THE R-F THERMAL RADIATION FROM THE SUN

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1952

ACKNOWLEDGMENT

The writer wishes to express his gratitude to Prof. P. S. Epstein under whose guidance and assistance this work was conducted. He is additionally indebted to Prof. J. L. Greenstein for his generous advice and assistance in its preparation.

Further appreciation is expressed to all those members of the staff with whom association has been a privilege and an inspiration in this and other endeavors.

ABSTRACT

The thermal component of the radio frequency radiation from the sun is derived from the laws of classical physics.

With the "velocity distribution" method of the kinetic theory of gas the mean number of collisions per second between the particles is found. From this the absorption coefficient for the radiation is obtained. For the intensity of the emitted radiation the equation of transfer is solved in a three dimensional medium. In the solution the emissivity is eliminated by means of a modified form of Kirchhoff's law of radiation where the index of refraction is generalized to include absorption as well. For the path of the rays the equation of the iconal of geometrical optics is considered in a refracting and absorbing medium. The solution does not exhibit the phenomenon of total reflection, present in purely refracting media.

Numerical calculations give the distribution of the radiation across the solar disk from 30 mc through 3000 mc. At 3000 mc the sun is of nearly uniform brightness with a sharply defined limb. At the lower frequencies a small central portion becomes quite bright and the limb less distinct and darker.

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I INTRODUCTION

1.1 Before the new field of radio astronomy could come into being, it was necessary that high gain, low noise receivers be developed. Ever since the days of Marconi, radio engineers have striven to improve the overall performance of their circuits and antennas. By the first years of the nineteen thirties, the art had reached such a point that weak extra-terrestrial radio frequency signals could be picked up. However, the great impetus to the development of high frequency, high gain systems for radar sets occurred during the last war. This was necessary before the field could make noticeable advances.

Before considering the observations of the early workers it would be well to mention what advances must be made before such observations could be undertaken with success. Consider that the general character of the emitted extra-terrestrial radio frequency radiation is the same as that of ordinary thermal agitation and tube noise. Thus to detect such signals, which are always weak, the development of high gain, low noise receivers is essential. This requirement must be coupled with that for highly directive antennas. Such antennas will have higher gains and hence be able to pick up weaker sources than broad beam ones.

It is necessary to use high frequency receivers not only to take advantage of highly directive antennas, but also to be able to receive the radiation at all. For just as the ionosphere is an aid to long distance radio transmission, it is equally a hindrance to the reception of extra-terrestrial radiation. Low frequency waves incident upon the upper layers of the ionosphere from above will be reflected and absorbed in much the same way as similar frequency waves are reflected and absorbed from below. Hence, it is possible to receive only those waves whose frequencies are so great that they will not be reflected by the ionosphere. In fact, in order to obtain reliable information as to the absolute intensity of the radiation, the frequency must be so high that the effects of refraction and absorption in the ionosphere may be considered to be negligible.

1.2 By 1932 the development of radio had reached such a point that K. G. Jansky (1) was able to discover the existence of galactic radiation at a wavelength of 14.6 meters (about 20 mc). As a result of continued observations, he (2) was able to locate that the most intense source of radiation came from Sagittarius at right ascension 18 hours and declination -30 degrees. This coincides with the direction of the center of the galaxy. Later work, principally by Reber (3), confirmed the general conclusions reached by Jansky.

No further significant progress was made until the postwar period. It should be mentioned, however, that in February 1942, the signals produced by the sun reached such an intensity that they were recorded by accident on numerous British army and air force receivers (4). Working in the 4 to 13 meter band (23 to 75 mc), they observed a high intensity disturbance in the direction of the sun. It had the characteristics of thermal or tube noise. Later correlation showed that there was intense solar activity (5) in the form of sunspots and flares during the same period. This correlation between optical and radio observations was later found to be generally true.

1.3 Upon the conclusion of World War II intensive study of both the sun and the galaxy has been undertaken by many workers in many lands. From this work considerable knowledge of the character of the radiation has been deduced over the entire frequency band. In particular the study of the sun soon showed that the rf intensity is highly variable with time, particularly at the lower frequencies.

However, at no time do the data indicate that the radiation drops below some constant minimum value. In the microwave region this quiescent component is the predominant part of the entire radiation received from the sun, while in the lower frequency region the variable components all but conceal the quiescent level. The results of numerous measurements of the quiet radiation have been summarized in Table I. For each observation the received intensity has been expressed in terms of the temperature that a black body would have if it were the size of the visual solar disk. From the table it is seen that the temperature varies from about 7000 degrees Kelvin at the shortest wavelengths to one million degrees Kelvin at the longest wavelengths. As indicated above, the accuracy of measurement in the long wavelength region is particularly poor. An indication of this may be seen from the scatter of the observed values.

Superimposed upon the above tabulated quiescent radiation are numerous time varying components. Payne-Scott (37) has classified them into bursts, outbursts, and enhanced radiation. The first kind lasts for only a few seconds, is unpolarized, and usually occurs on neighboring frequencies simultaneously. The outbursts are similar to the bursts but of longer duration, a matter of minutes, rather than seconds, are much more intense, and usually decay from their peak amplitudes exponentially. Finally there is the enchanced radiation in which the general background level changes over a period of hours or days. The last is circularly polarized and closely correlates with general solar activity.

TABLE I

INTENSITY OF THERMAL RADIATION

Frequency	Observer	Temperature
35,300 mc.	Hagen (6)	6,740 °K
24,000 "	Southworth (7)	2,000 "
	Piddington, Minnett (8)	10,000 "
	Dicke, Berringer (9)	10,000 "
9,500 "	Mayer (10)	12,000 "
9,440 "	Minnett, Labrum (11)	19,300 "
9,380 "	Southworth (12)	16,000 "
	Sander (13)	22,000 "
3,000 "	McCready, Pawsey, Payne-Scott (14)	25,000 "
	Southworth (15)	20,000 "
	Piddington, Hindman (16)	54,000 "
2,800 "	Covington (17)	56,000 "
	Covington (18)	79,000 "
1,200 "	Pawsey, Payne-Scott, McCready (19)	160,000 "
	Lehany, Yabsley (20)	100,000 "
600 "	Pawsey, Payne-Scott, McCready (21)	500,000 "
	Lehany, Yabsley (22)	500,000 "
480 "	Reber (23)	ca 1,000,000 "
	Reber (24)	ca 1,000,000 "
200 "	Pawsey, Payne-Scott McCready (25)	2,000,000 "

TABLE I (cont.)

Frequency	Observer	Temperature
200 mc.	Pawsey (26)	600,000 °K
	McCready, Pawsey, Payne-Scott (27)	500,000 "
	Payne-Scott, Yabsley, Bolton (28)	1,200,000 "
	Lehany, Yabsley (29)	800,000 "
	Allen (30)	120,000 "
	Pawsey, Yabsley (31)	700,000 "
175 "	Ryle, Vonberg (32)	2,000,000 "
	Ryle, Vonberg (33)	500,000 "
160 "	Reber (34)	1,800,000 "
	Blum, Denisse (35)	830,000 "
80 "	Ryle, Vonberg (36)	1,300,000 "

1.4 Of the above four components that make up the radio frequency radiation from the sun, the only one that does not defy explanation at present is the quiescent radiation. It may be explained by recalling that the sun is a hot body. Hence energy is emitted at all frequencies including the radio frequency region by the laws of thermodynamics. If just the solar disk is considered as the emitting source the values obtained do not correspond at all well with the observed values, particularly at the longer wavelengths. This presupposes that the solar corona and chromosphere are transparent to radio waves. Such, however, is not the case. By comparing the state of the gases there with those in our ionosphere, it is clear that as far as absorption is concerned the effect upon the passage of radio waves should be similar. In both cases the gases are higly ionized and so they impede the passage of the waves. It turns out that source for solar radio waves must be in the corona and chromosphere themselves. The million degree temperature of the coronal gases (38) readily permits the required explanation of the observed radiation. Higher frequency components originate in the somewhat cooler chromosphere. Observation and theory now tend to fall into line.

1.5 The first detailed explanation of the actual mechanism for the emission of radiation by the corona and chromosphere was advanced by Martyn (39) in

1946. A subsequent paper (40) clarified the details of his theory. It was based upon applying the theory of the earth's ionosphere to the solar corona and chromosphere in a direct manner. Thus any differences that may exist between radiation passing through a hot ionized gas and a cold one were neglected. The index of refraction was assumed to be independent of all absorptive processes. Thus the trajectories were computed on the basis of reciprocity between pencils of radiation entering and leaving the ionized corona. The intensity of the emergent radiation was obtained from Kirchhoff's law at the lowest point of the re-entrant rays.

Following a similar line of reasoning in connection with the index of refraction, Smerd and Westfold (42) solved the same problem. This time however the intensity of radiation was derived from the equation of transfer, but the path followed by a ray was still determined by neglecting all effects due to absorptive processes. In such a treatment of the index of refraction, there is always a totally reflecting layer for the frequency of radiation considered. For at that level the plasma frequency equals the frequency of the radiation under consideration.

1.6 In the following pages the theory for the thermal emission of radio frequency radiation will be developed on a somewhat different foundation. First the amount of radiation passing through each cubic centimeter of material will be determined by balancing the effects of emission and absorption with the equation of transfer. As the emission can be expressed in terms of inverse absorption, it will suffice to find the latter alone. Being in the long wavelength region of the spectrum, it will be safe to use the classical theory for absorption. This will be based upon the kinetic theory of gases and Maxwell's electromagnetic equations. A comparison of the results obtained bears this assumption out.

To complete the work it is necessary to compare theory with experiment. This requires a knowledge of the physical structure of the corona and chromosphere in terms of the electron density and temperature as a function of the solar radius. From applying this knowledge, the theoretical values for the thermal radiation are found to agree very well with the experimental ones.

II. HEAT RADIATION AND THE EQUATION OF TRANSFER

2.1 The amount of radiant energy that would be received by a radio antenna at some point on the earth's surface, when directed toward the sun, will be the integrated effect of the radiation emitted from each element of the surface of the sun. By the word sun it is meant that solar sphere which contributes to the radiation in question. In general its size is larger than the visual solar disk and is a function of wavelength. For the frequencies under discussion here, it includes parts of the corona and chromosphere. To compute the total radiation from the sun, consider that each element of area of the solar surface will emit $i_{fo}df$ ergs of energy per unit area per second into unit solid angle between the frequencies f and f+df in the direction of the earth. If α_o is the angle of emergence of the pencil of radiation with respect to the surface element, then

$$P_f df = df \int i_{fo} \cos\alpha_o \, dA \qquad (1)$$

is the total amount of observed radiation. From Figure I it can be seen that the element of area dA can be expressed in terms of the radius R_o and the colatitude angle α_o so that

$$P_f df = 2\pi R_o^2 df \int_0^{\pi/2} i_{fo} \cos\alpha_o \sin\alpha_o \, d\alpha_o$$

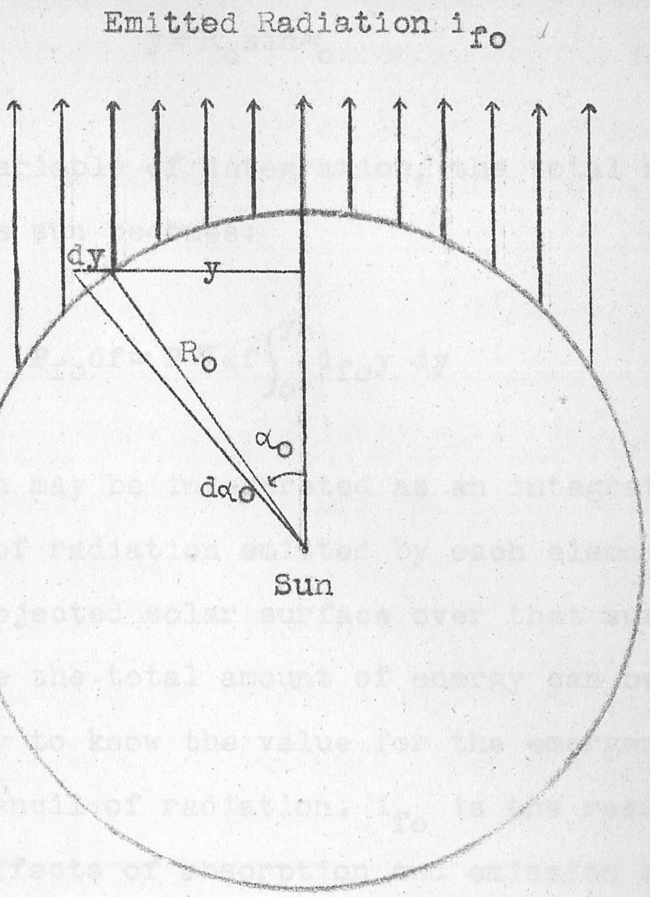


Figure I

It is convenient to change the variable of integration from the colatitude angle to the radius y of the projected solar sphere. From the figure

$$y = R_o \sin\alpha_o$$

Changing the variable of integration, the total radiation coming from the sun becomes:

$$P_{fo} df = 2\pi df \int_o^{y_o} i_{fo} y \, dy \qquad (2)$$

This expression may be interpreted as an integration of the intensity of radiation emitted by each element of area of the projected solar surface over that surface.

Before the total amount of energy can be computed, it is necessary to know the value for the emergent intensity i_{fo} for each pencil of radiation. i_{fo} is the resultant of the combined effects of absorption and emission along the optical path for that ray. On this path every element of volume will tend not only to add to the intensity by emission but also to reduce it by absorption. The net effect of these two processes, when integrated along the entire path, will be i_{fo}. In this chapter i_{fo} will be found in terms of the more specific properties of the corona and chromosphere.

2.2 The straightforward approach to the problem would be to compute the emission and absorption per unit volume and then integrate along the path. Unfortunately, however, this has been found to be impossible to do in terms of the temperature and other parameters of the body except for the hypothetical black body. But an equally satisfactory solution, for the purposes of this paper, would be to relate the emission and absorption of the physical corona and chromosphere to that of the black body. This method of attack has been successful.

2.3 Before proceeding further in relating the physical corona and chromosphere to the black body, it would be well to quote some of the results from the theory. From the work of Planck (43) the specific intensity i_f of radiation emitted by a black body of unit volume in thermodynamic equilibrium between the frequencies f and f+df into unit solid angle per unit time in a specified direction is

$$i_f df = B_f(T) df = \left\{ (2hf^3/c^2) \left[e^{hf/kT} - 1 \right]^{-1} \right\} df \qquad (3)$$

The symbol i_f in the above equation will be used henceforth to represent the specific intensity of radiation from any body, while the symbol $B_f(T)$ will be reserved for a black body. All the other letters have their usual meaning.

For the radio frequency part of the spectrum, the first term of the series expansion, that is the Rayleigh-

Jeans' law (44), (45), is valid.

$$B_f(T)df = (2kTf^2/c^2)df \tag{4}$$

The validity of this approximate form is particularly accurate considering the high temperatures encountered in the corona and chromosphere.

2.4 A convenient starting point in attacking the problem is Kirchhoff's law. This law states that the ratio of the emissivity e_f to the coefficient of absorption K_f in a body in local thermodynamic equilibrium is a constant, namely

$$e_f/K_f = \text{const.}$$

If the radiation from the body were to pass into a refracting but non-absorbing medium, then the law generalizes to

$$e_f/(K_f\mu^2) = \text{const.} \tag{5}$$

where μ is the coefficient of refraction. By considering radiative equilibrium between a general body and a black one, the constant of Equation 5 is found to be equal to the intensity of emission from a black body, namely

$$e_f/(K_f\mu^2) = B_f(T)$$

Likewise from the equilibrium conditions between two black bodies having a purely refracting medium between them,

$$B_f(T) = i_f/\mu^2 \qquad (6)$$

where i_f is the intensity of the radiation in the refracting medium.

These laws cannot rigorously be applied to the sun for the locations of the refracting and the emitting elements are indistinguishable. Kirchhoff's law is, however, a first approximation to the correct result in those regions where the absorption is small. This is what Woolley (47) used in deriving the equation of transfer for the radio frequency radiation emitted by the sun. For a more accurate theory the equation should be generalized to include the effects of absorption.

In 1910 Laue (48) attempted to broaden the coverage of Kirchhoff's law to include the effects of absorption. Although his results are not generally accepted, they are best extant. To derive the result, he had to assume that the radiation in an isotropic absorbing medium is both homogeneous and isotropic. However, it is generally believed that the radiation passing through such bodies is aeolotropic. Using Laue's hypothesis for the case of thermodynamic equilibrium, Equation 5 is replaced by

$$i_f/(\mu^2+k^2)=B_f(T) \tag{7}$$

where k is the imaginary part of the complex index of refraction,

$$n=\mu+ik \tag{8}$$

From the form of the above two expressions, it is plausible to assume (49) that the general formula is

$$B_f(T)=e_f/(K_f M_f^2) \tag{9}$$

where M_f is some function of the frequency of the radiation, the temperature and the physical properties of the material. For non-absorbing media it will reduce to μ. As the theory is developed, the conditions of the problem are such that it is eliminated from the final result. So its analytical form is irrelevant and need not be determined. To find the intensity within the medium, Equation 9 reduces to

$$B_f(T)=i_f/M_f^2 \tag{10}$$

2.5 With the above law replacing Kirchhoff's, consider how the intensity of radiation i_f will vary along the path of a ray. This path may be thought of as passing through a three dimensional medium similar to that treated by D. Hilbert (50). Such a consideration of the medium is necessary because of the large solar volume that contributes significantly to the radio frequency radiation. At optical wavelengths only a relatively shallow layer is involved in producing radiation, and hence a one-dimensional theory will be sufficiently accurate. With these facts in mind, consider a pencil of radiation of cross-sectional area ΔS and of intensity $i_f df$ entering a volume element of length Δl (see Figure II). The radiation leaves the element of volume with an intensity $i_f' df$ through an area $\Delta S'$.

While passing through the volume element, emission of the material will add

$$e_f \tfrac{1}{2}(\Delta S + \Delta S')\Delta l\, df\, dt \qquad (11)$$

ergs of radiation in time dt between the frequencies f and f+df. To a first approximation the volume of the element is taken as

$$\tfrac{1}{2}(\Delta S + \Delta S')\Delta l$$

At the same time

$$K_f\, i_f \tfrac{1}{2}(\Delta S + \Delta S')\Delta l\, df\, dt \qquad (12)$$

ergs will be absorbed, and

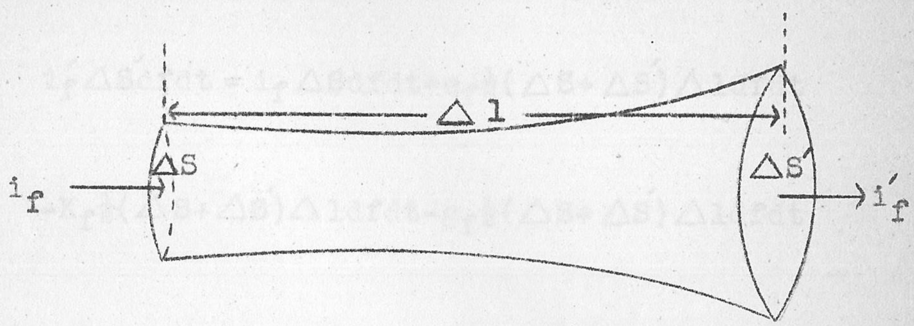


Figure II

$$g_f i_f \tfrac{1}{2}(\Delta S+\Delta S')\Delta l df dt \qquad (13)$$

ergs scattered and reflected, where g_f is coefficient of scattering and reflection. Collecting the various factors involved, the emergent energy will be

$$i_f' \Delta S' df dt = i_f \Delta S df dt + e_f \tfrac{1}{2}(\Delta S+\Delta S')\Delta l df dt$$

$$-K_f \tfrac{1}{2}(\Delta S+\Delta S')\Delta l df dt - g_f \tfrac{1}{2}(\Delta S+\Delta S')\Delta l df dt \qquad (14)$$

To a first approximation it will be sufficiently accurate to expand the primed quantities in a Taylor Series about the unprimed ones. For the intensity, the zeroth order terms cancel, leaving the first order equation

$$\frac{d}{dl}(i_f \Delta S)\Delta l df dt = e_f \Delta l \Delta S df dt - K_f i_f \Delta l \Delta S df dt$$

$$-g_f i_f \Delta l \Delta S df dt \qquad (15)$$

Cancel the common differentials and substitute Equation 9 for e_f:

$$d(i_f \Delta S)/dl = (K_f M_f^2 B_f(T) - K_f i_f - g_f i_f)\Delta S \qquad (16)$$

In order to express the coefficient of reflection in terms of the other quantities, apply the above equation of transfer to the case of equilibrium at a constant temperature. Then not only is Equation 10 valid, but also the cross-sectional areas of the pencils of radiation are no longer a function of the length of the volume element. The first two terms on the right are now equal leaving

$$di_f/dl = -g_f i_f \tag{17}$$

Taking the logarithmic derivative of Equation 10 to eliminate the intensity, the scattering coefficient becomes

$$g_f = -(1/M_f^2)(dM_f^2/dl) \tag{18}$$

This result is substituted into the equation of transfer (No. 16); where, after a re-arrangement of the terms,

$$d(i_f \Delta S/M_f^2)/dl = \left[K_f B_f(T) - K_f(i_f/M_f^2) \right] \Delta S \tag{19}$$

The above derivation of the equation of transfer has been made without reference to any particular set of coordinates. The actual path of the ray through the sun will be found in Chapter V to be a function of the solar radius and the pólar angle. So the element of path length dl in the above equation is a function of these two solar coordinates. But as long as a particular coordinate system is not explicitly required for the integration, it will be ignored.

2.6 The equation of transfer will be formally integrated in the usual manner. Introducing the optical depth

$$t = -\int_{l_o}^{l} K_f dl \qquad (20)$$

where the integration is along the pencil of radiation. Of the two limits l_o corresponds to the point of observation, in this case outside the sun, while l corresponds to some point within the medium. Thus the direction is opposite to that of the differential element of length dl in Equation 19. The choice is made so that t is essentially positive. To reduce Equation 20 to differential form, consider the optical depth from outside the sun of two neighboring points P and P′ separated by a distance Δl (see Figure III). If Equation 20 represents the case at P, the optical depth at P′ will be Δt smaller. So

$$t + \Delta t = -\int_{l_o}^{l+\Delta l} K_f dl \qquad (21)$$

Subtracting the two expressions

$$\Delta t = -\int_{l_o}^{l+\Delta l} K_f dl + \int_{l_o}^{l} K_f dl = -\int_{l}^{l+\Delta l} K_f dl \qquad (22)$$

In the limit as P′ approaches P,

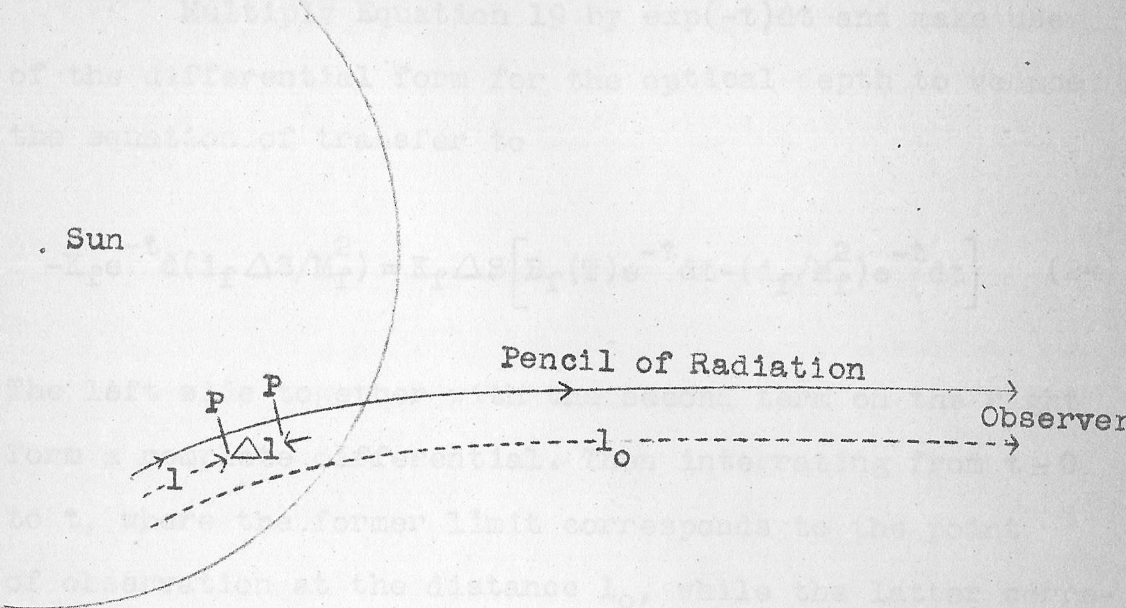


Figure III

$$d\tau = -K_f dl \qquad (23)$$

Thus a positive increment in the optical depth corresponds to a negative one in the distance l. (K_f is essentially positive.)

Multiply Equation 19 by exp($-\tau$)$d\tau$ and make use of the differential form for the optical depth to reduce the equation of transfer to

$$-K_f e^{-\tau} d(i_f \Delta S/M_f^2) = K_f \Delta S \left[B_f(T) e^{-\tau} dt - (i_f/M_f^2) e^{-\tau} dt \right] \qquad (24)$$

The left side together with the second term on the right form a complete differential. Then integrating from $\tau = 0$ to τ, where the former limit corresponds to the point of observation at the distance l_o, while the latter corresponds to l at point P, one finds

$$i_{fo} \Delta S_o = (i_f \Delta S e^{-\tau}/M_f^2)_l + \int_0^{\tau} B_f(T) \Delta S e^{-\tau} d\tau \qquad (25)$$

At points outside the sun the index of refraction and hence M_{f_o} is unity. This is the expression required by Equation 2 for computing the total intensity of radiation coming from the sun toward the earth.

