2.2}$$

To use the Papanicolaou and Kohler theorem on (2.2.1) we define

$$\underline{X}^{(\varepsilon)} = (\underline{p}, \underline{q})^T \tag{2.2.3}$$

$$N = 2M \tag{2.2.4}$$

$$F_i(t,\underline{X}^{(\varepsilon)},\frac{t}{\varepsilon^2}) = \begin{cases} 0 & i=1,2,\ldots,M \\ K_{i-M}(\underline{p},\frac{t}{\varepsilon^2}) & i=M+1,\ldots,2M \end{cases} \qquad (2.2.5)$$

$$G_i(t,\underline{X}^{(\varepsilon)},\frac{t}{\varepsilon^2}) = \begin{cases} q_i & i=1,2,\ldots,M \\ 0 & i=M+1,\ldots,2M \end{cases} \qquad (2.2.6)$$

Because $\underline{G}$ is non-stochastic and does not depend on t or t/ε^2, and $\underline{F}$ does not depend on t, many of the Papanicolaou and Kohler requirements will simplify. In this section we will write the requirements of the Papanicolaou and Kohler theorem in terms of $\underline{p}$, $\underline{q}$ and $\underline{K}$.

From (2.1.8) we require

$$E[\ \underline{K}(\underline{p},t)\ ] = 0 \qquad (2.2.7)$$

Conditions (2.1.10), (2.1.11), (2.1.12) are valid for $\underline{G}$, since $\underline{G}$ is linear in $\underline{X}$. These conditions, applied to $\underline{F}$, give us the following conditions on $\underline{K}$:

$$\left| K_i(\underline{p},T) \right| < C\ (1+|\underline{p}|)$$

$$\left| \frac{\partial K_i(\underline{p},t)}{\partial p_j} \right| < C$$

$$\left| \frac{\partial^{\alpha} K_i(\underline{p},t)}{\partial p_1^{\alpha_1}\ldots\partial p_M^{\alpha_M}} \right| < C\ (1+|\underline{p}|^r) \qquad (2.2.8)$$

$$2 \leq \alpha_1+\alpha_2+\ldots+\alpha_M = \alpha \leq 4 \qquad (2.2.9)$$

Conditions (2.1.13), (2.1.14) are valid for $\underline{F}$ and $\underline{G}$ since neither $\underline{F}$ nor $\underline{G}$ depends on t (the slow time scale).

Now we must check the conditions on the generator. From (2.2.3), (2.2.5) we have

$$F_i = 0 \qquad i=1,2,\ldots,M$$

$$\frac{\partial F_j}{\partial X_i} = 0 \qquad i=M+1,\ldots,2M; \quad \text{all } j \qquad (2.2.10)$$

Hence, the sum in (2.1.18) is vacuous, Also, the L' operator has no effect on the (2.2.6) definition of $\underline{G}$. Therefore, (2.1.18) becomes

$$b^i(\underline{X}) = G_i = \begin{cases} q_i & i=1,2,\ldots,M \\ 0 & i=M+1,\ldots,2M \end{cases} \qquad (2.2.11)$$

With (2.2.11), conditions (2.1.20), (2.1.25), (2.1.26), (2.1.28), (2.1.30) are all true ($\underline{b}$ is linear in $\underline{X}$).

Using (2.2.5) in (2.1.17) we have

$$a^{ij}(\underline{X}) = \begin{cases} \lim_{\varepsilon \to 0} L[E\ [K_{i-m}(\underline{p}, \frac{s}{\varepsilon^2})\ K_{j-M}(\underline{p}, \frac{\sigma}{\varepsilon^2})]] & \\ & M+1 \leq i,j \leq 2M \\ 0 & \text{otherwise} \end{cases} \qquad (2.2.12)$$

From (2.1.22) we observe $a^{ij} = a^{ji}$. This simplies the calculation of (2.2.12). Equation (2.1.19) becomes

$$|a^{ij}(X)-L[\ E[\ K_{i-m}(\underline{p},\frac{s}{\varepsilon^2})\ K_{j-m}(\underline{p},\frac{\sigma}{\varepsilon^2})]\ ]|\ <\ \varepsilon\ C\ (1+|\underline{p}|^2)$$

$$M+1 \leq i,j \leq 2M \qquad (2.2.13)$$

Equation (2.1.27) is vacuously true, since a^{ij} does not depend on t. However, equations (2.1.21), (2.1.22), (2.1.23), (2.1.24), (2.1.29) must still be verified. They cannot be verified without more knowledge of $\underline{K}$.

To summarize, we have found a^{ij}, b^i (in (2.2.11) and (2.2.12)). the conditions that must still be checked are:

Mixing rate:	(2.1.7)	
Conditions on $\underline{K}$:	(2.2.7), (2.2.9)	
Conditions on a^{ij}:	(2.2.13), (2.1.21), (2.1.22), (2.1.23), (2.1.24), (2.1.29)	(2.2.14)

2.3 Applying the Papanicolaou and Kohler Theorem to the Two Dimensional Equations of Motion: Deriving the Generator ($\mathfrak{L}_N^A$)

In this section we will use the Papanicolaou and Kohler theorem on the two dimensional equations of motion we have found: (1.4.18), (1.4.19). These equations are of the form (2.2.1) so we can use the results of section 2.2.

We will need the following fact to compute the infinitesimal generator: if

$$|\int_0^\infty d\zeta\ \zeta\ H(Z,\zeta)\ |\ <\ C\ <\ \infty \qquad (2.3.1)$$

then

$$L[H(Z, \frac{\sigma-s}{\varepsilon^2})] = \int_0^\infty H(Z,Y)\, dY + O(\varepsilon) \qquad (2.3.2)$$

To show this we take (2.1.13) (the definition of L[.])

$$I = L[H(Z, \frac{\sigma-s}{\varepsilon^2})] = \frac{1}{\varepsilon^3} \int_\tau^{\tau+\varepsilon} d\sigma \int_\tau^\sigma ds\ H(Z, \frac{\sigma-s}{\varepsilon^2})$$

and change variables to

$$Y = \frac{\sigma-s}{\varepsilon^2}, \quad X = \frac{\sigma-\tau}{\varepsilon^2}$$

for

$$I = \varepsilon \int_0^{1/\varepsilon} dX\ J(X) \qquad (2.3.3)$$

where

$$J(\eta) = \int_0^\eta dY\ H(Z,Y) \qquad (2.3.4)$$

we expand I as

$$I = \lim_{\varepsilon\to 0} \varepsilon \int_0^{1/\varepsilon} dX\ J(X) + \varepsilon \{\lim_{\varepsilon\to 0} \frac{\partial}{\partial\varepsilon}\, \varepsilon \int_0^{1/\varepsilon} dX\ J(X)\} + O(\varepsilon^2) \qquad (2.3.5)$$

Using L'Hopitals rule in the first term of (2.3.5) we obtain

$$I = \lim_{\varepsilon\to 0} J(1/\varepsilon) - \varepsilon \left\{ \lim_{\varepsilon\to 0} \int_0^{1/\varepsilon} dX\ [J(X)-J(1/\varepsilon)] \right\} + O(\varepsilon^2)$$

$$= \int_0^{\infty} H(Z,Y) - \varepsilon \int_0^{\infty} dY\ Y\ H(Z,Y) + O(\varepsilon^2) \qquad (2.3.6)$$

From (2.3.6) we conclude that (2.3.1) implies (2.3.2).

Now we copy (1.4.18), (1.4.19) for the case of N rays and keep only those terms that are order one or larger:

$$\frac{d}{dt}\begin{pmatrix} X_1^\sigma(t,\alpha_L) \\ X_2^\sigma(t,\alpha_L) \\ V_1^\sigma(t,\alpha_L) \\ A^\sigma(t,\alpha_L) \\ V_2^\sigma(t,\alpha_L) \\ \\ B^\sigma(t,\alpha_L) \end{pmatrix} = \begin{pmatrix} V_1^\sigma(t,\alpha_L) \\ V_2^\sigma(t,\alpha_L) \\ 0 \\ B^\sigma(t,\alpha_L) \\ -\frac{1}{\sigma^{1/3}}\hat{C}_{X_2}\left(\frac{t}{\sigma^{2/3}} + X_1^\sigma, X_2^\sigma, X_3^\sigma\right) \\ \\ -\frac{A^\sigma}{\sigma^{1/3}}\hat{C}_{X_2X_2}\left(\frac{t}{\sigma^{2/3}} + X_1^\sigma, X_2^\sigma, X_3^\sigma\right) \end{pmatrix}$$

$$L=1,2,\ldots,N \qquad (2.3.7)$$

$$\begin{pmatrix} X_1^\sigma(0,\alpha_L) \\ X_2^\sigma(0,\alpha_L) \\ A^\sigma(0,\alpha_L) \\ \\ V_1^\sigma(0,\alpha_L) \\ V_2^\sigma(0,\alpha_L) \\ B^\sigma(0,\alpha_L) \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_L \\ 1 \\ \\ 0 \\ h'(\alpha_L) \\ h''(\alpha_L) \end{pmatrix} \qquad (2.3.8)$$

Neglecting the terms in (1.4.18), (1.4.19) that are smaller than O(1) is the only non-rigorous step in the derivation of $\mathcal{L}_N^A$.

To this order of approximation, the solution for X_1^σ and V_1^σ is clearly

$$X_1^\sigma(t,\alpha_L) = 0$$

$$V_1^\sigma(t,\alpha_L) = 0 \qquad (2.3.9)$$

If we use (2.3.9) in (2.3.7) the system reduces to the 4N variables $\{X_2^\sigma(t,\alpha_L),\ V_2^\sigma(t,\alpha_L)\ ,\ A^\sigma(t,\alpha_L),\ B^\sigma(t,\alpha_L)\}$, for L=1, 2,...,N. We use the new variables

$$\begin{aligned} X^L(t,\alpha_L) &= X_2^\sigma(t,\alpha_L) \\ V^L(t,\alpha_L) &= V_2^\sigma(t,\alpha_L) \\ A^L(t,\alpha_L) &= A^\sigma(t,\alpha_L) \\ B^L(t,\alpha_L) &= B^\sigma(t,\alpha_L) \end{aligned} \qquad (2.3.10)$$

and (2.3.9) in (2.3.7) to find:

$$\frac{d}{dt}\begin{pmatrix} X^L \\ A^L \\ V^L \\ B^L \end{pmatrix} = \begin{pmatrix} V^L \\ B^L \\ -\frac{1}{\sigma^{1/3}}\hat{C}_{X_2}\left(\frac{t}{\sigma^{2/3}}, X^L\right) \\ -\frac{1}{\sigma^{1/3}} A^L \hat{C}_{X_2X_2}\left(\frac{t}{\sigma^{2/3}}, X^L\right) \end{pmatrix} \qquad (L=1,2,\ldots,N) \qquad (2.3.11)$$

$$\begin{pmatrix} X^L \\ A^L \\ V^L \\ B^L \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} \alpha_L \\ 1 \\ h'(\alpha_L) \\ h''(\alpha_L) \end{pmatrix} \qquad (2.3.12)$$

We now use the Papanicolaou and Kohler theorem on (2.3.11). Equation (2.3.11) has the same form as (2.2.1) if we identify:

$$\varepsilon = \sigma^{1/3}, \quad M = 2N$$

$$\underline{p} = (X^1, A^1, X^2, A^2, \ldots, X^N, A^N)^T$$

$$\underline{q} = (V^1, B^1, V^2, B^2, \ldots, V^N, B^N)^T$$

$$\underline{K}(\underline{p}, \frac{t}{\sigma^{2/3}}) = \begin{cases} -\hat{C}_{X_2}(\frac{t}{\sigma^{2/3}}, X^1) \\ -A^L \hat{C}_{X_2X_2}(\frac{t}{\sigma^{2/3}}, X^1) \\ \cdots\cdots \\ -\hat{C}_{X_2}(\frac{t}{\sigma^{2/3}}, X^N) \\ -A^N \hat{C}_{X_2X_2}(\frac{t}{\sigma^{2/3}}, X^N) \end{cases} \tag{2.3.13}$$

We also define $\overset{t}{s}$ to be the smallest sigma algebra with respect to which $\{\hat{C}_{X_2}(\tau,\underline{X}), \hat{C}_{X_2X_2}(\tau,\underline{X})|$ for all $\underline{X}$ and $s\leq\tau\leq t\}$ are measurable.

First we will find the infinitesimal generator, and then verify the requirements of the Papanicolaou and Kohler theorem. From (2.2.11) we have:

$$\underline{b} = \begin{pmatrix} \underline{q} \\ \underline{0} \end{pmatrix} = (V^1, B^1, \ldots, V^N, B^N, 0, \ldots, 0)^T \tag{2.3.14}$$

Now we calculate a^{ij} from (2.2.12) and (2.3.6) to obtain:

$$a^{ij} = \begin{cases} \lim_{\varepsilon\to 0} L\left[E\{\hat{C}_{X_2}(\frac{s}{\varepsilon^2},X^K)\ \hat{C}_{X_2}(\frac{\sigma}{\varepsilon^2},X^L)\}\right] \\ \qquad\qquad i = 2N + 2K-1,\ j = 2N + 2L-1 \\ \lim_{\varepsilon\to 0} L[E\{A^K\hat{C}_{X_2X_2}(\frac{s}{\varepsilon^2},X^K)\ \hat{C}_{X_2}(\frac{\sigma}{\varepsilon^2},X^L)\}] \\ \qquad\qquad i = 2N + 2K,\quad j = 2N + 2L-1 \\ \qquad\qquad i = 2N + 2K-1,\quad j = 2N + 2L \\ \lim_{\varepsilon\to 0} L[E\{\ A^K\ \hat{C}_{X_2X_2}(\frac{s}{\varepsilon^2},X^K)\ A^L\ \hat{C}_{X_2X_2}(\frac{\sigma}{\varepsilon^2},X^L)\}] \\ \qquad\qquad i = 2N + 2K,\quad j = 2N + 2L \\ 0 \qquad\qquad \text{otherwise} \end{cases} \tag{2.3.15}$$

We define the correlation function for $\hat{C}$ to be

$$R(x,y) = E[\hat{C}(\eta,\zeta)\ \hat{C}(x+\eta,y+\zeta)] \tag{2.3.16}$$

We assume that $\hat{C}$ is homogeneous and isotropic so that

$$B(r) = B(\sqrt{x^2+y^2}) = R(x,y) \tag{2.3.17}$$

From (2.3.8) we compute

$$-R_{X_2X_2}(x,y) = E\{\ \hat{C}_{X_2}(\eta,\zeta)\ \hat{C}_{X_2}(x+\eta,y+\zeta)\ \}$$

$$R_{X_2X_2X_2}(x,y) = E\{\ \hat{C}_{X_2X_2}(\eta,\zeta)\ \hat{C}_{X_2}(x+\eta,y+\zeta)\ \}$$

$$R_{X_2X_2X_2X_2}(x,y) = E\{\ \hat{C}_{X_2X_2}(\eta,\zeta)\ \hat{C}_{X_2X_2}(x+\eta,y+\zeta)\ \} \qquad (2.3.18)$$

Using (2.1.18) and the homogeneity of $\hat{C}$, (2.3.15) becomes

$$a^{ij}(\underline{X}) = \begin{cases} -\lim_{\varepsilon\to 0}\left[R_{X_2X_2}\left(\frac{\sigma-s}{\varepsilon^2},\ X^L-X^K\right)\right] & \\ \qquad i = 2N+2K-1,\quad j = 2N+2L-1 & \\ A^K \lim_{\varepsilon\to 0}\left[R_{X_2X_2X_2}\left(\frac{\sigma-s}{\varepsilon^2},\ X^L-X^K\right)\right] & \\ \qquad i = 2N+2K,\quad j = 2N+2L-1 & \\ \qquad j = 2N+2K-1,\quad j = 2N+2L & \\ A^KA^L \lim_{\varepsilon\to 0} L\left[R_{X_2X_2X_2X_2}\left(\frac{\sigma-s}{\varepsilon^2},\ X^L-X^k\right)\right] & \\ \qquad i = 2N+2K,\quad j - 2N+2L & \\ 0 \qquad \text{otherwise} & \end{cases} \qquad (2.3.19)$$

Using (2.3.1), (2.3.2) we can evaluate (2.3.19) as

$$a^{ij}(\underline{X}) = \begin{cases} g(X^L-X^K) & i=2N+2K-1,\ j=2N+2L-1 \\ -A^Kg'(X^L-X^K) & i=2N+2K,\ j=2N+2L-1 \\ & i=2N+2K-1,\ j=2N+2L \\ -A^KA^Lg''(X^L-X^K) & i=2N+2K,\ j=2N+2L \\ 0 & \text{otherwise} \end{cases} \qquad (2.3.20)$$

where we have defined

$$g(z) = -\int_0^\infty R_{X_2X_2}(y,z)\,dy \tag{2.3.21}$$

and we have the requirements from (2.3.1)

$$\left| \int_0^\infty \zeta\, R^*(\zeta,z)\,d\zeta \right| < C < \infty$$

$$R^* = \{ R_{X_2X_2},\ R_{X_2X_2X_2},\ R_{X_2X_2X_2X_2} \} \tag{2.3.22}$$

Chapter four of this thesis has representations, properties and examples of the g(z) function.

From (2.3.14), (2.3.20) we have found the infinitesimal generator corresponding to (2.3.7). It is (see (2.1.3))

$$\mathfrak{L}_N^A = \sum_{K=1}^N \{ V^K \frac{\partial}{\partial X^K} + B^K \frac{\partial}{\partial A^K} \} + \sum_{L,K=1}^N \{ g(X^L-X^K) \frac{\partial^2}{\partial V^K \partial V^L}$$

$$-A^K g'(X^L-X^K) \frac{\partial^2}{\partial B^K \partial V^L} - A^K A^L g''(X^L-X^K) \frac{\partial^2}{\partial B^K \partial B^L} \} \tag{2.3.23}$$

Now we must check the conditions of the Papanicolaou and Kohler theorem (see (2.2.14)). First we must check the mixing condition (2.1.7). At a minimum we must have that the correlation function of $\underline{K}$ decays to zero as the argument goes to infinity. This insures a weak form of mixing (see (13), p. 78, exercise 3) but not (2.1.7). We will assume the (stronger) required mixing rate. Equation (2.3.22) already assures us that the correlation function of $\underline{K}$ goes to zero, so no new conditions

are required.

Because we assumed $\hat{C}$ had mean zero in (1.4.1), equation (2.2.8) is satisifed. We also required $\hat{C}$ to have bounded derivatives in (1.4.3). Therefore (2.2.9) is satisfied (see (2.3.13)) if

$$\text{Probability } \{A^L > M\} \to 0 \quad \text{as } M \to \infty \tag{2.3.24}$$

We are only interested in applying the Papanicolaou and Kohler theorem to (2.3.7) when ε is small. Therefore we need only verify (2.2.13) for small ε. For small ε, (2.2.13) is verified by virtue of (2.3.2).

Equations (2.1.21), (2.1.22), (2.1.23), (2.1.24) and (2.1.29) must still be verified. We have not yet been able to show they are satisfied in the general case.

In conclusion, we have found the generator for the motion of N rays and their associated raytubes, $\mathfrak{L}_N^A$ (in (2.3.23)). The unverified requirements for the derivation are: (2.3.22), (2.3.24), (2.1.21) - (2.1.24), (2.1.29).

2.4 Forward Operator for N Rays, $\mathfrak{L}_N^*$

Here we find $\mathfrak{L}_N^*$, the forward operator for N rays.

To find the evaluation of the quantities $\{X^L, V^L, A^L, B^L\}$ we must solve the forward Kolmogorov equation

$$Q_t = \mathfrak{L}_N^{A^*} Q = \Big[-\sum_{K=1}^{N} \Big\{ V^K \frac{\partial}{\partial X^K} + B^K \frac{\partial}{\partial A^K} \Big\} + \sum_{L,K=1}^{N} \Big\{ g(X^L - X^K) \frac{\partial^2}{\partial V^K \partial V^L}$$

$$- A^K g'(X^L - X^K) \frac{\partial^2}{\partial B^K \partial V^L} - A^K A^L g''(X^L - X^K) \frac{\partial^2}{\partial B^K \partial B^L} \Big\} \Big] Q \qquad (2.4.1)$$

where a "*" of an operator will always mean the formal adjoint of that operator.

The value of Q, at a distance of t, will be the probability density for the variables $\{X^L(t), V^L(t), A^L(t), B^L(t)\}$. From (2.3.12) we know the initial values of $\{X^L, V^L, A^L, B^L\}$ exactly. Therefore the correct initial conditions for (2.4.1) are

$$Q(0, \underline{X}, \underline{V}, \underline{A}, \underline{B}; \underline{\alpha}) = \prod_{L=1}^{N} \delta(X^L - \alpha^L) \delta(V^L - h'(\alpha^L)) \delta(A^L - 1) \delta(B^L - h''(\alpha^L)) \qquad (2.4.2)$$

In (2.4.2) we introduce the practice of separating the initial conditions from the forward variables in the argument list by a semicolon.

If the initial wavefront was planar, so h=0, then (2.4.2) becomes

$$Q(0, \underline{X}, \underline{V}, \underline{A}, \underline{B}; \underline{\alpha}) = \prod_{L=1}^{N} \delta(X^L - \alpha^L) \, \delta(V^L) \, \delta(A^L - 1) \, \delta(B^L) \qquad (2.4.3)$$

From $\mathfrak{L}_N^A$ we can find the generator for N rays without their associated raytube areas. Because of the special form of $\mathfrak{L}_N^A$ we

can do this by integrating (2.4.1) and (2.4.2) with respect to $\underline{A}$ and $\underline{B}$. Define

$$H(t,\underline{X},\underline{V};\underline{\alpha}) = \int d\underline{A} \int d\underline{B}\ Q(t,\underline{X},\underline{V},\underline{A},\underline{B};\underline{\alpha}) \qquad (2.4.4)$$

Then, by integrating (2.4.1), (2.4.2) with respect to $\underline{A}$ and $\underline{B}$, we obtain

$$\mathfrak{L}_N^{*} H = H_t \qquad (2.4.5)$$

$$H(0,\underline{X},\underline{V};\underline{\alpha}) = \delta(\underline{X}-\underline{\alpha})\ \delta(\underline{V}-h'(\underline{\alpha})) \qquad (2.4.6)$$

where (see (2.3.23))

$$\mathfrak{L}_N = \sum_{K=1}^{N} V^K \frac{\partial}{\partial X^K} + \sum_{K,L=1}^{N} g(X^L-X^K) \frac{\partial^2}{\partial V^K \partial V^L} \qquad (2.4.7)$$

To obtain (2.4.5) from (2.4.1) we had to assume that Q and $\frac{\partial Q}{\partial V^L}$ are zero at $|A^L|=\infty$ and $|B^L|=\infty$. Alternately, we could have derived (2.4.7) from applying the Papanicolaou and Kohler theorem directly to (2.3.11a) and (2.3.11c).

The solution $H(t,\underline{X},\underline{V};\underline{\alpha})$ of (2.4.5) will be the joint density of $\{\underline{X}, \underline{V}\}$ at a distance t.

We define $g_0=g(0)$. Since $R(X,Y)$ is even, $g(z)$ is even (see (2.3.17), (2.3.21)). Therefore $g'(0)=0$. Using this we write out $\mathfrak{L}_1^{A*}$, $\mathfrak{L}_1^{*}$, $\mathfrak{L}_2^{*}$, $\mathfrak{L}_3^{*}$ for later reference.

$$\mathfrak{L}_1^{A*} = - V^1 \frac{\partial}{\partial X^1} - B^1 \frac{\partial}{\partial A^1} + g_0 \frac{\partial^2}{\partial V^{1^2}} - A^{1^2} g''(0) \frac{\partial^2}{\partial B^{1^2}} \qquad (2.4.8)$$

$$\mathfrak{L}_1^* = -V^1 \frac{\partial}{\partial X^1} + g_0 \frac{\partial^2}{\partial V^{1^2}} \tag{2.4.9}$$

$$\mathfrak{L}_2^* = -V^1 \frac{\partial}{\partial X^1} - V^2 \frac{\partial}{\partial X^2} + g_0\{ \frac{\partial^2}{\partial V^{1^2}} + \frac{\partial^2}{\partial V^{2^2}} \} + 2g(X^2-X^1) \frac{\partial^2}{\partial V^1 \partial V^2} \tag{2.4.10}$$

$$\mathfrak{L}_3^* = -V^1 \frac{\partial}{\partial X^1} - V^2 \frac{\partial}{\partial X^2} - V^3 \frac{\partial}{\partial X^3} + g_0\{\frac{\partial^2}{\partial V^{1^2}}+\frac{\partial^2}{\partial V^{2^2}}+\frac{\partial^2}{\partial V^{3^2}}\}$$
$$+ 2\{g(X^2-X^1) \frac{\partial^2}{\partial V^2 \partial V^1} + g(X^3-X^1) \frac{\partial^2}{\partial V^3 \partial V^1} + g(X^3-X^2) \frac{\partial^2}{\partial V^3 \partial V^2}\} \tag{2.4.11}$$

The initial conditions for (2.4.9) through (2.4.11) are given in (2.4.6). For the case of a plane initial wavefront (h=0) the initial conditions become:

$$H(0,\underline{X},\underline{V};\underline{\alpha}) = \delta(\underline{X}-\underline{\alpha})\ \delta(\underline{V}) \tag{2.4.12}$$

2.5 Taking Limits in the Initial Conditions

In this section we take limits in the initial conditions of $\mathfrak{L}_2$. We obtain the same answer that we would have obtained had we taken the limits in the initial conditions of the equations of motion and then applied the Papanicolaou and Kohler theorem.

We start with a plane initial wavefront and use $\mathfrak{L}_2$ to find the generator for:

I) 2 rays coalescing into one ray (i.e., $\mathfrak{L}_1$)

II) 2 rays starting infinitely far apart

III) 1 ray with its associated raytube area (i.e., $\mathfrak{L}_1^A$)

For a plane initial wavefront the initial conditions for $\mathfrak{L}_2 H = H_t$ are (see (2.3.12) or (2.4.12))

$$\text{at } t=0: \quad X^1=\alpha^1,\ X^2=\alpha^2,\ V^1=0,\ V^2=0 \tag{2.5.1}$$

to simplify notation, define

$$h = \alpha^1 - \alpha^2 \tag{2.5.2}$$

I: 2 Rays coalescing into one ray

We change variables in (2.4.10), (2.5.1) by

$$N=\frac{X^1+X^2}{2} \qquad V=\frac{V^1+V^2}{2}$$

$$M=\frac{X^1-X^2-h}{2} \qquad W=\frac{V^1-V^2}{2} \tag{2.5.3}$$

to find

$$\mathfrak{L}_2^* = -V\frac{\partial}{\partial N} - W\frac{\partial}{\partial M} + \frac{1}{2}\{g_0+g(2M+h)\}\frac{\partial^2}{\partial V^2} + \frac{1}{2}\{g_0-g(2M+h)\}\frac{\partial^2}{\partial W^2} \tag{2.5.4}$$

$$\text{at } t=0: \quad N=\frac{\alpha^1+\alpha^2}{2},\ M=0,\ V=0,\ W=0 \tag{2.5.5}$$

The Ito equations corresponding to (2.5.4) are:

$$\begin{aligned} dN &= V\,dt \\ dM &= W\,dt \\ dV &= \sqrt{g_0 + g(2M+h)}\;\; d\beta_1 \\ dW &= \sqrt{g_0 - g(2M+h)}\;\; d\beta_2 \end{aligned} \tag{2.5.6}$$

where β_1, β_2 are independent standard Brownian motions. In the limit of $h\to 0$, the two rays described by $\mathfrak{L}_2$ coalesce into one ray. Taking $h\to 0$ in (2.5.6) gives

$$\begin{aligned} dN &= V\,dt \\ dM &= W\,dt \\ dV &= \sqrt{g_0 + g(2M)}\;\; d\beta_1 \\ dW &= \sqrt{g_0 - g(2M)}\;\; d\beta_2 \end{aligned} \tag{2.5.7}$$

Recalling that $g_0=g(0)$ we see from (2.5.5), (2.5.7) that

$$M = 0, \quad W = 0 \tag{2.5.8}$$

is the solution for (M,W). Using (2.5.8) in (2.5.7) we find

$$\begin{aligned} dN &= V\,dt \\ dV &= \sqrt{2g_0}\;\; d\beta_1 \end{aligned} \tag{2.5.9}$$

The forward operator associated with (2.5.9) is

$$-V \frac{\partial}{\partial N} + g_0 \frac{\partial^2}{\partial V^2} \qquad (2.5.10)$$

which is $\mathfrak{L}_1^*$ (see (2.4.9)).

The result could also have been obtained by scaling the Ito equations corresponding to $\mathfrak{L}_2$ directly. These equations are

$$\begin{aligned} dX^1 &= V^1\, dt \\ dX^2 &= V^2\, dt \\ dV^1 &= \frac{1}{\sqrt{2}} (B+D)\, d\beta_3 + \frac{1}{\sqrt{2}} (B-D)\, d\beta_4 \\ dV^2 &= \frac{1}{\sqrt{2}} (B-D)\, d\beta_3 + \frac{1}{\sqrt{2}} (B+D)\, d\beta_4 \end{aligned} \qquad (2.5.11)$$

where

$$\begin{aligned} B &= \sqrt{g_0 + g(X^2-X^1)} \\ D &= \sqrt{g_0 - g(X^2-X^1)}\ \mathrm{sgn}(X^2-X^1) \end{aligned} \qquad (2.5.12)$$

and β_3, β_4 are independent standard Brownian motions (see section 3.4 for a derivation of (2.5.11), (2.5.12)).

Using (2.5.3) in (2.5.11) we find

$$\begin{aligned} dN &= V\ dt \\ dM &= W\ dt \\ dV &= \frac{B}{\sqrt{2}} (d\beta_3+d\beta_4) \\ dW &= \frac{D}{\sqrt{2}} (d\beta_3-d\beta_4) \end{aligned} \qquad (2.5.13)$$

where (2.5.12) has become:

$$B = \sqrt{g_0 + g(2M+h)}$$
$$D = \sqrt{g_0 - g(2M+h)}\ \mathrm{sgn}(2M+h) \qquad (2.5.14)$$

Now we define

$$\beta_5 = (\beta_3 - \beta_4)/\sqrt{2}$$
$$\beta_6 = (\beta_3 + \beta_4)/\sqrt{2} \qquad (2.5.15)$$

Since $d\beta_i d\beta_j = \delta_{ij} dt$, β_5 and β_6 are independent standard Brownian motions. Using (2.5.14), (2.5.15) in (2.5.13) we find

$$dN = V\ dt$$
$$dM = W\ dt$$
$$dV = \sqrt{g_0 + g(2M+h)}\ \ d\beta_5$$

$$dW = \sqrt{g_0 - g(2M+h)}\ \mathrm{sgn}(2M+h)\ d\beta_6$$
$$= \sqrt{g_0 - g(2M+h)}\ \ d\beta_7 \qquad (2.5.16)$$

Equations (2.5.16) are identical to (2.5.6)

II: 2 Rays starting infinitely far apart

We take the new variables

$$Y_1 = \sqrt{2}\ N \qquad U_1 = \sqrt{2}\ V$$

$$Y_2 = \sqrt{2}\, M \qquad U_2 = \sqrt{2}\, W \tag{2.5.17}$$

in (2.5.6) to find

$$\begin{aligned} dY_1 &= U_1\, dt \\ dY_2 &= U_2\, dt \\ dU_1 &= \sqrt{2}\,\sqrt{g_0 + g(\sqrt{2}\, Y_2 + h)}\; d\beta_1 \\ dU_2 &= \sqrt{2}\,\sqrt{g_0 - g(\sqrt{2}\, Y_2 + h)}\; d\beta_2 \end{aligned} \tag{2.5.18}$$

Taking $h\to\infty$ in (2.5.18) we obtain the Ito equations for the motion of two rays that start infinitely far apart. Taking this limit in (2.5.18) and using $g(\infty)=0$ (see chapter 3) we obtain

$$\begin{aligned} dY_1 &= U_1\, dt \\ dY_2 &= U_2\, dt \\ dU_1 &= \sqrt{g_0}\; d\beta_1 \\ dU_2 &= \sqrt{g_0}\; d\beta_2 \end{aligned} \tag{2.5.19}$$

From (2.5.19), the $\{U_1, Y_1\}$ variables are independent of the $\{U_2, Y_2\}$ variables, and vice-versa. The forward operator corresponding to (2.5.19) is

$$\left\{-U_1 \frac{\partial}{\partial Y_1} + g_0 \frac{\partial^2}{\partial U_1^{\,2}}\right\} + \left\{-U_2 \frac{\partial}{\partial Y_2} + g_0 \frac{\partial^2}{\partial U_2^{\,2}}\right\} \tag{2.5.20}$$

which has the same form as two independent rays (see (2.4.9)).

III: Recovering $\mathfrak{L}_1^A$ from $\mathfrak{L}_2$

Now we will let two rays coalesce in such a way that we will obtain $\mathfrak{L}_1^A$ from $\mathfrak{L}_2$. We do this by changing variables in $\mathfrak{L}_2$ to

$$A = \frac{X^1 - X^2}{h} \qquad B = \frac{V^1 - V^2}{h}$$

$$N = \frac{X^1 + X^2}{2} \qquad V = \frac{V^1 + V^2}{2} \tag{2.5.21}$$

Observe that, as $h \to 0$, A will become the vertical distance between two very close rays, normalized by their initial separation. We anticipate that horizontal deviations between two rays will be much smaller than vertical deviations (see (2.3.9)). Hence, we can approximate the area of a raytube by considering only vertical deviations. In this sense, as $h \to 0$, A becomes the approximate area of the raytube determined by rays 1 and 2. We will use this approximation again.

We can use (2.5.21) in $\mathfrak{L}_2$ by changing variables in (2.5.6) to (see (2.5.3)):

$$M = h\,(A-1)/2$$

$$W = h\,B/2 \tag{2.5.22}$$

We find:

$$dN = V\,dt$$

$$dA = B\,dt$$

$$dV = \sqrt{g_0 + g(A\,h)}\; d\beta_1$$

$$dB = \frac{2}{h}\sqrt{g_0 - g(A\ h)}\ d\beta_2 \qquad (2.5.23)$$

Taking the limit of $h \to 0$ in (2.5.23) yields (recall $g(z)$ is an even function of z):

$$\begin{aligned} dN &= V\ dt \\ dA &= B\ dt \\ dV &= 2\sqrt{g_0}\ d\beta_1 \\ dB &= \sqrt{-2g''(0)A^2}\ d\beta_2 \end{aligned} \qquad (2.5.24)$$

The forward generator corresponding to 2.5.24 is

$$\left\{-V\frac{\partial}{\partial N} + g_0\frac{\partial^2}{\partial V^2}\right\} + \left\{B\frac{\partial}{\partial A} - A^2 g''(0)\frac{\partial^2}{\partial B^2}\right\} \qquad (2.5.25)$$

which is precisely $\mathfrak{L}_1^{A^*}$ (see (2.4.8)). From (2.5.24) or (2.5.25) we also see that for one ray,

$$(A,B) \text{ are independent of } (N,V), \text{ and vice-versa} \qquad (2.5.26)$$

2.6 Deriving $\mathfrak{L}_N^A$ From $\mathfrak{L}_{2N}$

In this section we derive $\mathfrak{L}_N$ from $\mathfrak{L}_{2N}$ for a plane initial wavefront. This shows that all of the probabilistic information about raytube areas that can be derived from $\mathfrak{L}_N^A$ can also be found from $\mathfrak{L}_{2N}$. The technique used will confirm that the area of a raytube is well approximated by variations in the vertical position of a ray with respect to the initial conditions.

We write $\mathfrak{L}_{2N}$ from (2.4.7) as

$$\mathfrak{L}_{2N} = (\nabla_{\underline{V}}^T)\ \underline{g}\ \nabla_{\underline{V}} + \underline{V}\cdot\nabla_{\underline{Y}} \tag{2.6.1}$$

where

$$\underline{V} = (V^1, V^2, \ldots, V^{2N})^T \tag{2.6.2}$$

$$\underline{Y} = (Y^1, Y^2, \ldots, Y^{2N})^T \tag{2.6.3}$$

$$g_{ij} = g(Y^i - Y^j) \qquad 1 \leq i,j \leq 2N \tag{2.6.4}$$

The initial conditions for $\mathfrak{L}_{2N}^* H = H_t$ are (from (2.4.12))

$$H(0,\underline{V},\underline{Y};\underline{\alpha}) = \delta(\underline{Y}-\underline{\alpha})\ \delta(\underline{V}) \tag{2.6.5}$$

where

$$\underline{\alpha} = (\alpha^1, \alpha^2, \ldots, \alpha^{2N})^T \tag{2.6.6}$$

We choose

$$\alpha^{2m} = \alpha^{2m+1} + h \qquad m=1,2,\ldots,N \tag{2.6.7}$$

so that pairs of rays have an initial separation of h. Now define the following $2N \times 2N$ matrices:

$$R_{ij}^h = \begin{cases} \delta_{j,2i-1} & i=1,2,\ldots,N \\ \frac{1}{h}\left(\delta_{j,2(i-N)} - \delta_{j,2(i-N)-1}\right) & i=N+1,\ldots,2N \end{cases} \tag{2.6.8}$$

$$K_{ij}^h = \begin{cases} \delta_{i,2j} + \delta_{i,2j-1} & j=1,2,\ldots,N \\ h\ \delta_{i,2(j-N)} & j=N+1,\ldots,2n \end{cases}$$

For example, when N=2:

$$\underline{\underline{R}}^h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/h & -1/h & 0 & 0 \\ 0 & 0 & 1/h & 1/h \end{pmatrix}$$

$$\underline{\underline{K}}^h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & h & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & h \end{pmatrix}$$

Note that $(\underline{\underline{R}}^h)^{-1} = \underline{\underline{K}}^h$. We now define the new variables:

$$\underline{\eta} = \underline{\underline{R}}^h \underline{Y}$$

$$\underline{\zeta} = \underline{\underline{R}}^h \underline{V} \qquad (2.6.9)$$

Using (2.6.8) in (2.6.9) we have:

$$\eta_i = \begin{cases} Y^{2i-1} & i=1,2,\ldots,N \\ \dfrac{Y^{2(i-N)} - Y^{2(i-N)-1}}{h} & i=N+1,\ldots,2N \end{cases} \qquad (2.6.10)$$

$$\zeta_i = \begin{cases} V^{2i-1} & i=1,2,\ldots,N \\ \dfrac{V^{2(i-N)} - V^{2(i-N)-1}}{h} & i=N+1,\ldots,2N \end{cases} \qquad (2.6.11)$$

We define

$$A_i = \frac{Y^{2i} - Y^{2i-1}}{h}$$

$$B_i = \frac{V^{2i} - V^{2i-1}}{h} \qquad (2.6.12)$$

As $h \to 0$, A_i will become the approximate area of the raytube bounded by rays 2i and 2i-1. See the comments after (2.5.21).

