OPTIMAL CONTROLLER DESIGN METHODS FOR LINEAR SYSTEMS WITH UNCERTAIN PARAMETERS--DEVELOPMENT, EVALUATION, AND COMPARISON

Thesis by

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ABSTRACT

In many physical systems, an accurate knowledge of certain parameters is very difficult or very expensive to obtain. The designer of a remotely piloted vehicle flight control system, for example, frequently has available little data regarding aerodynamic coefficients, due to a lack of wind tunnel tests. Commonly used controller design methods, based on nominal values of plant parameters, often fail to achieve a satisfactory design in the face of parameter uncertainty.

In this work two methods have been developed for the design of linear, constant gain feedback controllers for systems with uncertain parameters:

1) The multistep guaranteed cost control method is based on the concept of minimizing an upper bound of a cost functional in the face of parameter uncertainty. An algorithm has been developed to analyze the effect of parameter uncertainties on closed-loop system stability. An extension of this algorithm results in a technique for choosing constant feedback gains which guarantee a stable closed-loop system that possesses some of the desirable features of optimally designed control systems.

2) The minimum discrete expected cost method is based on the concept of minimizing the expected value of a cost functional overa finite number of points in the range of parameter uncertainty. The design process makes use of statistical information about the uncertain parameters and incorporates in its cost functional whatever effects accompany a large departure in the plant parameters from their nominal values.

An extensive comparison of these two methods, together with the guaranteed cost control method, the minimax method, and the uncertainty weighting method, has been done in the context of the design of a fifth-order lateral autopilot for an RPV with uncertain aerodynamic coefficients. All five methods were evaluated on the bases of performance and design effort required. Both new methods were found to avoid some of the drawbacks associated with other techniques. The two newly developed methods are easy to implement and offer the designer tools for use in real control system design.

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Introduction

In many cases, a control problem can be represented as a system of ordinary differential equations characterized by a finite number of parameters. These parameters characterize various physical properties of the system which one would like to control. In practice, an accurate knowledge of parameters may be very difficult or expensive to obtain. In addition, certain system parameters may vary or may be changed during the period of operation. Modeling airplane dynamics, for example, involves the estimation or measurement of aerodynamic coefficients, as well as the making of assumptions regarding actuator dynamics and flight conditions. Accurate values of some of the aerodynamic coefficients (such as C_{n_p}, for example [1]) are very difficult to obtain by analysis. Many of the aerodynamic coefficients will vary with flight conditions such as dynamic pressure $\bar{q}$ and Mach number M. Some airplane parameters, such as mass and center of gravity, may change during the flight. A common objective of the control system designer is to achieve a robust closed-loop system, i.e., a system that maintains an acceptable performance in the face of uncertainties and variations in parameters.

Parameter uncertainties can be dealt with in several ways. They can often be reduced substantially through extensive testing or through real-time or non-real-time parameter identification techniques. Static and dynamic wind tunnel tests, while complicated and expensive, can produce a better estimate of flight vehicle parameters than that acquired through analysis only. The whole field of adaptive control is concerned with identifying system parameters in real-time and adapting the control law to cope with changes [2]. In the last decade, the subject of non-real-time system identification has been receiving considerable attention. It is possible to improve estimates of system parameters through flight tests and advanced identification techniques [3],[4]. Programming the control law (i.e., changing gains during operation) is another way to handle those cases where the parameters are varying during the period of operation, assuming that something is known about the variations.

A different approach to the problem is to accept parameter uncertainties at their a priori levels, and to design a control system that will be robust to parameter variations. It is this approach that is investigated in this work. Two basic directions may be taken in the design of feedback controllers for a system with some given level of parameter uncertainty.

The first is based on frequency domain design methods. A great deal is known about the design of a single-input, single-output feedback system in the presence of parameter uncertainty [5],[6]. The notions of gain and phase margin are well developed for those cases. The use of Bode or Nyquist plots provides a means of assessing the above quantities, and makes it easy to design the required control law. Unfortunately, in multivariable systems, the situation is not so simple. Considerable effort has been expended recently to develop frequency domain techniques which will work in multi-input, multi-output cases. The sequential return difference method [7], and methods based on the concepts of singular values and singular vectors [8] are only two examples of the work in this direction.

The second basic direction draws upon time domain techniques, specifically, the state space approach, to produce controller design methods. Optimal control concepts are common to all the methods using the state space approach. A central feature of these methods is a single figure of merit, called a cost functional (or performance index). The design objective is to find the control law so as to minimize this cost functional. The state space design methods can be grouped mainly into three categories: the stochastic approach, the sensitivity (first-order perturbation) approach, and the bounded cost approach.

In the stochastic approach, an a priori probability distribution is assumed for the uncertain parameters, and the control law is chosen to minimize the expected value of some performance index. Some methods have treated the variable parameters as random variables, and require the assumption that parameter perturbations are small [9]-[12]. The small perturbation assumption is waived in references [13]-[15]; but this causes an increase in the computational load and leaves open the question of the practicality of these methods.

Sensitivity methods are based on the assumption that the parameter uncertainty is small, so that first-order perturbation equations can be obtained, from which a controller is designed to minimize a performance index that includes some measure of trajectory sensitivity [16]-[18]. This approach is not valid in cases where the range of uncertainty is not small. Usually, the implementation of these methods requires additional state variables, an increase in computational complexity, and the use of extra trajectory sensitivity states for feedback [19].

In the bounded cost approach, a cost functional is defined, and a control law is sought so as to minimize an upper bound on the cost functional over the whole range of parameter uncertainty. The minimax design method [20] and the guaranteed cost control design method [21] are two of the methods in this category. A major drawback of this kind of approach is the tendency to produce large feedback gains and an overcontrolled closed-loop system [22],[23].

To date no one design technique for systems with uncertain parameters has received widespread acceptance from control system designers. In reference [19], a comparative assessment of seven methods based on an optimal control approach was made. Most of these methods were found to be at least somewhat burdensome computationally, and most did not produce control system designs judged to be significant improvements over designs which assume precisely known parameters.

The objective of this research was to investigate some of the above approaches and to develop easily implementable design methods for linear deterministic systems with uncertain (or varying) parameters. In particular, an optimal control approach was taken, and a state space notation was employed. Attention was limited to design methods that will produce a linear constant gain feedback controller, such that the closed-loop system behavior is acceptable for all values of the uncertain parameters within a specified bounded region (not necessarily small). Two such methods are suggested here, one from the bounded cost category and the other from the stochastic category. For each method, a general theory is presented and a design procedure is given. Special attention has been given to the areas of practicality, implementation, and computational load. Each design method is accompanied by a detailed algorithm and an example. A comparison of several existing techniques, together with the two new ones, in the context of the design of a lateral autopilot for a rudderless remotely piloted vehicle with uncertain aerodynamic coefficients was carried out. Qualitative, as well as quantitative results, are presented, and some conclusions as to the advantages and drawbacks of each method are given.

The structure of the thesis is as follows: In the first chapter the general guaranteed cost control design method is presented. Using the work by Chang and Peng [21] as a starting point, more general theorems are stated, and some limitations are removed. It is shown that this method produces not only a stable closed-loop system, but also an optimal one, for all values of the uncertain parameters. This optimality has the unique feature that different weighting matrices are associated with each value of the uncertain parameter vector. A multistep guaranteed cost control design method is suggested, and its advantages are demonstrated. Different design parameters and their influences on the design are described. A detailed algorithm for the multistep guaranteed cost control method is given, and a demonstrative example is presented. The minimum expected cost design method is described in the second chapter. The general approach follows works by Ly and Cannon [14] and Heath and Dillow [15]. Necessary conditions for a minimum of the expected cost are given, and a gradient of the cost functional with respect to the gain matrix is derived. The discrete minimum expected cost design method is developed, and its practicality demonstrated. A detailed algorithm for the discrete minimum expected cost design method is described, and an example is given. The third chapter is devoted to a comparative study of six design methods for systems with uncertain parameters. Some conclusions as to the relative advantages and disadvantages of each method are derived.

1. GUARANTEED COST CONTROL DESIGN METHOD

1.1 Statement of the Control Problem

Using state space notation, one may describe the dynamics of many systems by the linear differential equations

$$\dot{x}(t) = A[\omega(t)]\, x(t) + B[\theta(t)]\, u(t) \quad , \quad t \in [0,t_1] \tag{1.1.1}$$

$$y(t) = H\, x(t) \tag{1.1.2}$$

where

$x(t)$ = state vector $(n \times 1)$

$u(t)$ = control vector $(m \times 1)$

$\omega(t)$ = vector of uncertain parameters in open-loop system matrix, referenced to their nominal values $(n' \times 1)$

$\theta(t)$ = vector of uncertain parameters in control distribution matrix, referenced to their nominal values $(m' \times 1)$

$A[\omega(t)]$ = open-loop dynamics matrix $(n \times n)$

$B[\theta(t)]$ = control distribution matrix $(n \times m)$

$y(t)$ = vector of measured outputs $(r \times 1)$

H = output distribution matrix $(r \times n)$

The following assumptions are employed throughout this chapter:

1) The matrices A and B depend on time only implicitly through the uncertain parameter vectors ω and θ, which may vary with time. Nominal design conditions are $\omega(t) \equiv 0$, $\theta(t) \equiv 0$, $t \in [0,t_1]$.

2) The vector $\omega(t)$ lies somewhere within a closed bounded region $\Omega \in R^{n'}$ (n'-dimensional real cartesian space) for $t \in [0,t_1]$. The

vector $\theta(t)$ lies somewhere within a closed bounded region $\Theta \in R^{m'}$ (m'-dimensional real cartesian space) for $t \in [0,t_1]$. For simplicity, it is assumed that Ω and Θ are rectangular in shape and include the origin, so that each of the components of $\omega(t)$ and $\theta(t)$ are bounded as follows:

$$a_i \le \omega_i(t) \le b_i \quad , \quad i = 1,\cdots,n' \; , \quad t \in [0,t_1] \qquad (1.1.3)$$
$$a_i \le 0 \quad , \quad 0 \le b_i$$

$$c_i \le \theta_i(t) \le d_i \quad , \quad i = 1,\cdots,m' \; , \quad t \in [0,t_1] \qquad (1.1.4)$$
$$c_1 \le 0 \quad , \quad 0 \le d_i$$

3) The structure of $A[\omega(t)]$ is as follows:

$$A[\omega(t)] = A_o + \sum_{i=1}^{n'} \omega_i(t)\, A_i \qquad (1.1.5)$$

where A_i, $i = 0,1,\cdots,n'$, are constant $(n \times n)$ matrices, i.e., the uncertain parameters are assumed to enter A in a linear fashion.

4) The structure of $B[\theta(t)]$ is as follows:

$$B[\theta(t)] = B_o + \sum_{i=1}^{m'} \theta_i(t)\, B_i \qquad (1.1.6)$$

where B_i, $i = 0,1,\cdots,m'$, are constant $(n \times m)$ matrices.

5) $[A_o,B_o]$ form a controllable pair.

6) Control system performance is characterized by a quadratic cost functional J, defined as follows:

$$J = \int_0^{t_1} (x^T Q x + u^T R u)\, dt \tag{1.1.7}$$

where Q is an (n x n) constant positive semi-definite symmetric matrix, and R is a constant (m x m) positive definite symmetric matrix.

7) $\{A_0, Q^{1/2}\}$ form an observable pair, where $Q^{1/2}$ denotes any square root of the matrix Q, i.e., $(Q^{1/2})^T Q^{1/2} = Q$.

The objective of our discussion is to design a linear constant gain feedback controller that will guarantee the boundedness of J, of the form

$$u(t) = -Cy(t) \quad , \quad t \in [0,t_1] \tag{1.1.8}$$

to produce, in some sense, acceptable closed-loop behavior for all $\omega(t) \in \Omega$, $\theta(t) \in \Theta$, $t \in [0,t_1]$. The matrix C is a constant (m x r) matrix.

1.2 General Theorems of the Guaranteed Cost Control Method

We will now present and prove the general theorems of the guaranteed cost control design method.

Theorem 1.1: Let S(t) be an (n x n) symmetric matrix with continuously differentiable elements, and let $\eta[x(t),t]$ be a feedback control law (i.e., a particular form of u), which together satisfy the following inequality:

$$[x(t)^T Q x(t) + \eta^T[x(t),t]\, R\eta[x(t),t]] + x(t)^T \dot{S}(t) x(t) + 2x(t)^T S(t)\{A[\omega(t)]x(t) + B[\theta(t)]\, \eta[x(t),t]\} \leq 0 \tag{1.2.1}$$

$$\forall \omega \in \Omega, \quad \theta \in \Theta, \quad x(t) \in R^n, \quad t \in [0,t_1]$$

with

$$S(t_1) = 0 \tag{1.2.2}$$

Then, the value of J achieved using $u(t) = \eta[x(t),t]$ is bounded from above as follows:

$$J \leq x^T(0)\, S(0)\, x(0) \tag{1.2.3}$$

<u>Proof</u>: If $u(t) = \eta[x(t),t]$, Eq. (1.1.1) yields

$$\frac{d}{dt}[x(t)^T S(t)x(t)] = x(t)^T \dot{S}(t)x(t) + 2x(t)^T S(t)\{A[\omega(t)]x(t) + B[\theta(t)]\,\eta[x(t),t]\}\ , \qquad t \in [0,t_1] \tag{1.2.4}$$

Thus Eq. (1.2.1) yields

$$x(t)^T Qx(t) + \eta^T[x(t),t]\, R\eta[x(t),t] \leq -\frac{d}{dt}[x(t)^T S(t)x(t)]\ , \qquad t \in [0,t_1] \tag{1.2.5}$$

Integration of Eq. (1.2.5) from 0 to t_1, in conjunction with Eqs. (1.1.7), (1.2.2) yields Eq. (1.2.3).

Such an $S(t)$ and $\eta[x(t),t]$ are called a guaranteed cost matrix and a guaranteed cost control, respectively.

It is instructive to examine the specific case where Ω and Θ are reduced to a single point each (the origin). Then Eq.(1.2.1) will take the following form:

$$x(t)^T Qx(t) + \eta^T[x(t),t]R\eta[x(t),t] + x(t)^T \dot{S}(t)x(t) + 2x(t)^T S(t) \times [A_0 x(t) + B_0 \eta[x(t),t]] \leq 0, \quad \forall x(t) \in R^n,\ t \in [0,t_1] \tag{1.2.6}$$

Let us assume $\eta[x(t),t]$ to be a linear feedback control law of the form

$$\eta[x(t),t] = -Cy(t) = -CH\,x(t) \tag{1.2.7}$$

As $t_1 - t \to \infty$, $\dot{S}(t)$ will tend to zero in systems which satisfy the following assumptions [24]:

(i) $(A_0 - B_0 CH)$ is a stability matrix.

(ii) A_0, B_0, Q, R, H are constant matrices.

(iii) $Q + H^T C^T RCH$ is a positive-semidefinite matrix.

(iv) $[A_0, (Q+H^T C^T RCH)^{1/2}]$ form an observable pair.