2.7 The integrated equation must now be applied to the physical problem of the sun. For the problem at hand we are primarily interested in the radiation of an opaque medium with a temperature gradient. Two approaches for the solution appear at this point. In the first as one goes deeper into the sun, to smaller values of l, the optical depth τ increases without limit. The body becomes opaque as great depths are reached. The exponential factor of the first term becomes insignificant, so that

$$i_{fo} = (1/\Delta S_o)\int_0^{\infty} \Delta S B_f(T) e^{-\tau} d\tau \qquad (26)$$

The other approach to the physical problem rests upon the equilibrium conditions. The deeper one gets into the sun, the closer are the conditions to those of equilibrium. The exact level at which the equilibrium conditions are fulfilled depends upon the frequency of the radiation under consideration. Specifically the sun must become quite opaque for these conditions to hold true. It will be shown that the absorptive properties of the material are frequency sensitive. Thus at these lower layers, the validity of Equation 10 increases. So for great depths Equation 25 becomes

$$i_{fo}\Delta S_o = \Big[B_f(T)\Delta S e^{-\tau}\Big]_l + \int_0^{\tau} B_f(T)\Delta S e^{-\tau} d\tau \qquad (27)$$

The relative validity of the two equations will depend upon the physical conditions present although, in general, there should be little to choose between them.

It is interesting to note that in either case the unknown function M_f has disappeared. Its value would be important only if we wished to know the intensity of radiation at some point within the sun. For then it would be necessary to evaluate M_f at the point of interest.

Integrating Equation 27 by parts

$$i_{fo} = B_f(T_o) + (1/\Delta S_o)\int_{T_o}^{T} e^{-\tau}\left[dB_f(T)/dT\right] dT \qquad (28)$$

where T_o corresponds to the temperature at the point of observation outside the sun. It might at first be considered to be negligible; but upon realizing that the optical depth of the uppermost layers of the corona is essentially zero and the kinetic temperature there is very high, it is far from being unimportant.

2.8 To conform with the method used for presenting the experimental data (see Table I), it is found convenient to express the intensity of the observed radiation in terms of the temperature that a black body would have to have to produce an intensity of radiation equal to that observed. This temperature will be designated T_a. Since by Rayleigh-Jeans' law, Equation 4, T_a and i_f are linearly related, Equation 28 reduces to

$$T_a = T_s + (1/\Delta S_o)\int_{T_o}^{T} \Delta S e^{-\tau(T)} dT \qquad (29)$$

If the other approach to the physical problem had been followed through in like manner, Equation 26 would reduce to

$$T_a = (1/\Delta S_o)\int_o^{\infty} T(t)\Delta S e^{-\tau} d\tau \qquad (30)$$

The results obtained for the two cases will, in general, agree.

III. THE CONDUCTION CURRENT

3.1 In Chapter II the intensity of the thermal component of the radiation emitted by the sun was found to depend upon only two variables: the temperature and the optical depth. The former is a physical parameter determined by the equation of state of the matter, while the latter is some analytical function of the material and its properties. The first step in the determination of this function was given in Equation 2-23 by defining it in terms of the coefficient of absorption of the material. Before progress can be made in determining what form the coefficient of absorption will take, it will be well to describe the conditions that exist in both the corona and the chromosphere. Then, from a knowledge of the state of the matter, it will be possible to determine which forms of absorption will predominate. Knowing this the required coefficient may be worked out.

3.2 Both the corona and the chromosphere must have a very low density since they are normally completely transparent in the optical region of the spectrum. The corona is undoubtedly composed of almost pure hydrogen with but a slight admixture of helium (51). The heavier elements, which are the source of the coronal lines, make only a negligible contribution to the total. These gases will be completely ionized, by the million degree temperature that is believed to exist (52). Such temperatures are necessary to explain the coronal spectrum lines. To be specific it can be assumed that there are seven electrons and five hydrogen nuclei for each helium nucleus. As will be shown later on, the absorption is a function of the ratio of charge to mass. Thus the electrons will make the major contribution to the coefficient of absorption. The relative effects of the hydrogen and helium nuclei will be nearly the same, for a doubling of the charge is accompanied by a fourfold increase in mass when passing from hydrogen to helium. Hence it is assumed that there are equal numbers of electrons and hydrogen nuclei present, and no helium.

Between the corona and the photosphere lies the chromosphere. Data on its composition are less well known than for the corona. Usually its chemical composition is assumed to be similar to that of the corona, particularly at the greater altitudes. Regarding the temperature, it is definitely established that there is a strong positive temperature gradient between the photosphere and the corona. Redman's value of 35,000 degrees Kelvin (53) in the lower chromosphere is quite well established. From a consideration of the equilibrium conditions that must be maintained in the gas, it has been possible to predict values for both the electron density and the temperature as a function of altitude (54).

3.3 As the models adopted for the corona and chromosphere have the same general physical characteristics, it is safe to predict that the coefficient of absorption will be of the same form throughout. The chromosphere is just more dense and cooler than the corona. Regarding the absorptive processes, they may be looked at either from a quantum mechanical point of view or from a classical one with use of the kinetic theory of gases. In the former method of attack the elemental process of absorption may be considered to be transitions between two states within the continuum (free-free transitions). The bound-free transitions will also contribute to the coefficient of absorption. Gaunt (55), Menzel and Pekeris (56), Sommerfeld (57), Elwert (58) and others have obtained the atomic absorption coefficient by this method. Then, assuming that the particles in the gases obey a Maxwellian velocity distribution, the coefficient of absorption is evaluated. The other method of approach is along the lines of classical theory. From the kinetic theory of gases the number of collisions per second between ions can be computed. This is equivalent to a conduction current in the gas. Maxwell's electromagnetic equations then permit one to compute how much a wave will be attenuated due to this induced current. This is the required coefficient of absorption needed in the ray theory of Chapter II.

The results obtained from the two theories agree very closely. For the wavelengths under consideration the correspondence principle indicates that the classical approach should be valid to a high degree of approximation.

For this reason and also for being a more straightforward approach, the classical method was chosen. In this chapter the theory for finding the conduction current will be presented leaving the electromagnetic theory to Chapter IV.

3.4 The number of collisions per second between the electrons and protons (hydrogen nuclei) in the gas may be obtained by either the "free path method" or the "velocity distribution method". The former approach, used by Smerd and Westfold (59), assumes that the velocity-distribution function is unaffected by radiation or by any other non-uniform steady state condition. Thus the number of collisions per second is deduced from a pure Maxwell-Boltzmann distribution function. The conduction current is then derived by solving the equation of motion for a free electron under the combined influence of an external oscillating electric field and a damping force due to collisions. In this equation the coordinates of position do not take into account the rapid kinetic motion of the electron. This is considerable with the million degree coronal temperature.

On the other hand the "velocity distribution method" used by Hagen (60), takes into account the external forces and the density gradients in deriving the velocity-distribution function. The first approximation to this function is the Maxwell-Boltzmann function. Higher order terms retain the Boltzmann factor but also include the effects of the forces. Thus the above mentioned objection to the "free path method" is corrected. However, Hagen did not consider that the frequency of the radiation field would be greater than the number of collisions per second, but rather found the number of collisions for a static electric field and then proceeded to use the above mentioned equation of motion.

In the following it is proposed to consider the entire problem from the point of view of the "velocity distribution method" of the kinetic theory of gases. For this purpose consider that the field of radiation passing through the gas of the corona and chromosphere can be expanded by a Fourier series. For any term of this expansion the electric field will be the real part of

$$\vec{E} = \vec{E}_o \exp(i\omega t) \tag{1}$$

This will exert a force

$$\vec{F}_1 = e_1\vec{E} = e_1\vec{E}_o \exp(i\omega t) \tag{2}$$

upon the first component of the gas. (In the following theory the subscript 1 will denote either component of the gas. When necessary to speak of both the electrons and the protons, the subscript 2 will be used for the other component.)

The molecules of this component will be in random motion, whose overall characteristics may be expressed by a velocity-distribution function f_1. (The word molecule is used in a broad sense to indicate the electrons and protons that make up the gas.) The function f_1 will, in general, depend upon the velocity $\vec{v}_1$ and the position $\vec{r}$ of each molecule at every instant of time t. Physically the function f_1 denotes the number of molecules within a volume dV, having velocities between $\vec{v}_1$ and $\vec{v}_1+d\vec{v}_1$ at the time t. After a short interval dt the velocity of each molecule will have increased to

$$\vec{v}_1+(e_1/m_1)\vec{E}\ dt,$$

the position vector to $\vec{r}+\vec{v}_1 dt$, and the time to t+dt. The change that has occurred in f_1 during this time must be equal to the net change in the number of molecules present in dV $d\vec{v}_1$ due to collisions. This change is represented by

$$(\partial_e f_1/\partial t)d\vec{v}_1 dV\ dt$$

Thus equating the two changes,

$$\left[f_1(\vec{v}_1+(e_1/m_1)\vec{E}\ dt,\ \vec{r}+\vec{v}_1 dt,\ t+dt) - f_1(\vec{v}_1,\ \vec{r},\ t)\right]d\vec{v}_1 dV$$

$$= (\partial_e f_1/\partial t)d\vec{v}_1 dV\ dt \qquad (3)$$

On dividing through by $d\vec{v}_1 dV\, dt$ and passing to the limit as dt tends to zero, Boltzmann's equation for f_1 is obtained (61)

$$(\partial f_1/\partial t)+\vec{v}_1 \cdot \nabla_r f_1+(e_1/m_1)\vec{E}\cdot\nabla_{v_1} f_1 = (\partial_e f_1/\partial t) \qquad (4)$$

In this equation the operator

$$\nabla_r = \vec{i}(\partial\ /\partial x)+\vec{j}(\partial\ /\partial y)+\vec{k}(\partial\ /\partial z) \qquad (5)$$

has been introduced. A similar operator is indicated by ∇_{v_1}.

The external radiation field and the gradients within the gas will produce a general mass motion $\vec{v}_o$ of the gas as a whole. As a result of this motion it is found to be more convenient to re-express Boltzmann's equation in terms of the peculiar velocity,

$$\vec{c}_1 = \vec{v}_1 - \vec{v}_o \qquad (6)$$

and the mass velocity rather than the velocity $\vec{v}_1$ in inertial space. The velocity of mass motion is defined (62) by

$$\vec{v}_o = (1/\rho)(\rho_1\vec{v}_1+\rho_2\vec{v}_2) \qquad (7)$$

where ρ_1, ρ_2 and ρ are the densities of the first, second, and total gases respectively; and $\vec{\bar{v}}_1$ and $\vec{\bar{v}}_2$ are the mean values of the velocities. Since $\vec{v}_0$ and hence $\vec{c}_1$ is a function of $\vec{r}$ and t, the variables $\vec{r}$ and t appear not only explicitly in any function of $\vec{c}_1$, $\vec{r}$, and t but also implicitly through its dependence on $\vec{c}_1$. Hence the following changes in the derivatives are necessary (63)

$$(\partial\ /\partial t) \rightarrow (\partial\ /\partial t)+(\partial \vec{c}_1/\partial t)\cdot\nabla_{c_1} = (\partial\ /\partial t)-(\partial \vec{v}_o/\partial t)\cdot\nabla_{c_1} \tag{8}$$

$$(\partial\ /\partial x) \rightarrow (\partial\ /\partial x)-(\nabla_{c_1})\cdot(\partial \vec{v}_o/\partial x) \tag{9}$$

$$\nabla_{v_1} \rightarrow \nabla_{c_1} \tag{10}$$

Substituting these into Boltzmann's equation, one finds

$$(Df_1/Dt)+\vec{c}_1\cdot\nabla_r f_1+\left[(e_1/m_1)\vec{\bar{E}}-(D\vec{v}_o/Dt)\right]\cdot\nabla_{c_1} f_1$$

$$+\nabla_{c1} f_1 \vec{c}_1 : \nabla_r \vec{v}_o = (\partial_e f_1/\partial t) \tag{11}$$

where the mobile operator of hydrodynamics,

$$(D\ /Dt) = (\partial\ /\partial t)+\vec{v}_o\cdot\nabla_r \tag{12}$$

has been introduced. The last term on the left of Equation 11 is the double product of two dyadics.

3.5 The right hand side of Boltzmann's equation must now be interpreted in terms of the change in the population in the volume element dV and in the velocity range $d\vec{v}_1$ due to collisions. The collisions can be separated into two groups, those with like and those with unlike molecules. Hence

$$(\partial_e f_1/\partial t) = (\partial_e f_1/\partial t)_1 + (\partial_e f_2/\partial t)_2 \tag{13}$$

where the first term is with like and the second with unlike particles. Each of these terms is a balance between those molecules entering and those leaving dV $d\vec{v}_1$. During a time dt there will be

$$f_1 f_2 k_{12} d\vec{k}\, d\vec{v}_1 d\vec{v}_2 dV\, dt \tag{14}$$

molecules lost to the set as a result of encounters with molecules of the second kind having velocities between $\vec{v}_2$ and $\vec{v}_2+d\vec{v}_2$ and for certain values of the impact parameters k_{12} and $d\vec{k}$. These define an encounter by the initial conditions and the angle of deflection. The exact values will be discussed later on. The total number of such molecules lost due to collisions of this kind is the integral of the above over all values of $\vec{v}_2$ and the impact variables; viz.

$$d\vec{v}_1 dV\ dt \iint f_1 f_2 k_{12} d\vec{k}\ d\vec{v}_2 \qquad (15)$$

If f_1' and f_2' denote the velocity-distribution functions after a collision, it can be shown that

$$d\vec{v}_1 dV\ dt \iint f_1' f_2' k_{12} d\vec{k}\ d\vec{v}_2 \qquad (16)$$

indicates the number of molecules of the first kind entering $d\vec{v}_1\ dV$ in time dt as a result of encounters with particles of the first kind. In deriving this expression it is shown that the Jacobian relating $d\vec{v}_1 d\vec{v}_2$ before an encounter to $d\vec{v}_1' d\vec{v}_2'$ after an encounter is unity. Thus the net gain in $d\vec{v}_1\ dV$ as the result of such collisions is

$$(\partial_e f_1/\partial t)_2 d\vec{v}_1 dV\ dt = d\vec{v}_1 dV\ dt \iint (f_1' f_2' - f_1 f_2) k_{12} d\vec{k}\ d\vec{v}_2 \qquad (17)$$

A corresponding net gain for collisions of particles of the first kind with one another may be derived. Thus the first term of Equation 13 is

$$(\partial_e f_1/\partial t)_1 = \iint (f' f_1' - f f_1) k_1 d\vec{k}\ d\vec{v} \qquad (18)$$

The subscript 1 has been left off the variable of integration in order to be able to distinguish the two functions from one another. For subsequent work the introduction of the following notation for the above integrals will be useful:

$$(\partial_e f_1/\partial t)_1 = J_1(f_1 f) \qquad (19)$$

$$(\partial_e f_1/\partial t)_2 = J_{12}(f_1 f_2) \qquad (20)$$

With this the derivation of the Boltzmann's equation in terms of the velocity-distribution function is complete.

3.6 Before proceeding with the solution to Boltzmann's equation (Nos. 4 or 11), it will be well to derive certain useful relationships between the parameters of the problem. These relationships are used in the evaluation of various functions introduced in the solution to the equation.

First let ϕ_1 be any general molecular property. It may be a function of $\vec{v}_1$, $\vec{r}$ and t; or $\vec{c}_1$, $\vec{r}$ and t in either vector or scalar form. The mean value of such a property is defined (64) by

$$\bar{\phi}_1 = (1/N_1) \int \phi_1 f_1 d\vec{v}_1 = (1/N_1) \int \phi_1 f_1 d\vec{c}_1 \qquad (21)$$

where N_1 is the number density of the first gas. Note that the mass velocity $\vec{v}_o$ of Equation 7, was derived for

ϕ_1 equal $\vec{v}_1$. For the solution to Boltzmann's equation particular interest is focused on those parameters which remain invariant as a result of encounters. Specifically they are the number density, the linear momentum and the kinetic energy of translation of the gas. The relations required for the use of these quantities can best be obtained by studying the variation in the mean value of ϕ_1 due to collisions rather than the mean value itself. For the three summational invariants this variation is necessarily zero.

Now, to find the variation in the mean value of ϕ_1 due to encounters, multiply Boltzmann's equation (No. 11) by $\phi_1 d\vec{c}_1$ and then integrate throughout the range of $\vec{c}_1$.

$$\int \phi_1 \Big\{(Df_1/Dt)+\vec{c}_1 \cdot \nabla_r f_1+\Big[(e_1/m_1)\vec{E}-(D\vec{v}_o/Dt)\Big]\cdot\nabla_{c_1} f_1 + \nabla_{c_1} f_1 \vec{c}_1 : \nabla_r \vec{v}_o\Big\}\, d\vec{c}_1 = N_1\,\Delta\bar{\phi}_1 \qquad (22)$$

where

$$N_1\Delta\bar{\phi}_1 = \int \phi_1(\partial_e f_1/\partial t)\,d\vec{c}_1 = \int \phi_1(\partial_e f_1/\partial t)\,d\vec{v}_1 \qquad (23)$$

by the equality of the ranges of integration over $\vec{c}_1$ and $\vec{v}_1$. The significance of $\Delta\bar{\phi}_1$ may be seen from the second equation. $(\partial_e f_1/\partial t)d\vec{v}_1$ measures the rate of change in the number of molecules having velocities between $\vec{v}_1$ and

$\vec{v}_1 + d\vec{v}_1$ due to collisions. Consequently the above integral represents the variation in $\bar{\Phi}_1$ due to encounters. By means of certain transformations (65), such as

$$\int \Phi (\partial f_1/\partial c_{1x}) d\vec{c}_1 = \iint \Phi_1 f_1 \Big|_{c_{1x}=-\infty}^{c_{1x}=\infty} dc_{1y} dc_{1z}$$

$$- \int f_1 (\partial \Phi_1/\partial c_{1x}) d\vec{c}_1 = -N_1 (\overline{\partial \Phi_1/\partial c_{1x}}) \qquad (24)$$

on the components of the vector $\nabla_{c1} f_1$, Equation 22 may be expressed in terms of various mean values of Φ_1 in combination with other functions. In performing the partial integration above, it should be noted that $\Phi_1 f_1$ must tend to zero for large positive and negative values of $\vec{c}_1$, a general requirement on all functions Φ_1. The relation resulting from such transformations is

$$N_1 \Delta \bar{\Phi}_1 = (DN_1 \bar{\Phi}_1/Dt) + N_1 \bar{\Phi}_1 \nabla_r \cdot \vec{v}_o + \nabla_r \cdot (N_1 \overline{\Phi_1 \vec{c}_1}) - N_1 \Big\{ (D \bar{\Phi}_1/Dt)$$

$$+ \overline{\vec{c}_1 \cdot \nabla_r \Phi_1} + \Big[(e_1/m_1)\vec{E} - (D\vec{v}_o/Dt) \Big] \cdot \overline{\nabla_{c1} \Phi_1} - \overline{\nabla_{c1} \Phi_1 \vec{c}_1} : \nabla_r \vec{v}_o \Big\} \quad (25)$$

This expression will now be applied to each of the three summational invariants in turn. For the number density Φ_1 is unity as is also its mean value. This may be seen readily from Equation 21. The number density is, of course, unaltered for each gas separately.

Hence from Equation 25 and a similar equation for the second component, one obtains

$$(DN_1/Dt)+N_1\nabla_r\cdot\vec{v}_o+\nabla_r\cdot(N_1\vec{\bar{c}}_1)=0 \tag{26}$$

$$(DN_2/Dt)+N_2\nabla_r\cdot\vec{v}_o+\nabla_r\cdot(N_2\vec{\bar{c}}_2)=0 \tag{27}$$

The second summational invariant is the momentum. In this case

$$\vec{\phi}_1 = m_1\vec{c}_1 \tag{28}$$

and so the equation of change (No. 25) reduces to

$$N_1\Delta m_1\vec{\bar{c}}_1 = \left[D(\rho_1\vec{\bar{c}}_1)/Dt\right]+\rho_1\vec{\bar{c}}_1(\nabla_r\cdot\vec{v}_o)+\nabla_r\cdot(\rho_1\overline{\vec{c}_1\vec{c}_1})$$

$$-\rho_1\left[(e_1/m_1)\vec{E}-(D\vec{v}_o/Dt)\right]+\rho_1\vec{\bar{c}}_1\cdot\nabla_r\vec{v}_o \tag{29}$$

in terms of the mass density ρ. Differing from the number density the mean momentum of each kind of particle is not conserved separately but rather only the total. This may be expressed by

$$N_1\Delta m_1\vec{\bar{c}}_1-N_2\Delta m_2\vec{\bar{c}}_2=0 \tag{30}$$

Upon adding Equation 29 to a similar equation for the second gas and taking into account that the momentum of the gas as a whole is also zero, one finds that

$$0 = \nabla_r \cdot \underline{p} - (N_1 e_1 + N_2 e_2)\vec{E} + \rho(D\vec{v}_o/Dt) \tag{31}$$

where $\underline{p}$ is the pressure tensor (66). The first term of its series expansion is the hydrostatic pressure $(N_1+N_2)kT$.

The third independent summational invariant that is necessary and sufficient to describe a gas having purely translational energy is the kinetic energy. In this case

$$\phi_1 = \tfrac{1}{2} m_1 c_1^2 \tag{32}$$

The mean values of ϕ_1 and ϕ_2 are both $(3/2)kT$ where T is the kinetic temperature. The equation of change (No. 25) for the first gas is

$$N_1 \Delta \overline{\tfrac{1}{2} m_1 c_1^2} = (3/2)k(DN_1T/Dt) + (3/2)N_1 kT\nabla_r \cdot \vec{v}_o + \nabla_r \cdot \vec{q}_1$$

$$-\rho_1 \vec{c}_1 \cdot \left[(e_1/m_1)\vec{E} - (D\vec{v}_o/Dt)\right] + \rho_1 \overline{\vec{c}_1 \vec{c}_1} : \nabla_r \vec{v}_o \tag{33}$$

where $\vec{q}_1$ is the thermal flux vector. This equals $N_1 \overline{E_1 \vec{c}_1}$ (67). Like the momentum, the energy is conserved only for the gas as a whole. So

$$N_1 \overline{\Delta \tfrac{1}{2} m_1 c_1^2} + N_2 \overline{\Delta \tfrac{1}{2} m_2 c_2^2} = 0 \tag{34}$$

Upon adding Equation 33 to a similar one for the second gas, one finds that

$$0 = (3/2)(N_1+N_2)k(DT/Dt)+\nabla_r \cdot \vec{q} - (3/2)kT\nabla_r \cdot (N_1\overline{\vec{c}_1}+N_2\overline{\vec{c}_2})$$

$$-(N_1 e_1 \overline{\vec{c}_1}+N_2 e_2 \overline{\vec{c}_2}) \cdot \vec{E} + \underline{p} : \nabla_r \vec{v}_o \tag{35}$$

3.7 Having derived certain relationships for the summational invariants, the solution to Boltzmann's equation (Nos. 4 or 11) proceeds according to the theory of successive approximations developed by Enskog, as given by Chapman and Cowling (68). The zeroth order solution is for the unperturbed state of the gas with no forces acting upon it, namely the Maxwell-Boltzmann velocity-distribution formula. Successive orders of approximation reflect the effects of the external forces and of the space and time variations of the summational invariants upon the velocity-distribution function f_1. To do this expand f_1 into the infinite series

$$f_1 = f_1^{(o)} + f_1^{(1)} + \cdots \tag{36}$$

The functions $f_1^{(i)}$ are not as yet completely specified.

As the solution develops, certain subsidiary conditions will be added so that successive approximations to f_1 will follow the plan outlined above. The first step is to substitute the above series into Boltzmann's equation. Both the integral and differential parts are then suitably arranged into two infinite series. By equating terms of the same order, the successive terms of Equation 36 are evaluated.

First consider the integral or right hand side. Using the notation of Equations 19 and 20, the substitution of the above series yields

$$(\partial_e f_1/\partial t) = J_1\left[(\sum_i f_1^{(i)})(\sum_j f^{(j)})\right] + J_{12}\left[(\sum_i f_1^{(i)})(\sum_j f_2^{(j)})\right] \quad (37)$$

$$= \sum_i \sum_j \left[J_1(f_1^{(i)} f^{(j)}) + J_{12}(f_1^{(i)} f_2^{(j)})\right] \quad (38)$$

To reduce this to a single series, let each term of the new series correspond to the mode of grouping the terms in the expression for the product of two infinite series (69). Then

$$(\partial_e f_1/\partial t) = \sum J_1^{(i)} \quad (39)$$

where $J_1^{(i)} = J_1(f^{(o)} f_1^{(i)}) + J_1(f^{(1)} f_1^{(i-1)}) + \ldots + J_1(f^{(i)} f_1^{(o)})$

$$+ J_{12}(f_1^{(o)} f_2^{(i)}) + \ldots + J_{12}(f_1^{(i)} f_2^{(o)}) \quad (40)$$

The method for subdividing the differential or left hand side is not as obvious (70). It is so chosen that the space and time derivatives of the i'th term are not required for the evaluation of $f_1^{(i)}$. There is one important exception to this. Namely, since the frequency of the radiation field is much greater than the number of collisions per second between the molecules, it is necessary to include that part of the time derivative of $f_1^{(i)}$ which depends upon the radiation field. In this respect the following theory differs from earlier theories of the radiation from the solar corona and chromosphere. The peculiar choice of terms used in the series expansion for the differential part of Boltzmann's equation is made such that the time derivatives of the summational invariants are known to the same degree as is the approximation to f_1 from which they are determined. To simplify the following, that part of the time derivative of $f_1^{(i)}$ which depends upon the radiation field will be neglected. It will be introduced at the appropriate points in the actual solution to the problem. To effect this method of solution, let the time derivative in Boltzmann's equation (No. 11) and so in the mobile operator of Equation 12 be formally expanded into the series

$$(\partial /\partial t) = (\partial_o /\partial t) + (\partial_1 /\partial t) + \dots \qquad (41)$$

The various terms of the series are defined in terms of the derivatives with respect to N_1, $\vec{v}_o$ and T rather than implicitly through $f_1^{(i)}$. They are found by expanding the time derivatives in the equations of change listed above for the summational invariants. In each case the mobile operator will be introduced in the first term of the series to take care of the space derivatives; viz.

$$(D_o/Dt) = (\partial_o/\partial t) + \vec{v}_o \cdot \nabla_r \tag{42}$$

For the time derivative of the number density Equation 26 is divided into the following terms:

$$D_o N_1/Dt = -N_1 \nabla_r \cdot \vec{v}_o \tag{43}$$

$$\partial_i N_1/\partial t = -\nabla_r \cdot N_1 \vec{\bar{c}}_1^{(i)} \qquad (i>0) \tag{44}$$

For the mass velocity the results come from Equation 31

$$D_o \vec{v}_o/Dt = -(1/\rho)\nabla_r p + \vec{E}(N_1 e_1 + N_2 e_2) \tag{45}$$

$$\partial_i \vec{v}_o/\partial t = -(1/\rho)\nabla_r \cdot \underline{p}^{(i)} \qquad (i>0) \tag{46}$$

where p is the first term in the series expansion of the pressure tensor $\underline{p}$. This first term is equal to the hydrostatic pressure $(N_1+N_2)kT$.