From (2.6.5), (2.6.7), (2.6.12) the initial conditions for A_i and B_i are

$$\text{at } t=0: \quad A_i = 1$$
$$B_i = 0 \qquad (2.6.13)$$

The chain rule gives (from (2.6.9))

$$\nabla_{\underline{V}} = (\underline{\underline{R}}^h)^T \nabla_{\underline{\zeta}}$$

$$\nabla_{\underline{Y}} = (\underline{\underline{R}}^h)^T \nabla_{\underline{\eta}} \qquad (2.6.14)$$

Using (2.6.14) in (2.6.1)

$$\mathfrak{L}_{2N} = (\nabla_{\underline{\zeta}})^T \underline{\underline{D}} \, (\nabla_{\underline{\zeta}}) + \underline{\zeta} \cdot \nabla_{\underline{\eta}} \qquad (2.6.15)$$

where

$$\underline{\underline{D}} = \underline{\underline{R}}^h \, \underline{\underline{g}} \, (\underline{\underline{R}}^h)^T \qquad (2.6.16)$$

We would like to take the limit as $h \to 0$ in (2.6.15). First, we need to compute $\lim_{h\to 0} \underline{\underline{D}}$. Using (2.6.8) in (2.6.16) we find

$$\underline{\underline{D}} = \begin{pmatrix} \underline{\underline{D}}^1 & (\underline{\underline{D}}^2)^T \\ \underline{\underline{D}}^2 & \underline{\underline{D}}^3 \end{pmatrix} \tag{2.6.17}$$

where

$$\begin{aligned} D_{ij}^1 &= g_{2i-1,2j-1} \\ D_{ij}^2 &= (g_{2i-1,2j-1} - g_{2i,2j-1})/h \\ D_{ij}^3 &= (g_{2i-1,2j-1} - g_{2i-1,2j} + g_{2i,2j-1} + g_{2i,2j})/h^2 \end{aligned} \tag{2.6.18}$$

and we have used $g_{ij}=g_{ji}$ (since $g(z)$ is an even function).

We define:

$$\begin{aligned} \underline{A} &= (A_1, A_2, \ldots, A_N)^T \\ \underline{B} &= (B_1, B_2, \ldots, B_N)^T \\ \underline{X} &= (X_1, X_2, \ldots, X_N)^T = (Y^1, Y^3, \ldots, Y^{2N-1})^T \\ \underline{U} &= (U_1, U_2, \ldots, U_N)^T = (V^1, V^3, \ldots, V^{2N-1})^T \end{aligned} \tag{2.6.19}$$

From (2.6.10) - (2.6.12), (2.6.19)

$$\underline{\eta} = \begin{pmatrix} \underline{X} \\ \underline{A} \end{pmatrix} \qquad \underline{\zeta} = \begin{pmatrix} \underline{U} \\ \underline{B} \end{pmatrix} \tag{2.6.20}$$

Therefore (using (2.6.9))

$$\underline{Y} = (\underline{R}^h)^{-1}\underline{n} = \underline{K}^h \begin{matrix} \underline{X} \\ \underline{A} \end{matrix} \tag{2.6.21}$$

We use (2.6.21) to evaluate $g_{ij} = g(Y^i - Y^j)$ in the $\underline{X}$, $\underline{A}$ variables. We find:

$$\begin{aligned} g_{2i,2j} &= g(X_i - X_j + h\,A_j - h\,A_i) \\ g_{2i,2j-1} &= g(X_i - X_j - h\,A_i) \\ g_{2i-1,2j} &= g(X_i - X_j + h\,A_j) \\ g_{2i-1,2j-1} &= g(X_i - X_j) \end{aligned} \tag{2.6.22}$$

We define

$$\lim_{h\to 0} \underline{D}_m = \underline{E}_m \tag{2.6.23}$$

and use (2.6.18), (2.6.22) to find

$$\begin{aligned} E_{ij}^{1} &= g(X_j - X_i) \\ E_{ij}^{2} &= -A_j\, g'(X_j - X_i) \\ E_{ij}^{3} &= -A_i\, A_j\, g''(X_j - X_i) \end{aligned} \tag{2.6.24}$$

Using (2.6.20), (2.6.23), (2.6.24) in (2.6.15) we obtain

$$\mathfrak{L}_{2N,h=0} = \left[\nabla\begin{pmatrix}\underline{U}\\ \underline{B}\end{pmatrix}\right]^T \begin{pmatrix} \underline{E}^1 & (\underline{E}^2)^T \\ \underline{E}^2 & \underline{E}^3 \end{pmatrix} \nabla\begin{pmatrix}\underline{U}\\ \underline{B}\end{pmatrix} + \begin{pmatrix}\underline{U}\\ \underline{B}\end{pmatrix} \cdot \nabla\begin{pmatrix}\underline{X}\\ \underline{A}\end{pmatrix}$$

$$\mathfrak{L}_{2N,h=0} = \sum_{j=1}^{N} \left(B_j \frac{\partial}{\partial A_j} + U_j \frac{\partial}{\partial X_j}\right) + \sum_{i,j=1}^{N} \left(g(X_j-X_i) \frac{\partial^2}{\partial U_i \partial U_j}\right.$$

$$\left. -A_j g'(X_j-X_i) \frac{\partial^2}{\partial U_i \partial B_j} - A_j A_i g''(X_j-X_i) \frac{\partial^2}{\partial B_i \partial B_j}\right) \qquad (2.6.25)$$

We recognize (2.6.25) as being the same as $\mathfrak{L}_N^A$ (2.3.23). The initial conditions for (2.6.25) come from (2.6.5), (2.6.13), (2.6.19):

$$\text{at } t=0: \qquad X_i = \alpha_{2i-1}$$

$$V_i = 0$$

$$A_i = 1$$

$$B_i = 0 \qquad (2.6.26)$$

and are the same as (2.4.3).

3.1 g(z): Definition, Assumptions, Representations

In (2.3.21) we defined

$$g(z)=-\int_0^\infty R_{X_2X_2}(y,z)\,dy \tag{3.1.1}$$

where (see (2.3.16), (2.3.17))

$$B(r)=B(\sqrt{X^2+Y^2})=R(X,Y)=E[\hat{C}(X,Y)\ \hat{C}(0,0)] \tag{3.1.2}$$

We have assumed that (see (2.3.22), (2.3.20)):

$$\int_0^\infty \zeta\, R^*(\zeta,z)\,d\zeta \ < C < \infty$$

$$R^*=\{R_{X_2X_2},R_{X_2X_2X_2},R_{X_2X_2X_2X_2}\} \tag{3.1.3}$$

$$g(z) \text{ is twice differentiable} \tag{3.1.4}$$

We now give two alternative representations for g(z). We can write (3.1.1) for B(r) to find

$$g(z)= -\int_0^\infty \{B''(r)\,\frac{y^2}{r^2} + B'(r)\,\frac{z^2}{r^3}\}\,dY \tag{3.1.5}$$

We can find a third formula for g(z) by taking Fourier Transforms. First write (3.1.1) as

$$g(z)= -\tfrac{1}{2}\int_{-\infty}^\infty R_{X_2X_2}(y,z)\,dy \tag{3.1.6}$$

since R is an even function. Then define the Fourier Transform pair:

$$R(y,z) = \frac{1}{2\pi} \int\int_{-\infty}^{\infty} e^{i(y\omega_1 + z\omega)} S(\sqrt{\omega_1^2+\omega^2})\, d\omega_1 d\omega \quad (3.1.7)$$

$$S(\sqrt{\omega_1^2+\omega^2}) = \frac{1}{2\pi} \int\int_{-\infty}^{\infty} e^{-(y\omega_1 + z\omega)} B(\sqrt{y^2+z^2})\, dy\, dz \quad (3.1.8)$$

now use (3.1.7) in (3.1.6) for

$$g(z) = \frac{1}{2} \int_{-\infty}^{\infty} \omega^2\, e^{iz\omega}\, S(\omega)\, d\omega \quad (3.1.9)$$

If we use $y=r\cos\theta$, $z = r\sin\theta$ for $0\leq\theta\leq\infty$, $0\leq\theta\leq 2\pi$, then (3.1.8) with $\omega_1=0$ becomes

$$S(|\omega|) = \int_{0}^{\infty} dr\; r\, B(r)\, J_0(\omega r) \quad (3.1.10)$$

where we have used ((14), 9.1.18a). So $S(\omega)$ is the Hankel power spectrum of $B(r)$.

3.2 g(z): Properties

Since $S(\omega)\geq 0$ (being the power spectrum of a correlation function), $\omega^2 S(\omega)\geq 0$. Therefore, from (3.1.9), g(z) has all the properties of a correlation function. This gives us two immediate facts:

1) the matrix $g_{ij} = g(X_i - X_j)$ is non-negative definite (3.2.1)

2) $g(0) \geq g(m)$ (3.2.2)

Equation (3.2.2) can be strengthened to

$$g(0) > g(m) \qquad \text{for } m \neq 0 \tag{3.2.3}$$

because if $g(0)=g(m)$ and $m \neq 0$ then $S(\omega)$ must be a sum of delta functions, which violates the mixing condition.

From (4.2.1) we can classify $\mathfrak{L}_N P = P_t$ as degenerate parabolic. It is degenerate because the principal part of $\mathfrak{L}_N$ has (at least) N zero eigenvalues. From (3.2.2) we can classify the two point energy correlation equation (which we derive in section 4.4) as also being degenerate parabolic (see (4.4.20)).

From (3.1.9) we can compute

$$\int_{-\infty}^{\infty} g(z)\, dz = \frac{1}{2\pi} \int dz \int dw\, e^{iz\omega} \left[\pi\omega^2 S(\omega)\right]$$

$$= \pi\omega^2 S(\omega)\Big|_{\omega=0} = 0 \tag{3.2.4}$$

Since $g(z)$ is an even function of z, (3.2.4) becomes

$$\int_{0}^{\infty} g(z)\, dz = 0 \tag{3.2.5}$$

For use in the other chapters we define

$$g(z) = g_0 - g_2 z^2 + g_4 z^4 + O(z^6) \tag{3.2.6}$$

From (3.1.1), (3.1.2), (3.1.9) and (3.2.6) we find

$$g_0 = -\int_0^\infty \left(\frac{1}{r}\frac{\partial}{\partial r}\right) B(r)\, dr = \frac{1}{2}\int_{-\infty}^{\infty} \omega^2 S(\omega)\, d\omega$$

$$g_2 = \frac{3}{2}\int_0^\infty \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 B(r)\, dr = \int_{-\infty}^{\infty} \omega^4 S(\omega)\, d\omega$$

$$g_4 = -\frac{5}{8}\int_0^\infty \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3 B(r)\, dr = 12\int_{-\infty}^{\infty} \omega^6 S(\omega)\, d\omega \qquad (3.2.7)$$

For comparison with (6), they defined

$$\gamma_1^2 = 2g(0) = 2g_0$$

$$\gamma_2^2 = -2g''(0) = 4g_2 \qquad (3.2.8)$$

Recall that we use γ_2 to scale to universal time (see (I.2.2)).

If we define $f(z)=2(g_0-g(z))$ then the expansion of $f(z)$ about $z=0$ is:

$$f(z) = 2g_2 z^2 - 2g_4 z^4 + O(z^6) \qquad (3.2.9)$$

3.3 g(z): Examples

For all of the examples of g(z) that we give, equations (3.1.3), (3.1.4) have been verified.

Our first example is for a velocity correlation function of the form:

$$B_1(r) = e^{-r^2} \sum_{N=0}^{M} a_N r^N \qquad (3.3.1)$$

with M finite. For (3.3.1) to be a legitimate correlation function, the power spectral density must be non-negative. Using (3.1.10) this condition becomes (using (15) 6.631.1 and (19) 13.1.32, 13.6.9)

$$a_0 L_0(X) + \sum_{N=1}^{M} a_N \, N! \, (N-1)! \, L_N(X) > 0 \qquad (3.3.2)$$

for all positive X, where $L_N(X)$ is the Nth Laguerre polynomial. If (3.3.2) holds then the g(z) function corresponding to (3.3.1) is

$$g_1(z) = -\frac{e^{-z^2}}{z^2} \sum_{N=0}^{M} a_N \Big[\sum_{J=0}^{N} \binom{N}{J} \Gamma(N-J+\tfrac{1}{2}) \, z^{2J} \{ 2z^4 - (4J+1)z^2 + J(2J-1) \} \Big] \qquad (3.3.3)$$

For illustrations of (3.3.1), (3.3.3), consider:

$$B_2(r) = e^{-\frac{r^2}{2a^2}}$$

$$B_3(r) = e^{-r^2} \{ r^4 - 6r + 7 \} \qquad (3.3.4)$$

for which

$$g_2(z) = \sqrt{\frac{\pi}{2}} \frac{1}{a} \left\{ 1 - \frac{z^2}{a^2} \right\} e^{-\frac{z^2}{2a^2}}$$

$$g_3(z) = \frac{\sqrt{\pi}}{4} e^{-z^2} \{39-162z^2+72z^4-8z^6\} \qquad (3.3.5)$$

Another choice of B(r) might be

$$B_4(r) = \frac{1}{2^{\nu-1}\Gamma(\nu)} \left(\frac{r}{a}\right)^{\nu} K_{\nu}\left(\frac{r}{a}\right) \qquad (3.3.6)$$

where K_{υ} is the modified Bessel function of order υ. A correlation function of the form (3.3.6) was used by Von Karman to fit turbulence measurements. Using (3.1.1) we find (using (15) 6.592.4, 9.311.6, 9.34.3)

$$g_4(z) = - \frac{\sqrt{2\pi}}{\Gamma(\nu)(2a)^{\alpha+1}} z^{\alpha-2} \ell\left(\frac{z}{a}\right) \qquad (3.3.7)$$

where $\alpha=\nu+ 1/2$,

$$\ell(\eta) = K_{\alpha}(\eta) \left[4\alpha\nu+\eta^2\right] - 2\nu\eta K_{\alpha+1}(\eta) \qquad (3.3.8)$$

and we require $\upsilon>1/2$.

Another way to construct examples for g(z) is to choose a non-negative power spectrum and then use (3.1.9). If we take

$$S_5(\omega) = \omega^N e^{-\frac{\omega^2}{2\gamma^2}} \qquad (3.3.9)$$

with N a non-negative even integer, then we find (using (15) p. 121, #23)

$$g_5(z) = -(-1)^{\frac{N}{2}} \frac{\pi}{2} \gamma^{(N+3)} 2^{-\frac{N+2}{2}} e^{-\frac{z^2\gamma^2}{2}} H_{N+2}(z\gamma) \qquad (3.3.10)$$

where H_m is the mth Hermite polynomial. For N=0, N=2 we find from (3.3.10):

$$g_6(z) = \gamma^3 \sqrt{\frac{\pi}{2}} (1-\gamma^2 z^2) e^{-\frac{z^2\gamma^2}{2}}$$

$$g_7(z) = \gamma^5 \sqrt{\frac{\pi}{2}} (3-6\gamma^2 z^2+\gamma^4 z^4) e^{-\frac{z^2\gamma^2}{2}} \qquad (3.3.11)$$

We observe that $g_2(z)$ has the same form as $g_6(z)$. This is because the Hankel transform of a gaussian is another gaussian (so the spectrum of $B_2(r)$ has the same form as $S_5(\omega)$, for N=0)

In Figures 3.1-3.4 we have graphs of: $g_2(z)$ (with a=1), $g_3(z)$, $g_4(z)$ (with υ=2/3 and a=1) and $g_7(z)$ (with γ=.85).

3.4 Final Verification Of Papanicolaou and Kohler Requirements For $\mathfrak{L}_2$

We can now finish verifying the remaining requirements for the Papanicolaou and Kohler theorem, for $\mathfrak{L}_2$. At the end of section 2.3 is a list of the remaining requirements: (2.3.22), (2.1.21) - (2.1.24), (2.1.29).

Equation (2.3.22) says that R and its derivatives must be properly bounded. This was verified for all the examples in section 3.3. Equation (2.1.21) requires the principal part of $\mathfrak{L}_2$ to be non-negative definite; this is assured by (3.2.1).

Equation (2.1.22) requires the principal part of $\mathfrak{L}_2$ to have

a symmetric square root we find (see (2.4.10)):

$$\underline{\sigma}\,\underline{\sigma} = \begin{pmatrix} g_0 & g(\eta) \\ g(\eta) & g_0 \end{pmatrix}$$

for

$$\underline{\sigma} = 1/2 \begin{pmatrix} B+D & B-D \\ B-D & B+D \end{pmatrix}$$

where

$$B = \sqrt{g_0+g(\eta)}$$

$$D = \sqrt{g_0-g(\eta)}\ \ \mathrm{sgn}(\eta)$$

$$\mathrm{sgn}(\eta) = \begin{cases} 1 & \eta>0 \\ -1 & \eta<0 \end{cases} \qquad (3.4.1)$$

Note that (3.4.1) was used in (2.5.11), (2.5.12), to write the Ito equations corresponding to $\mathfrak{L}_2$.

The remaining conditions to be verified can be combined in the statement:

$$\left\{ \begin{array}{l} \underline{\sigma} \text{ is bounded and has bounded, continuous} \\ \quad \text{derivatives up to order 4.} \end{array} \right\} \qquad (3.4.2)$$

The examples we gave in section 3.3 are for g(z) functions that are bounded, and have 4 bounded, continuous derivatives.

From (3.2.3) we have $g_0>g(\eta)$ for $\eta\neq 0$. Hence every element in $\underline{\sigma}$ is bounded by $2g_0$. Likewise we see $B>0$ and that B and D are bounded. The only place then that $\underline{\sigma}$ could be discontinuous or have an unbounded derivative is where D=0, or (equivalently)

where $\eta=0$.

Around $z=0$, $g(z)$ has the expansion (see (3.2.6)):

$$g(z)=g_0 - g_2 z^2 + g_4 z^4 + O(z^6) \qquad (3.4.3)$$

Using (3.4.3) we can compute D as:

$$D= \mathrm{sgn}(\eta)\sqrt{g_0-g(z)}$$

$$= \sqrt{g_2}\,\eta - \frac{g_4}{2\sqrt{g_2}}\,\eta^3 + O(\eta^5) \qquad (3.4.4)$$

From (3.4.4) we see that D has 4 bounded, continuous derivations at $\eta=0$. So (3.4.2) is satisfied.

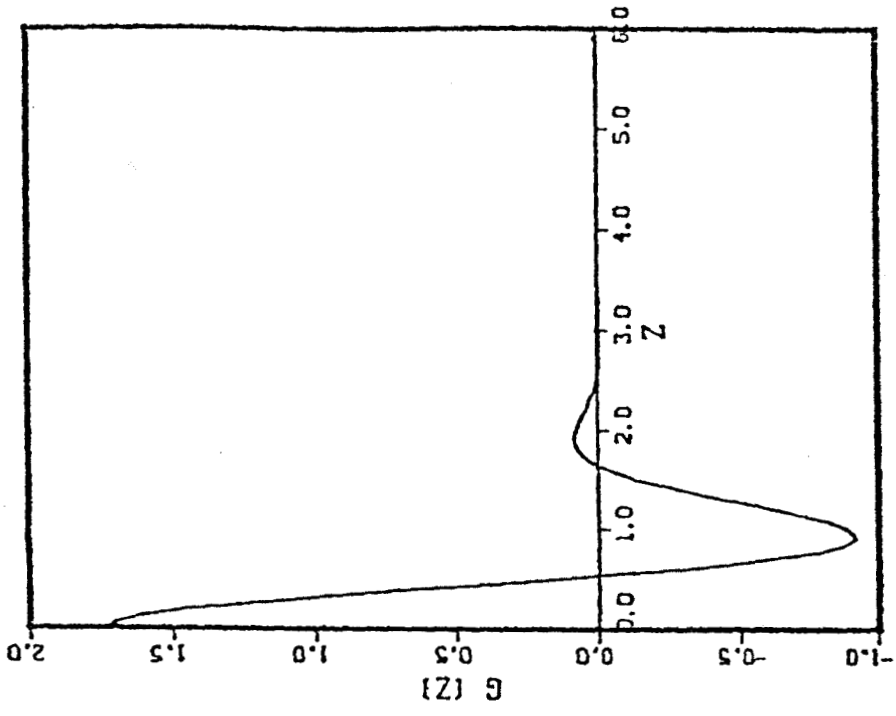

Figure 3.2 $g_3(z)$

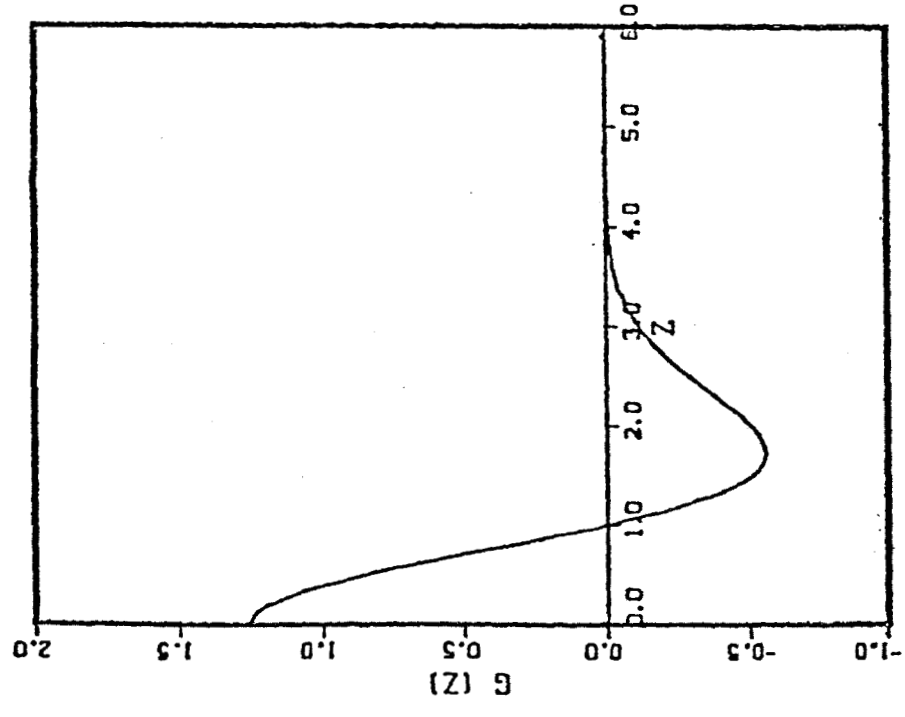

Figure 3.1 $g_2(z)$ (with a=1)

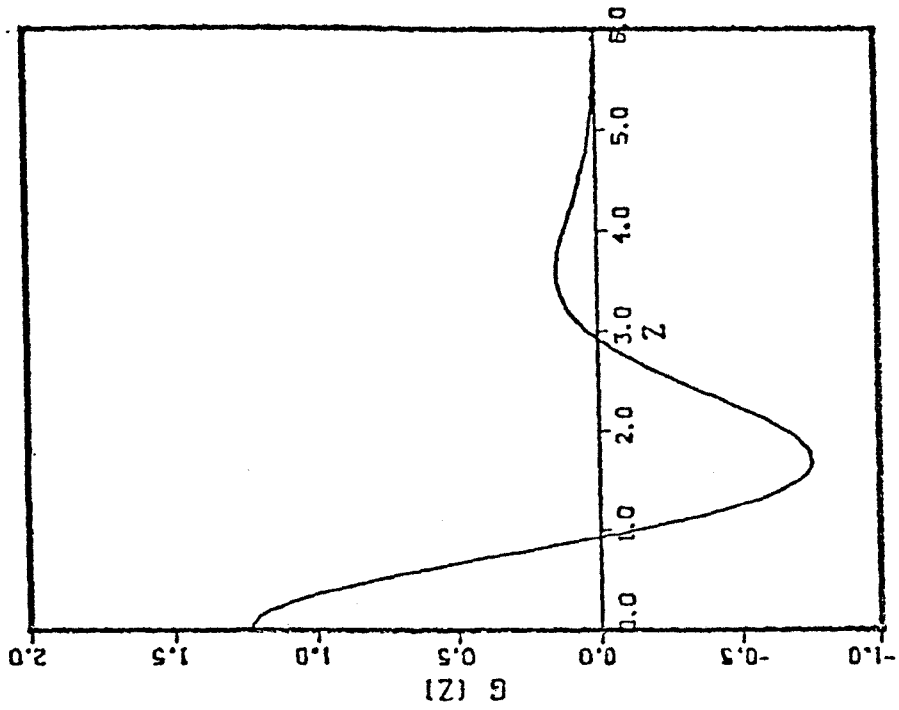

Figure 3.4 $g_7(z)$ (with $\upsilon = .8$)

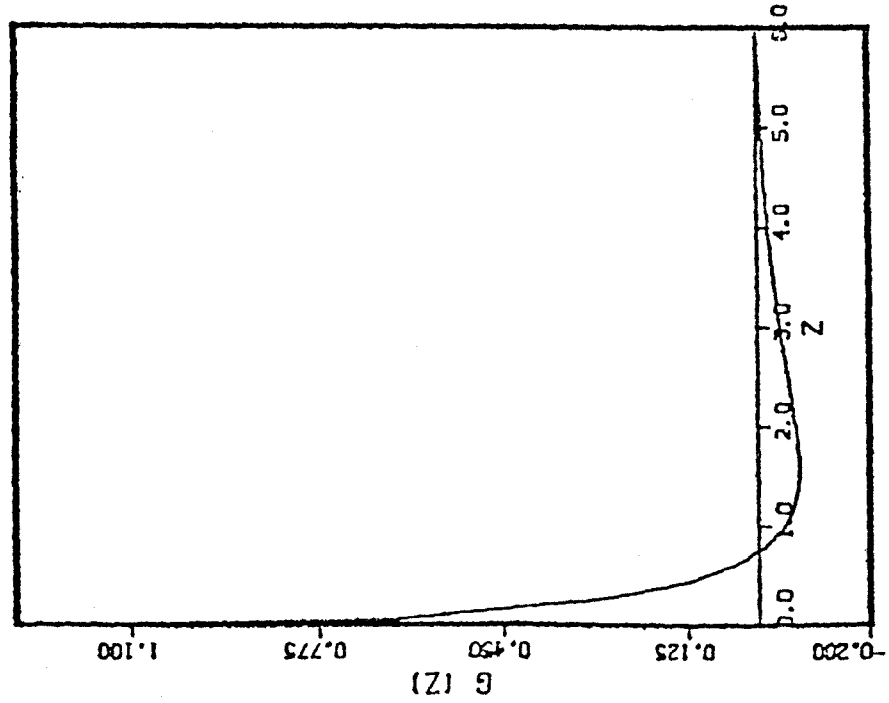

Figure 3.3 $g_4(z)$ (with $a=1$, $\gamma=2/3$)

4.1 Transformation To Physical Coordinates

For this entire chapter we restrict ourselves to two dimensions. Consider a continuum of rays leaving the initial wavefront. Let $X(t,\alpha)$ ($X_2(t,\alpha)$) be the scaled vertical (horizontal) position of a ray. From (1.2.18), (1.4.9):

$$|A(t,\alpha)|^2 = \left|\underline{X}_\alpha\right|^2 = \left(\frac{\partial X}{\partial \alpha}\right)^2 + \left(\frac{\partial X_2}{\partial \alpha}\right)^2 \tag{4.1.1}$$

We will approximate (4.1.1) by

$$A(t,\alpha) = \frac{\partial X}{\partial \alpha}(t,\alpha) \tag{4.1.2}$$

We have shown in section 2.6 that (4.1.2) is a good approximation in the sense that $\mathfrak{L}_N^A$ can be recovered from $\mathfrak{L}_{2N}$ and (4.1.2).

Now we write the obvious identity

$$\int d\alpha\, K(A(t,\alpha))\, \delta(X(t,\alpha)-X_0) =$$

$$\sum_{\substack{\{\alpha_i \mid \\ X(t,\alpha_i)=X_0\}}} K(A(t,\alpha_i)) \left|\frac{\partial \alpha}{\partial X(t,\alpha)}\right|\Bigg|_{\alpha=\alpha_i} \tag{4.1.3}$$

If we take $K(A) = f(A)|A|$ and use (4.1.2) in (4.1.3) we find:

$$\int d\alpha \; |A(t,\alpha)| \; f(A(t,\alpha)) \; \delta(X(t,\alpha)-X_0) =$$

$$\sum_{\substack{\{\alpha_i | \\ X(t,\alpha_i)=X_0\}}} f(A(t,\alpha_i)) \qquad (4.1.4)$$

Because we are tracking rays in a random medium, more than one ray may arrive at the same point in physical space. If we wish to sum a function of raytube area over all the rays that arrive at a fixed point in physical space then (4.1.4) tells us how.

We can generalize (4.1.4) to account for N fixed points in physical space. We have:

$$\int d\underline{\alpha} \left[\prod_{j=1}^{N} |A(t,\alpha_j)| \; \delta(X(t,\alpha_j)-Z_j) \right] f(A(t,\alpha_1),\ldots,A(t,\alpha_N))$$

$$= \sum_{\substack{\{\alpha_{ij} | \\ X(t,\alpha_{ij})=Z_i\}}} f(A(t,\alpha_{1j}),\ldots,A(t,\alpha_{Nj})) \qquad (4.1.5)$$

4.2 Expectations Taken in Physical Coordinates

In this section we relate statistics taken ray wise and statistics taken in physical space. We work out the case of one ray in detail.

Let $\underline{X}^0$ be a fixed point in physical space with coordinates $(t/\sigma^{2/3}, X_0)$. We define the expectation of a function of raytube area in physical space by:

$$E'_{X_0}[f(A(t,\alpha))] = E\Big[\sum_{\substack{\{\alpha_i \mid \\ X(t,\alpha_i)=X_0\}}} f(A(t,\alpha_i))\Big] \qquad (4.2.1)$$

The summation in (4.2.1) is over all those rays that arrive at the fixed point $\underline{X}^0$. We use (4.1.4) in the last expression for

$$E'_{X_0}[f(A(t,\alpha))] = E\Big[\int d\alpha\ |A(t,\alpha)|\ f(A(t,\alpha))\ \delta(X(t,\alpha)-X_0)\Big] \qquad (4.2.2)$$

For one ray, raytube area and ray position are independent random variables, see (2.5.26). Therefore, the expectation of the product in (4.2.2) is the product of the expectations:

$$E'_{X_0}[f(A(t,\alpha))] = \int d\alpha\ E[|A(t,\alpha)|\ f(A(t,\alpha))]\ E[\delta(X(t,\alpha)-X_0)] \qquad (4.2.3)$$

If the wave started with a planar initial wavefront then $E[|A(t,\alpha)|f(A(t,\alpha))]$ is independent of α, and can be removed from under the integral in (4.2.3). This yields

$$E'_{X_0}[f(A(t,\alpha))] = E[|A(t,\alpha)|f(A(t,\alpha))] \int d\alpha\ E[\delta(X(t,\alpha)-X_0)] \qquad (4.2.4)$$

To compute the second expectation in the right hand side of (4.2.4) we need to multiply by the probability density of $X(t,\alpha)$ and integrate over all possible valves for X. We do not know the probability density of $X(t,\alpha)$ alone, but we do know an equation for the joint probability density of $X(t,\alpha)$ and $V(t,\alpha)$; it is

described by $\mathfrak{L}_1^*$.

The joint probability density is given by $P(t,X,V;\alpha)$ where

$$P_t = \mathfrak{L}_1^* P = g_0 P_{VV} - V P_X$$

$$P(0,X,V;\alpha) = \delta(X-\alpha)\ \delta(V) \qquad (4.2.5)$$

Equation (4.2.5) can be solved by writing the Ito equations for (4.2.5):

$$dX = V\ dt$$

$$dV = \sqrt{2g_0}\ d\beta$$

$$\text{at } t=0: \quad X=\alpha,\ V=0 \qquad (4.2.6)$$

We solve (4.2.6) as

$$V(t) = \sqrt{2g_0} \int_0^t \beta(s)\ ds$$

$$X(t) = \alpha + \sqrt{2g_0} \int_0^t (t-s)\ \beta(s)\ ds \qquad (4.2.7)$$

Since V and X are integrals of the Wiener process, we expect them to have gaussian distributions. We calculate, from (4.2.7):

$$E[X]=\alpha$$

$$E[V]=0$$

$$E[V^2]=2g_0 E\left[\int_0^t \int_0^t \beta(s)\ \beta(u)\ ds\ du\right] = 2g_0 \int_0^t du = 2g_0 t$$

$$E[(X-\alpha)^2]=2g_0\frac{t^3}{3}$$

$$E[V(X-\alpha)]=2g_0\frac{t^2}{2} \qquad (4.2.8)$$

where we have used $E[\beta(s)\beta(u)]=\delta(s-u)$ and $E[\beta(s)]=0$. Knowing the means and covariances of a joint gaussian process allows us to write the probability density as:

$$P(t,X,V;\alpha) = \frac{1}{2\pi}\frac{1}{\sqrt{|\underline{C}|}}\exp\{-\frac{1}{2}\begin{pmatrix}X-\alpha\\V\end{pmatrix}^T\underline{\underline{C}}^{-1}\begin{pmatrix}X-\alpha\\V\end{pmatrix}\} \qquad (4.2.9)$$

where $\underline{\underline{C}}$ is the covariance matrix. Using (4.2.8) we find the solution;

$$P(t,X,V,\alpha) = \frac{\sqrt{3}}{2\pi}\frac{1}{g_0t^2}\exp\{-\frac{1}{g_0t^3}[t^2V^2-3tV(X-\alpha)+3(X-\alpha)^2]\} \qquad (4.2.10)$$

Now we can evaluate

$$\int d\alpha\, E[\delta(X(t,\alpha)-X_0)]=\int d\alpha\int dX\int dV\, P(t,X,V;\alpha)\,\delta(X-X_0)$$

$$=\int d\alpha\int dV\, P(t,X_0,V;\alpha)=\int dX_0\int dV\, P(t,X_0,V;\alpha)=1 \qquad (4.2.11)$$

The third equality follows from the symmetry of (4.2.10) in the α,X variables. The fourth equality comes from the fact that $P(t,X,V;\alpha)$ is a probability density, so it must integrate to one.

Using (4.2.11) in (4.2.4) we find

$$E'_{X_0}[f(A(t,\alpha))] = E[|A(t,\alpha)|f(A(t,\alpha))] \qquad (4.2.12)$$

when the initial wavefront is planar. The expectation on the right (left) hand side of (4.2.12) is the expectation taken raywise (in physical space).

We view the $|A(t,\alpha)|$ in (4.2.12) as the Jacobian of the mapping from α-space to X-space. That is, it relates the statistics along the initial wavefront to the statistics along the deformed wavefront in physical space.

Equation (4.2.12) and its derivation are essentially the same as in (6). Now we generalize (4.2.1) to account for N fixed points in physical space. We define

$$E'_{Z_1,\ldots,Z_N}[f(A(t,\alpha_1),\ldots,A(t,\alpha_N))]$$

$$= E\Big[\sum_{\substack{\{\alpha_{ij}|\\ X(t,\alpha_{ij})=Z_i\}}} f(A(t,\alpha_{ij}),\ldots,A(t,\alpha_{Nj}))\Big] \qquad (4.2.13)$$

Using (4.1.5) in (4.2.13) we find

$$E'_{Z_1,\ldots,Z_N}[f(A(t,\alpha_1),\ldots,A(t,\alpha_N))] =$$

$$E\Big[\int d\alpha \Big\{\prod_{j=1}^{N} |A(t,\alpha_j)|\, \delta(X(t,\alpha_j)-Z_j)\Big\} f[A(t,\alpha_1),\ldots,A(t,\alpha_N))]\Big] \qquad (4.2.14)$$

To evaluate the expectation on the right hand side of (4.2.14) we need to multiply by the joint density of $\{X(t,\alpha_j), A(t,\alpha_j)\}$ and integrate over all possible values. We do not know the joint density of $\{X(t,\alpha_j), A(t,\alpha_j)\}$, alone but we do know an equation for the joint density of $\{X(t,\alpha_j), A(t,\alpha_j), V(t,\alpha_j),$

$B(t,\alpha_j)\}$ - it is described by $\mathfrak{L}_N^A$.

So we evaluate (4.2.14) as

$$E'_{Z_1,\ldots,Z_N}[f(A(t,\alpha_1),\ldots A(t,\alpha_N))] = \int d\underline{X} \int d\underline{V} \int d\underline{A} \int d\underline{B} \int d\underline{\alpha}$$

$$P(t,\underline{X},\underline{V},\underline{A},\underline{B};\underline{\alpha})\ f(A_1,\ldots,A_N) \prod_{j=1}^{N} |A_j|\ \delta(X_j-Z_j)$$

$$= \int d\underline{V} \int d\underline{A} \int d\underline{B} \int d\underline{\alpha}\ P(t,\underline{Z},\underline{V},\underline{A},\underline{B};\underline{\alpha})\ f(A_1,\ldots,A_N) \prod_{j=1}^{N} |A_j| \qquad (4.2.15)$$

where

$$\mathfrak{L}_N^{A*} P = P_t$$

$$P(0,\underline{X},\underline{V},\underline{A},\underline{B};\underline{\alpha}) = \delta(\underline{X}-\underline{\alpha})\ \delta(\underline{V}-h'(\underline{\alpha}))\ \delta(\underline{A}-1)\ \delta(\underline{B}-h''(\underline{\alpha})) \qquad (4.2.16)$$

We can carry out the $\underline{\alpha}$ integration in (4.2.15) for

$$E'_{Z_1,\ldots,Z_N}[f(A(t,\alpha_1),\ldots,A(t,\alpha_N))] = \int d\underline{V} \int d\underline{A} \int d\underline{B}$$

$$Q(t,\underline{Z},\underline{V},\underline{A},\underline{B})\ f(A_1,\ldots,A_N) \prod_{j=1}^{N} |A_j| \qquad (4.2.17)$$

where

$$\mathfrak{L}_N^{A*} Q = Q_t$$

$$Q(0,\underline{X},\underline{V},\underline{A},\underline{B}) = \delta(\underline{V}-h'(\underline{X}))\ \delta(\underline{A}-1)\ \delta(\underline{B}-h''(\underline{X})) \qquad (4.2.18)$$

4.3 N-Point Energy Correlations

In this section we derive equations that describe the N-point energy correlation function.