Combining Eqs. (1.2.6) and (1.2.7) together with

$$\dot{S}(t) \to 0 \qquad t_1 - t \to \infty \tag{1.2.8}$$

yields

$$x^T[Q+H^T C^T RCH + S(A_0 - B_0 CH) + (A_0 - B_0 CH)^T S]\, x \leq 0 \tag{1.2.9}$$

$$\forall \quad x \in R^n$$

When Eq. (1.2.9) is an equality, i.e., the left-hand side is strictly equal to zero, the symmetric matrix inside the brackets should be zero identically. This yields the well known Lyapunov equation. It is known [25] that

$$S(A_0 - B_0 CH) + (A_0 - B_0 CH)^T S = -Q - H^T C^T RCH \tag{1.2.10}$$

has a positive definite symmetric matrix solution S, if the above stated assumptions hold. When H is assumed to be a square matrix of rank n (i.e., a full state feedback law is assumed), it is possible to show [24] that

$$\eta[x(t),t] = -R^{-1} B_0^T S\, x(t) \tag{1.2.11}$$

is the control law which minimizes J (as defined in Eq. (1.1.7)), so that Eq. (1.2.6) can be written as the following equality:

$$x^T[SA_0 + A_0^T S - SB_0 R^{-1} B_0^T S + Q]x = 0, \qquad x \in R^n \tag{1.2.12}$$

For Eq. (1.2.12) to hold, the symmetric matrix inside the brackets should be zero identically. This yields the familiar algebraic Riccati equation.

Conditions on the existence of $S(t)$ and $\eta[x(t),t]$ for the general case are not yet known. Even for systems which follow our assumptions of linear control logic and specific form of Ω and Θ, the existence of a gain matrix C such that $\{A[\omega(t)] - B[\theta(t)]CH\}$ is a stability matrix for every $\omega(t) \in \Omega$ and $\theta(t) \in \Theta$ is not assured. One may assert, based on the previous discussion and continuity properties of $A[\omega(t)]$, $B[\theta(t)]$, and $S(t)$ that when Ω and Θ are sufficiently small, such an $S(t)$ and $\eta[x(t),t]$ always exist. For the remainder of this chapter, we will assume $S(t)$ and $\eta[x(t),t]$ to exist, and will show some methods to evaluate them.

I. The Output Feedback Case

Following Eqs. (1.1.2) and (1.1.8), the control law $\eta[x(t),t]$ is assumed to be of the form

$$u(t) = \eta[x(t),t] = -CH\,x(t) \tag{1.2.13}$$

Let $t_1 - t \to \infty$ and assume that Eq. (1.2.8) holds. Substitute Eq. (1.2.13) into Eq. (1.2.1) and arrange to get

$$x^T[Q + H^TC^TRCH + A^T(\omega)S + SA(\omega) - SB(\theta)CH - H^TC^TB^T(\theta)S]x \leq 0 \;, \quad \forall\; x \in R^n \tag{1.2.14}$$

An equality consistent with Eq. (1.2.14) is the following:

$$x^T[Q + H^TC^TRCH + A_0^TS + SA_0 - SB_0CH - H^TC^TB_0^TS + U(S) - V(C,S)]x = 0 \quad \forall\; x \in R^n \tag{1.2.15}$$

where $U(S)$ and $V(C,S)$ are upper and lower bounds, respectively, on the effects of parameter uncertainty, in the sense that

$$x^T U(S) x \geq x^T [\sum_{i=1}^{n'} \omega_i(t)(A_i^T S + SA_i)] x \;, \quad \forall\, x \in R^n \;, \quad t \in [0,t_1] \tag{1.2.16}$$

$$x^T V(C,S) x \leq x^T [\sum_{i=1}^{m'} \theta_i(t)(SB_i CH + H^T C^T B_i^T S)] x, \quad \forall\, x \in R^n \;, \quad t \in [0,t_1] \tag{1.2.17}$$

There are a number of ways to construct U(S) and V(C,S) such that Eqs. (1.2.16) and (1.2.17) hold [26],[27]. We will present here one way, which is as follows. Let

$$U(S) = \sum_{i=1}^{n'} N_i E_i N_i^T \tag{1.2.18}$$

where N_i is the orthogonal transformation which diagonalizes the symmetric matrix $(SA_i + A_i^T S)$:

$$N_i^T (A_i^T S + SA_i) N_i = \Lambda_i \qquad i = 1,\cdots,n' \tag{1.2.19}$$

(Λ_i is diagonal and contains the eigenvalues of $A_i^T S + SA_i$, i.e., $(\Lambda_i)_{kk} = (\lambda_i)_k$, $(\Lambda_i)_{kj} = 0$, $k \neq j$, where $(\lambda_i)_k$ is the k^{th} eigenvalue of $A_i^T S + SA_i$). The matrix E_i is defined by

$$\left\{ \begin{array}{l} (E_i)_{kk} = \begin{cases} a_i(\lambda_i)_k \;, & (\lambda_i)_k < 0 \\ b_i(\lambda_i)_k \;, & (\lambda_i)_k \geq 0 \end{cases} \\ \\ (E_i)_{kj} = 0 \;, \quad k \neq j \end{array} \right\} \quad i = 1,\cdots,n' \tag{1.2.20}$$

The symmetric matrix V(C,S) will be constructed in a similar way, as follows:

$$V(C,S) = \sum_{i=1}^{m'} M_i F_i M_i^T \tag{1.2.21}$$

where M_i is the orthogonal transformation which diagonalizes the symmetric matrix $(SB_iCH + H^TC^TB_i^TS)$. The matrix F_i is defined by

$$\left\{\begin{array}{l} (F_i)_{kk} = \begin{cases} c_i(\sigma_i)_k \quad , & (\sigma_i)_k \geq 0 \\ d_i(\sigma_i)_k \quad , & (\sigma_i)_k < 0 \end{cases} \\ (F_i)_{kj} = 0 \quad , \quad k \neq j \end{array}\right\} i = 1,\cdots,m' \quad (1.2.22)$$

where $(\sigma_i)_k$ is the k^{th} eigenvalue of $(SB_iCH + H^TC^TB_i^TS)$. One should note that by construction, $U(S)$ is a symmetric positive semi-definite matrix, and $V(C,S)$ is a symmetric negative semi-definite matrix.

In order for Eq. (1.2.15) to hold for every $x(t) \in R^n$, the symmetric expression in the bracket should be identically zero. Hence, the following theorem, describing guaranteed cost control in the output feedback case:

Theorem 1.2: Let $S(t)$ be a symmetric positive definite matrix that satisfies

$$S(A_o - B_oCH) + (A_o - B_oCH)^TS + U(S) = -Q - H^TC^TRCH + V(C,S) \quad (1.2.23)$$

$$S(t_1) = 0 \quad , \quad \dot{S}(t) \to 0 \quad \text{as} \quad (t_1 - t) \to \infty \quad (1.2.24)$$

where U and V are consistent with Eqs. (1.2.16) and (1.2.17). Let $\mathcal{C}$ be a non-empty set, $\mathcal{C} \subset R^{mxr}$ such that for every $C \in \mathcal{C}$, $(A(\omega) - B(\theta)CH)$ is a stable matrix for every $\omega \in \Omega$ and $\theta \in \Theta$. Then, J as defined by Eq. (1.1.7), with $t_1 - t \to \infty$, is bounded from above as follows:

$$J \leq x^T(0)\, S\, x(0) \quad (1.2.25)$$

Proof: Follows the proof of Thm. 1.1 with the use of Eqs. (1.2.16), (1.2.17).

One can see from Eq. (1.2.25) that the bound on J depends on the initial condition x(0) as well as on S. In many engineering systems an exact knowledge of x(0) does not exist. In order to make the design process independent of initial conditions, let us define

$$\hat{J}(C) = E[x^T(0)\, S\, x(0)] = Tr[S(C)X_0] \tag{1.2.26}$$

where

$$X_0 = E[x(0)\ x^T(0)] \tag{1.2.27}$$

The operator E[·] in the above equations is the expected value with respect to the random variable x(0). When no statistical information on x(0) is known, one might assume that the initial condition vector lies with equal probability on the surface of a sphere of radius 1 in R^n. Hence, in this case, the following holds:

$$X_0 = I \tag{1.2.28}$$

and $\hat{J}(C)$ becomes

$$\hat{J}(C) = Tr[S(C)] \tag{1.2.29}$$

where S(C) satisfies Eq. (1.2.23). The objective now is to find $C^* \in \mathcal{C}$ so as to minimize $\hat{J}(C)$ (as defined by Eq. (1.2.26) or Eq. (1.2.29)).

By virtue of Thm. 1.2, the time-invariant closed-loop matrix $[A(\omega) - B(\theta)\hat{C}H]$ is a stable matrix for every constant $\omega \in \Omega$ and $\theta \in \Theta$. For the time-varying uncertain parameters, $\omega(t) \in \Omega$ and $\theta(t) \in \Theta$, the stability of the closed-loop system requires somewhat more complex conditions. Details may be found in [21] and [28].

II. The Full State Feedback Case

Whenever $y(t)$, as defined in Eq. (1.1.2), is an $(n \times 1)$ vector and H is nonsingular, the control law, Eq. (1.1.8), becomes a full state feedback law. Without loss of generality, one may then assume H to be the identity matrix.

Let us assume the control law to be of the following form:

$$u(t) = -R^{-1}B^T(\theta_0)S\, x(t) \quad , \qquad \theta_0 \in \Theta \tag{1.2.30}$$

and assume that Eq. (1.2.8) holds. Then Eqs. (1.2.1) and (1.2.8) yield

$$x^T[Q + SB(\theta_0)R^{-1}B^T(\theta_0)S + SA(\omega) + A^T(\omega)S - SB(\theta)R^{-1}B^T(\theta_0)S - SB(\theta_0)R^{-1}B^T(\theta)S]x \leq 0 \tag{1.2.31}$$

$$\forall\ \omega \in \Omega\ , \quad \theta \in \Theta\ , \quad x \in R^n$$

An equality which is consistent with Eq. (1.2.31) is the following:

$$x^T[Q - SB(\theta_0)R^{-1}B^T(\theta_0)S + SA_0 + A_0^T S + U(S)]x = 0\ , \quad \theta_0 \in \Theta \tag{1.2.32}$$

provided that Eq. (1.2.16) holds and that

$$2x^T[SB(\theta_0)R^{-1}B^T(\theta_0)S]x \leq x^T[SB(\theta)R^{-1}B^T(\theta_0)S + SB(\theta_0)R^{-1}B^T(\theta)S]x \tag{1.2.33}$$

for all $x \in R^n$, $\theta \in \Theta$, for some $\theta_0 \in \Theta$.

Hence, in the full state feedback case, Eq. (1.2.33) replaces Eq. (1.2.17) as a requirement for the inequality Eq. (1.2.31) to be replaceable by the equality Eq. (1.2.32). The question as to whether a point $\theta_0 \in \Theta$ exists such that Eq. (1.2.33) is satisfied can be answered in the process of searching for such a point. The matrix $B(\theta_0)$ is here called the control distribution design matrix. One can see that $u(t)$ in Eq. (1.2.30) will be of the familiar form for the LQR problem when $B(\theta_0)$ is used in place of the control distribution matrix.

When Eqs. (1.2.16) and (1.2.33) hold, in order for Eq. (1.2.32) to hold for an arbitrary x, the symmetric form within the bracket has to vanish identically, i.e.,

$$SA_0 + A_0^T S - SB(\theta_0)R^{-1}B^T(\theta_0)S + Q + U(S) = 0 \quad , \quad \theta_0 \in \Theta \qquad (1.2.34)$$

This equation is similar to the algebraic Riccati equation, with the inclusion of the additional term $U(S)$ [21],[22].

Let us consider the question of how to find a $\theta_0 \in \Theta$, such that Eq. (1.2.33) holds. Using Eqs. (1.1.6) and (1.2.33) one can write

$$0 \leq x^T S \{\sum_{i=1}^{m'} (\theta_i - \theta_{0i})[B_i R^{-1} B^T(\theta_0) + B(\theta_0) R^{-1} B_i^T]\} Sx \quad ,$$

$$x \in R^n, \ \theta \in \Theta, \text{ and fixed } \theta_0 \in \Theta \qquad (1.2.35)$$

where θ_{0i} denotes the i^{th} component of θ_0. Eq. (1.2.35) is a scalar inequality. An equivalent matrix inequality is

$$0 \leq \sum_{i=1}^{m'} (\theta_i - \theta_{0i})[B_i R^{-1} B^T(\theta_0) + B(\theta_0) R^{-1} B_i^T] \quad ,$$

$$\text{for all } \theta \in \Theta, \text{ and some } \theta_0 \in \Theta \qquad (1.2.36)$$

That Eq. (1.2.36) is equivalent to Eq. (1.2.35) can be observed when one recalls that S is a symmetric positive definite matrix which maps $R^n \to R^n$.

A necessary and sufficient condition for Eq. (1.2.36) to hold is that each of the symmetric matrices

$$(\theta_i - \theta_{0i})[B_i R^{-1} B^T(\theta_0) + B(\theta_0) R^{-1} B_i^T] \ , \ i = 1, \cdots, m' \ ,$$

$$\theta_i \in \Theta, \text{ and some } \theta_0 \in \Theta \qquad (1.2.37)$$

be positive semi-definite. From Eqs. (1.1.4) and (1.2.37) it is clear that if such an $\theta_0 \in \Theta$ exists, then

$$\theta_{oi} = c_i \qquad \text{or} \qquad \theta_{oi} = d_i \ , \qquad i = 1,\cdots,m' \tag{1.2.38}$$

i.e., the point θ_o Θ, such that Eq. (1.2.33) holds (if such a θ_o exists) will always be one of the $2^{m'}$ corners of θ .

Two interesting special cases are considered below:

1. The case $B(\theta) = \theta B_1$, $1 \leq \theta \leq b$,

which is treated in [21], is just a special case of that discussed here. Let us write $B(\theta)$ as follows:

$$B(\theta) = B_o + \theta_1 B_1 \tag{1.2.39}$$

where

$$B_o = B_1 \ ; \qquad \theta_1 = \theta - 1 \ ; \qquad 0 \leq \theta_1 \leq (b-1)$$

Now $B(\theta)$ in Eq. (1.2.39) complies with our notation. Following our previous discussion, we shall check Eq. (1.2.37) at the two corners of Θ.

Letting $\qquad \theta_o = 0$

yields $\qquad B(\theta_o) = B_1$

Substitution into Eq. (1.2.37) yields

$$2\theta_1 B_1 R^{-1} B_1^T \geq 0 \ , \quad \forall \ \theta_1 \in \Theta \tag{1.2.40}$$

Thus, Eq. (1.2.37) holds, and the control distribution design matrix is B_1, as was derived in [21].