For the time derivative of the temperature recourse is made to Equation 35, so that

$$\frac{D_oT}{D\,t} = -\left(\frac{2p}{3k(N_1+N_2)}\right)\nabla_r\cdot\vec{v}_o \qquad (47)$$

$$\frac{\partial_i T}{\partial t} = \frac{2}{3k(N_1+N_2)}\Big[\frac{3}{2}\,kT\nabla_r\cdot(N_1\vec{\bar{c}}_1^{(i)}+N_2\vec{\bar{c}}_2^{(i)})+(N_1e_1\vec{\bar{c}}_1^{(i)}$$

$$+N_2e_2\vec{\bar{c}}_2^{(i)})\cdot\vec{E}-\nabla_r\cdot\vec{q}^{(i)}-\underline{p}^{(i)}:\nabla_r\vec{v}_o\Big] \qquad (i>o) \quad (48)$$

where $\vec{q}^{(i)}$ is the i'th term in the series expansion of the thermal flux vector $\vec{q}$. The first term of this expansion (i equals 0) is zero.

Now by substituting the formal expansion for the time derivative (Equation 41) as well as that for the velocity-distribution function (Equation 36) into Boltzmann's equation (No. 4), the left hand side can be expanded into a series of the following terms

$$D_1^{(i)} = \frac{\partial_{i-1}f_1^{(o)}}{\partial t}+\frac{\partial_{i-2}f_1^{(1)}}{\partial t}+\cdots+\frac{\partial_o f_1^{(i-1)}}{\partial t}+\vec{v}_1\cdot\nabla_r f_1^{(i-1)}$$

$$+(e_1/m_1)\vec{E}\cdot\nabla_{v1}f_1^{(i-1)} \qquad (49)$$

The summation of these terms is the differential part of Boltzmann's equation, viz.

$$\sum D_1^{(i)} = (\partial f_1/\partial t) + \vec{v}_1 \cdot \nabla_r f_1 + (e_1/m_1)\vec{E}\cdot\nabla_{v1} f_1 \tag{50}$$

The equation has thus been expanded into two infinite series (Equations 39 and 50) such that

$$\sum D_1^{(i)} = \sum J_1^{(i)} \tag{51}$$

for the case where that part of the time derivative of f_1 dealing with the radiation field is neglected. Its inclusion would add another term on the left. From the arbitrariness of the definition of the functions $f_1^{(i)}$ in the series for f_1, it is possible to equate the two series term by term. Thus the zeroth and first order approximations to the velocity distribution function are

$$D_1^{(o)} = J_1^{(o)} \tag{52}$$

$$D_1^{(1)} = J_1^{(1)} \tag{53}$$

3.8 The zeroth order solution to Boltzmann's equation will now be evaluated. It should be noted from the definition of $D_1^{(i)}$ (Equation 49) that

$$D_1^{(o)} = 0 \tag{54}$$

Hence the zeroth order solution is a solution to the equation

$$J_1^{(o)} = J_1(f_1^{(o)} f^{(o)}) + J_{12}(f_1^{(o)} f_2^{(o)}) = 0 \qquad (55)$$

from the definition of $J_1^{(o)}$ (Equation 40). With regard to a remark made above that frequency of the radiation field is high, a term of the form $[\partial f_1^{(o)}/\partial t]_1$ should also be included. (By the square brackets around a time derivative is meant that part of the time derivative of f_1 which depends upon the radiation field only and not other parameters.)

But since the field does not appear in either of the above integrals, it may be logically assumed to have no effect on the form of $f_1^{(o)}$. So, expanding the above expressions as complete integrals (see Equations 17 through 20),

$$\iint (f_1'^{(o)} f'^{(o)} - f_1^{(o)} f^{(o)}) k_1 d\vec{k}\, d\vec{v} + \iint (f_1'^{(o)} f_2'^{(o)} - f_1^{(o)} f_2^{(o)}) k_{12} d\vec{k}\, d\vec{v}_2 = 0 \qquad (56)$$

Multiply this through by $\ln f_1^{(o)} d\vec{v}_1$ and then integrate over all values of $\vec{v}_1$ to obtain

$$\iiint (f_1'^{(o)} f'^{(o)} \ln f_1^{(o)} - f_1^{(o)} f^{(o)} \ln f_1^{(o)}) k_1 d\vec{k}\, d\vec{v}_1 d\vec{v}$$

$$+\iiint (f_1'^{(o)} f_2'^{(o)} \ln f_1^{(o)} - f_1^{(o)} f_2^{(o)} \ln f_1^{(o)}) k_{12} d\vec{k}\, d\vec{v}_1 d\vec{v}_2 = 0 \quad (57)$$

In the theory developed here the translational energy is conserved during an encounter. Hence every encounter may be considered to be the inverse of every other encounter (71). Thus the result of integrating over all velocities and values of the encounter variables yields the same as a similar integration over the variables of an inverse collision. By a use of this principle it can be shown the above integrals may be replaced by

$$0 = \tfrac{1}{4}\iiint \ln(f_1^{(o)} f^{(o)}/f_1'^{(o)} f'^{(o)})\,(f'^{(o)} f_1'^{(o)}$$

$$-f^{(o)} f_1^{(o)}) k_1 d\vec{k}\, d\vec{v}\, d\vec{v}_1 + \tfrac{1}{2}\iiint \ln(f_1^{(o)}/f_1'^{(o)})\,(f_1'^{(o)} f_2'^{(o)}$$

$$-f_1^{(o)} f_2^{(o)}) k_{12} d\vec{k}\, d\vec{v}_1 d\vec{v}_2 \quad (58)$$

A similar equation is obtained for the second component of the gas. When the two equations are added, the result has in it three integrals of the form of the first term in the above; i.e. one in $f_1^{(o)}$ and $f^{(o)}$; another in $f_1^{(o)}$ and $f_2^{(o)}$; and the third in $f_2^{(o)}$ and $f^{(o)}$. In each of these terms the logarithmic factor is always of opposite sign to the algebraic one. Hence each integrand is either negative or zero. As the variables of integration are all essentially positive, the only possible solution for the above equation is for each integrand to vanish separately, or for

$$f'^{(o)} f_1'^{(o)} = f^{(o)} f_1^{(o)} \tag{59}$$

$$f_1'^{(o)} f_2'^{(o)} = f_1^{(o)} f_2^{(o)} \tag{60}$$

$$f'^{(o)} f_2'^{(o)} = f^{(o)} f_2^{(o)} \tag{61}$$

The logarithms of both Equations 59 and 61 show that $\ln f_1^{(o)}$ and $\ln f_2^{(o)}$ must be summational invariants as each is conserved during encounters. In Section 3.6 it was shown that there are only three summational invariants. Thus $\ln f_1^{(o)}$ and $\ln f_2^{(o)}$ must be linear combinations of them, namely

$$\ln f_1^{(o)} = \alpha_1' + \vec{\alpha}'' \cdot m_1 \vec{v}_1 + \alpha''' \tfrac{1}{2} m_1 v_1^2 \tag{62}$$

$$\ln f_2^{(o)} = \alpha_2' + \vec{\alpha}'' \cdot m_2 \vec{v}_2 + \alpha''' \tfrac{1}{2} m_2 v_2^2 \tag{63}$$

There should be six constants in the solution except that the momentum and the energy for a gas mixture are conserved only for the gas as a whole. For this reason $\vec{\alpha}''$ and α''' are the same in both equations.

The constants of the above solution are evaluated (72) in the same manner as though they represented the complete solution to f_1 and f_2; i.e. in terms of the number density, mean temperature, and mass motion of the gas as a whole. Then

$$f_1^{(o)} = N_1(m_1/2\pi kT)^{3/2}\exp(-m_1c_1^2/2kT) \qquad (64)$$

Enough flexibility is still left in the definitions of the higher order terms of the series expansion of f_1, so that the number density N_1, the temperature T and the mass velocity $\vec{v}_o$ of the gas can be associated with the above constants in the zeroth order approximation. By doing this it is nesessary that the integrals defining similar properties in higher order approximations be zero. These integrals are those for the mean values of the three summational invariants. So from Equation 21 with ϕ_1 equal to each of the summational invariants in turn,

$$\int f_1^{(i)}d\vec{v}_1 = 0 \qquad (i>0) \quad (65)$$

$$\int f_1^{(i)}m_1\vec{c}_1d\vec{v}_1 + \int f_2^{(i)}m_2\vec{c}_2d\vec{v}_2 = 0 \qquad (i>0) \quad (66)$$

$$\int f_1^{(i)} m_1 c_1^2 d\vec{v}_1 + \int f_2^{(i)} m_2 c_2^2 d\vec{v}_2 = 0 \qquad (i>0) \qquad (67)$$

It should be noted that the zeroth order term for f_1 is the same as the uniform steady-state solution of Maxwell and Boltzmann.

3.9 Having completed the zeroth order solution, attention will now be focused on finding the first order solution to f_1. As long as the radiation field can be considered to be static, the equation to be solved is Number 53. The series expansion in Section 3.7 was made under this assumption. However, such is not the case here. Rather, the effect of the frequency of the radiation field upon the velocity distribution function must be included. In this respect this theory differs from earlier ones. The first order approximation to the time derivative of $f_1^{(1)}$ is added to the right side of Equation 53 to give

$$D_1^{(1)} + \left[\partial f_1^{(1)}/\partial t\right]_1 = J_1^{(1)} \qquad (68)$$

The values for the two sides of the equation are obtained from Equations 40 and 49. With the time variation of the electric field (Equation 1) explicitly expressed, one obtains

$$(\partial_o f_1^{(o)}/\partial t)+\vec{v}_1\cdot\nabla_r f_1^{(o)}+(e_1/m_1)\vec{E}_o e^{i\omega t}\cdot\nabla_{v_1} f_1^{(o)}+\left[\partial f_1^{(1)}/\partial t\right]_1 =$$

$$J_1(f^{(o)}f_1^{(1)})+J_1(f^{(1)}f_1^{(o)})+J_{12}(f_1^{(o)}f_2^{(1)})+J_{12}(f_1^{(1)}f_2^{(o)}) \quad (69)$$

It is seen that the time dependence of the electric field is exponential, so it is assumed that that part of the time derivative of $f_1^{(1)}$ which depends upon the radiation field will vary in the same manner. Hence the first approximation to the time derivative of $f_1^{(1)}$ is assumed to be

$$\left[\partial f_1^{(1)}/\partial t\right]_1 = i\omega f_1^{(1)} \quad (70)$$

Before proceeding further with the solution, it would be well to change the variables of the function f_1 from $\vec{v}_1$, $\vec{r}$ and t to $\vec{c}_1$, $\vec{r}$ and t. Making use of the transformations in Equations 8, 9 and 10,

$$(D_o f_1^{(o)}/Dt)+\vec{c}_1\cdot\nabla_r f_1^{(o)}+\left[(e_1/m_1)\vec{E}_o\exp(i\omega t)\right.$$

$$\left.-(D_o\vec{v}_o/Dt)\right]\cdot\nabla_{c_1} f_1^{(o)}-\nabla_{c_1} f_1^{(o)}\vec{c}_1:\nabla_r\vec{v}_o =$$

$$-i\omega f_1^{(1)}+J_1(f^{(o)}f_1^{(1)})+J_1(f^{(1)}f_1^{(o)})$$

$$+J_{12}(f_1^{(o)}f_2^{(1)})+J_{12}(f_1^{(1)}f_2^{(o)}) \quad (71)$$

The next step is the evaluation of the various derivatives on the left side of the above in terms of the zeroth approximation. From Equation 64

$$\frac{D_o f_1^{(o)}}{Dt} = f_1^{(o)} \left[\frac{D_o \ln(N_1 T^{-3/2})}{Dt} + \left(\frac{m_1 c_1^{\,2}}{2kT} \right) \left(\frac{D_o \ln T}{Dt} \right) \right] \qquad (72)$$

The first term in this is zero (Equations 43 and 47). The second term is re-expressed by the definition of $D_o\vec{v}_o/Dt$ (Equation 45), so that

$$(D_o f_1^{(o)}/Dt) = -f_1^{(o)}(m_1 c_1^2/3kT)\nabla_r \cdot \vec{v}_o \qquad (73)$$

The other two derivatives of $f_1^{(o)}$ are

$$\nabla_r f_1^{(o)} = f_1^{(o)} \left[\nabla_r \ln(N_1 T^{-3/2}) + (m_1 c_1/2kT)\nabla_r \ln T \right] \qquad (74)$$

$$\nabla_{c_1} f_1^{(o)} = -f_1^{(o)}(m_1 \vec{c}_1/kT) \qquad (75)$$

The formal time derivative of the mass velocity $\vec{v}_o$ in the square bracket of Equation 71 may be found from Equation 45. With this value the square bracket reduces to

$$(e_1/m_1)\vec{E}_o \exp(i\,\omega t) - (D_o\vec{v}_o/Dt) = (1/\rho)\Big\{ \nabla_r p$$

$$+ \rho_2 \left[(e_1/m_1) - (e_2/m_2) \right] \vec{E}_o \exp(i\,\omega t) \Big\} \qquad (76)$$

Having evaluated all of the various derivatives, the left hand side of Equation 71 may be expressed by

$$f_1^{(o)}\left\{\left(\frac{m_1c_1^2}{2kT}-\frac{5}{2}\right)\vec{c}_1\cdot\nabla_r \ln T+\left(\frac{N_1+N_2}{N_1}\right)\vec{d}_{12}\cdot\vec{c}_1+\left(\frac{m_1}{kT}\right)\vec{c}_1\overset{o}{\,}\vec{c}_1:\nabla_r\vec{v}_o\right\} \quad (77)$$

after a considerable rearrangement of the terms. The quantity $\vec{d}_{12}$ is defined by

$$\vec{d}_{12}=\nabla_r(N_1/(N_1+N_2))+\left[N_1N_2(m_2-m_1)/(N_1+N_2)\right]\nabla_r \ln p$$

$$-(\rho_1\rho_2/p\rho)\left[(e_1/m_1)-(e_2/m_2)\right]\vec{E}_o e^{i\omega t} \quad (78)$$

and $\vec{c}_1\overset{o}{\,}\vec{c}_2$ is a second order tensor whose divergence is zero. Notice that the other component of the gas is such that

$$\vec{d}_{12}=-\vec{d}_{21} \quad (79)$$

Having put the left hand side of Equation 71 into a convenient form for computation, consider the form of the solution. The right hand side contains the unknown function $f_1^{(1)}$ linearly and the left hand side is expressed in terms of known functions. Hence if $F_1^{(1)}$ is a solution to the equation, any other solution is

$$F_1^{(1)}+X_1^{(1)} \quad (80)$$

where $X_1^{(1)}$ is a solution of the right hand side when equated to zero. So the most general solution (73) is the general solution $X_1^{(1)}$ for the right hand side plus a particular solution $F_1^{(1)}$. To find such a general solution, let

$$X_1^{(1)} = f_1^{(o)} \Psi_1 \tag{81}$$

Then the first two integrals of Equation 71 (defined by Equations 18 and 19) may be combined

$$\begin{aligned} J_1(f^{(o)} f_1^{(1)}) + J_1(f^{(1)} f_1^{(o)}) \\ &= \iint f^{(o)} f_1^{(o)} (\Psi_1 + \Psi - \Psi_1' - \Psi') k_1 \, d\vec{k} \, d\vec{v} \\ &= N_1^2 I_1(\Psi_1) \end{aligned} \tag{82}$$

The theorem in Section 3.8 regarding inverse encounters was used in deriving the first part of this expression. In a similar manner let the other two integrals

$$J_{12}(f_1^{(o)} f_2^{(1)}) + J_{12}(f_1^{(1)} f_2^{(o)}) = N_1 N_2 I_{12}(\Psi_1 + \Psi_2) \tag{83}$$

Thus the equations to be solved for the two components of the gas are

$$-i\omega f_1^{(o)}\Psi_1+N_1^2 I_1(\Psi_1)+N_1N_2I_{12}(\Psi_1+\Psi_2)=0 \qquad (84)$$

$$-i\omega f_2^{(o)}\Psi_2+N_2^2 I_2(\Psi_2)+N_1N_2I_{21}(\Psi_1+\Psi_2)=0 \qquad (85)$$

By a series of arguments similar to the ones used in obtaining the zeroth order solution in Section 3.8, it can be shown that Ψ_1 must be a summational invariant. So

$$\Psi_1 = \alpha_1' + \vec{\alpha}'' \cdot m_1\vec{c}_1 + \alpha''' \tfrac{1}{2}m_1 c_1^2 \qquad (86)$$

$$\Psi_2 = \alpha_2' + \vec{\alpha}'' \cdot m_2\vec{c}_2 + \alpha''' \tfrac{1}{2}m_2 c_2^2 \qquad (87)$$

where the constants are arbitrary functions of $\vec{r}$ and t.

At this point the general solution to the first order approximation to f_1 may be written down. As it was found convenient to obtain the solution for the integral part in terms of the zeroth order solution, the general solution will probably be of the same form

$$f_1^{(1)} = f_1^{(o)}\Psi_1 \qquad (88)$$

For this to be so, it must be a solution of

$$f_1^{(o)}\left\{\left(\frac{m_1 c_1^2}{2kT}-\frac{5}{2}\right)\vec{c}_1\cdot\nabla_r \ln T+\left(\frac{N_1+N_2}{N_1}\right)\vec{d}_{12}\cdot\vec{c}_1+\left(\frac{m_1}{kT}\right)\overset{o}{\vec{c}_1}\vec{c}_1:\nabla_r\vec{v}_o\right\}$$

$$= -i\omega f_1^{(o)}\Psi_1+N_1^2 I_1(\Psi_1)+N_1N_2I_{12}(\Psi_1+\Psi_2) \qquad (89)$$

(Equations 71, 77, 82 and 83). A similar equation applies to the other component of the gas. Since Ψ_1 is a linear scalar funtion of the right hand side, only scalar solutions need be considered. Also, all of the derivatives on the left side are of first order, so the general solution must be the sum of four parts: (1) a linear combination of the components of $\nabla_r T$, (2) of $\vec{d}_{12}$, (3) of $\nabla_r\vec{v}_o$ and (4) the general solution to the integrals. Such a solution (74) can be obtained only by multiplying the first two by vector functions, the third by a tensor, and the fourth by use of Equation 86. Then

$$\Psi_1 = \vec{A}_1\cdot\nabla_r \ln T+(N_1+N_2)\vec{D}_1\cdot\vec{d}_{12}+\underline{B}_1:\nabla_r\vec{v}_o$$

$$+\alpha_1'+\vec{\alpha}''\cdot m_1\vec{c}_1+\alpha'''\tfrac{1}{2}m_1 c_1^2 \qquad (90)$$

The unknown functions $\vec{A}_1, \vec{D}_1$ and $\underline{B}_1$ are functions of N_1, T, and $\vec{c}_1$, and are determined by substituting back into the integral equation (No. 89). The constants from the general solution of the integrals are chosen in accordance with the boundary conditions of Equations 65, 66 and 67. It is found that

$$\alpha_1' = \alpha_2' = \alpha''' = 0 \qquad (91)$$

and that $\vec{\alpha}''$ may be absorbed into the functions $\vec{A}_1$ and $\vec{A}_2$. Substitute the above solution back into the integral equation (No. 89) and equate the coefficients of $\nabla_r T$, $\vec{d}_{12}$ and $\nabla_r \vec{v}_o$ to obtain the series of equations:

$$f_1^{(o)}\left(\frac{m_1 c_1^2}{2kT} - \frac{5}{2}\right)\vec{c}_1 = -i\,\omega f_1^{(o)}\vec{A}_1 + N_1^2 I_1(\vec{A}_1) + N_1 N_2 I_{12}(\vec{A}_1 + \vec{A}_2) \qquad (92)$$

$$(1/N_1) f_1^{(o)}\vec{c}_1 = -i\,\omega f_1^{(o)}\vec{D}_1 + N_1^2 I_1(\vec{D}_1) + N_1 N_2 I_{12}(\vec{D}_1 + \vec{D}_2) \qquad (93)$$

$$(m_1/2kT) f_1^{(o)}\overset{o}{\vec{c}_1}\vec{c}_2 = -i\,\omega f_1^{(o)}\underline{B}_1 + N_1^2 I_1(\underline{B}_1) + N_1 N_2 I_{12}(\underline{B}_1 + \underline{B}_2) \qquad (94)$$

From these it is obvious that the unknown vector functions can be written in terms of scalar functions of the form

$$\vec{A}_1 = \vec{c}_1 A_1(c_1) \qquad (95)$$

$$\vec{D}_1 = \vec{c}_1 D_1(c_1) \qquad (96)$$

$$\underline{B}_1 = \overset{o}{\vec{c}_1}\vec{c}_1 B_1(c_1) \qquad (97)$$

Thus the second term in the series solution for the velocity-distribution function is

$$f_1^{(1)} = f_1^{(o)} \left\{ \left[A_1(c_1) \nabla_r \ln T + (N_1+N_2) D_1(c_1) \vec{d}_{12} \right] \cdot \vec{c}_1 - B_1(c_1) \overset{o}{\vec{c}_1} \vec{c}_1 : \nabla_r \vec{v}_o \right\} \tag{98}$$

In order that the conditions of solubility (Equations 65, 66 and 67) be fulfilled, the functions $\vec{A}_1$ and $\vec{D}_1$ must be so chosen that

$$\int f_1^{(o)} m_1 \vec{c}_1 \cdot \vec{A}_1 d\vec{v}_1 + \int f_2^{(o)} m_2 \vec{c}_2 \cdot \vec{A}_2 d\vec{v}_2 = 0 \tag{99}$$

and

$$\int f_1^{(o)} m_1 \vec{c}_1 \cdot \vec{D}_1 d\vec{v}_1 - \int f_2^{(o)} m_2 \vec{c}_2 \cdot \vec{D}_2 d\vec{v}_2 = 0 \tag{100}$$

3.10 By using the above solution, it will be possible to find a first approximation to the conduction current induced and hence the conductivity present in the ionized gases. From the value of the conductivity the coefficient of absorption is computed in the next chapter. It will develop that the conductivity is actually complex. The real part corresponds principally to the damping effect of the collisions between the molecules while the imaginary part is due to the oscillations induced in the gas by the electric field. The results obtained here differ from those of the other theories in that the conduction current is obtained directly from the kinetic theory of gases rather than from the equation of motion of an electron in the field of force due to the electric field. For this problem the conduction current in an ionized gas is defined by

$$\vec{j} = N_1 e_1 \overline{\vec{c}}_1 + N_2 e_2 \overline{\vec{c}}_2 \qquad (101)$$

The mass velocity $\vec{v}_o$ will produce no net currents as the gas is considered to be electrically neutral. From the conservation of momentum of the gas as a whole,

$$N_1 m_1 \overline{\vec{c}}_1 + N_2 m_2 \overline{\vec{c}}_2 = 0 \qquad (102)$$

Using this expression, the current may be written in the form

$$\vec{j} = (N_1 N_2/\rho)(e_1 m_2 - e_2 m_1)(\overline{\vec{c}}_1 - \overline{\vec{c}}_2) \qquad (103)$$

where ρ is again the mass density of the gas. So the problem left to be solved is to compute the difference between the mean velocities of the two components from the above first order solution to the Boltzmann's equation.

3.11 First, the mean value of the peculiar velocity needed in computing the current is obtained by using the equation for the mean value of any quantity (No. 21) with Φ_1 equal to $\vec{c}_1$. In evaluating the integral for a first order solution to Boltzmann's equation, the zeroth order term and the tensor term contribute nothing as they are odd functions of $\vec{c}_1$. So

$$N_1\vec{c}_1 = (1/3)\left\{\nabla_r \ln T \cdot \int f_1^{(o)} A_1(c_1) c_1^2 d\vec{v}_1 + (N_1+N_2)\vec{d}_{12} \cdot \int f_1^{(o)} D_1(c_1) c_1^2 d\vec{v}_1 \right\} \qquad (104)$$

The part of the conduction current induced by the radiation field is of paramount interest. Since the field is included in the definition of $\vec{d}_{12}$, the function $D_1(c_1)$ must be determined. To do this, assume that

$$\nabla_r \ln T = 0 \qquad (105)$$

Then from Equation 104 and a similar one for the other component, remembering that $\vec{d}_{12}$ equals minus $\vec{d}_{21}$ (Equation 79),

$$\vec{c}_1 - \vec{c}_2 = (1/3)(N_1+N_2)\vec{d}_{12} \cdot \left[(1/N_1)\int f_1^{(o)} D_1(c_1) c_1^2 d\vec{v}_1 - (1/N_2)\int f_2^{(o)} D_2(c_2) c_2^2 d\vec{v}_2\right] \qquad (106)$$

The functions $D_1(c_1)$ and $D_2(c_2)$ are assumed to be expansible into two infinite series of the form (75)

$$\vec{D}_1 = \vec{c}_1 D_1(c_1) = \sum_{o}^{\infty} d_i \vec{a}_1^{(i)} \qquad (107)$$

$$\vec{D}_2 = \vec{c}_2 D_2(c_2) = -\sum_{0}^{-\infty} d_i \vec{a}_2^{(i)} \tag{108}$$

where:
$$\vec{a}_1^{(o)} = (\rho_2/\rho)(m_1/\sqrt{2kT})\vec{c}_1 \tag{109}$$

$$\vec{a}_1^{(i)} = \sqrt{m_1/2kT}\, S_{3/2}^{(i)}(m_1 c_1^2/2kT)\, \vec{c}_1 \qquad (i>0) \tag{110}$$

$$\vec{a}_2^{(o)} = -(\rho_1/\rho)(m_2/\sqrt{2kT})\vec{c}_2 \tag{111}$$

$$\vec{a}_2^{(-i)} = \sqrt{m_2/2kT}\, S_{3/2}^{(i)}(m_2 c_2^2/\ 2kT)\, \vec{c}_2 \qquad (i>0) \tag{112}$$

and the $S_{3/2}^{(i)}$ (x) are Sonine polynomials.