Recall $\mathcal{E}(t,\alpha)$ was defined to be the energy associated with the ray $\underline{X}(t,\alpha)$ (see section 1.3). Equation (1.3.12) with (1.4.1) gives

$$\mathcal{E}(t,\alpha) = \frac{\mathcal{E}(0,\alpha)}{|A(t,\alpha)|}\frac{C(\underline{X}(t,\alpha))}{C(\underline{X}(0,\alpha))} = \frac{\mathcal{E}(0,\alpha)}{|A(t,\alpha)|}\frac{1+\sigma\hat{C}(\underline{X}(t,\alpha))}{1+\sigma\hat{C}(\underline{X}(0,\alpha))}$$

$$= \frac{\mathcal{E}(0,\alpha)}{|A(t,\alpha)|}\,(1+0(\sigma)) \qquad (4.3.1)$$

We define $\mathcal{E}(0,\alpha)=1$, so the initial wavefront has a uniform energy density. We discard the $0(\sigma)$ term in (4.3.1) and do not investigate the effects of this error. We now have:

$$\mathcal{E}(t,\alpha) = \frac{1}{|A(t,\alpha)|} \qquad (4.3.2)$$

We define the N point energy correlation function, at an upstream distance of $t\sigma^{-2/3}$, and at the points $\{Z_1,\ldots,Z_N\}$ to be

$$R_N(t,Z_1,\ldots,Z_N) = E\Big[\sum_{\substack{\{\alpha_{ij}\,|\\ X(t,\alpha_{ij})=Z_i\}}}\ \prod_{i=1}^{N}\mathcal{E}(t,\alpha_{ij})\Big]$$

$$= E\Big[\sum_{\substack{\{\alpha_{ij}\,|\\ X(t,\alpha_{ij})=Z_i\}}}\ \prod_{i=1}^{N}\frac{1}{|A(t,\alpha_{ij})|}\Big] \qquad (4.3.3)$$

Now (4.3.3) is of the form (4.1.5) with

$$f(A(t,\alpha_1),\ldots,A(t,\alpha_N)) = \prod_{i=1}^{N} \frac{1}{|A(t,\alpha_i)|} \qquad (4.3.4)$$

so we have (using (4.1.5)

$$R_N(t,\underline{Z}) = E\left[\int d\underline{\alpha} \prod_{j=1}^{N} \delta(X(t,\alpha_j)-Z_j) \right] \qquad (4.3.5)$$

To evaluate the expectation in (4.3.3) we multiply by the probability density of $\{X(t,\alpha_j),\ V(t,\alpha_j)\}$ and integrate over all possible valves. The forward Kolmogorov equation for $\mathfrak{L}_N$ gives the joint probability density of $\{X(t,\alpha_j),\ V(t,\alpha_j)\}$. We have:

$$R_N(t,\underline{Z}) = \int d\underline{X} \int d\underline{V} \int d\underline{\alpha}\ P(t,\underline{X},\underline{V};\underline{\alpha})\ \delta(\underline{X}-\underline{Z})$$

$$= \int d\underline{V} \int d\underline{\alpha}\ P(t,\underline{Z},\underline{V};\underline{\alpha}) \qquad (4.3.6)$$

where

$$\mathfrak{L}_N^{*} P = P_t$$

$$P(0,\underline{X},\underline{V};\underline{\alpha}) = \delta(\underline{X}-\underline{\alpha})\ \delta(\underline{V}-h'(\underline{\alpha})) \qquad (4.3.7)$$

We can carry out the $\underline{\alpha}$ integration in (4.3.6), (4.3.7) for

$$\mathcal{L}_N{}^* Q = Q_t$$

$$R_N(t,\underline{Z}) = \int d\underline{V}\ Q(t,\underline{Z},\underline{V})$$

$$Q(0,\underline{X},\underline{V}) = \delta(\underline{V}-h'(\underline{X})) \qquad (4.3.8)$$

Equation (4.3.8) is the main result of this section.

If we use $f(A)=1/|A|$ in (4.2.12) then we find (using (4.3.2)):

$$E'_{X_0}[\mathcal{E}(t,\alpha)] = E'_{X_0}\left[\frac{1}{|A(t,\alpha)|}\right] = E[1] = 1 \qquad (4.3.9)$$

Equation (4.3.9) says that energy is conserved. We used (4.3.9) as a partial check on our computer routines when we simulated the motion of rays in a random medium.

If we use $f(A) = 1/A^2$ in (4.2.12) then we find (using (4.3.2)):

$$E'_{X_0}[\mathcal{E}^2(t,\alpha)] = E'_{X_0}\left[\frac{1}{A^2(t,\alpha)}\right] = E\left[\frac{1}{|A(t,\alpha)|}\right] \qquad (4.3.10)$$

The last quantity in (4.3.10) is infinite, because the probability density to first focus, $\mu(\tau)$, is positive for $\tau>0$ (see (I.2.1)). Hence, the two point energy correlation function is infinite at the origin. We will discuss this fact in more detail later.

4.4 N-Point Energy Correlations, Plane Initial Wavefront

For a plane initial wavefront we can remove the "center of mass" of the N rays from (4.3.8). For h=0 we write (4.3.8) as

$$R_N(t,\underline{X}) = \int d\underline{V}\ Q(t,\underline{X},\underline{V}) \tag{4.4.1}$$

$$P_t = \mathfrak{L}_N^* P = (\nabla_{\underline{V}})^T \underline{g}\ \nabla_{\underline{V}}\ P - \underline{V}\cdot\nabla_{\underline{X}}\ P$$

$$P(0,\underline{X},\underline{V}) = \delta(\underline{V}) \tag{4.4.2}$$

where

$$g_{ij} = g(X_i - X_j) \tag{4.4.3}$$

and $\underline{X}$, $\underline{V}$ are vectors with N components.

We define the new variables

$$\begin{aligned} M_1 &= X_1 - X_2 \\ M_2 &= X_2 - X_3 \\ &\cdots \\ M_{N-1} &= X_{N-1} - X_N \\ M_N &= X_1 + X_2 + \ldots + X_N \end{aligned} \tag{4.4.4}$$

or

$$\underline{M} = \underline{T}\ \underline{X} \tag{4.4.5}$$

where

$$T_{ij} = \begin{cases} 1 & i=j,\ i=N \\ -1 & i+1=j \\ 0 & \text{otherwise} \end{cases} \qquad (4.4.6)$$

For example, for N=4, $\underline{\underline{T}}$ is

$$\underline{\underline{T}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We also define

$$\underline{W} = \underline{\underline{T}}\ \underline{V}$$

$$S(t,M,W) = \frac{P(t,\underline{X},\underline{V})}{\left|\underline{\underline{T}}\right|} \qquad (4.4.7)$$

We use (4.4.5), (4.4.7) in (4.4.1), (4.4.2) to find (see (2.6.14))

$$S_t = (\nabla_{\underline{W}})^T(\underline{\underline{T}}\ \underline{\underline{g}}\ \underline{\underline{T}}^T)\ \nabla_{\underline{W}}\ S - \underline{W}\cdot\nabla_{\underline{M}}\ S$$

$$S(0,\underline{M},\underline{W}) = \delta(\underline{W}) \qquad (4.4.8)$$

$$R_N(t,\underline{M}) = \int d\underline{W}\ S(t,\underline{M},\underline{W}) \qquad (4.4.9)$$

Now assume j>i to write

$$g_{ij} = g(X_i - X_j) = g(\ (X_i - X_{i+1}) + (X_{i+1} - X_{i+2}) + \ldots + (X_{j-1} - X_j)\ ) = g(M_i + M_{i+1} + \ldots + M_{j-1}) \qquad (4.4.10)$$

From (4.4.10) (and $g_{ij}=g_{ji}$) we conclude that g_{ij}, when written in the '$\underline{M}$' variables, does not depend on M_N. The variable M_N represents the "center of mass" of the N rays.

We will remove M_N, W_N from (4.4.8), (4.4.9). First define

$$\underline{M}^* = (M_1, M_2, \ldots, M_{N-1})^T$$
$$\underline{W}^* = (W_1, W_2, \ldots, W_{N-1})^T \qquad (4.4.11)$$

and write (4.4.8) as

$$S_t = (\nabla_{\underline{W}})^T \, (\underline{\underline{I}} \; \underline{\underline{G}} \; \underline{\underline{I}}^T) \, \nabla_{\underline{W}} \, S - \underline{W}^* \cdot \nabla_{\underline{M}^*} S - W_N \frac{\partial S}{\partial M_N}$$

$$S(0, \underline{M}, \underline{W}) = \delta(\underline{W}) \qquad (4.4.12)$$

Now consider the new equation:

$$Q_t = (\nabla_{\underline{W}})^T \, (\underline{\underline{I}} \; \underline{\underline{g}} \; \underline{\underline{I}}^T) \, \nabla_{\underline{W}} \, Q - \underline{W}^* \cdot \nabla_{\underline{M}^*} Q$$

$$Q(0, \underline{M}^*, \underline{W}) = \delta(\underline{W}) \qquad (4.4.13)$$

If (4.4.13) has a solution, then that solution satisfies (4.4.12). If the solution to (4.4.12) is unique, then the solution to (4.4.13) is the unique solution to (4.4.12). We assume the required existence and uniqueness so we can write (see (4.4.9)):

$$R_N(t,\underline{M}) = R_N(t,\underline{M}^*) = \int d\underline{W}\ Q(0,\underline{M}^*,\underline{W}) \qquad (4.4.14)$$

We will very often use this type of uniqueness and existence argument.

Now we define:

$$U(t,\underline{M}^*,\underline{M}^*)=\int d\underline{W}_N\ Q(t,\underline{M}^*,\underline{W}) \qquad (4.4.15)$$

If we assume that

$$\left.\frac{\partial Q(t,\underline{M}^*,\underline{W})}{\partial W_j}\right|_{|W_N|=\infty} = 0 \qquad (4.4.16)$$

for $j=1,2,\ldots,N$, then we can integrate the system (4.4.13), (4.4.14) with respect to W_N to obtain:

$$U_t = (\nabla_{\underline{W}^*})^T\ (\underline{T}\ \underline{g}\ \underline{T}^T)\ \nabla_{\underline{W}^*}\ U\ -\underline{W}^*\cdot\nabla_{\underline{M}^*}U$$

$$U(0,\underline{M}^*,\underline{M}^*) = \delta(\underline{W}^*)$$

$$R_N(t,M^*)=\int d\underline{W}^*\ U(t,\underline{M}^*,\underline{W}^*) \qquad (4.4.17)$$

The equations in (4.4.17) are the final form for the N point correlation function.

We expand (3.4.17) for N=2 to find

$$U_t = 2(g_0-g(M_1))\ U_{W_1W_1} - W_1U_{M_1}$$

$$U(0,M_1,W_1) = \delta(W_1)$$

$$R_2(t,M_1) = \int dW_1\ U(t,M_1,W_1) \qquad (4.4.18)$$

To simplify notation we define

$$f(z) = 2(g_0 - g(z)) \qquad (4.4.19)$$

so that (4.4.18) becomes:

$$U_t = f(M)\ U_{WW} - W\ U_M$$

$$U(0,M,W) = \delta(W)$$

$$R_2(t,M) = \int U(t,M,W)\ dW \qquad (4.4.20)$$

The system in (4.4.20) is the basis for the rest of this chapter. In chapter 7 we solve (4.4.20) numerically. Because the variable 't' in (4.4.20) looks so much like a 'time', we will refer to 't' as both 'time' and 'distance'.

Note that R_2 (in (4.4.20)) only depends on $\{t,M\}$. The variable M ($=M_1=X_1-X_2$) is the distance between the two points at which the energy is being received. This agrees with our intuition: in an isotropic and homogeneous random medium it is only the relative position of the observing points that is important. The "center of mass" of the two rays should not affect the correlation function.

For N=3, (4.4.17) becomes (with (4.4.19)):

$$U_t = -W_1 U_{M_1} - W_2 U_{M_2} + f(M_1) U_{W_1 W_1} + f(M_2) U_{W_2 W_2}$$

$$+ [f(M_1)+f(M_2)-f(M_1+M_2)]\, U_{W_1 W_2}$$

$$U(0,M_1,M_2,W_1,W_2) = \delta(W_1)\delta(W_2)$$

$$R_3(t,M_1,M_2) = \int dW_1 \int dW_2\; U(t,M_1,M_2,W_1,W_2)$$

(4.4.21)

For N=4, (4.4.17) becomes (with (4.4.19)):

$$U_t = -W_1 U_{M_1} - W_2 U_{M_2} - W_3 U_{M_3} + f(M_1) U_{W_1 W_1} + f(M_2) U_{W_2 W_2} + f(M_3) U_{W_3 W_3}$$

$$+ [f(M_1)+f(M_2)-f(M_1+M_2)]\, U_{W_1 W_2} + [f(M_2)+f(M_3)-f(M_2+M_3)] U_{W_2 W_3}$$

$$+ [f(M_2)+f(M_1+M_2+M_3)-f(M_1+M_2)-f(M_2+M_3)] U_{W_1 W_3}$$

$$U(0,M_1,M_2,M_3,W_1,W_2,W_3) = \delta(W_1)\delta(W_2)\delta(W_3)$$

$$R_4(t,M_1,M_2,M_3) = \int dW_1 \int dW_2 \int dW_3\; U(t,M_1,M_2,M_3,W_1,W_2,W_3)$$

(4.4.22)

4.5 Short Distance Approximation To The Two Point Energy Correlation Function: Parametrix Method

In this section we find the short distance approximation to

(4.4.20) using the parametrix method. We copy (4.4.20) as

$$U_t = f(M)\, U_{WW} - W\, U_M$$
$$U(0,M,W) = \delta(W)$$
$$R_2(t,M) = \int dW\, U(t,M,W) \qquad (4.5.1)$$

We write (4.5.1) as

$$V_t = f(M)\, V_{WW} - W\, V_M$$
$$V(0,M,W;\alpha) = \delta(W)\, \delta(M-\alpha) \qquad (4.5.2)$$
$$R_2(t,M) = \int d\alpha \int dW\, V(t,M,W;\alpha) \qquad (4.5.3)$$

The systems in (4.5.1) and (4.5.2), (4.5.3) are exactly equivalent.

The Ito equations corresponding to (4.5.2) are

$$dM = W\, dt \qquad (4.5.4)$$
$$dW = \sqrt{2f(M)}\, d\beta \qquad (4.5.5)$$
$$\text{at } t=0: \quad M=\alpha,\ W=0 \qquad (4.5.6)$$

In a short time, we do not expect W to vary much from its initial value since (see (4.5.5)) f(M) is bounded and $E[\beta]=0$. So W will be "close" to zero for short times. If $|W| \ll 1$ then we expect M to remain close to its initial value for short times (from (4.5.4)). We conclude that M will be "close" to α for short times.

Our approximation then is to approximate f(M) in (4.5.5) by

$f(\alpha)$. This is equivalent to approximating $f(M)$ by $f(\alpha)$ in (4.5.2). For short times, our approximation is to use the solution of

$$V_t = f(\alpha)\, V_{WW} - W\, V_M$$
$$V(0,M,W;\alpha) = \delta(W)\, \delta(M-\alpha) \qquad (4.5.7)$$

in (4.5.3). This approximation is really the first step of the parametrix method (see (17)).

We have solved equation (4.5.7) in section 4.2. We compare (4.5.7) to (4.2.5) and so the solution from (4.2.10) becomes:

$$V(t,M,W;\alpha) = \frac{\sqrt{3}}{2\pi} \frac{1}{f(\alpha)t^2} \exp\{-\frac{1}{f(\alpha)t^3}\,[t^2W^2 - 3tW(M-\alpha) + 3(M-\alpha)^2]\} \qquad (4.5.8)$$

We use (4.5.8) in (4.5.3) and carry out the W integration for:

$$R_2(t,M) = \left(\frac{3}{4\pi t^3}\right)^{1/2} \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{f(\alpha)}} \exp\{-\frac{3(M-\alpha)^2}{4t^3 f(\alpha)}\} \qquad (4.5.9)$$

We define $X=3/4t^3$, $C=\sqrt{X/\pi}$ and write (4.5.9) as

$$R_2(t,M) = C \int_{-\infty}^{\infty} d\mu\, \frac{e^{-\frac{X\mu^2}{f(M+\mu)}}}{\sqrt{f(M+\mu)}} \qquad (4.5.10)$$

If we fix $M>0$ and take $t<<1$ then $X\to\infty$ and (4.5.10) is easy to approximate using by Laplace's method. To compute several terms it is easier to use theorem 8.1 of Olver (18) with $\lambda=1$, $\mu=2$, $a=0$,

$b=\infty$.

We write (4.5.10) as

$$R_2(t,M) = C\ (\ J_1 + J_2)$$

$$J_1 = \int_0^\infty d\alpha\ \sqrt{h(M+\alpha)}\ e^{-X\alpha^2 h(M+\alpha)}$$

$$J_2 = \int_0^\infty d\alpha\ \sqrt{h(M-\alpha)}\ e^{-X\alpha^2 h(M-\alpha)} \qquad (4.5.11)$$

where $h(\alpha) = 1/f(\alpha)$. Now we can apply Olver's theorem to each of the two integrals in (4.5.11).

Since $g(z)$ is even, $f(z)$ is even (see(4.4.19)). We showed in chapter 3 that $g_0=g(0) > |g(z)|$ for $z\neq 0$ (see (3.2.3)). Hence $f(z)=0$ only at $z=0$. Therefore, the point of dominant contribution to J_1 and J_2 comes from the region about $\alpha=0$.

If we expand

$$h(M+\alpha) = h_0 + \alpha h_1 + \alpha^2 h_2 + O(\alpha^3) \qquad (4.5.12)$$

we find after some fairly routine calculations

$$R_2(t,M) \sim C\sqrt{\frac{\pi}{X}}\ (\ 1 + \frac{1}{2Xh_0^2}\ [\ \frac{h_1^2}{h_0} - h_2\ ]\ ) + O(\frac{1}{X^2}) \qquad (4.5.13)$$

If we expand

$$f(M+\alpha) = f_0 + \alpha f_1 + \alpha^2 f_2 + O(\alpha^4) \qquad (4.5.14)$$

then (4.5.13) becomes

$$R_2(t,M) \sim C\sqrt{\frac{\pi}{X}}\left(1 + \frac{f_2}{2X}\right) + O\left(\frac{1}{X^2}\right) \qquad (4.5.15)$$

Using (4.4.19) and (4.5.14) we find that $f_2=-g''(M)$, so (4.5.15) becomes

$$R_2(t,M) \sim C\sqrt{\frac{\pi}{X}}\left(1 - \frac{g''(M)}{2X}\right) + O\left(\frac{1}{X^2}\right) \qquad (4.5.16)$$

substituting for C and X in (4.5.16) we obtain our final result:

$$R_2(t,M) \sim 1 - \frac{2}{3}\, g''(M)\, t^3 + O(t^6) \qquad (4.5.17)$$

valid for M fixed and greater than zero, and $t\to 0$. Equation (4.5.17) agrees with the result obtained by regular perturbation techniques, see the appendix (A.1.32).

If M=0 in (4.5.10) then

$$R_2(t,M) > C \int_{|\mu|\leq\delta<<1} d\mu\, \frac{e^{-\frac{X\mu^2}{f(\mu)}}}{\sqrt{f(\mu)}}$$

$$\approx \frac{C}{\sqrt{2g_2}} \int_{|\mu|\leq\delta} d\mu\, \frac{e^{-\frac{X}{2g_2}}}{|\mu|} = \infty \qquad (4.5.18)$$

since $f(\mu)\sim 2g_2\mu^2$ for $|\mu|<<1$.

The fact that we can not evaluate $R_2(t,M)$ at M=0 is not surprising. What $R_2(t,0)$ evaluates is

$$E\Big[\sum_{\substack{\{\alpha_1,\alpha_2| \\ X(t,\alpha_1)=X(t,\alpha_2)=Z\}}} \frac{1}{|A(t,\alpha_1)|\ |A(t,\alpha_2)|}\Big] \qquad (4.5.19)$$

which is greater than or equal to

$$E\Big[\sum_{\substack{\{\alpha | \\ X(t,\alpha)=Z\}}} \frac{1}{|A(t,\alpha)|^2} \Big] \qquad (4.5.20)$$

but (4.5.20) is infinite, see the end of section 4.3.

Our application of Olver's theorem does not work at M=0 because the correct expansion for $h(\alpha)$ is now (see (4.5.12), (3.2.9)):

$$h(\alpha) = \frac{h_{-1}}{\alpha^2} + h_0 + O(\alpha^2) \qquad (4.5.21)$$

In section 4.7 we start with equation (4.5.10) and demonstrate that $R_2(t,M)$ varies as log M as $M \downarrow 0$, for all (finite) t.

4.6 Short Distance Approximation to The Two Point Energy Correlation Function: Fourier Transform Method

Another way of obtaining the short distance approximation to (4.4.20), which also turns out to be easier for computing higher order correction terms, is by Fourier Transforms.

We define the Fourier Transforms:

$$v(t,x,y) = \int \int e^{i(xM+yW)} U(t,M,W)\, dM\, dW$$

$$\hat{R}(t,x) = \int e^{ixM} R_2(t,M)\, dM$$

$$\hat{g}(x) = \int e^{ixM} g(M)\, dM$$

$$\hat{f}(x) = \int e^{ixM} f(M)\, dM \qquad (4.6.1)$$

From (4.4.20c) and (4.6.1) we have

$$\hat{R}(t,M) = v(t,x,0) \qquad (4.6.2)$$

From (4.4.19) and (4.6.1)

$$\hat{f}(x) = 4\pi\, g_0 \delta(x) - 2\hat{g}(x) \qquad (4.6.3)$$

We take the Fourier Transform of (4.4.20a,b) to find

$$v_t = x\, v_y - \frac{y^2}{2\pi} \int_{-\infty}^{\infty} db\ \hat{f}(x-b)\, v(t,b,y)$$

$$v(0,x,y) = 2\pi\, \delta(x) \qquad (4.6.4)$$

To obtain (4.6.4) we had to assume

$$U\Big|_{|M|=\infty} = 0,\quad U\Big|_{|W|=\infty} = 0,\quad U_W\Big|_{|W|=\infty} = 0 \qquad (4.6.5)$$

Using (4.6.3) in (4.6.4) gives

$$v_t = x\, v_y - \frac{y^2}{2\pi} \left\{ 4\pi g_0 v - 2 \int_{-\infty}^{\infty} db\ \hat{g}(x-b)\, v(t,b,y) \right\}$$

$$v(0,x,y) = 2\pi\ \delta(x) \tag{4.6.6}$$

Now we make the change of dependent variable

$$v(t,x,y) = 2\pi\ e^{-2g_0 t y^2}\ \delta(x) + \phi(t,x,y) \tag{4.6.7}$$

in (4.6.6), (4.2.6) to find

$$\phi_t = x\ \phi_y - 2y^2[g_0\phi - e^{-2g_0 t y^2}\ \hat{g}(x)] + \frac{y^2}{\pi}\int_{-\infty}^{\infty} db\ \hat{g}(x-b)\ \phi(t,b,y)$$

$$\phi(0,x,y)=0 \tag{4.6.8}$$

$$\hat{R}(t,x)= 2\pi\ \delta(x) + \phi(t,x,0) \tag{4.6.9}$$

So we see that changing variables from V to ϕ has removed the delta function from the initial conditions.

The system (4.6.8), (4.6.9) is still exact. Now we look for a short distance approximation. We suppose that, for short times, ϕ can be expanded in a Taylor series in t:

$$\phi(t,x,y) = \sum_{N=0}^{\infty} \phi_N(x,y)\ t^N \tag{4.6.10}$$

Using (4.6.10) in (4.6.8b) gives

$$\phi_0(x,y)=0 \tag{4.6.11}$$

Using (4.6.10) and (4.6.11) in (4.6.8a) and taking a Taylor series in t gives the sequence of algebraic equations:

$$\phi_1 = 2\, y^2\, \hat{g}(x) \tag{4.6.12}$$

$$(N+1)\phi_{N+1} = x\, \phi_{N,y} - 2y^2 g_0 \phi_N + \frac{2}{N!}\, y^2\, \hat{g}(x)\, (-2g_0 y^2)^N + y^2\, K[\phi_N] \tag{4.6.13}$$

for $N \geq 1$ where

$$K[q(x)] = \frac{1}{\pi} \int_{-\infty}^{\infty} db\, \hat{g}(x-b)\, q(b) \tag{4.6.14}$$

If we define

$$h_1(x) = K[\, \hat{g}(x)\,]$$

$$h_2(x) = K[\, x\, \hat{g}(x)\,]$$

$$h_3(x) = K[\, x^2\, \hat{g}(x)\,] \tag{4.6.15}$$

then we find

$$\phi_2 = 2xy\, \hat{g}(x) + y^4\{ - 4g_0 \hat{g}(x) + 2h_1(x)\}$$

$$\phi_3 = \frac{2}{3}\, x^2\, \hat{g}(x) + y^3\{ \frac{4}{3}\, x[2h_1(x) - 5g_0 \hat{g}(x)] + \frac{2}{3}\, h_2(x)\} + O(y^4)$$

$$\phi_4 = \frac{y^2}{4}\{8x^2[h_1(x) - \frac{8}{3} g_0\hat{g}(x)] + 2xh_2(x) + \frac{2}{3} h_3(x)\} + O(y^3)$$

$$\phi_5 = \frac{4}{5} xy\{\frac{1}{3} h_3(x) + xh_2(x) + x^2[4h_1(x) - \frac{32}{3} g_0\hat{g}(x)]\} + O(y^2)$$

$$\phi_6 = \frac{2x^2}{15}\{\frac{1}{3} h_3(x) + xh_2(x) + 4x^2[h_1(x) - \frac{8}{3} g_0\hat{g}(x)]\} + O(y) \tag{4.6.16}$$

We use (4.6.10), (4.6.11), (4.6.12), (4.6.16) in (4.6.9) to find

$$\hat{R}(t,x) = 2\pi\,\delta(x) + \frac{2}{3} x^2\,\hat{g}(x)\,t^3 + \frac{2x^2t^6}{45}\{h_3(x) + 3xh_2(x) + 4x^2[3h_1(x) - 8g_0\hat{g}(x)]\} + O(t^7) \tag{4.6.17}$$

We recognize the first two terms in (4.6.17) as being the same as (4.5.17).

Since ϕ has powers of Y in its short distance expansion, then U (see (4.6.1), (4.6.7)) has generalized functions in the variable W in its short distance expansion. When we form R_2 (see (4.4.20)) the generalized functions are integrated out, and we obtain a result containing only ordinary functions.

4.7 Singularity In The Two Point Correlation Function

Here we show that

$$R_2(t,M) \sim \left(\frac{3}{\pi 2 g_2 t^3}\right)^{1/2} e^{-\frac{3}{8 g_2 t^3}} \log M \quad \text{as } M \to 0 \qquad (4.7.1)$$

First we show (4.7.1) to be valid for short times, then for all (finite) time.

We start with equation (4.5.10), valid for short times, which can be put in the form:

$$R_2(t,M) = C \int_{-\infty}^{\infty} d\mu \, \frac{e^{-\frac{X(\mu-M)^2}{f(\mu)}}}{\sqrt{f(\mu)}} \qquad (4.7.2)$$

where $X = 3/4t^3$, $C = \sqrt{X/\pi}$. We fix δ, $0<\delta<<1$, and write (4.7.2) as

$$R_2(t,M) = C \left\{ \int_{|\mu|<\delta} d\mu + \int_{|\mu|>\delta} d\mu \right\} \frac{e^{-\frac{X(\mu-M)^2}{f(\mu)}}}{\sqrt{f(\mu)}} \qquad (4.7.3)$$

The second integral in (4.7.3) is easy to approximate (and bound):

$$C \int_{|\mu|>\delta} d\mu \, \frac{e^{-\frac{X(\mu-M)^2}{f(\mu)}}}{\sqrt{f(\mu)}} < \frac{C}{\sqrt{f_L}} \int_{|\mu|>\delta} d\mu \, e^{-\frac{X(\mu-M)^2}{f_H}}$$

$$< \frac{C}{\sqrt{f_L}} \int_{-\infty}^{\infty} d\mu \, e^{-\frac{X(\mu-M)^2}{f_H}}$$

$$= \frac{C}{\sqrt{f_L}} \left(\frac{\pi f_H}{X}\right)^{1/2} = \left(\frac{f_H}{f_L}\right)^{1/2} < \infty \qquad (4.7.4)$$

where

$$f_L = \min_{|\alpha|\geq\delta} f(\alpha) > 0$$
$$f_H = \max_{|\alpha|\geq\delta} f(\alpha) < \infty \qquad (4.7.5)$$

The first integral in (4.7.3) is

$$C \int_{-\delta}^{\delta} d\mu \frac{e^{-\frac{X(\mu-M)^2}{f(\mu)}}}{\sqrt{f(\mu)}} \qquad (4.7.6)$$

For small $|\mu|$ we approximate $f(\mu) \sim 2g_2\mu^2$ (see (3.2.9)) so (4.7.6) becomes

$$\frac{C}{\sqrt{g_2}} \int_{-\delta}^{\delta} d\mu \frac{e^{-\frac{X(\mu-M)^2}{2g_2\mu^2}}}{|\mu|} = \frac{C}{\sqrt{2g_2}} (I_1+I_2) \qquad (4.7.7)$$

where

$$I_1 = \int_0^{\delta} \frac{d\mu}{\mu} e^{-\frac{X}{2g_2}(1-\frac{M}{\mu})^2}$$
$$I_2 = -\int_{-\delta}^{0} \frac{d\mu}{\mu} e^{-\frac{X}{2g_2}(1-\frac{M}{\mu})^2} \qquad (4.7.8)$$

We change variables in (4.7.8) by $\zeta = 1-\frac{M}{\mu}$ to obtain

$$I_1 = \int_{-\infty}^{1-\frac{M}{\delta}} \frac{d\zeta}{1-\zeta} e^{-\frac{X\zeta^2}{2g_2}}$$
$$I_2 = -\int_{1+\frac{M}{\delta}}^{\infty} \frac{d\zeta}{1-\zeta} e^{-\frac{X\zeta^2}{2g_2}} \qquad (4.7.9)$$

Recall that δ is fixed. As $M \downarrow 0$, both I_1 and I_2 are singular. We

can show they each have a logarithmic singularity by using L'Hopitals rule:

$$\lim_{M\downarrow 0} \frac{I_1}{\log M} = \lim_{M\downarrow 0} \frac{\frac{\partial}{\partial M} I_1}{\frac{\partial}{\partial M} \log M} = - e^{-\frac{X}{2g_2}} \tag{4.7.10}$$

Likewise

$$\lim_{M\downarrow 0} \frac{I_2}{\log M} = - e^{-\frac{X}{2g_2}} \tag{4.7.11}$$

Combining (4.7.4), (4.7.7), (4.7.10), (4.7.11) in (4.7.3) we have:

$$R_2(t,M) = C \int_{-\delta}^{\delta} d\mu \, \frac{e^{-\frac{X(\mu-M)^2}{f(\mu)}}}{\sqrt{f(\mu)}} + O(1)$$

$$= \frac{C}{\sqrt{2g_2}} (I_1+I_2)+O(1)$$

$$\sim \frac{-2C}{\sqrt{2g_2}} \, e^{-\frac{X}{2g_2}} \log M \qquad \text{as } M\downarrow 0$$

$$\sim \left(\frac{3}{2g_2\pi t^3}\right)^{1/2} e^{-\frac{3}{3g_2t^3}} \log M \qquad \text{as } M\downarrow 0 \tag{4.7.12}$$

Hence, we have demonstrated (4.7.1) for short times. We now argue that (4.7.1) is valid for all times.

The singularity is present because (from (4.5.1)) $U(t,0,0)=\infty$. This is true because $U(t,0,0)$ starts at infinity due to the delta function initial conditions. At $W=0$, $M=0$ the

equation in (4.5.1) becomes $U_t(t,0,0)=0$, because $f(0)=0$. Therefore we conclude $U(t,0,0)=\infty$.

When we write (4.5.1) in the form

$$V_t = f(M)\, V_{WW} - W\, V_M$$

$$V(0,M,W;\alpha) = \delta(W)\, \delta(M-\alpha) \tag{4.7.13}$$

$$R_2(t,M) = \int d\alpha \int dW\; V(t,M,W;\alpha) \tag{4.7.14}$$

the singularity now comes from the α integration (in (4.7.14)), around $\alpha=0$. We write (4.7.14) as

$$R_2(t,M) = \left\{ \int_{|\alpha|>\delta} d\alpha + \int_{|\alpha|<\delta} d\alpha \right\} \int dW\; V(t,M,W;\alpha) \tag{4.7.15}$$

where δ is fixed and much smaller than one. Because the singularity comes from the region around $\alpha=0$ we write (4.7.15) as

$$R_2(t,M) = \int_{|\alpha|<\delta} d\alpha \int dW\; V(t,M,W;\alpha) + O(1) \tag{4.7.16}$$

We now show that, to evaluate (4.7.16), we can use $f(\alpha)$ for $f(M)$ in (4.7.13).

We write the Ito equations corresponding to (4.7.13) as (see (4.2.5), (4.2.6)):

$$dM = W\, dt$$

$$dW = \sqrt{2f(M)}\; d\beta$$

$$\text{at } t=0: \quad M=\alpha,\ W=0 \tag{4.7.17}$$

If we approximate $f(M)$ by $f(\alpha)$, which is tantamount to saying M stays "near" α, then we can approximate the error made by computing $E[(M-\alpha)^2]$. Doing this and using (4.2.8c) we find

$$E[(M-\alpha)^2] = \frac{2}{3} f(\alpha)\, t^3 \qquad (4.7.18)$$

In section 4.5 we assumed $E[(M-\alpha)^2]$ was small if $0<t<<1$. Now we have α at our disposal. Note that in (4.7.16) we want the solution for small values of α. For $|\alpha|$ small, we can expand $f(\alpha)=2g_2\alpha^2+0(\alpha^4)$. Therefore, for $|\alpha|<\delta<<1$ we can bound (4.7.18) by

$$E[(M-\alpha)^2] < \frac{4g_2}{3} \delta^2 t^3 \qquad (4.7.19)$$

For any (finite) value of t, we can make (4.7.19) arbitrarily small by choosing δ small enough. Therefore, for small enough δ, we can use $f(\alpha)$ for $f(M)$ in (4.7.13) to obtain a good approximation to (4.7.16).

Carrying this out we obtain

$$R_2(t,M) = C \int_{|\mu|<\delta} d\mu \; \frac{e^{-\frac{X(\mu-M)^2}{f(\mu)}}}{\sqrt{f(\mu)}} + 0(1) \qquad (4.7.20)$$

which gives the same result as (4.7.3).

We conclude that (4.7.1) is valid for all (finite) time.

5.1 Interpretation Of Ray Angle, Interface Conditions

In the generators $\mathfrak{L}_N^A$, $\mathfrak{L}_N$ (see (2.3.23), (2.4.7)) we used the variables X^L, V^L. Here we interpret V^L as the scaled angle that ray L makes with the horizontal.

Let Ψ_L be the angle that ray L makes with the horizontal. For ray number L, let V_1^L (V_2^L) be the velocity in the $\underline{i}$ ($\underline{j}$) direction on the 's' scale.

The angle of the ray is then given by:

$$\Psi_L = \tan^{-1}\left(\frac{V_2^L}{V_1^L}\right) \tag{5.1.1}$$

From (1.4.9) and (1.4.16)

$$\underline{V}(s) = \underline{i} + \sigma^{2/3}\underline{V}^\sigma(t)$$

$$\underline{V}^\sigma(t) = V_1^\sigma(t)\ \underline{i} + V_2^\sigma(t)\ \underline{j} \tag{5.1.2}$$

From (5.1.2) we conclude

$$V_1^L = 1 + \sigma^{2/3}\ V_1^\sigma$$

$$V_2^L = \sigma^{2/3}\ V_2^\sigma \tag{5.1.3}$$

Using (5.1.3) in (5.1.1),

$$\psi_L = \tan^{-1} \frac{\sigma^{2/3} V_2^\sigma}{1+\sigma^{2/3} V_1^\sigma}$$

$$= \sigma^{2/3} V_2^\sigma + O(\sigma^{4/3}) \qquad (5.1.4)$$

We scale Ψ_L to form Θ_L and then use (5.1.4) for:

$$\theta_L \triangleq \frac{\psi_L}{\sigma^{2/3}}$$

$$= V_2^\sigma + O(\sigma^{2/3}) \qquad (5.1.5)$$

So Θ_L, the scaled angle relative to the horizontal, is given by V_2^σ (to leading order). It will be recalled that we dispensed with the V_1^σ variable in (2.3.9) and then defined V_2^σ as V^L in (2.3.10). So the V^L in $\mathfrak{L}_N$ represents the scaled angle that ray L makes with the horizontal.

This fact will enable us to trace rays through successive media. Since V^L is order one, then Ψ_L (the actual angle a ray makes with the horizontal) is very small (of order $\sigma^{2/3}$, see (5.1.5)).