2. Let us assume that when the control distribution matrix is of the following form:

$$B(\theta) = B_o + \sum_{i=1}^{m} \theta_i B_i \qquad , \qquad \theta \in \Theta \tag{1.2.41}$$

where

$$B_o = \begin{bmatrix} & 0 & \\ \alpha_1 & & \\ & \ddots & \\ & & \alpha_m \end{bmatrix} \qquad (1.2.42)$$

and

$$B_i = \begin{bmatrix} & 0 & \\ \delta_{im} & & \\ & \ddots & \\ & & \delta_{im} \end{bmatrix} \qquad i=1,\cdots,m \qquad (1.2.43)$$

α_i, $i=1,\cdots m$ in Eq. (1.2.42) are constants, and B_o is the nominal control distribution matrix. Furthermore, it is assumed that

$$\alpha_i + \theta_i \ , \ \theta \in \Theta_i \quad , \ i=1,\cdots,m \qquad (1.2.44)$$

does not change sign, and that R, the control weighting matrix, is diagonal. Then, it can be shown that there is always a point $\theta_o \in \Theta$ which satisfies Eq. (1.2.37). The form of $B(\theta)$ in Eqs. (1.2.41)-(1.2.43) is appropriate to the case in which actuator dynamics are included in the modeling of the system. Such a system is described in the fifth-order example in Sec. 3.2. In cases where some of our assumptions do not hold (R is not diagonal or $\alpha_i + \theta_i$ change sign for some i, for example) there will not be a point $\theta_o \in \Theta$ which satisfies Eq. (1.2.37). It is possible to show that when such a point exists, $B(\theta_o)$ is unique.

1.3 Multistep Guaranteed Cost Control Design Method

One of the drawbacks of the guaranteed cost control design method as presented in [21] is that it tends to result in overcontrolled behavior, i.e., large feedback gains and correspondingly relatively large control effort [22],[23]. The problem is due mainly to the fact that the term U(S) in Eq. (1.2.15) or Eq. (1.2.31) is a function of the matrix S, as well as the uncertain parameters. Thus, in cases where S increases due to open-loop instability ([29] , for example), the effect of parameter uncertainty is magnified and results in large feedback gains and large control effort. This can be shown as follows.

Rewrite Eq. (1.2.34), and assume that S_d is a constant symmetric positive-definite matrix which satisfies

$$S_dA_o + A_o^T S_d - S_dB(\theta_o)R^{-1}B^T(\theta_o)S_d + Q + U(S_d) = 0 \tag{1.3.1}$$

Define Q^* as

$$Q^* = Q + U(S_d) \tag{1.3.2}$$

S_d thus satisfies the following Riccati equation

$$S_dA_o + A_o^T S_d - S_dB(\theta_o)R^{-1}B^T(\theta_o)S_d + Q^* = 0 \tag{1.3.3}$$

U(S), by construction, is a positive-semidefinite matrix; thus

$$Q^* \geq Q \tag{1.3.4}$$

(For two symmetric matrices A,B with the same dimensions, $A \geq B$ means that the matrix C = A-B is positive-semidefinite. For A > B, C as defined above is positive-definite). It is well known [29] ,[30] that the feedback gains and control effort will generally increase as the state weighting matrix Q is increased. In Eq. (1.3.3), Q^* will increase as S_d increases and will tend to produce an overcontrolled design.

A means of overcoming this problem is suggested here. The essence of the approach is to perform the design in several steps. First, one stabilizes the open-loop system, using a linear-quadratic technique with no parameter uncertainty assumed. Then one tries to modify the resulting closed-loop system so as to accommodate the effects of uncertainty.

For simplicity and clarity of the presentation, the multistep guaranteed cost control design method will be described for systems which satisfy the assumptions stated in Section 1.1 as well as the following two assumptions:

1) A point $\theta_o \in \Theta$, such that Eq. (1.2.33) holds, exists. Thus, let

$$B = B(\theta_o) \tag{1.3.5}$$

for the remainder of the section.

2) The output distribution matrix H (in Eq. (1.1.2)), is assumed to be of rank n; i.e., full state feedback is assumed. Without loss of generality, one may assume H to be an identity matrix.

<u>Theorem 1.3</u>: Let us assume that there exists a matrix S_d, symmetric, constant, and positive definite, which satisfies the following equation:

$$S_dA_o + A_o^T S_d - S_d BR^{-1}B^TS_d + Q + \rho_d U(S_d) = 0 \;, \qquad \rho_d \geq 0 \tag{1.3.6}$$

where $U(S_d)$ is consistent with Eq. (1.2.16). Then,

i) The control law

$$u(t) = -C_d\, x(t) \tag{1.3.7}$$

$$C_d = R^{-1}B^T S_d \tag{1.3.8}$$

is an optimal control law for all constant ω such that

$$\rho_a a_i \leq \omega_i \leq \rho_a b_i \quad , \qquad i = 1,\cdots,n' \tag{1.3.9}$$

where ρ_a is that value of ρ which causes the quantity

$$Q_d(\rho) = Q + (\rho_d - \rho)\ U(S_d) \tag{1.3.10}$$

to change from positive-definite or -semidefinite to indefinite.

ii) The control law of Eqs. (1.3.7) and (1.3.8) will make the closed-loop system stable for all $\omega(t)$ such that

$$\rho_a a_i \leq \omega_i(t) \leq \rho_a b_i \quad , \quad i = 1,\cdots,n' \ , \ t \geq 0 \tag{1.3.11}$$

where ρ_a is as defined by Eq. (1.3.10).

<u>Proof</u>: For the first part, let us define

$$Q^*(\omega) = Q + \rho_d U(S_d) - \sum_{i=1}^{n'} \omega_i (S_d A_i + A_i^T S_d) \tag{1.3.12}$$

Eq. (1.3.6) may be written as

$$S_d A(\omega) + A^T(\omega)\ S_d - S_d B R^{-1} B^T S_d + Q^*(\omega) = 0 \tag{1.3.13}$$

where use has been made of Eqs. (1.1.5) and (1.3.12). In Eq. (1.3.13), S_d is the positive-definite solution to the Riccati equation for a precisely known system with dynamics matrix $A(\omega)$ (ω constant) and a state weighting matrix $Q^*(\omega)$. For Eq. (1.3.13) to be a meaningful Riccati equation, $Q^*(\omega)$ should be positive-definite or -semidefinite, and $[A(\omega),Q^{*1/2}(\omega)]$ should form an observable pair. Note that

$$\rho x^T U(S_d)x \leq x^T[\sum_{i=1}^{n'} \omega_i(S_d A_i + A_i^T S_d)]x \ , \quad \forall\ x \in R^n, \quad \text{where } \rho \geq 0 \qquad (1.3.14)$$

if ω is consistent with the inequalities

$$\rho a_i \leq \omega_i \leq \rho b_i \ , \qquad i = 1,\cdots,n' \qquad (1.3.15)$$

This follows from Eqs. (1.2.16). Thus, from Eqs. (1.3.10), (1.3.12), (1.3.14) follows the inequality

$$x^T Q^*(\omega)x \geq x^T Q_d(\omega)x \quad , \quad \forall\ x \in R^n \qquad (1.3.16)$$

for all ω consistent with Eq. (1.3.15). Thus, $Q^*(\omega)$ is positive-definite or -semidefinite whenever $Q_d(\omega)$ is positive-definite or -semidefinite, which is for $\rho \leq \rho_a$. Since S_d is positive-definite and satisfies an algebraic Riccati equation with a positive-definite control weighting matrix and a positive-definite or -semidefinite state weighting matrix, and controllability and observability conditions are satisfied, the controller described by Eqs. (1.3.7), (1.3.8) is an optimal controller for all constant ω consistent with Eq. (1.3.9).

For the proof of the second part, S_d can be viewed as the solution to the guaranteed cost matrix equation

$$S_d A_o + A_o^T S_d - S_d B R^{-1} B^T S_d + Q_d + \rho_a U(S_d) = 0 \qquad (1.3.17)$$

which follows from Eq. (1.2.1) with the use of Eqs. (1.3.7), (1.3.8), and (1.3.10). Equation (1.3.17) is a guaranteed cost matrix equation provided that Q_d is positive-semidefinite, which is for $\rho \leq \rho_a$. Thus, from Theorem 1.1, the cost functional

$$J = \int_0^\infty (x^T Q_d x + u^T R u)dt \tag{1.3.18}$$

is bounded as follows

$$J \leq x^T(0)\, S_d\, x(0) \tag{1.3.19}$$

The boundedness of J, together with the stated conditions in [21],[28], implies the stability of the closed-loop system for every $\omega(t)$ consistent with Eq. (1.3.11).

Theorem 1.4: Let S_o be a symmetric, constant, positive-definite solution to the equation

$$S_o A_o + A_o^T S_o - S_o B R^{-1} B^T S_o + Q + \rho_o U(S_o) = 0, \; \rho_o \geq 0 \tag{1.3.20}$$

and let the corresponding control law be

$$u(t) = -C_o\, x(t) \tag{1.3.21}$$

$$C_o = R^{-1} B^T S_o \tag{1.3.22}$$

Define

$$F_o = A_o - BC_o \tag{1.3.23}$$

and let S_1 be a symmetric, constant, positive definite matrix solution to

$$S_1 F_o + F_o^T S_1 - S_1 B R^{-1} B^T S_1 + Q_1 + \rho_1 U(S_1) = 0, \tag{1.3.24}$$

Then the control law

$$u(t) = -C_1 \; x(t) \tag{1.3.25}$$

$$C_1 = C_0 + R^{-1}B^T S_1 \tag{1.3.26}$$

is an optimal control law (i.e., minimizes some cost functional J) for all constant ω such that

$$\bar{\rho} a_i \leq \omega_i \leq \bar{\rho} b_i \;, \qquad i = 1,\cdots,n' \tag{1.3.27}$$

where $\bar{\rho}$ is that value of ρ which causes the quantity

$$\bar{Q} = Q + Q_1 + \rho_0 U(S_0) + \rho_1 U(S_1) - \rho U(S_0 + S_1) \tag{1.3.28}$$

to change from positive-definite or semidefinite to indefinite.

<u>Proof</u>: Let us define

$$\bar{S} = S_0 + S_1 \tag{1.3.29}$$

So defined, $\bar{S}$ is a symmetric, constant, positive-definite matrix solution to

$$\bar{S}A_0 + A_0^T\bar{S} - \bar{S}BR^{-1}B^T\bar{S} + \bar{Q} + \rho \; U(\bar{S}) = 0 \tag{1.3.30}$$

This can be shown to be true by the use of Eqs. (1.3.20),(1.3.22)-(1.3.24), (1.3.26), (1.3.28) and (1.3.29). But Eq. (1.3.30) is a meaningful guaranteed cost matrix equation provided that $\bar{Q}$ is positive-semidefinite, which is for $\rho \leq \bar{\rho}$. Since the matrix $\bar{S}$ is positive-definite and satisfies a guaranteed cost matrix equation, the optimality of the control law of Eqs. (1.3.25), (1.3.26) follows from Theorem 1.3.

The controller optimality described in the above two theorems should not be interpreted as optimality with respect to an a priori defined cost functional. The controller is optimal with respect to a cost function J (in particular a state weighting matrix Q) defined only after the controller has been determined. As shown above, the controller is optimal with respect to a cost functional J which includes a different state weighting matrix for different values of the uncertain parameter vector ω. Nevertheless, for each constant ω consistent with Eq. (1.3.9) or Eq. (1.3.27), the various desirable properties of an optimally designed control system described in [24,31,32] (e.g., global asymptotic stability, phase margin of at least 60°, infinite gain margin, tolerance of nonlinearities, etc.) hold.

A design method based on Theorems 1.3 and 1.4 is suggested here. The first step is to find $\theta_0 \in \Theta$ (if it exists) such that Eq. (1.2.33) holds. Assign $B(\theta_0)$ the role of the control distribution matrix B. Choose weighting matrices Q and R, so as to produce a satisfactory nominal closed-loop design. The selection of Q and R can be done by model matching techniques [33],[34], the pole placement technique [35], the total system damping approach [30], or trial and error methods. Solve Eq. (1.3.20) with $\rho_0 = 0$ (which yields the usual algebraic Riccati equation) to yield S_0 and C_0. The nominal ($\omega = 0$) closed-loop system matrix is F_0 (as defined in Eq. (1.3.23)). Beyond this stage of the design process, the effects of parameter uncertainties will be taken into account. Since the nominal closed-loop system has been designed to have some desired behavior (by the selection of Q and R), one would like to place a relatively large penalty on additional control effort required due to uncertainties. Thus, Q_1 in Eq. (1.3.24) is chosen to be μQ

with $0 < \mu \leq 1$. The value of ρ_1 can be selected by the designer or be assigned the value of ρ_d evaluated according to Eq. (1.3.10). A solution of Eq. (1.3.24) with the associated control law of Eqs. (1.3.25)and (1.3.26) yields a closed-loop system which is stable for some range of parameter uncertainties. This range can be determined by the use of Eq. (1.3.10); more-moreover, the resulting closed-loop design is an optimal one (i.e., has the desired properties described above) for a range of parameter uncertainty determined by the use of Eq. (1.3.28). If these ranges of uncertainty (for optimality or stability) are equal to or greater than the range over which the parameter vector may vary, the design process is terminated. Otherwise, define

$$F_{i-1} = A_0 - BC_{i-1} \quad , \qquad i = 1,\cdots, \tag{1.3.31}$$

where

$$C_i = C_{i-1} + R^{-1}B^TS_i \quad , \qquad i = 1,\cdots, \tag{1.3.32}$$

and go through the design process as described above. Once the range of uncertainty for which the closed-loop system is stable (optimal) has exceeded the predetermined range, a satisfactory controller design has been achieved. If the range of stability (optimality) tends toward a limiting value less than the desired one, one may conclude that constant feedback gains do not exist which will stabilize the closed-loop system over the entire range of parameter uncertainty.

The above design process is described in an algorithm in Section 1.4, and an example of using this approach is presented in Sec. 1.5.

1.4 Algorithm for the Multistep Guaranteed Cost Control Design Method

Throughout this section we will present an algorithm for the design of a constant gain feedback control law, using the multistep guaranteed cost control design technique.

The algorithm is as follows:

1. Input: A_0, $\{A_i, i=1,\cdots,n'\}$, B_0, $\{B_i, i=1,\cdots,m'\}$, Ω, Θ,

Q, R, μ .

2. Check to find θ_0 (Eq. (1.2.37)). If such a $\theta_0 \in \Theta$ exists, let $B = B(\theta_0)$. When θ_0 does not exist, stop.

3. Determine the positive-definite solution S_0 to the following algebraic Riccati equation:

$$S_0 A_0 + A_0^T S_0 - S_0 B R^{-1} B^T S_0 + Q = 0 \qquad (1.4.1)$$

The method of eigenvector decomposition [36],[37] is recommended.

4. Evaluate the controller feedback gain matrix

$$C_0 = R^{-1} B^T S_0 \qquad (1.4.2)$$

and set j equal to zero.

5. Evaluate $U(S_j)$ in accordance with Eqs. (1.2.18), (1.2.19), and (1.2.20), or in accordance with the procedure described in Appendix A, whichever is appropriate.