The first coefficients of the two series can be shown to be equal from the condition of solubility (Equation 100) by the following: Upon making the substitution of the series, those terms for which i does not equal zero vanish, leaving only

$$\int f_1^{(o)} m_1 \vec{c}_1 \cdot \vec{a}_1^{(o)} d\vec{v}_1 + \int f_2^{(o)} m_2 \vec{c}_2 \cdot \vec{a}_2^{(o)} d\vec{v}_2 = 0 \tag{113}$$

This may readily be seen to be satisfied by substituting the values for $\vec{a}_1^{(o)}$ and $\vec{a}_2^{(o)}$ and comparing the result with that for the mean kinetic energy, viz:

$$(1/N_1)\int f_1^{(o)} m_1 c_1^2 d\vec{v}_1 = (1/N_2)\int f_2^{(o)} m_2 c_2^2 d\vec{v}_2 \tag{114}$$

In this manner the condition on the series coefficients for $\vec{D}_1$ is satisfied.

To evaluate the constant d_o for the zeroth order term, multiply Equation 93, which is the coefficient of $\vec{d}_{12}$ in the integral equation, by $\vec{a}_1^{(i)} d\vec{v}_1$ and integrate over all values of $\vec{v}_1$ to give

$$(1/N_1)\int f_1^{(o)}\vec{c}_1 . \vec{a}_1^{(i)} d\vec{v}_1 = -i\omega \int f_1^{(o)}\vec{D}_1 . \vec{a}_1^{(i)} d\vec{v}_1$$

$$+N_1^2\int I_1(\vec{D}_1) . \vec{a}_1^{(i)} d\vec{v}_1 + N_1N_2\int I_{12}(\vec{D}_1+\vec{D}_2) . \vec{a}_1^{(i)} d\vec{v}_1 \qquad (115)$$

Subtracting this from a similar equation for the second component of the gas,

$$(1/N_1)\int f_1^{(o)}\vec{c}_1 . \vec{a}_1^{(i)} d\vec{v}_1 - (1/N_2)\int f_2^{(o)}\vec{c}_2 . \vec{a}_2^{(i)} d\vec{v}_2 =$$

$$-i\omega\left[\int f_1^{(o)}\vec{D}_1 . \vec{a}_1^{(i)} d\vec{v}_1 - \int f_2^{(o)}\vec{D}_2 . \vec{a}_2^{(i)} d\vec{v}_2\right] + N_1N_2\left\{\vec{D}, \vec{a}^{(i)}\right\} \qquad (116)$$

where $\left\{\vec{D}, \vec{a}^{(i)}\right\} = N_1^2\int I_1(\vec{D}_1) . \vec{a}_1^{(i)} d\vec{v}_1 + N_1N_2\int I_{12}(\vec{D}_1+\vec{D}_2) . \vec{a}_1^{(i)} d\vec{v}_1$

$$-N_1N_2\int I_{21}(\vec{D}_1+\vec{D}_2) . \vec{a}_2^{(i)} d\vec{v}_2 - N_2^2\int I_2(\vec{D}_2) . \vec{a}_2^{(i)} d\vec{v}_2 \qquad (117)$$

The unknown functions $\vec{D}_1$ and $\vec{D}_2$ in the above equation are replaced by their series expansions from Equations 107 and 108. Then

$$(1/N_1)\int f_1^{(o)}\vec{c}_1\cdot\vec{a}_1^{(i)}d\vec{v}_1-(1/N_2)\int f_2^{(o)}\vec{c}_2\cdot\vec{a}_2^{(i)}d\vec{v}_2 =$$

$$-i\omega\Big[\sum_j d_j\int f_1^{(o)}\vec{a}_1^{(j)}\cdot\vec{a}_1^{(i)}d\vec{v}_1-\sum_j d_j\int f_2^{(o)}\vec{a}_2^{(j)}\cdot\vec{a}_2^{(i)}d\vec{v}_2\Big]$$

$$+N_1N_2\sum_{-\infty}^{+\infty} d_j\{\vec{a}^{(j)},\ \vec{a}^{(i)}\} \qquad (118)$$

The range of i and j in the above integrals is from minus to plus infinity. All terms that have not been defined in the series are zero. Actually all of the integrals equal zero unless i equals j. The first two are also zero unless i is zero. This is seen from the relations between the Sonine polynomials (76) that

$$\int_0^\infty e^{-x}\ S_m^{(p)}(x)\ S_m^{(q)}(x)\ x dx=\begin{cases}0 & (p\neq q)\\ \Gamma(m+p+1)/p! & (p=q)\end{cases} \qquad (119)$$

and that $S_m^{(o)}(x)$ equals one. The variable of integration $\vec{v}$ in spherical coordinates is $v^2\sin\theta\, d\theta\, d\phi\, dv$. In this manner the first constant d_o may be found. Substituting the values for $\vec{a}_1^{(o)}$ (Equation 109) and $f_1^{(o)}$ (Equation 64) and working out the integrals, Equation 118 reduces to

$$3\sqrt{\tfrac{1}{2}kT}\ (\rho_1+\rho_2)(1/\rho)=-3i\omega\, d_o(\rho_1\rho_2/2\rho^2)(\rho_1+\rho_2)$$

$$+N_1N_2d_o\{\vec{a}^{(o)},\ \vec{a}^{(o)}\} \qquad (120)$$

Thus, a first approximation to the function $\vec{D}_1$ and hence to the current will be obtained, once the value of the curved brackets is determined. Actually this approximation to the complete result is a pretty good one. The theory of the velocity of diffusion is based on practically the same theory as the above. Results (77) indicate that the coefficient of diffusion changes around 15% when carried to higher orders. The accuracy of the results obtained from this theory are probably not that good when all the other approximations are taken into account.

Using the principle of momentum, it can be shown (78) that the first and last integrals in the definition of $\{\vec{a}^{(o)}, \vec{a}^{(o)}\}$ (Equation 117) are zero. The other two integrals may be combined by the principle of inverse encounters, enunciated in Section 3.8, to reduce the curved brackets to

$$\{\vec{a}^{(o)}, \vec{a}^{(o)}\} = (m_1 m_2/4kTN_1N_2) \iiint f_1^{(o)} f_2^{(o)} (\vec{c}_1 - \vec{c}_1') \cdot (\vec{c}_2 - \vec{c}_2') k_{12} d\vec{k}\, d\vec{v}_1 d\vec{v}_2 \tag{121}$$

To evaluate this integral the functions $f_1^{(o)}$ and $f_2^{(o)}$ are substituted (Equation 64) in the above. It is found convenient to change the coordinates of the peculiar velocities to those of the velocities of one particle relative to another in a collision. Specifically let

$$(m_1+m_2)\vec{G} = m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2' \tag{122}$$

be the momentum of mass motion of the two particles during a collision. The velocity of mass motion $\vec{G}$ is conserved in all encounters considered. Also let

$$\vec{g} = \vec{v}_2 - \vec{v}_1 \tag{123}$$

be the initial velocity at infinity of a particle of the second kind relative to one of the first kind during a collision, and let $\vec{g}'$ be the same for the final relative velocity. Using these velocity variables, the parameters of impact (see Figure IV) are

$$k_{12} d\vec{k} = g\ b\ db\ d\epsilon \tag{124}$$

They indicate the specific geometry of a particular impact. From the figure b is the impact parameter and ϵ represents the angle of rotation of the plane defined by the impact from some arbitrary reference plane. Now making all of the above substitutions, the multiple integral for the curved brackets may be replaced by the following series of equations:

$$\left\{ \vec{a}^{(0)}, \vec{a}^{(0)} \right\} = -(m_1+m_2)kT/E \tag{125}$$

$$E = kT(m_1+m_2)^2/(8m_1m_2\Omega_{12}(1)) \tag{126}$$

$$\Omega_{12}(1) = \left[\frac{m_1m_2}{(m_1+m_2)2kT}\right]^{5/2} \sqrt{\pi} \int_0^\infty \phi_{12} \exp\left[-\frac{m_1m_2g^2}{2(m_1+m_2)kT}\right] g^4 dg \tag{127}$$

and

$$\phi_{12} = \int (1-\cos\chi) gb\ db \tag{128}$$

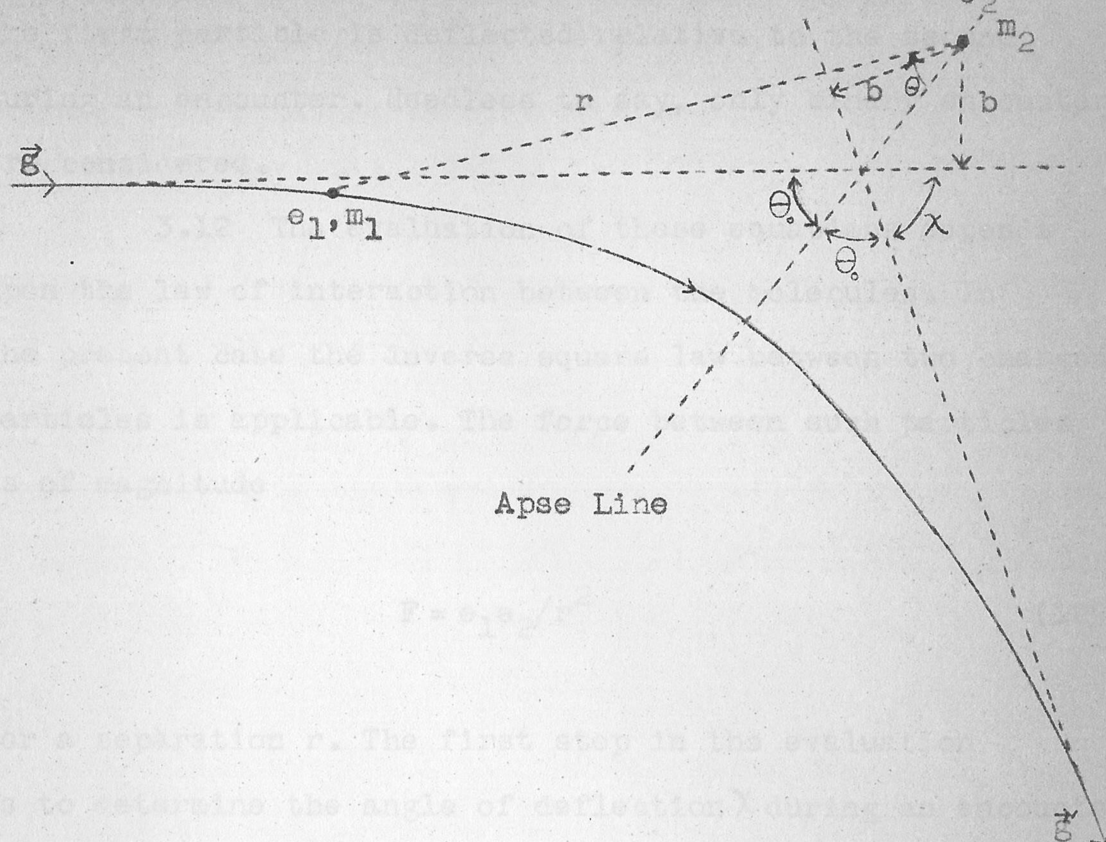


Figure IV

Two Body Collision

The only new quantity introduced is the angle χ. From the figure this is seen to be the angle through which the first particle is deflected relative to the second during an encounter. Needless to say, only binary encounters are considered.

3.12 The evaluation of these equations depends upon the law of interaction between the molecules. In the present case the inverse square law between two charged particles is applicable. The force between such particles is of magnitude

$$F = e_1 e_2 / r^2 \tag{129}$$

for a separation r. The first step in the evaluation is to determine the angle of deflection χ during an encounter. This requires a knowledge of the orbit of one particle about the other. In such an orbit the angular momentum is conserved, so in terms of the reduced mass

$$(m_1 m_2/(m_1+m_2))r^2\dot{\theta} = \text{const.} = (m_1 m_2/(m_1+m_2))gb \tag{130}$$

Energy is also conserved, or

$$\tfrac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] + \frac{e_1 e_2}{r} = \left(\frac{m_1 m_2}{2(m_1+m_2)}\right)g^2 \tag{131}$$

Upon eliminating the time between these, the differential equation of the orbit is obtained; viz.

$$(g^2b^2/2r^4)\left[(dr/d\theta)^2+r^2\right]= \tfrac{1}{2}g^2-(m_1+m_2)e_1e_2/m_1m_2r \qquad (132)$$

When solved by quadratures,

$$\theta = \int_{r_{oo}}^{\infty}\left[(r^4/b^2)-r^2-2(m_1+m_2)e_1e_2r^3/m_1m_2g^2b^2\right]^{-1/2}dr \qquad (133)$$

where r_{oo} is distance of closest approach. Thus θ measures the angle between the apse line and asymptote of particle No. 1 at infinity. Introducing two new variables

$$v = b/r \qquad (134)$$

and

$$v_o = bm_1m_2g^2/(m_1+m_2)e_1e_2 \qquad (135)$$

the integral reduces to an expression of the form

$$\theta_o = \int_{o}^{v_{oo}}\left[1-v^2-2(v/v_o)\right]^{-1/2}dv \qquad (136)$$

where v_{oo} is the real positive root of

$$1-v^2-2(v/v_o) = 0 \qquad (137)$$

This upper limit was determined from the fact that at the apse of the orbit

$$dr/d\theta = dv/d\theta = 0 \tag{138}$$

When solved, one finds

$$v_{oo} = v = (1/v_o)(\sqrt{1+v_o^2}-1) \tag{139}$$

Before evaluating the integral, the angle of deflection χ will be introduced. From the figure it is clear that

$$\chi = \pi - 2\theta_o = \pi - 2\int_0^{v_{oo}} \left[1-v^2-2(v/v_o)\right]^{-1/2} dv \tag{140}$$

Integrating by Pierce #161 (79), one obtains

$$\chi = 2 \sin^{-1}(1+v_o^2)^{-\frac{1}{2}} \tag{141}$$

From the relations of trigonometry, it follows that

$$\cos\chi = (v_o^2-1)/(v_o^2+1) \tag{142}$$

This can then be substituted into Equation 128; viz.

$$\phi_{12} = \int\left[1-(v_o^2-1)/(v_o^2+1)\right] g\ b\ db \tag{143}$$

Change the variable of integration by means of Equation 135 and simplifying the integrand to read

$$\phi_{12} = 2\left[(m_1+m_2)e_1e_2/m_1m_2g^2\right]^2 g\int_0^{v_{o1}} v_o/(v_o^2+1)dv_o \qquad (144)$$

Then by Pierce #53

$$\phi_{12} = \left[(m_1+m_2)e_1e_2/m_1m_2g^2\right]^2 g\ln(v_{o1}^2+1) \qquad (145)$$

where v_{o1} is the upper limit of integration. This limit should be taken as infinite; but by taking such a limit, the value of ϕ_{12} would likewise be infinite, giving zero value to the conduction current. The trouble lies in the electrostatic forces which do not fall off fast enough. So particles even when a great distance apart have appreciable effect upon one another. At these distances the encounter is no longer strictly a binary one (80). Rather the particle is essentially in a general field of force determined by the distribution of charge in its neighborhood. The magnitude of this charge should be included as a force acting on the particles of the gas in Boltzmann's equation. For the problem at hand this static force is on the average zero, as the numbers of positively and negatively charged particles are assumed to be equal. So, as long as the mean distance between the particles d is considerably greater than the impact parameter b, an encounter may be considered to be essentially binary. Indeed, being a logarithmic term, appreciable variations in the exact limit will have little effect upon the final result. Thus in Equation 134 for v let the impact parameter b have the upper limit d. For an upper limit to the distance r consider the energy equation

$$e_1e_2/r = \tfrac{1}{2}m_1m_2g^2/(m_1+m_2) \tag{146}$$

where the expression on the right is the kinetic energy of one particle relative to the other at infinity. The value of the left side corresponds to the potential energy at closest approach. This will maximize the value of v. In addition the above value for the kinetic energy should be replaced by its mean value.

This may be found in a manner analogous to that used in computing the mean values of quantities in inertial space (Equation 21). Equation 14 defines the number of collisions between two particles. If this were integrated over the encounter variables and the two velocity ranges, the mean number of collisions would be obtained. By analogy if a function ϕ were inserted, its mean value would be

$$\bar{\phi}\, N_{12} = \iiint f_1 f_2\, \phi\, k_{12}\, d\vec{k}\; d\vec{v}_1 d\vec{v}_2 \tag{147}$$

Substituting the zeroth approximations for f_1 and f_2 (Equation 64) and letting

$$\phi = \tfrac{1}{2}m_1m_2g^2/(m_1+m_2) \tag{148}$$

one finds $\bar{\phi}$ equals 2kT. For this calculation the zeroth order approximation is of sufficient accuracy. Hence in Equation 145

$$v_{01} = 4dkT/e_1e_2 \tag{149}$$

Returning to the evaluation of $\left\{\vec{a}^{(o)}, \vec{a}^{(o)}\right\}$, the result from Equation 145 is substituted into Equation 127.

$$\Omega_{12}(1) = \sqrt{\pi m_1m_2/(m_1+m_2)}\ (e_1^2e_2^2/(2kT)^{5/2})\ln\left[(4dkT/e_1e_2)^2 + 1\right]\int_0^\infty g\ \exp(-\tfrac{1}{2}m_1m_2g^2/(m_1+m_2)kT)dg \tag{150}$$

Integrating by Pierce #493,

$$\Omega_{12}(1) = \sqrt{\frac{\pi\,(m_1+m_2)}{m_1m_2}}\left(\frac{e_1^2e_2^2}{(2kT)^{3/2}}\right)\ln\left[(4dkT/e_1e_2)^2+1\right] \tag{151}$$

Finally substituting this into Equations 126 and then into No. 125,

$$\left\{\vec{a}^{(o)}, \vec{a}^{(o)}\right\} = -4\sqrt{\frac{\pi\ m_1m_2}{2kT(m_1+m_2)}}\left(\frac{e_1^2e_2^2}{kT}\right)\ln\left[\left(\frac{4dkT}{e_1e_2}\right)^2+1\right] \tag{152}$$

To simplify let

$$\nu = \frac{4\rho e_1^2 e_2^2}{3m_1 m_2 kT}\sqrt{\frac{\pi m_1 m_2}{2kT(m_1+m_2)}}\ \ln\left[\left(\frac{4dkT}{e_1 e_2}\right)^2+1\right] \tag{153}$$

so

$$\left\{\vec{a}^{(o)},\ \vec{a}^{(o)}\right\} = -(3m_1 m_2/\rho)\nu \tag{154}$$

The quantity ν introduced above has the dimensions of inverse time and is interpreted as the mean number of collisions per second between molecules of the two components of the gas.

The value of d_o will now be obtained. Substituting the above expression into Equation 120, d_o is found to be

$$-d_o(3\rho_1\rho_2/\rho)(\nu+\tfrac{1}{2}i\omega) = 3\sqrt{\tfrac{1}{2}kT} \tag{155}$$

3.13 The difference in the mean velocities, which is needed for computing the conduction current, was given in Equation 106 in terms of D_1 and D_2. Substitute the series expansions for these and keep only the first term to give

$$\overline{\vec{c}_1}-\overline{\vec{c}_2} = (1/3)(N_1+N_2)\vec{d}_{12}\cdot d_o\Big[(1/N_1)\int f_1^{(o)}\vec{a}_1^{(o)}\cdot\vec{c}_1 d\vec{v}_1$$

$$-(1/N_2)\int f_2^{(o)}\vec{a}_2^{(o)}\cdot\vec{c}_2 d\vec{v}_2\Big] \tag{156}$$

These integrals were evaluated in finding Equation 120. Substituting the value for d_o from above,

$$\vec{c}_1 - \vec{c}_2 = -\left[(N_1+N_2)\rho kT/\rho_1\rho_2(2\nu + i\omega)\right]\vec{d}_{12} \tag{157}$$

The conduction current of Equation 103 is now introduced to complete the solution

$$\vec{J} = -\left[\frac{(N_1+N_2)kT(e_1m_2-e_2m_1)}{2m_1m_2(\nu+i\frac{1}{2}\omega)}\right]\vec{d}_{12} \tag{158}$$

In the corona and chromosphere it can very probably be assumed that the gradients of temperature, density, and pressure are everywhere small over any region covering many wavelengths. Indeed these bodies are usually considered to be in equilibrium and not to be in general mass motion. Hence the assumption of neglecting the temperature gradient (Equation 105) in obtaining the first term in the series expansion for $\vec{D}_1$ may be supplemented by similar neglects with respect to density and pressure. Under these conditions the sole remaining term of $\vec{d}_{12}$ (Equation 78) is that due to the electric field in the field of radiation and thus

$$\vec{J} = (\rho_1\rho_2/\rho)(e_1/m_1-e_2/m_2)^2(2\nu - i\omega)/(4\nu^2+\omega^2)\vec{E}_o e^{i\omega t} \tag{159}$$

to complete the theory of this chapter. The exact form of this result should be compared to that obtained from the "free path method" of the kinetic theory. Under that method of attack two conduction currents would have been obtained, one for each component. Each of them would have been a function of its own properties and the forces acting upon it alone. Such is not the case here. Rather the current is essentially a mean value of the two currents in the other theory. It is the more logical result since the components of a physical gas undoubtedly do interact with one another.

IV THE COMPLEX INDEX OF REFRACTION

4.1 In Chapter II the equation of transfer was derived and solved on the basis of the ray theory of optics. The general expression was found to be a function of the temperature, the optical depth and a function M_f of the index of refraction of the medium. Upon application to the radio frequency case under discussion, the unknown function M_f was eliminated. Chapter III emphasized the properties of the material more than the field of radiation. There it was shown that a field of radiation of angular frequency ω would induce a conduction current in the ionized gases. Thus to complete the problem, it is necessary to relate the conduction current to the optical depth, or rather to the coefficient of absorption, as one is defined in terms of the other. This is done by continuing the work of the last chapter with the electromagnetic wave theory. From this theory it will be possible to derive a coefficient of absorption for a plane wave. The two coefficients are then said to be equal.

4.2 From various considerations Maxwell's electromagnetic equations will be used on a sub-macroscopic scale. For this scale of magnitudes the discreteness of the electric charge is still replaced by uniform charge densities, but such gross effects of matter as the dielectric constant and permeability are neglected. Using one term from a Fourier expansion of the field of radiation, Maxwell's equations in Gaussian units are

$$\mathrm{Div}\ \vec{E} = 0 \tag{1}$$

$$\mathrm{Div}\ \vec{B} = 0 \tag{2}$$

$$\mathrm{Curl}\ \vec{E} = -(i\omega/c)\vec{B} \tag{3}$$

$$\mathrm{Curl}\ \vec{B} = (i\omega/c)\vec{E} + (4\pi/c)\vec{j} \tag{4}$$

The value for the conduction current (Equation 3-159) should be substituted into the second of the curl equations to give

$$\mathrm{Curl}\ \vec{B} = (i\omega/c)\vec{E} + \left(\frac{\pi \rho_1 \rho_2}{\rho c}\right)\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \left(\frac{2\nu - i\omega}{4\nu^2 + \omega^2}\right)\vec{E} \tag{5}$$

To simplify the above expression, let

$$\Omega^2 = (4\pi \rho_1 \rho_2/\rho)\left[(e_1/m_1) - (e_2/m_2)\right]^2 \tag{6}$$

This quantity has the dimensions of angular frequency. It may be compared with the expression for the plasma frequency, first introduced by Langmuir for an electron gas. His expression was

$$\omega_p^2 = 4\pi N e^2/m \tag{7}$$

The two become identical if one component of the binary mixture is removed. This again clearly shows an essential difference between the "free path method" and the "velocity-distribution method" in the kinetic theory of gases. The former method derives quantities which are separate for the two components of the mixture while the latter method provides only one which is essentially a mean of the two separate results.

The mean plasma frequency of Equation 6, is seen to be dependent only upon the constants of the material and not upon the incident wave. Hence it will be a useful parameter with which to compare the frequency of the radiation. Some authors have even expressed the belief that the plasma frequency may be a natural period of vibration of the material. The calculations of the radiation in Chapter VII will indicate that the amount of damping encountered by a wave in this region is considerable. Making use of this parameter, Equation 5 becomes

$$\text{Curl } \vec{B} = \frac{i\omega}{c}\left(1 - \frac{\Omega^2}{4\nu^2+\omega^2}\right)\vec{E} + \frac{4\pi}{c}\left(\frac{\Omega^2 \nu}{2\pi(4\nu^2+\omega^2)}\right)\vec{E} \qquad (8)$$

Physical interpretation of the terms in the above equation is most easily made by comparing it with two other standard forms for the corresponding Maxwell equation.

4.3 First consider the propagation of a wave of frequency ω through a polarizable isotropic conducting medium. In such a material one of the field equations is

$$\text{Curl } \vec{B} = (i\omega/c)\vec{E} + (4\pi i\omega/c)\vec{P} + (4\pi/c)\vec{j} \tag{9}$$

where $\vec{P}$ is the polarization. Being an isotropic medium, the last two terms may be expressed in terms of the electric field, viz.

$$\text{Curl } \vec{B} = (i\omega/c)(1+4\pi\chi)\vec{E} + (4\pi/c)\sigma\vec{E} \tag{10}$$

where χ is the electric susceptibility and σ the conductivity. Comparing this with Equation 8, it is seen that the radiation field produces an equivalent susceptibility,

$$\chi = -\Omega^2/\left[4\pi(4\nu^2+\omega^2)\right] \tag{11}$$

and a conductivity

$$\sigma = \Omega^2\nu/\left[2\pi(4\nu^2+\omega^2)\right] \tag{12}$$

in the ionized gases.