Using the geometric optics approximation to the wave equation, across an interface the rays must satisfy Snell's law (see (19), A12.11):

$$\sin \alpha_1 = \frac{C_2}{C_1} \sin \alpha_2 \qquad (5.1.6)$$

where C_i (α_i) is the local wave speed (angle to the normal) in

medium i.

If the interface is along the $\underline{j}$ axis (so it is parallel to the mean initial wavefront), then the angles to the normal (α_i) will be the angle a ray makes with the horizontal (Ψ_i). Therefore we take $C_i=1+0(\sigma)$ (or $C_0=1$ in (1.4.1)), and $\alpha_i=0(\sigma^{2/3})$. Then (5.1.6) becomes

$$\alpha_1 = \alpha_2\left(1+ 0(\sigma \quad)\right) \tag{5.1.7}$$

This says that the angle of incidence equals the angle of refraction, for small angles.

Define S to be the interface between two media and let $\underline{N}$ be normal to S. Let U_i be the solution to the wave equation (1.1.1) in medium i. If we require

$$\begin{aligned} \underline{N}\cdot\nabla U_1 &= \underline{N}\cdot\nabla U_2 \\ U_1 &= U_2 \end{aligned} \tag{5.1.8}$$

on S then we need not consider reflected rays. If the parameter 'Z' is defined by (see (19), A12.13)

$$Z = \frac{C_1\cos\ \alpha_1}{C_2\cos\ \alpha_2} \tag{5.1.9}$$

then the leading order reflection (r) and transmission (t) coefficients are given by ((19), A12.16, A12.17)

$$r= \frac{1-Z}{1+Z} \quad , \quad t= \frac{2}{1+Z} \tag{5.1.10}$$

Using $C_i=1+0(\sigma)$, $\alpha_i=0(\sigma^{2/3})$ we find $Z=1+0(\sigma)$ and so

$r=0(\sigma)$, $t=1+0(\sigma)$. Therefore, to first order, there are no reflected rays.

Now suppose we find the probability density for the positions and scaled angles of N rays. From Snell's law, when the rays arrive at an interface, we use the value of V^L in the preceeding medium as the starting value for V^L in the new medium. That is (see (2.4.2), (2.4.6)):

$$h'(\alpha_L)\Big|_{\text{new medium}} = V^L\Big|_{\text{last medium}} \qquad \text{on } S \quad (5.1.11)$$

5.2 Two Point Energy Correlations, N Media

In this section we find the equations that describe the two point energy correlation for a wave that has travelled through N media, with planar and parallel interfaces.

We take the velocity field in medium i to be $C_i(\underline{X})=1+\sigma\hat{C}_i(\underline{X})$. We define $g^i(z)$ ($f_i(z)$) to be the $g(z)$ ($f(z)$) function in medium i. We suppose that medium i has a width of $s_i=t_i\sigma^{-2/3}$, and define $\underline{t}_k=(t_1, t_2,\ldots,t_k)$.

We require the boundary between the media to be planar and parallel (to each other and the mean initial wavefront). This is required to ensure that all the rays described by $\mathfrak{L}_N$ reach each medium at the same time (to leading order). This follows from the scaling in chapter one.

From (1.4.9), (1.4.8), (1.4.16), (2.3.9):

$$X(s) \cdot \underline{i} = (s\,\underline{i} + \underline{X}^{\sigma})\cdot\underline{i}$$

$$= \frac{t}{\sigma^{2/3}} + X_1^{\sigma}(t) = \frac{t}{\sigma^{2/3}} + o(1) \qquad (5.2.1)$$

Therefore, all the rays will be at a horizontal distance of $t/\sigma^{2/3}$ to leading order, independent of their starting angles.

To find the two point energy correlation for rays going through N media we can still use (4.3.6):

$$R_2(\underline{t}_N,\underline{Z}) = \int d\underline{\alpha} \int d\underline{V}\; P_N(\underline{t}_N,\underline{Z},\underline{V};\underline{\alpha}) \qquad (5.2.2)$$

where $P_N(\underline{t}_N,\underline{Z},\underline{V};\underline{\alpha})$ is the probability density of two rays starting from a wavefront at positions (α_1,α_2), travelling a scaled distance t_i in medium i (i=1, 2,..., N); and then the rays have positions (Z_1, Z_2) and scaled angles (V_1, V_2).

Clearly the initial condition for $P_N(\underline{t}_N,\underline{Z},\underline{V};\underline{\alpha})$ is:

$$P_N(\underline{0}_N,\underline{Z},\underline{V};\underline{\alpha}) = \delta(Z_1-\alpha_1)\delta(Z_2-\alpha_2)\delta(V_1-h'(\alpha_1))\delta(V_1-h'(\alpha_2)) \qquad (5.2.3)$$

Define ${}_i\mathfrak{L}_2$ to be $\mathfrak{L}_2$ with g(z) replaced by $g^i(z)$. While the rays are in medium i, ${}_i\mathfrak{L}_2$ describes the motion of the rays. Define $q_i(t_i,\underline{X},\underline{V};\underline{\eta},\underline{\zeta})$ to be the solution of:

$$(q_i)_{t_i} = {}_i\mathfrak{L}_2^{*}(q_i) \qquad (5.2.4)$$

$$q_i(t_i,\underline{X},\underline{V};\underline{\eta},\underline{\zeta}) = \delta(\underline{X}-\underline{\eta})\ \delta(\underline{V}-\underline{\zeta})$$

Hence, $q_i(t_i,\underline{X},\underline{V};\underline{\eta},\underline{\zeta})$ is the density for the position and angles of two rays travelling in medium i, that started from (η_1,η_2) with scaled angles (ζ_1,ζ_2). We will now find $P_N(\underline{t}_N,\underline{Z},\underline{V};\underline{\alpha})$ in terms of the $\{q_i\}_{i=1}^{N}$ by tracing two rays through the successive media.

The probability density for the positions, $(\underline{X}_1)$ and scaled angles $(\underline{V}_1)$ in the first medium is given by $q_1(t_1,\underline{X}_1,\underline{V}_1;\underline{\alpha},h'(\underline{\alpha}))$. When the rays arrive at the second medium (i.e., the first interface) we continue the motion of the rays by finding $q_2(t_2,\underline{X}_2,\underline{V}_2;\underline{X}_1,\underline{V}_1)$. What $q_2(t_2,\underline{X}_2,\underline{V}_2;\underline{X}_1,\underline{V}_1)$ represents is the probability density for the positions $(\underline{X}_2)$ and scaled angles $(\underline{V}_2)$ of two rays which started at the positions $(\underline{X}_1)$ and the scaled angles $(\underline{V}_1)$.

Now the Chapman-Kolmogorov equation can be used. To find the probability density for two rays that have gone through two media we integrate over all possible values of positions and scaled angles on the interface ($\underline{X}_1$ and $\underline{V}_1$). This yields:

$$P_2(\underline{t}_2,\underline{X}_2,\underline{V}_2;\underline{\alpha}) = \int d\underline{X}_1 \int d\underline{V}_1$$

$$q_1(t_1,\underline{X}_1,\underline{V}_1;\underline{\alpha},h'(\underline{\alpha}))\ q_2(t_2,\underline{X}_2,\underline{V}_2;\underline{X}_1,\underline{V}_1) \qquad (5.2.5)$$

Now we must solve (5.2.4) to find $q_3(t_3,\underline{X}_3,\underline{V}_3;\underline{X}_2,\underline{V}_2)$. Since P_2 has the probability density of two rays going through two media, and q_3 has the probability density of two rays going

through the third medium, we can again apply the Chapman-Kolmogorov equation to obtain:

$$P_3(\underline{t}_3,\underline{X}_3,\underline{V}_3;\underline{\alpha}) = \int d\underline{X}_2 \int d\underline{V}_2$$

$$P_2(\underline{t}_2,\underline{X}_2,\underline{V}_2;\underline{\alpha})\ q_3(t_3,\underline{X}_3,\underline{V}_3;\underline{X}_2,\underline{V}_2) \qquad (5.2.6)$$

In general we find:

$$P_k(\underline{t}_k,\underline{X}_k,\underline{V}_k;\underline{\alpha}) = \int d\underline{X}_{k-1} \int d\underline{V}_{k-1}$$

$$P_{k-1}(\underline{t}_{k-1},\underline{X}_{k-1},\underline{V}_{k-1};\underline{\alpha})\ q_k(t_k,\underline{X}_k,\underline{V}_k;\underline{X}_{k-1},\underline{V}_{k-1})$$

$$(\text{for } k=2,3,\ldots,N) \qquad (5.2.7)$$

$$P_1(\underline{t}_1,\underline{X}_1,\underline{V}_1;\underline{\alpha}) = q_1(t_1,\underline{X}_1,\underline{V}_1;\underline{\alpha},h'(\underline{\alpha})) \qquad (5.2.8)$$

where we still have to evaluate (5.2.2) to find $R_2(\underline{t}_N,\underline{Z})$.

We now simplify (5.2.7), (5.2.8). Note that ${}_k\mathfrak{L}_2$ is a differential operator with respect to the variables $(\underline{X}_k,\underline{V}_k)$. Applying ${}_k\mathfrak{L}_2^*$ to (5.2.7) and using (5.2.4) we find

$${}_k\mathfrak{L}_2^* P_k = {}_k\mathfrak{L}_2^* \int d\underline{X}_{k-1} \int d\underline{V}_{k-1}\ P_{k-1} q_k$$

$$= \int d\underline{X}_{k-1} \int d\underline{V}_{k-1}\ P_{k-1}\ {}_k\mathfrak{L}_2^*\ q_k$$

$$= \int d\underline{X}_{k-1} \int d\underline{V}_{k-1}\ P_{k-1}\ (q_k)_{t_k}$$

$$= \frac{\partial}{\partial t_k} \int d\underline{X}_{k-1} \int d\underline{V}_{k-1} \; P_{k-1} q_k$$

$$= (P_k)_{t_k} \qquad \text{(for } k = 2, 3, \ldots, N\text{)} \qquad (5.2.9)$$

To find the initial conditions for (5.2.9) we evaluate (5.2.7) at $t_k=0$ and use (5.2.4) for

$$P_k(\underline{t}_{k-1}, 0, \underline{X}_k, \underline{V}_k; \underline{\alpha}) = \int d\underline{X}_{k-1} \int d\underline{V}_{k-1}$$

$$P_{k-1}(\underline{t}_{k-1}, \underline{X}_{k-1}, \underline{V}_{k-1}; \underline{\alpha}) \; \delta(\underline{X}_k - \underline{X}_{k-1}) \delta(\underline{V}_k - \underline{V}_{k-1})$$

$$= P_{k-1}(\underline{t}_{k-1}, \underline{X}_k, \underline{V}_k; \underline{\alpha}) \qquad \text{(for } k=2,3,\ldots,N\text{)} \qquad (5.2.10)$$

From (5.2.8) and (5.2.4) we clearly have

$$1 \mathfrak{L}_2^* \; P_1 = (P_1)_{t_1}$$

$$P_1(0, \underline{X}_1, \underline{V}_1; \underline{\alpha}) = \delta(\underline{X}_1 - \underline{\alpha}) \; \delta(\underline{V}_1 - h'((\underline{\alpha})) \qquad (5.2.11)$$

To find $R_2(\underline{t}_N, \underline{Z})$ we solve (5.2.11) and then solve (5.2.9), (5.2.10) sucessively for k=2, 3,..., N. Then we can evaluate (5.2.2).

One further simplification is possible. We can integrate out the $\underline{\alpha}$ in (5.2.11), (5.2.9), (5.2.10), (5.2.2) to obtain the final form of our equations:

$$ {}_k\mathcal{L}_2^* \, Q_k = (Q_k)_{t_k} \qquad (\text{for } k=1,2,\ldots,N) $$

$$ Q_k(\underline{t}_{k-1},0,\underline{X}_k,\underline{V}_k) = Q_{k-1}(\underline{t}_{k-1},\underline{X}_k,\underline{V}_k) \qquad (\text{for } k=2,3,\ldots,N) $$

$$ Q_1(0,\underline{X}_1,\underline{V}_1) = \delta(\underline{V}_1 - h'(\underline{X}_1)) $$

$$ R_2(\underline{t}_N,\underline{Z}) = \int d\underline{V}_N \; Q_N(\underline{t}_N,\underline{Z},\underline{V}_N) \qquad (5.2.12) $$

5.3 Two Point Energy Correlations, Plane Initial Wavefront

We can simplify the final formulae in section 5.2 if we have a plane initial wavefront. Using h=0 in (5.2.12) we have

$$ {}_k\mathcal{L}_2^* \, P_k = (P_k)_{t_k} \qquad (\text{for } k=1,2,\ldots,N) $$

$$ P_k(\underline{t}_{k-1},0,\underline{X}_k,\underline{V}_k) = P_{k-1}(\underline{t}_{k-1},\underline{X}_k,\underline{V}_k) \qquad (\text{for } k=2,3,\ldots,N) $$

$$ P_1(0,\underline{X}_1,\underline{V}_1) = \delta(\underline{V}_1) $$

$$ R_2(\underline{t}_N,\underline{X}_N) = \int d\underline{V}_N \; P_N(\underline{t}_N,\underline{X}_N,\underline{V}_N) \qquad (5.3.1) $$

We change variables in (5.3.1) by

$$ \underline{Y}_k = \underline{\underline{T}} \, \underline{X}_k = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \underline{X}_k $$

$$ \underline{U}_k = \underline{\underline{T}} \, \underline{V}_k $$

$$Q_k(\underline{t}_k, \underline{X}_k, \underline{V}_k) = \tfrac{1}{2} P_k(\underline{t}_k, \underline{X}_k, \underline{V}_k) \qquad (5.3.2)$$

This is the same change of variables we used in section 3.4. Once again we will remove the "center of mass" coordinates from the system (5.3.1). Using (5.3.2) in (5.3.1) produces:

$$(Q_k)_{t_k} = {}_k\mathfrak{L}_2^{*}\, Q_k = -U_k^2 (Q_k)_{Y_k^2} + 2(g_0^k + g^k(Y_k^1))\, (Q_k)_{U_k^2 U^2{}_k} + \mathfrak{L}_k^{+}\, Q_k \qquad (\text{for } k=1,2,\ldots,N)$$

$$Q_k(\underline{t}_{k-1}, 0, \underline{Y}_k, \underline{U}_k) = Q_{k-1}(\underline{t}_{k-1}, \underline{Y}_k, \underline{U}_k) \qquad (\text{for } k=2,3,\ldots,N)$$

$$Q_1(0, \underline{Y}_1, \underline{U}_1) = \delta(\underline{U}_1)$$

$$R_2(\underline{t}_N, \underline{Y}_N) = \int d\underline{U}_N\, Q_N(\underline{t}_N, \underline{Y}_N, \underline{U}_N) \qquad (5.3.3)$$

where $g_0^k = g^k(0)$ and

$$\mathfrak{L}_k^{+} = -U_k^1 (Q_k)_{Y_k^1} + f_k (Y_k^1)_{U_k^1 U_k^1} \qquad (5.3.4)$$

We can now remove the $(Y_k^2,\ U_k^2)$ variables from (5.3.3). The procedure is as follows: (for $k=1,2,\ldots,N$)

1) Define $Q_k^{+}(\underline{t}_k, Y_k^1, \underline{U}_k)$ to be the solution of:

$$(Q_k^{+})_{t_k} = 2(g_0^k + g^k(Y_k^1))\, (Q_k^{+})_{U_k^2 U_k^2} + \mathfrak{L}_k^{+}\, Q_k^{+} \qquad (5.3.5)$$

with the initial conditions

$$\text{if } k=1 \qquad Q_1^+(0,Y_k^1,\underline{U}_1) = \delta(\underline{U}_1)$$

$$\text{if } k>1 \qquad Q_k^+(\underline{t}_{k-1},0,Y_k^1,\underline{U}_k) = Q_{k-1}^+(\underline{t}_{k-1},Y_k^1,\underline{U}_k) \tag{5.3.6}$$

Note that (5.3.5) and (5.3.6) do not depend on the variable Y_k^2. Observe that if (5.3.5), (5.3.6) has a solution, and if (5.3.3) has a unique solution for every k, then $Q_k=Q_k^+$. We assume the required existence and uniqueness.

2) Define $\Psi_k(\underline{t}_k,Y_k^1,U_K^1) = \int dU_k^2\ Q_k^+(\underline{t}_k,Y_k^1,\underline{U}_k)$.

Now integrate (5.3.5), (5.3.6) with respect to U_k^2. We assume that $(Q_k^+)\Big|_{|U_k^2|=\infty}=0$ so that (5.3.5), (5.3.6) become

$$(\Psi_k)_{t_k} = \mathfrak{L}_k^+\ \Psi_k$$

$$\text{if } k=1: \qquad \Psi_1(0,Y_1^1,U_1^1) = \delta(U_1^1)$$

$$\text{if } k>1: \qquad \Psi_k(\underline{t}_{k-1},0,Y_k^1,U_k^1) = \Psi_{k-1}(\underline{t}_{k-1},Y_k^1,U_k^1) \tag{5.3.7}$$

The final equations we obtain are:

$$(\Psi_k)_{t_k} = \mathcal{L}_k^+ \Psi_k = - U_k^1 (\Psi_k)_{Y_k^1} + f_k(Y_k^1)\ (\Psi_k)_{U_k^1 U_k^1}$$

$$(\text{for } k=1,2,\ldots,N)$$

$$\Psi_k(\underline{t}_{k-1}, 0, Y_k^1, U_k^1) = \Psi_{k-1}(\underline{t}_{k-1}, Y_k^1, U_k^1) \qquad (\text{for } k=2,3,\ldots,N)$$

$$\Psi_1(0, Y_1^1, U_1^1) = \delta(U_1^1)$$

$$R_2(\underline{t}_N, Y_N^1) = \int dU_N^1\ \Psi_N(\underline{t}_N, Y_N^1, U_N^1) \qquad (5.3.8)$$

The equations in (5.3.8) are the final form we obtain. Numerically they are no more difficult to solve than (4.4.20). We can use the computer code described in chapter 7 to solve for Ψ_1, for $0<t<t_1$. We use the results of this computation for Ψ_1 as the initial conditions for Ψ_2. To find Ψ_2 we change the f(z) function from $f_1(z)$ to $f_2(z)$, and solve for $0<t<t_2$. Then the value of Ψ_2 is used as the initial condition for Ψ_3, etc.

The only change required in the code described in chapter 7 is to permit different f(z) functions to be used at different times.

5.4 Shadowgraph Problem

The shadowgraph problem is to find the correlation of energy after a plane wave passes through a random medium and then a uniform medium. This is the way many experiments are set up; the

recording device is not in the random medium but separated by some distance. The shadowgraph method has been extensively employed in the analysis of the turbulent wakes of models of hypersonic vehicles.

In the shadowgraph problem, focusing of raytubes can occur either inside the random medium (this is what Kulkarny and White (6) investigated.) or outside the random medium, by purely geometrical considerations. This is because the rays leaving the random medium will have random angles.

Weyl (20) was one of the first investigators of the shadowgraph problem. Later results can be found in Uberoi and Kovasznay (21) and Taylor (10). All these results are restricted to regimes where focusing has not occurred in either the random medium or the uniform medium.

It is straightforward to use the regular perturbation method with geometrical optics to find the energy correlation, when focusing occurs either in the random medium or in the uniform medium. In section A.2 we present this method, which is just an extension of Tatarski's (5) analysis. Hesselink and White (11) analyzed their experimental data using an inversion scheme based on a procedure similiar to that of section A.2.

We can use the results of section 5.3 (with N=2) to obtain an expression for the correlation of energy obtained by the shadowgraph method.

We assume the random medium has a velocity field of the form $C=1+\sigma\hat{C}$ and has a width of $t_1\sigma^{-2/3}$. The second medium, which is uniform and has C=1, is of width $t_2\sigma^{-2/3}$ (see Figure 5.1).

We copy (5.3.8) for N=2 and remove the superscripts from the variables for:

$$\Psi_1(0,Y_1,U_1) = \delta(U_1)$$

$$\Psi_{1,t_1} = - U_1 \Psi_{1,Y_1} + f_1(Y_1) \Psi_{1,U_1U_1} \qquad (5.4.1)$$

$$\psi_2(t_1,0,Y_2,U_2) = \Psi_1(t_1,Y_2,U_2)$$

$$\Psi_{2,t_2} = -U_2 \Psi_{2,Y_2} + f_2(Y_2) \Psi_{2,U_2U_2} \qquad (5.4.2)$$

$$R_2(t_1,t_2,Y_2) = \int dU_2 \, \psi_2(t_1,t_2,Y_2,U_2) \qquad (5.4.3)$$

The second medium is uniform, so that $g^2(x)=0$ and $f_2(x)=0$. Using this in (5.4.2) we have

$$\Psi_{2,t_2} = - U_2 \Psi_{2,Y_2}$$

$$\psi_2(t_1,0,Y_2,U_2) = \Psi_1(t_1,Y_2,U_2) \qquad (5.4.4)$$

The equation in (5.4.4) is a wave equation. We have Ψ_2=constant on lines where $Y_2-U_2t_2$=constant. Hence, the solution of (5.4.4) is

$$\psi_2(t_1,t_2,Y_2,U_2) = \Psi_1(t_1,Y_2-U_2t_2,U_2) \qquad (5.4.5)$$

Using (5.4.5) in (5.4.3):

$$R_2(t_1,t_2,Y) = \int dU_2\ \Psi_1(t_1,Y_2-U_2t_2,U_2) \qquad (5.4.6)$$

When equation (5.4.1) is solved for Ψ_1, and the result used in (5.4.6) we obtain the shadowgraph energy correlation. In general, we can proceed no further.

5.5 Short distance approximation for the shadowgraph problem

In the special case when $t_1 \ll 1$ we can approximate the solution to (5.4.1), (5.4.6) and so find the energy correlation function for the shadowgraph problem.

We write equations (5.4.1), (5.4.6) in the equivalent form (with t for t_1):

$$\Psi_t = f(Y)\ \Psi_{UU} - U\ \Psi_Y$$
$$\Psi(0,Y,U;\alpha) = \delta(U)\ \delta(Y-\alpha) \qquad (5.5.1)$$
$$R_2(t,t_2,Y) = \int d\alpha \int dU\ \Psi(t,Y-Ut_2,U;\alpha) \qquad (5.5.2)$$

For short times we approximate f(Y) by $f(\alpha)$ in (5.5.1) (see section 4.5). The equation for Ψ becomes

$$\Psi_t = f(\alpha)\ \Psi_{UU} - U\ \Psi_Y$$
$$\Psi(0,Y,U;\alpha) = \delta(U)\ \delta(Y-\alpha) \qquad (5.5.3)$$

The solution to (5.5.3) is in (4.2.5), (4.2.10). We have

$$\Psi(t,Y,U;\alpha)= \frac{\sqrt{3}}{2\pi} \frac{1}{f(\alpha)t^2} \exp\{- \frac{1}{f(\alpha)t^3} [t^2U^2-3tU(Y-\alpha)+3(Y-\alpha)^2]\} \tag{5.5.4}$$

Using (5.5.4) in (5.5.2) we can evaluate the integral with respect to U exactly (the integrand is only a gaussian in U). We obtain:

$$R_2(t,t_2,Y)= (\frac{3}{4\pi Lt^3})^{1/2} \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{f(\alpha)}} \exp\{- \frac{(Y-\alpha)^2}{f(\alpha)} \frac{3}{4Lt^3}\} \tag{5.5.5}$$

where

$$L = 1 + 3 (\frac{t_2}{t}) + 3 (\frac{t_2}{t})^2 \tag{5.5.6}$$

For $t<<1$, we have $t^3L>>1$ and we can use Laplace's method on (5.5.5). In fact, (5.5.5) is the exact same integral as (4.5.9), with t replaced by $tL^{1/3}$. Therefore we can use the answer we obtained before, (4.5.17), with $t=tL^{1/3}$. We find

$$R_2(t,t_2,Y) = 1 - \frac{2}{3} g''(Y)\, t^3\{1+3(\frac{t_2}{t})+3(\frac{t_2}{t})^2\} + O(t^6+t_2{}^4t^2) \tag{5.5.7}$$

Equation (5.5.7) is the same answer we obtain by regular perturbation methods (see (A.2.20)).

Note that (5.5.7) is valid for arbitrarily large t_2, if t is sufficiently small. This explains the experimental results of Hesselink and White (11).

In their experiment, t is small enough for (5.5.7) to be valid. They used (5.5.7) to find the spectrum of the medium (which is the same as knowing g(Y), see (3.1.9)) at two values of t_2. They were concerned that they obtained the same spectrum, at

each value of t_2, while the shadowgraphs showed that much more focusing had occurred for larger values of t_2. We know now that (5.5.7) is valid for large t_2, and the increased focusing they observed must have been due to the geometrical focusing in the uniform medium.

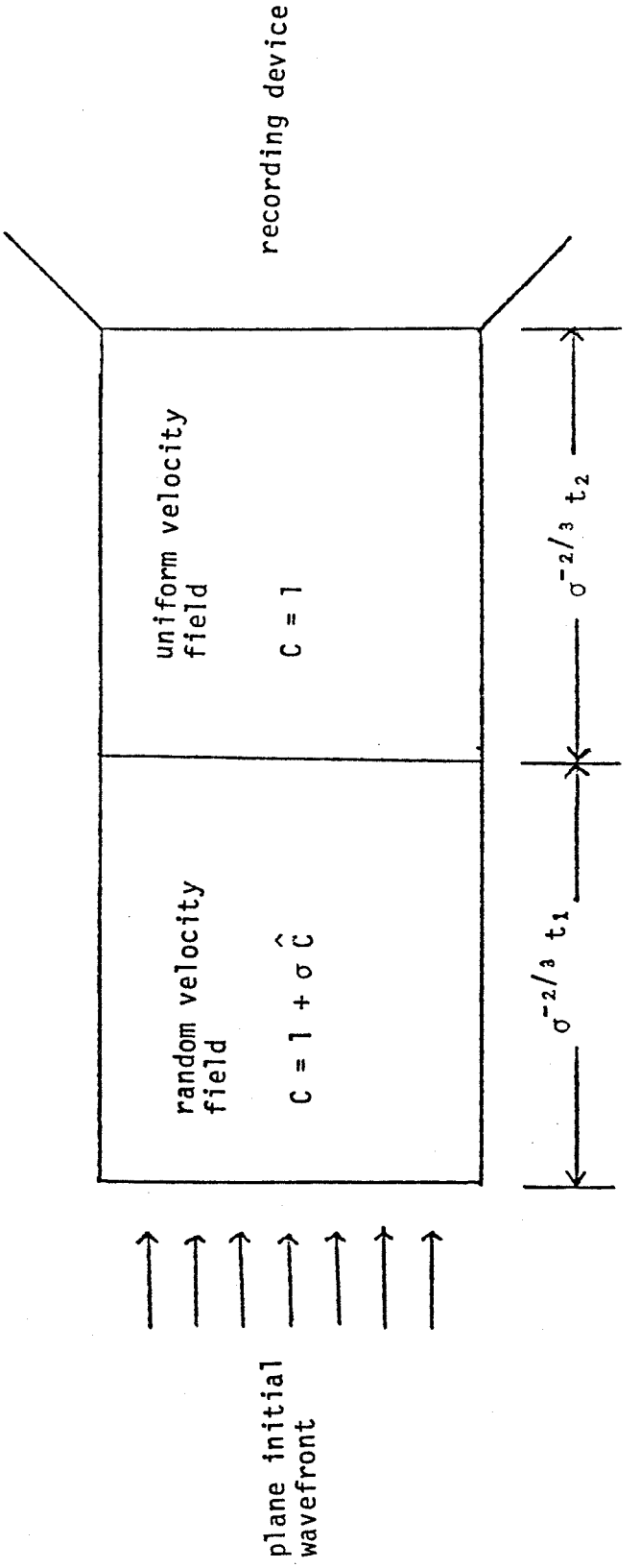

Figure 5.1 Geometry of the shadowgraph problem

6.1 Applying The Papanicolaou and Kohler Theorem To The Three Dimensional Equations Of Motion: Deriving The Generator ${}_3\mathfrak{L}_N$

In this section we will use the Papanicolaou and Kohler theorem on the three dimensional equations of motion: (1.5.13), (1.5.14). These equations are of the form (2.2.1) so we can use the results of section 2.2.

We copy (1.5.13), (1.5.14) for the case of N rays, and only keep only those terms that are of order one or larger:

$$\frac{d}{dt}\begin{pmatrix} X_1^\sigma(t, \underline{\alpha}^L) \\ X_2^\sigma(t, \underline{\alpha}^L) \\ X_3^\sigma(t, \underline{\alpha}^L) \\ V_1^\sigma(t, \underline{\alpha}^L) \\ V_2^\sigma(t, \underline{\alpha}^L) \\ V_3^\sigma(t, \underline{\alpha}^L) \end{pmatrix} = \begin{pmatrix} V_1^\sigma(t, \underline{\alpha}^L) \\ V_2^\sigma(t, \underline{\alpha}^L) \\ V_3^\sigma(t, \underline{\alpha}^L) \\ 0 \\ -\frac{1}{\sigma^{1/3}}\hat{C}_{X_2}\left(\frac{t}{\sigma^{2/3}} + X_1^\sigma, X_2^\sigma, X_3^\sigma\right) \\ -\frac{1}{\sigma^{1/3}}\hat{C}_{X_3}\left(\frac{t}{\sigma^{2/3}} + X_1^\sigma, X_2^\sigma, X_3^\sigma\right) \end{pmatrix}$$

$$L = 1, 2, \ldots, N.$$

$$\begin{pmatrix} X_1^\sigma(0, \underline{\alpha}^L) \\ X_2^\sigma(0, \underline{\alpha}^L) \\ X_3^\sigma(0, \underline{\alpha}^L) \\ V_1^\sigma(0, \underline{\alpha}^L) \\ V_2^\sigma(0, \underline{\alpha}^L) \\ V_3^\sigma(0, \underline{\alpha}^L) \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_2 \\ \alpha_3 \\ 0 \\ h_{\alpha_2}(\alpha_2, \alpha_3) \\ h_{\alpha_3}(\alpha_2, \alpha_3) \end{pmatrix} \qquad (6.1.1)$$

To this order of approximation, the solution for X_1^σ and V_1^σ is clearly

$$X_1^\sigma\ (0,\ \underline{\alpha}^L) = 0$$

$$V_1^\sigma\ (0,\ \underline{\alpha}^L) = 0 \qquad (6.1.2)$$

We use (6.1.2) in (6.1.1) and write the remaining equations as:

$$\frac{d}{dt}\begin{pmatrix} X_2^L \\ X_3^L \\ V_2^L \\ V_3^L \end{pmatrix} = \begin{pmatrix} V_2^L \\ V_3^L \\ -\frac{1}{\sigma^{1/3}}\hat{C}_{X_2}\left(\frac{t}{\sigma^{2/3}},\ X_2^L,\ X_3^L\right) \\ -\frac{1}{\sigma^{1/3}}\hat{C}_{X_3}\left(\frac{t}{\sigma^{2/3}},\ X_2^L,\ X_3^L\right) \end{pmatrix}$$

$$L = 1,\ 2,\ \ldots,\ N \qquad (6.1.3)$$

$$\begin{pmatrix} X_2^L \\ X_3^L \\ V_2^L \\ V_3^L \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} \alpha_2^L \\ \alpha_3^L \\ h_{\alpha_2}\left(\alpha_2^L,\ \alpha_3^L\right) \\ h_{\alpha_3}\left(\alpha_2^L,\ \alpha_3^L\right) \end{pmatrix} \qquad (6.1.4)$$

Equation (6.1.3) has the same form as (2.2.1) if we identify:

$$\varepsilon = \sigma^{1/3} \quad , \quad M = 2N$$

$$\underline{p} = (X_2^1,\ X_3^1,\ X_2^2,\ X_3^2,\ \ldots,\ X_2^N,\ X_3^N)^T$$
$$\underline{q} = (V_2^1,\ V_3^1,\ V_2^2,\ V_3^2,\ \ldots,\ V_2^N,\ V_3^N)^T$$

$$\underline{K}\left(\underline{p},\ \frac{t}{\sigma^{2/3}}\right) = \begin{cases} -\hat{C}_{X_2}\left(\frac{t}{\sigma^{2/3}},\ X_2^1,\ X_3^1\right) \\ -\hat{C}_{X_3}\left(\frac{t}{\sigma^{2/3}},\ X_2^1,\ X_3^1\right) \\ \cdots\cdots \\ -\hat{C}_{X_2}\left(\frac{t}{\sigma^{2/3}},\ X_2^N,\ X_3^N\right) \\ -\hat{C}_{X_3}\left(\frac{t}{\sigma^{2/3}},\ X_2^N,\ X_3^N\right) \end{cases} \qquad (6.1.5)$$

We also define $\mathcal{F}_s^t$ to be the smallest sigma algebra with respect to which $\{\hat{C}_{X_2}(\tau,X_2,X_3),\ \hat{C}_{X_3}(\tau,X_2,X_3) \mid$ for all X_2, X_3 and $s\leq\tau\leq t\}$ are measurable.

First we will find the infinitesimal generator, and then verify the requirements of the Papanicolaou and Kohler theorem. From (2.2.11) we have

$$\underline{b} = \begin{pmatrix} \underline{q} \\ \underline{0} \end{pmatrix} = (V_2^1,\ V_3^1,\ldots,\ V_2^N,\ V_3^N,\ 0,\ldots,\ 0)^T \qquad (6.1.6)$$

We define the three dimensional correlation function of $\hat{C}$ by

$$R(X,Y,Z) = E\left[\hat{C}(X,Y,Z)\ \hat{C}(0,0,0)\right] \qquad (6.1.7)$$

We assume that $\hat{C}$ is homogeneous and isotropic so that

$$R(X,Y,Z) = R(|X|, |Y|, |Z|) \tag{6.1.8}$$

From (6.1.7) we find

$$-R_{YY}(X,Y,Z) = E[\hat{C}_{X_2}(X,Y,Z)\ \hat{C}_{X_2}(0,0,0)]$$

$$-R_{ZZ}(X,Y,Z) = E[\hat{C}_{X_3}(X,Y,Z)\ \hat{C}_{X_3}(0,0,0)]$$

$$-R_{YZ}(X,Y,Z) = E[\hat{C}_{X_2}(X,Y,Z)\ \hat{C}_{X_3}(0,0,0)] \tag{6.1.9}$$

Using (6.1.5) and (6.1.9) we find:

$$E[K_{2i}(\underset{\varepsilon}{p}, \tfrac{s}{2})\ K_{2j}(\underset{\varepsilon}{p}, \tfrac{\sigma}{2})]$$

$$= E[\hat{C}_{X_3}(\tfrac{s}{2}, X_2^i, X_3^i)\ \hat{C}_{X_3}(\tfrac{\sigma}{2}, X_2^j, X_3^j)]$$

$$= -R_{ZZ}(\tfrac{s-\sigma}{2}, X_2^i - X_2^j, X_3^i - X_3^j) \tag{6.1.10}$$

Similarly,

$$E[K_{2i}(\underset{\varepsilon}{p}, \tfrac{s}{2})\ K_{2j-1}(\underset{\varepsilon}{p}, \tfrac{\sigma}{2})] = -R_{YZ}(\tfrac{s-\sigma}{2}, X_2^i - X_2^j, X_3^i - X_3^j)$$

$$E[K_{2i-1}(\underset{\varepsilon}{p}, \tfrac{s}{2})\ K_{2j-1}(\underset{\varepsilon}{p}, \tfrac{\sigma}{2})] = -R_{YY}(\tfrac{s-\sigma}{2}, X_2^i - X_2^j, X_3^i - X_3^j) \tag{6.1.11}$$

We define (analogous to (2.3.21))

$$G(Y,Z) = -\int_0^\infty R(X,Y,Z)\, dX \qquad (6.1.12)$$

By the isotropy of $\hat{C}$,

$$G(Y,Z) = H(\sqrt{Y^2 + Z^2}) \qquad (6.1.13)$$

Now we apply (2.3.1), (2.3.2) to (6.1.10) and (6.1.11) to calculate $\underline{a}$ by (2.2.12). We find

$$a^{LK} = \begin{cases} G_{ZZ}\,(X_2^i - X_2^j,\ X_3^i - X_3^j) & L = 2N + 2i,\ K = 2N + 2j \\ G_{YZ}\,(X_2^i - X_2^j,\ X_3^i - X_3^j) & L = 2N + 2i-1,\ K = 2N + 2j \\ & L = 2N + 2i,\ K = 2N + 2j-1 \\ G_{YY}\,(X_2^i - X_2^j,\ X_3^i - X_3^j) & L = 2N + 2i-1,\ K = 2N + 2j-1 \\ 0 & \text{otherwise} \end{cases} \qquad (6.1.14)$$

for $1 \leq i,j \leq N$. For this we had to require (from (2.3.2)):

$$\left| \int_0^\infty R^*(X,Y,Z)\ X\, dX \right| < C < \infty$$

$$R^* = \{R_{YY}, R_{YZ}, R_{ZZ}\} \qquad (6.1.15)$$

From (6.1.6), (6.1.14) we can immediately write the generator, ${}_3\mathfrak{L}_N$, for the evolution of the 4N quantities $\{X_2^L,\ X_3^L,\ V_2^L,\ V_3^L\}$. We have (see (2.1.3)):

$$3\mathcal{L}_N = \sum_{\substack{L=1,N \\ L=2,3}} V_i^L \frac{\partial}{\partial X_i^L} + \sum_{\substack{L=1,N \\ K=1,N}} \{G_{YY}(\cdot) \frac{\partial^2}{\partial V_2^L \partial V_2^K}$$

$$+ G_{YZ}(\cdot)\left[\frac{\partial^2}{\partial V_2^L \partial V_3^K} + \frac{\partial^2}{\partial V_2^K \partial V_3^L}\right] + G_{ZZ}(\cdot) \frac{\partial^2}{\partial V_3^L \partial V_3^K}\} \tag{6.1.16}$$

where the argument to the G function is

$$(X_2^L - X_2^K, X_3^L - X_3^K) \tag{6.1.17}$$

When $3\mathcal{L}_N$ is used in the forward Kolmogorov equation, $3\mathcal{L}_N^* Q = Q_t$, the initial conditions for Q come from (6.1.4). We have

$$Q(0, \underline{X}_2, \underline{X}_3, \underline{V}_2, \underline{V}_3; \underline{\alpha}_2, \underline{\alpha}_3) = \delta(\underline{X}_2 - \underline{\alpha}_2)\, \delta(\underline{X}_3 - \underline{\alpha}_3)$$

$$\delta(\underline{V}_2 - h_{\alpha_2}(\underline{\alpha}_2, \underline{\alpha}_3))\ \ \delta(\underline{V}_3 - h_{\alpha_3}(\underline{\alpha}_2, \underline{\alpha}_3)) \tag{6.1.18}$$

For an initially plane wavefront, $h \equiv 0$ (see (1.5.1)), so (6.1.18) becomes

$$Q(0, \underline{X}_2, \underline{X}_3, \underline{V}_2, \underline{V}_3; \underline{\alpha}_2, \underline{\alpha}_3) = \delta(\underline{X}_2 - \underline{\alpha}_2)\, \delta(\underline{X}_3 - \underline{\alpha}_3)\, \delta(\underline{V}_2)\, \delta(\underline{V}_3) \tag{6.1.19}$$

Now we must check the requirements of the Papanicolaou and

Kohler theorem (see (2.2.14). We cannot check the strong mixing rate, but (6.1.15) does insure a weak form of mixing.