6. For $j = 0$, determine the value of ρ_0 which makes

$$\overline{Q} = Q - \rho_0 U(S_0) \qquad (1.4.3)$$

indefinite. For $j > 0$, determine the value of ρ_j which makes either Q_s or Q_0 indefinite. Q_s is defined as follows:

$$Q_s = \mu Q - (\rho_j - \rho_{j-1})\, U(S_j) \tag{1.4.4}$$

Q_o is defined as follows:

$$Q_o = (1 + j\mu)Q + \sum_{i=1}^{j} \rho_i\, U(S_i) - \rho U(\sum_{i=0}^{j} S_i) \tag{1.4.5}$$

When ρ_j is determined according to Eq. (1.4.4) it gives a range of parameter uncertainty for which the closed-loop system is stable. When ρ_j is determined from Eq. (1.4.5), it gives a range of parameter uncertainty for which the closed-loop system is optimal.

7. If $\rho_j \geq 1$, a successful control system design has been achieved. If $j \geq$ maximum desired number of iterations, the design process terminates unsuccessfully.

8. Increase j by one and evaluate

$$F_{j-1} = A_o - BC_{j-1} \tag{1.4.6}$$

9. Assign to $\bar{\rho}_j$ the value of ρ_{j-1} obtained in step 6, or a predetermined value.

10. Obtain the positive-definite solution S_j to the following equation

$$S_jF_{j-1} + F_{j-1}^T S_j - S_jBR^{-1}B^TS_j + \mu Q + \bar{\rho}_j\, U(S_j) = 0 \tag{1.4.7}$$

A modified form of Kleinman's iterative technique [38] for solving this equation is described below. The Aitken acceleration method [39] can be used to improve the convergence rate.

11. Evaluate the feedback gain matrix

$$C_j = C_{j-1} + R^{-1}B^TS_j \qquad (1.4.8)$$

and go to step 5.

In [21], equations resembling Eq. (1.4.7) are solved by backward integration until a steady-state solution is achieved, using Eq. (1.2.2) as a starting condition. A modified version of Kleinman's iterative technique is suggested here for solving Eq. (1.4.7);

a) Set k = 1, $F^o_{j-1} = F_{j-1}$, $S^o_j = 0$, and $L_o = 0$, where L_k is an $(m \times n)$ matrix.

b) Evaluate

$$L_k = R^{-1}B^TS^{k-1}_j \qquad (1.4.9)$$

$$F^k_{j-1} = F_{j-1} - BL_k \qquad (1.4.10)$$

c) Obtain the positive-definite (unique) solution S^k_j to the following linear algebraic equation

$$S^k_j F^k_{j-1} + (F^k_{j-1})^TS^k_j + \mu Q + \bar{\rho}_j \quad U(S^{k-1}_j) + L^T_kRL_k = 0 \qquad (1.4.11)$$

The algorithm of Bartels and Stewart [40] is used to solve the equation above.

d) Check for convergence. Let $(S^k_j)_{pq}$ denote the pq element of the matrix S^k_j . If

$$\Delta_{pq} = \left| \frac{(S^k_j)_{pq} - (S^{k-1}_j)_{pq}}{(S^{k-1}_j)_{pq}} \right| \leq \varepsilon \qquad p = 1,\cdots,n;\ q = p,\cdots,n \qquad (1.4.12)$$

where ε is an accuracy parameter, the solution to Eq. (1.4.7) to the desired accuracy, is S_j^k. The convergence of $(S_j^k)_{pq}$ can be accelerated by using the Aitken method. If $\Delta_{pq} > \varepsilon$ for some p and q, increase k by one and go to step b.

1.5 Examples

In this section we will present two simple examples of the use of the methods described throughout the first chapter. Other examples of control system design using the guaranteed cost control method and the multistep guaranteed cost control method can be found in [22],[23] In Chapter 3 a more comprehensive study of these methods, together with a comparison of these and some other methods, will be presented.

a) A First-Order Example

Consider a first-order system of the form of Eq. (1.1.1)

$$\dot{x} = a(\omega)x + b(\theta)u \quad , \qquad t \geq 0 \tag{1.5.1}$$

where

$$a(\omega) = \omega_1 \quad , \qquad -1 \leq \omega_1 \leq 1 \tag{1.5.2}$$

and

$$b(\theta) = 1 + \theta_1 \quad , \qquad -1/2 \leq \theta_1 \leq 1/2 \tag{1.5.3}$$

Let the weighting matrices Q and R be

$$Q = 1 \quad , \qquad R = 1 \tag{1.5.4}$$

i) The output feedback case: Although in a first-order system output feedback is equivalent to full state feedback, we will treat this example as an output feedback problem and will compare the results to those obtained by a full state feedback approach.

Let

$$X_o = 1 \tag{1.5.5}$$

Clearly,

$$H = 1 \tag{1.5.6}$$

We seek to find C so as to minimize $\hat{J}(C)$ in Eq. (1.2.29), where Eq. (1.2.23) is satisfied.

In our example:

$$U(S) = 2S \tag{1.5.7}$$

$$V(C,S) = -CS \tag{1.5.8}$$

Eq. (1.2.23) can be written as follows:

$$-2SC + 2S = -1 - C^2 - CS \tag{1.5.9}$$

or

$$S(2-C) = -1 - C^2 \tag{1.5.10}$$

Clearly, for S positive it must be true that C > 2.

We can write

$$\hat{J}(C) = S = \frac{1 + C^2}{C - 2} \tag{1.5.11}$$

In this simple example we can find the minimum of $\hat{J}(C)$ as follows:

$$\frac{d\hat{J}}{dC} = \frac{(C-2)2C - (1 + C^2)}{(C-2)^2} \tag{1.5.12}$$

which yields

$$C^* = 2 + \sqrt{5} \tag{1.5.13}$$

and

$$\hat{J}(C^*) = 4 + 2\sqrt{5} \tag{1.5.14}$$

A quick test shows that the closed-loop system matrix $(\omega_1 - (1+\theta_1)C^*)$ is stable for all values of the uncertain parameters ω_1 and θ_1.

ii) The full state feedback case: Let us solve the same problem using Eq. (1.2.34). First use Eq. (1.2.37) to find that

$\theta_0 = 1/2$. Then, Eq. (1.2.34) yields

$$1 - \frac{1}{4} S^2 + 2S = 0 \tag{1.5.15}$$

$$S = 4 + 2\sqrt{5} \tag{1.5.16}$$

From Eq. (1.2.30),

$$C^* = \frac{1}{2}(4 + 2\sqrt{5}) = 2 + \sqrt{5} \tag{1.5.17}$$

Hence, the results obtained using the two approaches are equal, as must be the case.

b) A Second-Order Example

Consider a second-order system of the form of Eq. (1.1.1) with

$$A(\omega) = \begin{bmatrix} 0 & 1 \\ -2+\omega_1 & 1+\omega_2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1.5.18}$$

where

$$\begin{aligned} -1 &\leq \omega_1 \leq 1 \\ -1 &\leq \omega_2 \leq 1.5 \end{aligned} \tag{1.5.19}$$

and assume H of Eq. (1.1.2) to be the identity matrix. Let the weighting matrices be

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R = 10 \tag{1.5.20}$$

Using the notation of Eq. (1.1.5) with constant ω yields

$$A_0 = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad (1.5.21)$$

The design will be performed by three methods: the standard linear-quadratic regulator method (LQR), where the parameters are assumed to have their nominal values and the uncertainty is ignored; the guaranteed cost control design method (GCC); and the multistep guaranteed cost control method (MGCC). Feedback gains and closed-loop eigenvalues are given in Tables 1.1 and 1.2. The feedback gains and closed-loop eigenvalues for nominal parameter values obtained with the multistep guaranteed cost control with different design parameters ρ and μ are given in Table 1.3.

Table 1.1 Controller Feedback Gains Determined by GCC and MGCC Design Methods

Design Method	C_{11}	C_{12}
GCC	1.36	6.42
MGCC	.30	3.44
LQR	.03	2.07

Table 1.2 Closed-Loop Eigenvalues for Various Parameter Values Using Feedback Gains Calculated by Various Methods

Design Method	$\omega = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\omega = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$	$\omega = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}$	$\omega = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$	$\omega = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
LQR	-.54±1.32j	-1.04±1.4j	+.21±1.73j	+.21± .99j	-.82,-1.26
GCC	-.71,-4.71	-.77,-5.65	-1.96±.72j	-.74,-3.18	-.39,-6.03
MGCC	-1.22±.90j	-1.72±.58j	-.47±1.75j	-.47±1.04j	-.43,-3.01

Table 1.3 Feedback Gains and Closed-Loop Eigenvalues for Nominal Parameter Values Achieved Using the Multistep Guaranteed Cost Control Method

μ	Iterations of Main Algorithm	C_{11}	C_{12}	Closed-Loop Eigenvalues $\omega = 0$
.01	5 *	0.30	3.44	-1.22±.90j
.01	4 *	0.34	3.55	-1.28±.84j
.01	3 *	0.61	4.40	-2.23,-1.17
.5	5 *	0.37	3.55	-1.28±.86j
.5	9	0.36	3.40	-1.20±.96j
1.0	7	0.42	3.46	-1.23±.95j

*Sequence $\{\rho_i\}$ input to program.

Note the relatively large feedback gains and the real, overdamped closed-loop eigenvalues produced by the guaranteed cost control design method. One should note that the closed-loop system obtained with the LQR design is unstable at $\omega = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}$ and $\omega = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$.

2. MINIMUM EXPECTED COST DESIGN METHOD

2.1 Statement of the Control Problem

Using the state space notation, one may describe the dynamics of many systems by the linear differential equations

$$\dot{x}(t) = A(\omega)\, x(t) + B(\omega)\, u(t) \quad , \qquad t \geqslant 0 \tag{2.1.1}$$

$$y(t) = H(\omega)\, x(t) \tag{2.1.2}$$

where

$x(t)$ = state vector $(n \times 1)$

$u(t)$ = control vector $(m \times 1)$

ω = vector of uncertain parameters referenced to their nominal values $(n' \times 1)$

$A(\omega)$ = open-loop dynamics matrix $(n \times n)$

$B(\omega)$ = control distribution matrix $(n \times m)$

$y(t)$ = vector of measured outputs $(r \times 1)$

$H(\omega)$ = output distribution matrix $(r \times n)$

t = independent variable

The following assumptions are employed throughout this chapter:

1) The vector ω is constant and lies somewhere within a closed bounded region $\Omega \in R^{n'}$ (n'-dimensional real cartesian space).
2) $[A(\omega), B(\omega)]$ form a controllable pair for all $\omega \in \Omega$.
3) Control system performance is characterized by a quadratic cost functional J, defined as follows:

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt \tag{2.1.3}$$

where Q is an $(n \times n)$ constant positive semidefinite symmetric matrix, and R is an $(m \times m)$ constant positive definite symmetric matrix.

4) $[A(\omega), Q^{1/2}]$ form an observable pair for all $\omega \in \Omega$, where $Q^{1/2}$ denotes any square root of the matrix Q, i.e., $[Q^{1/2}]^T Q^{1/2} = Q$.

5) The distribution of ω within Ω is described by a probability density function $p(\omega)$, defined as follows:

$$p(\omega) = \begin{cases} p_0(\omega) \quad , & \omega \in \Omega \\ 0 \quad , & \omega \in \Omega \end{cases} \tag{2.1.4}$$

where $p_0(\omega)$ has the following properties:

$$\begin{aligned} & p_0(\omega) \geq 0 \\ & \int_\Omega p_0(\omega)\, d\omega = 1 \end{aligned} \tag{2.1.5}$$

The objective of our discussion is to design a linear constant gain feedback controller of the form:

$$u(t) = -Cy(t) \ , \qquad t \in [0,\infty] \tag{2.1.6}$$

which will produce acceptable closed-loop behavior for all $\omega \in \Omega$, $t \in [0,\infty]$. The matrix C is a constant $(m \times r)$ matrix.

2.2 Minimum Continuous Expected Cost Design Method

It is well known [24], [41], that for a system with perfectly known parameters and an exact measurement of the state vector available as a function of time, the minimization of J, as defined in Eq. (2.1.3), yields a linear constant gain feedback control law. This control law is independent of the initial conditions and possesses some attractive properties in the frequency domain, namely, an infinite gain margin, a phase margin of 60° (or more), and at least a 50% gain reduction tolerance [32]. In cases in which system parameters are not known exactly or only a portion of the state vector is available for feedback (i.e., rank $H < n$), the optimal set of gains C does not possess the above properties. For any given constant gain matrix C, the value of J, using u(t) as defined in Eq. (2.1.6), will be as follows:

$$J = x^T(0)\ S(\omega,C)\ x(0) \tag{2.2.1}$$

where x(0) is the state vector at time $t=0$, subject to the condition that the closed-loop system matrix $[A(\omega) - B(\omega)\ CH(\omega)]$ obtained using Eq. (2.1.6) be a stability matrix (i.e., all its eigenvalues lie in the open left-half of the complex plane). $S(\omega,C)$ in Eq. (2.2.1) is a symmetric positive semi-definite matrix which satisfies the following matrix equation:

$$\begin{aligned} S(\omega,C)[A(\omega) - B(\omega)CH(\omega)] + [A(\omega) - B(\omega)CH(\omega)]^T\ S(\omega,C) \\ = -Q - H^T(\omega)\, C^T RCH(\omega)\ , \quad \text{for} \quad \omega \in \Omega \end{aligned} \tag{2.2.2}$$

where Q and R are the weighting matrices given in Eq. (2.1.3).

Equation (2.2.2) has the form of a Lyapunov equation. Some conditions regarding the existence of solutions to this equation can be found in [24],[25]. From Eqs. (2.2.1) and (2.2.2) one can see that the cost functional J is a function of the initial condition vector x(0), the vector of uncertain parameters ω, and the set of feedback gains C. The boundedness of J is dependent upon the stability of the closed-loop system.

Given the probability density function $p(\omega)$ as defined in Eqs. (2.1.4), (2.1.5), it is possible to define a cost functional J^* as follows [42]-[46]:

$$J^*[x(0),C] = E_\omega[J[x(0),\omega,C]] = \int_{\omega\in\Omega} J[x(0),\omega,C]\; p(\omega)\; d\omega$$

$$= x^T(0)\Big[\int_{\omega\in\Omega} S(\omega,C)\; p(\omega)\; d\omega\Big]\; x(0) \qquad (2.2.3)$$

where $E_\omega[\cdot]$ stands for the expected value with respect to the uncertain vector ω [47]. Thus, $J^*[x(0),C]$ is the expected value of the cost functional $J[x(0),\omega,C]$ over all $\omega \in \Omega$. For J^* to be bounded, the set of feedback gains C must produce a closed-loop system matrix which is stable for every $\omega \in \Omega$. General conditions regarding the existence of such a C are still to be determined (see Section 1.2). It is clear that there are cases where such a gain matrix C can be found. In some other cases, where the region Ω is simply too large, no such gain matrix exists.