4.4 As the above physical interpretation does not lead further than that indicated, another approach is taken. In this case Equation 8 will be compared with the propagation of a wave through an isotropic non-conducting dielectric medium. For such a medium the corresponding Maxwell equation is

$$\text{Curl } \vec{B} = i\omega\vec{D}$$

$$= (i\omega\epsilon/c)\vec{E} \qquad (13)$$

where $\vec{D}$ is the displacement and ϵ the dielectric constant. This equation may likewise be compared to Equation 8. If so, the coefficient of $(i\omega/c)\vec{E}$ may be considered as a complex dielectric constant:

$$\epsilon = \epsilon_1 + i\epsilon_2 \qquad (14)$$

$$= 1 - \Omega^2/(4\gamma^2+\omega^2) - i(2\gamma/\omega)\left[\Omega^2/(4\gamma^2+\omega^2)\right] \qquad (15)$$

Consider for a moment the solution to Maxwell's equations for a plane wave traveling through a medium such as that described by Equation 13. The solution is of the form

$$\vec{E} = \vec{E}_o \exp\left\{i\omega\left[(nr/c) - t\right]\right\} \qquad (16)$$

where n is the index of refraction. This index is defined as the square root of the dielectric constant. So a complex dielectric constant, as in Equation 15, means a complex index of refraction and hence a damped plane wave solution. For if

$$n = \mu + ik = \sqrt{\epsilon} \tag{17}$$

then

$$\vec{E} = \vec{E}_o \exp(-\omega kr/c) \exp\left\{i\omega\left[(\mu r/c) - t\right]\right\} \tag{18}$$

This is just the solution desired to relate the ray theory of Chapter II to the wave theory of this chapter and the last one. The intensity of the radiation described by Equation 18 may be found from Poynting's vector,

$$\vec{i}_f = (c/8\pi)\vec{E} \times \vec{B}^* \tag{19}$$

As the induction field will be perpendicular to the electric field and of the same form, the magnitude of the intensity at any point is

$$i_f = (c/8\pi)E_oB_o\exp(-2\omega kr/c) \tag{20}$$

If this is compared with the equation for the attenuation of a ray passing through matter, one sees that the coefficient of absorption K_f is related to the imaginary part of the complex index of refraction (Equation 17) by

$$K_f = 2\omega k/c = 4\pi k/\lambda \tag{21}$$

The real and imaginary parts of the complex index of refraction have yet to be solved in terms of the real and imaginary parts of the dielectric constant. Substituting Equation 14 into Equation 17, squaring the result, and then equating the real and imaginary parts, one finds that

$$\mu^2 = \tfrac{1}{2}\left[\epsilon_1 \pm \sqrt{\epsilon_1^2 + \epsilon_2^2}\right]$$

$$k^2 = \tfrac{1}{2}\left[-\epsilon_1 \pm \sqrt{\epsilon_1^2 + \epsilon_2^2}\right]$$

To determine which sign should be used before the radical, consider the special case where there is no damping. Then both k and ϵ_2 are zero. These conditions require the upper sign. So

$$\mu = \sqrt{\tfrac{1}{2}\left[\epsilon_1 + \sqrt{\epsilon_1^2 + \epsilon_2^2}\right]} \qquad (22)$$

$$k = \sqrt{\tfrac{1}{2}\left[-\epsilon_1 + \sqrt{\epsilon_1^2 + \epsilon_2^2}\right]} \qquad (23)$$

This then completes the solution to the problem of computing the total amount of radiation emitted by the sun in the radio frequency spectrum. We have seen that from a knowledge of the temperature distribution and ion densities in the corona and chromosphere the mean number of collisions per second between the molecules of the two components of the gas may be found. The complex dielectric constant and hence the coefficient of absorption are then known. If the path of a ray is given, the optical depth and so the emergent intensity of radiation is easily found, thus completing the computation. In this manner only the paths of the rays are still unknown, the subject of the next chapter.

V PATHS OF THE RADIATION

5.1 To complete the theory for the thermal component of the radio frequency radiation emitted by the sun, it is necessary to know the path followed by a given pencil of radiation. The derivation of the path at these frequencies must be based upon slightly different postulates than at optical wavelengths. In the latter case the effects of refraction are quite negligible, and so the path may be regarded as rectilinear. Such, however, is not the case in the radio frequency spectrum. For here both refraction and absorption must be taken into account.

5.2 To obtain an idea of what effect refraction will have upon the path of a ray, first consider the simple case of radiation passing through a refracting but non-absorbing medium. This type of medium will closely approximate the upper parts of the corona where the absorption is quite small. For this case the path of the radiation may be derived from Snell's law of refraction. To apply the law consider two adjacent layers in the solar atmosphere at altitudes R and R+dR having indices of refraction μ and $\mu + d\mu$ respectively (see Figure V). A pencil of radiation ABCD will cross these two layers at point B and C respec-- tively. At point B the ray will be at an angle of incidence α with the radius vector. By the time the ray has reached C,

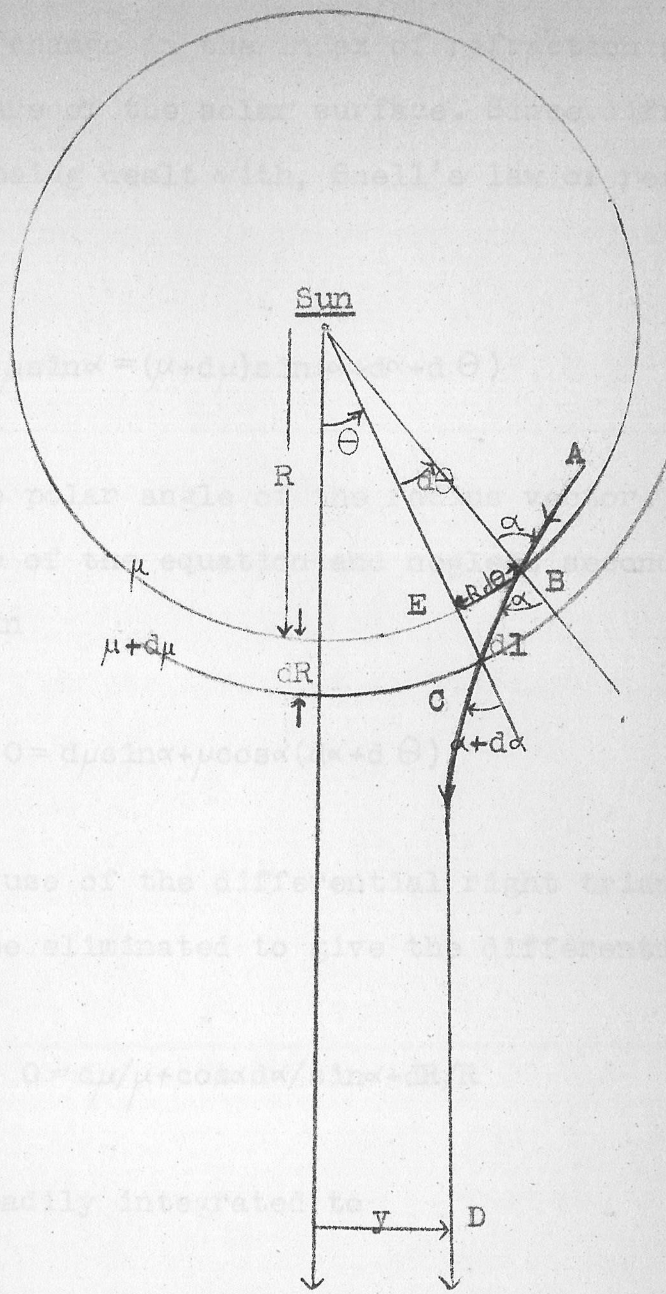


Figure V

Path of a Single Ray

the angle of incidence will have changed to $\alpha + d\alpha$ as a result of the change in the index of refraction and also of the curvature of the solar surface. Since differential elements are being dealt with, Snell's law of refraction is

$$\mu \sin\alpha = (\mu + d\mu)\sin(\alpha + d\alpha + d\,\Theta) \tag{1}$$

where Θ is the polar angle of the radius vector. Expand the right side of the equation and neglect second order terms to obtain

$$0 = d\mu \sin\alpha + \mu \cos\alpha\,(d\alpha + d\,\Theta)$$

Then with the use of the differential right triangle BCE, $d\Theta$ may be eliminated to give the differential equation

$$0 = d\mu/\mu + \cos\alpha\, d\alpha/\sin\alpha + dR/R$$

This may be readily integrated to

$$y = \mu R \sin\alpha \tag{2}$$

where y is the constant of integration. From the figure it is seen that for large R where μ is essentially unity, y is the perpendicular distance of the pencil of radiation from a parallel pencil passing out from the center of the sun. This is identical to the projected solar radius y used in Chapter II.

5.3 The above solution, of course, is not the required one. Instead it is necessary to consider both refraction and absorption in deriving the result. Snell's law of refraction is no longer applicable. Rather one must turn to a law of refraction which takes into account absorption as well as refraction. But under these conditions the very concept of a ray becomes questionable. For if the places of origin and absorption of the energy were taken into account, then the direction of its flow would no longer depend upon the local conditions, but rather would be a cooperative phenomenon of the whole system. And so it could be represented only as a non-holonomic system of rays.

Therefore simplifications are unavoidable. The best available approximation to the solution of the problem is contained in a paper by Epstein (81) in 1930. In this paper he treated the case of a finite pencil of rays passing from a non-absorbing medium into a refracting and absorbing one. The radiation in the sun is, of course, produced in the absorbing medium; and, in general, such radiation will be inhomogeneous and a function of its points of origin. It was ascertained, however, for special cases in plane stratified media that the difference between the two results was not significant. As the degree of curvature of the sun is quite small, it may be safely assumed that the difference due to the curvature will not materially affect the result. In addition it will be seen in the following chapters that the real part of the index of refraction is much greater than the imaginary part over much of the corona. This will also tend to reduce the degree of inhomogeneity present in the radiation.

5.4 In his paper Epstein showed that the equation of the iconal

$$(\nabla S)^2 = n^2 = (\mu + ik)^2 \tag{3}$$

was compatible with the general equation for a wave passing through a refracting and absorbing medium having a complex index of refraction n. The equation of the ray was shown to be

$$\mathrm{Re}\left[\frac{Udu}{(\partial S/\partial u)}\right] = \mathrm{Re}\left[\frac{Vdv}{(\partial S/\partial v)}\right] = \mathrm{Re}\left[\frac{Wdw}{(\partial S/\partial w)}\right] \tag{4}$$

for general orthogonal curvilinear coordinates having a line element

$$dl^2 = Udu^2 + Vdv^2 + Wdw^2$$

In the particular case where the constant of integration for Equation 3 is real (see below), the real and imaginary parts may be interpreted physically. The surfaces

$$\text{Re } S = \text{const.} \tag{5}$$

are those of equal phase while the ones

$$\text{Im } S = \text{const.} \tag{6}$$

are those of equal intensity. The Fermat principle cannot be used for this type of medium as the quantities t (time) and v (velocity) lose their simple physical meaning. Also strictly speaking the above equation may be applied only to the case where n changes appreciably in distances long compared to the wavelength of the radiation. This condition is well fulfilled throughout the corona and chromosphere except near the altitude where the mean plasma frequency (Equation 4-6) equals the frequency of the radiation. As this region is quite small, it is neglected.

The equation of the iconal will be applied to the problem, illustrated in Figure V, where n is a function of the solar radius only. Then the optical path will be a function of R and Θ only. Under these coordinates Equation 3 becomes

$$(\partial S/\partial R)^2 + (1/R^2)(\partial S/\partial \Theta)^2 = n^2 = (\mu + ik)^2 \tag{7}$$

To find the solution, let S be expressed as the sum of two functions

$$S = S'(R) + S''(\Theta) \tag{8}$$

Substituting and then re-arranging the terms,

$$(dS''/d\Theta)^2 = R^2\left[n^2 - (dS'/dR)^2\right]$$

If the separation constant is taken to be y, two first order linear differential equations result. Upon solving by quadratures,

$$S = y\Theta + \int_{R_o}^{R} \sqrt{n^2 - (y/R)^2}\, dR \tag{9}$$

The lower limit R_o of the integration is the radius of the top surface of the sun, above which the effects of refraction and absorption may be considered to be negligible. The other limit is the point in the sun under discussion at the moment.

With the above result the optical path length may be calculated but not its location in the sun. To do the latter, the equation of the rays must be found from Equation 4. For spherical coordinates this is

$$Re\left[\frac{dR}{(\partial S/\partial R)}\right] = Re\left[\frac{R^2 d\Theta}{\partial S/\partial \Theta}\right]$$

With reference to the differential triangle BCE in the figure

$$\tan\alpha = R(d\,\Theta/dR) \tag{10}$$

So upon substituting the values for the partial derivatives into the above, the angle of incidence of the ray becomes

$$\tan\alpha = \operatorname{Re}\left[(y/R)/\sqrt{n^2-(y/R)^2}\right] \tag{11}$$

To evaluate the separation constant y, consider the special case where there is no absorption. Then n is real and

$$\tan\alpha = (y/R)/\sqrt{n^2-(y/R)^2}$$

or

$$\sin\alpha = (y/R)/\mu$$

This is the same as Equation 2. It shows that y is again the projected solar radius since at these altitudes k is essentially zero.

For actual calculations it is necessary to take the real part of Equation 11. Substitute the complex value for n (Equation 3), and then upon evaluation one finds

$$\tan\alpha = \frac{(y/R)\sqrt{\sqrt{[\epsilon_1-(y/R)^2]^2+\epsilon_2^2}+[\epsilon_1-(y/R)^2]}}{\sqrt{2}\sqrt{[\epsilon_1-(y/R)^2]^2+\epsilon_2^2}} \tag{12}$$

This gives the angle of incidence α as a function of the radius R. To find the other coordinate Θ of a ray, consider Equation 10. When integrated,

$$\Theta = \int_R^{\infty} (1/R)\tan\alpha dR \tag{13}$$

The upper limit has been so chosen that θ is zero for infinite values of R. These two equations then define the path traveled by any given pencil of radiation in the corona or chromosphere.

To obtain a physical idea of the path follow by any ray, consider how α will vary as

$$\epsilon_1-(y/R)^2$$

varies. For negative values of the above, corresponding to layers deep in the sun, α is seen to be quite small. This means that the radiation is flowing almost parallel to a radius vector. Then as ϵ_1 becomes equal to $(y/R)^2$, α increases to almost 90 degrees. The exact value will depend upon the magnitude of ϵ_2. As it is a very small quantity in the present application, it means that the rays turn almost perpendicular to the radius vector for

$$\epsilon_1 = (y/R)^2$$

Finally at still greater elevations α gradually decreases to zero once more. For those regions where the absorption is quite small α varies essentially in the same manner as in a refracting but non-absorbing medium (see Section 5-2).

5.5 Having derived the path along which any given pencil of radiation will travel, it will be found convenient to express the path length l in terms of the radius vector R. From the differential right triangle BCE of Figure V,

$$\cos\alpha = dR/dl \tag{14}$$

This will be used to change the variables of integration in the equations for the optical depth and the intensity of emitted radiation to that of the radius vector. As was seen above, no trouble will occur from the zeros of $\cos\alpha$ as they do not exist in the physical problem. The optical depth (Equation 2-20) now becomes

$$\tau = -\int_{R_o}^{R} K_f \sec\alpha \, dR \tag{15}$$

The lower limit R_o represents the radius vector to the point of observation, replacing l_o. R has similarly replaced 1. If the absorption coefficient is expressed in terms of the imaginary part of the complex of refraction (Equation 4-21), then

$$\tau = -(4\pi/\lambda)\int_{R_o}^{R} k \sec\alpha \, dR \qquad (16)$$

In the case of the equation of transfer the various solutions given in Chapter II had the optical depth as the variable of integration. For the purposes of computation it is more convenient to re-express this in terms of the radius vector. So using Equations 2-23, 4-21, and 14, the emergent intensity of each pencil of radiation (Equation 2-26) is

$$i_{fo} = (4\pi/\lambda)(1/\Delta S_o)\int_{R}^{R_o} B_f(T)\ k \sec\alpha e^{-\tau}\Delta S \ dR \qquad (17)$$

The cross-sectional area ΔS of the pencil of radiation still needs to be determined in terms of known quantities. Consider a pencil of radiation having a differential area ABCD at an altitude R and colatitude angle θ (see Figure VI). This pencil will likewise intercept an area ABEF on a sphere of constant R. The common side $\overline{AB}$ of these two areas is

$$\overline{AB} = R \sin\theta\Delta\phi$$

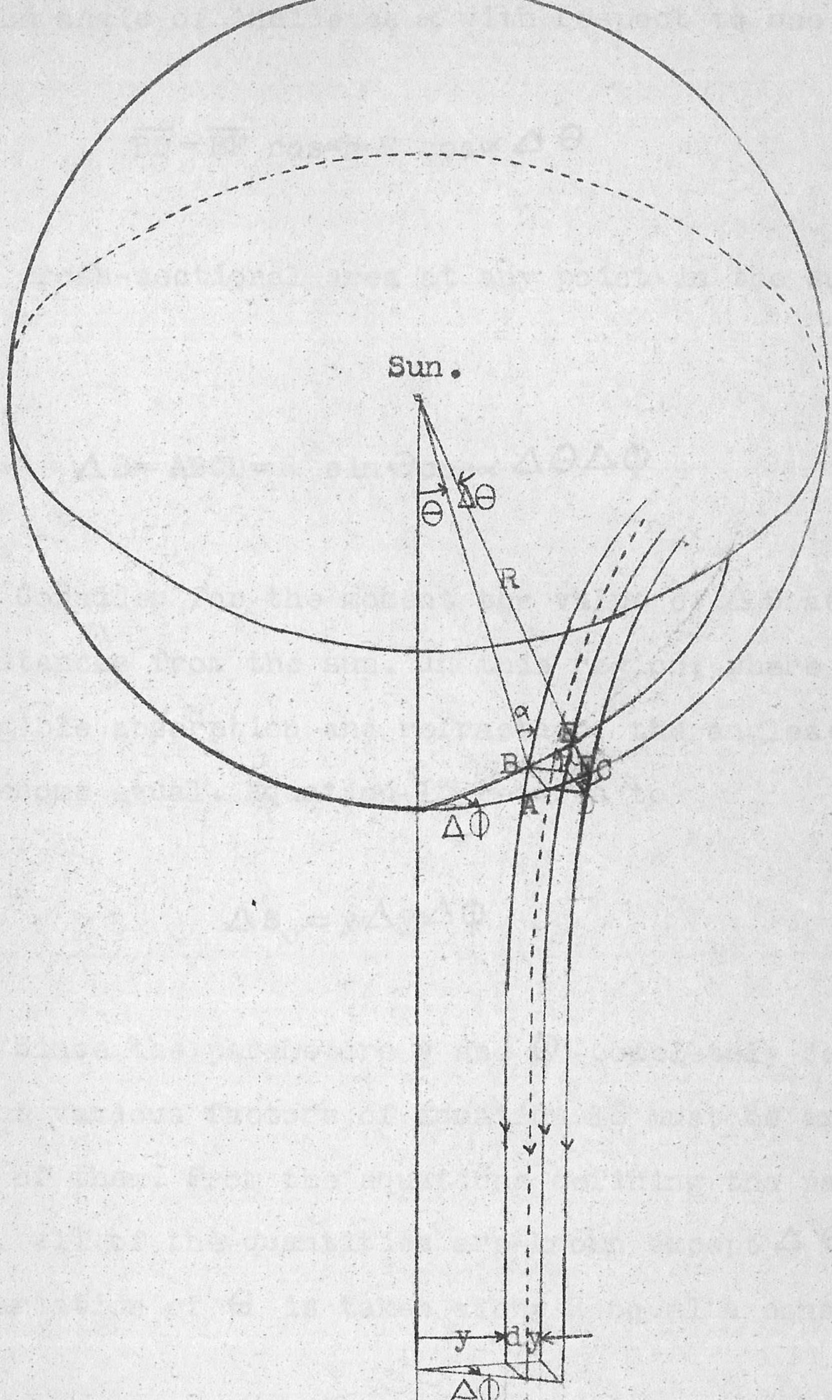


Figure VI

where ϕ is the azimuthal angle. The other dimension of the area ABEF is of magnitude $R\Delta\theta$. Since the two areas are at the angle of incidence α with respect to one another

$$\overline{BD}=\overline{BF}\cos\alpha=R\cos\alpha\,\Delta\theta$$

Thus the cross-sectional area at any point in the sun will be

$$\Delta S=ABCD=R^2\sin\theta\cos\alpha\,\Delta\theta\Delta\phi \tag{18}$$

Consider for the moment the value of ΔS at large distances from the sun. In this region, where there is negligible absorption and refraction, the angles α and θ become equal. Equation 18 reduces to

$$\Delta S_o=y\Delta y\Delta\phi \tag{19}$$

Since the parameters y and ϕ completely define a ray, the various factors of Equation 18 must be expressed in terms of them. From the equations defining the path of a ray, all of the quantities are known except $\Delta\theta$. As the variation of θ is taken along R equal a constant

$$\Delta\theta=(\partial\theta/\partial y)_R\Delta y \tag{20}$$

To evaluate the partial derivative, the angle α must be eliminated from the integral defining Θ. (Substitute Equation 12 into 13.) Then upon carrying out the required differentiation, one finds

$$\left(\frac{\partial \Theta}{\partial y}\right)_R = \int_R^{\infty} \left\{ \frac{\tan\alpha}{yR} + \frac{y^3}{2R^5 \tan\alpha} \left[\frac{\epsilon_1 - (y/R)^2}{\left\{\left[\epsilon_1 - (y/R)^2\right]^2 + \epsilon_2^2\right\}^{3/2}} + \frac{\left[\epsilon_1 - (y/R)^2\right]^2 - \epsilon_2^2}{\left\{\left[\epsilon_1 - (y/R)^2\right]^2 + \epsilon_2^2\right\}^2} \right] \right\} dR \qquad (21)$$

This expression should also be evaluated for normal emergence (y equal zero). Substituting the value for tan α and going to the limit as y tends to zero, one finds

$$(\partial \Theta / \partial y)_R = \int_R^{\infty} \left[\mu / \left(R^2 \sqrt{\epsilon_1^2 + \epsilon_2^2}\right) \right] dR \qquad (22)$$

where the value for μ was found from Equation 4-22.

The emergent intensity of the radiation is now completely determined. Substituting Equations 18, 19, and 20 into 17, one finds for a general ray emerging with a projected solar radius y that

$$i_{fo} = (4\pi / y\lambda) \int_R^{R_o} B_f(T) k e^{-t} (\partial \Theta / \partial y)_R R^2 \sin\Theta \, dR \qquad (23)$$

For the case of normal emergence this reduces to

$$i_{fo} = (4\pi/\lambda)\int_{R}^{R_o} B_f(T)ke^{-\tau}(\partial\Theta/\partial y)_R R \, dR \qquad (24)$$

If one wishes to express the emergent intensity in terms of an equivalent black body temperature T_a (see Section 2.8), then the last two equations become

$$T_a = (4\pi/y\lambda)\int_{R}^{R_o} Tke^{-\tau}(\partial\Theta/\partial y)_R R^2 \sin\Theta \, dR \qquad (25)$$

and

$$T_a = (4\pi/\lambda)\int_{R}^{R_o} Tke^{-\tau}(\partial\Theta/\partial y)_R R \, dR \qquad (26)$$

5.7 With these equations the theory for finding the thermal component of the radio frequency radiation generated by the sun is concluded. In the past chapters it was shown how the kinetic theory of gases will predict the number of collisions per second in a binary gas, a good approximation to the gases in the solar corona and chromosphere. Then with the use of the electromagnetic wave theory a complex dielectric constant and an index of refraction were found. These quantities determine the path of a pencil of radiation through the solar atmosphere. Finally the solution to equation of transfer for a three-dimensional medium gives the emergent intensity over the solar sphere.

VI PHYSICAL PROPERTIES OF THE CORONA AND CHROMOSPHERE

6.1 The test of any theory is whether it will predict the results obtained from experiments. The above presented theory was developed on just this assumption. But before such a comparison can be made, it will be necessary to have adequate physical models for both the corona and the chromosphere. These models must be able to present both the temperature and particle distributions as functions of the solar coordinates. In addition it would be desirable if they were made up of binary gases (electrons and protons) having spherical symmetry and if they would represent the quiet conditions in the sun. By fulfilling these conditions the above theory would closely fit the models. Those that have been developed to describe the parts of the sun that are of interest do indeed have the necessary qualifications.

6.2 Of the two parts of the sun that are of interest, the outer one has the more satisfactory model. The one adopted for this paper was derived by van de Hulst (82) in 1950. It represents the mean quiet condition in the sun between the maximum and minimum phase. He assumes that the corona is composed of almost pure hydrogen with but a slight admixture of helium. The small contribution of the heavier elements that produce the coronal lines was neglected. To be specific he assumed that there are five hydrogen atoms present for each helium atom. Due to the high kinetic temperature of about one million degrees Kelvin (83), these atoms are assumed to be completely ionized. Thus there are seven electrons for each helium atom. Under this assumption the mean atomic weight μ in the corona will be 9/13 due to 13 particles but only 9 mass units.

Thus, as soon as the absolute abundance of any one of the three components is known as a function of altitude, those of the others are known. The problem of determining these abundances was first solved by Baumbach (84) in 1937 on the assumption that the light emitted by the corona is due to scattering by the free electrons. Subsequent work (85) has shown his work to be in error by some 15%. This was the result of bringing out the importance of the inner zodiacal light or F-corona. The theory adopted by van de Hulst considers that the light from the corona is due to (a) the emission lines or L-corona, (b) the scattering of sunlight by the free electrons or K-component, and (c) the F-corona.