If we assume $E[\hat{C}] = 0$, then (2.2.7) is satisfied. We assumed that $\hat{C}$ is bounded and has bounded derivatives in (1.4.3), so (2.2.9) is satisfied.

Because we used (2.3.1), (2.3.2) to evaluate the a^{ij} coefficients, conditions (2.2.13) is satisfied.

We have not verified conditions (2.1.21) - (2.1.24), (2.1.29) for ${}_3\mathfrak{L}_N$, but believe them to be satisfied. These are conditions on the continuity and differentiability of the principal part of ${}_3\mathfrak{L}_N$.

6.2 N-Point Energy Correlations

In this section we will find the equations that describe the N-point energy correlation in three dimensions.

We define $J(t,\alpha_2,\alpha_3)$ to be the Jacobian of the mapping from the wavefront to physical space. We assume that variations in the $\underline{i}$ direction are small compared to variations in the ($\underline{j}$, $\underline{k}$) directions. This is the same assumption we made in two dimensions. With this assumption, we can approximate the Jacobian by:

$$J(t,\alpha_2,\alpha_3) = \left| \frac{\partial(X_2, X_3)}{\partial(\alpha_2,\alpha_3)} \right| \qquad (6.2.1)$$

The Jacobian is the ratio of infinitesimal areas, so we use $J(t,\alpha_2,\alpha_3)$ (defined by (6.2.1)) as the approximate raytube area

in three dimensions.

We define the expectation in physical space, the same way we did in two dimensions (see (3.2.1)):

$$E_{(Z_2,\ Z_3)}\ [\ f(J(t,\alpha_2,\alpha_3))\ ]\ =\ E\ [\ \sum_{\{\alpha_2,\ \alpha_3 |\ X_2(t,\alpha_2,\alpha_3)\ =\ Z_2,\ X_3(t,\alpha_2,\alpha_3)\ =\ Z_3\}} f(J(t,\alpha_2,\alpha_3))\ ] \qquad (6.2.2)$$

We generalize (6.2.2) to N points in physical space as follows:

$$E_{(\underline{Z}_2,\underline{Z}_3)}\ [\ f(J(t,\alpha_2^1,\alpha_3^1),\ \ldots,\ J(t,\alpha_2^N,\alpha_3^N))\ ]$$

$$=\ E\ [\ \sum_{\{\underline{\alpha}_2,\underline{\alpha}_3 |\ \underline{X}_2(t,\underline{\alpha}_2,\underline{\alpha}_3)\ =\ \underline{Z}_2,\ \underline{X}_3(t,\underline{\alpha}_2,\underline{\alpha}_3)\ =\ \underline{Z}_3\}} f(J(t,\alpha_2^1,\alpha_3^1),\ \ldots,\ f(t,\alpha_2^N,\alpha_3^N))] \qquad (6.2.3)$$

Since $J(t,\alpha_2^L,\alpha_3^L)$ is an approximate area of the L^{th} raytube, and ray energy is approximately the reciprocal of the raytube area (see (4.3.1)), we define

$${}_3R_N(t,\underline{Z}_2,\underline{Z}_3)\ =\ E_{(\underline{Z}_2,\underline{Z}_3)}\ [\ \prod_{L=1}^{N}\ \frac{1}{J(t,\alpha_2^L,\alpha_3^L)}\ ] \qquad (6.2.4)$$

to be the energy correlation function for N points in three dimensions. Using (6.2.3) in (6.2.4)

$$
{}_3R_N(t,\underline{Z}_2,\underline{Z}_3) = E\Big[\sum_{\substack{\{\underline{\alpha}_2,\underline{\alpha}_3 | \\ \underline{X}_2(t,\underline{\alpha}_2,\underline{\alpha}_3)=\underline{Z}_2 \\ \underline{X}_3(t,\underline{\alpha}_2,\underline{\alpha}_3)=\underline{Z}_3\}}} \prod_{L=1}^{N} \frac{1}{J(t,\alpha_2^L,\alpha_3^L)}\Big]
$$

$$
= E\Big[\sum_{\substack{\{\underline{\alpha}_2,\underline{\alpha}_3 | \\ \underline{X}_2(t,\underline{\alpha}_2,\underline{\alpha}_3)=\underline{Z}_2 \\ \underline{X}_3(t,\underline{\alpha}_2,\underline{\alpha}_3)=\underline{Z}_3\}}} \prod_{L=1}^{N} \left| \frac{\partial(\alpha_2,\alpha_3)}{\partial(X_2^L,X_3^L)} \right| \Big]
$$

$$
= E\left[\int d\underline{\alpha}_2 \int d\underline{\alpha}_3\ \delta(\underline{X}_2-\underline{Z}_2)\ \delta(\underline{X}_3-\underline{Z}_3)\right] \qquad (6.2.5)
$$

where we have used (6.2.1) for the second equality. The third equality is a generalization of (4.1.3).

We can evaluate the right hand side of (6.2.5) by multiplying by the joint probability density of $\{\underline{X}_2, \underline{X}_3, \underline{V}_2, \underline{V}_3\}$ (which we can find from ${}_3f_N$) and integrating over all values of $\{\underline{X}_2, \underline{X}_3, \underline{V}_2, \underline{V}_3\}$. Carrying this out we find:

$$
{}_3R_N(t,\underline{Z}_2,\underline{Z}_3) = \int d\underline{\alpha}_2 \int d\underline{\alpha}_3 \int d\underline{X}_2 \int d\underline{X}_3 \int d\underline{V}_2 \int d\underline{V}_3\ P(t,\underline{X}_2,\underline{X}_3,\underline{V}_2,\underline{V}_3;\underline{\alpha}_2,\underline{\alpha}_3)
$$

$$
\delta(\underline{X}_2-\underline{Z}_2)\delta(\underline{X}_3-\underline{Z}_3) = \int d\underline{\alpha}_2 \int d\underline{\alpha}_3 \int d\underline{V}_2 \int d\underline{V}_3\ P(t,\underline{Z}_2,\underline{Z}_3,\underline{V}_2,\underline{V}_3;\underline{\alpha}_2,\underline{\alpha}_3) \qquad (6.2.6)
$$

where

$$
{}_3f_N^*\ P = P_t
$$

$$
P(0,\underline{X}_2,\underline{X}_3,\underline{V}_2,\underline{V}_3;\underline{\alpha}_2,\underline{\alpha}_3) = \delta(\underline{X}_2-\underline{\alpha}_2)\ \delta(\underline{X}_3-\underline{\alpha}_3)\ \delta(\underline{V}_2-h_{\alpha_2}(\underline{\alpha}_2,\underline{\alpha}_3))
$$

$$\delta(\underline{V}_3 - h_{\alpha_3}(\underline{\alpha}_2, \underline{\alpha}_3)) \tag{6.2.7}$$

We can carry out the $\underline{\alpha}_2$, $\underline{\alpha}_3$ integrations in (6.2.6), (6.2.7) because ${}_3f_N$ does not depend on $\underline{\alpha}_2$ or $\underline{\alpha}_3$. We find

$${}_3R_N(t, \underline{X}_2, \underline{X}_3) = \int d\underline{V}_2 \int d\underline{V}_3 \; S(t, \underline{X}_2, \underline{X}_3, \underline{V}_2, \underline{V}_3) \tag{6.2.8}$$

$${}_3f_N^* \; S = S_t \tag{6.2.9}$$

$$S(0, \underline{X}_2, \underline{X}_3, \underline{V}_2, \underline{V}_3) = \delta(\underline{V}_2 - h_{\alpha_2}(\underline{X}_2, \underline{X}_3)) \; \delta(\underline{V}_3 - h_{\alpha_3}(\underline{X}_2, \underline{X}_3)) \tag{6.2.10}$$

For a plane initial wavefront, (6.2.10) is replaced by

$$S(0, \underline{X}_2, \underline{X}_3, \underline{V}_2, \underline{V}_3) = \delta(\underline{V}_2) \; \delta(\underline{V}_3) \tag{6.2.11}$$

Equations (6.2.8) through (6.2.11) are the main results of this section.

6.3 Two Point Energy Correlations, Plane Inital Wavefront

Here we simplify the two point energy correlation equations when the intial wavefront is planar (equations (6.2.8), (6.2.9), (6.2.11)).

We know G(Y,Z) is a function of (Y^2+Z^2) from (6.1.13), so we

have

$$G_{YZ}(0,0) = 0 \tag{6.3.1}$$

We define G_0 by:

$$G_0 = G_{YY}(0,0) = G_{ZZ}(0,0) \tag{6.3.2}$$

using (6.3.1), (6.3.2) and N=2 in (6.1.16) we find

$$\begin{aligned} {}_3\mathfrak{L}_2 = & \sum_{\substack{L=1,2 \\ i=2,3}} V_i^L \frac{\partial}{\partial X_i^L} + \sum_{\substack{L=1,2 \\ i=2,3}} G_0 \frac{\partial^2}{\partial V_i^{L\,2}} \\ & + 2G_{YY}(\cdot) \frac{\partial^2}{\partial V_2^1 \partial V_2^2} + 2G_{ZZ}(\cdot) \frac{\partial^2}{\partial V_3^1 \partial V_3^2} \\ & + 2G_{YZ}(\cdot) \left\{ \frac{\partial^2}{\partial V_2^1 \partial V_3^2} + \frac{\partial^2}{\partial V_3^1 \partial V_2^2} \right\} \end{aligned} \tag{6.3.3}$$

where the argument to G(•) in (6.3.3) is

$$(X_2^1 - X_2^2,\ X_3^1 - X_3^2) \tag{6.3.4}$$

We copy equations (6.2.8), (6.2.9), (6.2.11) for reference:

$${}_3R_2(t,\underline{X}_2,\underline{X}_3) = \int d\underline{V}_2 \int d\underline{V}_3\ S(t,\underline{X}_2,\underline{X}_3,\underline{V}_2,\underline{V}_3)$$

$$3\mathfrak{L}_2^{\,*}\, S = S_t$$

$$S(0,\underline{X}_2,\underline{X}_3,\underline{V}_2,\underline{V}_3) = \delta(\underline{V}_2)\ \delta(\underline{V}_3) \qquad (6.3.5)$$

We now change variables in (6.3.3) and (6.3.5). In analogy with what we did in two dimensions, we choose new variables to be the sum and difference of the old variables. We define

$$Y_1 = X_2^1 - X_2^2 \qquad r_1 = X_2^1 + X_2^2$$
$$Y_2 = X_3^1 - X_3^2 \qquad r_2 = X_3^1 + X_3^2$$

$$V_1 = V_2^1 - V_2^2 \qquad P_1 = V_2^1 + V_2^2$$
$$V_2 = V_3^1 - V_3^2 \qquad P_2 = V_3^2 + V_3^2$$

$$H(t,\underline{Y},\underline{V},\underline{r},\underline{P}) = 4\ S(t,\underline{X}_2,\underline{X}_3,\underline{V}_1,\underline{V}_2)$$

Routine calculations give

$$3\mathfrak{L}_2^{\,*}\, H = \mathfrak{L}^+ H + \{-P_1 \frac{\partial}{\partial r_1} - P_2 \frac{\partial}{\partial r_2} + 2[G_0 + G_{YY}(Y_1,Y_2)] \frac{\partial^2}{\partial P_1^2}$$
$$+ 2[G_0 + G_{ZZ}(Y_1,Y_2)] \frac{\partial^2}{\partial P_2^2} - 4\ G_{YZ}(Y_1,Y_2) \frac{\partial^2}{\partial P_1 \partial P_2}\} H = H_t \qquad (6.3.6)$$

where

$$\mathcal{L}^{+} H = \{-V_1 \frac{\partial}{\partial Y_1} - V_2 \frac{\partial}{\partial Y_2} + 2[G_0 - G_{YY}(Y_1,Y_2)] \frac{\partial^2}{\partial V_1^2}$$

$$+ 2[G_0 - G_{ZZ}(Y_1,Y_2)] \frac{\partial^2}{\partial V_2^2} + 4\, G_{YZ}(Y_1,Y_2) \frac{\partial^2}{\partial V_1 \partial V_2}\} H \qquad (6.3.7)$$

and

$${}_3R_2(t,\underline{Y},\underline{r}) = \int d\underline{V} \int d\underline{P}\; H(0,\underline{Y},\underline{r},\underline{V},\underline{P}) \qquad (6.3.8)$$

$$H(0,\underline{Y},\underline{r},\underline{V},\underline{P}) = \delta(\underline{V})\,\delta(\underline{P}) \qquad (6.3.9)$$

Now we will eliminate $\underline{r}$, $\underline{P}$ from (6.3.6) - (6.3.9) the same way we removed M_N, W_N in two dimensions (see section 3.4).

Consider the equation:

$$Q_t = \mathcal{L}^{+} Q + \{ 2[G_0 + G_{YY}(Y_1,Y_2)] \frac{\partial^2}{\partial P_1^2}$$

$$2[G_0 + G_{ZZ}(Y_1,Y_2)] \frac{\partial^2}{\partial P_2^2} - 4\, G_{YZ}(Y_1,Y_2) \frac{\partial^2}{\partial P_1 \partial P_2} \} Q$$

$$Q(0,\underline{Y},\underline{V},\underline{P}) = \delta(\underline{V})\,\delta(\underline{P}) \qquad (6.3.10)$$

If (6.3.10) has a solution, then it satisfies (6.3.6), (6.3.9). If (6.3.6), (6.3.9) has a unique solution, then the solution to (6.3.10) is the unique solution to (6.3.6), (6.3.9).

We assume existence and uniqueness so H=Q and we can write (6.3.8) as:

$${}_3R_2(t,\underline{Y}) = \int d\underline{V} \int d\underline{P}\; Q(t,\underline{Y},\underline{V},\underline{P}) \qquad (6.3.11)$$

Now we define

$$S(t,\underline{Y},\underline{V}) = \int d\underline{P}\ Q(t,\underline{Y},\underline{V},\underline{P}) \qquad (6.3.12)$$

If we assume

$$Q_{P_1}\Big|_{|\underline{P}| = \infty} = Q_{P_2}\Big|_{|\underline{P}| = \infty} = 0 \qquad (6.3.13)$$

then (6.3.10), (6.3.11) become (using (6.3.7),(6.3.12)):

$$-V_1 S_{Y_1} - V_2 S_{Y_2} + 2[G_0 - G_{YY}(Y_1,Y_2)] S_{V_1V_1} + 2[G_0 - G_{ZZ}(Y_1,Y_2)]\ S_{V_2V_2}$$

$$+ 4\ G_{YZ}(Y_1,Y_2)\ S_{V_1V_2} = \mathfrak{L}^+ S = S_t$$

$$S(0,\underline{Y},\underline{V}) = \delta(\underline{V}) \qquad (6.3.14)$$

$${}_3R_2(t,\underline{Y}) = \int d\underline{V}\ S(t,\underline{Y},\underline{V}) \qquad (6.3.15)$$

The system {(6.3.14), (6.3.14), (6.3.15)} is the three dimensional analog of the system in (4.4.20).

The equation (6.3.14) depends on four space variables and t. By taking the new variables

$$r = \sqrt{Y_1^2 + Y_2^2}$$

$$\Theta = \tan^{-1}\left(\frac{Y_2}{Y_1}\right)$$

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \qquad (6.3.16)$$

we can reduce the system to only three space variables. Using (6.3.16) in (6.3.14), (6.3.15) we find

$$P_t = 2[G_0 - 2B'] \{P_{U_1U_1} + P_{U_2U_2}\}$$

$$-8\, r^2 B''\, P_{U_1U_1} - U_1 P_r - \frac{U_2}{r} P_\Theta - \frac{U_2^2}{r} P_{U_1}$$

$$+ \frac{U_1U_2}{r} P_{U_2}$$

$$P(0,r,\Theta,U_1,U_2) = \delta(U_1)\ \delta(U_2)$$

$${}_3R_2(t,r,\Theta) = \int dU_1 \int dU_2\ P(t,r,\Theta,U_1,U_2) \qquad (6.3.17)$$

where

$$H(r) = B(r^2) = B(X^2+Y^2) = G(X,Y) \qquad (6.3.18)$$

The dependence of ${}_3R_2$ on Θ is artificial. We can remove Θ from (6.3.17) the same way we removed $\underline{r}$ from (6.3.6) to obtain:

$$P_t = 2[H''(0) - H''(r)]\, P_{U_1U_2} + 2\left[H''(0) - \frac{H'(r)}{r}\right] P_{U_2U_2}$$

$$-U_1 P_r - \frac{U_2^2}{r} P_{U_1} + \frac{U_1U_2}{r} P_{U_2}$$

$$P(0,r,U_1,U_2) = \delta(U_1)\ \delta(U_2)$$

$$_3R_2(t,r) = \int dU_1 \int dU_2\ P(t,r,U_1,U_2) \qquad (6.3.19)$$

In the system (6.3.19) we see that $_3R_2$ only depends on $\{t,r\}$. Here,

$$r = \sqrt{Y_1^2+Y_2^2} = \sqrt{(X_2^1-X_2^2)^2 + (X_3^1-X_3^2)^2}$$

is the distance between the two points at which the energy is being received. This agrees with our intuition: in an isotropic and homogeneous random field the correlation function should only depend on the distance between the observation points.

The system in (6.3.19) is the main result of this section.

6.4 Short Distance Approximation to the Two Point Energy Correlation Function: Fourier Transform Method

In the section we use Fourier Transforms to obtain a short distance approximation to the two point energy correlation function. We use the system {(6.3.14), (6.3.15)} instead of (6.3.18) because the former exhibits greater symmetry. The techniques used in this section are completely analogous to the techniques used in section 4.6.

We define the Fourier Transforms:

$$Q(t,X,Y,U,V) = \int_{-\infty}^{\infty}\int\int\int dY_1\, dY_2\, dV_1\, dV_2$$

$$e^{i(XY_1+YY_2+UV_1+VV_2)}\, P(t,Y_1,Y_2,V_1,V_2)$$

$$\hat{R}(t,X,Y) = \int_{-\infty}^{\infty}\int dY_1\, dY_2\, e^{i(XY_1+YY_2)}\, {}_3R_2(t,Y_1,Y_2)$$

$$\hat{G}(X,Y) = \int_{-\infty}^{\infty}\int dY_1\, dY_2\, e^{i(XY_1+YY_2)}\, G(Y_1,Y_2) \qquad (6.4.1)$$

If we assume that

$$P\Big|_{|\underline{V}|=\infty} = P_{V_1}\Big|_{|\underline{V}|=\infty} = P_{V_2}\Big|_{|\underline{V}|=\infty} = P\Big|_{|\underline{Y}|=\infty} = 0 \qquad (6.4.2)$$

then the system (6.3.14), (6.3.15) becomes (using (6.4.1))

$$Q_t = -XQ_U - YQ_V - 2G_0(U^2+V^2)Q$$

$$\frac{-2(XU+YV)^2}{4\pi^2} \int\int db_1\, db_2\, \hat{G}(x-b_1,Y-b_2)\, Q(t,b_1,b_2,U,V)$$

$$Q(0,X,Y,U,V) = 4\,\pi^2\, \delta(X)\, \delta(Y)$$

$$\hat{R}(t,X,Y) = Q(t,X,Y,0,0) \qquad (6.4.3)$$

We subtract out the delta functions in the intial conditions of Q by forming:

$$Q(t,X,Y,U,V) = \phi(t,X,Y,U,V) + \delta(X)\,\delta(Y)\,4\pi^2\, e^{-2G_0(U^2+V^2)t} \tag{6.4.4}$$

Using (6.4.4) in (6.4.3) and setting all terms with $X\delta(X)$ or $Y\delta(Y)$ to zero we obtain:

$$\phi_t = -X\phi_U - Y\phi_V - 2G_0(U^2+V^2)\,\phi$$

$$- 2\,(XU+YV)^2\,\hat{G}(X,Y)\, e^{-2G_0(U^2+V^2)t}$$

$$- \frac{(XU+YV)^2}{2\pi^2} \iint db_1\, db_2\, \hat{G}(X-b_1,Y-b_2)\,\phi(t,b_1,b_2,U,V)$$

$$\phi(0,X,Y,U,V) = 0 \tag{6.4.5}$$

$$\hat{R}(t,X,Y) = \phi(t,X,Y,0,0) + 4\pi^2\,\delta(X)\,\delta(Y) \tag{6.4.6}$$

The system (6.4.5), (6.4.6) is still exact. To approximate this system we look for a solution of the form

$$\phi(t,X,Y,U,V) = \sum_{N=0}^{\infty} \phi_N(X,Y,U,V)\, t^N \tag{6.4.7}$$

From (6.4.5) we conclude

$$\phi_0 \equiv 0 \tag{6.4.8}$$

If we substitute (6.4.7), (6.4.8) in (6.4.5), take a Taylor

series in t, and equate powers of t, we obtain the sequence of algebraic equations:

$$(N+1)\ \phi_{N+1} = -X\ \phi_{N,U} - Y\ \phi_{N,V} - 2G_0(U^2+V^2)\ \phi_N$$

$$- \frac{(XU+YV)^2}{2\pi^2} \iint db_1\ db_2\ \hat{G}(X-b_1, Y-b_2)\ \phi_N(b_1,b_2,U,V)$$

$$- \frac{2}{N!}\ (XU+YV)^2\ \hat{G}(X,Y)\ [-2G_0(U^2+V^2)]^N \qquad (6.4.9)$$

for N=0,1,2 . . . . It is straightforward to find:

$$\phi_1 = -2(XU+YV)^2\ \hat{G}(X,Y)$$

$$\phi_2 = 2\hat{G}(X,Y)\ (XU+YV)\ (X^2+Y^2) + 4\ G_0\hat{G}(X,Y)\ (U^2+V^2)\ (XU+YV)$$

$$+ \frac{(XU+YV)^2}{\pi} \iint db_1\ db_2\ \hat{G}(X-b_1, Y-b_2)\ \hat{G}(b_1,b_2)\ (b_1U+b_2V)^2$$

$$\phi_3(X,Y,0,0) = \frac{2}{3}\ \hat{G}(X,Y)\ (X^2+Y^2)^2 \qquad (6.4.10)$$

Using (6.4.7), (6.4.8), (6.4.10) in (6.4.6) we obtain

$$\hat{R}(t,X,Y) = 4\pi^2\ \delta(X)\ \delta(Y) - \frac{2}{3}\ \hat{G}(X,Y)\ (X^2+Y^2)^2 + O(t^4) \qquad (6.4.11)$$

Taking the inverse transform of (6.4.11) yields (see (6.4.1))

$$3R_2(t,Y_1,Y_2) = 1 - \frac{2}{3}\ t^3\left(\frac{\partial^2}{\partial Y_1^2} + \frac{\partial^2}{\partial Y_2^2}\right)^2\ G(Y_1,Y_2) + O(t^4) \qquad (6.4.12)$$

Equation (6.4.12) is the same as (A.1.30) which was derived by regular perturbation techniques.

6.5 G(Y,Z)

Here we give an alternate representation for G(Y,Z), give an example of G(Y,Z), and check the requirements from section 6.1.

In section 6.1 we defined G(Y,Z) by (see (6.1.12))

$$G(Y,Z) = -\int_0^\infty R(X,Y,Z)\, dX \tag{6.5.1}$$

where R(X,Y,Z) is the three dimensional correlation of $\hat{C}$. From the homogeneity and isotropy of $\hat{C}$, we have (see (6.1.13), (6.3.18)):

$$R(X,Y,Z) = B(\sqrt{X^2 + Y^2 + Z^2}) \tag{6.5.2}$$

$$G(Y,Z) = H(\sqrt{Y^2 + Z^2}) \tag{6.5.3}$$

We define the Fourier Transform of B by:

$$S(\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}) = -\frac{1}{8\pi^2} \iiint_{-\infty}^{\infty} e^{-i(X\omega_1 + Y\omega_2 + Z\omega_3)} B(\sqrt{X^2+Y^2+Z^2})\, dX\, dY\, dZ \tag{6.5.4}$$

so that

$$B\left(\sqrt{X^2+Y^2+Z^2}\right) = -\frac{1}{\pi}\iiint_{-\infty}^{\infty} e^{\,i(X\omega_1+\,Y\omega_2\,+\,Z\omega_3)}$$

$$S\left(\sqrt{{\omega_1}^2+\,{\omega_2}^2\,+\,{\omega_3}^2}\right)\,d\omega_1 d\omega_2 d\omega_3 \qquad (6.5.5)$$

Using (6.5.1) (note the range of integration is $[0,\infty)$) with (6.5.2) and (6.5.5) produces

$$G(Y,Z) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{\,i(Y\omega_1\,+\,Z\omega_2)}\; S\left(\sqrt{{\omega_1}^2+\,{\omega_2}^2}\right)\,d\omega_1 d\omega_2 \qquad (6.5.6)$$

In (6.5.6) we change to polar coordinates, $\omega_1 = \omega \cos\Theta$, $\omega_2 = \omega \sin\Theta$, and set $Z = 0$ to find

$$G(Y,0) = \frac{1}{2\pi}\int_0^{\infty} d\omega \int_0^{2\pi} d\theta\; e^{i\omega\, Y\, \cos\theta}\;\; \omega\; S(\omega)$$

$$= \int_0^{\infty} d\omega \;\; \omega\; S(\omega)\; J_0(\omega Y) \qquad (6.5.7)$$

From (6.5.3), (6.5.7)

$$G(Y,Z) = \int_0^{\infty} d\omega \;\; \omega\; S(\omega)\; J_0\left(\omega\,\sqrt{Y^2\,+\,Z^2}\right) \qquad (6.5.8)$$

To simplify (6.5.4) we change to spherical polar coordinates, $X = r \sin\Theta \cos\Phi$, $Y = r \sin\Theta \sin\Phi$, $Z = r \cos\Theta$, and set $\omega_1 = \omega_2 = 0$ to find:

$$S(|\omega_3|) = -\frac{1}{8\pi^2}\int_0^{\infty} dr\; r^2\; B(r) \int_0^{\pi} d\Theta\; e^{-i\; r\; \omega_3 \cos\Theta}\; \sin\Theta \int_0^{2\pi} d\Phi$$

$$= -\frac{1}{8\pi^2}\int_0^\infty dr\ r^2\ B(r)\ \left(\left(\frac{2\pi}{r\omega_3}\right)^{1/2} J_{1/2}\ (-r\omega_3)\right)\ (2\pi)$$

$$= -\frac{1}{2\pi}\int_0^\infty dr\ r^2\ B(r)\ j_0(-r\omega_3)$$

$$= -\frac{1}{2\pi\omega_3}\int_0^\infty dr\ r\ B(r)\ \sin(r\omega_3) \qquad (6.5.9)$$

where we have used (15) 3.715.21.

Given a correlation function for $\hat{C}$, we can compute the power spectrum by (6.5.9). Then, by using (6.5.8), we can find G(Y,Z).

For example, if we take

$$B(r) = e^{-\frac{r^2}{2a^2}} \qquad (6.5.10)$$

then we find:

$$G(Y,Z) = -\sqrt{2\pi}\ a\ e^{-\frac{Y^2+Z^2}{2a^2}} \qquad (6.5.11)$$

For this example the conditions in (6.1.15) are satisfied.

7.1 Numerical Technique

In this section we describe the numerical technique used to solve the two point energy correlation system, (3.4.20).

We place (4.4.20) on the universal time scale (see (I.2.2)) by the change of variables:

$$\begin{aligned} \tau &= (\gamma_2)^{2/3} t \\ X &= M \\ Y &= W/(\gamma_2)^{2/3} \end{aligned} \tag{7.1.1}$$

$$V(\tau, X, Y) = U(t, M, W)\ (\gamma_2)^{2/3}$$

to obtain:

$$V_\tau = \frac{f(X)}{\gamma_2^2} V_{YY} - Y V_X \tag{7.1.2}$$

$$V(0, X, Y) = \delta(Y) \tag{7.1.3}$$

$$R_2(\tau, X) = \int_{-\infty}^{\infty} V(\tau, X, Y)\, dY \tag{7.1.4}$$

Up until now we have solved (4.4.20) with no real regard for what the correct boundary conditions for (4.4.20). To calculate the solution numerically, we need to know the "proper" boundary conditions because we must adapt (7.1.2) to a numerical grid, and the information on the boundaries of the grid will greatly determine the solution.

Fichera (22) has given results for the existence of a weak solution for an equation of the form (7.1.2), if the initial conditions and boundary conditions are L^2 and some estimates on the solution can be found. Also, in such cases, he states what the proper initial conditions and boundary conditions are.

The initial conditions to (7.1.2) are not L^2, but we can still use Fichera's result to suggest the proper conditions. His results suggest that V or its first derivative should be given on Σ, where Σ is defined by:

$$\sum = \begin{cases} \tau = 0 & \\ X = \infty & Y \leq 0 \\ X = -\infty & Y \geq 0 \\ Y = \pm\infty & X \neq 0 \end{cases} \qquad (7.1.5)$$

The reason that conditions need to be given on ($X=\infty$, $Y>0$) and ($X=-\infty$, $Y<0$) is because these are the entrance boundaries of the problem. This can be realized by considering the subcharacteristics of (7.1.2) (that is, (7.1.2) with the diffusion term removed). The solution to this new equation is $V=h(X-Y\tau)$, where h is an arbitrary function.

For $Y<0$ this represents a wave traveling to the left in the (X,Y) plane, whose inital value must be given at $X=\infty$. For $Y<0$, the initial condition must be prescribed at $X=-\infty$.

For $Y<0$, $X>0$ and $Y>0$, $X<0$ the subcharacteristics are pointing into the region surrounding ($X=0$, $Y=0$). Hence, the upstream solution is convected into the region of the numerical

grid. Upstream, with $|X|>>1$, we can approximate $f(\underline{X})$ by $f_\infty = f(\infty)$. The solution of (7.1.2), (7.1.3) with $f(X)$ replaced by f_∞ is given by integrating (4.2.5), (4.2.10) with respect to α. We find

$$V(\tau,X,Y) = \left(\frac{f_\infty}{4\pi\tau}\right)^{1/2} \exp\left\{-\frac{Y^2}{f_\infty \tau}\right\} \qquad (7.1.6)$$

This solution is independent of X or, $V_X = 0$. Hence, for the boundaries $Y<0$, $X<<0$ and $Y>0$, $X<<0$, we used $V_X = 0$.

The $f(\underline{X})$ function we used in (7.1.2) corresponded to a gaussian correlation function for $\hat{C}$. Using (3.3.4.a) (3.3.5.a) with a=1, and (4.4.19) we have:

$$\begin{aligned} B(r) &= e^{-r^2/2} \\ g(z) &= \sqrt{\frac{\pi}{2}}\ (1-z^2)\ e^{-z^2/2} \\ f(z) &= \sqrt{2\pi}\ [1-(1-z^2)\ e^{-z^2/2}] \end{aligned} \qquad (7.1.7)$$

From (3.28), (3.2.7) we compute from (7.1.7):

$$(\gamma_2)^2 = 3\sqrt{2\pi} \simeq 7.52 \qquad (7.1.8)$$

Let δX, δY be the grid spacings in the X, Y directions. Let δt be the time separation between grids. To solve (7.1.2) numerically we used a uniform grid with 131 points in the X direction (with $|X|<6$, so $\delta X=.092$) and 60 points in the Y

direction (with $-3 \leq Y \leq 2\ \delta Y$, so $\delta Y = .053$). Since equation (7.1.2) is invariant under the change of variables $(X,Y) \to (-X,-Y)$ we only solved the equation, at each time step, for $Y<0$ and then reflected the solution to the upper half plane. We choose $|X| \leq 6$ since the $f(X)$ we used is essentially constant at $X = \pm 6$.

The delta function initial condition in (7.1.3) is hard to implement. We approximated the delta function by a gaussian with a standard deviation (ε) of .20. If the standard deviation of a gaussian is very small, then it "simulates" a delta function. The grid spacing puts a constraint on the width of the gaussian: there must be enough grid points to resolve the gaussian or the form of the inital conditions will be lost. For our uniform grid, we have 3.8 grid points per correlation length and the (simulated) mesh has a width of of 30 correlation lengths.

To solve (7.1.2) we used an explicit method in time. For increased accuracy we used a second order Runge-Kutta method ((14), 25.5.6) in time. That is, if $L[V]$ represents a discrete approximaton to $f(X)U_{YY} - YU_X$ we found

$$A = V(t) + (\delta t)\ L[V(t)]$$

$$B = A + \delta t\ L[A] \qquad (7.1.9)$$

and then formed

$$V(t+\delta t) = \frac{1}{2}(A + B) \qquad (7.1.10)$$

This produces errors of order $(\delta t)^3$. We choose $\delta t = \frac{1}{2}(\delta Y)^2 = .0014$.

Now we describe what L[V] looks like. In the center of the grid we used a third order accurate formula for V_{YY} ((14), 25.3.24, $\partial^2/\partial Y^2 = (4D_h D_{-h} - D_{2h}D_{-2h})/3 + O(h^4)$ where $D_h V(Y) = \{V(Y+h)-V(Y-h)\}/2h$, $h=\delta Y$) and a third order accurate method for V_X ($\partial/\partial X = (D_h - 4D_{2h})/3 + O(h^4)$, $h=\delta X$).

For $Y<0$ and $X<<1$, the subcharacteristics of the equation are pointing out of the numerical domain. For $X=-6$, $-3<Y\leq 0$ we used centered differencing for the V_{YY} term ((14), 25.3.23, $\partial^2/\partial Y^2 = D_h D_{-h} + O(h^2)$, $h=\delta Y$) and upstream differencing for the V_X term ($V_X = (V(X+h)-V(X))/h + O(h^2)$, $h=\delta X$). For $X=-6+\delta X$, $-3+\delta Y < Y \leq 0$ and $X=6-\delta X$, $-3+\delta Y < Y \leq 0$ we used centered differencing for both the V_{YY} ((14), 25.3.24) and the V_X ((14), 25.3.21, $\partial/\partial X = D_h + O(h^2)$, $h=\delta X$) terms.

For $Y=-3+\delta Y$, $X=-6+\delta x$ and $Y=-3+\delta Y$, $X=6-\delta X$, we used centered formulae for both V_X and V_{YY} ((14), 25.3.21 and 25.3.23). For $Y=-3+\delta Y$, $-6+\delta X < X < 6-\delta X$ we again used centered formulae ((14), 25.3.23 for V_{YY} and $V_X = (1/3\ D_h - 4/3\ D_{2h})\ V + O(h^4)$, $h=\delta X$).

A boundary condition must also be imposed for $|Y|>>1$. Because of the analogy of (17.1.2) with the heat equation, we expect that the delta function initial conditions will quickly take on a gaussian appearance. In the heat equation these gaussians will have finite mass beyond any fixed value $|Y|=Y_0$, but the total mass will be exponentially small for a large value of Y_0 (if τ is order one). We took $V\equiv 0$ at $Y=-3$ and only found the solution for $\tau=O(1)$. To integrate (7.1.2) to a larger value of τ

would necessitate a wider grid in the Y direction.

For $Y>0$ we reflected the values obtained for $Y<0$.

It will be noticed that the approximations made to V_X and V_{YY} are of different orders in different regions of the numerical grid. This is not efficient programming (the order of the solution will be the worst order used in approximating the derivatives). This was done in our calculation because we did not know a priori what accuracy requirements would be required. We had planned to increase the accuracy of the inaccurate approximations, if it was needed. As it turned out the program executed as written, in double precision on the IBM 4341.

Since the solution of (7.1.2) is the integral of a probability density, the solution $V(\tau,X,Y)$ must never be negative. Local roundoff errors could cause V to be negative, so we adopted the following scheme. Let $\eta = (\delta t)^2 + (\delta x)^4 + (\delta 4)^4$. If, during execution, a value V occurred with $-\eta < V < 0$, then that value was set to zero. No value was ever smaller than $-\eta$ during execution.

7.2 Checks of the Numerical Solution

The program described in the last section was checked by varying the parameters. We did two checks.

In the first check we reduced δt from .0014 to .0007, and kept all the other parameters as they were.

For the second check we used 180 points in the X direction, with $|X|<7.5$ and used 100 points in the Y direction, with $|Y|<4$

(so $\delta X = .083$ and $\delta Y = .04$). We took $\delta t = 1/2\ (\delta Y)^2 = .00085$ and took $\varepsilon = .18$. In this run there were 4.5 grids points per correlation length of the gaussian, and the grid had a width of 44 correlation lengths.

In each case the program was run for $\tau \leq 1.6$, and the results agreed with the results obtained in the last section.

7.3 Numerical Results

Figures 7.1 through 7.3 contain pictures of the values contained by the numerical grid at different instants of time. The peak around the point X=0, Y=0 is due to the singularity of the two point energy correlation system. The value is finite in our pictures because of the gaussian approximation used for the initial conditions.