Using the same arguments as in Section 1.2, we will define a new cost functional $\hat{J}(C)$ to be independent of the initial conditions

$$\hat{J}(C) = Tr[S^*(C)X_0] \qquad (2.2.4)$$

where

$$X_o = E[x(0)\ x^T(0)] \tag{2.2.5}$$

$$S^*(C) = \int_{\omega \in \Omega} S(\omega,C)\ p(\omega)\ d\omega \tag{2.2.6}$$

In Eq. (2.2.5) the expected value is taken with respect to the random variable, x(0). Clearly, $\hat{J}(C)$ depends on statistical knowledge of the initial conditions. In many cases, no specific data on the initial conditions exist. Following the discussion in Section 1.2, Eq. (1.2.28), one might then assume the following to hold

$$X_o = I \tag{2.2.7}$$

so that $\hat{J}(C)$ becomes

$$\hat{J}(C) = Tr[S^*(C)] \tag{2.2.8}$$

Let us assume that there exists a nonempty subset $\mathcal{C}$ of R^{mxr} such that $[A(\omega) - B(\omega)CH(\omega)]$ is stable for every $\omega \in \Omega$ and all $C \in \mathcal{C}$. The design objective is to determine $C \in \mathcal{C}$ so as to minimize $\hat{J}(C)$. The evaluation of $\hat{J}(C)$ is quite complicated in general. It involves obtaining $S(\omega,C)$ as a function of $\omega \in \Omega$ for fixed C, through Eq. (2.2.2), and then evaluating the integral in Eq. (2.2.6). Both these processes can be done analytically only in simple cases, such as a low order system expressed in phase canonical form [13].

In order to minimize $\hat{J}(C)$ one should derive the gradient of $\hat{J}(C)$ with respect to C. An expression for the gradient is given in [14], based upon a derivation in [48] for problems with known parameters. We will follow a derivation suggested by Dr. L. J. Wood.

Let us define the matrix $D(\omega,C)$ as follows:

$$D(\omega,C) = S(\omega,C)[A(\omega) - B(\omega)CH(\omega)] + [A(\omega) - B(\omega)CH(\omega)]^T S(\omega,C) + Q + H^T(\omega)C^T RC\, H(\omega) \qquad (2.2.9)$$

$$\forall\ \omega \in \Omega\ ,\ C \in \mathcal{C}$$

From Eq. (2.2.2) it is clear that

$$D(\omega,C) \equiv 0\ ,\quad \forall\ \omega \in \Omega\ ,\quad C \in \mathcal{C} \qquad (2.2.10)$$

Equation (2.2.4) can be written as follows:

$$\hat{J}(C) = E_\omega\{Tr[S(\omega,C)X_o]\} \qquad (2.2.11)$$

where use has been made of Eq. (2.2.6) and the fact that the order of the expected value operator and any linear operator can be interchanged [47].

The objective now is to minimize $\hat{J}(C)$ subject to the constraints on $S(\omega,C)$, Eq. (2.2.2). This can be done by adjoining the constraints to $\hat{J}(C)$ with a matrix of Lagrange multipliers. Thus, let us define $\mathcal{P}$ as

$$\mathcal{P} = \int_{\omega\in\Omega} I(\omega,C) = \hat{J}(C) + Tr\{E_\omega[L(\omega,C)\ D(\omega,C)]\} \qquad (2.2.12)$$

where $L(\omega,C)$ is an $(n \times n)$ symmetric matrix of Lagrange multipliers, $\mathcal{P}$ now will be minimized with C and $S(\omega,C)$ treated as independent. Use will be made below of the following four lemmas pertaining to matrix traces:

Lemma 1: For A an arbitrary square matrix

$$Tr(A) = Tr(A^T) \tag{2.2.13}$$

Lemma 2: For arbitrary square matrices A and B of consistent dimension

$$Tr(A+B) = Tr(A) + Tr(B) \tag{2.2.14}$$

Lemma 3: For arbitrary A (n x m) and B (m x n) matrices

$$Tr(AB) = Tr(BA) \tag{2.2.15}$$

Lemma 4: For the above matrices A and B

$$\frac{Tr(AB)}{B} = A^T \tag{2.2.16}$$

Proofs of the above lemmas are based on the definitions of the trace operator and of the differentiation of a scalar with respect to a matrix.

Two stationarity conditions for $\mathcal{P}$ are

$$\frac{\partial \mathcal{P}}{\partial C} = 0 \tag{2.2.17}$$

and Eq. (2.2.10). Using the above lemmas together with the fact that differentiation is a linear operator, one gets:

$$\frac{\partial \mathcal{P}}{\partial C} - 2\{RCE_\omega[H(\omega)\ L(\omega,C)H^T(\omega)] - E_\omega[B^T(\omega)\ S(\omega,C)\ L(\omega,C)\ H^T(\omega)]\} = 0, \quad C \in \mathcal{C} \tag{2.2.18}$$

Now define $I(\omega,C)$ (the integrand of $\mathcal{P}$) as

$$I(\omega,C) = Tr[S(\omega,C)X_0] + Tr[L(\omega,C)D(\omega,C)] \ \forall \omega \in \Omega, \quad C \in C \tag{2.2.19}$$

At the stationary point, variations of the integrand of $\mathcal{P}$ with respect to S(ω,C) should vanish. Thus, let us choose the matrix of Lagrange multipliers L(ω,C) so as to make $\frac{\partial I(\omega,C)}{\partial S} = 0$. Then,

$$L(\omega,C)[A(\omega) - B(\omega)CH(\omega)]^T + [A(\omega) - BCH(\omega)]L(\omega,C) = -X_0 \qquad (2.2.20)$$

$$\omega \in \Omega \,, \quad C \in \mathcal{C}$$

This is the third stationarity condition for $\mathcal{P}$.

Thus, in order to find the set of gains C so as to minimize $\hat{J}(C)$, Eqs. (2.2.2), (2.2.18), and (2.2.20) should be solved. It is clear that solving these equations is not practical in the general case. In some special cases, where the order of the system is low (first or second order), and p(ω) has a simple form (such as a uniform density function of a set of δ-functions), an analytical solution might be found. In order to implement the idea of minimum expected cost as a useful design method, one must modify the approach described above [23].

2.3 Minimum Discrete Expected Cost Design Method

Determining the set of feedback gains C so as to minimize $\hat{J}(C)$ (defined in Eq. (2.2.4)), requires the solution of one integral equation (Eq. (2.2.18)), and two linear equations (Eqs. (2.2.2),(2.2.20)), over all $\omega \in \Omega$. In general, an analytical solution does not exist when a continuous probability density function is assumed. When numerical minimization techniques are used to find C, the computational requirements are excessive [13]. Each evaluation of $\hat{J}(C)$ requires an integration over all $\omega \in \Omega$. Each integration step requires the solution of an n^{th} order algebraic Lyapunov equation, while the gradient evaluation requires the solution of an adjoint n^{th} order Lyapunov equation (Eq. 2.2.21) for all $\omega \in \Omega$.

These computational obstacles can be reduced to a manageable level, however, if the point of view is taken that the inclusion of all $\omega \in \Omega$ in the definition of $\hat{J}(C)$ is unnecessary in practical terms. If only a fairly small number of points in Ω, which are in some sense representative of the entire region, are considered (for example, the corners and selected intermediate points), much computational effort can be avoided at little cost in performance. In effect, the continuous probability density function $p(\omega)$ in Eq. (2.2.6) is replaced by a collection of Dirac delta functions throughout Ω, whose amplitudes add to unity [23].

A more rigorous argument to justify the above point of view will be presented here. We will assume the following statements to hold:

1) The region Ω, defined in Section 2.1, is assumed to be a convex polygon.
2) The system matrices $A(\omega)$, $B(\omega)$, $H(\omega)$ defined in Eqs. (2.1.1) and (2.1.2) have the special form

$$\left.\begin{aligned} A(\omega) &= A_o + \sum_{i=1}^{n'} \omega_i A_i \\ B(\omega) &= B \\ H(\omega) &= H \end{aligned}\right\} \quad \forall\ \omega \in \Omega \qquad (2.3.1)$$

where A_o, A_i, $i=1,\cdots,n'$, B, and H are constant matrices.

3) $$p(\omega) = \alpha \text{ (constant)} \qquad \forall\ \omega \in \Omega \qquad (2.3.2)$$

Although the three assumptions stated restrict the generality of our discussion, many engineering systems satisfy these conditions.

Theorem 2.1: Let $S_k^*(C)$ be defined as follows:

$$S_k^*(C) = \frac{1}{N_k} \sum_{i=1}^{N_k} S(\omega_i, C) \quad , \quad \omega_i \in \Omega,\ i=1,\cdots,N_k \quad C \in \mathcal{C} \qquad (2.3.3)$$

where

$$N_k = (2^k + 1)^{n'} \qquad (2.3.4)$$

and $S(\omega_i, C)$ satisfies Eq. (2.2.2) for each ω_i. Let us assume the k^{th} set of points $\{\omega_i;\ i=1,\cdots,N_k\}$, to be distributed uniformly over Ω and to include all the vertices of Ω. The function $Tr[S(\omega,C)]$ is assumed to be a convex function over Ω. Then:

i) $$\lim_{k \to \infty} S_k^*(C) = S^*(C) = \int_{\omega \in \Omega} S(\omega, C)\ p(\omega)\ d\omega \ , \quad C \in \mathcal{C} \qquad (2.3.5)$$

ii) The sequence $Tr[S_k^*(C)]$ is bounded and monotone decreasing.

Proof: Use of Eqs. (2.3.2), (2.3.3), and (2.3.4), together with the definition of an integral [49], proves that the limiting process on the left-hand side of Eq. (2.3.5) converges to the value of the integral on the right-hand side. Hence, the first part of the theorem has been proved.

For the proof of the second part of this theorem we will make use of the assumption that $Tr[S(\omega,C)]$ is a convex function. One can find a definition of convex functions and some of their properties in [50]. We cannot establish this hypothesis as a fact yet, but we have been unable to find an example where it was not true. Let us state here two properties of a convex function defined on bounded, closed convex region Ω .

a) Let $f(x)$ be a convex function over Ω and let $x_1, x_2 \in \Omega$. Then for every $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)\ f(x_2) \tag{2.3.6}$$

holds.

b) If $f(x)$ has a maximum over Ω, it is achieved at an extreme point of Ω .

Clearly, from our assumptions and hypotheses

$$Tr[S_0^*(C)] \geq Tr[S_k^*(C)]_{k>0} \geq Tr[S^*(C)] \quad , \qquad C \in \mathcal{C} \tag{2.3.7}$$

where use has been made of property (b) stated above. (When $k = 0$, the set $\{\omega_i,\ i=1,\cdots,2^{n'}\}$ includes only the vertices of Ω).

For simplicity and clarity of the presentation, we will demonstrate the fact that $Tr[S_k^*(C)]$ is a monotone decreasing sequence for $n' = 1$ (i.e., the closed, bounded, convex region is a line in R). We will make use of Eq. (2.3.6), with $\lambda = 1/2$, to correspond to an intermediate point with two equidistant neighboring points. The value of the function at an intermediate point can be represented by

$$f(x_3) = f(\frac{x_1+x_2}{2}) = \frac{1}{2}[f(x_1) + f(x_2)] - \varepsilon_{1,2}\ , \quad \varepsilon_{1,2} \geq 0 \qquad (2.3.8)$$

To prove the monotone decreasing feature of the sequence, $Tr[S_k^*(C)]$, one would have to show that

$$Tr[S_k^*(C)] \geq Tr[S_{k+1}^*(C)]\ , \quad k = 0,1,\cdots,\ C \in \mathcal{C} \qquad (2.3.9)$$

Let us define

$$f_i = Tr[S(\omega_i,C)]\ , \qquad i = 1,\cdots,N_k,\ C \in \mathcal{C} \qquad (2.3.10)$$

Let us define

$$g_i = Tr[S(\omega_i,C)]\ , \qquad i = N_k+1,\ \cdots,N_{k+1} \qquad (2.3.11)$$

as the value of $Tr[S(\omega,C)]$ evaluated at the points included in the $k+1^{st}$ set but not in the k^{th} set. Now define Δ_k as

$$\Delta_k = Tr[S_k^*(C)] - Tr[S_{k+1}^*(C)]\ , \qquad C \in C \qquad (2.3.12)$$

Eqs. (2.3.3), (2.3.4), (2.3.10), (2.3.11), and (2.3.12) yield

$$\Delta_k = \frac{1}{N_k}\sum_{i=1}^{N_k} f_i - \frac{1}{N_{k+1}}\sum_{i=1}^{N_k} f_i + \sum_{i=N_k+1}^{N_{k+1}} g_i] \qquad (2.3.13)$$

Making use of Eq. (2.3.8) yields

$$\Delta_k = \frac{1}{N_k}\sum_{i=1}^{N_k} f_i - \frac{1}{N_{k+1}}[\sum_{i=1}^{N_k} f_i + \sum_{i=1}^{N_k-1} (\frac{f_i+f_{i+1}}{2} - \varepsilon_{i,i+1})] \qquad (2.3.14)$$

Rearrangement yields

$$\Delta_k = \frac{1}{N_k \cdot N_{k+1}}[(2^k-1)\frac{f_1+f_{Nk}}{2} - \sum_{i=1}^{2^k} f_i + N_k E_1] \qquad (2.3.15)$$

where

$$E_1 = \sum_{i=1}^{2^k} \varepsilon_{i,i+1} \geq 0 \tag{2.3.16}$$

Eq. (2.3.8) yields

$$f_i = \lambda_i f_1 + (1-\lambda_i) f_{N_k} - \varepsilon_i \ , \qquad \varepsilon_i \geq 0, \ i = 2,\cdots,2^k \tag{2.3.17}$$

where

$$\lambda_i = \frac{i-1}{2^k} \ , \qquad i = 2,\cdots,2^k \tag{2.3.18}$$

Eqs. (2.3.15), (2.3.16), (2.3.17), and (2.3.18) yield

$$\Delta_k = \frac{E_1}{N_{k+1}} + \frac{E_2}{N_k \cdot N_{k+1}} \tag{2.3.19}$$

where

$$E_2 = \sum_{i=2}^{2^k} \varepsilon_i \geq 0 \tag{2.3.20}$$

Hence, from Eqs. (2.3.15), (2.3.19), and (2.3.20)

$$\Delta_k \geq 0 \quad , \qquad k = 0,1,\cdots, \tag{2.3.21}$$

which proves that $Tr[S_k^*(C)]$ is a monotone decreasing sequence.

We have shown that $Tr[S^*(C)]$ (as defined by Eq. (2.2.6)) is bounded from above by $Tr[S_k^*(\omega,C)]$ and that this upper bound decreases as k increases. Our experience shows that the relative difference $\Delta_k/Tr[S^*(C)]$, becomes small even for fairly small values of k.

An algorithm that makes use of the above theorem as a way to overcome the computational difficulties involved in solving Eqs. (2.2.2), (2.2.18), and (2.2.20) is presented in Section 2.4.

2.4 Algorithm for the Minimum Discrete Expected Cost Design Method

In this section we will present an algorithm for the design of a constant gain feedback control law, using the minimum expected cost design technique.