In order to determine the electron density it was necessary for him to know the magnitude of the K-component. The correction due to the L-component was considered to be negligible, being less than than $\frac{1}{2}$% of the total. This was not true of the F-component. It was removed from the photometric data on the assumption that the

Fraunhofer lines are completely obliterated by the K-corona but are present in the F-corona.

Applying this reasoning to the experimental data, he was able to obtain the values of the transverse component of the K-component of the coronal emission as a function of altitude. From the results the following empirical formula was derived

$$K_t(\rho) = -17.5\rho^{-34} + 132.0\rho^{-17} - 22.5\rho^{-9.7} + 85.2\rho^{-7} + 0.31\rho^{-2.5} \qquad (1)$$

where ρ is relative solar radius $R/R_\odot$. This agrees to better than 2% with the experimental data. Then the electron density N may be found from the following series of formulae together with Equation 1:

$$\sin\gamma = 1/\rho \qquad (2)$$

$$2A+B = 1-(\cos^2\gamma/\sin\gamma)\ln\left[(1+\sin\gamma)/\cos\gamma\right] + (2/3)(1-\cos\gamma) \qquad (3)$$

$$2A-B = \tfrac{1}{4}\Big\{1+\sin^2\gamma-(\cos^4\gamma/\sin\gamma)\ln\Big[(1+\sin\gamma)/\cos\gamma\Big]\Big\}+(2/9)(1-\cos^3\gamma) \qquad (4)$$

$$N = K_t(\rho)/\left[CC_t(\rho)\rho A(\rho)\right] \qquad (5)$$

where C is a proportionality constant equal to 3.44×10^{-6} cc and $C_t(\rho)$ is an empirical correction factor. Its value as a function of ρ may be found from Table II.

Table II

ρ	1.00	1.03	1.06	1.1	1.2	1.3	1.5
C_t	0.380	0.383	0.390	0.404	0.445	0.474	0.495
ρ	1.7	2.0	2.6	3.0	4.0	5	6
C_t	0.504	0.510	0.542	0.577	0.690	0.770	0.820

In addition to a knowledge of the electron density, it is necessary to find the kinetic temperature. Van de Hulst proceeded on the assumption that (a) the corona is in hydrostatic equilibrium and (b) the temperature gradients are all small. Then from the equation of hydrostatic equilibrium he found

$$T_1/T = d(\ln N)/d(1/\rho)+d(\ln T)/d(1/\rho) \qquad (6)$$

where: $T_1 = \mu M_{\odot} G m_H / R_{\odot} k = 1.58 \times 10^6 \ ^oK$

$M_{\odot}$ = mass of the sun

$R_{\odot}$ = radius of the sun (6.97×10^5 km)

G = gravitational constant

m_H = mass of the proton.

6.3 Next the physical model adopted for the chromosphere will be presented. Its exact nature has still not been completely understood. However, it has been definitely established that there must be a strong temperature gradient between the six thousand degree photospheric temperature and the million degree value in the corona. Among the various theories that have been presented on the structure, the one developed by Thomas (86) in 1949 appears to be the most satisfactory. In its formulation he tried to avoid the difficulties of other theories by interpreting simultaneously the intensity as a function of height and the Balmer decrement at each height in the solar flash spectrum. By doing this, the optical depth in the lines was found to be greater than that previously supposed, and thus the effect of self-absorption becomes important. Since most of the observed intensity probably comes from the base of the observed sector, this region was treated as being isothermal. Using these considerations, he found that the number of atoms in the ground state of the Balmer series lying along the line of sight could be expressed as a known function of the electron density and kinetic temperature aside from a proportionality constant.

In addition to the above considerations, Thomas assumed that, to a first approximation, the chromosphere was in hydrostatic equilibrium. For a model of a quiet chromosphere this is a reasonable assumption. The decision was also made to set the electron temperature equal to the kinetic temperature, a novel idea. To eliminate the unknown constant in the radiation problem, the logarithmic derivatives of the solutions for both phases of the work were taken. The resulting equations were then solved simultaneously for the gradients of the kinetic temperature and electron density. However, the integrations required for a complete solution can only be obtained numerically. In a form convenient for computation, the final results are:

$$\frac{d}{dh}\left[\frac{7}{2}\ln T - \frac{h\nu_2}{kT} - \ln b_2\right] = -\frac{d\ln N_2}{dh} - \frac{2\mu MGm_H}{R^2 kT} \qquad (7)$$

$$\ln N = \ln(N_o T_o/T) - (\mu MGm_H/kR_\odot^2)\int_{h_o}^{h}(\rho^2 T)^{-1}dh \qquad (8)$$

where: $d(\ln N_2/dh) = -1.86\times10^{-8}/\text{cm}$ is the gradient of the population of the Balmer ground state.

$3900 \leqq b_2 \leqq 7900$ is the actual ratio of the population of the Balmer ground state to that in a distribution in which the radiative temperature is assumed to equal the kinetic temperature. (For purposes of calculation a mean value of $\ln b_2 = 9.00$ was chosen.)

ν_2 is the series limit for the Balmer series.

N_o, T_o are the electron density and temperature at the altitude h_o.

All of the other quantities have already been defined.

As the accuracy of the theory is probably not as good as that in the corona, a mean molecular weight of $\frac{1}{2}$ was adopted by Thomas. To make a numerical integration of Equations 7 and 8, it is necessary to know N and T at some height h. Thomas (87) considered that for $h_o = 500$ km above the photosphere

$$N_o = 1.74 \times 10^{11} \text{ /cc}$$

$$T_o = 35{,}000 \text{ deg Kelvin}$$

6.4 With the adoption of the above models the electron density and kinetic temperature for both the corona and chromosphere were calculated. The results are given in Table III and in Figure VII as a function of the

TABLE III

ELECTRON DENSITY AND TEMPERATURE OF THE CORONA AND CHROMOSPHERE

Relative Solar Radius ρ	Altitude h	Electron Density N	Temperature T
0.9835	500 km	1.74×10^{11}/cc	35.0×10^{3} °K
0.9839	750 "	1.475 "	36.8 "
0.9842	1000 "	1.250 "	38.9 "
0.9846	1250 "	1.0662 "	41.3 "
0.9849	1500 "	9.0878×10^{10}/cc	43.9 "
0.9853	1750 "	7.7840 "	46.9 "
0.9857	2000 "	6.6665 "	50.3 "
0.9860	2250 "	5.7090 "	54.3 "
0.9864	2500 "	4.9330 "	58.4 "
0.9867	2750 "	4.2603 "	63.2 "
0.9871	3000 "	3.6771 "	68.8 "
0.9874	3250 "	3.1859 "	75.1 "
0.9878	3500 "	2.7585 "	82.3 "
0.9882	3750 "	2.3962 "	90.4 "
0.9885	4000 "	2.0792 "	99.7 "
0.9889	4250 "	1.8099 "	110 "
0.9892	4500 "	1.5737 "	122 "
0.9896	4750 "	1.3717 "	136 "
0.9900	5000 "	1.1945 "	152 "
0.9903	5250 "	1.0421 "	170 "
0.9907	5500 "	9.0777×10^{9}/cc	191 "

TABLE III (cont.)

ρ	h	N	T
0.9910	5750 km	7.9313×10^{9}/cc	214×10^{3} °K
0.9914	6000 "	6.9172 "	241 "
0.9918	6250 "	6.0440 "	271 "
0.9921	6500 "	5.2717 "	306 "
0.9925	6750 "	4.6101 "	346 "
0.9928	7000 "	4.0219 "	392 "
0.9931	7250 "	3.5185 "	445 "
0.9935	7500 "	3.0702 "	505 "
0.9939	7750 "	2.6886 "	573 "
0.9943	8000 "	2.3446 "	652 "
0.9946	8250 "	2.0537 "	741 "
0.9950	8500 "	1.7909 "	844 "
0.9953	8750 "	1.5687 "	961 "
0.9957	9000 "	1.3682 "	1.10×10^{6} °K
0.9961	9250 "	1.1985 "	1.18 "
0.9964	9500 "	1.0454 "	1.27 "
0.9968	9750 "	9.158×10^{8}/cc	1.34 "
0.9971	10,000"	7.988 "	1.40 "
0.9975	10,250"	7.15 "	1.45 "
0.9978	10,500"	6.25 "	1.48 "
0.9981	10,750"	5.51 "	1.52 "
0.9986	11,000"	4.90 "	1.55 "
0.9989	11,250"	4.35 "	1.57 "

TABLE III (cont.)

ρ	h	N	T
0.9993	11.50×10^3 km	3.93×10^8/cc	1.58×10^6 °K
0.9996	11.75 "	3.57 "	1.60 "
1.0000	12.00 "	3.22 "	1.62 "
1.0007	12.50 "	2.78 "	"
1.0014	13.00 "	2.46 "	"
1.0021	13.50 "	2.28 "	"
1.0029	14.00 "	2.22 "	"
1.01	18.97 "	2.1388 "	"
1.02	25.94 "	1.9490 "	"
1.03	32.91 "	1.7825 "	"
1.04	39.88 "	1.6177 "	"
1.05	46.85 "	1.4672 "	"
1.06	53.8 "	1.3291 "	"
1.065	57.3 "	1.2649 "	"
1.070	60.8 "	1.2056 "	"
1.075	64.2 "	1.1488 "	"
1.08	67.8 "	1.0952 "	"
1.09	74.7 "	9.9439×10^7/cc	"
1.10	81.7 "	9.0504 "	"
1.11	88.7 "	8.2511 "	"
1.12	95.6 "	7.5444 "	"
1.13	102.6 "	6.813 "	"
1.14	109.6 "	6.273 "	"

TABLE III (cont.)

ρ	h	N	T
1.15	116.6x10^3 km	5.782x10^7/cc	1.62x10^6 ^{o}K
1.16	123.5 "	5.324 "	"
1.17	130.5 "	4.945 "	"
1.18	137.5 "	4.588 "	"
1.19	144.4 "	4.262 "	"
1.20	151.4 "	3.958 "	"
1.22	165 "	3.435 "	"
1.24	179 "	3.006 "	"
1.25	186 "	2.822 "	"
1.26	193 "	2.653 "	"
1.28	207 "	2.361 "	"
1.30	221 "	2.115 "	"
1.32	235 "	1.897 "	"
1.34	249 "	1.713 "	"
1.35	256 "	1.631 "	"
1.36	263 "	1.549 "	"
1.38	277 "	1.412 "	"
1.40	291 "	1.287 "	"
1.41	298 "	1.230 "	"
1.42	305 "	1.175 "	"
1.43	312 "	1.125 "	"
1.44	319 "	1.077 "	"
1.45	326 "	1.032 "	"

TABLE III (cont.)

ρ	h	N	T
1.46	333×10^3 km	9.871×10^6/cc	1.62×10^6 °K
1.47	340 "	9.461 "	"
1.48	347 "	9.071 "	"
1.49	354 "	8.717 "	"
1.50	360 "	8.364 "	"
1.55	395 "	6.864 "	"
1.60	430 "	5.681 "	"
1.65	465 "	4.741 "	"
1.70	500 "	3.995 "	"
1.75	535 "	3.374 "	"
1.80	570 "	2.879 "	"
1.85	604 "	2.456 "	"
1.90	639 "	2.102 "	"
1.95	674 "	1.816 "	"
2.00	709 "	1.574 "	"
2.3	918 "	7.324×10^5/cc	1.35×10^6 °K
2.6	1.13×10^6 km	3.74 "	1.07 "
3.0	1.41 "	1.76 "	1.07 "
4.0	2.10 "	5.0×10^4/cc	1.07 "
5	2.80 "	2.5 "	1.07 "
6	3.50 "	1.6 "	1.07 "

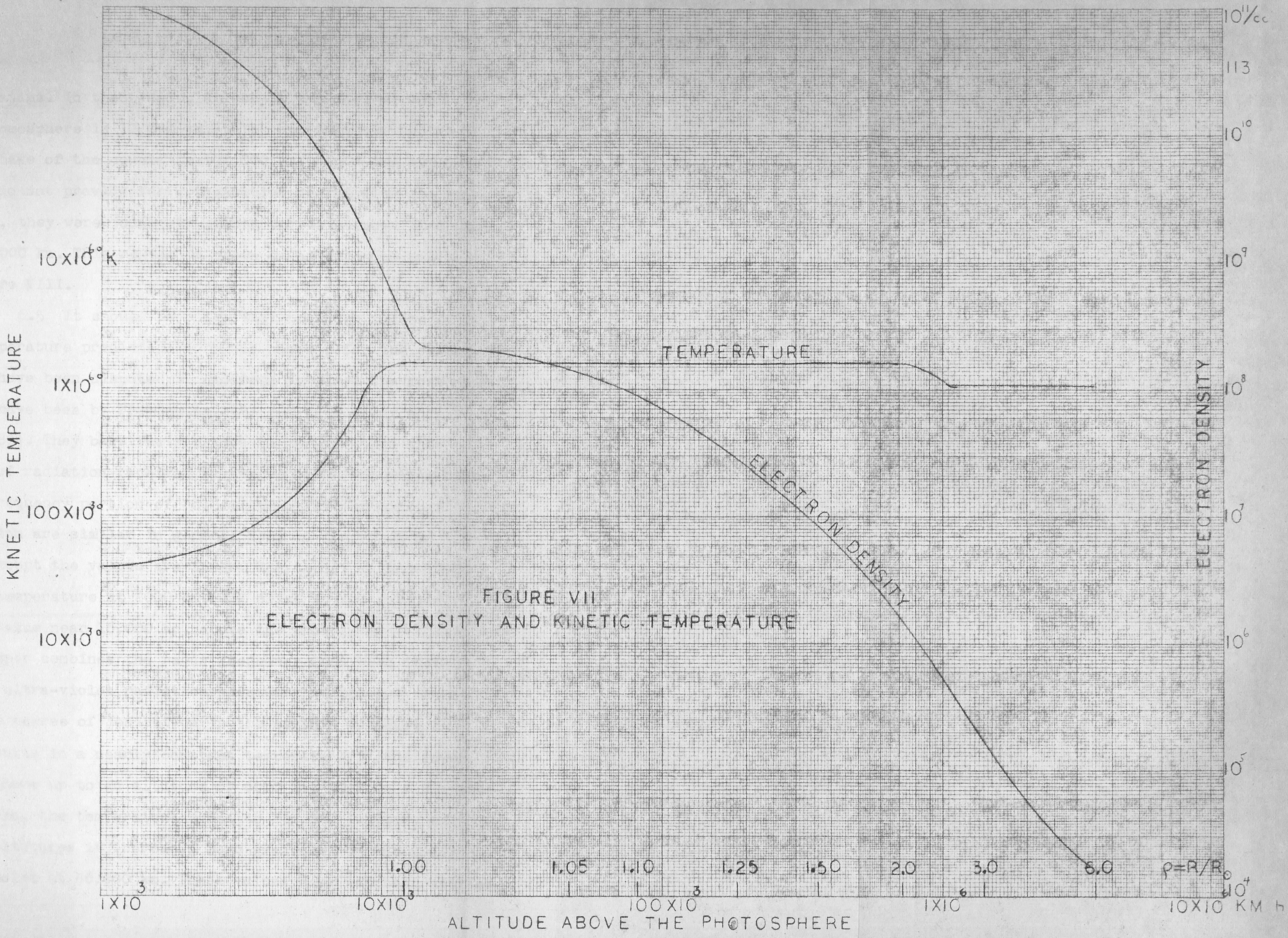


FIGURE VII
ELECTRON DENSITY AND KINETIC TEMPERATURE

solar radius. In these calculations it was assumed that the chromosphere is 12,000 km. thick and that the radius of the base of the corona is 697,000 km. Since the two models do not provide for a transition region from one to another, they were joined by smooth curves between 9000 km. and 14,000 km. This region is shown on an expanded scale in Figure VIII.

6.5 It should be noted that the high values of the temperature predicted by Thomas, above, for the chromosphere have been disputed by numerous authors. The most recent values have been by Piddington (88) and by Woolley and Allen (89). They base their arguments in part upon the amount of radiation received by the earth in the ultra-high radio frequency portion of the spectrum. The results of Piddington are similar in general form to those quoted above except the values are lower. They start with a 5000 degree temperature at 2000 km. and climb to the million degree value near 16,000 km. above the photosphere. The other paper combines the radio frequency data with that for the ultra-violet radiation necessary to produce the observed degree of ionization in the earth's ionosphere. This results in a nearly constant temperature in the lower chromosphere up to an altitude of 6000 km. Then in a distance of 1000 km. the temperature rises to 230,000 degrees. At higher altitudes it increases more gradually to the million degree point at 46,000 km. These values have not been adopted

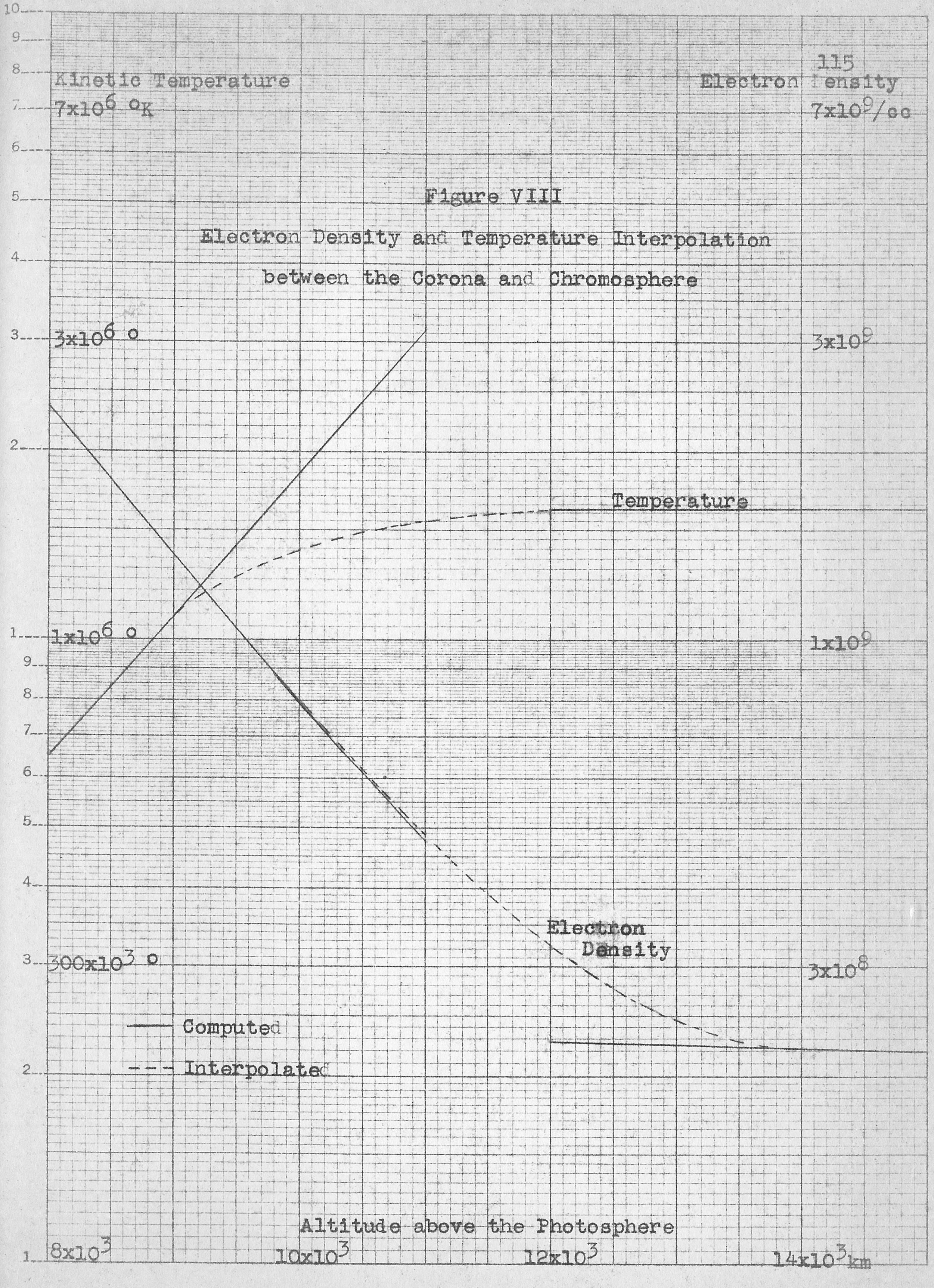


Figure VIII

Electron Density and Temperature Interpolation between the Corona and Chromosphere for this work as it is felt that the theory cannot be checked using data that are obtained in part as a result of the theory.

VII CALCULATIONS OF THE EMITTED RADIATION

7.1 In the last chapter the physical properties of the corona and chromosphere were presented. Here it was shown that these bodies can be considered to be composed of completely ionized hydrogen with a slight admixture of helium. In the corona the high kinetic temperature of one and one half million degrees will completely ionize the helium as well. For the somewhat cooler chromosphere the degree of ionization of the helium will depend upon the kinetic temperature and hence upon the altitude. Thus, only in the corona and in the top of the chromosphere may all the atoms be regarded as hydrogen like.

To find the coefficient of absorption by the classical method, as was done in this paper, it is necessary that the atoms must be treated in this manner. If they are not, then quantum mechanics must be used. When applying the kinetic theory of gases to completely ionized atoms, certain simplifications may be introduced. These represent the difference in mass between the electron and the heavier particles and the discreteness of electric charge. If the electrons are assumed to be the first component of the gas and either the protons or the helium nuclei the second, then the following are valid:

$$N = N_1 \geq N_2 \qquad (1)$$

$$m = m_1 \ll m_2 = m_H \text{ or } m_{He} \qquad (2)$$

$$Ze = -Ze_1 = e_2 \qquad (3)$$

where Z is the atomic number. For the above relation on the number densities, it was assumed that the gas as a whole is electrically neutral. Equality in this relation represents the case where none of the helium is ionized.

First consider the effect of these relations upon the mean number of collisions per second between the particles. These collisions will cause the field of radiation passing through the matter to be damped. The field induces an in-phase oscillation of the particles. Any collisions between them will destroy the in-phase motion and so attenuate the radiation. Equation 3-153 for the mean number of collisions per second can be simplified to

$$\nu = (4N_2 Z^2 e^4/3kT)\sqrt{\pi/2mkT}\ \ln\left[(4dkT/Ze^2)^2+1\right] \qquad (4)$$

In this expression it should be recalled that d represents the mean distance between the particles. To a good approximation it may be expressed in terms of the number density of the electrons. Since there are nearly 2N particles per cubic centimeter, each particle must occupy 1/2N cubic centimeters. If these volumetric elements are taken as cubic, then

$$d=\sqrt[3]{1/2N} \qquad (5)$$

The exact shape used is not critical since (a) the value for the expression in the logarithmic term is itself an approximation and (b) any variation in ν with different factors will be small as it is a logarithmic term. Making this substitution and neglecting the second term in the logarithmic factor as being negligible, one finds

$$\nu=(4N_2Z^2e^4/3kT)\sqrt{2\pi/mkT}\ \ln(4kT/Ze^2\sqrt[3]{2N}) \qquad (6)$$

(A substitution of numerical values shows the last to be true.) In the corona where van de Hulst assumed that there are 5 hydrogen atoms for every helium one, the number of collisions resulting from electron-proton collisions will be 5/4 the number of electron-helium nuclei collisions. Thus the damping from the two sources will be nearly equal.

It would be of interest at this point to compare the above result with that from other classical theories on the same subject. Smerd and Westfold (90) in their paper used the "free path method" for deriving the number of collisions per second between the particles rather than the "velocity distribution method" used here. In this approach to the problem each component is treated independently rather than collectively. Their result is

$$\nu = (NZ^2e^4/kT)\sqrt{2\pi/mkT}\ \ln(4kT\sqrt[3]{6}/Ze^2\ \sqrt[3]{N}) \qquad (7)$$

This differs from Equation 6 in two respects. As a result of using the "free path method" the factor 4/3 does not appear. Also in the logarithmic term they assumed that each molecule occupied a spherical volume element

$$(4\pi/3)(d/2)^3 = 1/N \qquad (8)$$

rather than a cubic one. It would appear from this that the two approaches to the problem yield essentially the same result. Hagen's derivation (91) of this quantity was based upon the "velocity distribution method" as is that in this paper. Hence his solution is identical to Equation 6.

Next consider the effect of the simplifying conditions (Nos. 1, 2 and 3) upon the mean plasma frequency Ω(Equation 4-6). This quantity now reduces to

$$\Omega^2 = 4\pi Ne^2/m \qquad (9)$$

which is the same as that of the mean plasma frequency introduced by Langmuir for an electron gas (Equation 4-7).

7.2 In addition to the simplifications given above, others are applicable over large parts of the sun. In Chapter IV it was seen how the conduction current induced by the oscillating radiation field could be expressed in terms of a complex dielectric constant. The real part of this constant represents the in-phase component of the motion of the particles with respect to the radiation field, while the imaginary part represents out-of-phase component or damping factor. In both parts of the formula for the dielectric constant the factor

$$\nu^2 + \omega^2$$

appears, where ω is the angular frequency of the radiation. Sample calculations show that

$$\nu \ll \omega \qquad (10)$$

for all regions and frequencies that are of interest. Under this condition and with the simplified form for Ω (Equation 9), the real and imaginary parts of the dielectric constant (Equations 4-14 and 4-15) become:

$$\epsilon_1 = 1 - Ne^2/\pi mf^2 \qquad (11)$$

$$\epsilon_2 = -Ne^2\nu/\pi^2 mf^3 \qquad (12)$$

If in addition the relation between ν and ω given above be applied to the two parts of the dielectric constant, it can be seen that

$$\epsilon_1^2 \gg \epsilon_2^2 \tag{13}$$

except in the region where the two terms in ϵ_1 are nearly equal.