From an investigation of the universal focusing curve (Figure I.1) we find $\mu(\tau) < .001$ for $\tau < .62$. We define $\tau^* = .62$ to be the "onset of focusing". After $\tau = \tau^*$, the $\mu(\tau)$ curve rises sharply and focusing becomes significant. Before the onset of focusing, when focusing is not significant, we anticipate that the short distance approximations (in sections 4.5 and 4.6) will be a good approximation to $R_2(\tau, X)$. After the onset of focusing we anticipate that the short distance approximaton will no longer be accurate.

In Figures 7.4 through 7.8 we have graphed the numerical solution for $R_2(\tau, X)-1$ (using squares) versus the short time approximation (3.5.17) minus one (using circles). Figures 7.4

through 7.8 show these curves for τ = .4, .6, .8, 1.2, 1.6.

These pictures show that the regular perturbation solution (i.e., the solution in (3.5.17) or (A.1.32)) is a good approximation only before the onset of focusing.

After the onset of focusing, the $R_2(\tau,X)$-1 curve retains the same general shape, but appears to flatten out. This means that the energy distribution is becoming less well correlated with itself.

7.4 Loss of singularity in the numerical solution

In section 4.7 we found that the two point energy correlation system had a logarithmic singularity in M, as M↓0. We do not observe from Figures 7.4 - 7.8 any such singularity. This is consistent with our numerical technique.

The equation we solved numerically was

$$V_t = f(M)\, V_{WW} - W\, V_M$$

$$V(0,M,W) = \frac{1}{\sqrt{2\pi}\varepsilon} \exp\{-\frac{W^2}{2\varepsilon^2}\}$$

$$\tilde{R}_2(t,M,\varepsilon) = \int V(t,M,W)\, dW \qquad (7.4.1)$$

which is exactly equivalent to:

$$U_t = f(M)\, U_{WW} - W\, U_M$$

$$U(0,M,W;\alpha) = \frac{1}{\sqrt{2\pi}\ \varepsilon} \exp\{-\frac{W^2}{2\varepsilon^2}\}\ \delta(M-\alpha)$$

$$\tilde{R}_2(t,M,\varepsilon) = \int d\alpha \int dW\ U(t,M,W;\alpha) \qquad (7.4.2)$$

The difference between (7.4.2) and (3.5.2) (which is what we wanted to solve), is that a gaussian was used in the initial conditions for (7.4.2) instead of a delta function. We claimed in section 7.1 that, for ε small, the gaussian would "simulate" a delta function and so $\tilde{R}_2(t,M)$ would be a good approximation to $R_2(t,M)$.

However, using (7.4.2) instead of (3.5.2), removes the singularity at M=0 and creates a singularity in ε (for M=0), for all time. We now show this.

For short times we approximate the solution of (7.4.2) by replacing f(M) by f(α) (see section 4.7). We obtain

$$U_t = f(\alpha)\ U_{WW} - W\ U_M$$

$$U(0,M,W;\alpha) = \frac{1}{\sqrt{2\pi}\ \varepsilon} \exp\{-\frac{W^2}{2\varepsilon^2}\}\ \delta(M-\alpha) \qquad (7.4.3)$$

$$\tilde{R}_2(t,M,\varepsilon) = \int dW \int d\alpha\ U(t,M,W;\alpha) \qquad (7.4.4)$$

If we now define $q(t,M,W;\alpha,\beta)$ to be the solution of

$$q_t = f(\alpha)\ q_{WW} - W\ q_M$$

$$q(0,M,W;\alpha,\beta) = \delta(W-\beta)\ \delta(M-\alpha) \qquad (7.4.5)$$

then we can write U (from (7.4.3)) as:

$$U(t,M,W;\alpha) = \int_{-\infty}^{\infty} \frac{d\beta}{\sqrt{2\pi}\varepsilon} \exp\{-\frac{\beta^2}{2\varepsilon^2}\} \, q(0,M,W;\alpha,\beta) \qquad (7.4.6)$$

So q is the Green's function for (7.4.3). The change of variables:

$$\begin{aligned} W' &= W - \beta \\ M' &= M - \beta t \\ S(0,M',W';\alpha) &= q(t,M,W;\alpha,\beta) \end{aligned} \qquad (7.4.7)$$

changes (7.4.5) into

$$\begin{aligned} S_t &= f(\alpha)\, S_{W'W'} - W'\, S_{M'} \\ S(0,M',W';\alpha) &= \delta(W')\, \delta(M'-\alpha) \end{aligned} \qquad (7.4.8)$$

The solution to (7.4.8) is in section 7.2 (see (7.2.5) and (4.2.10)). The solution is:

$$S(t,M,W;\alpha) = \frac{\sqrt{3}}{2\pi} \frac{1}{g_0 t^2} \exp\{-\frac{1}{g_0 t^3}\,[t^2W^2 - 3tW(M-\alpha) + 3(M-\alpha)^2]\} \qquad (7.4.9)$$

Now we use (7.4.9) to find q from (7.4.7) and then use q in (7.4.6) to find U. Since U is only a gaussian in W, we use U in (7.4.4) to find

$$\tilde{R}_2(t,M,\varepsilon) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\frac{2}{3}f(\alpha)t^3+\varepsilon^2t^2}} \exp\{-\frac{1}{2}\frac{(M-\alpha)^2}{\frac{2}{3}f(\alpha)t^3+\varepsilon^2t^3}\}$$

(7.4.10)

For any value of M (including M=0), the value of (7.4.10) is finite for $\varepsilon\neq 0$. This is why there is no singularity near M=0 observed in Figures 7.4 through 7.8.

Since $f(\alpha) \geq 0$, it is easy to bound $\tilde{R}_2(t,M,\varepsilon)$ (for $\varepsilon\neq 0$) by:

$$R(t,M,\varepsilon) \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha \frac{1}{|\varepsilon t|} \exp\{-\frac{1}{2}\frac{(M-\alpha)^2}{\frac{2}{3}f_0t^3+\varepsilon^2t^3}\}$$

$$= \frac{1}{|\varepsilon t|} (\frac{2}{3}f_0t^3+\varepsilon^2t^3)^{1/2} < \infty \qquad (7.4.11)$$

where $f_0 = \max_{\alpha} f(\alpha)$.

The singularity in (7.4.10) is a logarithmic singularity in ε, when M=0. To show this we fix δ ($0<\delta<<1$) and write (7.4.10) as:

$$\tilde{R}(t,0,\varepsilon) = \frac{1}{\sqrt{2\pi}} \{ \int_{|\alpha|>\delta} d\alpha + \int_{|\alpha|\leq\delta} d\alpha \} \frac{1}{\sqrt{\frac{2}{3}f(\alpha)t^3+\varepsilon^2t^3}} \exp\{$$

$$- \frac{1}{2}\frac{\alpha^2}{\frac{2}{3}f(\alpha)t^3+\varepsilon^2t^2} \} \qquad (7.4.12)$$

The first integral in (7.4.12) can be bounded by:

$$\frac{1}{\sqrt{2\pi}} \int_{|\alpha|>\delta} d\alpha \frac{1}{\sqrt{\frac{2}{3}f_\delta t^3}} \exp\{-\frac{1}{2}\frac{\alpha^2}{\frac{2}{3}f_\delta t^3+\varepsilon^2 t^3}\}$$

$$= (1+\frac{3}{2}\frac{\varepsilon^2}{f_\delta t})^{1/2} < \infty \qquad (7.4.13)$$

where $f_\delta = \max_{|\alpha|>\delta} f(\alpha) > 0$. We conclude the singularity in (7.4.12) comes from the region around $\alpha=0$.

In the second integral of (7.4.12), we expand $f(\alpha) = g_2\alpha^2 + O(\alpha^4)$ (from (3.2.9)) for:

$$\tilde{R}(0,t,\varepsilon) = O(1) + \frac{1}{\sqrt{2\pi}} \int_{-\delta}^{\delta} d\alpha \frac{1}{\sqrt{\frac{4}{3}g_2 t^3\alpha^2+\varepsilon^2 t^2}} \exp\{ - \frac{1}{2}\frac{\alpha^2}{\frac{4}{3}g_2 t^3\alpha^2+\varepsilon^2 t^2}\}$$

$$= O(1) + \sqrt{\frac{2}{\pi}} \int_0^{\delta/\varepsilon t} \frac{d\beta}{\sqrt{C\beta^2+1}} \exp\{-\frac{1}{2}\frac{\beta^2}{C\beta^2+1}\} \qquad (7.4.14)$$

where $C = \frac{4g_2 t^3}{3}$, $\beta=\alpha/\varepsilon t$. We now use L'Hopitals rule to show $\tilde{R}(t,0,\varepsilon)$ varies as $\log \varepsilon$ as $\varepsilon \downarrow 0$:

$$\lim_{\varepsilon\downarrow 0} \frac{\tilde{R}(0,t,\varepsilon)}{\log \varepsilon} = -(\frac{3}{2g_2\pi t^3})^{1/2} \exp\{-\frac{3}{8g_2 t^3}\} \qquad (7.4.15)$$

We conclude from (7.4.14), (7.4.15):

$$\tilde{R}(t,0,\varepsilon) \sim -(\frac{3}{2g_2\pi t^3})^{1/2} \exp\{-\frac{3}{8g_2 t^3}\} \log \varepsilon \quad \text{as } \varepsilon\downarrow 0 \qquad (7.4.16)$$

Since the dominant contribution to $\tilde{R}(t,0,\varepsilon)$ comes from the region around $\alpha=0$ as $\varepsilon\downarrow 0$, the expression in (7.4.16) is valid for all time. This is because the approximation of using $f(\alpha)$ for $f(M)$ is valid for large times, if α is sufficiently small (see section 4.7).

Conceivably we could observe this singularity in ε by taking ε very small in section 7.1. But, as we mentioned there, taking ε very small puts a constraint on the grid spacing.

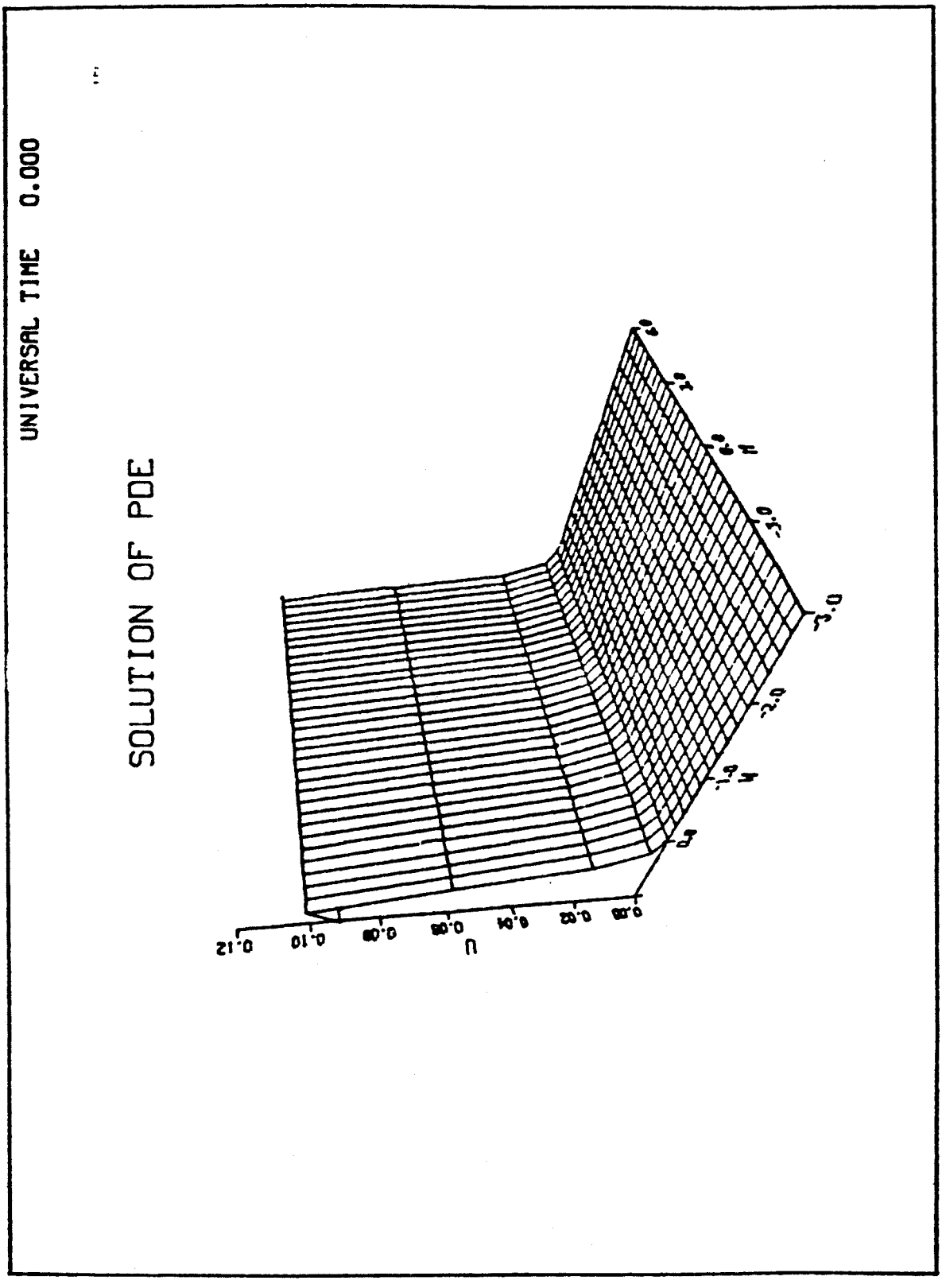
UNIVERSAL TIME 0.000
SOLUTION OF PDE
U
0.12
0.10
0.08
0.06
0.04
0.02
0.0
-3.0
-2.0
-1.0
0.0

Figure 7.1

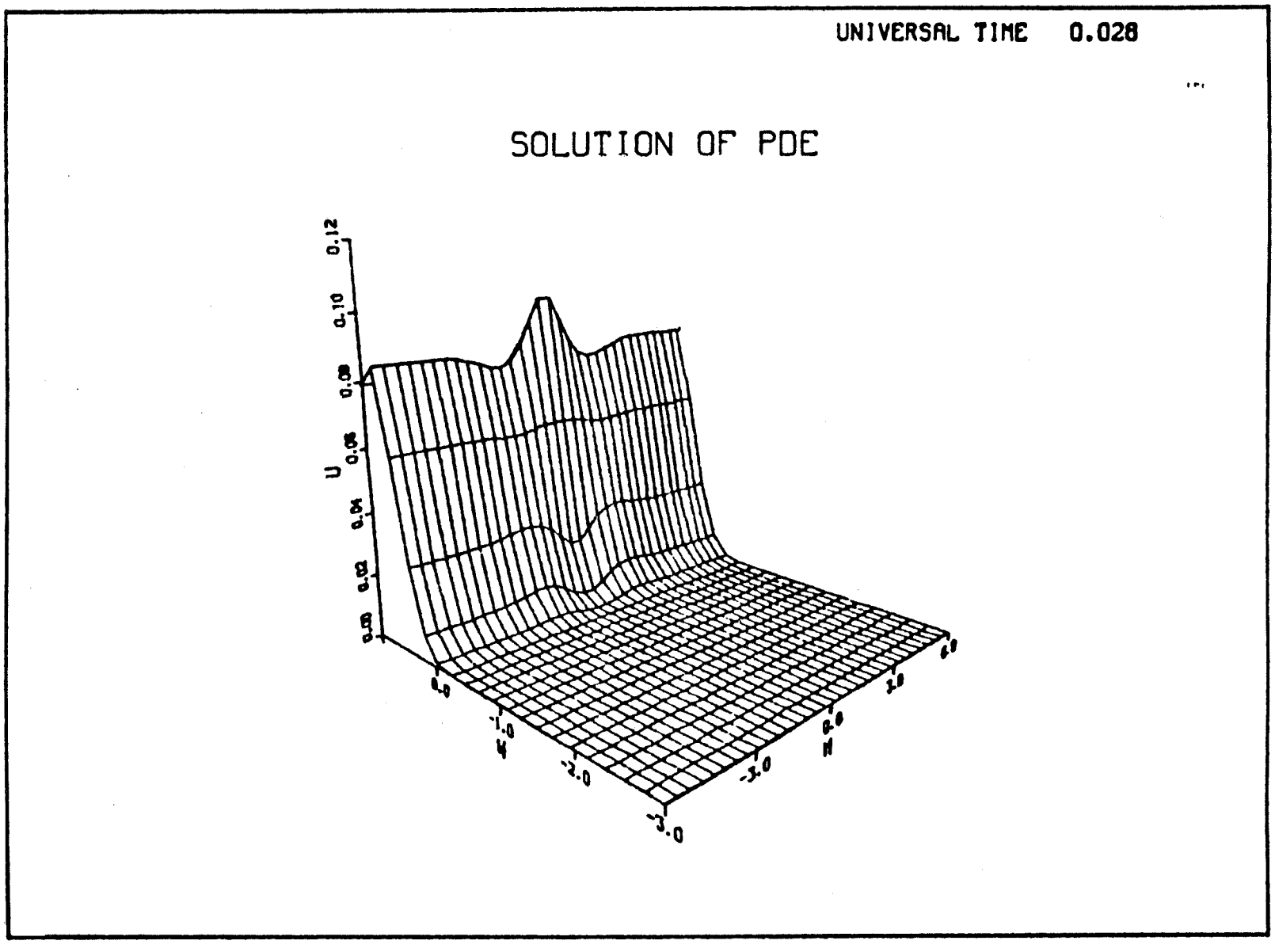
UNIVERSAL TIME 0.028
SOLUTION OF PDE
U
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0.0
-1.0
-2.0
-3.0
H
-3.0
-2.0
-1.0
0.0
1.0

Figure 7.2

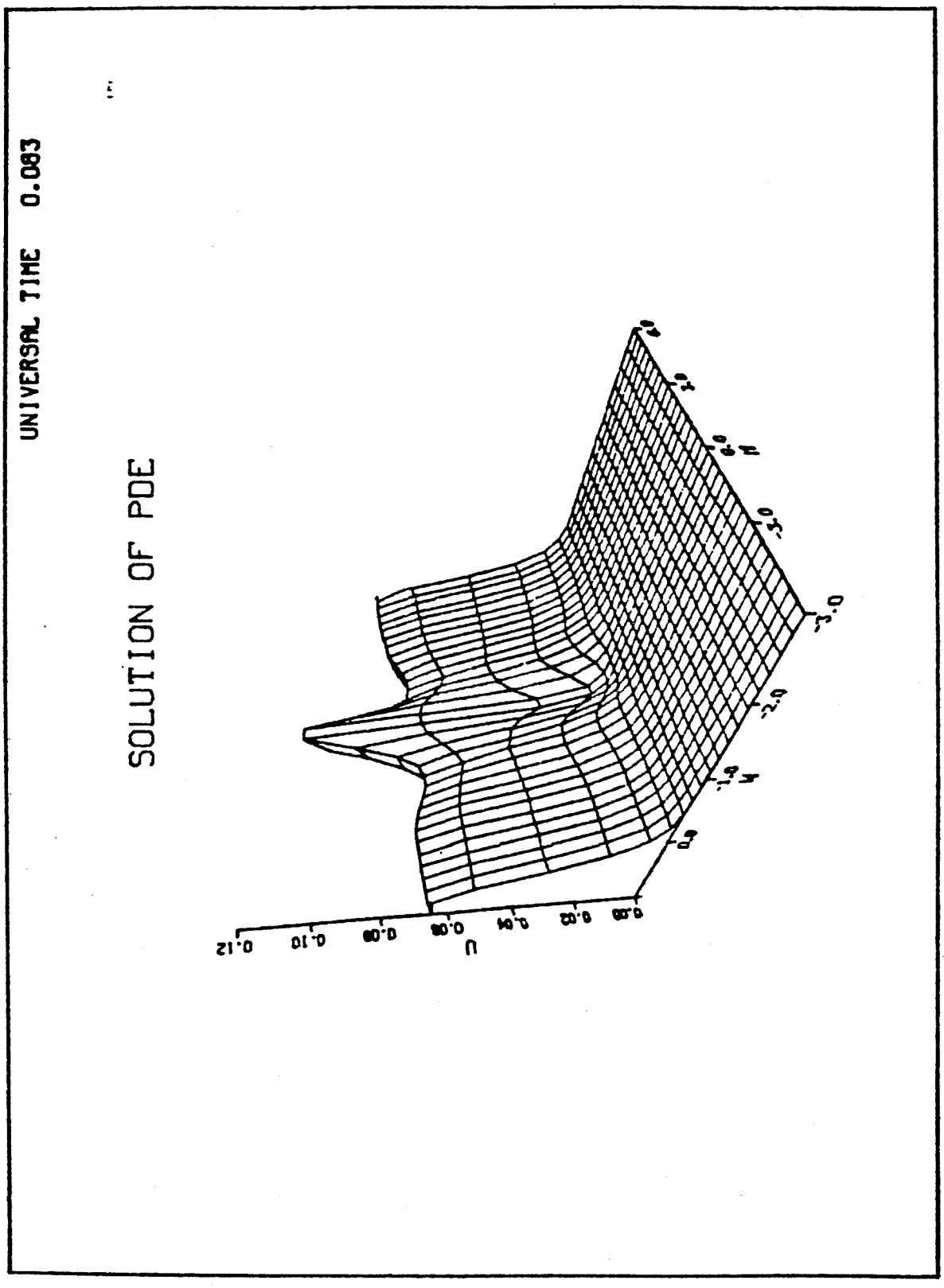
UNIVERSAL TIME 0.083
SOLUTION OF PDE
U
0.12
0.10
0.08
0.06
0.04
0.02
0.00

Figure 7.3

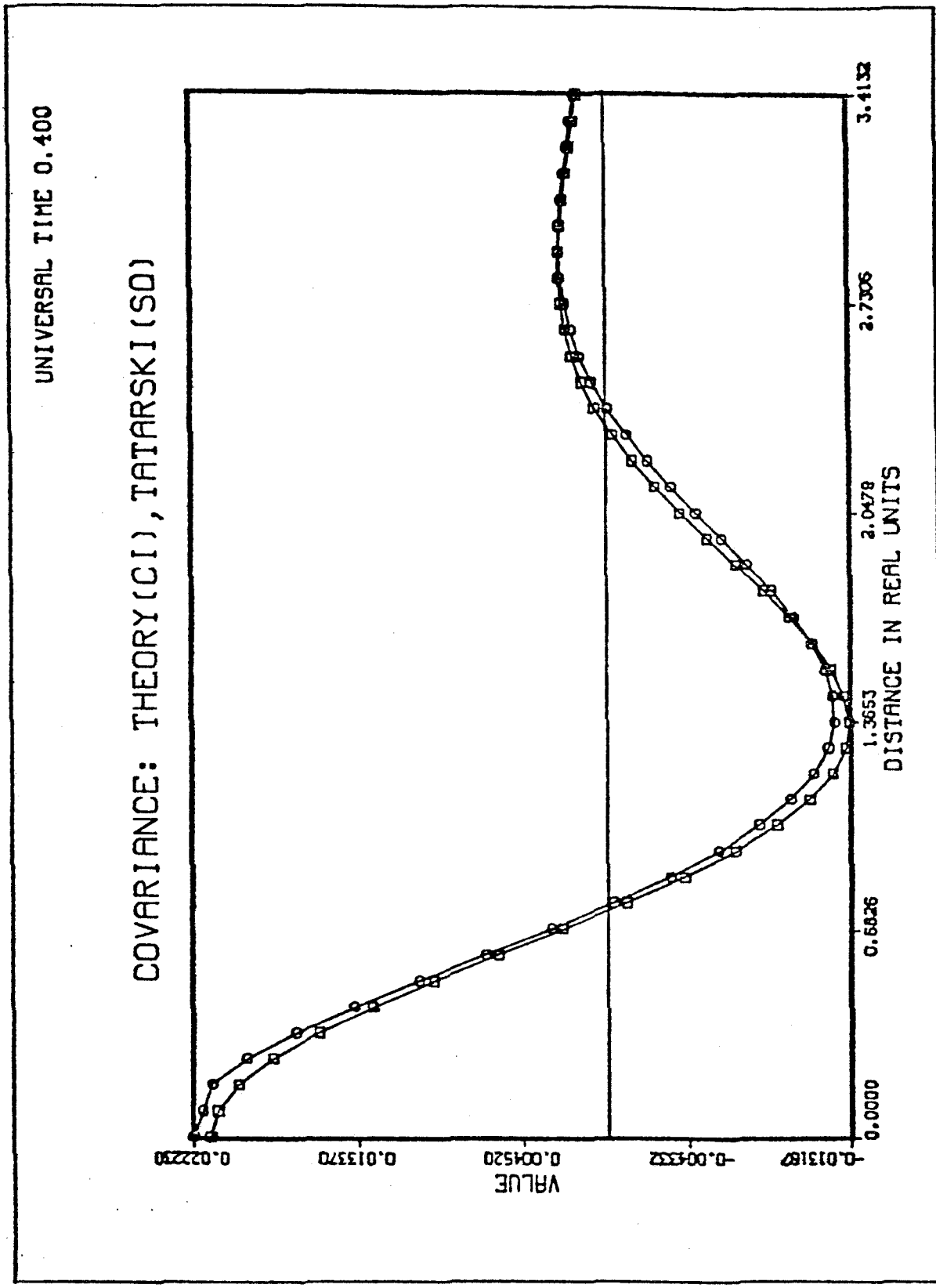
UNIVERSAL TIME 0.400
COVARIANCE: THEORY(CI),TATARSKI(SD)
VALUE
0.022290
0.013370
0.004520
-0.004252
-0.013187
0.0000
0.6826
1.3653
2.0478
2.7306
3.4132
DISTANCE IN REAL UNITS

Figure 7.4

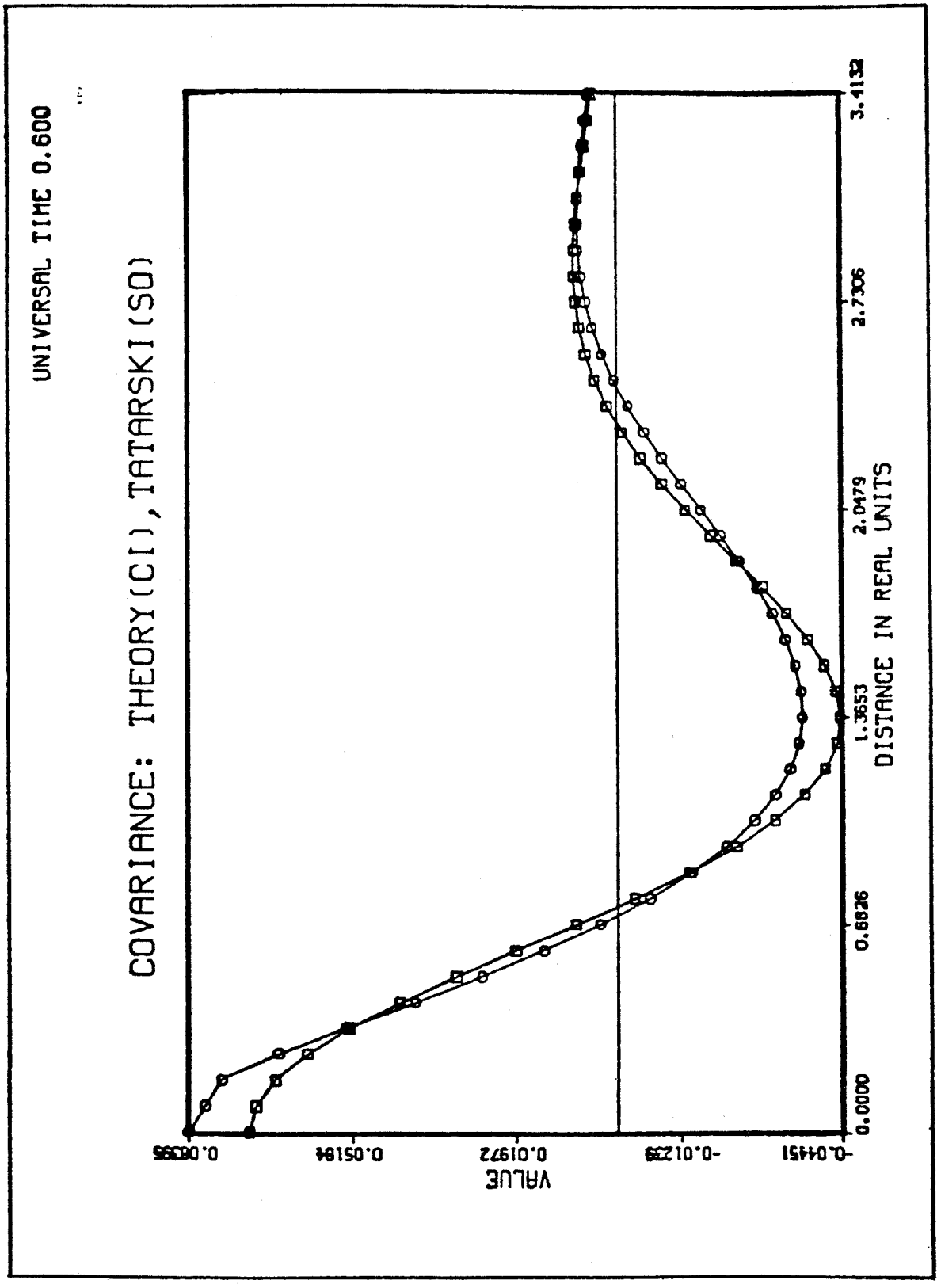
UNIVERSAL TIME 0.600
COVARIANCE: THEORY(CI),TATARSKI(SO)
VALUE
0.06056
0.05184
0.01972
-0.01239
-0.04151
0.0000
0.6826
1.3653
2.0479
2.7306
3.4132
DISTANCE IN REAL UNITS

Figure 7.5

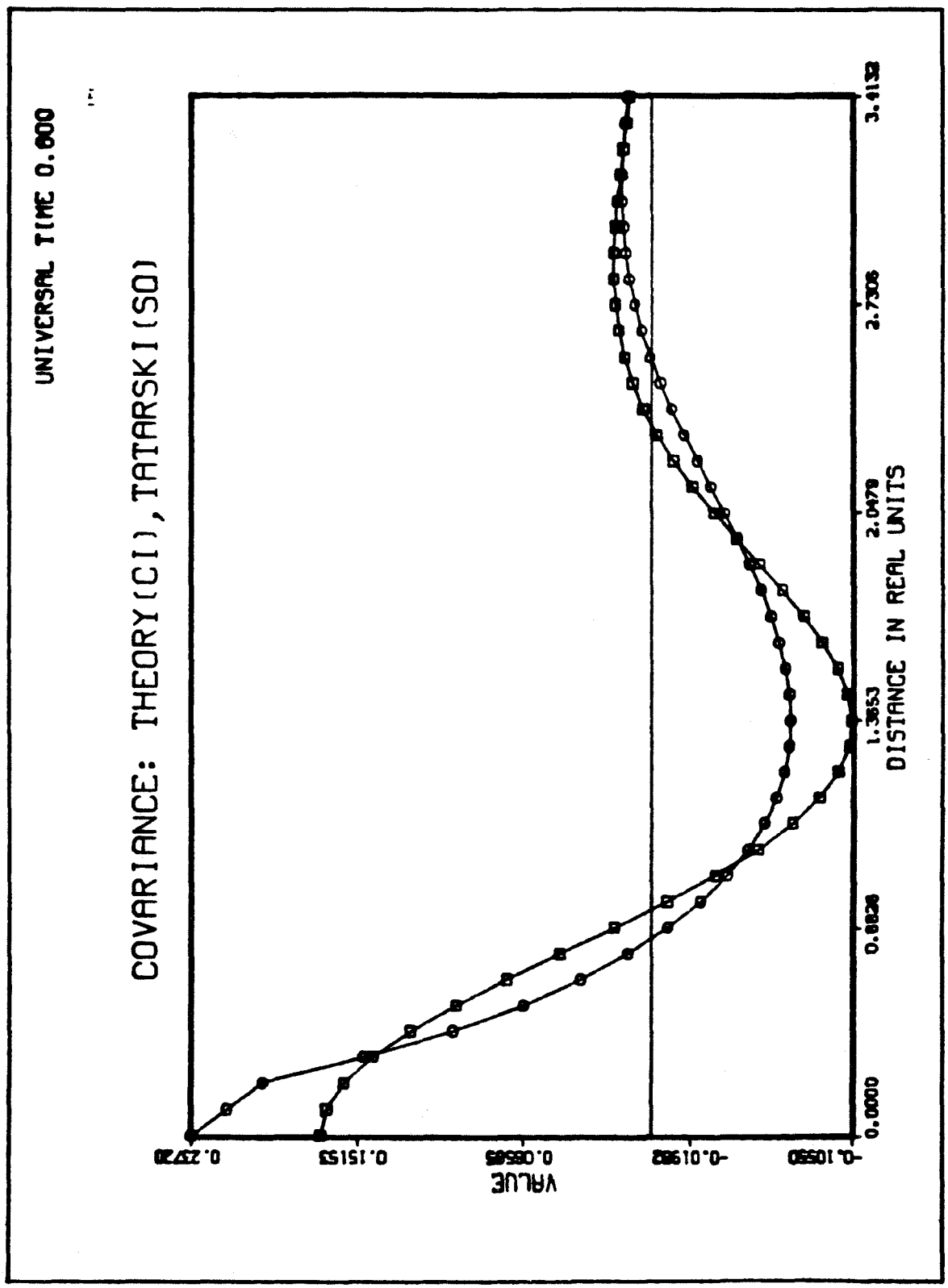
UNIVERSAL TIME 0.000
COVARIANCE: THEORY(CI),TATARSKI(SO)
VALUE
0.23720
0.15153
0.06585
-0.01982
-0.10580
0.0000
0.6826
1.3653
2.0479
2.7306
3.4132
DISTANCE IN REAL UNITS

Figure 7.6

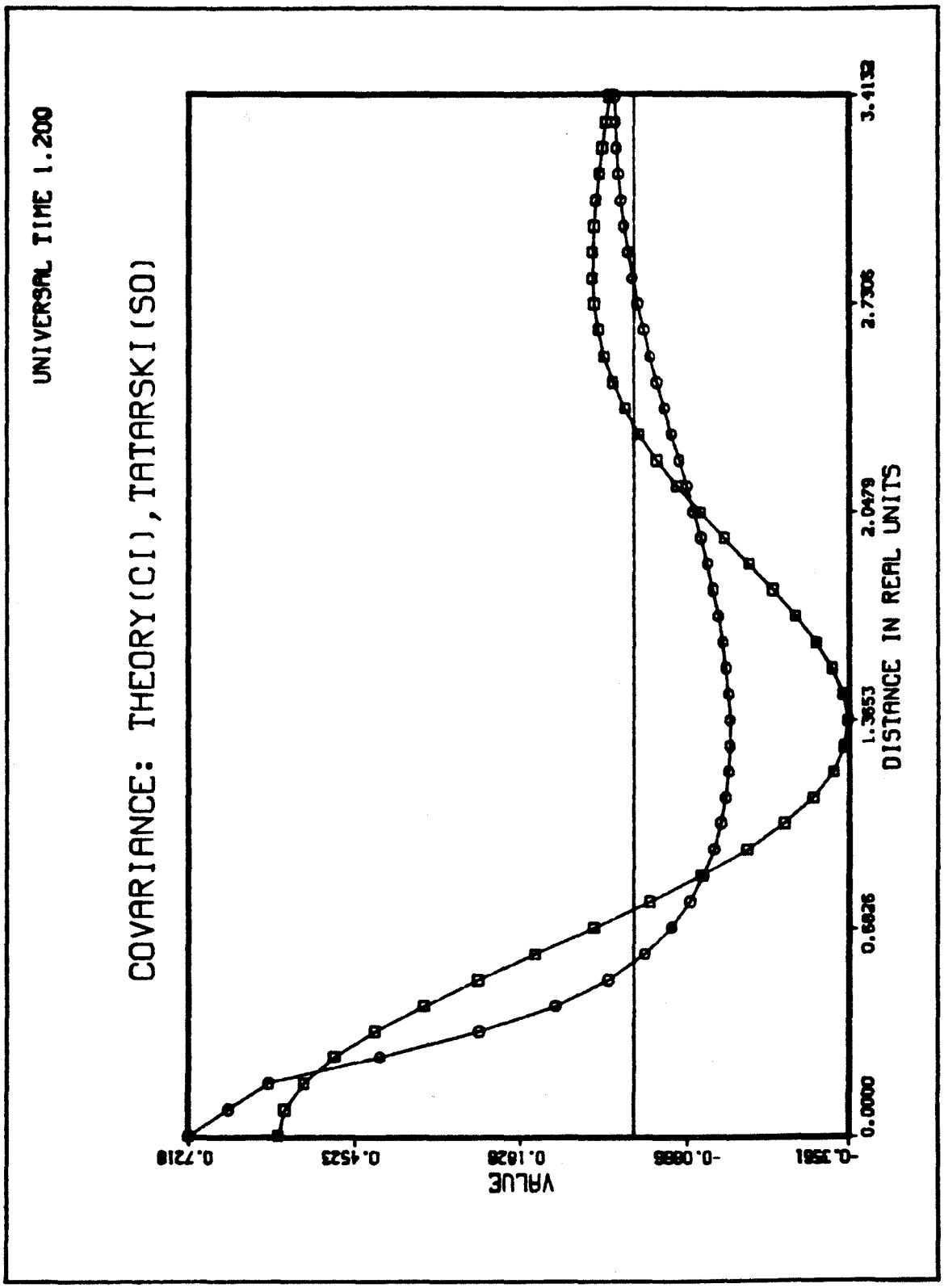
UNIVERSAL TIME 1.200
COVARIANCE: THEORY(CI), TATARSKI(SO)
VALUE
0.7218
0.4523
0.1828
-0.0868
-0.3581
0.0000
0.6826
1.3653
2.0479
2.7305
3.4132
DISTANCE IN REAL UNITS

Figure 7.7

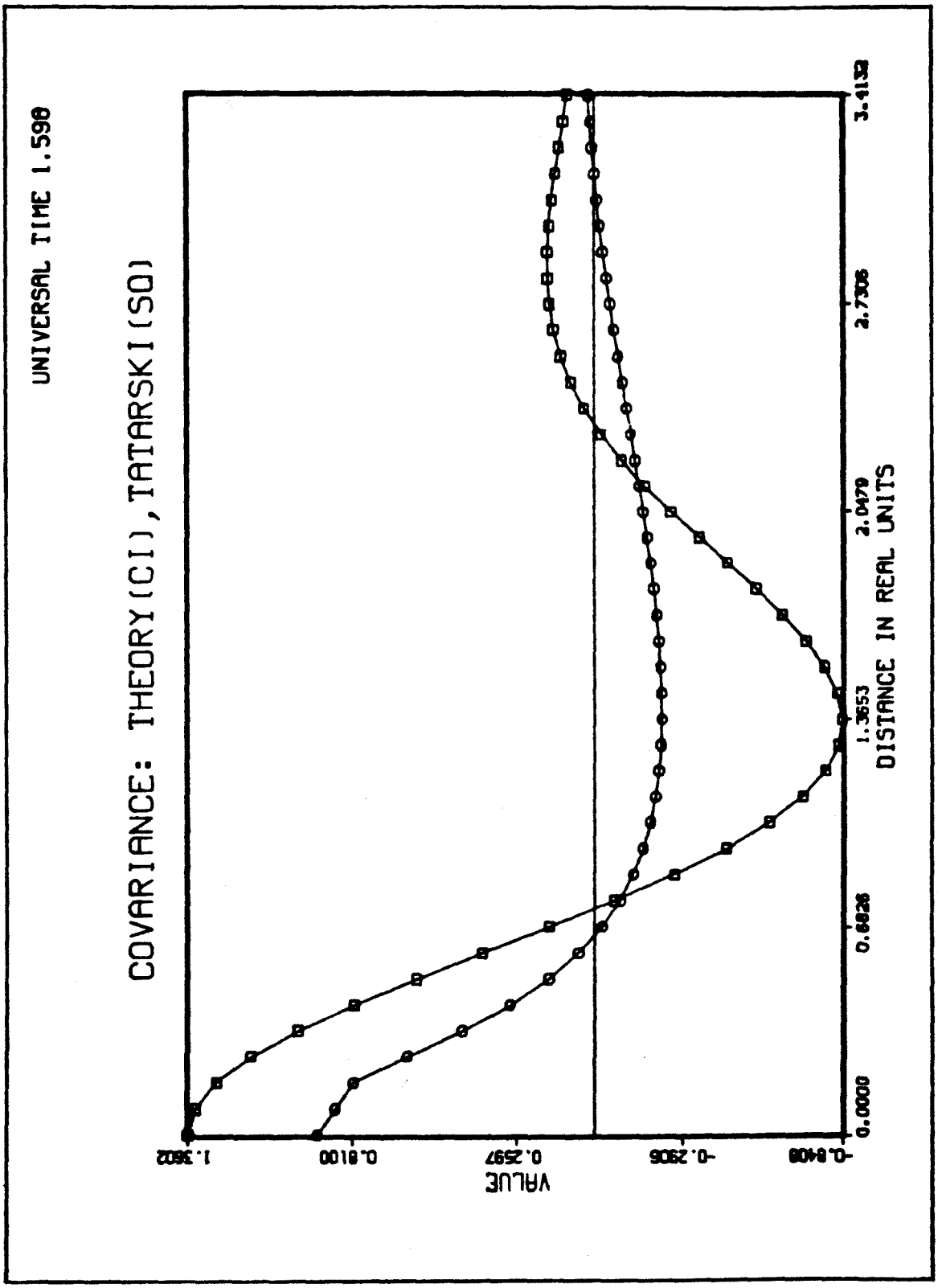
UNIVERSAL TIME 1.590
COVARIANCE: THEORY(CI),TATARSKI(SQ)
VALUE
1.3602
0.8100
0.2597
-0.2906
-0.8408
0.0000
0.6826
1.3653
2.0479
2.7306
3.4132
DISTANCE IN REAL UNITS

Figure 7.8

8.1 Simulation Techiuque

As a check of our analysis, we simulated the motion of rays in a random medium for σ=.01.