The algorithm is as follows:

1. Input: $A(\omega)$, $B(\omega)$, $H(\omega)$, Ω, $p(\omega)$, X_0, Q , R, N.

2. Select points $\omega_i \in \Omega$, $i = 1,\cdots,N$ distributed over Ω. (Points on the boundary of Ω should be included.)

3. Evaluate
$$\alpha_i = \frac{p(\omega_i)}{M} , \qquad i = 1,\cdots,N \tag{2.4.1}$$
where M is defined as
$$M = \sum_{i=1}^{N} p(\omega_i) \tag{2.4.2}$$

4. Choose $C_0 \in \mathcal{C}$ such that $[A(\omega_i) - B(\omega_i)C_0H(\omega_i)]$ is a stability matrix for every $\omega_i \in \Omega$, $i = 1,\cdots,N$. The choice of C_0 can be done by an alternate design technique (the multistep guaranteed cost control, for example) or by trial and error. Set j equal to zero.

5. Use a quasi-Newton method [52],[53] to find the value of $C \in C$ which minimizes $\hat{J}_N(C)$. The program [54] makes use of the gradient $D_N(C)$ and requires the evaluation of $\hat{J}_N(C)$. Determination of $D_N(C)$, $\hat{J}_N(C)$ requires the positive-definite solutions $S(\omega_i,C_j)$ and $L(\omega_i,C_j)$ to the following Lyapunov equations:
$$S(\omega_i,C_j)[A(\omega_i) - B(\omega_i)C_jH(\omega_i)] + [A(\omega_i) - B(\omega_i)C_jH(\omega_i)]^T S(\omega_i,C_j)$$
$$= -Q - H^T(\omega_i)\, C_j^T R C_j H(\omega_i), \quad i=1,\cdots,N , \quad C_j \in C \tag{2.4.3}$$

$$L(\omega_i,C_j)[A(\omega_i)-B(\omega_i)C_jH(\omega_i)]^T+[A(\omega_i)-B(\omega_i)C_jH(\omega_i)]\,L(\omega_i,C_j)$$

$$=-X_o \qquad , \quad i=1,\cdots,N \ , \quad C_j\in \mathcal{C} \qquad (2.4.4)$$

Eqs. (2.4.3) and (2.4.4) are adjoint. Thus, when the algorithm of Bartels and Stewart [40] is used to solve Eq. (2.4.3), a transformation suggested in [51] enables the efficient solution of Eq. (2.4.4) at the same time. Evaluate

$$\hat{J}_N(C_j) = \sum_{i=1}^{N} Tr[S(\omega_i,C_j)X_o]\cdot\alpha_i \qquad , \qquad C_j\in \mathcal{C} \qquad (2.4.5)$$

$$D_N(C_j) = 2\{RC_j[\sum_{i=1}^{N} H(\omega_i)L(\omega_i,C_j)H^T(\omega_i)\cdot\alpha_i]$$

$$-[\sum_{i=1}^{N} B^T(\omega_i)S(\omega_i,C_j)\,L(\omega_i,C_j)H^T(\omega_i)\cdot\alpha_i]\} \ , \ C_j\in \mathcal{C} \qquad (2.4.6)$$

$\hat{J}_N(C_j)$, it follows from Theorem 2.1, is an approximation to $\hat{J}(C_j)$ in Eq. (2.2.12). $D_N(C_j)$ is the gradient of $\hat{J}_N(C)$ with respect to C, evaluated at C_j. This follows from Eqs. (2.2.18) and (2.2.20).

6. Check the closed-loop system behavior over Ω, achieved with the use of the control law

$$u(t) = -Cy(t) \qquad (2.4.7)$$

If the results are satisfactory stop, otherwise increase N and go to step 2.

The algorithm described above for the discrete-valued uncertain parameter minimum expected cost design method is similar in philosophy to the design concept described in [15].

The choice of N and the selection of $\omega_i \in \Omega$, $i = 1,\cdots,N$, although important, are not critical for a successful design. Our experience has shown that the sensitivity of the controller design and closed-loop system performance with respect to N is not great [23]. This is demonstrated in our examples in Section 2.5 and Section 3.2.

2.5 Examples

Two examples of the use of the minimum discrete expected cost method will be presented here, a second-order system that was considered in [13] and was presented in Section 1.5, and a third-order system, similar to a problem considered in [21]. The application of this design method to a higher order system was presented in [23] and will be discussed in Chapter 3.

i) A Second-Order System

Consider a second-order system of the form (2.1.1) with

$$A(\omega) = \begin{bmatrix} 0 & 1 \\ -2+\omega_1 & 1+\omega_2 \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.5.1}$$

where

$$\begin{aligned} p(\omega_1) &= .5 , \quad -1 \le \omega_1 \le 1 \\ p(\omega_2) &= .4 , \quad -1 \le \omega_2 \le 1.5 \end{aligned} \tag{2.5.2}$$

The weighting matrices Q and R are

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad R = 10 \tag{2.5.3}$$

The controller gain matrix

$$C = [C_{11} \quad C_{12}] \tag{2.5.4}$$

was calculated using various values of N to evaluate $\hat{J}_N(C)$. An analytical expression for $\hat{J}(C)$ can be obtained in this particular example

[13]. Numerical techniques are needed only for minimizing $\hat{J}(C)$ with respect to C. Results generated by this means are denoted MEC, while MDEC(30) and MDEC(99) denote results generated by the minimum discrete expected cost method using 30 and 99 points, respectively. For comparison, results obtained using the linear-quadratic regulator approach (denoted as LQR) are also presented. In this approach, the parameters are assumed to have their nominal values--uncertainty is ignored. In Table 2.1, the controller gains and the trace of the cost function for $\omega = 0$ are presented. In Table 2.2, the eigenvalues of the closed-loop system matrix $A(\omega)-BC$ are evaluated for five values of the uncertain parameter vector ω . In Section 1.5, the same problem was solved using the guaranteed cost control method and the multistep guaranteed cost control method.

Table 2.1 Controller Feedback Gains and Trace of Cost Matrix for $\omega = 0$ for the Various Design Methods — Second-Order Example

Design Method	c_{11}	c_{12}	Tr[S(0,C)]
LQR	0.02	2.07	62
MEC	0.38	3.69	74
MDEC(99)	0.42	3.84	76
MDEC(30)	0.45	3.97	77

Table 2.2 Closed-Loop Eigenvalues for Various Parameter Values Using Feedback Gains Calculated by the Various Methods — Second-Order Example

Design Method	$\omega = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\omega = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$	$\omega = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}$	$\omega = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$	$\omega = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
LQR	-.54±1.32j	-1.04±1.4j	+.21±1.73j	+.21± .99j	-.82,-1.26
MEC	-1.34± .76j	-1.66,-2.02	-.59±1.74j	-.59±1.01j	-.42,-3.27
MDEC(99)	-1.42± .63j	-1.40,-2.44	-.67±1.72j	-.67± .98j	-.41,-3.43
MDEC(30)	-1.49±..49j	-1.28,-2.69	-.74±1.70j	-.74± .95j	-.41,-3.57

Note that the standard linear-quadratic regulator approach produces an unstable closed-loop system for $\omega = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}$ or $\omega = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$.

ii) A Third-Order System

Consider a third-order system of the form (2.1.1) with

$$A(\omega) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \omega \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad H = I \tag{2.5.5}$$

where

$$p(\omega) = 1/3 , \quad -1.5 \leq \omega \leq 1.5 \tag{2.5.6}$$

The weighting matrices Q and R are:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = 1 \tag{2.5.7}$$

The controller gain matrix

$$C = [C_{11} \quad C_{12} \quad C_{13}] \tag{2.5.8}$$

was calculated using various numbers of points to evaluate $\hat{J}_N(C)$. In Table 2.3, the controller gains and the value of $\hat{J}_N(C)$ are presented. In Table 2.4, the eigenvalues of the closed-loop system matrix when the uncertain parameter assumes its nominal value are presented.

Table 2.3 Controller Feedback Gains and $\hat{J}_N(C)$ Using Various Numbers of Points in Parameter Space

No. of Points	C_{11}	C_{12}	C_{13}	$\hat{J}_N(C)$
3	1.52	2.96	3.94	10.293
7	1.44	2.76	3.53	9.414
11	1.41	2.71	3.42	9.214
31	1.38	2.65	3.30	9.005

Table 2.4 Closed-Loop Eigenvalues for Nominal Parameter Values Using Various Numbers of Points in Parameter Space

No. of Points	Closed-Loop Eigenvalues for $\omega = 0$	
3	$-.39 \pm .57j$	-3.15
7	$-.41 \pm .60j$	-2.71
11	$-.42 \pm .61j$	-2.59
31	$-.43 \pm .62j$	-2.45

Note the relatively small changes in the feedback gains and the cost function when the number of points is increased.

3. A COMPARISON OF SEVERAL DESIGN METHODS FOR SYSTEMS WITH UNCERTAIN PARAMETERS

3.1 Description of the Various Design Methods

In Chapters 1 and 2 we presented the multistep guaranteed cost control design method and the minimum discrete expected cost design method. A survey of several design methods for systems with uncertain parameters was done by Harvey and Pope [19]. A comparative assessment of seven such methods was made in the context of wing load alleviation for the C-5A, with uncertainties assumed to exist in dynamic pressure, structural damping and frequency, and the stability derivative M_w. The techniques investigated were referred to as the additive noise design, the minimax design [20] the multiplant design, the sensitivity vector augmentation design ([16], for example), the state dependent noise design, the mismatch estimation design, and the uncertainty weighting design. Most of the methods were found to be at least somewhat burdensome computationally, and most did not produce control system designs judged to be significant improvements over a standard linear-quadratic synthesis design [41]which assumes precisely known parameters. The uncertainty weighting and minimax techniques were judged to be generally superior to the other approaches. In this chapter the minimax method, the uncertainty weighting method, and the standard linear-quadratic synthesis method, the guaranteed cost control method, the multistep guaranteed cost control method, and the minimum discrete expected cost design method are compared. The notation used follows that in Sections 1.1 and 2.1.

Let us describe the various design methods:

Standard Linear-Quadratic Regulator Design Method

The standard linear-quadratic regulator design, as described in [41], bases the choice of controller feedback gains on nominal parameter values. Parameter uncertainties are not explicitly taken into account in the design process. The controller gains are determined by finding the positive definite solution to the algebraic Riccati equation

$$SA_0 + A_0^T S - SBR^{-1}B^T S + Q = 0 \tag{3.1.1}$$

and then evaluating

$$C = -R^{-1}B^T S \tag{3.1.2}$$

Under the controllability, observability, and positivity assumptions made above, a unique positive definite solution to the algebraic Riccati equation is guaranteed to exist [25]. It is easily determined using the method of eigenvector decomposition [37]. The resulting closed-loop system is guaranteed to be stable when the uncertain parameters lie within some neighborhood of their nominal values. In many situations, though, this neighborhood does not include all of Ω.

Minimax Method

The goal of the minimax design method [19], [20] is to choose feedback gains so as to optimize the performance when the uncertain parameters assume their most unfavorable possible values. Specifically, we seek the gain set $C^* \in R^{mxn}$ such that

$$Tr[X_0 S(\omega_w, C^*)] \leq Tr[X_0 S(\omega_w, C)], \quad \forall\, C \in R^{mxn} \tag{3.1.3}$$

where $\omega_W \in \Omega$ is such that

$$\mathrm{Tr}[X_o S(\omega_W, C^*)] \geq \mathrm{Tr}[X_o S(\omega, C^*)], \quad \forall\ \omega \in \Omega \tag{3.1.4}$$

If ω_W has been determined subject to no constraints on C, determination of C^* is simple--the problem reduces to a standard linear-quadratic regulator design with $\omega = \omega_W$. Determination of ω_W is quite expensive computationally, in general. In cases where Ω, the region of parameter uncertainty, is a polygon in $R^{n'}$ space, and $A(\omega)$ has the structure of Eq. (1.1.5), ω_W will lie at one of the vertices of Ω [55]. When there are no constraints imposed on C, ω_W might not exist, as will be shown later. When C is restricted to be in a subset $C \in R^{mxn}$, ω_W may exist, but then the determination of C^* at the point ω_W becomes a complicated process [19],[20].

Uncertainty Weighting Method

The uncertainty weighting method was suggested by Porter in [19]. The basic idea is to minimize the difference in output occurring when parameters assume their worst possible values, i.e., $\omega = \omega_W$, relative to when they assume their nominal values.

One can describe the dynamics of the corresponding state variations as follows:

$$\Delta\dot{x} = A(\omega_W)\Delta x + (A(\omega_W) - A_o)x_o \tag{3.1.5}$$

where x_o denotes the state vector when the parameters assume their nominal values.

In order to keep Δx small one would like to keep $[A(\omega_W) - A_o]x_o$ small. Thus, let us define

$$\tilde{x} \triangleq [A(\omega_W) - A_o]x \tag{3.1.6}$$

and augment the cost functional J by including the term

$$\tilde{x}^T \tilde{Q} \tilde{x} \tag{3.1.7}$$

in it. Hence, once ω_w is known (it can be found with the minimax method), a standard linear-quadratic regulator design is carried out with $\omega = 0$. The state weighting matrix Q, used in the design process is replaced by the matrix

$$Q + \lambda[A(\omega_w) - A_0]^T \tilde{Q}[A(\omega_w) - A_0] \tag{3.1.8}$$

where $\lambda > 0$ is a design parameter to be selected. The usefulness of this method in any given situation depends upon the existence of ω_w and upon one's ability to identify it.

Guaranteed Cost Control Method

With the guaranteed cost control design method [21], controller gains are determined by

$$C = -R^{-1}B^TS \tag{3.1.9}$$

where S is the solution to the matrix equation

$$SA_0 + A_0^T S - SBR^{-1}B^TS + Q + U(S) = 0 \tag{3.1.10}$$

U(S) can be constructed in accordance with Eqs. (1.2.18)-(1.2.20) or in accordance with the procedure described in Appendix A, whichever is suitable. Eq. (3.1.10) can be solved by an extension of Kleinman's method described in Section 1.5.

New Methods

The multistep guaranteed cost control design method and the minimum discrete expected cost are presented in Chapters 1 and 2, respectively.