The behavior of ϵ_1 with solar altitude is of interest. At very great heights N will be small and so ϵ_1 will be near unity. As the distance from the center of the sun decreases, ϵ_1 will decrease through zero to negative values. The value of zero will occur at that altitude where the mean plasma frequency equals the frequency of radiation. Thus there are two regions one above and one below the plasma frequency that must be considered.

Using the usual relationship between the index of refraction and the dielectric constant, it was shown in Section 4-4 that a complex dielectric constant implies a complex index of refraction. The real part is identical to the normal index of refraction while the imaginary part is proportional to the coefficient of absorption. In the two regions divided by the plasma frequency, the index of refraction simplifies to different values. In the high altitude region of positive ϵ_1 where Equation 13 is valid, the index of refraction (Equations 4-22 and 4-23) reduces to

$$\mu = \sqrt{\epsilon_1} \tag{14}$$

$$k = |\epsilon_2|/2\mu \tag{15}$$

In the region near the plasma frequency where the order of magnitude condition between the two parts of the dielectric constant is no longer valid, no simplification is possible. Passing to the region below the plasma frequency where ϵ_1 is negative and Equation 13 is again valid, the index of refraction may again be simplified to

$$\mu = |\epsilon_2|/2k \tag{16}$$

$$k = \sqrt{|\epsilon_1|} \tag{17}$$

At this point it will be of interest to compare these results with those from other theories. These other theories considered that all of the observed radiation came from layers above the plasma frequency. They assumed that there would be total reflection of any radiation that was incident upon this layer from either above or below. This is only true in the case of a refracting but non-absorbing medium. It was shown in the derivation of the path followed by a given pencil of rays in Chapter V that with absorption rays can never become perpendicular to the solar radius vector. Thus total reflection cannot occur. In comparing the results derived in this paper with those of others, only the region above the plasma frequency will be considered. For this region the coefficient of absorption on the basis of the preceding theory becomes (Equations 4-21, 6, 12 and 15)

$$K_f = (8NN_2Z^2e^6/3\mu cmkTf^2)\sqrt{2/\pi mkT}\,\ln(4kT/Ze^2\sqrt[3]{2N}) \qquad (18)$$

If this result is compared with that found by Smerd and Westfold (92) and by Hagen (93), it is found to be twice theirs. In addition it has already been noted that Smerd and Westfold have a different numerical factor in the logarithmic term (Equation 8). The factor of two comes from the different method of handling the effect of the radiation field upon the motion of the particles in the gas. It was assumed in Chapter III that the frequency of this field was high compared with the number of collisions between the particles. Others have treated it as a d-c field.

The work by Burkhardt, Elwert and Unsöld (94) also differs from the above by a slight amount. They find

$$K_f = (8NN_2Z^2e^6/3mkTcf^2\sqrt{2\pi mkT})\ln(kT/1.44Ze^2\sqrt[3]{N}) \qquad (19)$$

There is again the same factor of two and also a different numerical coefficient in the logarithmic term. Since they did not consider refraction this factor is also missing. Their result was based upon the atomic coefficient of absorption rather than the number of collisions per second from kinetic theory. A fourth paper on this subject by Townes (95) gives the same result as Burkhardt et al. except for the numerical factor in the logarithmic term. He has 1.64 instead of 1.44. His approach to the problem was also from the atomic absorption coefficient.

7.3 To simplify the formulae for the path followed by the radiation, the condition on the real and imaginary parts of the dielectric constant (No. 13) is replaced by

$$\left[\epsilon_1 - (y/R)^2\right]^2 \gg \epsilon_2^2 \tag{20}$$

over large parts of the sun. The quantity on the left side of the above varies in the same manner as does ϵ_1 with altitude. In the high altitude region the angle of incidence α (Equation 5-12) will reduce to

$$\sin\alpha = y/\mu R \tag{21}$$

This is the same as the expression for a ray in a refracting but non-absorbing medium (see Section 5-2). As the index of refraction is less than one, the ray will bend away from the center of the sun as one follows it back into the sun. Then at the level where

$$\epsilon_1 = (y/R)^2$$

the ray will turn sharply in along a radius vector. The angle with respect to the radius vector when Equation 20 is again valid will be

$$\tan\alpha = |\epsilon_2| (y/R)/2 \left[(y/R)^2 - \epsilon_1\right]^{3/2} \qquad (22)$$

The formula (No. 5-13) for the other coordinate, the colatitude angle Θ, of the ray does not simplify.

Finally turning to the expressions for the emergent intensity of the radiation as derived from the equation of transfer (see Section 5-5), considerable simplifications result in the partial derivative (Equation 5-21) of Θ with respect to y. This is used in finding the change in the cross-sectional area of a pencil of rays as it travels through the sun. At the high altitudes where

$$\epsilon_1 > (y/R)^2 \qquad (23)$$

and Equation 20 is valid, one finds

$$(\partial \Theta / \partial y)_R = \int_R^\infty (\tan\alpha / yR\cos^2\alpha)\,dR \tag{24}$$

In the lower regions where the inverse of No. 23 is true, the integral in that region becomes

$$(\partial \Theta / \partial y)_R = -\int (\tan\alpha / yR)\left\{\left[2(y/R)^2 + \epsilon_1\right] / \left[(y/R)^2 - \epsilon_1\right]\right\} dR \tag{25}$$

Thus the total integral for $(\partial \Theta / \partial y)_R$ is the sum of two parts, a positive one covering the high altitude region and a negative one in the lower region. The sum of these two parts becomes equal to zero at an altitude only a little below the turning point of the ray. This means that essentially all of the observed radiation comes from layers above the maximum value for α.

7.4 These simplified formulae will be used in the actual calculation of the theoretical emission from the sun. To make a comparison with the observed intensities reported by experimental workers, it is necessary to know the total amount of radiation emitted by the sun. A convenient measure of this radiation is the temperature that a black body of the size of the visual solar disk would have to be at in order to produce the observed intensity. Calling this temperature T_{fo}, it can be easily shown that (see Equation 2-2)

$$T_{fo} = (2/R_\odot^2)\int_0^{y_o} T_a y\,dy \tag{26}$$

where y is the projected solar radius.

7.5 Looking at Table I in Chapter I it is seen that the radio frequency spectrum covers the range from 30 megacycles per second through 3000 megacycles per second. As a result the theoretical radiation was computed at the frequencies of 30 mc, 100 mc, 300 mc, 1000 mc and 3000 mc (wavelengths of 10 m, 3 m, 1 m, 30 cm and 10 cm respectively). For each of these frequencies the emergent intensity of the central ray ($y = 0$) was found. This indicated, qualitatively, how the emergent intensity from the sun varies with frequency. Then to obtain a picture of the distribution of the emergent radiation across the solar disk, the intensity for various rays was calculated at 30 mc, 300 mc and 3000 mc. Finally a summation of the intensities emitted by the various rays gave the total emission. This agrees quite well with the experimental data in Table I.

7.6 The steps taken in finding the emission along any given ray were the same in every case. For the purposes of simplifying the calculation it was assumed that the corona and chromosphere were composed of pure hydrogen. The first step is to calculate the mean number of collisions per second ν between the electrons and protons as a function of solar altitude (Equation 6 with Z equal 1). For this purpose the values of N and T listed in Table III (Chapter VI) were used. The altitude was measured in terms of h (the height above the photosphere) in the chromosphere and in terms of ρ equal $R/R_{\odot}$ (the relative solar radius) in the corona. The results are shown in Figure IX. The bend at an altitude of about 12,000 km reflects the similar change in N and T there.

From these calculations it was possible to find the complex dielectric constant,

$$\epsilon = \epsilon_1 + i\epsilon_2$$

(Equations 9 and 10). At this point it became evident that additional values of h or ρ would be needed to evaluate ϵ_1 near its zero point. Only by so doing, was it possible to obtain a continuous variation of the complex index of refraction through this region. For this purpose it was assumed that N and T varied linearly between the values listed in Table III. Although this is not strictly accurate, it was felt that the errors introduced would not materially affect the end result.

The values for the complex index of refraction, $n=\mu+ik$, quickly followed (Equations 4-22 and 4-23, 14 and 15, or 16 and 17 depending upon the magnitudes of ϵ_1 and ϵ_2). A summary of these results are shown in Figures X and XI. In looking at these figures it should be realized that μ does not go to zero but merely approaches that

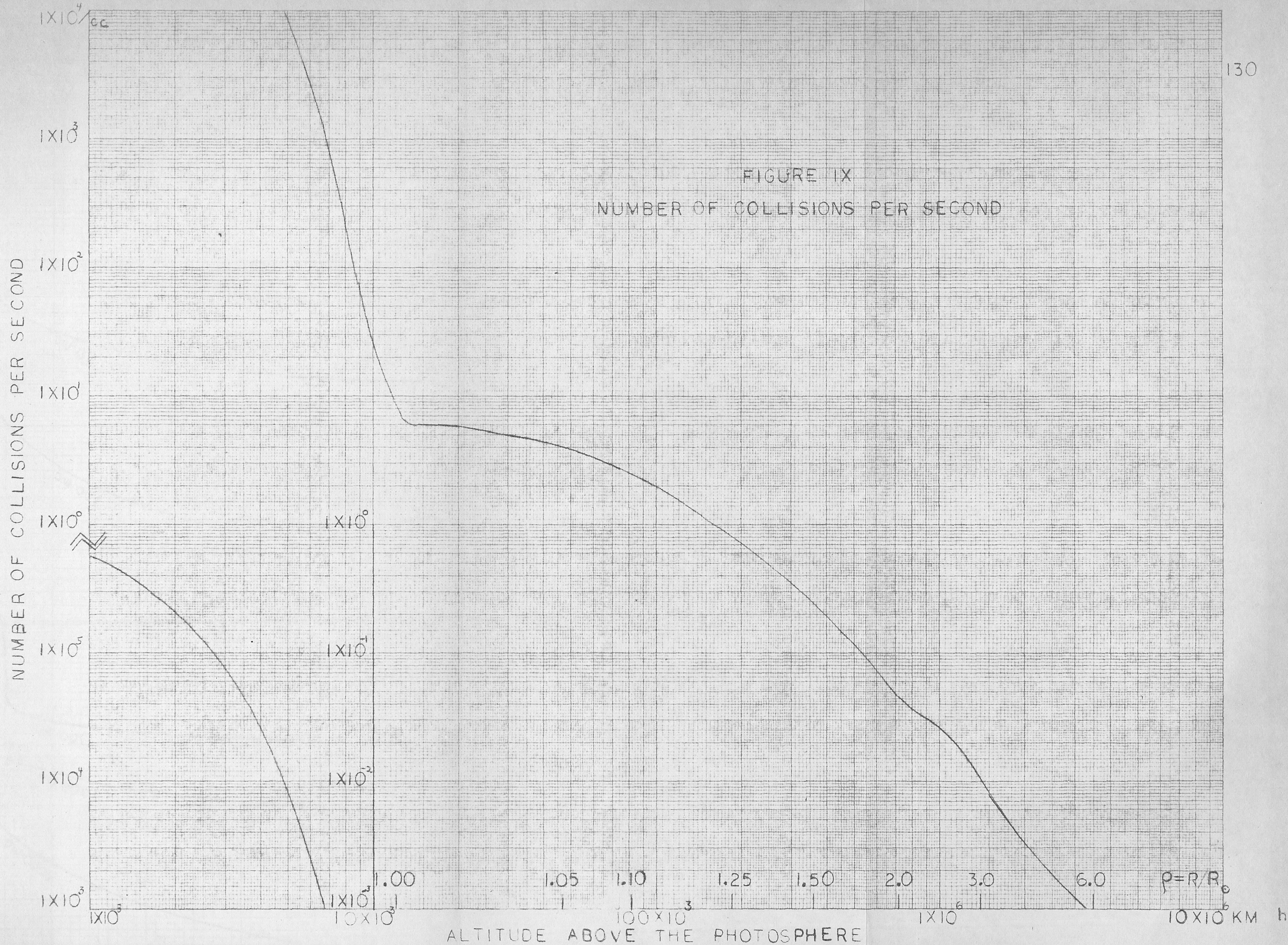


FIGURE IX
NUMBER OF COLLISIONS PER SECOND
NUMBER OF COLLISIONS PER SECOND
1X10⁴/cc
1X10³
1X10²
1X10¹
1X10⁰
1X10⁵
1X10⁴
1X10³
1X10⁰
1X10⁻¹
1X10⁻²
1X10⁻³
1.00
1.05
1.10
1.25
1.50
2.0
3.0
6.0
ρ=R/R☉
1X10³
10X10³
100X10³
1X10⁶
10X10⁶ KM
h
ALTITUDE ABOVE THE PHOTOSPHERE

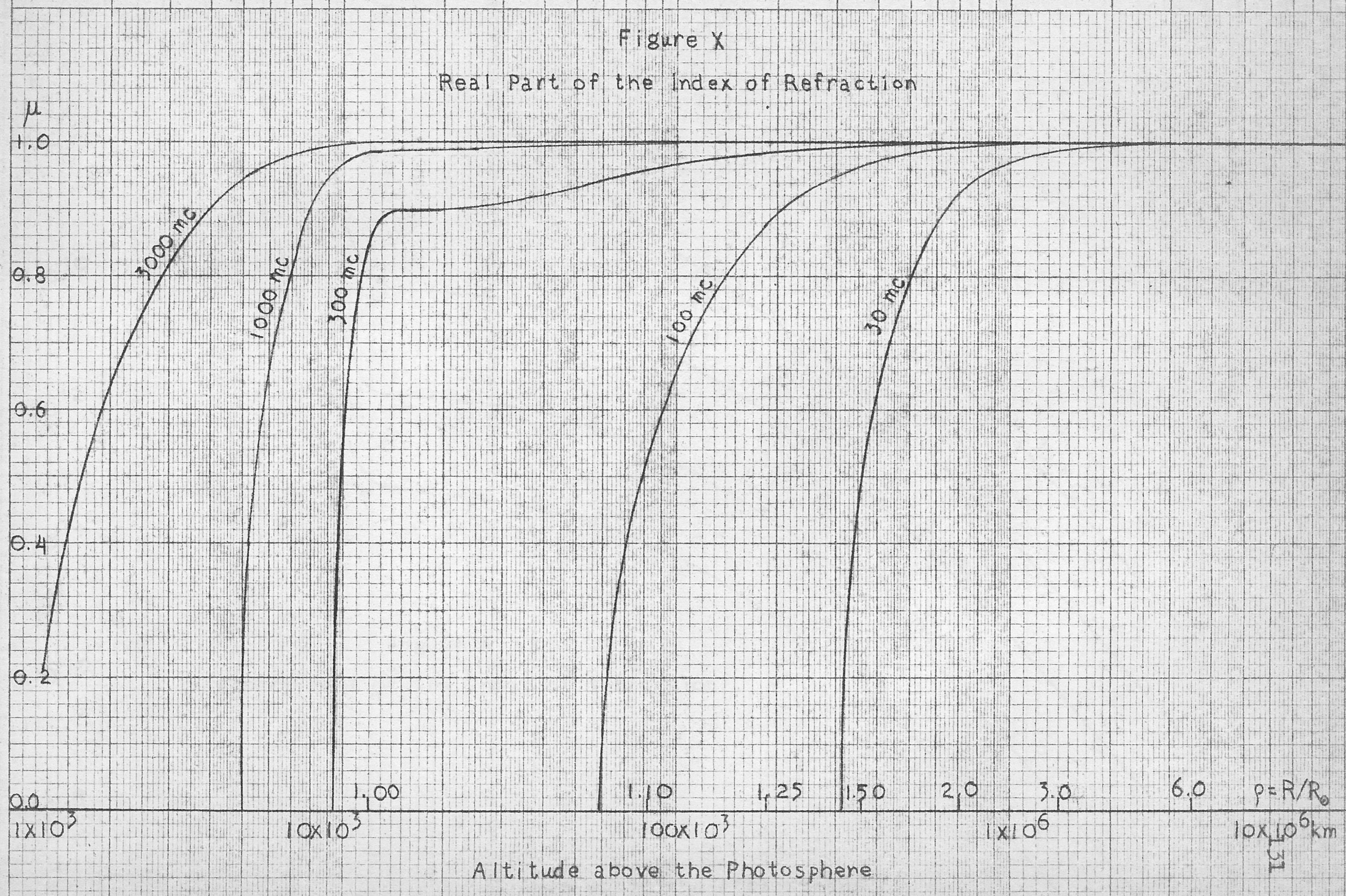


Figure X
Real Part of the Index of Refraction
μ
1.0
0.8
0.6
0.4
0.2
0.0
3000 mc
1000 mc
300 mc
100 mc
30 mc
1.00
1.10
1.25
1.50
2.0
3.0
6.0
ρ = R/R₀
1X10³
10X10³
100X10³
1X10⁶
10X10⁶ km
Altitude above the Photosphere

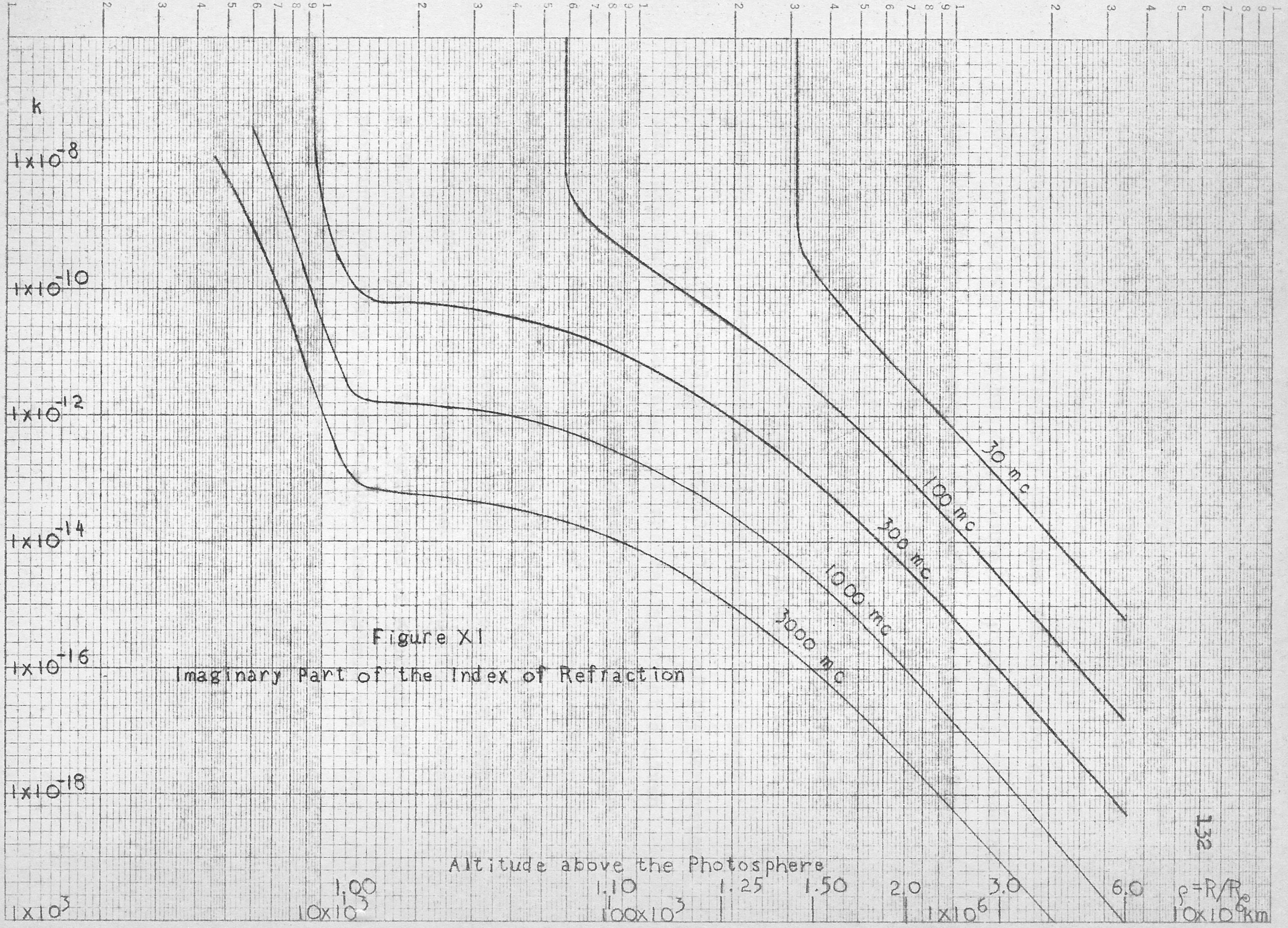


Figure XI

Imaginary Part of the Index of Refraction value. The sharp decrease observed starts just above the altitude where the frequency of the radiation equals the plasma frequency. It continues to decrease very sharply through the plasma frequency and then changes more slowly at still greater depths. The figure for μ clearly indicates how the plasma frequency is a function of altitude. The curves of the imaginary part of the index of refraction also show the effect of the plasma frequency. The almost perpendicular rise occurs at this point. The shape of the higher frequency curves again reflect the electron density and temperature variations used in this paper.

Having completed the calculations of the parameters that are basic to the study of the radiation at a given frequency, attention is now turned to the paths followed by the rays. In finding the angle of incidence α along a given ray as a function of altitude (Equations 5-12, 21 or 22), it again became necessary to add more steps to the path. Only by so doing was it possible to obtain a complete picture of the variations of α about its maximum. For every ray calculated the maximum value of α was within a few minutes of arc of 90 degrees. This may be seen by substituting the values for ϵ_2 into the equation for maximum α, namely

$$\tan\alpha = y/R\sqrt{|\epsilon_2|} \qquad (27)$$

(from Equation 5-12). Taking a typical large value for ϵ_2 from Figure XI, it is seen that tan α will be greater than about 10,000 for reasonable values of y/R. In computing the path followed by each pencil of radiation, it was followed down from the top of the corona ($\rho = 6$) to a level only a little below the maximum value for α. Calculations on each ray showed that the contribution to the observed intensity from still lower levels was insignificant.

The other coordinate of the path, namely θ (Equation 5-13), was easily found by numerical integration. For this purpose, and also in subsequent integrations, Simpson's one third rule or occasionally the trapezoidal rule was employed. It was felt that the inaccuracies of these laws were not great enough to justify the employment of more complicated rules.

The results from this determination of the paths are portrayed in Figure XII. On the graph the value of the projected solar radius for each of the rays is expressed in relative terms,

$$\eta = y/R_{\odot}$$

A study of the graph will show that the rays for 30 mc radiation bend quite sharply towards large values of α just preceeding the point where the angle decreases again. This same phenomenon occurs at the higher frequencies but

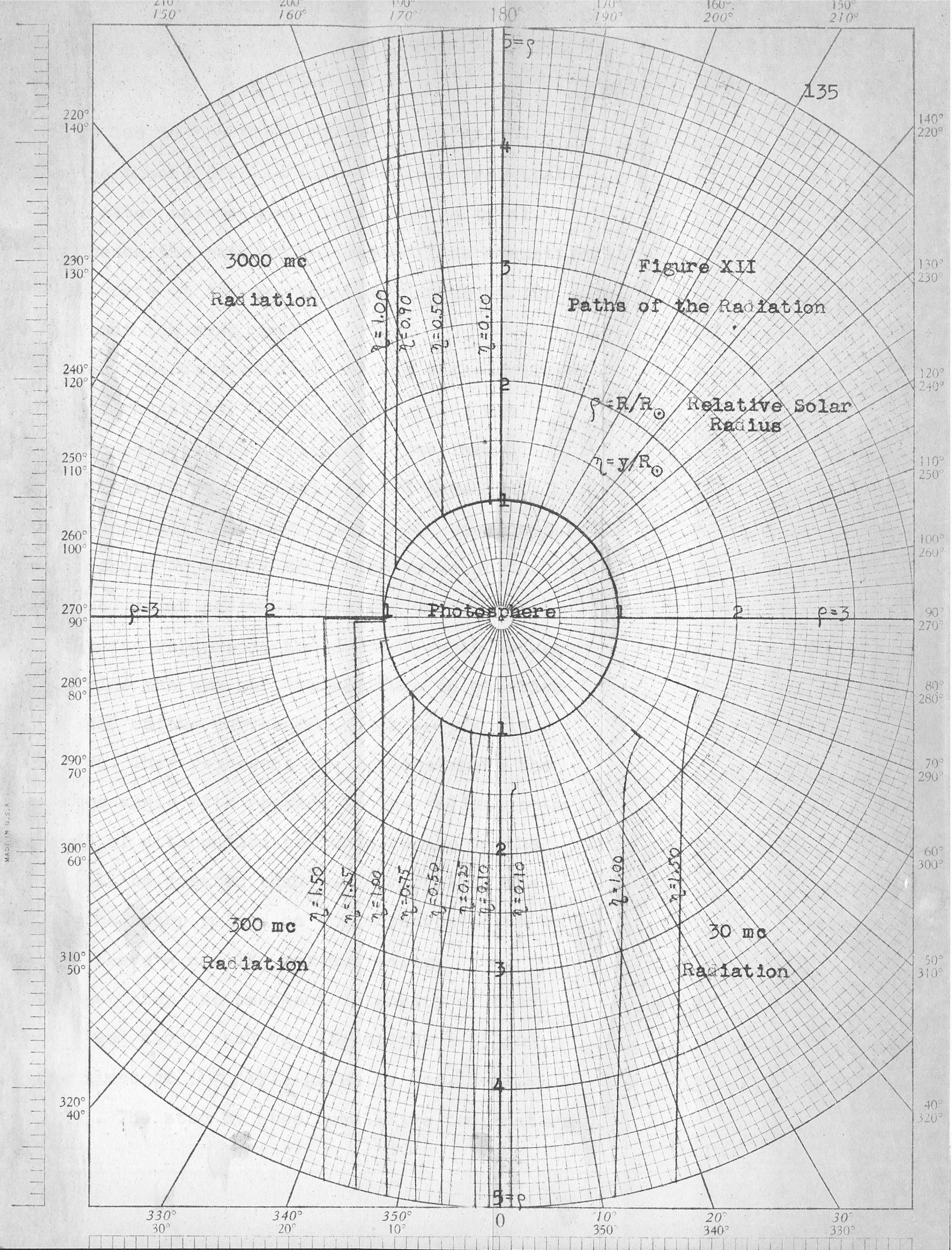


Figure XII

Paths of the Radiation

does not show up due to the small scale on the graph. In this point the theory of this paper differs from that in others. The latter assume that there will be total reflection and hence the path of the ray will be symmetrical about the point where it is perpendicular to the radius vector. Such is not the case here.