We took a grid of 256 x 256 points, representing a physical region of size 17.2 x 17.2. We used the IMSL (23) routine GGUBFS (and then subtracted $\frac{1}{2}$) to determine random numbers at the grid points.

Using fast Fourier Transforms (FFTs), we convolved the grid of random numbers with a two dimensional gaussian having a correlation length of $1/\sqrt{2}$. This produced a random field having approximately a gaussian correlation with an expected standard deviation of one. All the numbers were scaled by a constant, so that the correlation at the origin had a value of one. This was our $\hat{C}$ field.

As we calculated $\hat{C}$, we used FFTs to also calculate $\hat{C}_X$, $\hat{C}_Y$, $\hat{C}_{XX}$, $\hat{C}_{XY}$, $\hat{C}_{YY}$.

The random field had 256 points over a distance of 17.2 correlation lengths. This means there were almost 15 points per correlation length; hence, the random field should have been well sampled. We calculated the correlation of $\hat{C}$, it is shown in Figure 8.1. It looks very much like a gaussian.

We traced the path of 200 rays in the random medium. The wavefront was taken to be initially planar. To find the ray positions, the raytube areas, and the phase (ϕ) along each ray we integrated equations (1.1.12), (1.2.20) and (1.2.22) using a

variable order Runge-Kutta-Verner integrator (IMSL (23) routine DVERK).

We list the equations we integrated (1.1.12), (1.2.20), (1.2.22):

$$\frac{d}{ds}\begin{pmatrix} X \\ Y \\ P_X \\ P_Y \\ \phi \\ A \\ B \end{pmatrix} = \begin{pmatrix} C\ P_X \\ C\ P_Y \\ -C_X/C^2 \\ -C_Y/C^2 \\ 1/C \\ B \\ -A\,[C_{XX}(P_X)^2 - 2\,C_{XY}P_XP_Y + C_{YY}(P_Y)^2] \\ \qquad + B\,[P_xC_x + P_yC_y] \end{pmatrix} \tag{8.1.1}$$

where $C=1+\sigma\hat{C}$. We always have $|P|^2 = P_X^2 + P_Y^2 = 1/C^2$ (see (1.1.17)) so we took as initial conditions for (8.1.1):

$$\begin{pmatrix} X \\ Y \\ P_X \\ P_Y \\ \phi \\ A \\ B \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} X_0 \\ Y_0 \\ 1/C(X_0,Y_0) \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Because the position of a ray was never exactly at a grid point, we interpolated C, C_X, C_Y, C_{XX}, C_{XY}, C_{YY} from the four points closest to the ray position (using (14), 25.2.66).

Since we took:

$$B(r) = e^{-\frac{r^2}{2}} \tag{8.1.3}$$

we have (see (7.1.7), (7.1.8))

$$\gamma_2 = 2.82 \tag{8.1.4}$$

To scale the region of the grid to be on universal time, we multiply the length of the grid by $(\sigma\gamma_2)^{2/3}$. With σ=0.1, $(\sigma\gamma_2)^{2/3} = 0.93$ and the grid has a universal length of $s_{max}\,(\sigma\gamma_2)^{2/3} = (17.2)\,(0.93) = 1.6$.

From the ray positions, the values of ϕ, and the areas of the raytubes, we calculated the two point energy correlation function, to compare with the solution of (4.4.20), and the short distance approximation calculated in section 4.5.

To calculate the correlation, we divided the numerical region width wise into 200 equally spaced points. We discarded the 20 points closest to the top and bottom of the box to remove "end effects." Then we choose 6 different values of universal time (.4, .6, .., 1.4).

At each of these universal times, we constructed the wavefront using the ϕ values. We then investigated each of the 160 equally spaced points, along the calculated wavefront. For each point we found those pairs of rays that surrounded the point. From each pair we interpolated the raytube area at the point. We approximated the energy at a point to be the sum of

the recipricals of the areas at that point. Knowing the energy at the 160 points we calculated the average energy and the two point correlation.

This describes the steps that occurred to obtain one sample of $R_2(t,M)$. We repeated this calculation 15 times, each identical except for the seed for the random number generator. We then averaged the 15 curves to obtain our final estimate of $R_2(t,M)$.

8.2 Agreement of Simulations With Assumptions

A typical picture of the rays is shown in Figure 8.2. A series of wavefronts is shown in Figure 8.3. For the most part the rays are straight and the wavefronts are planar and parallel.

This is what we had assumed in the scaling in chapter one. Recall (1.4.9), (1.4.8), (1.4.16) which together give:

$$\underline{X}(s) = \frac{t}{\sigma^{2/3}}\ \underline{i} + \underline{X}^{\sigma}$$

$$= \left(\frac{t}{\sigma^{2/3}} + X_1^{\sigma}\right)\underline{i} + X_2^{\sigma}\,\underline{j} \qquad (8.2.1)$$

From (2.3.9) we know $X_1^{\sigma} = O(1)$. The horizontal location of the wavefronts, given by $\underline{X}(s)\cdot\underline{i}$, then satisfies

$$\underline{X}(s)\cdot\underline{i} = \frac{t}{\sigma^{2/3}} + O(1) \qquad (8.2.2)$$

From (8.2.2) the wavefronts are flat to leading order. From (8.2.1) we also find that the total angle that a ray subtends is given by

$$\frac{\underline{X}(s)\cdot\underline{j}}{\underline{X}(s)\cdot\underline{i}} = \frac{X_2^\sigma}{\frac{t}{\sigma^{2/3}} + 0(1)} = \frac{X_2^\sigma}{t}\,\sigma^{2/3} + \; 0(\sigma^{4/3}) \qquad (8.2.3)$$

From (8.2.3) the total angular deviation is small (of order $\sigma^{2/3}$).

The wavefront also forms loops at the occurrence of a focus, as predicted in (6). Figure 8.4 is a blown up portion of the waves and wavefronts shown in Figures 8.2 and 8.3. In Figure 8.4, is a very narrow loop of the type predicted.

In (3.3.9) we stated that $E'_{\chi_0}[\mathcal{E}(t,\alpha)] = 1$, or that energy is conserved. For each of the 15 trials we ran, we computed the expected value of energy along every wavefront (τ = .4, .6, ..., 1.4). In Table 8.1, we summarize these numbers, they are all very close to one.

8.3 Agreement of Simulations With Numerical Solution

In Figures 8.5 through 8.10 are graphs of the simulated covariance function (with all 15 samples averaged), the numerical solution from chapter 7 ($R_2(\tau,\alpha)$-1), and the short distance approximation (A.1.32).

For universal times of .8, 1.0, 1.2, 1.4 there is very good agreement between the simulation results and the numerical solution of equation (4.4.20). Because the short time solution

is only correct up to the onset of focusing ($\tau < 6.2$) the simulations do not agree with the short time solution in this range.

The agreement at universal times of .4 and .6 is not very good. This is because the two point correlation system (3.4.20) and the regular perturbation result both required the rays to go through many correlation lengths (see the comment after (A.3.1)). For $\tau = .4$, a ray has only gone $\frac{.4}{(\gamma_2 \tau)^{2/3}} = 4.4$ units in physical space; or 4.4 correlation lengths. This is not a long enough distance for a "central limit theorem" to be valid. In particular, (A.2.15) is not a good approximation when P,Q are not large.

From Figures 8.5 through 8.10 we see that the simulated $R_2(t,M)-1$ has a large value near M=0. This is the logarithmic singularity we found in Section 3.7.

We conclude that the system in (3.4.20) accurately predicts the two point energy correlation function.

8.4 Spacing of Caustic Bundles

From Figure 8.2, the "caustic bundles" (regions surrounding each focus) appear to be quite regularly spaced. This was to have been expected.

The short time solution of (4.4.20) says that $R_2(t,M)-1$ is proportional to the fourth derivative of $\int R(M,Y)\, dY$. If R has a gaussian form then $R_2(t,M)-1$ is proportional to the fourth

derivative of a gaussian.

The fourth derivative of a gaussian (with unit standard deviation), has a large negative dip at x = 1.4. This means that there is a "regular" spacing of alternately high and low energies, with a spacing of about 1.4.

Within the caustic bundles the rays are close together, the areas are small, and the energies are large. Between the caustic bundles the energies are low because the rays are further apart. Hence, for the correlation function to have a dip at X=1.4, we anticipate the distance between caustic bundles to be about 2.8. This is approximately the average distance we find between caustic bundles in the simulations.

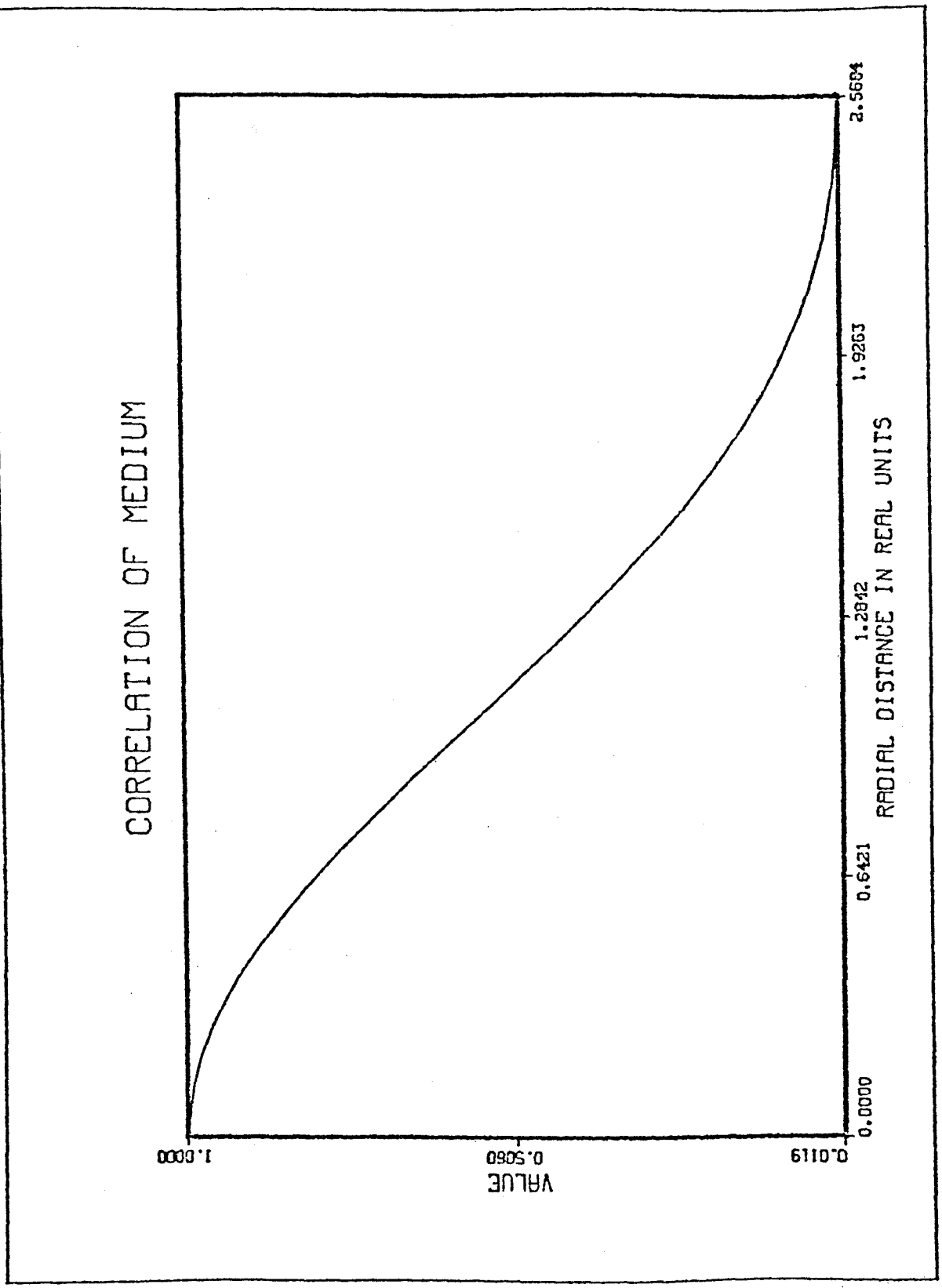
CORRELATION OF MEDIUM
VALUE
1.0000
0.5080
0.0119
0.0000
0.6421
1.2842
1.9263
2.5684
RADIAL DISTANCE IN REAL UNITS

Figure 8.1

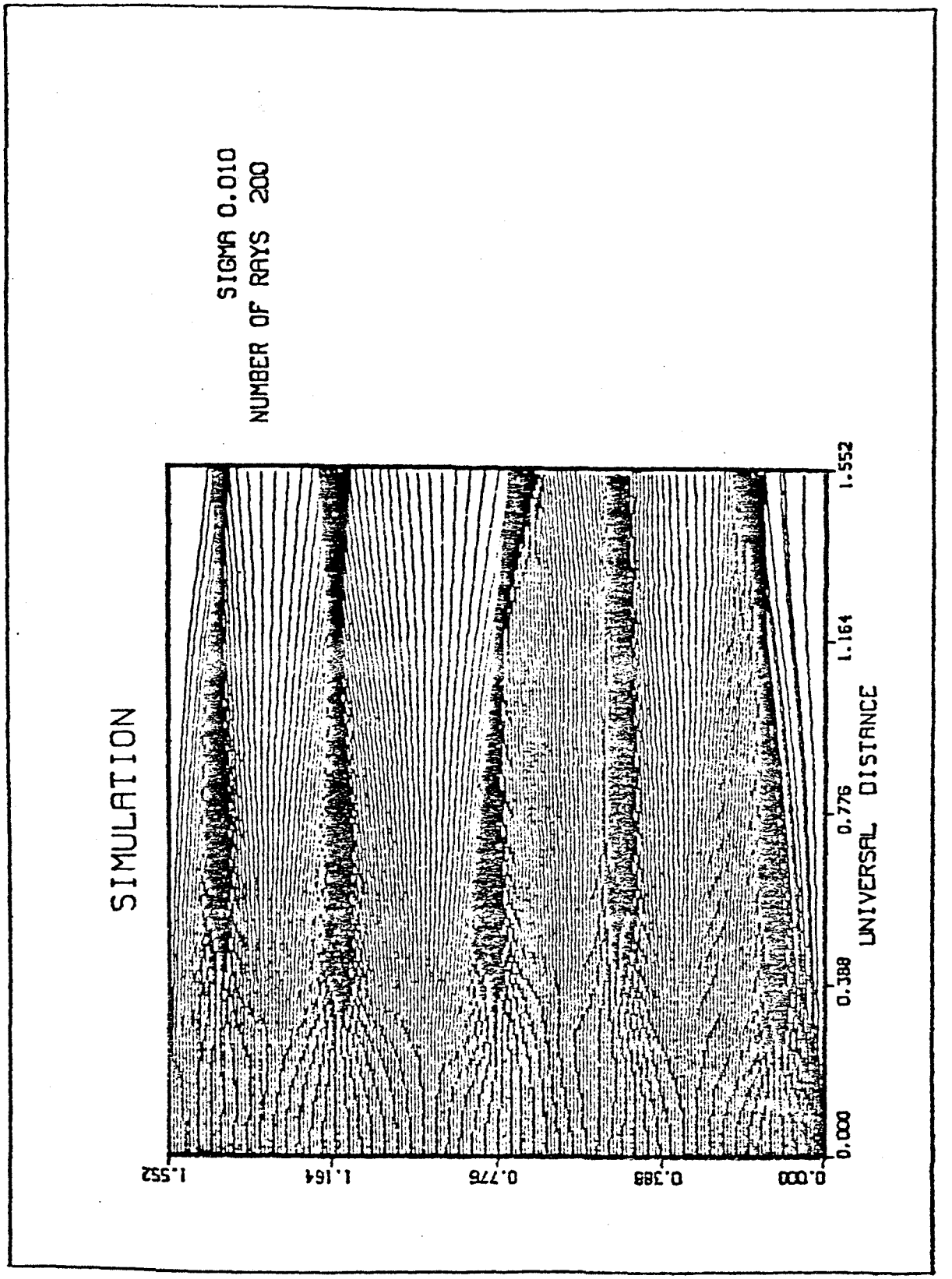

Figure 8.2 Path of rays in a random medium

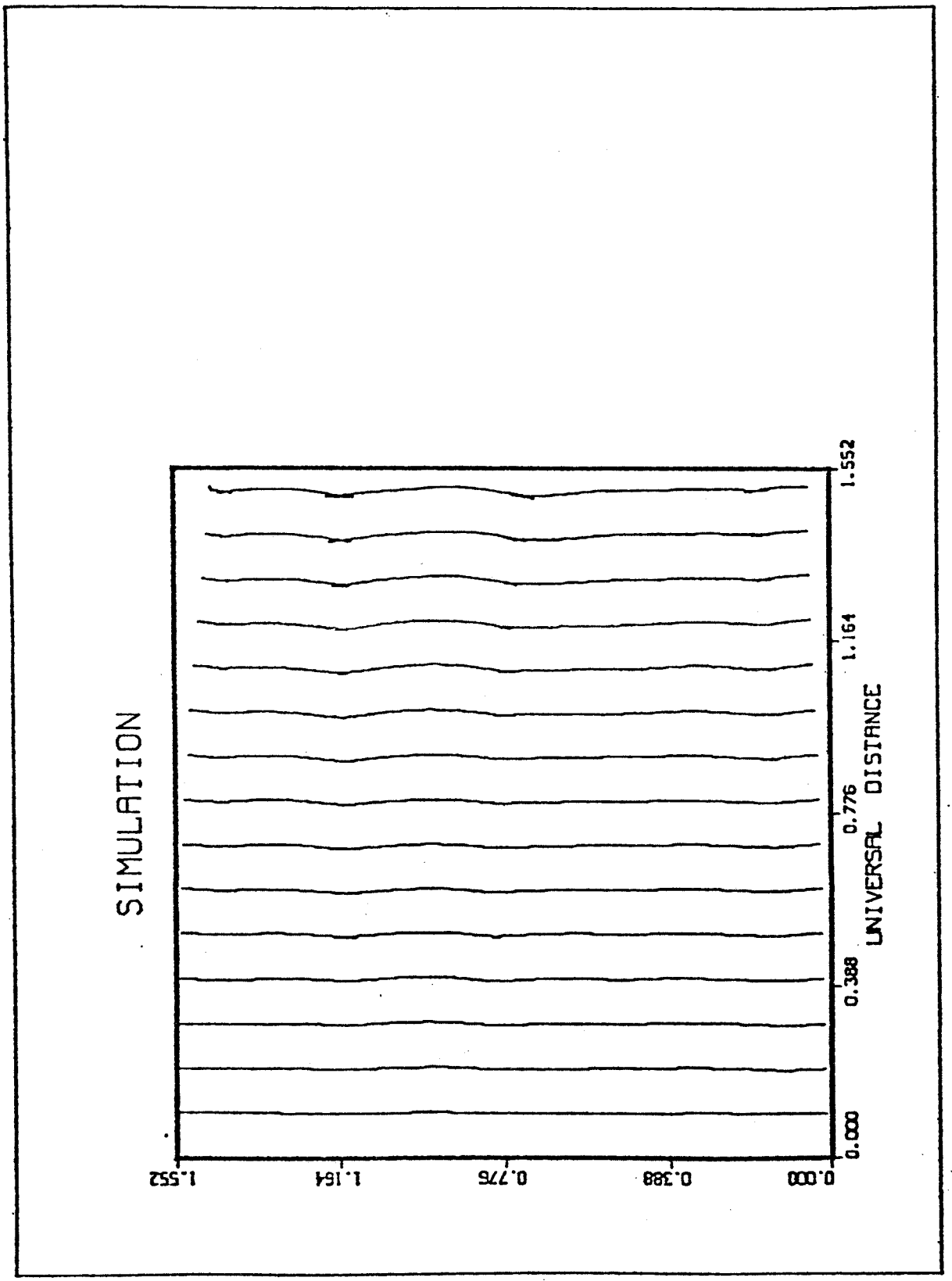

Figure 8.3 Wavefronts associated with the rays in figure 8.2

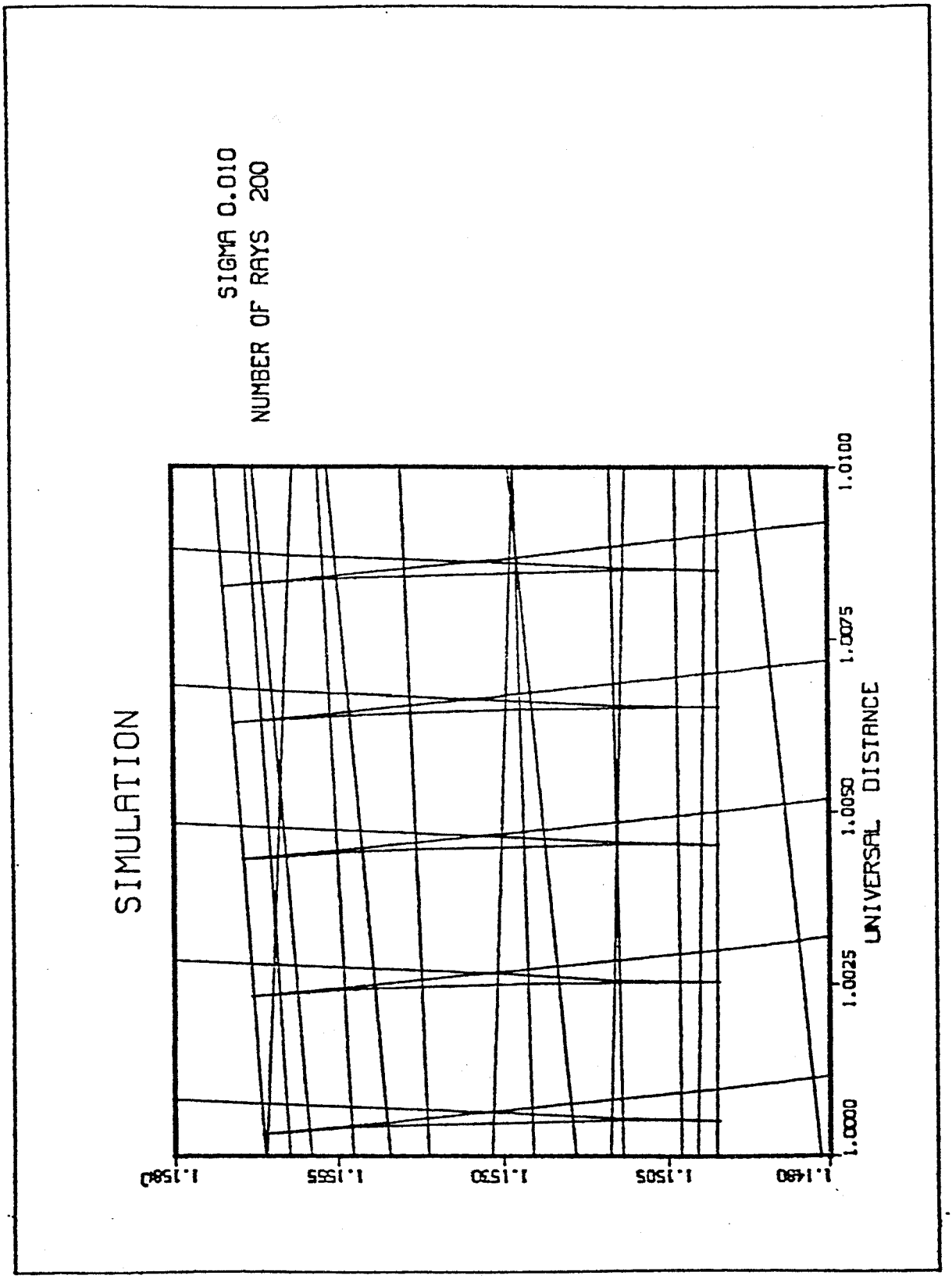

Figure 8.4 Looped wavefront (blown up portion of figure 8.3)

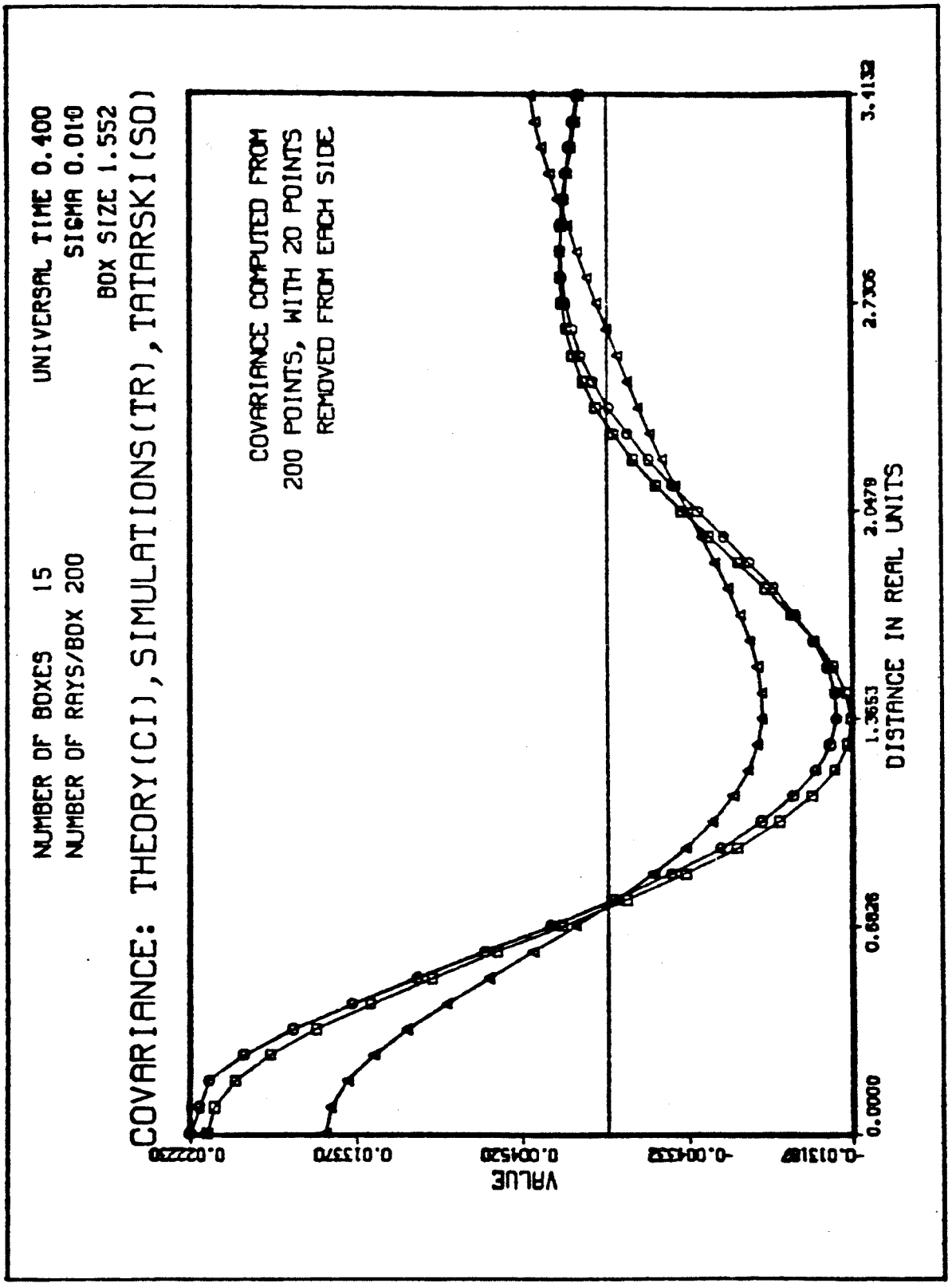
COVARIANCE: THEORY(CI), SIMULATIONS(TR), TATARSKI(SO)
NUMBER OF BOXES 15
NUMBER OF RAYS/BOX 200
UNIVERSAL TIME 0.400
SIGMA 0.010
BOX SIZE 1.552
COVARIANCE COMPUTED FROM
200 POINTS, WITH 20 POINTS
REMOVED FROM EACH SIDE
VALUE
0.022220
0.013770
0.005320
-0.003132
-0.013180
0.0000
0.6826
1.3653
2.0479
2.7306
3.4132
DISTANCE IN REAL UNITS

Figure 8.5

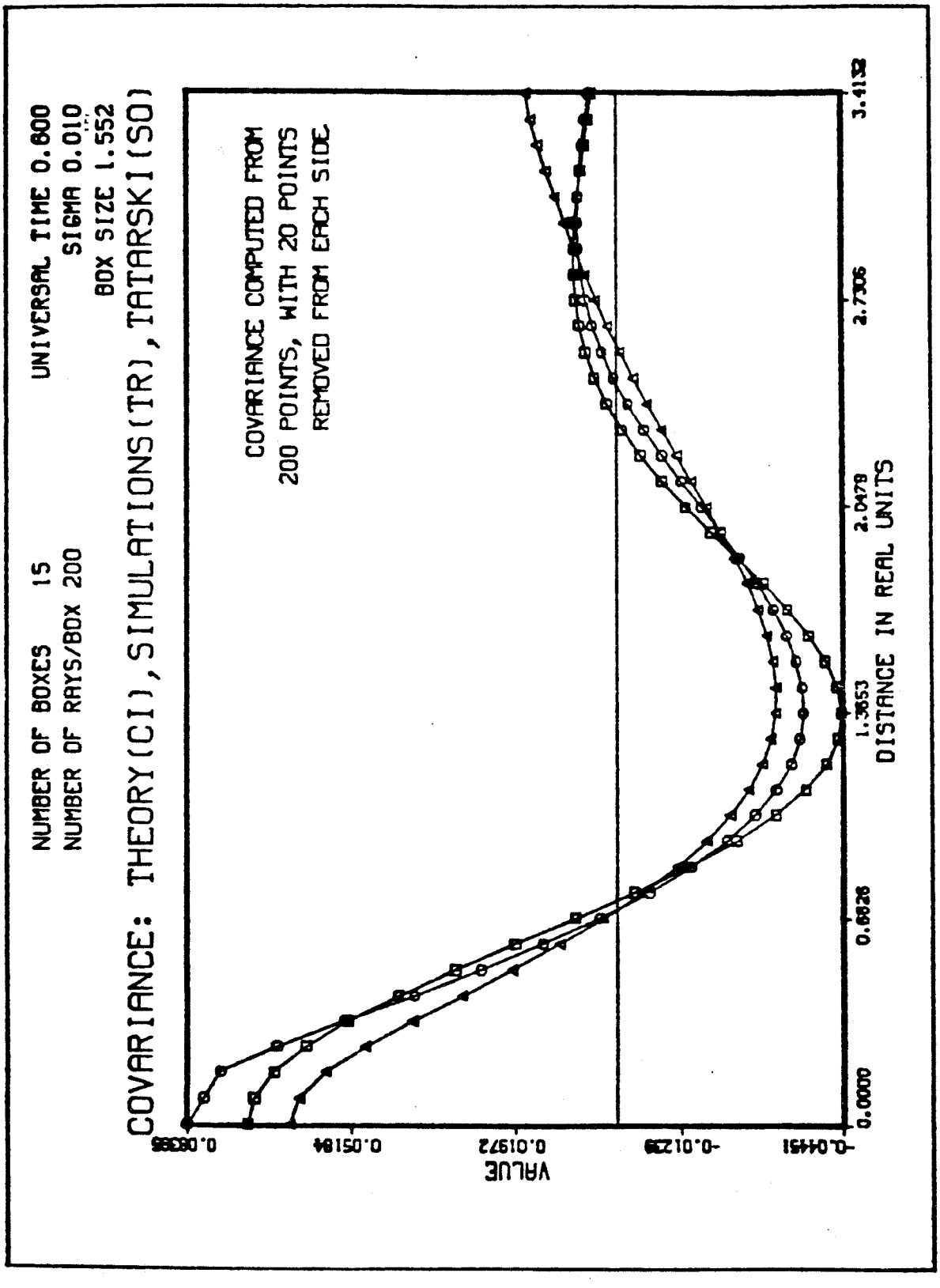
COVARIANCE: THEORY(CI), SIMULATIONS(TR), TATARSKI(SO)
NUMBER OF BOXES 15
NUMBER OF RAYS/BOX 200
UNIVERSAL TIME 0.600
SIGMA 0.010
BOX SIZE 1.552
COVARIANCE COMPUTED FROM
200 POINTS, WITH 20 POINTS
REMOVED FROM EACH SIDE
VALUE
0.0688
0.0184
0.0172
-0.0129
-0.0145
0.0000
0.6828
1.3653
2.0479
2.7306
3.4132
DISTANCE IN REAL UNITS