3.2 Design of a Fifth-Order Lateral Autopilot for a Remotely Piloted Vehicle

One may describe the lateral dynamics of a remotely piloted vehicle by

$$\dot{x} = A(\omega)x + Bu \quad , \quad t \geq 0 \tag{3.2.1}$$

$$A(\omega) = A_o + \omega_1 A_1 \tag{3.2.2}$$

with

$$x^T = [v,p,r,\phi,\delta_a], u = \delta_{a_c}, \omega = \frac{C_{n_{\delta_a}} - 2.0}{2.0} \tag{3.2.3}$$

where

v = component of vehicle velocity parallel to pitch axis

p = vehicle roll rate

r = vehicle yaw rate

ϕ = vehicle roll angle

δ_a = aileron deflection

δ_{a_c} = commanded aileron deflection

$C_{n_{\delta_a}}$ = dimensionless partial derivative of moment about vehicle yaw axis with respect to aileron deflection

and

$$A_o = \begin{bmatrix} -.85 & 25.47 & -979.5 & 32.14 & 0 \\ -.339 & -8.789 & 1.765 & 0 & 59.89 \\ .021 & -.547 & -1.407 & 0 & 6.477 \\ 0 & 1 & .0256 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix} \tag{3.2.4}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20 \end{bmatrix} , \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.71 \\ 0 & 0 & 0 & 0 & 3.22 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.2.5}$$

$$-1.5 \leq \omega \leq 0.5 \qquad (3.2.6)$$

The above perturbation equations describe the lateral dynamics of a particular remotely piloted vehicle which has no rudder. The aerodynamic derivative $C_{n_{\delta_a}}$ is assumed to be an unknown constant within the range

$$-1.0 \leq C_{n_{\delta_a}} \leq 3.0 \ , \qquad p(C_{n_{\delta_a}}) = .25 \qquad (3.2.7)$$

for this flight condition. Its nominal value is assumed to be 2.0 ($\omega = 0$). The open-loop eigenvalues at nominal ω are: -20.0, -9.90, -.56 ± 6.28j, -.035. The following features are desired of the closed-loop system:

1. The closed-loop system must be stable for all $C_{n_{\delta_a}}$ consistent with Eq. (3.2.7).
2. The closed-loop poles corresponding to vehicle (as opposed to actuator) dynamics should be in the vicinity of -5.0, -0.2, and -1.4±3.1j when $C_{n_{\delta_a}}$ takes on its nominal value.
3. The sensitivity of closed-loop pole locations to changes in $C_{n_{\delta_a}}$ should be as small as possible.

In accordance with requirement 2, the desired closed-loop behavior was modeled as

$$\dot{y} = Ly \qquad , \qquad y = Hx \qquad (3.2.8)$$

where

$$L = \begin{bmatrix} -2.6 & 0 & 2 & 0 \\ 0 & -2.8 & 0 & -11.57 \\ 2.88 & 0 & -2.6 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad (3.2.9)$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad (3.2.10)$$

The performance functional was chosen to be of the same form as Eq. (1.1.7), with

$$Q = (HA_o - LH)^T Q_y(HA_o - LH) \tag{3.2.11}$$

$$Q_y = \text{Diag}(.01, .1, .01, .1) \tag{3.2.12}$$

$$R = 2500 \tag{3.2.13}$$

The controller gain matrices

$$C = [C_{11} \quad C_{12} \quad C_{13} \quad C_{14} \quad C_{15}] \tag{3.2.14}$$

calculated using each of the design methods along with the required CPU times,are presented in Table 3.1. MDEC(21) and MDEC(6) denote the minimum discrete expected cost method using 21 and 6 equally spaced and equally weighted points in the region of parameter uncertainty Ω .

Table 3.1 Controller Feedback Gains and CPU Time for the Various Design Methods — Fifth-Order Example, One Uncertain Parameter

Design Method	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	CPU Time (Sec)
LQR	-.0029	-.084	1.80	.013	.34	0.8
GCC	-.017	-.10	2.84	-.023	4.96	3.7
MGCC	-.0065	-.075	1.83	.012	1.24	33.3
MDEC(21)	-.0091	.093	.95	.081	.26	16.4
MDEC(6)	-.0090	.11	.84	.083	.28	7.3
MM			No Solution			
UW			No Solution			

Closed-loop eigenvalues and the trace of the cost matrix when $C_{n_{\delta_a}}$ assumes its nominal value (ω = 0) associated with the various design methods are presented in Table 3.2. In Table 3.3, the three dominant closed-loop eigenvalues are given for the extreme values of ω for the various design methods.

Table 3.2 Closed-Loop Eigenvalues and Trace of Cost Matrix at Nominal ω for the Various Design Methods

Design Method	Closed-Loop Eigenvalues	Tr[S(0,C)]
LQR	-19.74, -10.96, -3.30±5.23j, -.62	1660
GCC	-117.1, - 9.90, -1.57±6.51j, -.10	2359
MGCC	-41.28, -10.05, -2.14±6.22j, -.29	1803
MDEC(21)	-15.92±10.60j, -2.00±7.31j, -.36	1998
MDEC(6)	-16.27±10.78j, -1.86±7.38j, -.34	2096

The standard linear-quadratic regulator approach again produces an unstable closed-loop system at an extreme value of the uncertain parameter. The guaranteed cost control, multistep guaranteed cost control, and expected cost methods produce closed-loop systems which are stable at the nominal and extreme parameter values. The guaranteed cost control approach produces larger feedback gains than the other methods and requires a very fast actuator (pole at -117.1). For the multistep guaranteed cost control method, the parameter μ was chosen to be 0.0001, and the sequence (ρ_j, $j=1,\cdots,6$) was chosen to be 0.375, 0.75, 0.8125, 0.875, 0.9375, 1.0. The undesirable features of the guaranteed cost control design were avoided using the multistep guaranteed cost control approach. The discrete expected cost designs based on 6 and 21 points were both quite satisfactory and not significantly different from each other. The minimum discrete expected cost methods were less costly

Table 3.3 Dominant Closed-Loop Eigenvalues at Extreme Values of ω for the Various Design Methods

Design Method	Dominant Closed-Loop Eigenvalues $\omega = -1.5$	$\omega = 0.5$
LQR	+.13±5.08j, -.38	-5.17±4.27j, -.70
GCC	-.48±5.65j, -.08	-1.94±6.74j, -.11
MGCC	-.34±5.40j, -.21	-2.86±6.36j, -.30
MDEC(21)	-.91±5.10j, -.39	-2.41±7.87j, -.35
MDEC(6)	-.97±5.17j, -.36	-2.19±7.95j, -.33

computationally than the multistep guaranteed cost control approach in this example. The 33 seconds of CPU time associated with the latter approach is still not particularly expensive, though. The minimax and uncertainty weighting methods failed when applied to the problem, since there is no $\omega_w \in \Omega$ consistent with Eqs. (3.1.3) and (3.1.4).

The discussion of the results obtained by the guaranteed cost control method and the multistep guaranteed cost control method was based on the assumption that $C_{n_{\delta_a}}$ is an unknown constant within certain bounds.

It should be noted that the multistep guaranteed cost control algorithm would have produced the same feedback gains if $C_{n_{\delta_a}}$ were assumed to vary in some arbitrary manner within this range. The resulting closed-loop system would be asymptotically stable for all such variations. Thus, for cases in which $C_{n_{\delta_a}}$ is a function of angle of attack, for example, which varies with time, this controller design would still be applicable (as long as the small perturbation model for the lateral dynamics is valid.)

Let us reconsider the design of a lateral autopilot for the remotely piloted vehicle described above, with two uncertain parameters now, instead of one. In addition to uncertainty in $C_{n_{\delta_a}}$, we assume now Y_p, the side force due to roll rate, to be uncertain.

Let

$$22.90 \leq Y_p \leq 28.00 \qquad (3.2.15)$$

Thus, Eq. (3.2.2) is replaced by

$$A(\omega) = A_o + \omega_1 A_1 + \omega_2 A_2 \qquad (3.2.16)$$

with A_o, A_1, ω_1 remaining as before, and

$$A_2 = \begin{bmatrix} 0 & 2.55 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3.2.17)$$

and

$$-1 \leq \omega_2 \leq 1 \quad , \qquad p(\omega_2) = .5 \qquad (3.2.18)$$

The weighting matrices Q and R remain the same as in Eqs. (3.2.11)-(3.2.13).

The controller gain matrix calculated using the standard linear quadratic approach remains as in Table 3.1 . In this approach the addition of a second uncertain parameter makes no difference.

The minimax method and the uncertainty method failed again, since there is no $\omega_w \in \Omega$ consistent with Eqs. (3.1.3) and (3.1.4). In Table 3.4, the controller gain matrices, of the form (3.2.14), calculated using the guaranteed cost control design method, the multistep guaranteed cost control method, and the minimum discrete expected cost method, are presented. MDEC(36) and MDEC(4) denote the minimum discrete expected cost method using 36 and 4 equally spaced and equally weighted points in the region of parameter uncertainty Ω .

Table 3.4 Controller Feedback Gains and CPU Time for the Various Design Methods — Fifth-Order Example with Two Uncertain Parameters

Design Method	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	CPU Time (sec)
GCC	-.037	-.18	5.55	-.10	10.69	14.8
MGCC	-.012	-.084	2.37	-.006	2.93	69.5
MDEC(36)	-.0089	.11	.84	.084	.27	74.6
MDEC(4)	-.0082	.15	.57	.11	.37	13.4

Closed-loop eigenvalues and the trace of the cost matrix, when $C_{n_{\delta_a}}$ and Y_p assume their nominal values ($\omega = 0$), associated with the various design methods, are presented in Table 3.5. The three dominant closed-loop eigenvalues are given in Table 3.6 for the extreme values of ω for the various design methods.

In carrying out the multistep guaranteed cost control design, a predetermined sequence ρ_j, $j=1,\cdots,6$ was chosen: 0.375, 0.75, 0.8125, 0.875, 0.9375, 1.0. The parameter μ was taken to be 10^{-4}.

Table 3.5 Closed-Loop Eigenvalues and Trace of Cost Matrix at Nominal ω for the Various Design Methods

Design Method	Closed-Loop Eigenvalues	Tr[S(0,C)]
GCC	-231.6, -9.95 -1.60±6.56j, -.06	2801
MGCC	- 76.0, -10.0, -1.79±6.47j, -.15	2057
MDEC(36)	-16.23±10.88j, -1.86±7.38j, -.34	2093
MDEC(4)	-17.55±11.39j, -1.47±7.43j, -.33	2443

Table 3.6 Dominant Closed-Loop Eigenvalues at Two Extreme Values of ω for the Various Design Methods

Design Method	Dominant Closed-Loop Eigenvalues $\omega = \begin{bmatrix} -1.5 \\ -1 \end{bmatrix}$	$\omega = \begin{bmatrix} .5 \\ 1 \end{bmatrix}$
GCC	-.49±5.58j, -.05	-1.98±6.77j, -.06
MGCC	-.42±5.50j, -.12	-2.26±6.74j, -.15
MDEC(36)	-.98±5.16j, -.37	-2.16±7.95j, -.33
MDEC(4)	-1.06±5.38j, -.34	-1.63±7.98j, -.34

The CPU time for the minimum discrete expected cost method with 36 points in Ω is 10 times larger than the time for the same design with one uncertain parameter and 6 points in Ω. The CPU time for the multistep guaranteed cost control method grew by a factor of 2. The guaranteed cost control method produces larger feedback gains than the other methods and results in an overcontrolled closed-loop system with one pole at -232 and another at -.06. The minimum discrete expected cost method produces the smallest feedback gains and the best closed-loop pole locations. One should note that the feedback gains produced by MDEC(36) are very similar to the feedback gains produced by MDEC(21) in Table 3.1, which may indicate that the second uncertain parameter has no great influence on the design. Although the feedback gains of the multistep guaranteed cost control method are smaller than those of the guaranteed cost control method, they are larger than those produced by the minimum discrete expected cost method and require a faster actuator (pole at -76.0). The minimum discrete expected cost designs based on 4 (only the corner points included) and 36 points were both satisfactory and not significantly different from each other.

In applying the multistep guaranteed cost control approach to this example, a predetermined sequence of ρ_j was used in step 9 in the MGCC algorithm (Section 1.4), instead of ρ_j produced by step 6. In order to use step 6, the state weighting matrix Q was modified according to the procedure described in Appendix B. The relative influence of $C_{n_{\delta_a}}$ and Y_p on the nominal closed-loop design was evaluated, and it was found that while $C_{n_{\delta_a}}$ is an important parameter for the design, Y_p has no great influence on the closed-loop design. Thus, a design based on uncertainty in $C_{n_{\delta_a}}$ only, will be applicable for cases when there is uncertainty in Y_p within certain bounds.

It should be noted that an analysis of the lateral dynamics of this specific vehicle shows that Y_p is not an influential parameter. Thus, the multistep guaranteed cost control method enables us to confirm this analysis, and can be used to determine the relative importance of uncertain parameters in complex systems.

3.3 Evaluation of Design Methods

In this section we describe the results of a qualitative and quantitative evaluation of the five controller design methods that were compared in this work. A design method will be judged to be superior to another one when it produces a more desirable closed-loop system behavior, and the level of effort that is required in the design does not compromise its usefulness.

Several criteria which reflect important capabilities that a design method should possess are defined below for our evaluation. Throughout the following discussion we refer to systems which follow the representation of Section 1.1 or Section 2.1.

1. Information Required Regarding Uncertainty

This item refers to what type of information about parameter uncertainty is required by the design method. The guaranteed cost control method, the multistep guaranteed cost control method, the minimax method, and the uncertainty weighting method require only an a priori knowledge of the bounded region of parameter uncertainty Ω. The minimum discrete expected cost method requires and makes explicit use of a weighting function throughout Ω, which may be chosen to correspond to a probability density function.

The guaranteed cost control method and the multistep guaranteed cost control method accept a time-varying uncertain parameter vector $\omega(t) \in \Omega$ without requiring explicit knowledge of $\omega(t)$. All the other methods accept only a time invariant vector ω.

2. Computational Load

All the design methods discussed here deal with an optimal controller design for multivariable systems via digital computer, this item refers to the computational load required by each method. In cases where ω_W exists, the minimax and the uncertainty weighting methods require the least amount of computation. The identification of ω_W requires the solution of an algebraic Riccati equation at each vertex of Ω, in conjunction with solution of a Lyapunov equation at all the other vertices. The solution of these matrix equations is well documented and is very efficient. Clearly, the computational load will increase considerably as the number of uncertain parameters n' increases.

The computational load is considerably higher for the guaranteed cost control and the multistep guaranteed cost control methods, especially for the latter. The design process requires the iterative solution of a modified Riccati equation with an extra term U(S) that needs to be evaluated each iteration. For the multistep guaranteed cost control method this procedure is repeated several times for an increasing range of parameter uncertainty. The number of uncertain parameters n' influences only the construction of U(S), which is not very costly computationally. Thus, those two methods are more sensitive, from the point of view of computational effort, to the order of the system n than to the number of uncertain parameters n'.

The minimum discrete expected cost method seems to require the largest computational load in general. It requires the solution of a Lyapunov equation and an adjoint Lyapunov equation at each point considered in parameter space, for the cost and gradient evaluation. Then, a numerical minimization procedure must be used in order to determine C*. The computational load for this method increases considerably as the number of points selected to represent the range of uncertainty is increased. Thus, this method will be more sensitive to the number of uncertain parameters n' than to the order of the system n.

3. Treatment of Engineering Design Criteria

As mentioned above, for a design method to be acceptable, it has to produce a closed-loop system that has various desired features. These features include stability of the closed-loop system for all possible values of uncertain parameters, a satisfactory transient response at nominal and off-nominal points, and an acceptable level of control effort. Of all the methods considered here, only the uncertainty weighting method does not explicitly ensure stability of the closed-loop system for all possible values of ω. For the minimax method, when ω_W exists the stability criterion is satisfied. As was shown in our fifth-order example, however, such an ω_W does not always exist. The guaranteed cost control, multistep guaranteed cost control, and minimum discrete expected cost methods provide a stable closed-loop system whenever a solution to the design problem exists.