Following the determination of the paths followed by the pencils of radiation, the optical depth τ was computed (Equation 5-16). The integration was inward along the ray either to such a depth that τ exceeded 7.0 or to the lowest altitude for which α had been computed. The results are shown in Figure XIII. The value of the relative projected solar radius η is given on each of the rays. Each curve represents the optical depth between an observer outside the sun and the altitude in question. Notice the "cusps" on the curves for which $\eta = y/R_{\odot}$ is greater than zero. They represent the regions where the ray is at a large angle with respect to the radius vector. Hence the amount of attenuation per unit change in the radius vector will be greater than for a ray along the radius vector. The sharp points on these curves occur at the altitude where α has its maximum value.

As a last step the emergent intensity of the radiation was found along each ray (Equations 5-25 and 5-26 with the values of $(\partial \Theta / \partial y)_R$ given by Nos. 5-21, 5-22, 24 and 25). The step by step cumulative temperature

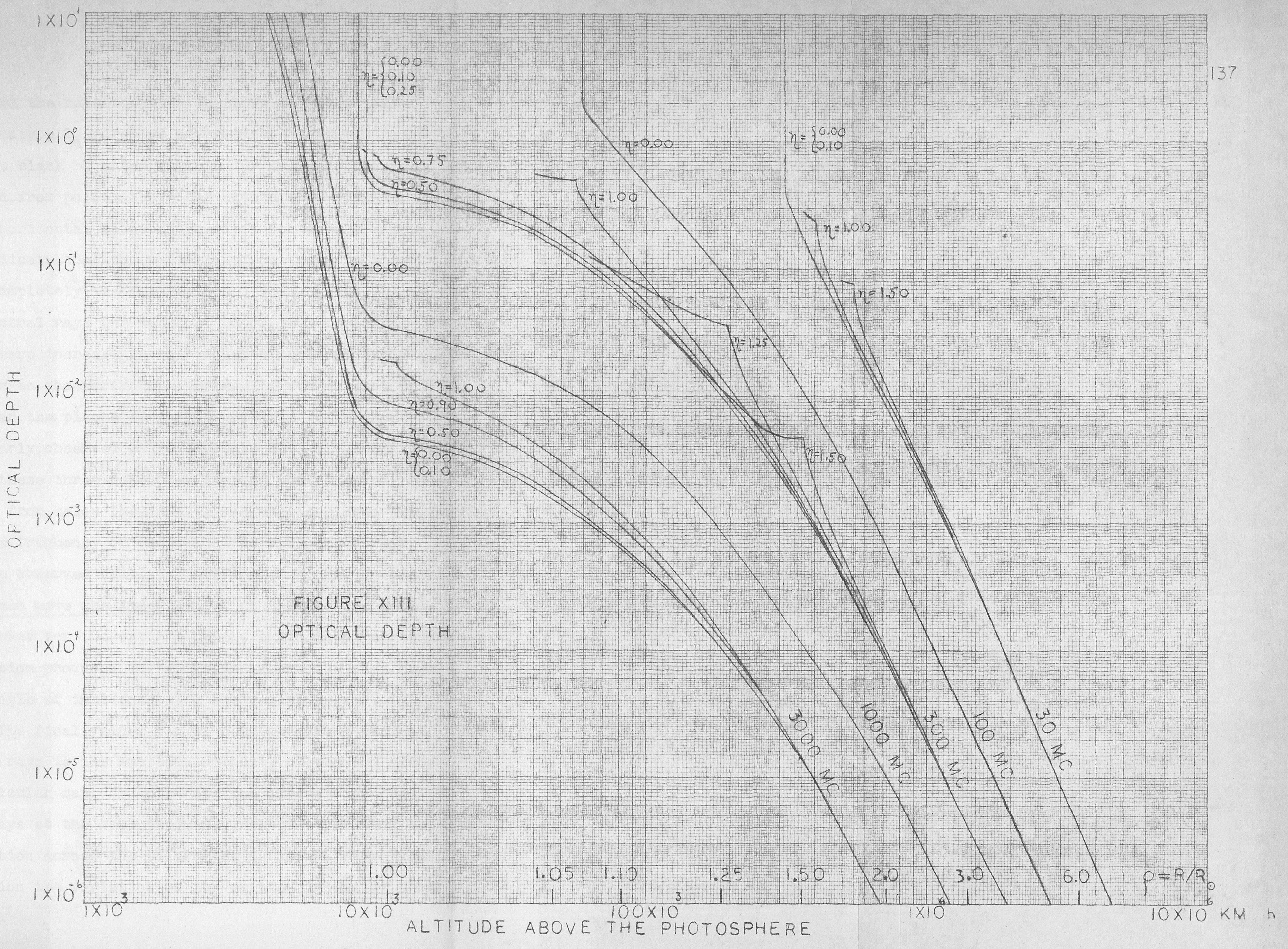


FIGURE XIII
OPTICAL DEPTH
OPTICAL DEPTH
ALTITUDE ABOVE THE PHOTOSPHERE
1X10^1
1X10^0
1X10^-1
1X10^-2
1X10^-3
1X10^-4
1X10^-5
1X10^-6
1X10^3
10X10^3
100X10^3
1X10^6
10X10^6 KM
h
1.00
1.05
1.10
1.25
1.50
2.0
3.0
6.0
ρ=R/R☉
3000 MC
1000 MC
300 MC
100 MC
30 MC
η=0.00
η=0.10
η=0.25
η=0.75
η=0.50
η=1.00
η=0.90
η=1.25
η=1.50

for each of the rays computed is shown in Figure XIV. The temperature T_a given at any altitude represents the equivalent black body temperature for the radiation emitted by the sun from points above the altitude in question. Thus the horizontal lines on the left portions of the curves indicate that the radiation from these regions will be completely absorbed before it can escape. A study of the central rays ($\eta = 0.00$) at 30 mc, 100 mc and 300 mc shows a sharp increase in the amount of emitted radiation just prior to the horizontal portion of the curves. This occurs near the plasma frequency and is a reflection of the similarly observed sharp increase in k at these levels. Only for these three rays does some of the observed radiation come from below the level at which the plasma frequency equals the frequency of radiation. For all other rays all of the observed radiation comes from higher levels. These curves have the same "cusps" on them as was noted in the curves for the optical depth. Again they represent the radiation produced in the region where the rays have a large angle of incidence with the radius vector.

The final values for T_a obtained along each of the above rays is the emitted intensity of the sun along that particular ray. By comparing the intensities from several rays at the same frequency, the distribution of the radiation across the solar disk is known. From this distribution the variation of the emitted intensity with

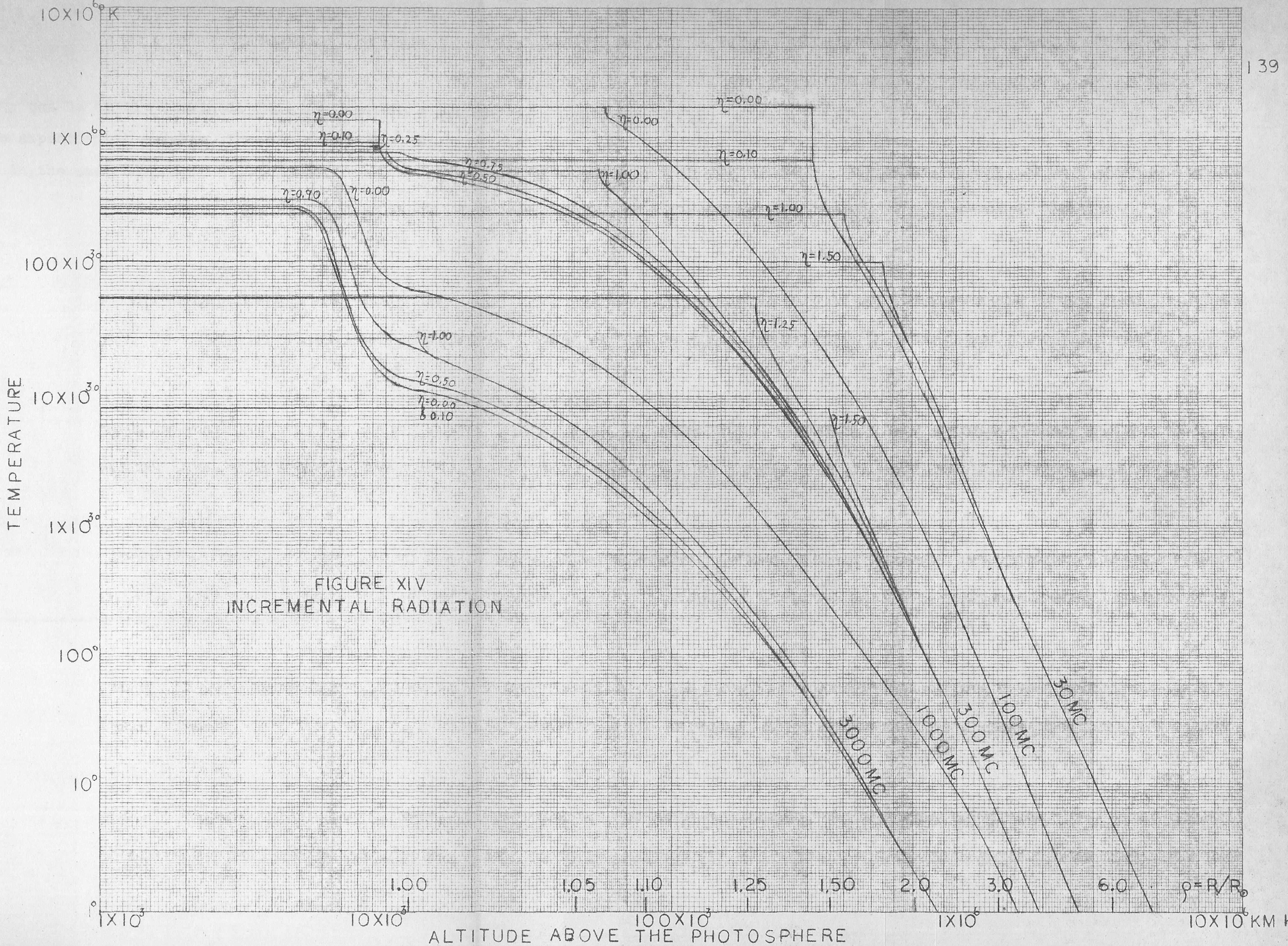


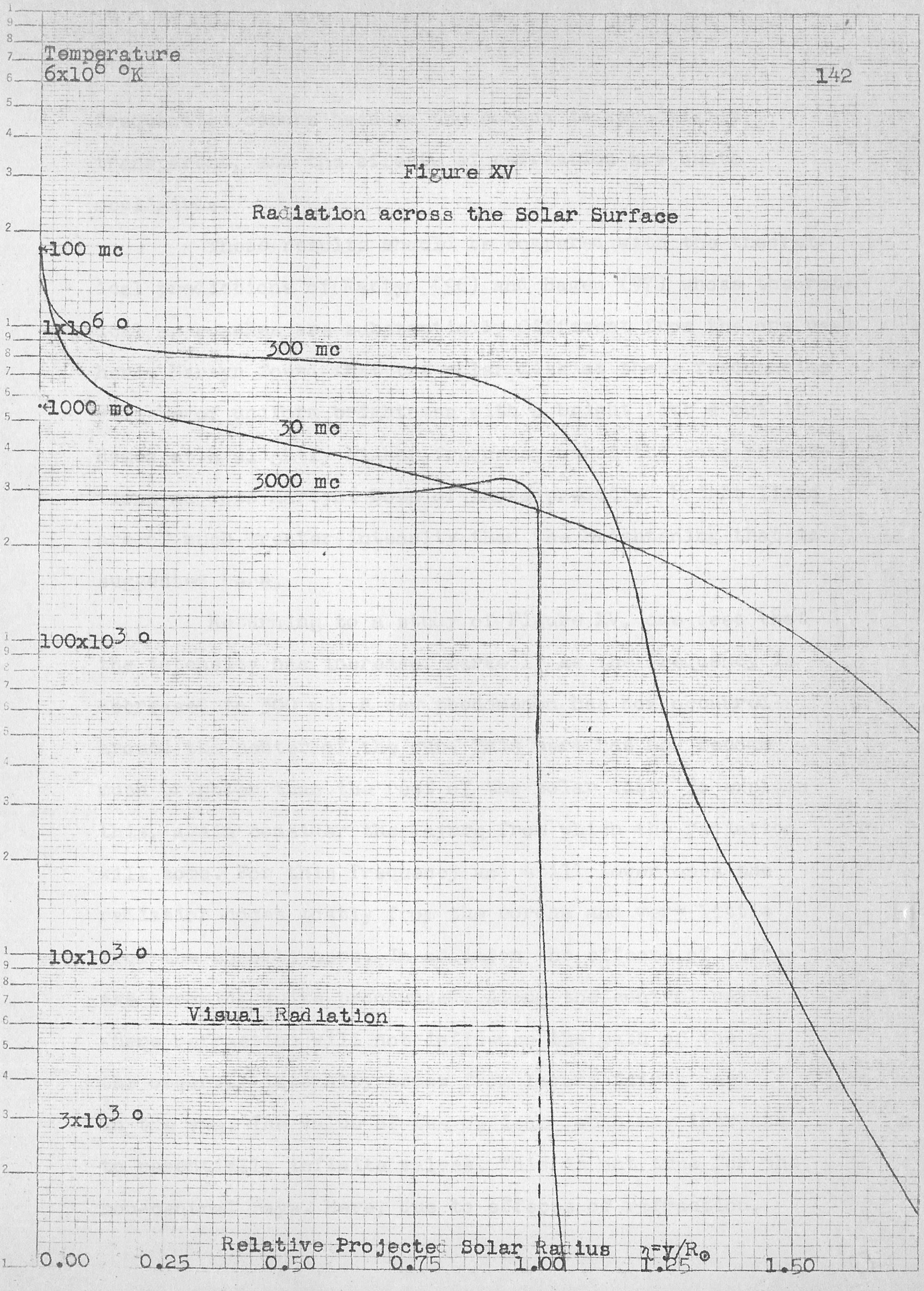
FIGURE XIV
INCREMENTAL RADIATION

frequency can be easily obtained. This may then be compared with the experimental results. These subjects will be treated in the next chapter.

VIII COMPARISON OF THE THEORETICAL AND OBSERVED RADIATION

8.1 From Figure XIV in the last chapter one saw how the emitted intensity was produced along the various rays in the sun. The maximum value shown for each of these rays equals the intensity that will be observed as coming out of the sun along that ray. From a series of these rays at the same frequency the distribution of intensity across the solar sphere is known. The results are shown in Figure XV at 30 mc, 300 mc, and 3000 mc. The center of the sun is on the left edge of the graph. The value of η equal 1.00 corresponds to the limb of the visual solar disk. A study of these curves reveals how the distribution of emitted intensity varies with frequency.

At 3000 mc the disk appears to be of nearly uniform brightness. The center is observed to be only a little darker than the edge. This dark center may be explained by considering the altitude from which the radiation comes. On the average the energy along the central ray comes from a somewhat lower altitude than does the radiation from the edges of the disk. Since there is a positive temperature gradient within the chromosphere, the energy for the central ray comes, on the average, from a somewhat cooler region than does the energy from near the limb. The sharp limb at these


Temperature
6x10^6 °K
Figure XV
Radiation across the Solar Surface
100 mc
1x10^6 °
300 mc
1000 mc
30 mc
3000 mc
100x10^3 °
10x10^3 °
Visual Radiation
3x10^3 °
Relative Projected Solar Radius
0.00
0.25
0.50
0.75
1.00
1.25
1.50

frequencies occurs because the corona is essentially transparent, and the effects of refraction are quite negligible.

These results should be compared with the theoretical predictions of Martyn (96) and Hagen (97) whose theoretical approaches are substantially different from the one used here. Martyn found that the solar disk at these frequencies will be of uniform brightness with a bright limb about it. Hagen at a slightly higher frequency makes the same predictions as does Martyn. Both of these predicted bright limbs are of much greater intensity than the brightening that is suggested here.

Returning to a study of Figure XV, one sees that the intensity has increased markedly as the frequency is decreased to 300 mc. A new phenomenon has now occurred around the center of the sun. This part has now become much brighter than the rest of the solar disk. To explain this, again consider the layers from which the radiation will come. For this frequency and still lower ones the radiation comes mostly from the corona and very little from the chromosphere. In addition the effects of refraction are becoming quite pronounced. Along the central ray where refraction will not influence the path of the ray, the observed intensity comes from layers fairly deep in the sun. The value of the optical depth may increase to essentially infinite values. This is not true for the non-central rays. Here, due to effects of refraction, a ray followed back into the sun will bend away from the center of the sun until it is almost perpendicular to the radius vector. It then turns sharply in along the radius vector. However, it was shown in Section 7.3 that essentially all of the observed radiation comes from layers above this sharp bend. For these rays the effective optical depth does not reach as high a value as along the central ray. Thus it is seen that the intensity emitted along the central ray will be greater than that along any other ray. A part of this effect is cancelled by the obliqueness of the non-central rays to the radius vector. This does not, however, completely cancel the increase in absorption with depth. For the region in the center of the sun, where most of the radiation originates in regions near the plasma frequency, the coefficient of absorption and so the optical depth change rapidly with altitude. Hence the change in the observed intensity between the central ray and one only a short distance away will be quite great.

Near the limb the intensity is seen to fall off fairly rapidly. However the limb is seen to be less well defined than at the higher frequencies. This ill-defined limb is a property of radiation from the corona. For here the change in density with altitude is much slower than in the chromosphere. As a result the observed intensity comes from a much larger region than in the chromosphere.

The above results agree reasonably well with Martyn's except at the center of the disk. He obtains a solar disk having a slightly dark center, much like the 3000 mc distribution predicted here. His limb has the same general properties as the one described here.

The bright central portion of the curve described here does not occur in those theories (Martyn's, Smerd and Westfold's (98) etc.) which consider the radiation to be totally reflected. For in the theory presented in this paper the added intensity that produces the very sharp central region comes mostly from below the plasma frequency. By not considering this region, only a slight increase would be noticed at the center.

This discrepancy could be resolved experimentally by use of interferometer type antennas. With a series of narrow lobes sweeping the sun an increase in intensity should be expected at the center of the sun according to the present theory. This increase would correspond to the highly intense central region.

Finally turning to the curve for 30 mc in Figure XV one sees that the observed radiation has decreased once more. The two items that were noted in the curve for 300 mc have become more pronounced. The high intensity central portion has become much stronger than what it was before. In addition the limb is no longer well defined. In fact the edge of the sun gradually trails off as greater and greater solar radii are reached. Looking at Martyn's results once more, it may be noted that the two agree except at the center. As was noted above, a bright center would not be expected in a theory that considers total reflection.

Experimental data taken during eclipses give the best information on the distribution of energy across the solar disk. Reber (99) working on 480 mc found that the decrease in observed intensity during the partial eclipse of November 23, 1946 was equal to the fraction of the visual solar disk covered. This result is in agreement with those predicted above at 300 mc. Sanders' observations (100) of the eclipse of July 2, 1945 on 9000 mc seem to indicate that the radio frequency disk again equals the visual disk. He also obtained some evidence of limb brightening. As the magnitude was not stated, it is not possible to find out whether the radiation is in agreement with the theory predicted here or with that of Martyn and of Hagen.

8.2 To obtain qualitative confirmation of the theory, the distribution of the total emitted radiation with frequency must be considered. This was found by integrating the distribution shown in Figure XV over the solar surface (Equation 7-26). The results are shown on Figure XVI together with a number of the experimental results quoted in Table I (Chapter I). The number in

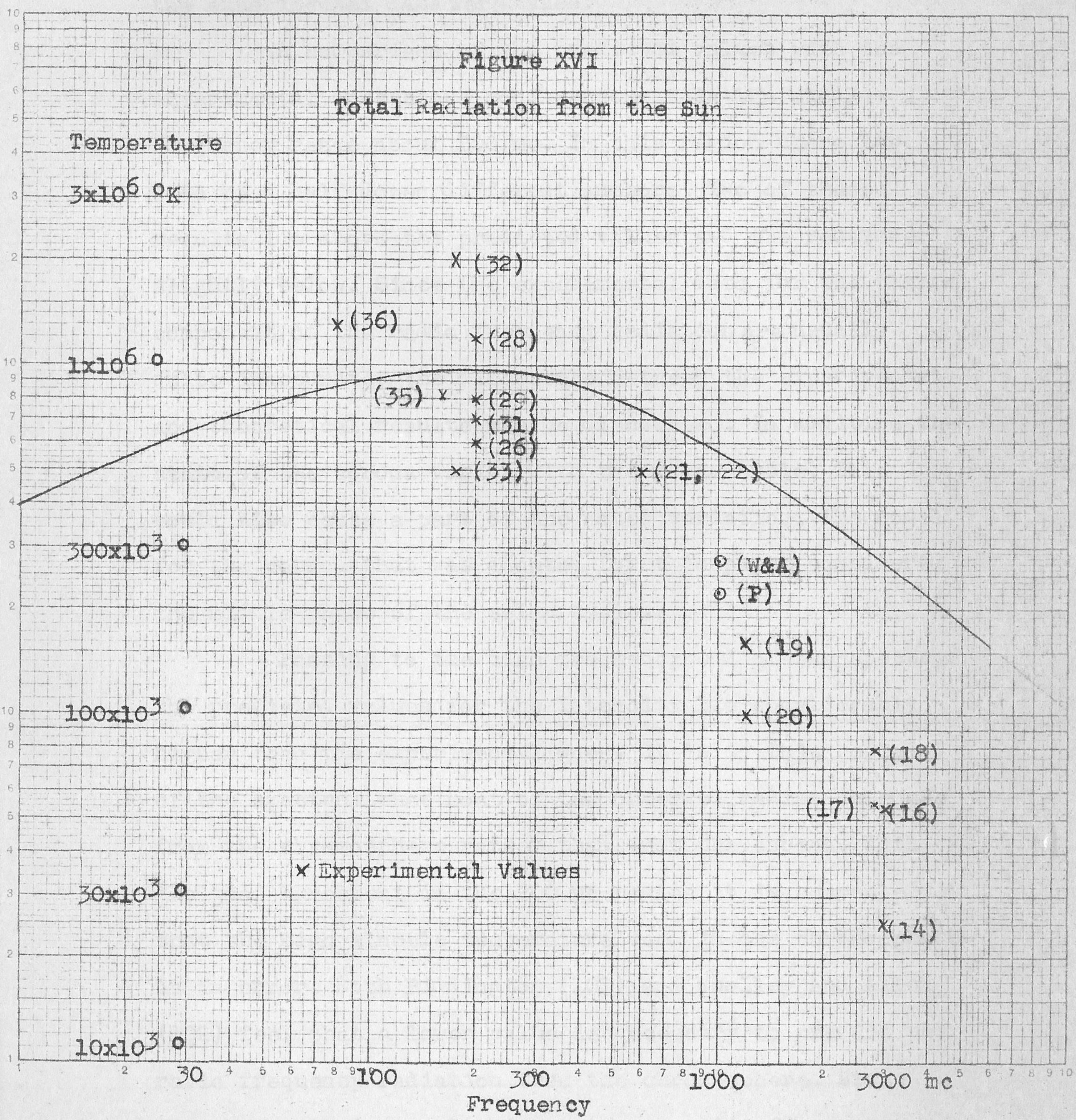


Figure XVI
Total Radiation from the Sun
Temperature
3x10^6 °K
1x10^6
300x10^3
100x10^3
30x10^3
10x10^3
(32)
(36)
(28)
(35)
(29)
(31)
(26)
(33)
(21, 22)
(W&A)
(P)
(19)
(20)
(18)
(17)
(16)
(14)
Experimental Values
30
100
300
1000
3000 mc
Frequency

parenthesis by each cross is the reference from which the experimental data were taken.

A comparison of the predicted values with the experimental ones shows good agreement in the middle frequency range about 200 mc. In this region there are data by a half dozen different authors. Two of these results lie above the predicted values and the remainder slightly below. Since the experimental data for the thermal component of the radio frequency radiation are usually quite inaccurate, the difference between the experimental and theoretical results may be discounted. In addition the state of the corona varies with the solar cycle. This would also cause a part of the noted variation. The model for the corona that was adopted for this paper considered the middle phase of the solar cycle.

Passing to the high frequency end of the spectrum, the predicted values for the intensity lie considerably higher than the experimental ones. It is in this region of the spectrum that most of the observed radiation comes from the chromosphere rather than from the corona. With the good agreement in the 200 mc region it is suspected that the high kinetic temperature used in the chromosphere is in error. Both Piddington (101) and Woolley and Allen (102) feel that a lower value is necessary to predict the radio frequency radiation from the chromosphere. Since they based their results in part upon radio frequency observations, it was felt that they could not be used in the present paper. However, the emitted intensity on 1000 mc was computed for the central ray on the basis of their theories. The results, marked by their initials an Figure XVI, show close agreement with the experimental points. If a curve were drawn through the above computed results at 300 mc and then through the values obtained from their theory at 1000 mc, it would intersect the experimental points in this region of the spectrum. Thus it must be concluded that the values of temperature suggest by Thomas (103), and used in this paper, are too great for predicting the observed radio frequency radiation.

It is regretable that no experimental data is available on the low frequency end of the spectrum. It would be of interest to obtain a confirmation of the predicted falling off of the intensities at these frequencies. This same drop in intensity is also predicted by Martyn.

In conclusion it may be said that the radiation predicted by this paper agrees in the middle frequency region with the experimental results but not at the high frequency end. This is undoubtedly due to the high kinetic temperature chosen for the chromosphere. The somewhat lower value required to explain the radio frequency radiation is at variance with certain observations in the optical region of the spectrum.

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