Figure 8.6

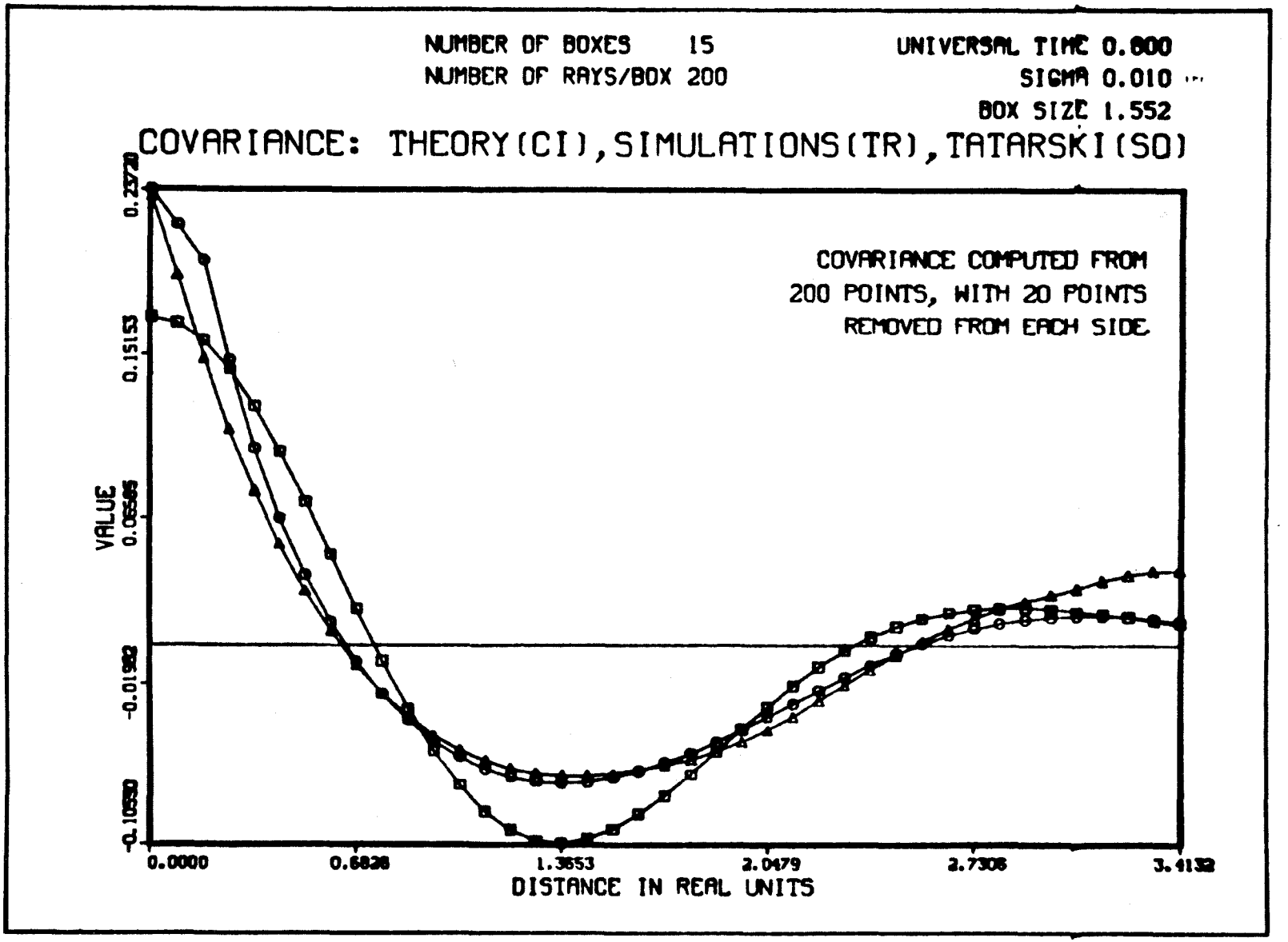
NUMBER OF BOXES 15
NUMBER OF RAYS/BOX 200
UNIVERSAL TIME 0.800
SIGMA 0.010
BOX SIZE 1.552
COVARIANCE: THEORY(CI),SIMULATIONS(TR),TATARSKI(SO)
COVARIANCE COMPUTED FROM
200 POINTS, WITH 20 POINTS
REMOVED FROM EACH SIDE
VALUE
0.15153
-0.01982
0.0000
0.6828
1.3653
2.0479
2.7306
DISTANCE IN REAL UNITS

Figure 8.7

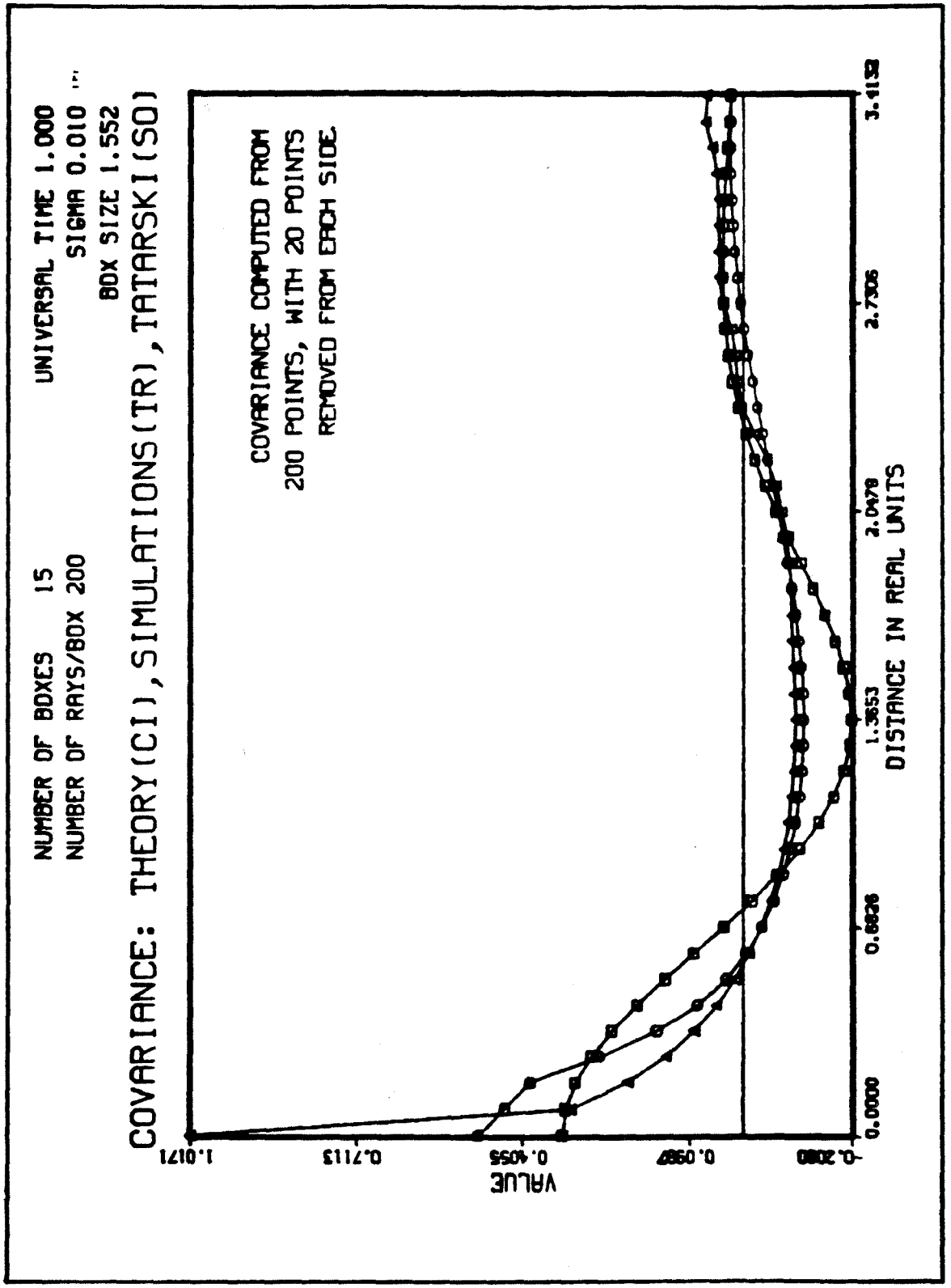
COVARIANCE: THEORY(CI), SIMULATIONS(TR), TATARSKI(SO)
NUMBER OF BOXES 15
NUMBER OF RAYS/BOX 200
UNIVERSAL TIME 1.000
SIGMA 0.010
BOX SIZE 1.552
COVARIANCE COMPUTED FROM
200 POINTS, WITH 20 POINTS
REMOVED FROM EACH SIDE
VALUE
1.0171
0.7113
0.4056
0.0997
-0.2060
0.0000
0.6826
1.3653
2.0478
2.7306
3.4132
DISTANCE IN REAL UNITS

Figure 8.8

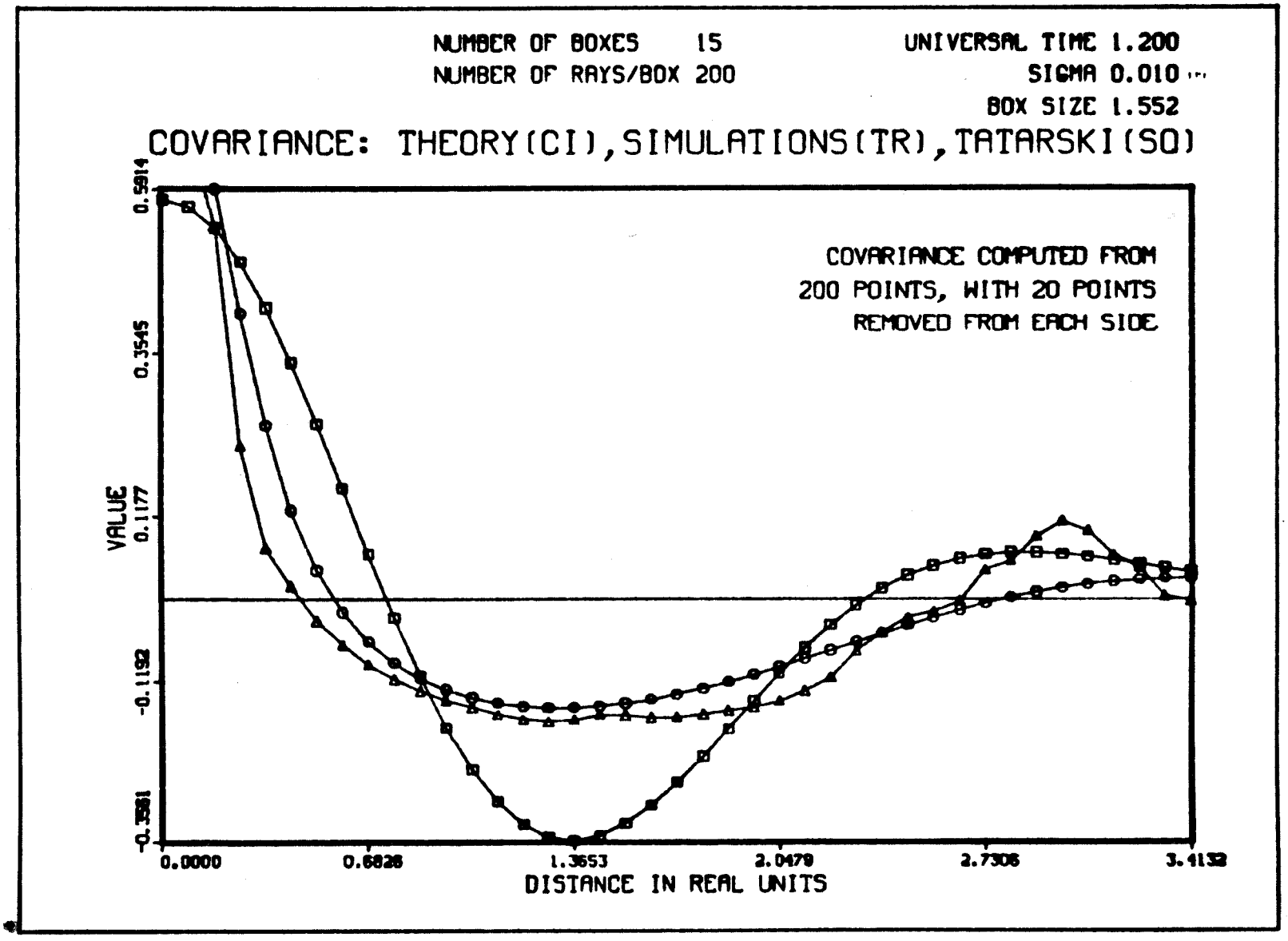
NUMBER OF BOXES 15
NUMBER OF RAYS/BOX 200
UNIVERSAL TIME 1.200
SIGMA 0.010
BOX SIZE 1.552
COVARIANCE: THEORY(CI), SIMULATIONS(TR), TATARSKI(SQ)
COVARIANCE COMPUTED FROM
200 POINTS, WITH 20 POINTS
REMOVED FROM EACH SIDE
VALUE
0.5914
0.3545
0.1177
-0.1192
-0.3581
0.0000
0.6826
1.3653
2.0479
2.7306
3.4132
DISTANCE IN REAL UNITS

Figure 8.9

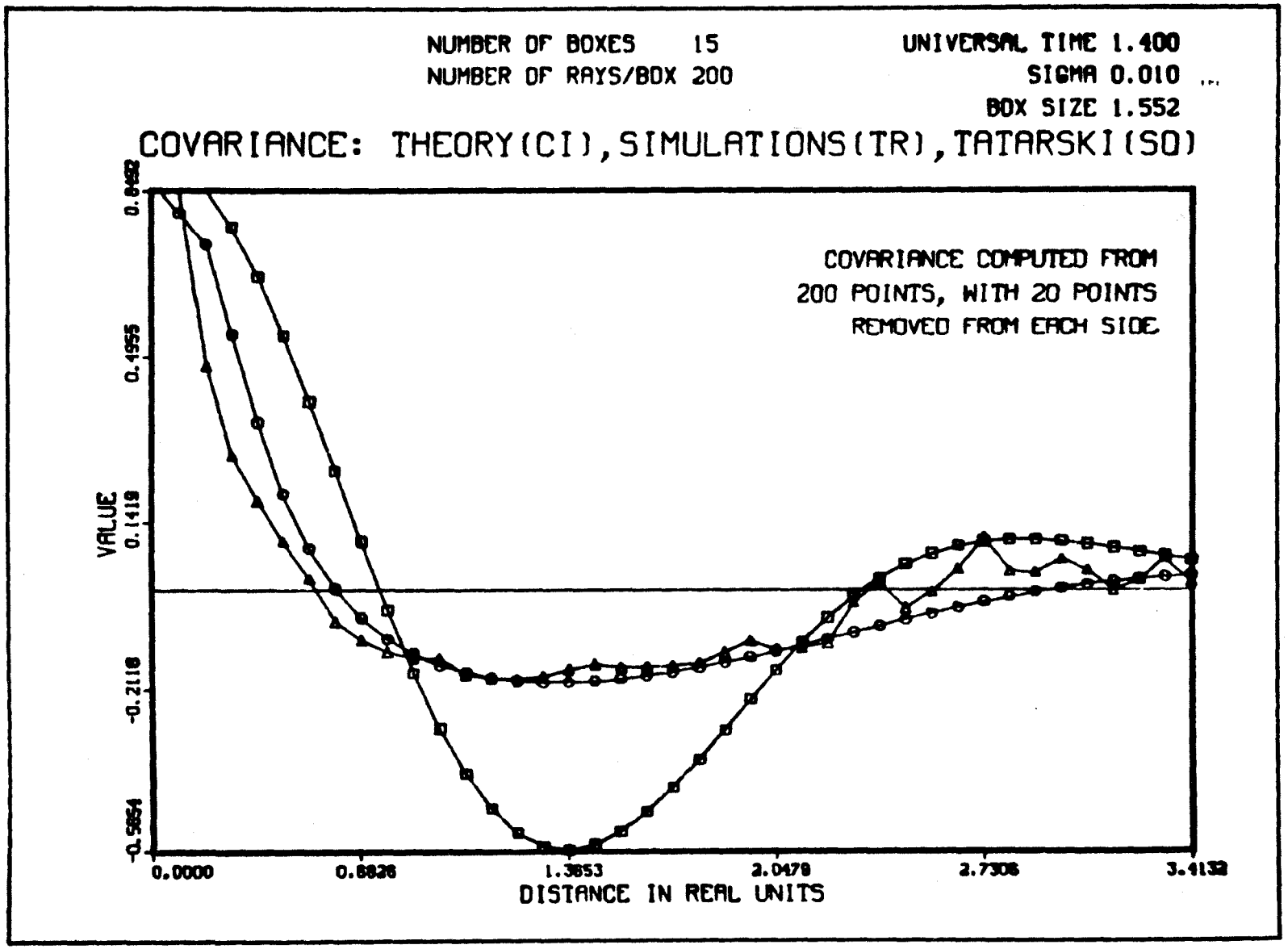
NUMBER OF BOXES 15
NUMBER OF RAYS/BOX 200
UNIVERSAL TIME 1.400
SIGMA 0.010
BOX SIZE 1.552
COVARIANCE: THEORY(CI), SIMULATIONS(TR), TATARSKI(SQ)
COVARIANCE COMPUTED FROM
200 POINTS, WITH 20 POINTS
REMOVED FROM EACH SIDE
VALUE
0.8452
0.4955
0.1419
-0.2118
-0.5654
0.0000
0.8826
1.3653
2.0479
2.7306
3.4132
DISTANCE IN REAL UNITS

Figure 8.10

Table 8.1

Average value of $\mathcal{E}(\tau,\alpha)$

τ	Average value
.2	1.00018
.4	1.00056
.6	1.00196
.8	1.00496
1.0	.99508
1.2	1.01255
1.4	.99052

A.1 Two Point Energy Correlation Function: Plane Initial Wavefront

In this section we will use reglar perturbation techniques to find the two point energy correlation function. The development here is modeled after Tatarski (5).

We start with the wave equation (see (1.1.1)):

$$Q_{TT} = C^2(\underline{X})\ \Delta Q \tag{A.1.1}$$

and look for a solution of the form

$$Q = \Psi(\underline{X},K)\ e^{iK(T+\phi(\underline{X}))} \tag{A.1.2}$$

where:

$$K>>1$$

$$\Psi(\underline{X},K) \sim \sum_{m=0}^{\infty} \Phi_m(\underline{X})\ (iK)^{-m} \tag{A.1.3}$$

We choose a plane wave entering the medium at X=0. Therefore, the initial conditions for Ψ and ϕ are:

$$\text{at } X=0\text{: } \Psi=1,\ \phi=0 \tag{A.1.4}$$

Using (A.1.2), (A.1.3) in (A.1.1) and equating the coefficients of powers of K yields:

$$(\nabla\phi)^2 = \frac{1}{C^2(\underline{X})} \tag{A.1.5}$$

$$2\nabla\phi\cdot\nabla\Phi_m + \Phi\Delta\phi = -\Delta\Phi_{m-1} \qquad m=0,1,2,\ldots \qquad (A.1.6)$$

where $\Phi_{-1}=0$. We now assume:

$$C(\underline{X}) = 1 + \sigma\,\hat{C}(\underline{X}) \qquad (A.1.7)$$

where $\hat{C}$ is mean zero, homogeneous and isotropic. We also expand ϕ, Φ_m in a regular perturbation series in σ:

$$\begin{aligned} \phi &= \phi_0 + \sigma\,\phi_1 + O(\alpha^2) \\ \Phi_m &= \Phi_m^0 + \sigma\,\Phi_m^1 + O(\alpha^2) \end{aligned} \qquad (A.1.8)$$

Using (A.1.7), (A.1.8) in (A.1.5) produces

$$\begin{aligned} O(1): &\qquad (\nabla\phi)_0^2 = 1 \\ O(\sigma): &\quad \nabla\phi_0\cdot\nabla\phi_1 = -\hat{C}(\underline{X}) \end{aligned} \qquad (A.1.9)$$

Using (A.1.8) in (A.1.6) for m=0 produces

$$O(1): \quad 2\,\nabla\phi_0\cdot\nabla\Phi_0^0 + \Phi_0^0\,\Delta\phi_0 = 0 \qquad (A.1.10)$$

$$O(\sigma): \quad 2\,(\nabla\phi_0\cdot\nabla\Phi_0^1 + \nabla\phi_1\cdot\nabla\Phi_0^0) + (\Phi_0^0\,\Delta\phi_1 + \Phi_0^1\,\Delta\phi_0) = 0$$

Using (A.1.3), (A.1.8) in (A.1.4) gives the initial

conditions for ϕ_m and Φ_m^k:

at X=0: $\phi_m = 0 \qquad m > 0$

$$\Phi_0 = 1 \tag{A.1.11}$$

$$\Phi_m^k = 0 \quad m > 0;\ m = 0,\ k > 0$$

From (A.1.9), (A.1.10), (A.1.11) it is easy to find the solutions:

$$\phi_0 = X$$

$$\phi_1 = -\int_0^X d\zeta\ C(\zeta,Y,Z)$$

$$\Phi_0^0 = 1 \tag{A.1.12}$$

$$\Phi_0^1 = \frac{1}{2}\int_0^X d\eta \int_0^\eta d\zeta\ \Delta_2\ \hat{C}(\zeta,Y,Z) + \frac{1}{2}[\hat{C}(X,Y,Z)-\hat{C}(0,Y,Z)]$$

where

$$\Delta_2 \equiv \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} = \Delta - \frac{\partial^2}{\partial X^2} \tag{A.1.13}$$

The solution $\phi_0 = X$ says the wave is propagating along the X axis to first order.

Now we write (A.1.2) as, using (A.1.3), (A.1.8)

$$Q = [\Phi_0 + \sigma\Phi_0^1 + O(\sigma^2, 1/K)]\ e^{iK(T+\phi)} \tag{A.1.14}$$

Since Q is the amplitude of the wave, $|Q^2|$ is the local energy of the wave. From (A.1.4)

$$|Q^2| = \Phi_0^{\,2} + 2\,\sigma\,\Phi_0^0\;\;\Phi_0^1 + O(\sigma^2, 1/K) \qquad \text{(A.1.15)}$$

Suppressing the $O(\sigma^2, 1/K)$ terms and using (A.1.12), expression (A.1.15) becomes

$$|Q^2| = 1 - \sigma\,\{\int_0^X d\eta \int_0^\eta d\zeta\;\Delta_2\,\hat{C}(\zeta,Y,Z) + [\hat{C}(X,Y,Z) - \hat{C}(0,Y,Z)]\} \qquad \text{(A.1.16)}$$

We are interested in the correlation of energy. Since $\hat{C}$ has mean zero, $|Q^2|$ has mean one to first order. We define $|Q^2| = 1 + \Lambda(X,Y,Z)$ to find:

$$\Lambda(X,Y,Z) = -\,\sigma\{\int_0^X d\eta \int_0^\eta d\zeta\;\Delta_2\,\hat{C}(\zeta,Y,Z) + [\hat{C}(X,Y,Z) - \hat{C}(0,0,0)]\} \qquad \text{(A.1.17)}$$

Since $E[\hat{C}] = 0$ we have, to first order,

$$\text{correlation of energy} = 1 + \text{correlation of } \Lambda \qquad \text{(A.1.18)}$$

Now we will use (A.1.17) to find the correlation of Λ. Then we use (A.1.18) for the correlation of energy. First we neglect the $\hat{C}$ terms in (A.1.17), in comparison with the $\int\int \Delta_2 \hat{C}$ term to find

$$\Lambda(X,Y,Z) = -\sigma \int_0^X d\eta \int_0^\eta d\zeta \; \Delta_2 \hat{C}(\zeta,Y,Z) \qquad (A.1.19)$$

We define the correlation of $\hat{C}$ by (see (2.3.16)):

$$R(\zeta-\zeta',Y,Z) = E[\hat{C}(\zeta,Y,Z)\,\hat{C}(\zeta',0,0)] \qquad (A.1.20)$$

and we define the transverse correlation of Λ, at a distance X into the medium, by:

$$R_2(X,Y,Z) = E\,[\Lambda(X,Y,Z)\,\Lambda(X,0,0)] \qquad (A.1.21)$$

Now we use (A.1.19), (A.1.20) in (A.1.21) to find

$$\begin{aligned} R_2(X,Y,Z) &= E\Big[\,\sigma^2 \int_0^X d\eta \int_0^\eta d\zeta \; \Delta_2 \hat{C}(\zeta,Y,Z) \\ &\qquad \int_0^X d\eta' \int_0^{\eta'} d\zeta' \; \Delta_2 \hat{C}(\zeta',0,0)\,\Big] \\ &= \sigma^2\Delta_2^2 \int_0^X d\eta \int_0^\eta d\zeta \int_0^X d\eta' \int_0^{\eta'} d\zeta' \; E[\hat{C}(\zeta,Y,Z)\hat{C}(\zeta',0,0)] \\ &= \sigma^2 \Delta_2^2 \int_0^X d\eta \int_0^X d\eta' \int_0^\eta d\zeta \int_0^{\eta'} d\zeta' \; R(\zeta-\zeta',Y,Z) \end{aligned} \qquad (A.1.22)$$

Now we assume X>>1 and approximate (A.1.22). Since R is a correlation function for a mixing process, we anticipate that R(X,Y,Z) will "quickly" go to zero as $X^2+Y^2+Z^2 \to 0$. If X>>1, then η and η' will generally be large in (A.1.22). We approximate

(A.1.22) by

$$R_2(X,Y,Z) = \sigma^2 \Delta_2^2 \int_0^X d\eta \int_0^X d\eta' \int_0^\gamma d\zeta \int_0^\gamma d\zeta' \; R(\zeta-\zeta',Y,Z) \tag{A.1.23}$$

where $\gamma=\min(\eta,\eta')$. Now $R(X,Y,Z)$ is an even function with respect to all of its arguments since $\hat{C}$ is homogeneous and isotropic.

We note the formula

$$\int_0^a dX \int_0^a dY \; H(X-Y) = 2 \int_0^a d\mu \; (a-\mu) \; H(\mu) \tag{A.1.24}$$

valid for all even functions H.

Using (A.1.24) in (A.1.23) yields

$$R_2(X,Y,Z) = 2\sigma^2 \Delta_2^2 \int_0^X d\eta \int_0^X d\eta' \int_0^\gamma d\mu \; (\gamma-\mu) \; R(\mu,Y,Z) \tag{A.1.25}$$

If $X>>1$ then η,η' will generally be "large" in (A.1.25). Hence, $\gamma=\min(\eta,\eta')$ will generally be "large." Therefore, we approximate $(\gamma-\mu)$ by γ. When μ is "large", so $(\gamma-\mu)$ is not well approximated by γ, then $R(\mu,Y,Z)$ will be "small" (if $R(X,Y,Z)$ goes to zero "quickly" as $X^2+Y^2+Z^2\to\infty$).
This gives

$$R_2(X,Y,Z) = 2\sigma^2 \Delta_2^2 \int_0^X d\eta \int_0^X d\eta' \; \gamma \int_0^\gamma d\mu \; R(\mu,Y,Z) \tag{A.1.26}$$

Now we increase the upper limit of the μ integral from γ to ∞ to find:

$$R_2(X,Y,Z) = 2\sigma^2 \Delta_2^2 \left(\int_0^X d\eta \int_0^X d\eta' \min(\eta,\eta')\right)\left(\int_0^\infty d\mu\, R(\mu,Y,Z)\right) \tag{A.1.27}$$

$$= 2\sigma^2 \Delta_2^2 \left(\frac{X^3}{3}\right)\left(-G(Y,Z)\right)$$

$$= -\frac{2}{3}\sigma^2 \left(\frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right)^2 G(Y,Z) \tag{A.1.28}$$

where we have used (6.1.12)

If we scale X to be our long distance scale (see (1.4.8)) we have

$$t = \sigma^{2/3} X \tag{A.1.29}$$

Using (A.1.29) in (A.1.28) gives

$$R_2(t,Y,Z) = -\frac{2}{3} t^3 \left(\frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right)^2 G(Y,Z) \tag{A.1.30}$$

When (A.1.30) is used with (A.1.18) we have the same answer we found in (6.4.12)

In two dimensions, (A.1.27) becomes (see (2.3.21))

$$R_2(X,Y) = 2\sigma^2 \left(\frac{\partial^2}{\partial Y^2}\right)^2 \left(\int_0^X d\eta \int_0^X d\eta' \min(\eta,\eta')\right)\left(\int_0^\infty d\mu\, R(\mu,Y)\right)$$

$$= 2\sigma^2 \frac{\partial^2}{\partial Y^2}\left(\frac{X^3}{3}\right) \int_0^\infty d\mu\, R_{YY}(\mu,Y)$$

$$= -\frac{2}{3}\sigma^2 X^3 \frac{\partial^2}{\partial Y^2} g(Y) = -\frac{2}{3}\sigma^2 X^3 g''(Y) \tag{A.1.31}$$

Using (A.1.29) in (A.1.31):

$$R_2(t,Y) = -\frac{2}{3} t^3 g''(Y) \qquad \text{(A.1.32)}$$

Equation (A.1.32), with (A.1.18), is the same answer we obtained in (4.5.17). Note that there is no singularity in (A.1.32) as $Y \downarrow 0$, as we found in the solution of (4.4.20) (see section 4.7).

A.2 Two Part Energy Correlation Function in Two Media

In this section, we use regular perturbation methods to find the two point energy correlation function, after rays have gone through two media. We require the initial wavefront to be plane and the interface between the two media to be parallel to the initial wavefront.

The first medium extends from $X = 0$ to $X = \frac{t_1}{\sigma^{2/3}} = X_1$ and has a wave velocity of $C(\underline{X}) = 1+\sigma\hat{C}(\underline{X})$ (where $\hat{C}$ is mean zero, homogeneous and isotropic). The second medium extends from $X = X_1$ to $X = \frac{t_2 - t_1}{\sigma^{2/3}}$. In the second medium the wave speed is identically one.

What we have described is the shadowgraph problem (see section 5.4). We will obtain the same answer here, using regular perturbation techiques, as we obtained using Papanicolaou and Kohler theorem (see (5.5.7)).

To solve the shadowgraph problem we look for a geometrical

optics solution in each of the two media. In the first media we look for a solution of the wave equation.

$$Q_{TT} = C^2(\underline{X})\ \Delta Q = (1+\sigma\hat{C}(\underline{X}))^2\ \Delta Q \qquad (A.2.1)$$

in the form

$$Q = [\Phi_0^0 + \sigma\ \Phi_0^1 + O(\sigma^2,1/K)]\ e^{iK(T+\phi_0 + \sigma\phi_1 + O(\sigma^2))} \qquad (A.2.2)$$

From (A.1.12), (A.1.14) we have

$$\phi_0 = X, \quad \phi_1 = -\int_0^X \hat{C}(\zeta,Y,Z)\ d\zeta$$

$$\Phi_0^0 = 1, \quad \Phi_0^1 = \frac{1}{2}\int_0^X d\eta \int_0^\eta d\zeta\ \ \Delta_2\ \hat{C}(\zeta,Y,Z) \qquad (A.2.3)$$
$$+ \frac{1}{2}\ [\hat{C}(X,Y,Z) - \hat{C}(X,0,0)]$$

In the second medium we look for a solution of the wave equation

$$Q_{TT} = C^2(\underline{X})\ \Delta Q = \Delta Q \qquad (A.2.4)$$

in the form

$$Q = [\Gamma_0^0 + \sigma\ \Gamma_0^1 + O(\sigma^2,1/K)]\ e^{iK(T + \Theta_0 + \sigma\Theta + O(\sigma^2))} \qquad (A.2.5)$$

Substituting (A.2.5) in (A.2.4) we find (analogous to

(A.1.9), (A.1.10)):

$$(\nabla\Theta_0)^2 = 1$$

$$\nabla\Theta_0 \cdot \nabla\Theta_1 = -1 \qquad \text{(A.2.6)}$$

$$2\ \nabla\Theta_0 \cdot\ \nabla\Gamma_0^0 + \Gamma_0^0\ \Delta\Theta_0 = 0$$

$$2(\nabla\Theta_0 \cdot \nabla\Gamma_0^1 + \nabla\Theta_1 \cdot \nabla\Gamma_0^0) + (\Gamma_0^0\ \Delta\Theta_1 + \Gamma_0^1\ \Delta\Theta_0) = 0 \qquad \text{(A.2.7)}$$

The initial conditions for Θ_0, Θ_1, Γ_0^0, Γ_0^1 come from the boundary, at $X = X_1$. If we require, continuity of Q and $\frac{\partial Q}{\partial X}$ across the boundary then, from (A.2.4), (A.2.5)

$$\Phi_0^0\Big|_{X_1} = \Gamma_0^0\Big|_{X_1} \qquad \phi_0\Big|_{X_1} = \Theta_0\Big|_{X_1}$$
$$\Phi_0^1\Big|_{X_1} = \Gamma_0^1\Big|_{X_1} \qquad \phi_1\Big|_{X_1} = \Theta_1\Big|_{X_1} \qquad \text{(A.2.8)}$$

Using (A.2.3), (A.2.8) we can solve (A.2.6) for

$$\Theta_0 = X$$

$$\Theta_1 = -\int_0^{X_1} d\zeta\ \hat{C}(\zeta, Y, Z) - (X - X_1)$$

$$\Gamma_0^0 = 1 \qquad \text{(A.2.9)}$$

Using (A.2.9), equation (A.2.7) becomes:

$$2\ \frac{\partial \Gamma_0^1}{\partial X} = \int_0^{X_1} d\zeta\ \Delta_2\ \hat{C}(\zeta,Y,) \qquad (A.2.10)$$

The solution of (A.2.10) with (A.2.3), (A.2.8) is

$$\Gamma_0^1 = \frac{(X-X_1)}{2} \int_0^{X_1} d\zeta\ \Delta_2\hat{C}(\zeta,Y,Z) + \frac{1}{2} \int_0^{X_1} d\eta \int_0^{\eta} d\zeta\ \Delta_2\ \hat{C}(\zeta,Y,Z)$$

$$+ \frac{1}{2}\ [\hat{C}(X,Y,Z) - \hat{C}(0,Y,Z,)] \qquad (A.2.11)$$

From (A.2.5), (A.2.9), (A.2.11) we have $E\ [|Q^2|\,] = 1$ to first order. We define $|Q^2| = 1 + \Lambda(X,Y,Z)$ so that

$$\Lambda(X,Y,Z) = 2\sigma\Gamma_0^1 + O(\sigma^2,1/K) \qquad (A.2.12)$$

and (A.1.18) follows.

Now we use (A.2.11) in (A.2.12) and neglect the $O(\sigma^2,1/K)$ terms and the $\hat{C}$ terms in comparison with $\int\Delta_2\hat{C}$ for:

$$\Lambda(X,Y,Z) = \sigma\ [(X-X_1) \int_0^{X_1} d\zeta + \int_0^{X_1} d\eta \int_0^{\eta} d\zeta]\ \Delta_2\ \hat{C}(\zeta,Y,Z) \qquad (A.2.13)$$

Now the procedure we follow is identical to what we did in section A.1. We assume the definitions of R and R_2 given in (A.1.20), (A.1.21) to find (from (A.2.13)):

$$R_2(X,Y,Z) = \sigma^2 \Delta_2^2 \Big[(X-X_1)^2 \int_0^{X_1} d\zeta \int_0^{X} d\zeta' + 2(X_1-X) \int_0^{X_1} d\zeta \int_0^{X_1} d\eta' \int_0^{\eta'} d\zeta' + \int_0^{X_1} d\eta \int_0^{\eta} d\zeta \int_0^{X_1} d\eta' \int_0^{\eta'} d\zeta' \Big] R(\zeta-\zeta',Y,Z) \qquad (A.2.14)$$

We assume $X, X_1 >> 1$ and approximate the integrals in (A.2.14) the same way we approximated (A.1.22). The approximations used in going from (A.1.23) to (A.1.27) can be summarized by

$$\int_0^P d\zeta \int_0^Q d\zeta' \, R(\zeta-\zeta',Y,Z) = 2 \min(P,Q) \int_0^\infty d\mu \, R(\mu,Y,Z) \qquad (A.2.15)$$

when P, Q are "large." Using (A.2.15) in (A.2.14) we obtain our approximation to R_2:

$$R_2(X,Y,Z) = \sigma^2\Delta_2^2 \Big[2(X-X_1)^2 \min(X_1,X_1) + 4(X-X_1)\int_0^{X_1} d\eta' \min(X_1,\eta') + 2 \int_0^{X_1} d\eta \int_0^{X_1} d\eta' \min(\eta,\eta') \Big]$$

$$\int_0^\infty d\mu \, R(\mu,Y,Z) = \sigma^2 \Delta_2^2 \Big[2 (X-X_1)^2 X_1 + 2 (X-X_1) X_1^2 + \frac{2}{3} X_1^3 \Big] \int_0^\infty d\mu \, R(\mu,Y,Z) \qquad (A.2.16)$$

Now we scale X, X_1 to be on our long distance scale. We define:

$$t_1 = \sigma^{2/3} X_1 \qquad (A.2.17)$$

$$t_2 = \sigma^{2/3}(X-X_1) = \sigma^{2/3}\, X - t_1$$

From (A.2.17) the rays travel a scaled distance t_1 in the first medium and then a scaled distance t_2 in the second medium. Using (A.2.17) in (A.2.16):

$$R_2(t_1,t_2,Y,Z) = \frac{2}{3}\,(t_1^3 + 3\,t_1^2 t_2 + 3\,t_1 t_2^2)\,\Delta_2^2 \int_0^\infty d\mu\; R(\mu,Y,Z) \qquad (A.2.18)$$

In three dimensions (A.2.18) becomes

$$R_2(t_1,t_2 Y,Z) = -\frac{2}{3}\,(t_1^3 + 3\,t_1^2 t_2 + 3\,t_1 t_2^2)\,\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial X^2}\right)^2 G(Y,Z) \qquad (A.2.19)$$

In two dimensions (A.2.18) becomes

$$R_2(t_1,t_2,Y) = -\frac{2}{3}\,(t_1^3 + 3\,t_1^2 t_2 + 3\,t_1 t_2^2)\; g''(Y) \qquad (A.2.20)$$

Equation (A.2.20) (with (A.1.18)) is the same answer we obtained in section 5.5 (see 5.5.7).

A.3 Restriction on the Regular Perturbation Method

As we have seen in section 7.3, the short time approximation is not valid for $\tau > \tau^*$. This enables us to restrict the regime for σ for which the regular perturbation method is valid.

To have $\tau \leq \tau^*$ requires (see (7.1.1.a), (1.4.8)):

$$(\gamma_2\sigma)^{2/3}s = \gamma_2^{2/3}\, t = \tau \leq \tau^* = .62 \qquad (A.3.1)$$

Now the regular perturbation solution (presented in section A.1) requires a ray to go a long distance for (A.2.15) to be valid. Suppose we have a random velocity field $\hat{C}$ whose correlation function has a standard deviation of $\sum$. For a "central limit theorem" to apply, the distance travelled, s, must be much larger than $\sum$.

If we suppose that 5 correlation lengths is sufficient, then we need:

$$\left(\frac{s}{\Sigma}\right) \geq 5 \qquad (A.3.2)$$

We can combine (A.3.1) and (A.3.2.) for:

$$5\Sigma \leq s \leq \frac{.62}{(\gamma_2\sigma)^{2/3}} \qquad (A.3.3)$$

or

$$\sigma \leq \frac{.04}{\gamma_2\Sigma^{3/2}} \qquad (A.3.4)$$

We conclude that the regular perturbation solution will not be valid unless σ satisfies (A.3.4). For the typical numbers (see (7.1.7), (7.1.8)): $\sum=1$, $\gamma_2=2.7$, we require $\sigma \leq 0.1$.

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