The transient response criterion is dealt with only at the ω_W point in the minimax design method. The uncertainty weighting method deals with the transient response only at the nominal point. The multistep guaranteed cost control method can emphasize transient response at nominal operating conditions if μ is chosen to be small. The minimum discrete expected cost method can be used to shape the transient response in any one portion of the parameter uncertainty range. The guaranteed cost control design method does not directly deal with transient response.

As was shown in our examples in Section 3.2 and in [22],[23], the minimax method and the guaranteed cost control method tend to produce relatively large controller feedback gains, relatively large control effort, and overdamped closed-loop poles. The three other methods, when a successful design can be achieved, tend to produce reasonable controller feedback gains and control effort.

4. Insight into Design Problem

We feel that a good design method should provide the designer with some data that will shed light on critical design problems. The multistep guaranteed cost control method is the only method among those considered here which satisfies this criterion. Evaluation of ρ_j (Eqs. (1.4.3)-(1.4.5)) gives the range of parameter uncertainty for which the design is stable or optimal. The determination of ρ_j can be done for each uncertain parameter individually, assuming the other parameters to be precisely known. This enables the designer to evaluate the relative importance of each parameter among a group of parameters. Among the other methods, the minimax method gives the worst operating condition of the system, which may be of interest to the designer.

5. Generality of Design Problem

Certain assumptions must hold in order to apply each design method. The more general a problem that can be handled by a design method, the greater its versatility. Among the methods considered here, the minimum discrete expected cost method will accept the most general problem. There are no restrictions on the structure of the system, nor on the way the uncertain parameters influence the system dynamics. The range of parameter uncertainty Ω can have any shape in $R^{n'}$. The design method can handle full state feedback and output feedback without any modification or change in the procedure. The minimax method, and consequently the uncertainty weighting method that uses ω_w found by the minimax procedure, require the structure of $A(\omega)$ to follow Eq. (1.1.5) and restrict Ω to be a polygon in $R^{n'}$. These requirements are needed in order to reduce the computational burden in determining ω_w [19]. These methods handle the full state feedback cases, but the output feedback case requires modification of the procedures. The guaranteed cost control method and the multistep guaranteed cost control method require $A(\omega)$ to satisfy Eq. (1.1.5) and restrict Ω to be a polygon in $R^{n'}$. These requirements are needed in order to construct $U(S)$. Full state feedback cases are handled by these two methods. Although output feedback cases can be handled also (Section 1.2), this seems impractical because of computational requirements.

6. General Comments

It is clear that each of the design methods discussed here has advantages and disadvantages. We will comment here on some of the features of each method that have not been discussed above.

The minimax method: In one of our examples, a point ω_w which satisfies Eqs. (3.1.3) and (3.1.4) does not exist. If there exists a subset $\mathcal{C} \in R^{m \times n}$, such that for every $C \in \mathcal{C}$ the closed-loop system matrix $[A(\omega) - BC]$ is stable for every $\omega \in \Omega$, and one seeks $C^* \in \mathcal{C}$ so as to satisfy Eqs. (3.1.3) and (3.1.4), then such an ω_w may exist. When this restriction in the allowable gain set is imposed, the solution becomes more complicated. One cannot use the standard linear-quadratic approach at each vertex of Ω, but must instead use a numerical minimization technique which increases the computational load considerably.

The uncertainty weighting method: The use of this method requires the existence of and determination of ω_W. This method usually [19],[23] produces a better closed-loop design than the minimax method, without substantial increase in the computational load.

The guaranteed cost control method: As has been demonstrated throughout this work, this method tends to produce an overcontrolled closed-loop system. Computationally it requires less time than the multistep guaranteed cost control method and can be used as a way to get a set of gains that stabilize the closed-loop system for all possible values of ω (assuming that such a set exists). This method does not require any initial gain matrix for starting the optimization procedure, and can treat constant uncertain parameter vector as well as time-varying vector.

The multistep guaranteed cost control method: In all the examples we have studied, this method has been found to perform well. Although the computational load is high compared to that of some of the other methods, it is by no means objectionable. This method seems to be attractive for low dimensional systems, but with a large number of uncertain parameters. Like the guaranteed cost control method, this method does not require an initial gain and can treat constant as well as time-varying uncertain parameter vectors.

The minimum discrete expected cost method: The main problem with this method is the requirement for an initial gain matrix C_0 for starting the minimization procedure. This gain matrix C_0 should stabilize the closed-loop system for all possible values of ω. It is not clear how to find such a matrix, in general. This method seems more practical for high dimensional systems with a small number of uncertain parameters.

A summary of the evaluation of the five design methods investigated here versus these five criteria is shown in Table 3.7.

Table 3.7 Evaluation of Design Methods

Criterion / Method	(1) Information Regarding Uncertainty Required	(2) Computational Load	(3) Treats Engineering Design Criteria			(4) Provides Insight	(5) Generality	(6) Comments
			Stability	Good Transient Response	Control Effort			
Minimax	Range	Low	Yes	Only at ω_w	High	Only about ω_w	Restrictions on Ω, $A(\omega)$	ω_w does not always exist
UW	Range	Low (+)	No	Only at $\omega=0$	Reasonable	No	Restrictions on Ω, $A(\omega)$	Requires knowledge of ω_w. May have to try several times before stability achieved
GCC	Range	Moderate	Yes	No	Very high	No	Restrictions on $\Omega, A(\omega)$; Accepts $\omega(t)$	Suitable for producing initial gain for MDEC
MGCC	Range	High	Yes	at $\omega=0$; some over the wide range	Reasonable	Yes	Restrictions on $\Omega, A(\omega)$; Accepts $\omega(t)$	Does not need initial gains Useful especially at low n, high n′
MDEC	Range, Weighting function	High	Yes	At any point	Reasonable	Some	Almost no restrictions	Needs initial gains; Useful especially at high n, low n′

Conclusion

In this thesis two new techniques for optimally designing constant gain feedback controllers for linear systems with large parameter uncertainty have been developed. The first is a generalized form of the guaranteed cost control method, based on general theorems derived here, which allows treatment of more general systems than can be handled with the previously published guaranteed cost control approach of Chang and Peng [21]. Output feedback cases can be handled and the control distribution matrix is allowed to have a more general form of parameter uncertainty. This generalization eliminates certain limitations such as restrictions on the shape of the uncertain parameter region, for example, and thus makes the method more readily usable for engineering designs.

The new theory is accompanied by development and demonstration of a computational procedure. An algorithm has been developed to analyze the effect of parameter uncertainty on closed-loop system stability. An extension of this algorithm, based on the multistep guaranteed cost control method, produces constant gain feedback controllers which result in a stable closed-loop system for all values of the uncertain parameters within some bounded range, the extent of which is easily determinable. Multistep guaranteed cost control designs are shown to overcome the over-controlled behavior often associated with the guaranteed cost control designs and to possess some of the desirable features (infinite gain margin, phase margin of at least 60°, etc.) associated with conventional optimally designed systems assuming fixed parameters.

The minimum discrete expected cost method is the second method developed in this work. It is based on the minimum expected cost approach due to Ly and Cannon [14]. The minimum discrete expected cost method is shown to be an easily implementable method. Only a modest number of points in the range of parameter uncertainty is needed to achieve a satisfactory design. An algorithm for the implementation of this method is developed, and two examples are carried out to demonstrate its usefulness.

Chapter 3 presents an extensive comparison and evaluation of the two new methods, together with other recommended techniques for control law design for systems with uncertain parameters: the minimax method, the uncertainty weighting method, and the guaranteed cost control method. Specifically, the design of constant gain feedback controllers for a fifth-order lateral autopilot for an RPV with one or two uncertain parameters is performed using the five different methods. Two of the methods, the minimax method and the uncertainty weighting method, which had been recommended based on earlier tests [19], failed when applied to the fifth-order examples in this study. The reason is that in our tests there is no point in the specified region of parameter uncertainty with the desired minimax property. Of the three other methods considered in the comparison, the guaranteed cost control method produced an unacceptable control system design due to the large gains and large control effort required. The two new methods, the multistep guaranteed cost control method and the minimum discrete expected cost method, both produced acceptable designs without any objectionable computational load.

A detailed evaluation of each of the five methods with regard to five criteria is done in Chapter 3. Recommendations regarding the use of each method are presented.

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Appendix A

CONSTRUCTION OF U(S) FOR A NON-RECTANGULAR REGION Ω

One limitation of the guaranteed cost control design method and the multistep guaranteed cost control method is the requirement that Ω be a rectangle in $R^{n'}$, Eq. (1.1.3). This requirement is needed for the construction of U(S) as defined by Eqs. (1.2.18)-(1.2.20). This construction method was suggested in [21]. Another method was suggested in [28], for which Eq. (1.1.3) must also be satisfied. In some engineering systems this restriction on the shape of Ω can limit the applicability of these design methods. In this appendix we will relax the restriction on Ω and allow it to be any convex polygon in $R^{n'}$.

Throughout this appendix the assumptions of Section 1.1 hold. Let us assume Ω to be a closed bounded convex polygon in $R^{n'}$ with ℓ vertices which includes the origin. Thus Eq. (1.1.3) is replaced by

$$P^T\omega(t) + b \leq \underset{\sim}{0} \qquad t \in [0,t_1] \tag{A.1}$$

where P is a constant $(n' \times \ell)$ matrix and b is a constant $(\ell \times 1)$ vector and

$$b_i \leq 0 \ , \qquad i = 1,\cdots,\ell \tag{A.2}$$

Let us seek a matrix functional U(S) such that

$$x^T U(S) x \geq x^T [\sum_{i=1}^{n'} \omega_i(t)(SA_i + A_i^T S)]x \ , \quad \omega(t) \in \Omega, \quad t \in [0,t_1] \tag{A.3}$$

holds for every $x \in R^n$. Let us define

$$L(\omega,x) = x^T[\sum_{i=1}^{n'} \omega_i(t)\ (SA_i + A_i{}^T S)]\, x \tag{A.4}$$

Clearly, $L(\omega,x)$ is a scalar function, linear in ω_1 and quadratic in x. It is known [50], that the maximum of a linear function defined over a polygon will occur at one of the vertices of the polygon. Thus, the maximum of $L(\omega,x)$ with respect to ω will be at one of the vertices of Ω. For different x's, the maxima can occur at different vertices. In order to construct U(S) such that (A.3) holds for every $x \in R^n$, the following procedure is suggested:

Let us denote the vertices of Ω as ω^j, $j=1,\cdots,\ell$. Let

$$K^j = \sum_{i=1}^{n'} \omega_i^j(t)\ [SA_i + A_i^T S]\ , \qquad j=1,\cdots,\ell \tag{A.5}$$

thus

$$L(\omega^j,x) = x^T K^j x\ , \qquad \forall\ x \in R^n \tag{A.6}$$

Let us define

$$U^o = 0 \tag{A.7}$$

$$U^j = K^j + N_j E_j N_j^T\ , \qquad j=1,\cdots,\ell \tag{A.8}$$

where N_j is the orthogonal transformation which diagonalizes the symmetric matrix $(U^{j-1} - K^j)$:

$$N_j^T(U^{j-1} - K^j)N_j = \Lambda_j, \qquad j=1,\cdots,\ell \tag{A.9}$$

The matrix E_j is defined by

$$\left\{\begin{array}{ll} (E_j)_{kk} = \begin{cases} 0 & , \ (\lambda_j)_k < 0 \\ (\lambda_j)_k & , \ (\lambda_j)_k \geq 0 \end{cases} \\ \\ (E_j)_{ik} = 0 & , \ i \neq k \end{array}\right\} \qquad (A.10)$$

where $(\lambda_j)_k$ is the k^{th} eigenvalue of $(U^{j-1} - K^j)$. Let

$$U(S) = U^{\ell} \qquad (A.11)$$

It remains to be shown that U(S) so constructed will satisfy Eq. (A.3) for all $\omega \in \Omega$ and all $x \in R^n$. By construction,

$$U(S) = U^{\ell} \geq K^j \qquad , \quad j = 1, \cdots, \ell \qquad (A.12)$$

(For two symmetric natrices A and B with the same dimension, $A \geq B$ means that $C = A-B$ is positive-semidefinite matrix). This can be seen when one notes that $N_j E_j N_j^T$ is a positive-semidefinite matrix for $j = 1, \cdots, \ell$ by construction.

Eqs. (A.5), (A.6), and (A.12) yield

$$x^T U(S) x \geq L(\omega^j, x) \qquad , \quad j = 1, \cdots, \ell \ , \qquad x \in R^n \qquad (A.13)$$

But for some j

$$L(\omega^j, x) \geq x^T [\sum_{i=1} \omega_i(t)(SA_i + A_i^T S)] x \ , \quad \omega(t) \in \Omega, \quad x \in R^n \qquad (A.14)$$

Eqs. (A.13) and (A.14) prove that U(S) as constructed here satisfies Eq. (A.3).

Appendix B

A STATE WEIGHTING MATRIX FOR THE MULTISTEP GUARANTEED COST CONTROL ALGORITHM

One of the features of the multistep guaranteed cost control algorithm is the ability to investigate the effects of parameter uncertainty on closed-loop system stability. Determination of ρ in Eq. (1.4.3) provides this information. A difficulty in this procedure may occur when Q, the state weighting matrix, is a positive-semidefinite matrix (instead of being a positive-definite one). In these cases, it may well be that the smallest value of $\rho \geq 0$ which makes $\bar{Q}$ in Eq. (1.4.3) indefinite, will be found to be zero. One should recall that the choice of Q was done so as to produce a satisfactory nominal closed-loop design. A way to modify this state weighting matrix so that the design algorithm can provide useful information about closed-loop stability is suggested here.

Let Q be a symmetric matrix defined by

$$Q = Q + \varepsilon NEN^T \quad , \quad 1 \gg \varepsilon > 0 \tag{B.1}$$

where ε is an arbitrary chosen real constant, and N is the orthogonal transformation which diagonalizes the symmetric matrix $[U(S_0) - Q]$:

$$N^T[U(S_0) - Q]N = \Lambda \tag{B.2}$$

The matrix E is defined by

$$\left\{\begin{array}{l} (E)_{kk} = \begin{cases} 0 , & \lambda_k < 0 \\ \lambda_k , & \lambda_k \geq 0 \end{cases} \\ \\ (E)_{ij} = 0 , \quad i \neq j \end{array}\right\} \tag{B.3}$$

where λ_k is the k^{th} eigenvalue of $[U(S_o) - Q]$.

$\hat{Q}$ so defined will assume the role of the state weighting matrix $\hat{Q}$ in the multistep guaranteed cost control algorithm. It should be noted that $\hat{Q}$ is a positive-definite matrix whenever $U(S_o)$ is such a matrix. This enables us to use in step 9, the values of ρ produced by step 6 of the algorithm (Section 1.4).